# Dijkstra's Algorithm Verification

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## 1 Dijkstra's Algorithm

### 1.1 Pseudocode

Given input graph $g$ and source node $s$ with types:

g : Graph gsize weight
s : Node gsize

We denote $(u, v)$ as an edge from node $u$ to $v$, $weight(u, v)$ as the weight of edge $(u, v)$. For any two nodes $u, w$ that are not connected by an edge in the graph, we let $weight(u, w)$ equals infinity. We define $unexplored$ as the list of unexplored nodes, and $dist$ as the list storing distance from $s$ to each node $n \in g$

(initially $unexplored$ contains all nodes in graph $g$)
$unexplored : List(Node\ \ gsize)$
$unexplored = \{v : v \in g\}$

(node value is used to index $dist$, initially distance of all nodes are infinity except the source node)
$dist : List\ \ weight$
$dist[s] = 0, dist[a] = infinity, \forall a \in g, a \neq s$

The Dijkstra's Algorithm runs as follows: Given graph $g$ and source node $s$, $dist$ stores the distance value from $s$ to all nodes in $g$ calculated by the Dijkstra's algorithm, $dist[v]$ gives the corresponding distance value of $v$ from $s$. We index $unexplored$ and $dist$ by the number of iterations. Specifically, denote $u_i$ as the node being explored at the $i^th$ iteration, and denote $dist_i$, $unexplored_i$ as the value of distance list and unexplored list at the beginning of the $i^{th}$ iteration. Then during each iteration the Dijkstra's Algorithm calculates $dist, unexplored, explored$ as follows:

$\forall k \geq 1$
`choose` $u_k \in unexplored_k$ `and` $\forall u' \in unexplored_k, dist_k[u_k] \leq dist_k[u']$
$unexplored_{k+1} = unexplored_k - \{u_k\}$
$\forall v \in g$ `,`
$\quad dist_{k+1}[v] = min(dist_k[v], (dist_k[u_k] + weight(u_k, v)))$

### 1.2 Assumptions

1. Weight of edges in graph are non-negative and smaller than infinity
2. Distance value can only be zero, infinity, or summation of edge weights
3. All nodes $n$ and edge $e$ are valid: $n, e \in g$

## 2 Definition

**Definition 2.1. Path**
*(We adopt the definition of path presented in the `Discrete Mathematics with Applications` book by `SUSANNA S. EPP`.)*

A path from node $v$ to $w$ is a finite alternating sequence of adjacent vertices and edges of G, which does not contain any repeated edge or vertex. A path from $v$ to $w$ has the form:

$$ve_0v_0e_1v_2....v_{n-1}e_nw$$

where $e_i$ is an edge in $g$ with endpoints $v_{i-1}, v_i$. We denote the set of paths from $v$ to $w$ as $path(v, w)$.

**Definition 2.2. Prefix of Path**
Given a path from node $v$ to $w$ : $p(v, w) = ve_0v_0e_1v_2....v_{n-1}e_nw$, the prefix of this $v - w$ path is defined as the subsequence of $p(v, w)$ that starts with $v$ and ends with some node $w' \in p(v, w)$ ($w'$ is a vertex in the sequence $p(v, w)$).

**Definition 2.3. Length of Path**
The length of a path $p = ve_0v_0e_1v_2....v_{n-1}e_nw$ is the sum of the weights of all edges in $p$. We write:

$$length(p) = \sum weight(e_i), \forall e_i \in p.$$

**Definition 2.4. Shortest Path**
Denote $\Delta(s, v)$ as the shortest path from $s$ to $v$, and $\delta(v)$ as the length of $\Delta(s, v)$. $\Delta(s, v)$ must fulfills:

$$\Delta(s, v) \in path(s, v)$$
and
$$\forall p' \in path(s, v),\ \delta(v) = length(\Delta(s, v)) \leq length(p')$$

## 3 Proof of Correctness

### 3.1 Proof of Termination

The inner for loop is guaranteed to terminate as the algorithm goes through each adjacent node exactly once. As the size of list `unexplored` decreases by one during each iteration of the while loop, the algorithm is guaranteed to terminate.

### 3.2 Proof of Correctness

Denote *explored* as the list of nodes in $g$ but not in *unexplored*, i.e., *explored* stored all nodes whose neighbors have been updated by the algorithm. We index *explored* by the number of iterations, such that $explored_i$ denotes the value of *explored* at the beginning of the $i^{th}$ iteration.

