Dijkstra's Algorithm Verification

Yazhe Feng

February 17, 2019

1 Dijkstra's Algorithm

1.1 Pseudocode

Given input graph g and source node s with types:

g : Graph gsize weight
s : Node gsize

We denote (u, v) as an edge from node u to v, $weight(u, v)$ as the weight of edge (u, v). We define *unexplored* as the list of unexplored nodes, and *dist* as the list storing distance from s to each node $n \in g$

(initially *unexplored* contains all nodes in graph g)
$unexplored : List(Node\ \ gsize)$
$unexplored = \{v : v \in g\}$

(node value is used to index *dist*, initially distance of all nodes are infinity except the source node)
$dist : List\ \ weight$
$dist[s] = 0, dist[a] = infinity, \forall a \in g, a \neq s$

The Dijkstra's Algorithm runs as follows:

```
while (unexplored is not Nil)      {
     (At the k^th iteration of the while loop)
     choose u ∈ unexplored s.t.∀u' ∈ unexplored, dist[u] ≤ dist[u']
     let unexplored' be the list after removing u from unexplored
     for(∀v ∈ g s.t.(u, v) ∈ g) {
          (At the p^th iteration of this for loop)
          if(dist[u] + weight(u, v) < dist[v]) {
               let dist' = dist with dist'[v] = dist[u] + weight(u, v)
          }
          input the new dist' to the (p + 1)^th iteration of the for loop
     }
     input the new unexplored' and dist' to the k^th iteration of the while loop
}
```

1.2 Assumptions

1. Weight of edges are positive
2. Distance value can only be zero, infinity, or summation of edge weights
3. All nodes n and edge e are valid: $n, e \in g$

2 Definition

Definition 2.1. Path
(We adopt the definition of path presented in the `Discrete Mathematics with Applications` book by `SUSANNA S. EPP`.)

A path from node v to w is a finite alternating sequence of adjacent vertices and edges of G, which does not contain any repeated edge or vertex. A path from v to w has the form:

$$ve_0v_0e_1v_2....v_{n-1}e_nw$$

where e_i is an edge in g with endpoints v_{i-1}, v_i. We denote the set of paths from v to w as $path(v, w)$.

Definition 2.2. Length of Path
The length of a path $p = ve_0v_0e_1v_2....v_{n-1}e_nw$ is the sum of the weights of all edges in p. We write:

$$length(p) = \sum weight(e_i), \forall e_i \in p.$$

Definition 2.3. Shortest Path
Denote $\Delta(s, v)$ as the shortest path from s to v, and $\delta(v)$ as the length of $\Delta(s, v)$. $\Delta(s, v)$ must fulfills:

$$\Delta(s, v) \in path(s, v)$$
and
$$\forall p' \in path(s, v),\ \delta(v) = length(\Delta(s, v)) \leq length(p')$$

3 Proof of Correctness

3.1 Proof of Termination

The inner for loop is guaranteed to terminate as the algorithm goes through each adjacent node exactly once. As the size of list `unexplored` decreases by one during each iteration of the while loop, the algorithm is guaranteed to terminate.

3.2 Proof of Correctness

Given graph g and source node s, *dist* stores the distance value from s to all nodes in g calculated by the Dijkstra's algorithm, $dist[v]$ gives the corresponding distance value of v from s. Denote *explored* as the list of nodes in g but not in *unexplored*, i.e., *explored* stored all nodes whose neighbors have been updated by the algorithm, and $dist_k[v]$ as the value of $dist[v]$ during the k^{th} iteration of the algorithm.

Lemma (1)**.** During the n^{th} iteration of the algorithm for $n \geq 1$, forall node $v \in explored$, we have:

1. $\delta(v) \leq \delta(v')$, $\forall v' \in unexplored$.
2. $dist_n[v] = \delta(v)$

Proof. We will prove this by inducting on the number of iterations.

Let P(n) be: during the n^{th} iteration of the algroithm for $n \geq 1$, forall node $v \in explored$: (1) $\delta(v) \leq \delta(v')$, $\forall v' \in unexplored$; and (2) $dist_n[v] = \delta(v)$.

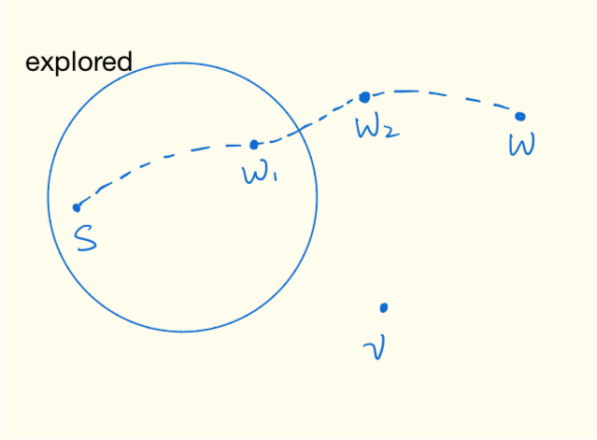
Base Case: We shall show P(1) holds
Based on the algorithm, during the first iteration, the node with minimum distance value is the source node s with $dist_1[s] = 0$. Hence during the first iteration, only s is removed from *unexplored* and added to *explored*. Since all edge weights are positive, then the shortest distance value from s to s is indeed 0, hence $dist_1[s] = 0 = \delta(s)$ and $\delta(s) \leq \delta(v')$, $\forall v' \in unexplored$. P(1) holds.

