Bavarian Graduate School of Computational Engineering

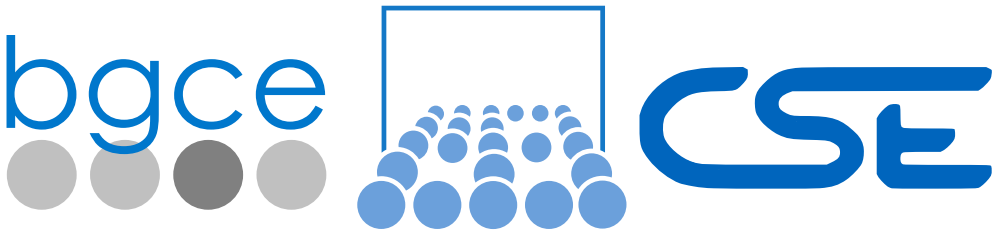
Technische Universität München

BGCE Honours project report

CAD-integrated topology optimization

Authors: S. Joshi,
J.C. Medina,
F. Menhorn,
S. Reiz,
B. Rüth,
E. Wannerberg,
A. Yurova

Advisors: Arash Bakhtiari (TUM),
Dirk Hartmann (Siemens),
Utz Wever (Siemens)



Preface

NOTE: always write the long version of acronyms before using them

The Bavarian Graduate School of Computational Engineering (BGCE) honours project at the Computational Science and Engineering (CSE) Institute of Technische Universitaet Muenchen (TUM) is

This report and the work described herein is the result from the 2015-2016 BGCE Honours Project within CSE at TUM, a 10-month project where students attending BGCE, the Bavarian Graduate School of Computational Engineering Elite Master's Programme, conduct a research project on cutting-edge topics in within the field of Computational Engineering, in cooperation with a partner in industry or academia. This project, *CAD-Integrated Topology Optimization*, was carried out by the BGCE students at the Computational Science and Engineering Master's Programme (CSE) at the Technische Universität München (TUM), initiated and supervised in a cooperation between TUM and Siemens in Munich.

The 2015-16 project is titled "..."

and is

AG

Acknowledgments

This Honour's project is carried out under the supervision of Dr. Dirk Hartmann, (Topical Advisor), Dr. Utz Wever (Topical Advisor) from the side of the customer (Siemens AG) and Arash Bakhtiari (Team Advisor) from the side of the Technische Universität München. We would like to thank Bavarian Graduate School of Computational Engineering for providing us an opportunity to participate in a project closely related to the industry in a highly relevant and challenging topic.

Contents

Outline and Overview

Purpose of the document

The purpose of this document is to describe the software *CAD-integrated Topology Optimization* software tool, as well as for the interested reader or developer describe how it was implemented, along with the theoretical bakcground that it relies on.

background

Document overview

The document is arranged in chapters. The current and first chapter provides a an overview of the document structure, brief introduction to the topic, motivation and administrative details of the project. The second chapter provides a thorough review of the theory and techniques used within the project.

Part I: Introduction If only one chapter in each part -> no need for parts

CHAPTER 1: INTRODUCTION

This chapter presents an overview of the motivation behind *CAD–integrated Topology Optimization*, including the current state of this field. It also provides general organizational information about the project execution, such as timeline and structure.

Part II: Background theory

CHAPTER 2: BACKGROUND THEORY

This chapter provides the theoretical background for the implementation of *CAD–integrated Topology Optimization* tool. It consists of the five parts, describing the essential steps of the *Topology Optimization* pipeline. Furthermore, the detailed description of algorithms and libraries used in each step is given.

Part I.

Introduction

1. Introduction

In this chapter a motivation for the project and a short description of the problem task is stated. A brief introduction to topology optimisation and the project structure is also given.

1.1. Motivation

A common problem in product design is to create a functional structure using as few material as possible. Three decades ago engineering design versions were drawn, prototypes created and experimental test performed. Nowadays, the field of topology optimisation simplifies this process and has become a great help in all fields of engineering.