**Lemma 3.1.** Given any two nodes $v, w$, the prefix of the shortest path $\Delta(v, w)$ is also a shortest path.

*Proof.* We will prove Lemma 3.1 by contradiction.
Consider any node $q$ in the sequence of $\Delta(v, w)$, we have $\Delta(v, w) = ve_0v_0e_1v_2...v_iqv_j....v_{n-1}e_nw$. Suppose the prefix of $\Delta(v, w)$ from $v$ to $q$, denote as $p(v, q)$, is not the shortest path from $v$ to $q$. Then we know $p(v, q) = ve_0v_0e_1v_2...v_iq$ is a path from $v$ to $q$ and $length(p(v, q)) > length(\Delta(v, q))$.
Based on the definition of shortest path, we know:

$$length(\Delta(v, w)) \leq length(p), \forall p \in path(v, w)$$

Fenote the path after the node $q$ as $p(q, w) = qv_j....v_{n-1}e_nw$, since $\Delta(v, w) = ve_0v_0e_1v_2...v_iqv_j....v_{n-1}e_nw$, then $\Delta(v, w) = p(v, q)+p(q, w)$, and that $length(\Delta(v, w)) = length(p(v, q))+length(p(q, w))$. Then we have:

$$length(\Delta(v, w)) = length(p(v, q)) + length(p(q, w)) \leq length(p), \forall p \in path(v, w)$$

Since $p(v, q)$ is not the shortest path from $v$ to $q$ by assumption, there exists another $v - w$ path $p'(v, w)$ such that:

$$\begin{aligned}
&p'(v, w) \in path(v, w) \\
&p'(v, w) = \Delta(v, q) + p(q, w) \\
&length(p'(v, w)) = length(\Delta(v, q)) + length(p(q, w)) \\
&\qquad\qquad\qquad < length(p(v, q)) + length(p(q, w)) \\
&\text{i.e. } length(p'(v, w)) < length(\Delta(v, w))
\end{aligned}$$

Hence we have reached a contradiction. Thus by the principle of prove by contradiction, for any the prefix $p(v, q)$ of $\Delta(v, w)$ is the shortest path from $v$ to $q$. Lemma 3.1 holds. □

**Lemma 3.2.** After the $n^{th}$ iteration for $n \geq 1$, forall node $v \in explored_{n+1}$, if $dist_{n+1}[v] \neq infinity$, then $dist_{n+1}[v]$ is the length of some $s - v$ path, i.e, $path(s, v) \neq \emptyset$.

*Proof.* We will prove `Lemma 3.2` by inducting on the number of iterations.
Let P(n) be: After the $n^{th}$ iteration, $n \geq 1$, for all node $v \in g$, if $dist_{n+1}[v] \neq infinity$, then $dist_{n+1}[v]$ is the length of some $s - v$ path.
**Base Case**: We shall show P(1) holds.
Based on the algorithm, initially $dist_1[s] = 0$ and for all node $v \in g, v \neq s, dist_1[v] = infinity$, then $s$ is the only node whose distance value is not infinity. Based on the definition of path, the path from the source node $s$ to itself is $s$, $path(s, s) = \{s\}$. Hence P(1) holds.
**Inductive Hypothesis**: Suppose $\forall i, 1 \leq i \leq k$, P(i) holds. That is, after the $i^{th}$ iteration, $1 \leq i \leq k$, for all nodes $v \in g$, if $dist_{i+1}[v] \neq infinity$, then $dist_{n+1}[v]$ is the length of some $s - v$ path.
**Inductive Step**: We shall show P(k+1) holds.

For node $u_{k+1}$ being explored during the $(k+1)^{th}$ iteration, based on the algorithm, $dist_{k+1}[u_{k+1}]$ is calculated as:

$dist_{k+2}[u_{k+1}] = min(dist_{k+1}[u_{k+1}], dist_{k+1}[u_{k+1}] + weight(u_{k+1}, u_{k+1}))$

Since the distance value from $u_{k+1}$ to itself is 0, then $dist_{k+2}[u_{k+1}] = dist_{k+1}[u_{k+1}]$, and that $dist_{k+2}[u_{k+1}]$ and $dist_{k+1}[u_{k+1}]$ are the length of the same $s - u_{k+1}$ path if there exists one.
If $dist_{k+2}[u_{k+1}] \neq infinity$, then $dist_{k+1}[u_{k+1}] = dist_{k+2}[u_{k+1}] \neq infinity$. Since $k \leq k$ and $dist_{k+1}[u_{k+1}] \neq infinity$, then based on the inductive hypothesis, $dist_{k+1}[u_{k+1}]$ is the length of some $s - u_{k+1}$ path, and hence $dist_{k+2}[u_{k+1}]$ is the length of some $s - u_{k+1}$ path.

Based on the algorithm, we have $dist_{k+2}[v] = min(dist_{k+1}[v], dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v))$. There are two cases:

- `Case 1:` $dist_{k+1}[v] < dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v)$.
  In this case, $dist_{k+2}[v] = dist_{k+1}[v]$. Then if $dist_{k+2}[v] \neq infinity$, we have $dist_{k+1}[v] \neq infinity$, and that $dist_{k+2}[v]$ and $dist_{k+1}[v]$ are the length of the same $s - v$ path if there exists one. Since $dist_{k+1}[v] \neq infinity$, the inductive hypothesis implies that $dist_{k+1}[v]$ is the length of some $s-v$ path, hence $dist_{k+2}[v]$ is the length of some $s - v$ path. P(k+1) holds.

