MAT 455 Midterm Formula Sheet

Poisson Pdf, μ, σ^2: $\frac{e^{-\lambda}\lambda^x}{x!}, \lambda, \lambda$
Uniform Pdf, μ, σ^2: $\frac{1}{b-a}, \frac{b+a}{2}, \frac{(b-a)^2}{12}$
Binomial Pdf, μ, σ^2: $\binom{n}{x}p^x(1-p)^{n-x}, np, np(1-p)$
Geometric Pdf, μ, σ^2: $(1-p)^{x-1}p, \frac{1}{p}, \frac{1-p}{p^2}$
Exponential | parameter λ Pdf, μ, σ^2: $\lambda e^{-\lambda x}, 1/\lambda, 1/\lambda^2$ AND **Exponential $P(X > a)$:** $e^{-a\lambda}$
Normal Pdf: $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Bayes' Rule: $P(B|A) = \frac{P(A|B)P(B)}{\sum_j P(A|B_j)P(B_j)}$
Marginal pmf: $P(X = x) = \sum_y P(X, Y)$
Law of Total Probability: $P(A) = \sum P(A|B_i)P(B_i)$
Conditional Exp: $E(Y|X = x) = \sum yP(Y = y|X = x)$ or $E(Y|X = x) = \int yP(Y = y|X = x)dy$
Properties of Conditional Exp:
1) $E(aY + bZ|X = x) = aE(Y|X = x) + bE(Z|X = x)$
2) $E(g(Y)|X = x) = \sum g(y)P(Y = y|X = x)$ or $E(g(Y)|X = x) = \int g(y)f_{y|x}(y|x)dy$
3) If X and Y are independent, $E(Y|X = x) = E(Y)$
4) If Y is a function of X, Y=g(X), $E(Y|X = x) = g(x)$
Conditional Expectation of an Event: Let $I_A = 1$, when A occurs, then $E(Y|A) = \frac{E(YI_A)}{P(A)}$
Conditional Dist: $P(Y = y|X = x) = \frac{P(X=x,Y=y)}{P(X=x)}$ or $f_{y|x}(y|x) = \frac{f(x,y)}{f(x)}$, where $f(x) = \int f(x, y)dy$
Conditional Variance: $Var(Y|X = x) = E(Y^2|X = x) - (E(Y|X = x))^2$
Law of Total Expectation: $E(Y) = \sum E(Y|X = x)P(X = x)$ or $E(Y) = \int E(Y|X = x)f_X(x)dx$
Law of Total Variance: $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$
Random Sums of Random Variables:
If your random variable has mean, μ_x, and variance, σ_x^2, and you have a sum of N values where $E(N) = \mu_N$ and $Var(N) = \sigma_N^2$. Then
$E(T) = \mu_x\mu_N$ and $Var(T) = \sigma_X^2\mu_N + \sigma_N^2\mu_X^2$.

Markov Chain: $(P(X_{n+1} = j|_{X_0=i_0,\ldots,X_{n-1}=i_{n-1},X_n=i}) = P(X_{n+1} = j|X_n = i) = (P(X_1 = j|X_0 = i)$
Time-homogeneous. Transition Matrix with non-negative entries and rows sum to 1.
Stochastic Matrix: $P_{ij} \geq 0$ for all i, j. And each row, i, $\sum_j P_{ij} = 1$
Graphs: (Weighted $w(i, j)$, Directed): $P_{ij} = \frac{w(i,j)}{w(i)}$
N-Step Transition Matrix:
P^n is the N step transition matrix of the chain. $P_{ij}^n = P(X_n = j|X_0 = i)$, for all i, j.
Distrib of X_n: $P(X_n = j) = \alpha\mathbb{P}_j^n$: Like the "Marginal Distribution" & Initial Dist α is Row vector
Joint Distribution of $\mathbb{P}^n$: For all $0 \geq n_1 \geq n_2 \geq n_3$ and states i_1, i_2, i_3
$P(X_{n_1} = i_1, X_{n_2} = i_2, X_{n_3} = i_3) = (\alpha P^{n_1})_{i_1} * (P^{n_2-n_1})_{i_1i_2} * (P^{n_3-n_2})_{i_2i_3}$ Can Generalize.
Chapman-Kolmogorov Relationship: If $P^{m+n} = P^mP^n$, then $P_{ij}^{m+n} = \sum_k P_{ik}^m P_{kj}^n$ for all i, j.

