VERIFICATION OF DIJKSTRA'S ALGORITHM IN IDRIS

Yazhe Feng

A thesis submitted in partial fulfillment of the requirements for the degree of Bachelor of Arts

in the

Department of Computer Science

at Bryn Mawr College

Advisor: Richard A. Eisenberg

Acknowledgments

(To be finished...)

Abstract

Contents

1 Introduction

Shortest path problems are concerned with finding the path with minimum distance value between two nodes in a given graph. One variation of shortest path problem is single-source shortest path problem, which focuses on finding the path with minimum distance value from one source to all other vertices within the graph. Dijkstra's algorithm [1] is one of the most well-known single-source shortest path algorithms, and are implemented in various fields including network protocols and aritificial intelligence.

Given the importance of Dijkstra's in real-life applications, we are interested in verifying the implementation of both algorithms. We first provide concrete implementation for Dijkstra's algorithm, and then define functions with precise type signatures which carry specifications that should hold for the correct implementation, for instance returning the minimum distance value from the source to each node in the graph. Having these functions type checked will then ensure the correctness of our algorithm implementation. Our implementation uses the Idris functional programming language, which embraces powerful tools and features that makes program verification possible.

Specifically, our contributions are:

- Provide a concrete implementation of Dijkstra's algorithm in Idris.
- Offer a verification program for Dijkstra's algorithm written in Idris, which is available on this (`https://github.com/EileenFeng/algorithm_verification`). Although the proof of some lemmas are incomplete, we are confident that we can provide the complete implementation if granted more time.

The structure of this thesis is as follows. Section 2 describes the significance and value of algorithm verification, and reasons of choosing Idris as the language for verifying programs. Section 3 provides some background on Dijkstra's algorithm, follows up by briefly introduction on the Idris functional programming language. Section 4 includes an overview of our verification program, including definition of key concepts, assumptions made by our program, and details on the pseudocode and mathematical proof of Dijkstra's, which serves as important guideline in implementation our verification program. Section 5 covers more details of our verification program, including function type signatures and code of the proof for key lemmas. Section 6 discusses future work. Section 7 presents and compares related work, and section 8 gives a brief conclusion.

2 Motivation

Software bugs are generally undesirable, especially in safety-critical and mission-critical systems. Back in 1985, errors in programs that controlled the Therac-25 radiation therapy machine were responsible for causing patience death by giving massive overdose of raidations [1]. The Northeast Blackout in 2003 due to race condition in power control systems has affected more than 50 million people in 8 states, causing an estimated loss of over 4 billion dollars [2]. In practice,

[1] Therac-25 Wikipedia page

[2] (1) Northeast Blackout 2003 Wikipedia Page (2) The Economic Impacts of the August 2003 Blackout

people usually convince themselves that a program is probably correct through testing, however as Dijkstras emphasized back in 1970s, "Program testing can be used to show the presence of bugs, but never to show their absence!" [2]. Concerning the serious consequences that might be caused by software errors in real life applications, it is importnat to validate the actual behaviors of programs.

As computer programs can be considered as formal mathematical objects whose properties are subject to mathematical proofs, program verification aims to provide proofs of correctness for programs by using formal, mathematical techniques [3]. Common techniques in program verification include using proof systems, for instance the Why3 Platform [4] applies the SMT solver[3], and automatic verification techniques. Applications of program verification include the Compcert C Compiler, which is verified using machine-assisted mathematical proofs, and is considered exempt from miscompilation issues[4].

In this thesis we aim to present verification as a programming issue. We want to show that with certain functional programming languages, we can specify the expected bahaviors in function type signatures, and any incorrect function definitions will fail to type check. This not only indicates that program verification can be achieved at compilation level, but more importantly, presents a technique that enforces programmers to write programs that are correct by construction. We choose Dijkstra's algorithm as our target as since it is widely applied in many fields, such as artificial intelligence.

Based on the above motivations, we choose the Idris programming language for implementing our verification program [5]. Compare to other proof management systems, the Idris type checker is based on a smaller code base, which reduces the chance of introducing unexpected bugs into our verification program. Idris is a functional programming language with dependent types, which allows programmers to provide more precise description of function's expected behaviors through its type signature. As we plan to achieve verification with type checking, this feature is essential to our verification process. In addition, the compiler-supported interactive editing feature in Idris allows programmers to inspect functions based on their type and thus to use type as guidance for writing programs, which offers considerable assistance during our implementation. Section 3 covers more backgrounds on the Idris programming language.

3 Background

3.1 Introduction of Idris

Idris is a general-purpose functional programming language with dependent types. Many aspects of Idris are influenced by Haskell and ML. Features of Idris include but not limit to dependent types, `with` rule, `case` expression, and interactive editing.

Variables and Types

Idris requires type declarations for all variables and functions defined. To define a variable, we provide the type on one line, and specify the value on the next line. Below presents the syntax

[3] information on SMT solver
[4] main page of Compcert C

for variable declaration.

```
<variable_name> : <type>
<variable_name> = <value>
```

The example below defines a variable `n` of type `Int` with value 37.

```
n : Int
n = 37
```

Types in Idris are first-class values, which means types can be operated as any other values. Type declaration is the same as declaring any other variables, with exactly the same syntax, except that the type of a type is `Type`. By convention, variables that represent types are capitalized. Below example declares a type `CharList`, which denotes the type of list of characters.

```
CharList : Type
CharList = List Char
```

`CharList` is a type that stands for `List` of `Char`s, and declaring a variable of type `CharList` is the same way as we declare a variable of type `List Char`. The following example declares a variable `lisChar` of type `CharList`. `lisChar` contains the characters for the English word "hello".

```
lisChar : CharList
lisChar = ‘h’ :: ‘e’ :: ‘l’ :: ‘l’ :: ‘o’ :: Nil
```

Function

To define a function a Idris, the types for all input values and output values must be specified in the function type signature, connecting by right arrows. Specifically, function type is of the form:

<func_name> : x_1 -> x_2 -> ... -> x_n

where $x_1, x_2, ..., x_{n-1}$ are types for the input values, and x_n is the output type of the function. Input values can be named to provide more information, and also allows each input to be referred to easily later. For instance the type of the `reverse` function below names the first input of type `Type` as `elem`, which specifies that the input and output lists contain elements of same type.

```
-- "reverse" reverse a list
reverse : (elem : Type) -> List elem -> List elem
```

An example of calling `reverse` is provided below. The variable `nats` has type `List Nat`. When calling `reverse` on `nats`, the first argument of `reverse` denotes the type of the input list and output list, which is `Nat` in this case, then the output of `(reverse Nat nats)` is also of type `List Nat`, as specified by the type of `reverse_nats`.

```
nats : List Nat
nats = 3 :: 2 :: 1 :: Nil

reverse_nats : List Nat
reverse_nats = reverse Nat nats
```

A function definition is provided on the line below the function type. In Idris, functions are defined by pattern matching, which will be elaborated on later. Here we provide an example for function definition that requires little experience with pattern matching, only aiming to illustrate the syntax for defining functions. The `mult` function defined below multiplies the two input integers.

```
-- calculates the multiplication of two input integers 'n' and 'm'
mult : Int -> Int -> Int
mult n m = n * m
```

Data Types

User defined data types are supported in Idris. To define a data type, we need to provide the name and type of the data type starting with the keyword `data`, followed by the id and the type of the data type. On the next few lines we define the constructors for this data type. Below provides the definition of the natural number type `Nat` in Idris.

```
-- natural number can be either zero, written as 'Z', or the
   successor of another natural number 'n', written as 'S n'
data Nat : Type where
  Z : Nat
  S : (n : Nat) -> Nat
```

Idris allows data types to be parameterized. The data type defined below shows that the type constructor `List` takes in a parameter `elem` of type `Type`, which stands for the type of elements in the list, and the type constructed is a list of elements of type `elem`. `List` type has two data constructor, `Nil` and `(::)`. `Nil` builds an empty list of type `List elem`. `(::)` append a new element `x` of type `elem` to the head of an existing list `xs` of type `List elem`, and builds a new list `x :: xs` of the same type as `xs`.

```
-- declaration of List data type in Idris standard library
data List : (elem : Type) -> Type where
  Nil : List elem
  (::) : (x : elem) -> (xs : List elem) -> List elem
```

Dependent Types

Dependent types are types that depend on elements of other types[6]. They allow programmers to specify certain properties of data types explicitly in their type. The following example provides a definition of a vector data type, which is indexed by the vector length `len` and parameterized over the element type `elem`.

```
-- declaration of Vect data type in Idris standard library
data Vect : (len : Nat) -> (elem : Type) -> Type where
  Nil   : Vect Z elem
  (::) : (x : elem) ->
         (xs : Vect len elem) ->
         Vect (S len) elem
```

The type `Vect len elem` is dependent on the value of type variables `len` and `elem`, which means two `Vect`s of length 3 and 4 are considered as different types, and two `Vect`s of same length but with element type `Nat` and `Char` are considered as different types. Dependent types allow programmers to obtain more confidence in a function's correctness by specifying its expected behaviors in its type. For instance, consider a function `concat` that concatenates two `Vect`, whose type signature is presented below.

```
concat : Vect n elem -> Vect m elem -> resultType
```

The output value of `concat` is a vector that concatenates both input vectors, which means its length should be the sum of the length of the two input vectors, i.e., (n+m), hence `resultType` has the type `Vect (n+m)elem`. The dependent type system helps to ensure the function correctness of `concat` through the Idris type checker. By providing a function type for `concat` that specifies the length of the output `Vect`, if the definition of `concat` does not return a vector of length (n+m), `concat` would fail type check. Take the following definition of `concat` as an example.

```
concat : Vect n elem -> Vect m elem -> Vect (n+m) elem
concat Nil v2 = v2
concat (x :: xs) ys = concat xs ys
```

The type of `concat` specifies that the output value should be a `Vect` of length `(n+m)`, where `n`, `m` are the length of the two input `Vect`, however the definition of `concat` eliminates one element from the input vector `x :: xs` during each recursive call, which is not the expected function behavior. Idris gives the following error message when compiling this function definition:

```
Type checking ./Example.idr
Example.idr:6:23-34:
  |
6 | concat (x :: xs) ys = concat xs ys
  |                       ~~~~~~~~~~~~
When checking right hand side of Example.concat with expected type
        Vect (S (plus len m)) Nat

Type mismatch between
        Vect (plus len m) Nat (Type of concat xs ys)
and
        Vect (S (plus len m)) Nat (Expected type)

Specifically:
        Type mismatch between
                plus len m
        and
                S (plus len m)
```

The error message clearly indicates that the expected return type is `Vect (S (plus len m)) Nat (Expected type)`, which is a vector of length `S (plus len m)`, however the type of `concat xs ys` is `Vect (plus len m)Nat`, whose length is one less than the length of the expected type. As the return type of this definition fail to match with the return type specified in the type of `concat`, it fails to be type checked. A correct implementation of `concat` is provided below.

```
concat : Vect n Nat -> Vect m Nat -> Vect (plus n m) Nat
concat Nil v2 = v2
concat (x :: xs) ys = x :: (concat xs ys)

-- definition of 'plus' in Idris
total plus : (n, m : Nat) -> Nat
plus Z right        = right
plus (S left) right = S (plus left right)
```

Under the case where the first input argument is `(x :: xs)` (i.e., vector is not empty), the length of the first vector `n` should be the successor of some other natural number `n'`, i.e. `n = S n'`, then `(x :: xs)` has type `Vect (S n') Nat`, and `xs` has type `Vect n' Nat`. The `concat` function is defined by appending the head of the first input argument, `x`, to the result of `concat xs ys`. As the types of `xs, ys` are `Vect n' Nat`, `Vect m Nat`, the type of `concat xs ys` is `Vect (plus n' m) Nat`, hence the vector obtained by appending `x` to `concat xs ys` has type `Vect (S (plus n' m))Nat`. Based on the definition of `plus` in Idris (which is provided above), we see that `S (plus n' m)= plus (S n')m`, which is exactly the expected output type `Vect (plus n m)Nat`, which indicates that the above definition of `concat` type checks.

The `concat` example above illustrates how dependent types help programmers to ensure function correctness with the Idris type checker. In program verification, dependent types can be used to specify intended behaviors of a program, and thus allowing us to verify its correctness.