- `Case 2:` $dist_{k+1}[v] \geq dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v)$
  Under this case, $dist_{k+2}[v] = dist_{k+1}[u_{k+1}] + weight(w, v)$. If $dist_{k+2}[v] \neq infinity$, then it follows that $dist_{k+1}[u_{k+1}] = dist_{k+2}[v] - weight(u_{k+1}, v) \neq infinity$. Then the inductive hypothesis implies that $dist_{k+1}[u_{k+1}]$ must be the length of some $s - u_{k+1}$ path, denote as $p(s, u_{k+1})$. Since there is an edge $(u_{k+1}, v) \in g$, then $dist_{k+2}[v] = dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v)$ must be the length of the $s - v$ path through $u_{k+1}$. P(k+1) holds.

Hence P(k+1) holds for $u_{k+1}$ and all nodes $v \in g$ other than $u_{k+1}$. By the principle of prove by induction, P(n) holds. `Lemma 3.2` proved. □

**Lemma 3.3.** For any node $v \in g$, if after the $i^{th}$ iteration, $dist_{i+1}[v] = \delta(v)$, then for each proceeding $j^{th}$ iteration, $j > i$, $dist_{j+1}[v] = dist_{i+1}[v] = \delta(v)$.

*Proof.* We will prove `Lemma 3.3` by induction on the number iterations after the $i^{th}$ iteration.
Let P(n) be: For any node $v \in g$, if after the $i^{th}$ iteration, $dist_{i+1}[v] = \delta(v)$, then for the $(i+n)^{th}$ iteration, $n \geq 1$, $dist_{i+n+1}[v] = dist_{i+1}[v] = \delta(v)$
**Base Case**: We shall show P(1) holds.
During the $(i+1)^{th}$ iteration, suppose $u_{i+1}$ is the node being explored, then $dist_{i+2}[v]$ is calculated as:

$$dist_{i+2}[v] = min(dist_{i+1}[v], dist_{i+1}[u_{i+1}] + weight(u_{i+1}, v))$$

If $(u_{i+1}, v) \in g$, $dist_{i+1}[u_{i+1}]$ is the length of some $s - u_{i+1}$ path, then $(dist_{i+1}[u_{i+1}] + weight(u_{i+1}, v))$ is the length of some $s - v$ path. Since $dist_{i+1}[v] = \delta(v)$, based on the definition of shortest path, $dist_{i+1}[v] \leq dist_{i+1}[u_{i+1}] + weight(u_{i+1}, v)$, hence $dist_{i+2}[v] = dist_{i+1}[v] = \delta(v)$.

If $u_{i+1}$ does not have an edge to $v$, $weight(u_{i+1}, v) = \infty$, we have:

$$\begin{aligned} dist_{i+2}[v] &= min(dist_{i+1}[v], dist_{i+1}[u_{i+1}] + weight(u_{i+1}, v)) \\ &= min(dist_{i+1}[v], dist_{i+1}[u_{i+1}] + \infty) \\ &= dist_{i+1}[v] = \delta(v). \end{aligned}$$

Hence $dist_{i+2}[v] = dist_{i+1}[v] = \delta(v)$. P(1) holds.

**Inductive Hypothesis**: Suppose P(k) holds, that is, if after the $i^{th}$ iteration, $dist_{i+1}[v] = \delta(v)$, then for the $(i+k)^{th}$ iteration, $n \geq 1$, $dist_{i+k+1}[v] = dist_{i+1}[v] = \delta(v)$.

**Inductive Step**: We shall show P(k+1) holds.
Based on the algorithm, for $dist_{i+k+2}[v]$, we have:

$$dist_{i+k+2}[v] = min(dist_{i+k+1}[v], dist_{i+k+1}[u_{i+k+1}] + weight(u_{i+k+1}, v))$$

If $u_{i+k+1}$ does not have an edge to $v$, then $weight(u_{i+k+1}, v) = \infty$, we have:

$$\begin{aligned} dist_{i+k+2}[v] &= min(dist_{i+k+1}[v], dist_{i+k+1}[u_{i+k+1}] + weight(u_{i+k+1}, v)) \\ &= min(dist_{i+k+1}[v], dist_{i+k+1}[u_{i+k+1}] + \infty) \end{aligned}$$

Based on our inductive hypothesis, $dist_{i+k+1}[v] = dist_{i+1}[v] = \delta(v)$, then if $weight(u_{i+k+1}, v) = \infty$, we have $dist_{i+k+2}[v] = dist_{i+k+1}[v] = \delta(v)$.
If $weight(u_{i+k+1}, v) \neq \infty$, $dist_{i+k+1}[u_{i+k+1}]$ is the length of some $s - u_{i+k+1}$ path, then $(dist_{i+k+1}[u_{i+1}] + weight(u_{i+k+1}, v))$ is the length of some $s - v$ path. Based on the definition of shortest path distance, $dist_{i+k+1}[v] = \delta(v) \leq (dist_{i+k+1}[u_{i+1}] + weight(u_{i+k+1}, v))$. Hence:

$$\begin{aligned} dist_{i+k+2}[v] &= min(dist_{i+k+1}[v], dist_{i+k+1}[u_{i+k+1}] + weight(u_{i+k+1}, v)) \\ &= min(\delta(v), dist_{i+k+1}[u_{i+k+1}] + weight(u_{i+k+1}, v)) \\ &= \delta(v) = dist_{i+1}[v] \end{aligned}$$

Thus P(k+1) holds. By the principle of prove by induction, P(n) holds. `Lemma 3.3` proved. □

**Lemma 3.4.** For any node $v \in g$, for each $u_i \in explored_{n+1}$, $n \geq 1, 1 \leq i \leq n$, $dist_{n+1}[v] \leq dist_i[u_i] + weight(u_i, v)$.