Limiting Distribution: The prob. dist. λ with the property that for all i & j:
0) $\lim_{n\to\infty} \mathbb{P}_{ij}^n = \lambda_j$
i) $\lim_{n\to\infty} P(X_n = j) = \lambda_j$
ii) For any initial distribution α: $\lim\limits_{n\to\infty} \alpha\mathbb{P}^n = \lambda$
iii) $\lim\limits_{n\to\infty} \mathbb{P}^n = \Lambda$, where Λ is a stochastic matrix whose rows are equal to λ.

Stationary Distribution: $\pi\mathbb{P} = \pi$ and $\pi_j = \sum\limits_i \pi_i\mathbb{P}_{ij}$, for all j.

Lemma 3.1: Assume that π is the limiting dist. of a M.C., then π is a stationary dist.
Limit Theorem for Regular Markov Chain: A Markov Chain whose transition matrix, P, is regular has a limiting distribution. Equivalently, there exists a positive stochastic matrix Π, such that $\lim_{n\to\infty} P^n = \Pi$
Random Walk on a Weighted Graph: $\pi_v = \frac{w(v)}{\sum_z w(z)}$ for all vertices v,
where $w(v) = \sum_{z\sim v} w(v,z)$ is the sum of the weights on all edges incident to v.
Random Walk on a Non-weighted Graph: $\pi_v = \frac{deg(v)}{\sum_z deg(z)} = \frac{deg(v)}{2e}$ where e is the number of edges in the graph.

Accessible $\{p_{ij}^n > 0\} \to$ Communicate {Reflexive, Symmetric, Transitive}.
Irreducible MC: Only 1 communication class.
$T_j = min\{n > 0 : X_n = j\}$ is first passage time to j, $f_j = P(T_j < \infty | X_0 = j)$
Recurrent: State j is recurrent if a MC started in j eventually revisits j, $f_j = 1$.
Transient: State j is transient if $f_j < 1$.
j Recurrent: $\iff \sum_{n=0}^{\infty} P_{jj}^n = \infty$; **j Transient:** $\iff \sum_n^{\infty} P_{jj}^{n=0} < \infty$
All states in a FINITE IRREDUCIBLE MARKOV CHAIN are RECURRENT
Positive Recurrent: $E(T_j|X_0 = j) < \infty$ and **Null Recurrent:** $E(T_j|X_0 = j) = \infty$
Lemma: All states in a recurrent class are either null recurrent or positive recurrent
Limit Theorem for Finite Irreducible MCs: Given a finite, irreducible MC, for each state j, let $\mu_j = E(T_j|X_0 = j)$ be th expected return time to j. Then μ_j is finite, and there exists a unique stationary distribution π, such that $\pi_j = \frac{1}{\mu_j}$ and $\pi_j = lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} P_{ij}^m$
You can also find E(Return Time) with First-Step Analysis $e_x = E(T_i|X_0 = x)$ creates a system.
Lemma: A communication class is closed if it consists of all recurrent states. A finite communication class is closed if it consists of all recurrent states.
Period: $d(i) = gcd(n > 0; P_{ij}^n > 0)$ and Aperiodic: $d(i) = 1$ and $d(i) = \infty$ if no return
Canonical Decomposition: Arrange the matrix into closed & transient classes, $\mathbb{P}$ is broken down into mini-irreducible chains.

Ergodic Markov Chains:
A MC is ergodic if it is irreducible, aperiodic, and all states have finite return times (all finite MC's have finite return times).
Given an ergodic MC, then there exists a unique, positive, stationary distribution.
Ergodic Chains and Regular Matrices: The Markov chain is ergodic $\iff$ P is regular
Lemma: If i is an aperiodic state, there exists a pos. integer N such that $P_{ii}^n > 0$ for all $n \geq N$

Absorbing Chains:
State i is an absorbing state if $P_{ii} = 1$. A M.C. is called an *absorbing chain* if it has at least one absorbing state.
Expected # of Visits to Transient States: $F = (I - Q)^{-1}$. $P = [\frac{Q|R}{0|I}] \to P^n = [\frac{0|(I-Q)^{-1}R}{0| \quad I}]$
Absorption Probability: $(FR)_{ij}$ and **Absorption Time:** $(F1)_i$ (sum of the row)

Strong Markov Property:
Let X_0, X_1,... be a MC. Let S be a stopping time. Then, X_S, X_{S+1},... is a MC with transition P. i.e. The chain regenerates itself and starts anew when it reaches S.