Topology optimisation tackles the problem of material distribution in a structure in order to fulfil certain target loads. Several topology optimisation open-source tools exist that are ready to use, however it is still a challenge to incorporate these tools handily in the design process. The idea of this project is to allow these tools to work directly from CAD files and to transfer the obtained mesh based solution back to the CAD world. Unfortunately, at the moment, there is no open source solution for the conversion mesh based geometry to the spline-based CAD format (where in our case NURBS are used). The would-be straightforward approach – to convert each triangle of mesh geometry directly to CAD format – results in enormous file sizes. One of the biggest challenges of this project is thus to develop a conversion tool that feasibly provides a useful CAD-representation of the optimized surface.

1.2. Project structure

1.2.1. Aims and Goals

The aim of the project is to provide a tool that allows ~~to~~ *Topology Optimisation* without leaving CAD–framework. ~~This includes the reaching of the following goals:~~

Therefore main goals of the project is as follows:

- Implementation of *Topology Optimisation* using available open source libraries

by and extending the

- Development of a flexible tool for the conversion of the optimized surface ~~to~~ back to the CAD format.

1.2.2. Timeline and Structure

Hence, project

The duration of the project has been set to ~~be~~ 10 months. This ~~period~~ has been divided into 4 phases:

Phase 1: Getting familiar with the topic and agreement on the project specification.

Phase 2: Implementation of the first part of the pipeline (Topology Optimisation from CAD surface using existing tools), investigating the tools and algorithms available for the conversion of the geometry generated after topology optimisation back to CAD format (later referred as *NURBS fitting pipeline*), prototyping (using MATLAB) and evaluating of ~~found solutions~~ results.

Phase 3: ~~Reimplementing~~ Implementing the prototypes, developed on ~~a~~ the previous stage, ~~using an open source language,~~ in non-proprietary langauges such as Python or C++, extension of the NURBS fitting algorithm to more complex cases, finalising the first part of the pipeline.

Phase 4: Implementation of the extended NURBS fitting algorithm, integrating it with the *Topology Optimisation* part and ~~forming of final deliverables~~ delivering the final product to costumer.

Part II.

Background Theory

2. Background Theory

2.1. CAD overview

2.1.1. CAD in engineering

Computer Aided Design (CAD) refers to the process of designing a product using a computer system. Before CAD applications were used, products were designed using a sketch board. It was a challenge to incorporate changes in the construction drafts as well as to keep documentations up to date; hence, it was no surprise that CAD systems spread rapidly across all design development branches. Computer aided design is now irreplaceable used in architecture, mechanical, electrical and civil engineering.

Depending on the discipline different requirements are set on the virtual model. One may imagine that in a civil engineering model of a building a 2D floor plan is often sufficient; however in the design of a mechanical motor a 3D model is always necessary. Given these circumstances, various CAD software bundles evolved in the different disciplines with completely different modelling approaches. Besides the geometry representation additional parameters, such as material properties or manufacturing information, are stored. In order to move between different data structures standardized exchange interfaces are commonly used.

2.1.2. Geometry representations

In general, two different ways of describing a geometry are used in CAD systems: a constructive solid geometry or a boundary representation (BREP). Other approaches, such as a complete voxelised geometry are not common due to extensive memory consumption.

Constructive solid geometry

One way of representing a geometry in CAD is the approach of *constructive solid geometry* (CSG). The basic idea is to start from a set of primitives, e.g. a sphere, cylinder and cube. Basic Boolean operations link these primitives towards a complex geometry. This procedure can be seen in figure 2.1.

Key advantages of this format is the precise representation using very little storage memory. However, not all desired forms can be represented by CSG and hence, a second type of geometry description is needed.