*Proof.* We will prove Lemma 3.4 by inducting on the number $n$.
Let P(n) be: for any node $v \in g$, for each $u_i \in explored_{n+1}$, $n \geq 1, 1 \leq i \leq n$, $dist_{n+1}[v] \leq dist_i[u_i] + weight(u_i, v)$.
**Base Case**: We shall show P(1) holds.
Based on the algorithm, $dist_1[s] = 0$, and for all node $v \in g$ other than $s$, $dist_1[v] = \infty$, and $explored_2$ only contains $s$. For node $s$, $dist_2[s] = 0 \leq dist_1[s] + weight(s, s) = 0$. For all node $v \in g$ other than $s$, we have:

$$\begin{aligned} dist_2[v] &= min(dist_1[v], dist_1[s] + weight(s, v)) \\ &\leq dist_1[s] + weight(s, v) \end{aligned}$$

Since $s$ is the only node in $explored_2$, then the above equation directly shows that P(1) holds.

**Induction Hypothesis**: Suppose P(k) holds for $k > 1$. That is, for any node $v \in g$, for each $u_i \in explored_{k+1}$, $k > 1, 1 \leq i \leq k$, $dist_{k+1}[v] \leq dist_i[u_i] + weight(u_i, v)$.

**Inductive Step**: we shall show P(k+1) holds. That is, for $k + 1 > 1$, forall nodes $v \in g$, for each $u_i \in explored_{k+2}$, $k > 1, 1 \leq i \leq k+1$, $dist_{k+2}[v] \leq dist_i[u_i] + weight(u_i, v)$.
Suppose $u_{k+1}$ is the node being explored during the $(k+1)^{th}$ iteration, then $explored_{k+2} = explored_{k+1} \cup \{u_{k+1}\}$. Forall node $v \in g$, we have:

$$dist_{k+2}[v] = min(dist_{k+1}[v], dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v))$$

Hence we have:

$$\begin{aligned} dist_{k+2}[v] &\leq dist_{k+1}[v]([E3.4.1]) \\ dist_{k+2} &\leq dist_{k+1}[u_{k+1}] + weight(u_{k+1}, v)([E3.4.2]) \end{aligned}$$

The induction hypothesis implies that $dist_{k+1}[v] \leq dist_i[u_i] + weight(u_i, v), \forall u_i \in explored_{k+1}$. Combining with [E3.4.1], we have:

$$dist_{k+2}[v] \leq dist_i[u_i] + weight(u_i, v), \forall u_i \in explored_{k+1}[E3.4.3]$$

Since $explored_{k+2} = explored_{k+1} \cup \{u_{k+1}\}$, then equation [E3.4.2] and equation [E3.4.3] implies that $dist_{k+2}[v] \leq dist_i[u_i] + weight(u_i, v), \forall u_i \in explored_{k+1} \cup \{u_{k+1}\} = explored_{k+2}$. P(k+1) holds. By the principle of prove by induction, P(n) holds. `Lemma 3.4` proved. □

**Lemma 3.5.** Assume $g$ is a connected graph. Forall node $v \in explored_{n+1}$:

1. $dist_{n+1}[v] < \infty$
2. $dist_{n+1}[v] \leq \delta(v')$, $\forall v' \in unexplored_{n+1}$.
3. $dist_{n+1}[v] = \delta(v)$

*Proof.* We will prove `Lemma 3.5` by inducting on the number of iterations.

Let P(n) be: For a connected graph $g$, for $n \geq 1$, forall node $w \in explored_{n+1}$: (L1) $dist_{n+1}[w] < \infty$; (L2) $dist_{n+1}[w] \leq \delta(w')$, $\forall w' \in unexplored_{n+1}$; (L3) $dist_{n+1}[w] = \delta(w)$.

**Base Case**: We shall show P(1) holds
Based on the algorithm, during the first iteration, the node with minimum distance value is the source node $s$ with $dist_1[s] = 0$. Hence during the first iteration, only $s$ is removed from $unexplored_1$ and added to $explored_2$. Since $dist_2[s] = 0 < \infty$, then (L1) holds for P(1). Since all edge weights are non-negative, then the shortest distance value from $s$ to $s$ is indeed 0, hence $dist_2[s] = 0 = \delta(s)$ and $dist_2[s] \leq \delta(v')$, $\forall v' \in unexplored_2$. Thus (L2) and (L3) holds for P(1). Hence P(1) holds.