Pattern Matching and Totality Checking

Pattern matching is the process of matching values against specific patterns. In Idris, functions are implemented by pattern matching on possible values of inputs. Continuing with the above example of `concate` function that concatenates two vectors, to define `concate`, we need to provide definitions on all possible values of `Vect`, which can either be `Nil`, i.e., a vector of length zero, or a non-empty vector of the pattern `(x :: xs)`.

```
concat : Vect n Nat -> Vect m Nat -> Vect (n+m) Nat
concat Nil v2 = v2
concat (x :: xs) v2 = x :: concat xs v2
```

Total function are defined for all possible input values and are guaranteed to terminate. Partial functions are not total, and hence might crash for some inputs. To secure the termination of programs, every function definition in Idris is checked for totality after type checking. However, due to the undecidability of the halting problem, the Idris totality checker is conservative, i.e., is never certain on whether a function is total or not. Based on the Idris Tutorial, Idris decides a function `f` is total if all of the following holds [7]:

- Cover all possible inputs
- Be well-founded — i.e. by the time a sequence of (possibly mutually) recursive calls reaches f again, it must be possible to show that one of its arguments has decreased.
- Not use any data types which are not strictly positive
- Not call any non-total functions

Specifically, `f` is considered as total if it is defined for all possible input values, for instance given an input of type `Nat`, `f` must cover the cases where it is either `Z` or the successor of another `Nat` (of the form `S n'`); and must have at least one argument that has a property, for instance its value (the `Nat` data type) or length (the `Vect` data type), that is strictly decreasing during each recursive call; the strictly positive restriction is a technical restriction that does not really concern us here, and lastly, `f` cannot call any non-total functions, otherwise `f` might fail to terminate due to the non-total functions called. To illustrate totality checking in Idris, continue with our `concat` function (the definition of `concat` below is not total):

```
concat : Vect n Nat -> Vect m Nat -> Vect (n+m) Nat
concat (x :: xs) ys = x :: (concat xs ys)
```

We use the `:total` command to check whether the above definition of `concat` is total, and we get the following message:

```
*Example> :total Example.concat
Example.concat is not total as there are missing cases
```

As `concat` is not defined for the case where the first input vector is `Nil`, hence the Idris totality checker marks `concat` as not total. If we check totality for the correct implementation of `concat` provided under the `Dependent Types` section, we see that Idris considers it as total:

```
concat : Vect n Nat -> Vect m Nat -> Vect (n+m) Nat
concat Nil v2 = v2
concat (x :: xs) ys = x :: (concat xs ys)

-- totality checking result for concat
Type checking ./Example.idr
*Example> :total Example.concat
Example.concat is Total
```

`case` expressions

`case` expression can be used to inspect a data value by matching on several cases. The syntax for case expression is as follow:

```
case <test> of
    <case 1>   => <expr>
    <case 2>   => <expr>
    ...
    otherwise => <expr>
```

where `<test>` is the expression being matched on, followed by all cases in the next few lines. Consider the following example that defines a function `findNat` with `case` expressions. `findNat` checks whether a given number `n` is an element of the input vector of `Nat`s.

```
findNat : Nat -> Vect m Nat -> Bool
findNat _ Nil = False
findNat n (x :: xs) = case (n == x) of
                        True => True
                        False => findNat n xs
```

The base case is when input vector is `Nil`, which indicates that `n` is not an element in the vector. Otherwise we check whether the head of the input vector `(x ::xs)` is equal to `n` with `(n == x)`. Using `case` expression, we can match on the value of `(n == x)`, that if `(n == x)` is True, then `n` is an element of the input vector, `findNat` returns `True`; otherwise we recur on the remaining of the vector `xs` to keep searching.

The `with` Rule

In a dependently typed language, matching on the resulting value of an intermediate computation can affect what we know about other values. In program implementation and theorem proving, it is a common technique to match on intermediate value in order to obtain more information. Idris provides the `with` rule for this purpose. Consider the following example `checkEvenPrf`:

```
checkEven : Nat -> Bool
checkEven Z = True
checkEven (S n) = case (checkEven n) of
                      True => False
                      False => True


checkEvenPrf : (n : Nat) ->
               (checkEven n = True) ->
               checkEven (S n) = False
checkEvenPrf n prf = ?check
```

The `checkEven` function checks whether a given `Nat` is even or not. It returns `True` if the input `Nat` is an even number, and returns `False` otherwise. The `checkEvenPrf` function is a proof that if a natural number is even, then its successor must not be even. The type of `checkEvenPrf` describes the premise and conclusion of this proof: given a natural number `n`, if the result of calling `checkEven` on `n` is true (as specified by `checkEven n = True`), then the successor of `n` must not be even, and the result of calling `checkEven` on `(S n)` must be `False`, which is specified by the output type `checkEven (S n)= False`.

Idris allows holes in a proof which stands for incomplete parts of a program, for instance `?check` in the example above is a hole. Idris allows programmers to inspect the type of holes and write functions incrementally. Inspecting the type of `check` we get the following:

```
*Example> :t check
  n : Nat
  prf : checkEven n = True
--------------------------------------
check : (case (checkEven n) of
          True => False
          False => True) = False
Holes: Example.check
```

The types of arguments of `checkEven` is presented above the dash line in the terminal output, and the expected return type, which is the type of the `check` hole, is presented below the dash line. The information provided by the terminal output shows that the value of `(checkEven n)` might effect the type of `check`, which indicates that matching on the value of `(checkEven n)` with `with` rule might provide more insights in writing this proof, as presented below.

```
checkEvenPrf : (n : Nat) ->
               (checkEven n = True) ->
               checkEven (S n) = False
checkEvenPrf n prf with (checkEven n) proof nIsEven
  | True = ?checkT
  | False = ?checkF
```

In the `checkEvenPrf` definition above we use the `with` rule to match on the value of `checkEven n`, which can be either `True` or `False` (as `checkEven` has return type `Bool`). By postfix the `with` clause with `proof nIsEven`, a proof named `nIsEven` generated by the pattern match will be in scope. By inspecting the type of `checkT` under the cases where `(checkEven n)` is matched as `True`, we get the following information.

```
*Example> :t checkT
  n : Nat
  prf : True = True
  nIsEven : True = checkEven n
--------------------------------------
checkT : False = False
Holes: Example.checkF, Example.checkT
```

Notice that `nIsEven` is a proof of `True = checkEven n` generated by the pattern match directly. As the `with` rule matches the value of `(checkEven n)` to `True`, and based on the definition of `checkEven`, Idris is able to deduce that the value of `checkEven (S n)` should be `False`, and hence the expected type of `checkT` is `False = False` as presented above. When `(checkEven n)` is matched to `False`, the type of `checkF` is as follows:

```
*Example> :t checkF
```

```
  n : Nat
  prf : False = True
  nIsEven : False = checkEven n
--------------------------------------
checkF : True = False
Holes: Example.checkF, Example.checkT
```

As the second argument of `checkEvenPrf` indicates that the value of `(checkEven n)` should be `True`, Idris is able to deduce that under this case the type of `prf` should be `(False = True)`, which is an absurdity, indicating that the value of `(checkEven n)` cannot be `False`. Hence we call the built-in function `absurd` on `prf` to mark that the case where `(checkEven n)` is matched to `False` is impossible. `Refl` is the data constructor for the equality data type `(=)`. `sym` and `trueNotFalse` are built-in functions in Idris that helps with constructing proof with impossible cases in Idris. The complete `checkEvenPrf` proof is presented below.

```
checkEvenPrf : (n : Nat) ->
               (checkEven n = True) ->
               checkEven (S n) = False
checkEvenPrf n prf with (checkEven n) proof nIsEven
  | True = Refl
  | False = absurd $ trueNotFalse (sym prf)
```

On the other hand, Idris also restricts programmers from proving something that is not true. Consider the following proof `checkEven_wrong`.

```
  predN : Nat -> Nat
  predN Z = Z
  predN (S n) = n


  checkEven_wrong : (n : Nat) ->
                    (checkEven n = True) ->
                    checkEven (predN n) = False
  checkEven_wrong Z prf = ?caseZ
  checkEven_wrong (S pn) prf with (checkEven pn) proof pnIsEven
    | True = absurd $ trueNotFalse (sym prf)
    | False = Refl
```

The `predN` function calculates the predecessor of a natural number (of type `Nat`). The predecessor of zero `Z` is `Z` itself, and the predecessor of `(S n)` is `n`. Given the definition of `predN`, the function `checkEven_wrong` attempts to prove that for a natural number `n`, if `(checkEven n)` is `True`, as specified by `(checkEven n = True)`, then the predecessor of `n` must not be even, as specified by the output type `checkEven (predN n)= False`. Similar to the `checkEvenPrf` function, the implementation of `checkEven_wrong` under the case where input value `n` is (S pn) (the second case) is straightforward, however as we inspect the hole `?caseZ` in the first case where `n` is `Z`, we notice that it is impossible to complete this proof:

```
*Example> :t caseZ
  prf : True = True
--------------------------------------
caseZ : True = False
```

As the type of `caseZ` is `True = False`, which is an absurdity, and there is no information available (above the dash line is what we know for approaching the proof) for us to reach this absurdity, there is no way for us to complete this hole, that the implementation for `checkEven_wrong` can never be completed, which indicates that Idris restricts programmers from writing proofs that are not true.

3.2 Dijkstra's algorithms

Dijkstra's Algorithm

Dijkstra's algorithm is a greedy algorithm that finds the shortest path from a given source to all other nodes in a directed graph with weighted edges. It was first introduced in 1959 by Edsger Wybe Dijkstra[1], and it is widely applied in many real-life applications, for instance Internet routing protocols such as the Open Shortest Path First protocol, and a variant of Dijkstra's algorithm is formulated as an instance of the best-first search algorithm in artificial intelligence.

Dijkstra's algorithm takes in a directed graph with non-negative edge weights, and computes the shortest path distance from one single source node to all other reachable nodes in the graph. The algorithm maintains a list of unexplored nodes and their distance values to the source node. Initially, the list of unexplored nodes contains all nodes in the input graph, and the distance value of all node are set as infinity except for the source node itself, which is set to zero. The algorithm extracts the node v with minimum distance value from the unexplored list during each iteration, and for each neighbor v' of v, if the path from source to v' via v contributes a smaller distance value, then the distance value of v' is updated.

4 Overview of Dijkstra's Implementation and Proof of Correctness

4.1 Data Structures

Dijkstra's algorithm requires non-negative edge weights and valid input graph, and the data structures in our implementation are designed to ensure these properties of input values. An overview of data structures in our implementation is presented below, and a detailed description is provided under Section 5.

Denote `gsize` as the size of graph, i.e. the number of vertices in a graph. A graph g is defined as a vector containing `gsize` number of adjacent lists, one for each node in the graph, and a node is defined as a data structure carrying a value of type `Fin gsize`. An adjacent list for a node $n \in g$ is defined as a list of tuples $(n', edge_w)$, where the first element n' in each tuple is a neighbor of n in g, and the second element $edge_w$ is the weight of the edge (n, n') in g. To access the adjacent list for a particularly node, the `Fin gsize` type value carried by this node is used to index the graph g. As the graph is defined as a vector of length `gsize`, the definition of node data type ensures that every well-typed node is a valid vertex in the graph, and that each indexing to the graph data structure are guaranteed to be in-bound.

The type of edge weight is user-defined in our implementation. Specifically, we define a `WeightOps` data type, which carries a user-specified type for the edge weight, along with operators and properties proofs for this type, which includes arithmetic operators, proof of non-negative value, and proof of plus associativity. As all edge weight are non-negative, and we assume a connected input graph, all edge weight should be non-negative and not equal infinity, whereas Dijkstra's algorithm initialize the distance value of all nodes in the graph (except the source node) as infinity. Based on this consideration we defined a `Distance` data type in addition to the user-defined edge weight type. `Distance` is parameterized over the user-defined weight type and can have value of either infinity, or the sum of edge weights.

4.2 Definition

Our implementation and correctness proof are based on the following definitions of key concepts used in Dijkstra's algorithm.

Definition 4.1. Path
(We adopt the definition of $path$ presented in the `Discrete Mathematics with Applications` book by `SUSANNA S. EPP` [8].)

A path from node v to w is a finite alternating sequence of adjacent vertices of G, which does not contain any repeated edge or vertex. A path from v to w has the form:

$$vv_0v_1....v_{n-1}w$$

where each adjacent nodes v_{i-1}, v_i has an edge from v_{i-1} to v_i in G. We denote the set of paths from v to w as $path(v, w)$.

Definition 4.2. Prefix of Path
Given a path from node v to w : $path(v, w) = vv_0v_1....v_{n-1}w$, the prefix of this $v-w$ path is defined as a subsequence of $path(v, w)$ that starts with v and ends with some node $w' \in path(v, w)$ (w' is a vertex in the sequence $path(v, w)$).