Boundary representation

A different kind of modelling approach is the so-called *boundary representation* (BREP). Instead of storing the geometry information at every single point, *BREP* formats only save the boundary surface of the body. The interior is assumed to be uniformly filled. Especially in complex geometries, this approach simplifies the model to such an extent, that the amount of data becomes

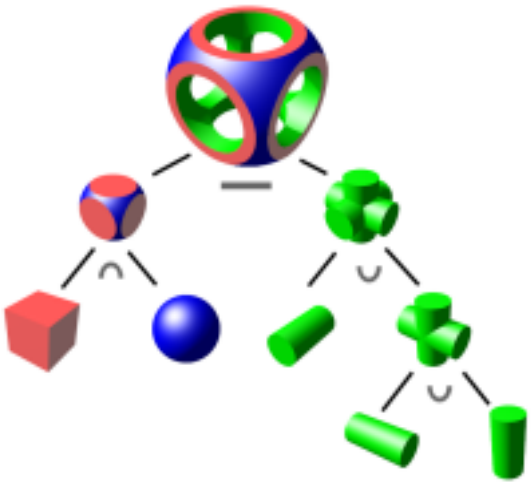


Figure 2.1.: CSG object tree IS THIS PICTURE YOURS? Otherwise cite!

much easier to handle. Surfaces can then be for example stored as a set of triangles (as in STL files, see section 2.1.3) or in NURBS patches (see section 2.5). Furthermore, holes in the body are possible by saving the surface normal of the respective boundary.

By the boundary representation arbitrary geometries can be created. Data amounts to fulfill a certain precision are larger than by the CSG representation, but BREP files are usually easier to work with. One also has to keep in mind, that non-physical geometries can result from BREP formats through a not closed surface.

2.1.3. Data exchange interfaces

CAD software programs usually use their own data formats; in order to exchange models standardized interface formats have been developed. Geometric models are compressed to certain geometry descriptions; transferring additional information, such as material properties or manufacturing information, is general a difficult task and in some exchange file formats even prohibited. A few common exchange file types are described below.

STL file format

The STL (from *ST*ereo*L*ithography) file format describes the model only by its boundary amd is thus a BREP format. Because of only taking this into account, only geometric information can be transferred.

The idea behind the STL files is simple: the geometric model is discretized into a cloud of points. Sets of three vertices form a triangle; hence, a connected surface of triangles emerges which describes the geometry. This procedure is shown in Figure 2.2 for a two dimensional circle. The aforementioned triangles boil down to lines in two dimensions. The advantages and disadvantages of this approach becomes clear: It can be applied to an arbitrary geometry, but accuracy causes difficulties. In order to ~~to~~ transfer high precision geometries many vertices are necessary. This will result in big files; nevertheless, a precise circle can never be represented.

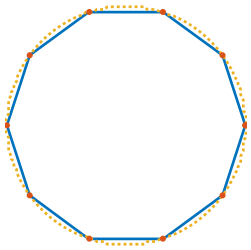


Figure 2.2.: STL discretization for a circle

ASCII STL files begin with a name and data on the triangles are constructed as follows:

- a facet normal pointing outward
- a sequence of vertex coordinates

Note that no additional information such as material properties are transferred through STL files.

IGES file format

To overcome issues of unsufficient precision there exist also more elaborate exchange formats that save e.g. a circle as a parameter where no discretisation step is involved. Also, the possibility of passing additional parameter information is required by certain users. Popular file types that offer these two functionalities are STEP and IGES files.

The IGES file format contains five different sections: a *Start*, *Global*, *Directory Entry*, *Parameter Data* and *Terminate* section. The *Start* and *Global* section are used for naming and part information. In the *Directory Entry* section additional information like the node color is saved. The *Parameter Data* section is used for storing the coordinate points; the *Terminate* section signals the end of the file.

2.2. Topology Optimisation

2.2.1. Definition and motivation

Topology Optimization describes the process of finding the optimal distribution of a limited amount of material for a given area or volume based on a predefined constraint/minimization problem. Possible optimization goals are for example:

- **Minimum compliance** which seeks to find the optimal distribution of material that returns the stiffest possible structure. The structure is thereby subjected to loads (forces) and supports (boundary conditions). By maximizing the stiffness, the complianceis minimized. This is also analogous to minimizing the stress energy stored by the applied loads.
- **Heat conduction** tries to optimize the domain of a conductive material with respect to conductivity for the purpose of heat transfer. This maximization problem is the same as minimizing the temperature gradient over the domain– a poor conductor will create a large gradient.