**Induction Hypothesis**: Suppose P(i) is true for all $1 \leq i \leq k$. That is, forall $1 < i \leq k$, forall node $w \in explored_{i+1}$: (L1) $dist_{i+1}[w] < \infty$; (L2) $dist_{i+1}[w] \leq \delta(w'), \forall w' \in unexplored_{i+1}$; (L3) $dist_{i+1}[w] = \delta(w)$;

**Inductive Step**: We shall show P(k+1) holds. That is, forall node $w \in explored_{k+2}$, (L1) $dist_{k+2}[w] \neq \infty$; (L2) $dist_{k+2}[w] \leq \delta(w'), \forall w' \in unexplored_{k+2}$; (L3) $dist_{k+2}[w] = \delta(w)$;

Suppose $u_{k+1}$ is the node added into *explored* during the $(k+1)^{th}$ iteration, then $explored_{k+2} = explored_{k+1} \cup \{u_{k+1}\}$. We will show that (L1)(L2) and (L3) holds for all nodes in $explored_{k+1}$ in `Part (a)`, and `Part (b)` proves (L1)(L2)(L3) holds for $u_{k+1}$, so that the statements holds forall nodes in $explored_{k+2}$.

- Part(a): `WTP: After the` $(k+1)^{th}$ `iteration,` $\forall w \in explored_{k+1}$`, (L1)(L2)(L3) holds.`

  Consider each node $q \in (explored_{k+1} \cap explored_{k+2}) = explored_{k+1}$, $q$ must be explored before the $(k+1)^{th}$ iteration. Suppose $q$ is explored during the $i^{th}$ iteration for some $i < k+1$, then based on our induction hypothesis, $dist_{i+1}[q] = \delta(q)$, and $\delta(q) \leq \delta(q'), \forall q' \in unexplored_{i+1}$.

  `Proof of (L3)`: Since for each node $q \in explored_{k+1}$, the induction hypothesis implies that $dist_{k+1}[q] = \delta(q)$, then `Lemma 3.3` imples that $dist_{k+2}[q] = dist_{k+1}[q] = \delta(q)$. (L3) holds for $explored_{k+1}$.

  `Proof of (L2)`: Based on the algorithm, for each iteration, the algorithm explores exactly one node and never revisits any explored nodes. For each node $q \in explored_{k+1}$ mentioned above, since $q$ is explored before the $(k+1)^{th}$ iteration, then $unexplored_{k+1} \subseteq unexplored_{i+1}$. Since $\delta(q) \leq \delta(q'), \forall q' \in unexplored_{i+1}$, and $unexplored_{i+1}$ includes all node in $unexplored_{k+1}$, then $\delta(q) \leq \delta(q'), \forall q' \in unexplored_{k+1}$. Since proof of (L3) above shows that $dist_{k+2}[q] = \delta(q)$, then $dist_{k+2}[q] \leq \delta(q'), \forall q' \in unexplored_{k+1}$. (L2) holds for $explored_{k+1}$.

  `Proof of (L1)`: Since the induction hypothesis implies that $\forall q \in explored_{k+1}, dist_{k+1}[q] < \infty$, and the proof of (L3) above shows that $dist_{k+2}[q] = dist_{k+1}[q]$, then $dist_{k+2}[q] < \infty$. (L1) holds for $explored_{k+1}$.

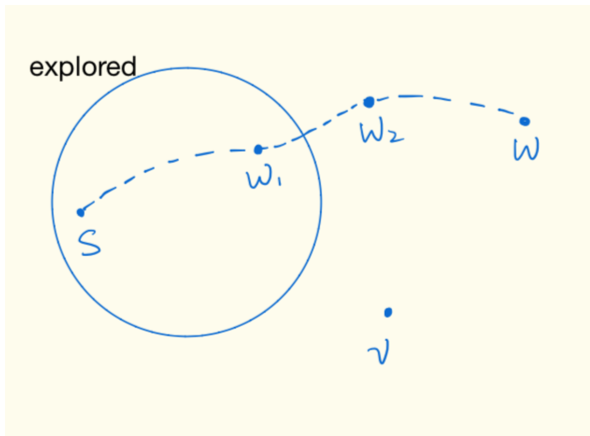
  Hence we have proved that both (1) and (2) holds for all nodes in $explored_{k+1}$.

- Part(b): (L1)(L2)(L3) holds for $\{u_{k+1}\}$.
  Specifically, we want to show: (L1) $dist_{k+2}[u_{k+1}] < \infty$; (L2) $dist_{k+2}[u_{k+1}] \leq \delta(v'), \forall v' \in unexplored_{k+2}$, and (L2) $dist_{k+2}[u_{k+1}] = \delta(u_{k+1})$.