Definition 4.3. Length of Path
The length of a path $p = vv_0v_1....v_{n-1}w$ is the sum of the weights of all edges in p. We write:

$$length(p) = \sum weight(v_{i-1}, v_i), \forall v_{i-1}, v_i \in p \text{ where } (v_{i-1}, v_i) \in G.$$

Definition 4.4. Shortest Path
Denote $\Delta(s, v)$ as a shortest path from s to v, and $\delta(v)$ as the length of $\Delta(s, v)$. $\Delta(s, v)$ must fulfills:

$$\Delta(s, v) \in path(s, v)$$
$$\text{and}$$
$$\forall p' \in path(s, v), length(\Delta(s, v)) = \delta(v) \leq length(p')$$

4.3 Pseudocode

We denote (u, v) as an edge from node u to v, $weight(u, v)$ as the weight of edge (u, v). Let `gsize` denote the size of the input graph, i.e., the number of nodes in the graph. The type `Graph gsize weight` specifies a graph with `gsize` nodes and edge weight of type `weight`.
Given input graph g and source node s with types:

```
g : Graph gsize weight
s : Node gsize
```

Define $unexplored$ as the list of unexplored nodes, and $dist$ as a list storing the distance value[5] from s to all nodes in g calculated by the Dijkstra's algorithm. $dist[v]$ gives the corresponding distance value of v from s. Initially, $unexplroed$ contains all node in g, and the distance value from s to every node $v \in g$ is ∞ except for s itself, whose distance value to s is 0, as shown below:

(initially $unexplored$ contains all nodes in graph g)
$unexplored = \{v : v \in g\}$

(node value is used to index $dist$, initially distance of all nodes are infinity except the source node)
$dist[s] = 0, dist[a] = \infty, \forall a \in g, a \neq s$

We index $unexplored$ and $dist$ by the number of iterations. Specifically, denote u_i as the node being explored at the i^{th} iteration, and denote $dist_i$, $unexplored_i$ as the value of distance list and unexplored list at the beginning of the i^{th} iteration. Then during each iteration the Dijkstra's Algorithm calculates $dist, unexplored, explored$ as follows:

choose $u_k \in unexplored_k$ and $\forall u' \in unexplored_k, dist_k[u_k] \leq dist_k[u']$
$unexplored_{k+1} = unexplored_k - \{u_k\}$
for($\forall v \in g$) {

$$dist_{k+1}[v] = \begin{Bmatrix} min(dist_k[v], (dist_k[u_k] + weight(u_k, v))), & (u_k, v) \in g \\ dist_k[v] & otherwise \end{Bmatrix}$$

}

This implementation of Dijkstra's algorithm can be viewed as generating a matrix, where the i^{th} column in the matrix stores the value of $unexplored_i$ and $dist_i$. After calculating a matrix with n columns, the $(n+1)^{th}$ column can be calculated based on the value of $unexplored_n$ and $dist_n$ stored in the last column, i.e., the n^{th} column in the matrix. This reprensentation provides a clear recursive structure for the implementation of Dijkstra's algorithm, and the correctness of the program can be verified by proving that certain properties, for instance distance value of explored nodes stored in each column is the minimum distance value, hold for every column generated.

4.4 Proof of Correctness

This section provides a mathematical proof for our Dijkstra's implementation, which includes proof of program termination and proof of correct program behavior.

[5]For convenience purpose, in this thesis we denotes the 'distance value' for a node 'n' in a graph 'g' as the distance from the source node to n in 'g'

4.4.1 Lemmas

Denote $explored$ as the list of nodes in g but not in $unexplored$, i.e., $explored$ stored all nodes whose neighbors have been updated by the algorithm. We index $explored$ by the number of iterations, such that $explored_i$ denotes the value of $explored$ at the beginning of the i^{th} iteration.

Lemma 4.1. Given any two nodes v, w, the prefix of the shortest path $\Delta(v, w)$ is also a shortest path.

Proof. We will prove Lemma 4.1 by contradiction.
Consider any node q in the sequence of $\Delta(v, w)$, we have $\Delta(v, w) = ve_0v_0e_1v_2...v_iqv_j....v_{n-1}e_nw$. Suppose the prefix of $\Delta(v, w)$ from v to q, denote as $p(v, q)$, is not the shortest path from v to q. Then we know $p(v, q) = ve_0v_0e_1v_2...v_iq$ is a path from v to q and $length(p(v, q)) > length(\Delta(v, q))$.
Based on the definition of shortest path, we know:

$$length(\Delta(v, w)) \leq length(p), \forall p \in path(v, w)$$

Fenote the path after the node q as $p(q, w) = qv_j....v_{n-1}e_nw$, since $\Delta(v, w) = ve_0v_0e_1v_2...v_iqv_j....v_{n-1}e_nw$, then $\Delta(v, w) = p(v, q) + p(q, w)$, and that $length(\Delta(v, w)) = length(p(v, q)) + length(p(q, w))$. Then we have:

$$length(\Delta(v, w)) = length(p(v, q)) + length(p(q, w)) \leq length(p), \forall p \in path(v, w)$$

Since $p(v, q)$ is not the shortest path from v to q by assumption, then based on the definition of shortest path, $length(p(v, q)) < length(\Delta(v, w))$. Hence there exists another $v - w$ path $p'(v, w)$ such that:

$$\begin{aligned}
&p'(v, w) \in path(v, w) \\
&p'(v, w) = \Delta(v, q) + p(q, w) \\
&length(p'(v, w)) = length(\Delta(v, q)) + length(p(q, w)) \\
&\qquad\qquad\qquad < length(p(v, q)) + length(p(q, w)) \\
&\text{i.e. } length(p'(v, w)) < length(\Delta(v, w))
\end{aligned}$$

Hence we have reached a contradiction. Thus by the principle of prove by contradiction, for any the prefix $p(v, q)$ of $\Delta(v, w)$ is the shortest path from v to q. Lemma 4.1 holds. □

Lemma 4.2. Forall node $v \in g$, $n \geq 0$, if $dist_{n+1}[v] \neq \infty$, then $dist_{n+1}[v]$ is the length of some $s - v$ path, i.e, $path(s, v) \neq \emptyset$.

Proof. We will prove `Lemma 4.2` by inducting on the number of iterations.
Let P(n) be: After the n^{th} iteration, $n \geq 1$, for all node $v \in g$, if $dist_{n+1}[v] \neq \infty$, then $dist_{n+1}[v]$ is the length of some $s - v$ path.

Base Case : We shall show P(1) holds.
Based on the algorithm, initially $dist_1[s] = 0$ and for all node $v \in g, v \neq s, dist_1[v] = \infty$, then s is the only node whose distance value is not infinity. Based on the definition of path, the path from the source node s to itself is s, $path(s, s) = \{s\}$. Hence P(1) holds.

Inductive Hypothesis : Suppose $\forall i, 1 \leq i \leq k$, P(i) holds. That is, for all nodes $v \in g$, if $dist_{i+1}[v] \neq \infty$, then $dist_{n+1}[v]$ is the length of some $s - v$ path.

Inductive Step : We shall show P(k+1) holds.
For node u_{k+1} being explored during the $(k+1)^{th}$ iteration, based on the algorithm, $dist_{k+1}[u_{k+1}]$ is calculated as:

$$dist_{k+2}[u_{k+1}] = \begin{cases} min(dist_{k+1}[u_{k+1}], dist_{k+1}[u_{k+1}] + weight(u_{k+1}, u_{k+1})), & (u_{k+1}, u_{k+1}) \in g \\ dist_{k+1}[u_{k+1}] & otherwise \end{cases}$$

Since the distance value from u_{k+1} to itself is 0, then $dist_{k+2}[u_{k+1}] = dist_{k+1}[u_{k+1}]$, and that $dist_{k+2}[u_{k+1}]$ and $dist_{k+1}[u_{k+1}]$ are the length of the same $s - u_{k+1}$ path if there exists one.
If $dist_{k+2}[u_{k+1}] \neq \infty$, then $dist_{k+1}[u_{k+1}] = dist_{k+2}[u_{k+1}] \neq \infty$. Since $k \leq k$ and $dist_{k+1}[u_{k+1}] \neq \infty$, then based on the inductive hypothesis, $dist_{k+1}[u_{k+1}]$ is the length of some $s - u_{k+1}$ path, and hence $dist_{k+2}[u_{k+1}]$ is the length of some $s - u_{k+1}$ path.
Then for all node $v \in g$ other than u_{k+1}, there are two cases: (1) $(u_{k+1}, v) \in g$; (2) u_{k+1} does not have an edge to v. We will prove P(k+1) holds in both cases separately.

Case (1): $(u_{k+1}, v) \in g$
Based on the algorithm, as $(u_{k+1}, v) \in g$, $dist_{k+2}[v] = min(dist_{k+1}[v], dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v))$.

- If $dist_{k+1}[v] < dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v)$, then $dist_{k+2}[v] = dist_{k+1}[v]$. Then if $dist_{k+2}[v] \neq \infty$, we have $dist_{k+1}[v] \neq \infty$, and that $dist_{k+2}[v]$ and $dist_{k+1}[v]$ are the length of the same $s - v$ path if there exists one. Since $dist_{k+1}[v] \neq \infty$, the inductive hypothesis implies that $dist_{k+1}[v]$ is the length of some $s - v$ path, hence $dist_{k+2}[v]$ is the length of some $s - v$ path. P(k+1) holds.

- If $dist_{k+1}[v] \geq dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v)$, then $dist_{k+2}[v] = dist_{k+1}[u_{k+1}] + weight(w, v)$. If $dist_{k+2}[v] \neq \infty$, then it follows that $dist_{k+1}[u_{k+1}] = dist_{k+2}[v] - weight(u_{k+1}, v) \neq \infty$. Then the inductive hypothesis implies that $dist_{k+1}[u_{k+1}]$ must be the length of some $s - u_{k+1}$ path, denote as $p(s, u_{k+1})$. Since there is an edge $(u_{k+1}, v) \in g$, then $dist_{k+2}[v] = dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v)$ must be the length of the $s - v$ path through u_{k+1}. P(k+1) holds.

Hence P(k+1) holds under under `Case(1)`.

Case (2): u_{k+1} **does not have an edge to** v
Under this case, our algorithm indicates that $dist_{k+2}[v] = dist_{k+1}[v]$, and that $dist_{k+1}[v]$ and $dist_{k+2}[v]$ are the length of the same $s - v$ path if there exists one. If $dist_{k+1}[v] = dist_{k+2}[v] \neq \infty$,

then based on the inductive hypothesis, $dist_{k+1}[v]$ is the length of some $s - v$ path, and hence $dist_{k+2}[v]$ is the length of some $s - v$ path. P(k+1) holds under `Case(2)`.

We have proved P(k+1) holds for u_{k+1} and both cases for all nodes $v \in g$ other than u_{k+1}. Hence by the principle of prove by induction, P(n) holds. Thus `Lemma 4.2` holds. □

Lemma 4.3. For any node $v \in g$, if $dist_{i+1}[v] = \delta(v)$, then $\forall j > i$, $dist_{j+1}[v] = dist_{i+1}[v] = \delta(v)$.

Proof. We will prove `Lemma 4.3` by induction on the number iterations after the i^{th} iteration. Let P(n) be: For any node $v \in g$, if after the i^{th} iteration, $dist_{i+1}[v] = \delta(v)$, then for the $(i+n)^{th}$ iteration, $n \geq 1$, $dist_{i+n+1}[v] = dist_{i+1}[v] = \delta(v)$

Base Case : We shall show P(1) holds.
During the $(i+1)^{th}$ iteration, suppose u_{i+1} is the node being explored, then $dist_{i+2}[v]$ is calculated as:

$$dist_{i+2}[v] = \begin{cases} min(dist_{i+1}[v], dist_{i+1}[u_{i+1}] + weight(u_{i+1}, v)), & (u_{i+1}, v)) \in g \\ dist_{i+1}[v] & otherwise \end{cases}$$

If $(u_{i+1}, v)) \in g$, then if $dist_{i+1}[u_{i+1}]$ is the length of some $s - u_{i+1}$ path, then $(dist_{i+1}[u_{i+1}] + weight(u_{i+1}, v))$ is the length of some $s - v$ path. Since $dist_{i+1}[v] = \delta(v)$, then based on the definition of shortest path, $dist_{i+1}[v] \leq dist_{i+1}[u_{i+1}] + weight(u_{i+1}, v)$, and hence $dist_{i+2}[v] = dist_{i+1}[v] = \delta(v)$.
If u_{i+1} does not have an edge to v, then $dist_{i+2}[v] = dist_{i+1}[v] = \delta(v)$.
Hence in either cases, $dist_{i+2}[v] = dist_{i+1}[v] = \delta(v)$. P(1) holds.

Inductive Hypothesis : Suppose P(k) holds, that is, for $i > 0$, if $dist_{i+1}[v] = \delta(v)$, then for the $(i+k)^{th}$ iteration, $k \geq 1$, $dist_{i+k+1}[v] = dist_{i+1}[v] = \delta(v)$.