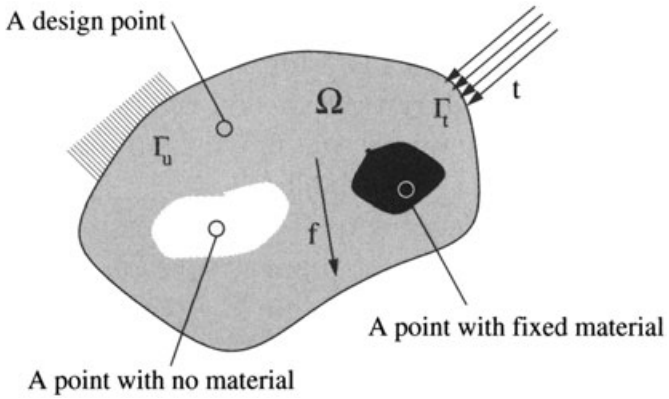


Figure 2.3.: The reference domain Ω for the minimum compliance problem. The problem is formulated such that for a set of external loads t on boundaries λ_t, body forces f and a set of fixed support points λ_u, the material distribution within Ω is such that the stiffness with regards to these loads and forces is maximal and the energy stored by the application of those forces is minimal. The problem also allows defining areas which either cannot or must be filled with material. Figure taken from [2].

- **Mechanism synthesis**' objective is to obtain a device that can convert an input displacement in one location to an output displacement in another location. Topology Optimization hereby seeks the optimal design which maximizes the output force for a given input or, respectively, minimizes the input force for a given output.

As one can imagine by this short list of optimization goals, Topology Optimization has a wide field of possible applications. Hence, it has become a well established technology used by engineers in the fields of aeronautics, civil, materials, mechanical and structural optimization. Furthermore, the rising significance of additive manufacturing techniques in industry, the realisation of complex optimized designs is now much easier.

2.2.2. Theory

Minimum compliance: Problem formulation

In order to constrain the resulting structure as little as possible, the formulation of the topology optimization problem is generally given as follows: for a given set of external fixture points, external loads and/or body forces, the distribution of material within the reference domain should be found such that the structure has maximum stiffness. This is obtained when the structure has the minimum energy stored by external work for the applied forces. The problem is also usually formed to allow for regions in the domain to be specified as filled or empty of material (see Figure 2.3).

The formulation allows the problem to be cast as finding a displacement field u and a stiffness tensor field E that is in equilibrium with the applied loads, and that minimizes the external work done by the forces.

Physical and mathematical simplifications

To turn this into a more tractable mathematical problem, a few physical assumptions are also typically made: that the material is isotropic, and that it is linearly elastic. From the assumptions of isotropy and linear elasticity of the material, the stiffness field becomes a constant of the material, defined where there is material in the domain.

The problem is also easy to cast into a weak form. First of all, we compute the integrated internal virtual work and external work. The former is the work of deforming the elastic material from equilibrium by an admissible displacement. External work is done by the loads and forces to bring out this displacement. Having computed these, we set them equal to one another in order to conserve energy. As a result we obtain an equation that relates the equilibrium displacement, stiffness tensor, and the forces and loads. We then cast this into the weak form, which can be solved using Finite Element Methods (FEM). These can also incorporate the calculation of the external work done.

SIMP: Solid Isotropic Material with Penalization

However, in trying to minimise this external work done by looking at different material distributions, the usual problem of finding an optimum arises: where to look? After discretising the domain with FEM, the possibilities of where to put material at least aren't infinite - but they still grow exponentially with the number of elements, so trying out one-by-one is not going to prove efficient. One popular way of recasting the problem to allow for easier solving is the SIMP model, where instead of either being present or not at a point, the material presence can take a continous set of values between one and zero. The total final volume is then obtained and fixed by integrating this persence variable over the domain, instead of constraining the allowed occupied space, which allows for the interpretation as some kind of density.

In order to still obtain topologies where material is predominantly in certain areas – of densities one, with the rest being empty at densities close to zero – a "penalty" is applied to the intermediate values. This is effected by raising the density to a power > 1 in the elastic energy calculation, but not in the volume calculation, such that an intermediate density value provided less elastic support, but still "costs" as much volume, and will thus be suboptimal.