  1. (L1) $dist_{k+2}[u_{k+1}] \neq \infty$
     Since $g$ is a connected graph, then $s$ must have a path to $u_{k+1}$. Since $u_{k+1}$ is the node currently being explored, then we know there must exists a $s - u_{k+1}$ path, denote as $p(s, u_{k+1})$, such any node proceeding $u_{k+1}$ in $p(s, u_{k+1})$ are explored before $u_{k+1}$, i.e., in $explored_{k+1}$.
     Denote the node right before $u_{k+1}$ in $p(s, u_{k+1})$ as $u'$, $u' \in explored_{k+1}$. Suppose $u'$ is explored during the $i^{th}$ iteration, $i < k+1$. The induction hypothesis implies that $dist_{i+1}[u'] < \infty$. Since $dist_{i+1}[u'] = min(dist_i[u'], dist_i[u'] + weight(u', u')) = min(dist_i[u'], dist_i[u'] + 0) = dist_i[u']$, then $dist_i[u'] < \infty$. `Lemma 3.4` implies $dist_{k+2}[u_{k+1}] \leq dist_i[u'] + weight(u', u_{k+1}]$, then it follows that $dist_{k+1}[u_{k+1}] < \infty$. (L1) holds for $u_{k+1}$.

  2. (L2) $dist_{k+2}[u_{k+1}] \leq \delta(v'), \forall v' \in unexplored_{k+2}$

We will prove (L2) by contradiction. Suppose there exists $w \in unexplored_{k+2}$, such that $dist_{k+2}[u_{k+1}] > \delta(w)([E3.5.1])$.
Consider the shortest path $\Delta(s,w)$ from $s$ to $w$, $\delta(w) = length(\Delta(s,w))$. Since $w \notin explored_{k+2}$, then there must exists some node in $\Delta(s,w)$ that are not in $explored_{k+2}$. Suppose the first node along $\Delta(s,w)$ that is not in the $explored_{k+2}$ list is $w_2$, and the node right before $w_2$ in the $s$ to $w_2$ subpath is $w_1$, thus $w_1 \in explored_{k+2}$. The image below illustrates this construction:




Denote the subpath from $s$ to $w_1$ in $\Delta(s,w)$ as $p(s,w_1)$, subpath from $s$ to $w_2$ in $\Delta(s,w)$ as $p(s,w_2)$, and subpath $w_2$ to $w$ as $p(w_2,w)$. Based on `Definition 2.2 Prefix of Path`, $p(s,w_1)$ is a prefix of $\Delta(s,w)$. Since $p(s,w_1)$ is the prefix of the shortest $s-w$ path, then based on `Lemma 3.1`, $p(s,w_1)$ is the shortest path from $s$ to $w_1$, $\Delta(s,w_1) = p(s,w_1)$, $length(p(s,w_1)) = \delta(w_1)$.
Similarly, since $p(s,w_2) = p(s,w_1) + (w_1,w_2)$, then $p(s,w_2)$ is a prefix of $\Delta(s,w)$, and hence `Lemma 3.1` implies that $p(s,w_2)$ is the shortest path from $s$ to $w_2$. Then we have:

$$\begin{aligned}
\Delta(s,w_2) &= p(s,w_2) = p(s,w_1) + (w_1,w_2) \\
\delta(w_2) &= length(\Delta(s,w_2)) \\
&= length(p(s,w_2)) \\
&= length(p(s,w_1)) + weight(w_1,w_2) \\
&= \delta(w_1) + weight(w_1,w_2)([E3.5.2])
\end{aligned}$$

For $\Delta(s,w)$ we have:

$$\begin{aligned}
\delta(w) &= length(p_w) \\
&= length(p(s,w_1)) + weight(w_1,w_2) + length(p(w_2,w)) \\
&= \delta(w_1) + weight(w_1,w_2) + length(p(w_2,w))
\end{aligned}$$

Since all edge weights are non-negative, then:

$$\delta(w_2) = \delta(w_1) + weight(w_1,w_2) \leq \delta(w) \ \ ([E3.5.3])$$

Since $w_1 \in explored_{k+2}$, there are two cases to consider: $w_1 = u_{k+1}$ and $w_1 \neq u_{k+1}$. We will prove P(k+1) under both cases below.

`Case 1:` $w_1 = u_{k+1}$
Since $\delta(w_2) = \delta(w_1) + weight(w_1, w_2) \leq \delta(w)$ and all edge weights are non-negative, then $\delta(w_1) \leq \delta(w)$. When $w_1 = u_{k+1}$, we have $\delta(u_{k+1}) \leq \delta(w)$. Since $dist_{k+2}[u_{k+1}] > \delta(w)$ and $\delta(u_{k+1}) \leq \delta(w)$, we have $\delta(u_{k+1}) < dist_{k+2}[u_{k+1}]$.
Suppose the node right before $u_{k+1}$ in $\Delta(s, u_{k+1})$ is $w_3$. We know $length(\Delta(s, u_{k+1})) = length(p(s, w_3)) + weight(w_3, u_{k+1}))$, where $p(s, w_3)$ is the prefix of $\Delta(s, u_{k+1})$. Based on `Lemma 3.1`, we know $length(p(s, w_3)) = \delta(w_3)$. Hence:

$$\begin{aligned}\delta(u_{k+1}) &= length(p(s, w_3)) + weight(w_3, u_{k+1})) \\ &= \delta(w_3) + weight(w_3, u_{k+1}) \\ &< dist_{k+2}[u_{k+1}]\end{aligned}$$

i.e.