Inductive Step : We shall show P(k+1) holds.
For the node u_{i+k+1} being explored during the $(i+k+1)^{th}$ iteration, there are two cases: (1) $(u_{i+k+1}, v) \in g$; (2) u_{i+k+1} does not have an edge to v. We will show that P(k+1) holds under both cases separately.
Case 1: $(u_{i+k+1}, v) \in g$
If u_{i+k+1} has an edge to v, then based on the algorithm, for $dist_{i+k+2}[v]$, we have:

$$dist_{i+k+2}[v] = min(dist_{i+k+1}[v], dist_{i+k+1}[u_{i+k+1}] + weight(u_{i+k+1}, v))$$

Since based on our inductive hypothesis, $dist_{i+k+1}[v] = dist_{i+1}[v] = \delta(v)$, then if $dist_{i+k+1}[u_{i+k+1}]$ is the length of some $s - u_{i+k+1}$ path, then $(dist_{i+k+1}[u_{i+1}] + weight(u_{i+k+1}, v))$ is the length of some $s - v$ path, and hence $dist_{i+k+1}[v] = \delta(v) \leq (dist_{i+k+1}[u_{i+1}] + weight(u_{i+k+1}, v))$. Then:

$$
\begin{aligned}
dist_{i+k+2}[v] &= min(dist_{i+k+1}[v], dist_{i+k+1}[u_{i+k+1}] + weight(u_{i+k+1}, v)) \\
&= dist_{i+k+1}[v] \\
&= dist_{i+1}[v] = \delta(v)
\end{aligned}
$$

P(k+1) holds under `Case 1`.
Case 2: u_{i+k+1} does not have an edge to v
Since u_{i+k+1} does not have an edge to v, then $dist_{i+k+2}[v] = dist_{i+k+1}[v]$. Based on the inductive hypothesis, $dist_{i+k+1}[v] = dist_{i+1}[v] = \delta(v)$. then $dist_{i+k+2}[v] = dist_{i+1}[v] = \delta(v)$. P(k+1) holds for `Case (2)`.
Thus P(k+1) holds. By the principle of prove by induction, P(n) holds. `Lemma 4.3` proved.

□

Lemma 4.4. For any node $v \in g$, for each $u_i \in explored_{n+1}$, $n \geq 1, 1 \leq i \leq n$, $dist_{n+1}[v] \leq dist_i[u_i] + weight(u_i, v)$.

Proof. We will prove Lemma 4.4 by inducting on the number n.
Let P(n) be: for any node $v \in g$, for each $u_i \in explored_{n+1}$, $n \geq 1, 1 \leq i \leq n$, $dist_{n+1}[v] \leq dist_i[u_i] + weight(u_i, v)$.
Base Case: We shall show P(1) holds.
Based on the algorithm, $dist_1[s] = 0$, and for all node $v \in g$ other than s, $dist_1[v] = \infty$, and $explored_2$ only contains s. For node s, $dist_2[s] = 0 \leq dist_1[s] + weight(s, s) = 0$. For all node $v \in g$ other than s, we have:

$$
\begin{aligned}
dist_2[v] &= min(dist_1[v], dist_1[s] + weight(s, v)) \\
&\leq dist_1[s] + weight(s, v)
\end{aligned}
$$

Since s is the only node in $explored_2$, then the above equation directly shows that P(1) holds.

Induction Hypothesis: Suppose P(k) holds for $k > 1$. That is, for any node $v \in g$, for each $u_i \in explored_{k+1}$, $k > 1, 1 \leq i \leq k$, $dist_{k+1}[v] \leq dist_i[u_i] + weight(u_i, v)$.

Inductive Step: we shall show P(k+1) holds. That is, for $k + 1 > 1$, forall nodes $v \in g$, for each $u_i \in explored_{k+2}$, $k > 1, 1 \leq i \leq k + 1$, $dist_{k+2}[v] \leq dist_i[u_i] + weight(u_i, v)$.
Suppose u_{k+1} is the node being explored during the $(k + 1)^{th}$ iteration, then $explored_{k+2} = explored_{k+1} \cup \{u_{k+1}\}$. Forall node $v \in g$, we have:

$$
dist_{k+2}[v] = min(dist_{k+1}[v], dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v))
$$

Hence we have:

$$
\begin{aligned}
dist_{k+2}[v] &\leq dist_{k+1}[v]([E4.4.1]) \\
dist_{k+2} &\leq dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v)([E4.4.2])
\end{aligned}
$$

The induction hypothesis implies that $dist_{k+1}[v] \leq dist_i[u_i] + weight(u_i, v), \forall u_i \in explored_{k+1}$. Combining with [E4.4.1], we have:

$$
dist_{k+2}[v] \leq dist_i[u_i] + weight(u_i, v), \forall u_i \in explored_{k+1}[E4.4.3]
$$

Since $explored_{k+2} = explored_{k+1} \cup \{u_{k+1}\}$, then equation [E4.4.2] and equation [E4.4.3] implies that $dist_{k+2}[v] \leq dist_i[u_i] + weight(u_i, v), \forall u_i \in explored_{k+1} \cup \{u_{k+1}\} = explored_{k+2}$. P(k+1) holds. By the principle of prove by induction, P(n) holds. `Lemma 4.4` proved. □

Lemma 4.5. Assume g is a connected graph. Forall node $v \in explored_{n+1}$:

1. $dist_{n+1}[v] < \infty$
2. $dist_{n+1}[v] \leq \delta(v'), \forall v' \in unexplored_{n+1}$.
3. $dist_{n+1}[v] = \delta(v)$

Proof. We will prove `Lemma 4.5` by inducting on the number of iterations.

Let P(n) be: For a connected graph g, for $n \geq 1$, forall node $w \in explored_{n+1}$: (L1) $dist_{n+1}[w] < \infty$; (L2) $dist_{n+1}[w] \leq \delta(w'), \forall w' \in unexplored_{n+1}$; (L3) $dist_{n+1}[w] = \delta(w)$.

Base Case : We shall show P(1) holds
Based on the algorithm, during the first iteration, the node with minimum distance value is the source node s with $dist_1[s] = 0$. Hence during the first iteration, only s is removed from $unexplored_1$ and added to $explored_2$. Since $dist_2[s] = 0 < \infty$, then (L1) holds for P(1). Since all edge weights are non-negative, then the shortest distance value from s to s is indeed 0, hence $dist_2[s] = 0 = \delta(s)$ and $dist_2[s] \leq \delta(v')$, $\forall v' \in unexplored_2$. Thus (L2) and (L3) holds for P(1). Hence P(1) holds.

Induction Hypothesis : Suppose P(i) is true for all $1 \leq i \leq k$. That is, forall $1 < i \leq k$, forall node $w \in explored_{i+1}$: (L1) $dist_{i+1}[w] < \infty$; (L2) $dist_{i+1}[w] \leq \delta(w'), \forall w' \in unexplored_{i+1}$; (L3) $dist_{i+1}[w] = \delta(w)$;

Inductive Step : We shall show P(k+1) holds. That is, forall node $w \in explored_{k+2}$, (L1) $dist_{k+2}[w] \neq \infty$; (L2) $dist_{k+2}[w] \leq \delta(w')$, $\forall w' \in unexplored_{k+2}$; (L3) $dist_{k+2}[w] = \delta(w)$;

Suppose u_{k+1} is the node added into $explored$ during the $(k+1)^{th}$ iteration, then $explored_{k+2} = explored_{k+1} \cup \{u_{k+1}\}$. We will show that (L1)(L2) and (L3) holds for all nodes in $explored_{k+1}$ in `Part (a)`, and `Part (b)` proves (L1)(L2)(L3) holds for u_{k+1}, so that the statements holds forall nodes in $explored_{k+2}$.

- Part(a): `WTP: After the` $(k+1)^{th}$ `iteration,` $\forall w \in explored_{k+1}$`, (L1)(L2)(L3) holds.`

  Consider each node $q \in (explored_{k+1} \cap explored_{k+2}) = explored_{k+1}$, q must be explored before the $(k+1)^{th}$ iteration. Suppose q is explored during the i^{th} iteration for some $i < k+1$, then based on our induction hypothesis, $dist_{i+1}[q] = \delta(q)$, and $\delta(q) \leq \delta(q'), \forall q' \in unexplored_{i+1}$.

  `Proof of (L3)`: Since for each node $q \in explored_{k+1}$, the induction hypothesis implies that $dist_{k+1}[q] = \delta(q)$, then `Lemma 3.3` imples that $dist_{k+2}[q] = dist_{k+1}[q] = \delta(q)$. (L3) holds for $explored_{k+1}$.

`Proof of (L2)`: Based on the algorithm, for each iteration, the algorithm explores exactly one node and never revisits any explored nodes. For each node $q \in explored_{k+1}$ mentioned above, since q is explored before the $(k+1)^{th}$ iteration, then $unexplored_{k+1} \subseteq unexplored_{i+1}$. Since $\delta(q) \leq \delta(q'), \forall q' \in unexplored_{i+1}$, and $unexplored_{i+1}$ includes all node in $unexplored_{k+1}$, then $\delta(q) \leq \delta(q'), \forall q' \in unexplored_{k+1}$. Since proof of (L3) above shows that $dist_{k+2}[q] = \delta(q)$, then $dist_{k+2}[q] \leq \delta(q'), \forall q' \in unexplored_{k+1}$. (L2) holds for $explored_{k+1}$.

`Proof of (L1)`: Since the induction hypothesis implies that $\forall q \in explored_{k+1}, dist_{k+1}[q] < \infty$, and the proof of (L3) above shows that $dist_{k+2}[q] = dist_{k+1}[q]$, then $dist_{k+2}[q] < \infty$. (L1) holds for $explored_{k+1}$.

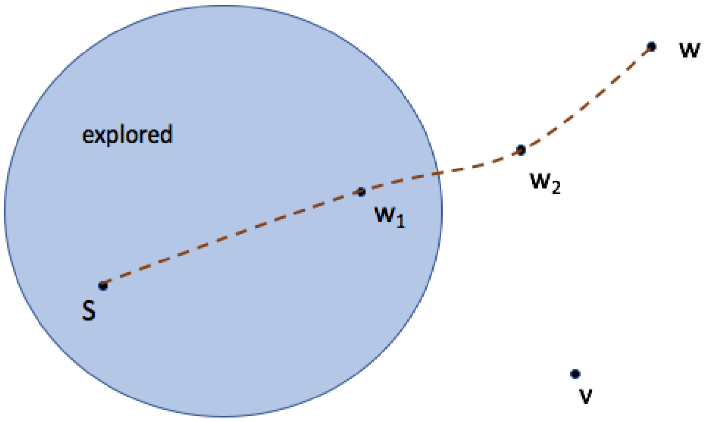
Hence we have proved that both (1) and (2) holds for all nodes in $explored_{k+1}$.

- Part(b): (L1)(L2)(L3) holds for $\{u_{k+1}\}$.
  Specifically, we want to show: (L1) $dist_{k+2}[u_{k+1}] < \infty$; (L2) $dist_{k+2}[u_{k+1}] \leq \delta(v')$, $\forall v' \in unexplored_{k+2}$, and (L2) $dist_{k+2}[u_{k+1}] = \delta(u_{k+1})$.

  1. (L1) $dist_{k+2}[u_{k+1}] \neq \infty$
     Since g is a connected graph, then s must have a path to u_{k+1}. Since u_{k+1} is the node currently being explored, then we know there must exists a $s - u_{k+1}$ path, denote as $p(s, u_{k+1})$, such any node proceeding u_{k+1} in $p(s, u_{k+1})$ are explored before u_{k+1}, i.e., in $explored_{k+1}$.
     Denote the node right before u_{k+1} in $p(s, u_{k+1})$ as u', $u' \in explored_{k+1}$. Suppose u' is explored during the i^{th} iteration, $i < k+1$. The induction hypothesis implies that $dist_{i+1}[u'] < \infty$. Since $dist_{i+1}[u'] = min(dist_i[u'], dist_i[u'] + weight(u', u')) = min(dist_i[u'], dist_i[u'] + 0) = dist_i[u']$, then $dist_i[u'] < \infty$. `Lemma 4.4` implies $dist_{k+2}[u_{k+1}] \leq dist_i[u'] + weight(u', u_{k+1}]$, then it follows that $dist_{k+1}[u_{k+1}] < \infty$. (L1) holds for u_{k+1}.