Solution and implementation

In typical implementations, a heuristic iterative scheme is then used for finding a solution. The optimal solution is assumed to have all present parts stressed (as they would otherwise be unnecessary, not providing any support). Thus, at places where the elastic energy is high, material is added if possible, and where it is low, material is likewise removed, with the values "high" and "low" being determined dynamically to keep the total volume constraint.

This whole scheme is one of the simpler topology optimisation schemes to implement, and has been done so in several pieces of open-source software, including a known 99-line Matlab code by Sigmund [6]and ToPy described in subsection 2.2.3 below. For an extended explanation and discussion, as well as further alternative methods for topology optimisation, the interested reader is referred to [2].

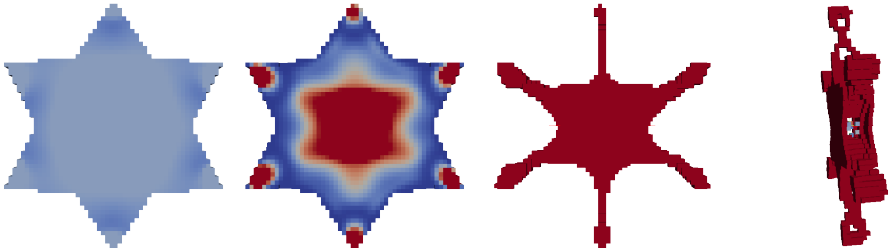


Figure 2.4.: Topology Optimization with minimum compliance of a star structure, given by an stl-file. The fixtures were applied in the corners of the star, while a load was set in the middle.

2.2.3. ToPy library

ToPy [5] is a python library/program, written by William Hunter and documented in [4], implementing the SIMP model and method described above. It is based on the 99-line Matlab code by Sigmund's for minimum compliance [6]. The program can optimize the three above named problem types, minimum compliance, heat conduction and mechanism synthesis– in 2D as well as 3D. It uses available open source python software, as for example Pysparse and Numpy, leading to improved speed, porta- and scalability. The whole program is steered by an input file which– with the help of the documentation– is straightforward to use and easy to adapt.

2.2.4. Implementation

In terms of our implementation, we use ToPy as a blackbox topology optimizer. This means, we launch the program with an input file based on our scenario, let ToPy run and proceed by working with the output of ToPy. The intention is to touch the solver itself as less as possible to be able to just plug in different solvers later on. Implementation-wise that means, that we wrote a program which takes as input a voxelized CAD design in, for example, stl-format and outputs a tpd-file which can be used by ToPy. Results of the process can be seen in figure 2.4. Here, a star was given as input from a stl-file. We fixed the voxels in the corners of the structure, while we set a load in the middle, pointing into the structure. As can be seen, the optimization process "cuts" away unnecessary material in-between the corners and even in the middle of the material and returns stiff structure for a minimal amount of material.

2.3. From CAD to Voxels

The main hurdle with most state-of-the-art open source topology optimization tools is their input format, where many of them require input to be specified as a 3-dimensional voxel grid. Presence (or absence) of material in these voxels is defined by a boolean variable, and boundary conditions are imposed on the appropriate locations.

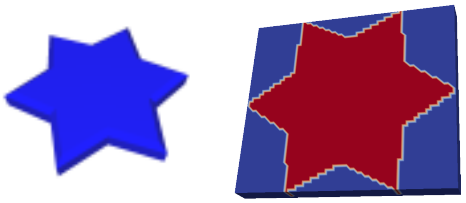


Figure 2.5.: The STL geometry of a star (left) and its voxelized form (right) obtained via the CVMLCPP voxelizer