$$dist_{k+2}[u_{k+1}] > \delta(w_3) + weight(w_3, u_{k+1})([E3.5.6])$$

Based on the construction, $w_2$ is the first node along $\Delta(s, w)$, $w_1$ is right before $w_2$ in the path, $w_3$ is right before $w_1 = u_{k+1}$ in the path, then $w_3 \in explored_{k+2}$. Assume $w_3$ is explored during the $j^{th}$ iteration. Then based on `Lemma 3.4`, we have:

$$dist_{k+2}[u_{k+1}] \leq dist_j[w_3] + weight(w_3, u_{k+1})([E3.5.7])$$

The induction hypothesis implies $dist_{j+1}[w_3] = \delta(w_3)$. For $dist_{j+1}[w_3]$ we have:

$$\begin{aligned}dist_{j+1}[w_3] &= min(dist_j[w_3], dist_j[w_3] + weight(w_3, w_3)) \\ &= min(dist_j[w_3], dist_j[w_3] + 0) \\ &= dist_j[w_3]\end{aligned}$$

Hence $dist_j[w_3] = \delta(w_3)$, combine with [E3.5.7], we have:

$$dist_{k+2}[u_{k+1}] \leq \delta(w_3) + weight(w_3, u_{k+1})([E3.5.8])$$

The equation [E3.5.8] contradicts with equation[E3.5.6]. Hence by the principle of prove by contradiction, (L2) holds when $w_1 = u_{k+1}$.

`Case 2:` $w_1 \neq u_{k+1}$
Since $w_1 \in explored_{k+2}$ and $w_1 \neq u_{k+1}$, $w_1$ is explored before the $(k+1)^{th}$ iteration. i.e., $w_1 \in explored_{k+1}$. Suppose $w_1$ is being explored during the $i^{th}$ iteration, $i < k+1$, then based on the algorithm, the value of $dist_{i+1}[w_1]$ is calculated as:

$$\begin{aligned}dist_{i+1}[w_1] &= min(dist_i[w_1], dist_i[w_1] + weight(w_1, w_1)) \\ &= min(dist_i[w_1], dist_i[w_1] + 0) \\ &= min(dist_i[w_1], dist_i[w_1]) \\ &= dist_i[w_1]\end{aligned}$$

Since the induction hypothesis implies that $dist_{i+1}[w_1] = \delta(w_1)$, then $dist_i[w_1] = \delta(w_1)$.
Since $w_1$ has an edge to $w_2$, then $dist_{i+1}[w_2]$ must have been updated according as follows:

$$\begin{aligned} dist_{i+1}[w_2] &= min(dist_i[w_2], dist_i[w_1] + weight(w_1, w_2)) \\ &= min(dist_i[w_2], \delta(w_1) + weight(w_1, w_2)) \end{aligned}$$

Based on [E3.5.2] we know that $\delta(w_2) = \delta(w_1)+weight(w_1, w_2)$, then $dist_{i+1}[w_2] = min(dist_i[w_2], \delta(w_2))$.
If $dist_i[w_2] = \infty$, then $dist_{i+1}[w_2] = min(dist_i[w_2], \delta(w_2)) = \delta(w_2)$. If $dist_i[w_2] \neq \infty$, then based on `Lemma 3.2`, $dist_i[w_2]$ is the length of some $s - w_2$ path. Since $\delta(w_2) \leq length(p), \forall p \in path(s, w_2)$, then $dist_{i+1}[w_2] = min(dist_i[w_2], \delta(w_2)) = \delta(w_2)$. Hence in either cases, we conclude that $dist_{i+1}[w_2] = \delta(w_2)$.
Since $dist_{i+1}[w_2] = \delta(w_2)$ and $i < k + 1$, then based on `Lemma 3.3`, we have:

$$dist_{k+1}[w_2] = dist_{i+1} = \delta(w_2)([E3.5.4])$$

Based on our assumption, at the beginning of the $(k + 1)^{th}$ generation, $u_{k+1}, w_2 \notin explored_{k+1}$ and $u_{k+1}$ is selected by the algorithm, then we must have $dist_{k+1}[w_2] \geq dist_{k+1}[u_{k+1}]$. For $dist_{k+2}[u_{k+1}]$ we have:

$$\begin{aligned} dist_{k+2}[u_{k+1}] &= min(dist_{k+1}[u_{k+1}], dist_{k+1}[u_{k+1}] + weight(u_{k+1}, u_{k+1})) \\ &= min(dist_{k+1}[u_{k+1}], dist_{k+1}[u_{k+1}] + 0) \\ &= dist_{k+1}[u_{k+1}] \end{aligned}$$