  2. (L2) $dist_{k+2}[u_{k+1}] \leq \delta(v')$, $\forall v' \in unexplored_{k+2}$

     We will prove (L2) by contradiction. Suppose there exists $w \in unexplored_{k+2}$, such that $dist_{k+2}[u_{k+1}] > \delta(w)$([E4.5.1]).
     Consider the shortest path $\Delta(s, w)$ from s to w, $\delta(w) = length(\Delta(s, w))$. Since $w \notin explored_{k+2}$, then there must exists some node in $\Delta(s, w)$ that are not in $explored_{k+2}$. Suppose the first node along $\Delta(s, w)$ that is not in the $explored_{k+2}$ list is w_2, and the node right before w_2 in the s to w_2 subpath is w_1, thus $w_1 \in explored_{k+2}$. The image below illustrates this construction:



Denote the subpath from s to w_1 in $\Delta(s, w)$ as $p(s, w_1)$, subpath from s to w_2 in $\Delta(s, w)$ as $p(s, w_2)$, and subpath w_2 to w as $p(w_2, w)$. Based on `Definition 2.2 Prefix of Path`, $p(s, w_1)$ is a prefix of $\Delta(s, w)$. Since $p(s, w_1)$ is the prefix of the shortest $s - w$ path, then based on `Lemma 3.1`, $p(s, w_1)$ is the shortest path from s to w_1, $\Delta(s, w_1) = p(s, w_1)$, $length(p(s, w_1)) = \delta(w_1)$.
Similarly, since $p(s, w_2) = p(s, w_1) + (w_1, w_2)$, then $p(s, w_2)$ is a prefix of $\Delta(s, w)$, and hence `Lemma 3.1` implies that $p(s, w_2)$ is the shortest path from s to w_2. Then we have:

$$\begin{aligned}
\Delta(s, w_2) &= p(s, w_2) = p(s, w_1) + (w_1, w_2) \\
\delta(w_2) &= length(\Delta(s, w_2)) \\
&= length(p(s, w_2)) \\
&= length(p(s, w_1)) + weight(w_1, w_2) \\
&= \delta(w_1) + weight(w_1, w_2)([E4.5.2])
\end{aligned}$$

For $\Delta(s, w)$ we have:

$$\begin{aligned}
\delta(w) &= length(p_w) \\
&= length(p(s, w_1)) + weight(w_1, w_2) + length(p(w_2, w)) \\
&= \delta(w_1) + weight(w_1, w_2) + length(p(w_2, w))
\end{aligned}$$

Since all edge weights are non-negative, then:

$$\delta(w_2) = \delta(w_1) + weight(w_1, w_2) \leq \delta(w) \text{ ([E4.5.3])}$$

Since $w_1 \in explored_{k+2}$, there are two cases to consider: $w_1 = u_{k+1}$ and $w_1 \neq u_{k+1}$. We will prove P(k+1) under both cases below.

Case 1: $w_1 = u_{k+1}$
Since $\delta(w_2) = \delta(w_1) + weight(w_1, w_2) \leq \delta(w)$ and all edge weights are non-negative, then $\delta(w_1) \leq \delta(w)$. When $w_1 = u_{k+1}$, we have $\delta(u_{k+1}) \leq \delta(w)$. Since $dist_{k+2}[u_{k+1}] >$

$\delta(w)$ and $\delta(u_{k+1}) \leq \delta(w)$, we have $\delta(u_{k+1}) < dist_{k+2}[u_{k+1}]$.
Suppose the node right before u_{k+1} in $\Delta(s, u_{k+1})$ is w_3. We know $length(\Delta(s, u_{k+1})) = length(p(s, w_3)) + weight(w_3, u_{k+1}))$, where $p(s, w_3)$ is the prefix of $\Delta(s, u_{k+1})$. Based on `Lemma 3.1`, we know $length(p(s, w_3)) = \delta(w_3)$. Hence:

$$\begin{aligned}
\delta(u_{k+1}) &= length(p(s, w_3)) + weight(w_3, u_{k+1})) \\
&= \delta(w_3) + weight(w_3, u_{k+1}) \\
&< dist_{k+2}[u_{k+1}]
\end{aligned}$$

i.e.

$$dist_{k+2}[u_{k+1}] > \delta(w_3) + weight(w_3, u_{k+1})([E4.5.6])$$

Based on the construction, w_2 is the first node along $\Delta(s, w)$, w_1 is right before w_2 in the path, w_3 is right before $w_1 = u_{k+1}$ in the path, then $w_3 \in explored_{k+2}$. Assume w_3 is explored during the j^{th} iteration. Then based on `Lemma 4.4`, we have:

$$dist_{k+2}[u_{k+1}] \leq dist_j[w_3] + weight(w_3, u_{k+1})([E4.5.7])$$

The induction hypothesis implies $dist_{j+1}[w_3] = \delta(w_3)$. For $dist_{j+1}[w_3]$ we have:

$$\begin{aligned}
dist_{j+1}[w_3] &= min(dist_j[w_3], dist_j[w_3] + weight(w_3, w_3)) \\
&= min(dist_j[w_3], dist_j[w_3] + 0) \\
&= dist_j[w_3]
\end{aligned}$$

Hence $dist_j[w_3] = \delta(w_3)$, combine with [E4.5.7], we have:

$$dist_{k+2}[u_{k+1}] \leq \delta(w_3) + weight(w_3, u_{k+1})([E4.5.8])$$

The equation [E4.5.8] contradicts with equation[E4.5.6]. Hence by the principle of prove by contradiction, (L2) holds when $w_1 = u_{k+1}$.

Case 2: $w_1 \neq u_{k+1}$
Since $w_1 \in explored_{k+2}$ and $w_1 \neq u_{k+1}$, w_1 is explored before the $(k+1)^{th}$ iteration. i.e., $w_1 \in explored_{k+1}$. Suppose w_1 is being explored during the i^{th} iteration, $i < k+1$, then based on the algorithm, the value of $dist_{i+1}[w_1]$ is calculated as:

$$\begin{aligned}
dist_{i+1}[w_1] &= min(dist_i[w_1], dist_i[w_1] + weight(w_1, w_1)) \\
&= min(dist_i[w_1], dist_i[w_1] + 0) \\
&= min(dist_i[w_1], dist_i[w_1]) \\
&= dist_i[w_1]
\end{aligned}$$

Since the induction hypothesis implies that $dist_{i+1}[w_1] = \delta(w_1)$, then $dist_i[w_1] = \delta(w_1)$.
Since w_1 has an edge to w_2, then $dist_{i+1}[w_2]$ must have been updated according as

follows:

$$\begin{aligned}dist_{i+1}[w_2] &= min(dist_i[w_2], dist_i[w_1] + weight(w_1, w_2)) \\ &= min(dist_i[w_2], \delta(w_1) + weight(w_1, w_2))\end{aligned}$$

Based on [E4.5.2] we know that $\delta(w_2) = \delta(w_1) + weight(w_1, w_2)$, then $dist_{i+1}[w_2] = min(dist_i[w_2], \delta(w_2))$. If $dist_i[w_2] = \infty$, then $dist_{i+1}[w_2] = min(dist_i[w_2], \delta(w_2)) = \delta(w_2)$. If $dist_i[w_2] \neq \infty$, then based on `Lemma 3.2`, $dist_i[w_2]$ is the length of some $s - w_2$ path. Since $\delta(w_2) \leq length(p), \forall p \in path(s, w_2)$, then $dist_{i+1}[w_2] = min(dist_i[w_2], \delta(w_2)) = \delta(w_2)$. Hence in either cases, we conclude that $dist_{i+1}[w_2] = \delta(w_2)$.
Since $dist_{i+1}[w_2] = \delta(w_2)$ and $i < k + 1$, then based on `Lemma 3.3`, we have:

$$dist_{k+1}[w_2] = dist_{i+1} = \delta(w_2)([E4.5.4])$$

Based on our assumption, at the beginning of the $(k + 1)^{th}$ generation, $u_{k+1}, w_2 \notin explored_{k+1}$ and u_{k+1} is selected by the algorithm, then we must have $dist_{k+1}[w_2] \geq dist_{k+1}[u_{k+1}]$. For $dist_{k+2}[u_{k+1}]$ we have:

$$\begin{aligned}dist_{k+2}[u_{k+1}] &= min(dist_{k+1}[u_{k+1}], dist_{k+1}[u_{k+1}] + weight(u_{k+1}, u_{k+1})) \\ &= min(dist_{k+1}[u_{k+1}], dist_{k+1}[u_{k+1}] + 0) \\ &= dist_{k+1}[u_{k+1}]\end{aligned}$$

Hence $dist_{k+1}[w_2] \geq dist_{k+2}[u_{k+1}]$. Combine with [E4.5.4], [E4.5.3] we have:

$$\begin{aligned}&dist_{k+1}[w_2] \geq dist_{k+2}[u_{k+1}](\\ &dist_{k+1}[w_2] = dist_{i+1} = \delta(w_2)(from[E4.5.4]) \\ &\qquad \delta(w) \geq \delta(w_2) = \delta(w_1) + weight(w_1, w_2)(from[E4.5.3])\end{aligned}$$

Hence $\delta(w) \geq dist_{k+2}[u_{k+1}]$, which contradicts with [E4.5.1]. Hence by the principle of prove by contradiction, when $w_1 \neq u_{k+1}$, $dist_{k+2}[u_{k+1}] \leq \delta(w), \forall w \in unexplored_{k+2}$. (L2) holds for u_{k+1}.

3. (L3) $dist_{k+2}[u_{k+1}] = \delta(u_{k+1})$

   We will prove this by contradiction.
   Since (L1) proves $dist_{k+2}[u_{k+1}] \neq \infty$, then `Lemma 3.2` implies that $dist_{k+2}[u_{k+1}]$ is the length of some $s - u_{k+1}$ path, denote as p. Suppose there is a $s - u_{k+1}$ path p' that's shorter than p, i.e, $dist_{k+2}[u_{k+1}] > length(p')$([E4.5.9]). Suppose the node right before u_{k+1} in p' is v'. Then we know:

$$\begin{aligned}&length(p') = length(p(s, v')) + weight(v', u_{k+1}) \\ &length(p') < dist_{k+2}[u_{k+1}]\end{aligned}$$

   , where $p(s, v')$ is the prefix of p' from s to v'. Hence:

$$dist_{k+2}[u_{k+1}] > length(p(s, v')) + weight(v', u_{k+1})$$

Based on the definition of shortest path, $length(p(s, v')) \geq \delta(v')$, then we have:

$$dist_{k+2}[u_{k+1}] > \delta(v') + weight(v', u_{k+1})([E4.5.10])$$

There are two cases to consider: (1) $v' \in explored_{k+2}$; (2) $v' \notin explored_{k+2}$

Case(1): $v' \in explored_{k+2}$
Suppose v' is explored during the i^{th} iteration. Then `Lemma 4.4` implies:

$$dist_{k+2}[u_{k+1}] \leq dist_i[v'] + weight(v', u_{k+1})([E4.5.11])$$

The induction hypothesis implies $dist_{i+1}[v'] = \delta(v')$, and for $dist_{i+1}[v']$ we have:

$$\begin{aligned} dist_{i+1}[v'] &= min(dist_i[v'], dist_i[v'] + weight(v', v')) \\ &= min(dist_i[v'], dist_i[v'] + 0) \\ &= dist_i[v'] \end{aligned}$$

Hence $dist_i[v'] = \delta(v')$. Combining [E4.5.11], we have:

$$dist_{k+2}[u_{k+1}] \leq \delta(v') + weight(v', u_{k+1})([E4.5.12])$$

Hence equation [E4.5.12] contradicts with equaltion [E4.5.10]. By the principle of prove by contradiction, (L3) holds when $v' \in explored_{k+2}$.

Case(2): $v' \notin explored_{k+2}$
Since $length(p') = length(p(s, v')) + weight(v', u_{k+1})$, $p(s, v)$ is the prefix of p' from s to v', then based on the definition of shortest path, $length(p(s, v')) \leq \delta(v')$, and thus $\delta(v') + weight(v', u_{k+1}) \leq length(p(s, v')) + weight(v', u_{k+1}) = length(p')$. Since all edge weights are non-negative, then $\delta(v') \leq length(p')$.
Since $v' \notin explored_{k+2}$, i.e., $v' \in unexplored_{k+2}$, based on proof of (L2), $dist_{k+2}[u_{k+1}] \leq \delta(v')$. Since $dist_{k+2}[u_{k+1}] \leq \delta(v')$ and $\delta(v') \leq length(p')$, then $dist_{k+2}[u_{k+1}] \leq length(p')$, which contradicts with our assumption ([E4.5.9]). Hence by the principle of prove by contradiction, (L3) holds when $v' \notin explored_{k+2}$.
Since we have proved (L3) for both cases, then (L3) holds for P(K+1).

Since we have proved (L1)(L2)(L3) forall nodes in $explored_{k+1}$ after the $(k+1)^{th}$ iteration, P(k+1) holds. Then by the principle of prove by induction, `Lemma 4.5` holds. □

4.4.2 Proof of Termination

Proof. The inner for loop is guaranteed to terminate as the algorithm goes through each adjacent node exactly once. As the size of list `unexplored` decreases by one during each iteration of the while loop, the algorithm is guaranteed to terminate. □

4.4.3 Prove of Correctness

Proof. By applying `Lemma 4.5` to the last iteration, denote as m^{th} iteration, of the algorithm, we obtained that for all nodes n in the explored list, $dist_{m+1}[n]$ is indeed the shortest path distance value from source s to n, hence Dijkstra's algorithm indeed calculates the shortest path distance value from the source s to each node $n \in g$. □

5 Concrete Implementation of Dijkstra's Verification

5.1 Data Structures

5.1.1 The `WeightOps` data type

Our implementation of Dijkstra's algorithm allows user-defined edge weight type, with a `WeightOps` data type specifying the operations and properties of the edge weight type that user needs to provide. Below presents part of the definition of `WeightOps`.

```
using (weight : type)
  record WeightOps weight where
    constructor MKWeight
    -- zero value of weight
    zero : weight
    -- greater than or equal to
    gtew : weight -> weight -> Bool
    -- equality
    eq : weight -> weight -> Bool
    -- addition
    add : weight -> weight -> weight
    ...
    triangle_ineq : (a : weight) ->
            (b : weight) -> gtew (add a b) a = True
    ...
    addComm : (a : weight) ->
          (b : weight) ->
          add a b = add b a
```

`WeightOps` is defined as a record data type, which allows programmers to collect several values (referred as record's fields) together. `WeightOps` is parameterized over the user-defined edge weight type `weight`. The `MKWeight` constructor takes in all the fields and build a `WeightOps weight` type. The field name can be used to access the field value. For instance given a value `ops` of `WeightOps weight` type, `add ops` will gives the addition operator for the `weight` data type.