Inductive Hypothesis: Suppose P(i) is true for all $1 < i \leq k$. That is, during the i^{th} iteration forall $1 < i \leq k$, forall node $v \in explored$: (1) $\delta(v) \leq \delta(v')$, $\forall v' \in unexplored$; and (2) $dist_i[v] = \delta(v)$.

Inductive Step: We shall show P(k+1) holds.
Suppose v is the node added into *explored* during the $(k+1)^{th}$ iteration. We need to show (1) $\delta(v) \leq \delta(v')$, $\forall v' \in unexplored$, and (2) $dist_{k+1}[v] = \delta(v)$.

1. $\delta(v) \leq \delta(v')$, $\forall v' \in unexplored, v' \neq v$

   We will prove (1) by contradiction. Suppose there exists $w \in unexplored$, such that $\delta(v) > \delta(w)$.
   Based on the definition of shortest path, $\delta(v) \leq dist_{k+1}[v]$. Since $\delta(v) > \delta(w)$ and $\delta(v) \leq dist_{k+1}[v]$, then we have $\delta(w) < dist_{k+1}[v]$([1]).
   Let p_w be the shortest path from s to w, i.e., $\delta(w) = length(p_w)$. Since $w \notin explored$ during the $(k+1)^{th}$ iteration, then there must exists some node along p_w that are not in *explored*. Suppose during the $(k+1)^{th}$ iteration, the first node in p_w that is not in the *explored* list is w_2, and the node right before w_2 in the s to w_2 subpath is w_1, thus $w_1 \in explored$ during the $(k+1)^{th}$ iteration. The image below illustrates this construction:



Denote the subpath from s to w_1 in p_w as $p(s, w_1)$, subpath w_1 to w_2 as $p(w_1, w_2)$, and subpath w_2 to w as $p(w_2, w)$. Since w_1 is right before w_2 in p_w, then $length(p(w_1, w_2)) = weight(w_1, w_2)$. Then:

$$\delta(w) = length(p_w) = length(\Delta(s, w_1)) + weight(w_1, w_2) + length(w_2, w)$$

Since all edge weights are positive, then:

$$length(\Delta(s, w_1)) + weight(w_1, w_2) \leq \delta(w)$$
i.e., $\delta(w_1) + weight(w_1, w_2) \leq \delta(w)$

Since during the $(k+1)^{th}$ iteration, $w_1 \in explored$, then w_1 must be added into $explored$ with all neighbors of w_1 updated during the i^{th} iteration for some $i < k+1$. Then based on our inductive hypothesis, $dist_{k+1}[w_1] = \delta(w_1)$. Since the value of $dist[w_1]$ remains unchanged after adding w_1 into $explored$, then $dist_i[w_1] = dist_{k+1}[w_1] = \delta(w_1)$.
Since w_1 is the node right before w_2 in p_w during the $(k+1)^{th}$ iteration, then $dist_{k+1}[w_2] = dist_i[w_1] + weight(w_1, w_2) = \delta(w_1) + weight(w_1, w_2)$. Since $\delta(w_1) + weight(w_1, w_2) \leq \delta(w)$, then $dist_{k+1}[w_2] \leq \delta(w)$([2]).
Combining [1] and [2], we have:

$$\delta(w) < dist_{k+1}[v]([1])$$
$$dist_{k+1}[w_2] \leq \delta(w)([2])$$

Hence $dist_{k+1}[w_2] < dist_{k+2}[v]([3])$.
Based on our assumption, during the $(k+1)^{th}$ generation, $w_2 \notin explored$ and v is selected by the algorithm, then we must have $dist_{k+1}[w_2] \geq dist_{k+1}[v]$, which contradicts with [3]. Hence by the principle of prove by contradiction, there does not exsist $w \in unexplored$, such that $\delta(v) > \delta(w)$. (1) holds for the $(k+1)^{th}$ iteration.

(Proof below are not modified yet)

2. $dist[v] = \delta(v)$

Suppose $dist[v]$ is associates with path $p \in path(s, v)$ during the k^{th} iteration, and assume the shortest path from s to v is some path $p' \in path(s, v)$ different than p, $length(p') = \delta(v) < dist[v]([b])$. Suppose v' is the node just before v in p'.

$$\delta(v) = dist[v'] + weight(v', v)$$

Since all edge weights are non-negative, then $dist[v'] < \delta(v)$. Based on (1), since $\delta(v) < \delta(w) \forall w \in unexplored$, then v' must be in $explored$. Since v' is in $explored$ and has an edge to v, then the algorithm must have compared $dist[v'] + weight(v', v)$ to the current $dist[v]$ and chose $dist[v]$. Hence it must be $dist[v'] + weight(v', v) \geq dist[v]$, i.e. $\delta(v) \geq dist[v]$, which contradicts with [b]. Hence by the principle of prove by contradiction, p is the shortest path from s to v, and that $dist[v] = \delta(v)$.

Since we proved both (1) and (2) for the k^{th} iteration, forall $k \leq 1$, we have proved that Lemma (1) holds. □

Proof. **Prove of Correctness**

By applying Lemma (1) to each iteration of the algorithm, we obtained that for all nodes n in the explored list, $dist[n]$ is indeed the shortest path distance value from source s to n, hence Dijkstra's algorithm indeed calculates the shortest path distance value from the source s to each node $n \in g$. $\square$