MAT 455 Midterm Formula Sheet

Poisson Pdf, μ, σ^2: $\frac{e^{-\lambda}\lambda^x}{x!}, \lambda, \lambda$

Uniform Pdf, μ, σ^2: $\frac{1}{b-a}$, $\frac{b+a}{2}$, $\frac{(b-a)^2}{12}$

Binomial Pdf, μ, σ^2: $\binom{n}{x}p^x(1-p)^{n-x}, np, np(1-p)$

Geometric Pdf, μ, σ^2: $(1-p)^{x-1}p$, $\frac{1}{p}$, $\frac{1-p}{p^2}$

Normal Pdf, μ, σ^2: $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \mu, \sigma^2$

Gamma Pdf, μ, σ^2: $\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$, $\frac{\alpha}{\beta}$, $\frac{\alpha}{\beta^2}$

Exponentia dist. with parameter λ:

1) density function $f(x) = \lambda e^{-\lambda x}$. $\mu = 1/\lambda$, $\quad \sigma^2 = 1/\lambda^2 \quad P(X > x) = e^{-\lambda x}$

2) Memoryless: $P(X > t + s|X > s) = P(X > t)$.

3) Minimum of independent Exp r.v.s: Let $X_1, ..., X_n$ be indep. exp. r.v.s with parameters $\lambda_1, ..., \lambda_n$. Let $M = \min(X_1, ..., X_n)$. Then $M \sim Exp(\lambda_1 + ... + \lambda_n)$ and $P(M = X_k) = \lambda_k/(\lambda_1 + ... + \lambda_n)$.

Law of Total Probability: $P(A) = \sum P(A|B_i)P(B_i)$

Law of Total Expectation: $E(Y) = \sum E(Y|X = x)P(X = x)$ or $E(Y) = \int E(Y|X = x)f_X(x)dx$

Law of Total Variance: $Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$

Random Sums of Random Variables: If your random variable has mean, μ_x, and variance, σ_x^2, and you have a sum of N values where $E(N) = \mu_N$ and $Var(N) = \sigma_N^2$. Then $E(T) = \mu_x\mu_N$ and $Var(T) = \sigma_X^2\mu_N + \sigma_N^2\mu_X^2$.

Markov Chain: $P(X_{n+1} = j|_{X_0=i_0,...,X_{n-1}=i_{n-1},X_n=i}) = P(X_{n+1} = j|X_n = i) = (P(X_1 = j|X_0 = i)$

Chapman-Kolmogorov Relationship: $P^{m+n} = P^mP^n$, OR $P_{ij}^{m+n} = \sum_k P_{ik}^m P_{kj}^n$ for all i, j.

Limiting Distribution: The prob. dist. λ with the property that for all i, j: $\lim_{n\to\infty} \mathbb{P}_{ij}^n = \lambda_j$.

Stationary Distribution: $\pi\mathbb{P} = \pi$ and $\pi_j = \sum_i \pi_i\mathbb{P}_{ij}$, for all j.

First passage time to j: $T_j = min\{n > 0 : X_n = j\}$, $f_j = P(T_j < \infty|X_0 = j)$

Recurrent: State j is recurrent if a MC started in j eventually revisits j, $f_j = 1$.

Transient: State j is transient if $f_j < 1$.

j Recurrent: $\iff \sum_{n=0}^{\infty} P_{jj}^n = \infty$; **j Transient:** $\iff \sum_{n=0}^{\infty} P_{jj}^n < \infty$

Period: $d(i) = gcd(n > 0; P_{ij}^n > 0)$ and Aperiodic: $d(i) = 1$ and $d(i) = \infty$ if no return

Ergodic Markov Chains: A MC is ergodic if it is irreducible, aperiodic, and all states have finite return times (all finite MC's have finite return times).

Lemma: Assume that π is the limiting dist. of a M.C., then π is a stationary dist.

Limit Theorem for Regular Markov Chain: A M.C. whose transition matrix, P, is regular has a limiting distribution, which is the unique, positive, stationary distribution of the chain.