The `zero` field stands for the zero value for the `weight` type, and `gtew, eq, add` are basic operators for `weight`. The `triangle_ineq` field in `WeightOps` ensures that the value of `weight` data type can only be non-negative. Given any two values `a, b` of type `weight`, `triangle_ineq` specifies that the sum of `a, b` is greater than or equal to either of them, which guarantees that both `a, b` have non-negative values. The remaining fields in `WeightOps` are required for Dijkstra's implementation and verification.

As we assume the input graph is a connected graph, the value of edge weight between two adjacent nodes are considered as not infinity, whereas Dijkstra's algorithm initializes the distance value from source node to all other nodes in the graph as infinity. Based on this consideration, we define a `Distance` type to represent the distance value between two nodes. `Distance` is parameterized over the user-defined `weight` type, and the value of `Distance weight` is either infinity, or sum of `weight`s. The definition of `Distance` data type is provided below.

```
data Distance : Type -> Type where
  DInf : Distance weight
  DVal : (val : weight) -> Distance weight
```

The data constructor `DInf` builds a value of `Distance weight` that represents infinity distance, and `DVal` carries a value `val` of type `weight`, which is the sum of one or more `weight`s. Arithmetic operators for the `Distance weight` type is defined based on operators of `weight`.

5.1.2 Data Types for `Node`, `nodeset`, and `Graph`

The size of a graph is defined as the number of nodes in the graph, and all nodes in a graph is enumerated by natural numbers starting from zero. For instance given a graph of size 3, each node in the graph is represented by a natural number in the range from zero to two, with three nodes in total. Given a graph `G` of size `gsize`, the `Node` type is indexed by graph size `gsize` and carries a value that is strictly less than `gsize`. A `nodeset` is defined as a `List` of pairs of adjacent nodes and corresponding edge weights for each node in the graph, and the definition of `Graph` data type contains a vector of `nodeset`s for all nodes in the graph. The definition of `Node`, `nodeset` and `Graph` data structure are presented below.

```
-- definition of Node
data Node : (gsize : Nat) -> Type where
  MKNode : (nodeVal : Fin gsize) -> Node gsize

-- definition of nodeset
nodeset : (gsize : Nat) -> (weight : Type) -> Type
nodeset gsize weight = List (Node gsize, weight)

-- data type for Graph
data Graph : Nat -> (w : Type) -> (WeightOps w)-> Type where
  MKGraph : (gsize : Nat) ->
            (weight : Type) ->
            (ops : WeightOps weight) ->
            (edges : Vect gsize (nodeset gsize weight)) ->
            Graph gsize weight ops
```

The `Node gsize` type is indexed by a `Nat`, `gsize`, and stands for a node in a graph of size `gsize`. The data constructor `MKNode` takes in a parameter of type `Fin gsize`, which captures a `Nat` value that is greater than or equal to `Z` and strictly smaller than `gsize`. `nodeset` is defined as a type synonym for the type `(List (Node gsize, weight))`. As the edge weight type is user defined, the `Graph` data type is parameterized over the edge weight type `weight`, and index by the size of the graph `gsize`. Operators and properties of `weight` are carried in the `Graph` data type by the `ops` parameter. The `edges` parameter is a vector of length `gsize` with element type nodeset.

Such construction ensures that, given a graph `G` of type '`Graph gsize weight ops`' (graph size is `gsize` and edge weight type is `weight`), any well-typed '`Node gsize`' value is a valid node in `G`, and the that there are only `gsize` possible values of the type '`Node gsize`' as restricted by the '`Fin gsize`' type, which ensures that a graph of size `gsize` indeed has `gsize` nodes. In addition, since the elements of vector `edges` in `G` has type '`nodeset gsize weight`', all `nodeset`s in `G` can only contain valid nodes. More importantly, as our implementation uses the value carried by each `Node` to index its `nodeset` in the graph, for each node with type '`Node gsize`' in `G`, the value with type '`Fin gsize`' carried by this node is guaranteed to be a bounded index for the vector `edges` in `G` (as `edges` has length `gsize`).

Based on the above construction, a node `m` is considered as adjacent to a node `n` in a graph `g` if `m` is in the `nodeset` of `n`. The definition of adjancent nodes is provided below.

```
adj : (g : Graph gsize weight ops) ->
      (n, m : Node gsize) ->
      Type
adj g n m = (inNodeset m (getNeighbors g n) = True)
```

The `getNeighbors` function takes in a graph `g` and a node `n`, and gets the `nodeset` of `n` in `g`, and the `inNodeset` function takes in a node and a `nodeset`, and returns true fi the input node is in the input nodeset, returns false otherwise. The definition of `adj` above states that `m` is adjacent to `n` in `g` if we can find `m` in the `nodeset` of `n` in `g`.

5.1.3 `Path` and shortest `Path`

A path in a graph is defined as a sequence of non-repeating nodes, where each two adjacent nodes have an edge in this graph. A path can contain only one node, as specified by the `Unit` data constructor below, or multiple nodes, as the `Cons` data constructor allows a new path to be constructed from an existing path. Specifically, given a path from node `s` to `v`, if `n` is an adjacent to `v` (`adj g v n` specifies that there is an edge from `v` to `n` in the graph `g`), then we can obtain a new path from `s` to `n` by appending the node `n` to the end of the existing `s`-to-`v` path.

```
data Path : Node gsize ->
            Node gsize ->
            Graph gsize weight ops -> Type where
  Unit : (g : Graph gsize weight ops) ->
         (n : Node gsize) ->
         Path n n g
  Cons : Path s v g ->
         (n : Node gsize) ->
         (adj : adj g v n) ->
         Path s n g
```

To implement a shortest path in a graph, recall in section 4.2, we define the length of a path as the sum of the weights of all edges in the path, and define a shortest path as follows:

Denote $\Delta(s, v)$ as a shortest path from s to v, and $\delta(v)$ as the length of $\Delta(s, v)$. $\Delta(s, v)$ must fulfills:

$$\Delta(s, v) \in path(s, v)$$

and

$$\forall p' \in path(s, v), length(\Delta(s, v)) = \delta(v) \leq length(p')$$

The above definition specifies that given a shortest path $\Delta(s, v)$, the length of $\Delta(s, v)$ is smaller than or equal to the length of any other $s-to-v$ path in the graph. We then provide the following implementation of shortest path based on the above definition.

```
shortestPath : (g : Graph gsize weight ops) ->
               (sp : Path s v g) ->
               Type
shortestPath g sp {ops} {v}
  = (lp : Path s v g) ->
    dgte ops (length lp) (length sp) = True
```

The statement stated by the return type of `shortestPath` is highly similar to our mathematical definition of shortest path above. Specifically, given a graph `g`, and a path `sp` from node `s` to inlv in `g`, the return type of `shortestPath` specifies that, given any path `lp` from `s` to `v` in `g`, the length of `lp` must be greater than or equal to the length of `sp`. `dgte` is the greater than or equal to operator for `Distance` data type.

5.1.4 The `Column` data type

As we mentioned back in section 4.3, our implementation viewed Dijkstra's algorithm as generating a matrix, where each column in the matrix represents one state of the algorithm, we define a `Column` data type for this purpose. The definition of `Column` type is provided below.

```
data Column : (len : Nat) -> (g : Graph gsize weight ops) -> (src
   : Node gsize) -> Type where
  MKColumn : (g : Graph gsize weight ops) ->
             (src : Node gsize) ->
             (len : Nat) ->
             (unexp : Vect len (Node gsize)) ->
             (dist : Vect gsize (Distance weight)) ->
             Column len g src
```

The `Column` data type takes in the input graph `g`, the source node `src`, the number of unexplored nodes `len`, which is also the length of `unexp`, the vector of unexplored nodes, and a vector of distance values from source for all nodes in the graph. The `Column` type is indexed by the number of unexplored nodes. As Dijkstra's requires a source node for running the algorithm, the type `Column` is also dependent on the input graph as well as the source node `src`.

Such definition of `Column` data type provides enough information for us to calculate a new column in the matrix, as the `unexp` and `dist` vectors provides enough information for calculating the current unexplored node with minimum distance value and generating the new distance vector with updated distance values for all nodes in the graph. Our implementation of Dijkstra's algorithm has a recursive structgure that generates a new `Column` during each recursive call. With an input graph of size `gsize`, the first column should have length `gsize` as all nodes are unexplored, and the last column generated contains an empty vector for unexplored nodes, and a vector of the minimum value from source to all nodes in the graph.

5.2 Implementation of Dijkstra's Algorithm

Our implementation of Dijkstra's algorithm can be viewed as generating a matrix, where one column of the matrix represents one state during the execution of the algorithm. Each column stores the original input graph, source node, a vector of currently unexplored node, and a vector of current distance value for all nodes in the graph, and a new column is calculated based on the value stored in the last column generated. The `Column` data type in the data structure section (section 5.1.4) is defined for this column representation. However, since the calculation of a new column does not requires all previous columns calculated, and in order to simplify the data structures, the implementation does not maintain the whole matrix, rather, only one variable of the `Column` data type is maintained to store the last column calculated. Even though the whole matrix is not presented, we can still visualize this implementation as generating a matrix representation of Dijkstra's algorithm, where the columns shows how distance value of all nodes in the graph is gradually updated, hence in the following sections we still refers= to this matrix representation of Dijkstra's algorithm, based on the above clarification that the actual matrix is not presented in the implementation. Eliminating the matrix structure not only reduces some redundancy in our implementation, but also allows us to verify Dijkstra's algorithm by proving properties over each `Column` calculated, which provides a clear structure for our verification program.

The implementation can be divided into three layers, where each layer breaks down the calculation to deal with a smaller structure. Specifically, the first layer calculates the whole matrix representation by calling function from the second layer. The second layer is responsible for generating a new `Column` data based on the last `Column` calculated, where the updated distance value for each node in the graph is calculated by the third layer. The remaining of this section provides more details on our three layers of calculation.

5.2.1 `dijkstras` and `runDijkstras`

The first layer involves two functions, `dijkstras` and `runDijkstras`, where `dijkstras` takes in the input graph and the source node, generate the first `Column` of the matrix representation, and calls `runDijkstras` on the first `Column` to calculate the whole matrix and returns the last column generated. The definition of `dijkstras` and `runDijkstras` are provided below.

```
runDijkstras : {g : Graph gsize weight ops} ->
               (cl : Column len g src) ->
               Column Z g src
runDijkstras {len = Z} {src} cl = cl
runDijkstras {len = S l'} cl@(MKColumn g src (S l') _ _ ) =
   runDijkstras $ runHelper cl


dijkstras : (gsize : Nat) ->
            (g : Graph gsize weight ops) ->
            (src : Node gsize) ->
            (nadj : ((n : Node gsize) ->
                  inNodeset n (getNeighbors g n) = False)) ->
            (Vect gsize (Distance weight))
dijkstras gsize g src nadj {weight} {ops} = cdist $ runDijkstras
   cl
  where
    nodes : Vect gsize (Node gsize)
    nodes = mkNodes gsize
    dist : Vect gsize (Distance weight)
    dist = mkdists gsize src ops
    cl : Column gsize g src
    cl = (MKColumn g src gsize nodes dist)
```

The `dijkstras` function takes in the size of graph `gsize`, the input graph `g`, the source node `src`, and a function `nadj` stating that any node in the graph cannot be in its own `nodeset`. `nadj` ensures that when constructing a `Path` data with the `Cons` constructor, it is not possible to append a node `n` to itself, as `adj g n n` cannot hold for any node in the input graph(definitions of `adj` and `Path` is illustrated in section 5.1.3). `dijkstras` function constructs the first `Column` `cl` and calls the `runDijkstras` on `cl`. `runDijkstras` traverses all unexplored nodes and returns the last `Column` calculated, which should contain an empty vector of unexplored nodes. The `dijkstras` function then returns the vector of distance values in the `Column` returned by `runDijkstras`, which contains the minimum distance values for all nodes in the graph.

