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2 Semiring

A semiring $K = (\mathbb{K}, +_K, \times_K, 0_K, 1_K)$ is a set $\mathbb{K}$ equipped with a commutative binary operation $+_K$ (called "addition") with identity element 0_K and an associative binary operation $\times_K$ (called "multiplication") with identity element 1_K that distributes over $+$.

Here are commonly used semirings:

name	$\mathbb{K}$	$x +_K y$	$x \times_K y$	0_K	1_K
log-semiring	$\mathbb{R}$	$\ln(e^x + e^y)$	$x + y$	$-\infty$	0
tropical-semiring	$\mathbb{R}$	max(x, y)	$x + y$	$-\infty$	0
concat-semiring	sets of strings	alternation	concatenation	{}	{""}

3 Finite State Machine

We define a Finite State Machine (FSM) $\mathcal{M}$ by the tuple $\mathcal{M} = (Q, \Sigma, K, L, \boldsymbol{\alpha},\ \mathbf{T},\ \boldsymbol{\omega}, \boldsymbol{\lambda})$ where:

- $Q = \{1, \dots, d\}$ is the set of states (with cardinality d) identified as integers
- Σ is the set of symbols
- K is FSM's weights semiring
- L is the FSM's labels semiring defined over a subset of Σ^*
- $\boldsymbol{\alpha} \in K^d$ is a vector such that α_i is the initial weight of the state $i \in Q$
- $\mathbf{T} \in K^{d \times d}$ is a matrix such that T_{ij} is the transition weight from the state $i \in Q$ to the state $j \in Q$
- $\boldsymbol{\omega} \in K^d$ is a vector such that ω_i is the final weight of the state $i \in Q$
- $\boldsymbol{\lambda} \in L^d$ is a vector of symbol such that λ_i is the symbol of the state $i \in Q$

4 FSM operations

We describe here some operations over FSMs that have the same weights and labels semiring K and L. We write $\mathcal{M}_m$ the FSM defined as

$$\mathcal{M}_m = (Q_m, \Sigma, K, L, \boldsymbol{\alpha}_m, \mathbf{T}_m, \boldsymbol{\omega}_m, \boldsymbol{\lambda}_m); \tag{1}$$

$\mathcal{M}'$ is defined similarly. Finally, we write Π_m (respectively Π') the set of all strings and their associated weight generated by the FSM $\mathcal{M}$ (respectively $\mathcal{M}'$).

4.1 Renormalization

4.2 Union

The union of two FSMs $\mathcal{M}' = \mathcal{M}_1 \cup \mathcal{M}_2$ gives a FSM such that $\Pi' = \Pi_1 \cup \Pi_2$. $\mathcal{M}'$ can be obtained with the following construction:

$$Q' = \{1, \dots, d_1, d_1 + 1, \dots, d_1 + d_2\} \tag{2}$$

$$\boldsymbol{\alpha}' = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix} \tag{3}$$

$$\mathbf{T}' = \begin{bmatrix} \mathbf{T}_1 & \\ & \mathbf{T}_2 \end{bmatrix} \tag{4}$$

$$\boldsymbol{\omega}' = \begin{bmatrix} \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{bmatrix} \tag{5}$$

$$\boldsymbol{\lambda}' = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{bmatrix} \tag{6}$$

4.3 Concatenation

The concatenation of two FSMs $\mathcal{M}' = \text{concat}(M_1, \mathcal{M}_2)$ gives a FSM such that $\Pi' = \{\pi_1\pi_2 : \pi_1 \in \Pi_1,\ \pi_2 \in \Pi_2.\ \mathcal{M}'\}$ can be obtained with the following construction:

$$Q' = \{1, \dots, d_1, d_1 + 1, \dots, d_1 + d_2\} \tag{7}$$

$$\boldsymbol{\alpha}' = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ 0_K \boldsymbol{\alpha}_2 \end{bmatrix} \tag{8}$$

$$\mathbf{T}' = \begin{bmatrix} \mathbf{T}_1 & \\ & \mathbf{T}_2 \end{bmatrix} \tag{9}$$

$$\boldsymbol{\omega}' = \begin{bmatrix} 0_K \boldsymbol{\omega}_1 \\ \boldsymbol{\omega}_2 \end{bmatrix} \tag{10}$$

$$\boldsymbol{\lambda}' = \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{bmatrix} \tag{11}$$

4.4 Reversal

The reversal (denoted $^\top$) of a FSM $\mathcal{M}$ yields a FSM $\mathcal{M}' = \mathcal{M}^\top$ such that $\Pi' = \{\overleftarrow{\pi} : \pi \in \Pi\}$ where $\overleftarrow{\pi}$ is the reversed of π.

$$Q' = Q \tag{12}$$
$$\boldsymbol{\alpha}' = \boldsymbol{\omega} \tag{13}$$
$$\mathbf{T}' = \mathbf{T}^\top \tag{14}$$
$$\boldsymbol{\omega}' = \boldsymbol{\alpha} \tag{15}$$
$$\boldsymbol{\lambda}' = \boldsymbol{\lambda} \tag{16}$$