Hence $dist_{k+1}[w_2] \geq dist_{k+2}[u_{k+1}]$. Combine with [E3.5.4], [E3.5.3] we have:

$$\begin{aligned} &dist_{k+1}[w_2] \geq dist_{k+2}[u_{k+1}]( \\ &dist_{k+1}[w_2] = dist_{i+1} = \delta(w_2)(from[E3.5.4]) \\ &\quad \delta(w) \geq \delta(w_2) = \delta(w_1) + weight(w_1, w_2)(from[E3.5.3]) \end{aligned}$$

Hence $\delta(w) \geq dist_{k+2}[u_{k+1}]$, which contradicts with [E3.5.1]. Hence by the principle of prove by contradiction, when $w_1 \neq u_{k+1}$, $dist_{k+2}[u_{k+1}] \leq \delta(w), \forall w \in unexplored_{k+2}$.
(L2) holds for $u_{k+1}$.

3. (L3) $dist_{k+2}[u_{k+1}] = \delta(u_{k+1})$

   We will prove this by contradiction.
   Since (L1) proves $dist_{k+2}[u_{k+1}] \neq \infty$, then `Lemma 3.2` implies that $dist_{k+2}[u_{k+1}]$ is the length of some $s - u_{k+1}$ path, denote as $p$. Suppose there is a $s - u_{k+1}$ path $p'$ that's shorter than $p$, i.e, $dist_{k+2}[u_{k+1}] > length(p')([E3.5.9])$. Suppose the node right before $u_{k+1}$ in $p'$ is $v'$. Then we know:

$$\begin{aligned} &length(p') = length(p(s, v')) + weight(v', u_{k+1}) \\ &length(p') < dist_{k+2}[u_{k+1}] \end{aligned}$$

   , where $p(s, v')$ is the prefix of $p'$ from $s$ to $v'$. Hence:

$$dist_{k+2}[u_{k+1}] > length(p(s, v')) + weight(v', u_{k+1})$$

Based on the definition of shortest path, $length(p(s, v')) \geq \delta(v')$, then we have:

$$dist_{k+2}[u_{k+1}] > \delta(v') + weight(v', u_{k+1})([E3.5.10])$$

There are two cases to consider: (1) $v' \in explored_{k+2}$; (2) $v' \notin explored_{k+2}$

`Case(1):` $v' \in explored_{k+2}$
Suppose $v'$ is explored during the $i^{th}$ iteration. Then `Lemma 3.4` implies:

$$dist_{k+2}[u_{k+1}] \leq dist_i[v'] + weight(v', u_{k+1})([E3.5.11])$$

The induction hypothesis implies $dist_{i+1}[v'] = \delta(v')$, and for $dist_{i+1}[v']$ we have:

$$\begin{aligned} dist_{i+1}[v'] &= min(dist_i[v'], dist_i[v'] + weight(v', v')) \\ &= min(dist_i[v'], dist_i[v'] + 0) \\ &= dist_i[v'] \end{aligned}$$

Hence $dist_i[v'] = \delta(v')$. Combining [E3.5.11], we have:

$$dist_{k+2}[u_{k+1}] \leq \delta(v') + weight(v', u_{k+1})([E3.5.12])$$

Hence equation [E3.5.12] contradicts with equaltion [E3.5.10]. By the principle of prove by contradiction, (L3) holds when $v' \in explored_{k+2}$.

`Case(2):` $v' \notin explored_{k+2}$
Since $length(p') = length(p(s, v')) + weight(v', u_{k+1})$, $p(s, v)$ is the prefix of $p'$ from $s$ to $v'$, then based on the definition of shortest path, $length(p(s, v')) \leq \delta(v')$, and thus $\delta(v') + weight(v', u_{k+1}) \leq length(p(s, v')) + weight(v', u_{k+1}) = length(p')$. Since all edge weights are non-negative, then $\delta(v') \leq length(p')$.
Since $v' \notin explored_{k+2}$, i.e., $v' \in unexplored_{k+2}$, based on proof of (L2), $dist_{k+2}[u_{k+1}] \leq \delta(v')$. Since $dist_{k+2}[u_{k+1}] \leq \delta(v')$ and $\delta(v') \leq length(p')$, then $dist_{k+2}[u_{k+1}] \leq length(p')$, which contradicts with our assumption ([E3.5.9]). Hence by the principle of prove by contradiction, (L3) holds when $v' \notin explored_{k+2}$.
Since we have proved (L3) for both cases, then (L3) holds for P(K+1).

Since we have proved (L1)(L2)(L3) forall nodes in $explored_{k+1}$ after the $(k+1)^{th}$ iteration, P(k+1) holds. Then by the principle of prove by induction, `Lemma 3.5` holds. □

*Proof.* **Prove of Correctness**

By applying `Lemma 3.5(L3)` to the last iteration of the algorithm, we obtained that for all nodes $n$ in the explored list, $dist[n]$ is indeed the shortest path distance value from source $s$ to $n$, hence Dijkstra's algorithm indeed calculates the shortest path distance value from the source $s$ to each node $n \in g$. □