The `mkNodes` function takes in a `Nat` and generates a vector `nodes` with type '`Vect gsize (Node gsize)`', where the '`Fin gsize`' value carried by each node in `nodes` is increasing in order. Specifically, suppose `gsize` is not zero, i.e., `gsize = S n`, then the first node in `nodes` `FZ` with type '`Fin gsize`', the second node carries '`FS FZ`', ..., and the last element in `nodes` carries a value of type `Fin gsize` that captures the natural number `n`, which the largest `Nat` value that falls into the range of `Z` to `n`. Since we enumerate all nodes in the graph by natural numbers starting from `Z` (as mentioned in section 5.1.2), the vector generated by `mkNodes` contains all nodes in the input graph. Since initially all nodes are unexplored, when constructing the first `Column cl` , the `dijkstras` function calls the `mkNodes` function to generate the initial vector of unexplored nodes. The `mkdists` function generates the initial vector of distance values for all nodes in the graph(distance value for all nodes are infinity except for the source node, which is 0), in the same ordering as the vector generated by `mkNodes`. In the definition of `dijkstras`, both `mkNodes` and `mkdists` return a vector of length `gsize` (named as `dists` and `nodes` correspondingly), which ensures that the i^{th} element `dists` is the initial distance value for the i^{th} node in `nodes`. Later paragraphs shows how this constructions gives a clear recursive structure for the implementation of Dijkstra's algorithm. s

The `runDijkstras` algorithm takes in a parameter `cl` of type '`Column len g src`', traverses all unexplored nodes in `cl`(if there is any), and returns a value of type '`Column Z g src`'. `runDijkstras` is defined recursively: if the input value `cl` contains an empty vector of unexplored nodes, then simply returns `cl`, otherwise we extracts the unexplored nodes in `cl` with minimum distance value, calculate a new `Column` with updated vectors of unexplored nodes and distance values, and recurs on the new `Column` calculated. The calculation of new `Column` is completed with the `runHelper` function, which is elaborated in the following.

5.2.2 `runHelper` and `updateDist`

The second layer includes two functions, `runHelper` and `updateDist`, which calculate a new `Column` value based on the last column generated. `runHelper` takes in a column with non-empty unexplored list type and returns the new `Column` calculated with one less unexplored nodes, and the `updateDist` function calculates the updated distance vector for the new `Column`. The implementation of `runHelper` and `updateDist` are provided below.

```
updateDist :  (g : Graph gsize weight ops) ->
              (min_node : Node gsize) ->
              (min_dist : Distance weight) ->
              (nodes : Vect m (Node gsize)) ->
              (dist : Vect m (Distance weight)) ->
              Vect m (Distance weight)
updateDist g min_node min_dist Nil Nil = Nil
updateDist g min_node min_dist (x :: xs) (d :: ds)
  = (calcDist g min_node x min_dist d) :: (updateDist g min_node
     min_dist xs ds)


runHelper : {g : Graph gsize weight ops} ->
```

```
              (cl : Column (S len) g src) ->
              Column len g src
runHelper cl@(MKColumn g src (S len) unexp dist) {gsize} {weight}
   {ops}
  = MKColumn g src len (deleteMinNode min_node unexp (minCElem cl)
     ) newds
  where
    min_node : Node gsize
    min_node = getMin cl
    ...
    newds : Vect gsize (Distance weight)
    newds = updateDist g min_node min_dist (mkNodes gsize) dist
```

The input value `cl` of `runHelper` has type `Column (S len)g src`, which is a `Column` with non-empty vector of unexplored nodes, as specified by `S len` in the type. `runHelper` extracts the unexplored node with minimum distance value from the unexplored vector in `cl`, and calculates a new column with the updated unexplored and distance vectors. The currently unexplored node with minimum distance value is named as `min_node` and calculated by calling `getMin cl` under the `where` clause. The return value of `runHelper` has type `Column len g src`, indicating the unexplored vector in the new `Column` has one less element than that `cl`. The `deleteMinNode` is responsible for calculating the updated vector of unexplored nodes based on that of `cl`, but with `min_node` removed. `deleteMinNode` requires a proof that the targeting node to be removed is in the input vector, as specified by `(minCElem cl)`.

The updated vector of distance values for the new `Column`(named as `newds` in definition of `runHelper`) is calculated by `updateDist`, which takes in a graph `g`, the minimum node `min_node` and its distance value `min_dist`, and vectors of nodes and distances, both have the same length. Notice that in `runHelper`, `updateDist` is called on the vector generated by `mkNodes gsize` (which contains all nodes in the graph) rather than the vector of unexplored nodes, and the initial input distance vector of `updateDist` is calculated by `mkdists` and passed down from the `dijkstras` function. The definition of `mkNodes` and `mkdists` mentioned in previous paragraphs allows `updateDist` to recur on the nodes and distances vectors in parallel and update the distance value for every node in the graph, as long as the elements ordering in both vectors remains the same during each recursive step. Since `updateDist` never changes the ordering of the input nodes vector, and during each recursive step, the new distance value for the current head node `x` calculated is append to the result of calling `updateDist` on the remaining nodes `xs` and their distance values `ds`, the element ordering of the distance vector is also unchanged. This definition of `updateDist` again provides a clear recursive structure in implementing our verification program, which is expanded in more details in section TODO.

5.2.3 `calcDist`

The third layer contains the function `calcDist`, which is called by `updateDist` to calculate the updated distance value for one specific node in the graph. Below presents the implementation of `calcDist`.

```
calcDist : (g : Graph gsize weight ops) ->
           (min_node : Node gsize) ->
           (cur : Node gsize) ->
           (min_dist : Distance weight) ->
```

```
                (cur_dist : Distance weight) ->
                Distance weight
      calcDist g min_node cur min_dist cur_dist
        = min ops cur_dist (dplus ops (edgeW g min_node cur) min_dist)
```

Given the input graph `g`, the current node being explored(named as `min_node`), the distance value of `min_node`(named as `min_dist`), a node `cur` and its distance value `cur_dist`, `calcDist` compares the distance value from source node to `cur` through `min_node` in `g` with `cur_dist` and returns the smaller value. The distance value of `cur` passing `min_node` is calculated by adding `min_dist` with the weight of edge from `min_node` to `cur`. If there is no edge between `min_node` and `cur` in `g`, the edge weight will be infinity, which is already greater than or equal to the original distance of `cur`, then `cur_dist` will be returned by `calcDist` in this case. Otherwise, `calcDist` calls the `min` function to find and return the smaller value between `cur_dist`, and the sum of `min_dist` and weight of edge from `min_node` to `cur`.

5.3 Verification of Dijkstra's Algorithm

Our verification of Dijkstra's algorithm is based on and has a similar structure as the implementation in section 5.2. Instead of proving Dijkstra's correctness based on the `dijkstras` and `runDijkstras` functions directly, we approach the verification by proving that certain properties maintain for each new `Column` value generated by every call to the `runHelper` function. Specifically, since the `Column` structure carries information on the unexplored nodes and distance values calculated for all nodes in the graph, we can re-state Lemma **??** to Lemma **??** in the mathematical proof of Dijkstra's correctness(in Section **??**) as properties on `Column`, and prove that if these properties preserve after calling `runHelper`. As `runHelper` is called by the `runDijkstras` function, the implementation of our verification follows the same structure by defining a function that recursively applies the above proof of properties preservation, and shows that same properties also hold for the last `Column` value calculated, i.e., the value returned by the `runDijkstras` function, which verifies the correctness of Dijkstra's algorithm.

In the following sections, we first provide proofs of lemmas that state the properties preservation of each new `Column` generated by `runHelper`, and then present the functions that directly verifies the correctness of Dijkstra's algorithm. As the implementation of all proofs are highly complicated and involves significantly amount of details, the following only elaborates on the implementation of two lemma proofs for the purpose of presenting some techniques on how proofs are approached in our verification, and discuss on the types of other lemmas instead. As this thesis aims to verify Dijkstra's algorithm with the Idris type checker, the types of proofs should provide sufficient information on our verification program.

5.3.1 Lemmas

The mathematical proof of Dijkstra's algorithm in section 4.4 includes five lemmas, however in implementing our verification program, we found it easier to approach by merging Lemma 4.4 into Lemma 4.5 as one of its statements. Lemma 4.1 to 4.5 are defined in order, meaning that the proof of Lemma 4.2 is built on Lemma 4.1, and proof of Lemma 4.3 is built on Lemma 4.1 and 4.2 etc. The implementation of Lemma 4.5 (function `l5_spath`) is directly applied in verifying

Dijkstra's correctness, and implementations of Lemma 4.1 to Lemma 4.4 are helper proofs for proving Lemma 4.5

The following first presents the types for all lemmas of our verification program, and then elaborate on the implementation of one of the lemma proofs, in order to provide more insights into how we approach proofs generally. We choose to expand on the proofs of Lemma **??** (which corresponds to function `l1_prefixSP`) and Lemma 4.3 (which corresponds to the `l3_preserveDelta` function), as compare to other lemma proofs, proof of Lemma 4.3 involves less details but presents enough information on our techniques in implementing proofs.

5.3.1.1 Lemma 1 - `l1_prefixSP`

Lemma 4.1 states that the prefix of a shortest path is also a shortest path. In section 4.2, we provide the following definition for the prefix of a path.

Definition 5.1. Prefix of Path
Given a path from node v to w : $path(v, w) = vv_0v_1....v_{n-1}w$, the prefix of this $v-w$ path is defined as a subsequence of $path(v, w)$ that starts with v and ends with some node $w' \in path(v, w)$ (w' is a vertex in the sequence $path(v, w)$).

Specifically, a prefix of a path is a subsequence of this path, and has the same start node (i.e., the first node in a path) as the path. Based on the above mathematical definition of path prefix, and our `Path` data type defined in section 5.1.3, we first define a `append` function that concatenates two paths by appending one path to the beginning of the other path, and then implement the prefix of a path based on the `append` function. The following presents the implementation of `append` and `pathPrefix`.

```
append : (p1 : Path s v g) ->
         (p2 : Path v w g) ->
         Path s w g

pathPrefix : (pprefix : Path s w g) ->
             (p : Path s v g) ->
             Type
pathPrefix pprefix p {w} {v} {g}
  = (ppost : Path w v g ** append pprefix ppost = p)
```

The type of `append` function specifies that, given a path `p1` from node `s` to `v` in `g`, and a path `p2` from node `v` to `w` in `g`, the result of appending `p1` to the head of `p2` is a path from node `s` to `w` in `g`. Notice that the ending node `v` in `p1` is exactly the starting node of `p2`, and the resulting path of appending `p1` to `p2` (i.e., return value of `append p1 p2`) starts from the same node as `p1`, and ends at the same node as `p2`. Then according to our definition of prefix of a path above, the first input path `p1` is actually a prefix of the return value of `append p1 p2`.

The `pathPrefix` function is a predicate stating that the first input path `pprefix` is a prefix of the second input path `p`. `(v ** P)` is the syntax for dependent pairs, which states that the second element `P` in the pair is dependent on the value of the first element `v`. Dependent pairs are used to represent existential quantification in Idris. For instance, the dependent pair `(n : Nat ** Vect n Nat)` states the existence of a natural number `n`, such that `n` is the length of the `Vect` included as the second element of the pair. In the definition of `pathPrefix`, as `pprefix` is the prefix of `p`, then there only exists one path (with type `Path w v g`) such that the result of appending `pprefix` to this path is `p`. This is specified by the dependent pair `(ppost : Path w v g ** append pprefix ppost = p)` in our definition, which quantified a specific path `ppost` with type `Path w v g` such that the result of `append pprefix ppost` is `p`, and hence the path `pprefix` is a prefix of `p`.

Given the definition of `pathPrefix` above and definition of shortest path in section 5.1.3, the implementation of Lemma 4.1 is provided below.

```
shorter_trans : {g : Graph gsize weight ops} ->
               (p1 : Path s w g) ->
               (p2 : Path s w g) ->
               (p3 : Path w v g) ->
               (p : dgte ops (length p1) (length p2) = False) ->
               dgte ops (length $ append p1 p3)
                        (length $ append p2 p3) = False


l1_prefixSP : {g: Graph gsize weight ops} ->
              {s, v, w : Node gsize} ->
              {sp : Path s v g} ->
              {sp_pre : Path s w g} ->
              (shortestPath g sp) ->
              (pathPrefix sp_pre sp) ->
              (shortestPath g sp_pre)
l1_prefixSP spath (post ** appendRefl) lp_pre {ops} {sp_pre}
  with (dgte ops (length lp_pre) (length sp_pre)) proof lpsp
    | True = Refl
    | False = absurd $ contradict (spath (append lp_pre post))
                                  (rewrite (sym appendRefl) in
                                    (shorter_trans
                                      lp_pre sp_pre post (sym lpsp)))
```

The type of the `l1_prefixSP` states that, given an input graph `g`, nodes `s, v, w`, a path `sp` from `s` to `v` in `g` (as specifies by the type `Path s v g`), the prefix of `sp` from `s` to `w` (named as `sp_pre`), if `sp` is a shortest path from `s` to `v`, as specifies by `shortestPath g sp`, then the prefix `sp_pre` of `sp` is also a shortest path from `s` to `w` in `g`.