Limit Theorem for Finite Irreducible MCs: Given a finite, irreducible MC, for each state j, let $\mu_j = E(T_j|X_0 = j)$ be th expected return time to j. Then μ_j is finite, and there exists a unique stationary distribution π, such that $\pi_j = \frac{1}{\mu_j}$ and $\pi_j = lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1} P_{ij}^m$

Ergodic Chains and Regular Matrices: The Markov chain is ergodic $\iff$ P is regular

Lemma: If i is an aperiodic state, there exists a pos. integer N such that $P_{ii}^n > 0$ for all $n \geq N$

Poisson process with λ: 1) $N_0 = 0$. 2) For all $t > 0$, $N_t \sim Poisson(\lambda t)$.
3) Stationary increment: $P(N_{t+s} - N_s)$ has the same distribution as N_t.
4) Independent increment: For $0 < q < r \leq s < t$, $N_t - N_s$ and $N_r - N_q$ are independent.
Inter-arrival time between $(i-1)$th arrival and ith arrival X_i: $X_i \sim Exp(\lambda)$. All inter-arrival times are independent.
Arrival times $S_i = X_1 + ... + X_i$:
1) S_i has a gamma distribution with parameters n and λ.
2) Conditional on $N_t = n$, the joint dist. of $(S_1, ..., S_n)$ is the same as $(U_{(1)}, ..., U_{(n)})$, where the latter are order statistics of n i.i.d. uniform r.v.s on $[0, t]$, i.e., the density function is $f(s_1, ..., s_n) = \frac{n!}{t^n}$, for $0 < s_1 < ... < s_n < t$.
Thinned Poisson process: Let N_t be a Poisson process with λ. Assume each arrival, independent of other arrivals, can be marked as a type k event with prob. p_k, $k = 1, ..., K$, where $p_1 + ... + p_K = 1$. Then $N_t^{(k)}$, the number of type k events until time t, is a Poisson process with parameter λp_k.
Superposition process: Assume that $N_t^{(1)}, ..., N_t^{(K)}$ are K indep. Poisson processes with $\lambda_1, ..., \lambda_K$. Then $N_t = N_t^{(1)} + ... + N_t^{(K)}$ is a Poisson process with parameter $\lambda_1 + ... + \lambda_K$.

Continuous-time Markov property:
$P(X_{t+s} = j | X_s = i, X_u = x_u, 0 \leq u < s) = P(X_{t+s} = j | X_s = i) = P(X_t = j | X_0 = i)$.
Chapman-Kolmogorov Relationship: $\mathbb{P}(s + t) = \mathbb{P}(s)\mathbb{P}(t)$.
Holding time: For a C.M.C., the holding time at state i has an exponential distribution with parameter $q_i = \sum_k q_{ik}$, q_{ik} are transition rates from state i to state k.
Transition probability: Given transition rates q_{ij}, transition prob. $p_{ij} = \frac{q_{ij}}{q_i}$.
Kolmogorov equation: Forward: $\mathbb{P}'(t) = \mathbb{P}(t)Q$. Backward: $\mathbb{P}'(t) = Q\mathbb{P}(t)$.
Limiting Distribution: The prob. dist. π with the property that for all i, j: $\lim_{t\to\infty} \mathbb{P}_{ij}(t) = \lambda_j$.
Stationary Distribution: $\pi\mathbb{P}(t) = \pi$ for all $t \geq 0$. $\iff$ $\pi Q = 0$.
Limit theorem: If X_t is a finite, irreducible C.M.C. with $\mathbb{P}(t)$. Then X_t has a unique stationary dist. which is the limiting dist.
Stationary dist. for Birth-and-Death process: For a B-and-D process with birth rates λ_i and death rates μ_i, the unique stationary dist. π is
$\pi_k = \pi_0 \prod_{i=1}^k \frac{\lambda_{i-1}}{\mu_i}$, for $k = 1, 2, ...$ where $\pi_0 = \left(1 + \sum_{k=1}^\infty \prod_{i=1}^k \frac{\lambda_{i-1}}{\mu_i}\right)^{-1}$, given $\pi_0 > 0$.
Little's Formula: In a queueing system, $L = \lambda W$, where L is the long-term average number of customers in the system, λ is the arrival rate and W is the long-term average time that a customer is in the system.