The definition of `shortestPath` allows us to bring into scope a variable `lp_pre` with type `Path s w g` that quantifies over any path from `s` to `w` in `g`. We approach the proof of `l1_prefixSP` by matching on the value of `(dgte ops (length lp_pre)(length sp_pre))`, which compares the length of `lp_pre` against the length of the prefix `sp_pre` of `sp`.

When `(dgte ops (length lp_pre)(length sp_pre))` is matched to `True`, this indicates that the length of any path from `s` to `w` is longer than or equal to the length of `sp_pre`, then `sp_pre` is a shortest path from `s` to `w` based on the definition of shortest path.

When `(dgte ops (length lp_pre)(length sp_pre))` is matched to `False`, this indicates that the length of `lp_pre` is smaller than the length of `sp_pre`, i.e., `length(lp_pre)< length(sp_pre)`. Since `sp_pre`, `lp_pre` are both paths from `s` in `w` in `g`, and appending `sp_pre` to `ppost` gives us the path `sp` from `s` to `v` (`sp_pre` is the prefix of `sp`), then we can construct another path `p' : Path s v g` from `s` to `v` by appending `lp_pre` to `ppost`, whose length is smaller than that of `sp` as we conclude `length(lp_pre)< length(sp_pre)` before. As indicated by the type of `shorter_trans` provided above, `(shorter_trans lp_pre sp_pre post (sym lpsp))` is a proof that shows, since we know `length(lp_pre)< length(sp_pre)`, then the length of the path obtained by appending `lp_pre` to `ppost` (the length `p'`), is smaller than the length of the path obtained by appending `sp_pre` to `ppost` (the length of `p`). This contradicts with the statement that `p` is a shortest path from `s` to `v` in `g`(specified by `shortestPath sp p`). Hence with prove by contradiction we can show that the case when `(dgte ops (length lp_pre)(length sp_pre))` is matched to `False` is impossible, i.e., the length of `sp_pre` is smaller than or equal to the length of any other `s` to `w` path in `g`, and that `sp_pre` is a shortest path from `s` to `v` in `g`. Proof of `l1_prefixSP` is completed.

The structure of the implementation of Lemma 4.2 and Lemma 4.5 is as follows: we first define functions that specifies the `Column` properties stated by each lemma, and then implement a function that proves the preservation of these properties. This structure provides a more clear and straightforward type signatures for our functions in the verification program by separating the types that specifies `Column` properties from the types of the proofs.

5.3.1.2 Lemma 2 - `l2_existPath`

In our mathematical proof of Dijkstra's correctness, Lemma 4.2 states that given an input graph `g`, for all nodes `v` in `g`, if $dist_{n+1}[v] \neq \infty$, then $dist_{n+1}[v]$ is the length of some `s-to-v` path in `g`. As mentioned at the beginning of this section, in our verification program, we state Lemma 4.2 as a `Column` property and prove these properties preserve after calling `runHelper`. The function `neDInfPath` provided below specifies the `Column` property stated by Lemma 4.2.

```
neDInfPath : {g : Graph gsize weight ops} ->
             (cl : Column len g src) ->
             Type
neDInfPath cl {g} {src} {ops} {gsize}
  = (v : Node gsize) ->
    (ne : dgte ops (nodeDistN v cl) DInf = False) ->
    (psv : Path src v g ** dEq ops (nodeDistN v cl) (length psv) =
       True)
```

Given `cl` with type `Column len g src`, the function `neDInfPath` specifies that for any node `v` in the graph, if the distance value of `v` stored in `cl` is smaller than infinity, then it is the length of some path from `src` to `v` in `g`. `nodeDistN` is a function that indexes the distance value for a specific node in a `Column`, and in the definition of `neDInfPath`, `nodeDistN v cl` gets the distance value of `v` stored in `cl`, and the dependent pair `(psv : Path src v g ** dEq ops (nodeDistN v cl)(length psv)= True)` specifies the existence of a path `psv` from `src` to `v` in `g`, such that the distance value of `v` stored in `cl` is the length of `psv`. We then define the type of the function `l2_existPath` that states the preservation of the `neDInfPath` property.

```
l2_existPath : {g : Graph gsize weight ops} ->
               (cl : Column (S len) g src) ->
               (l2_ih : neDInfPath cl) ->
               neDInfPath (runHelper cl)
```

`l2_existPath` states that given `cl` with type `Column (S len)g src`, if `neDInfPath` holds for `cl` (specified by `l2_ih`), then it also holds for the column generated by `(runHelper cl)`. Notice that the input `cl` of `l2_existPath` contains a non-empty vector of unexplored nodes, which is restricted by the `runHelper` function. Similar to the previous proof on `l1_prefixSP`, we can bring the node `v` and statement `ne : dgte ops (nodeDistN v cl)DInf = False` in `neDInfPath` into scope. The proof of `l2_existPath` is approached by matching on the distance value of `v` stored in the `Column` generated by `runHelper cl`. If the distance value of `v` in `runHelper cl` is the same as that in `cl`, then the proof is given by `l2_ih`. Otherwise we check whether `v` is equal to `getMin cl` (the current unexplored node with minimum distance value, mentioned in Section 5.2.2), and prove both cases by applying `l2_ih` on `(getMin cl)`. The detailed proof is provided in the Appendix (TODO: cross reference).

5.3.1.3 Lemma 3 - `l3_preserveDelta`

In verifying Dijkstra's correctness, it is important to show that forall nodes `v` in the input graph, once the distance value calculated for `v` is equal to the minimum distance value from the source node to `v`, then the distance value of `v` does not change through the execution of the algorithm. This is proved by Lemma **??** in the mathematical proof of Dijkstra's algorithm, and implemented by the function `l3_preserveDelta` below (the proof of `l3_preserveDelta` is provided in later paragraphs).

```
l3_preserveDelta : {g : Graph gsize weight ops} ->
                   (cl : Column (S len) g src) ->
                   (l2_ih : neDInfPath cl) ->
                   (v : Node gsize) ->
                   (psv : Path src v g) ->
                   (spsv : shortestPath g psv) ->
                   (eq : dEq ops (nodeDistN v cl) (length psv) =
                      True) ->
                   dEq ops (nodeDistN v (runHelper cl)) (length psv)
                       = True
```

`l3_preserveDelta` states that given a `Column` named `cl`, for any node `v` in graph, if the distance value of `v` stored in `cl` is equal to the length of a shortest path (named as `psv`) from source node `src` to `v` in `g` (stated by the input `eq : dEq ops (nodeDistN v cl)(length psv)= True`), then the distance value of `v` stored in `runHelper cl` is also equal to the length of `psv`. Since the proof of Lemma 4.3 is based on Lemma 4.2 as we mentioned at the beginning of Section 5.3.1, the proof of property `neDInPath` on `cl` is provided by the input `l2_ih : neDInfPath cl`.

We implement the proof of `l3_preservDelta` by contradiction, which requires a proof that shows the distance value stored for all node is non-increasing after each call of `runHelper`. The function `runDecre` provided below states this property. We provide a detailed discussion on the implementation of `runDecre` as it presents how we approach the proofs of some key lemmas in our verification. Specifically, we break the statement that we want to prove into smaller ones by destructing the data structures involved in the statement, so that the implementation of functions that involve more complex data types can be built on functions that deal with simpler data types. The following explanation on the implementation of `runDecre` illustrates this technique.

```
runDecre : {g : Graph gsize weight ops} ->
           (cl : Column (S len) g src) ->
           (v : Node gsize) ->
           dgte ops (nodeDistN v cl)
                (nodeDistN v (runHelper cl)) = True
```

```
runDecre (MKColumn g src (S len) unexp dist) (MKNode nv) {gsize} {
    ops} {weight}
  = distDecre g min_node min_dist (mkNodes gsize) dist (finToNat nv)
      {p=nvLTE nv}
  where
   ...
```

The return type of the runDecre function specifies that for all node v, the distance value stored for v in cl is either decreasing, or maintains the same after each call of runHelper on cl. Since runDecre involves the Column data type, and the main field in Column that concerns us here is the Vect of distance values, the implementation of runDecre is built on a function distDecre, which states the same non-increasing property of distance values calculated, however involves the Vect of distance values instead. The implementation of distDecre is presented below. (explain finToNat)

```
distDecre : (g : Graph gsize weight ops) ->
            (mn : Node gsize) ->
            (min_dist : Distance weight) ->
            (nodes : Vect m (Node gsize)) ->
            (dist : Vect m (Distance weight)) ->
            (nv : Nat) ->
            {auto p : LT nv m} ->
            dgte ops (indexN nv dist)
                 (indexN nv (updateDist g mn min_dist nodes dist)) =
                      True
...
distDecre g mn min_dist (n :: ns) (d :: ds) Z
  = calcDistEq g mn n min_dist d
```

Similarly, the property specified by distDecre is again break down into a statement on the distance value of a specific node in the graph, as specified by the calcistEq function.

5.3.1.4 Lemma 5 - l5_spath

5.3.2 Verification of Correctness

The implementation of lemma proofs in the previous section shows that if certain properties, such as those specified by the function l5_stms, holds for the current column cl, then they must hold for the new column generated based on cl. With the proofs of the lemmas, we are able to define the below recursive function, correctness, which specifies that given a column cl relating to an input graph g and source node src, if all properties stated by neDInfPath and l5_stms hold for cl (specified by l2_ih and l5_ih inputs), then the properties should also hold after calling runDijkstras on cl. We updates the inputs to the next recursive call by applying lemmas to l2_ih and l5_ih, which is indeed equivalent to the inductive steps in our theoretical proofs of Dijkstra's algorithm provided back in Section 4.

```
correctness : {g : Graph gsize weight ops} ->
              (cl : Column len g src) ->
              (nadj : ((n : Node gsize) -> inNodeset n (getNeighbors g n
                 ) = False)) ->
```

```
              (l2_ih : neDInfPath cl) ->
              (l5_ih : l5_stms cl) ->
              l5_stms (runDijkstras cl)
correctness {len = Z} cl nadj l2_ih l5_ih = l5_ih
correctness {len=S n} cl@(MKColumn g src (S n) unexp dist) nadj l2_ih
   l5_ih
  = correctness (runHelper {len=n} cl) nadj
          (l2_existPath cl l2_ih)
          (l5_spath cl nadj l2_ih l5_ih)
```

We then defined a dijkstras_correctness function that wraps up all proofs and verify the minimum distance property for all nodes in the input graph.

```
dijkstras_correctness : (gsize : Nat) ->
                        (g : Graph gsize weight ops) ->
                        (src : Node gsize) ->
                        (v : Node gsize) ->
                        (psv : Path src v g) ->
                        (spsv : shortestPath g psv) ->
                        (nadj : ((n : Node gsize) ->
                            inNodeset n (getNeighbors g n) = False)) ->
                        dEq ops (indexN (finToNat (getVal v))
                                (dijkstras gsize g src nadj)
                                {p=nvLTE {gsize=gsize} (getVal v)})
                            (length psv) = True
```

(To be continued....)

6 Discussion

7 Related Work

The increasing importance of Dijkstra's algorithm in many real-world applications has raised an interest on verifying it's implementation. Mange and Kuhn provide a project that verifies a Java implementation of Dijkstra's algorithm with the Jahob verification system in their report on efficient proving of Java programs [9]. Although the concrete implementation of this work is unavailable, the report demonstrates the verification process. Function behaviors are specified with preconditions, postconditions, and invariants, and Jahob allows programmers to provide these specifications in high-order logic(HOL), which reduces the problem of program verification to the validity of HOL formulas.

Klasen et. al. verifies Dijkstra's algorithm with the KeY system [10], an interactive theorem prover for Java. Concrete implementations of Dijkstra's algorithm with different variants are provided, and all of them are written in Java. Similarly to the work by Mange and Kuhn, the verification process in the work by Klasen involves describing the behavior of each function with preconditions, postconditions and modifies clause. Loop invariants are specified to support the verification. A function is then verified as correct by the KeY system, with respect to its behavior specifications, if the postconditions specified hold after execution. A similar implementation is provided by Filliâtre, a senior researcher from the National Center for Scientific Research(CNRS), which verifies Dijkstra's implementation with Why3, a deductive program verification platform that relies on external theorem provers [11][4].

All works presented above are largely dependent on theorem proving systems, however our work relies on a significantly smaller trusted code base. Most proofs in our work will be implemented from scratches, and considerable amount of details on verification is presented explicitly. This reduces the chance of introducing errors into our verification program due to bugs in the proof management systems, and additionally, provides an example of how program verification can be achieved with a general-purpose programming language, and that the implementation is highly similar to that of any other programs.

8 Conclusion

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Appendix

Statutory Declaration