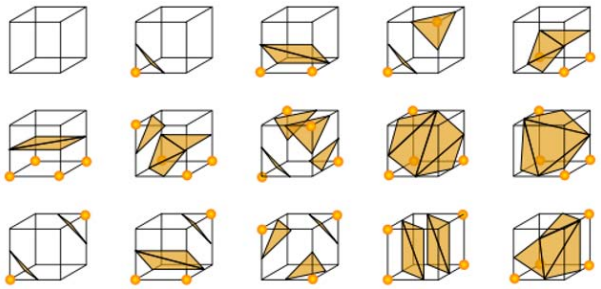


Figure 2.6.: Base cases of Marching Cubes

2.3.1. CVMLCPP library

The Common Versatile Multi-purpose Library for C++ ("abbreviated" to CVMLCPP) is a collection of mathematical algorithms whose objective is "to eliminate this redundancy by offering high-quality implementations of commonly needed functionality". The library offers an easy-to-use voxelizer, which we use for conversion of CAD input to a boolean voxel grid.

2.4. From Voxels to a surface representation

2.4.1. Isosurface Contouring

After obtaining the optimized voxel data, ...

Now when the optimized voxel data was obtained, the next step is to generate a *mesh based geometry*. It will be useful in the further conversion of our surface back to CAD format, in particular, to NURBS representation. In order to achieve it, the data will be represented by a contour of a smooth function, rendering an isosurface. The isosurface allows to visualize Scalar Volumetric Data in 3D. It furthermore permits a mesh representation of the volume data. The mesh can be composed of triangles or quads, according to the algorithm used. There are two main approaches to solve this problem, the most famous one is Marching Cubes.

This will be used in

and , furthermore, it

Based on the using alg. the .

A well-know approach to tackle this problem is the Marching Cubes technique.

Marching Cubes

The *Marching Cubes* algorithm takes as an input a regular volumetric data set and extracts a polygonal mesh. It divides the space into cubes, which are defined by the volume information.

Each cube has scalar information on its vertices, the value is equal or above a marked isovalue. Therefore each of the eight vertices of a cube can be marked or unmarked. According to these values vertices are drawn on the edges of the cube at calculated points with the use of interpolation. A cube that contains an edge is called active. Non active cubes are not further considered in the algorithm.

By connecting the vertices we obtain a polygon on each cube. There are 256 possible scenarios, but most of them are just reflections or rotationally symmetric cases of each other. Therefore there are 15 base cases which represent all the possibilities of the marching cubes (see Figure 2.6). The original algorithm presents two main problems. Firstly it does not guarantee neither correctness nor topological consistency, which means that holes may appear on the surface due to inaccurate base case selection. The second problem is ambiguity, which appears when two base cases are possible and the algorithm chooses the incorrect one. These cases can be grouped into face ambiguities and internal ambiguities. There are many extended Marching Cubes algorithms that tackle the problems of the original one, getting rid of the ambiguities and providing correctness.

Dual Contouring

The idea of the *Dual Contouring* algorithm is similar to Marching Cubes, but instead of generating vertices on the edges of the cubes, it locates them inside the cube. Figure 2.7 shows the basic differences in both approaches. The vertices associated with the four contigous cubes are joined and form a quad. The question now is which place inside the cube is the ideal one to insert each vertex. Different dual algorithms are classified according to the answer for this question. Dual contouring generates a vertex positioned at the minimizer of a quadratic function which depends on the intersection points and normals. Therefore the method needs Hermite data to work with.

$$E(x) = x^T A^T A x - 2x^T A^T b + b^T b$$

Where A is a matrix whose rows are the normals and b is a vector whose entries are the product of normals and intersection points. To solve this system, a numerical treatment is needed. As proposed in [1] the best approach is to compute the SVD decomposition of A and form the pseudo-inverse by truncating its small singular values.

The main advantage of this method over MC is the acquisition of better aspect ratios. On the other hand the need of Hermite Data represents a disadvantage. Furthermore there is no open source algorithm that implements the Dual Contouring scheme.

2.4.2. VTK Toolbox

The VTK Toolbox is an open–source tool, providing algorithms for 3D computer graphics, image processing, and visualization. Among the variety of tools, VTK offers algorithms, allowing us to obtain a surface representation from voxel data.

Since *Marching Cubes* algorithm only works with ImageData and PolyData, it is inapplicable to our case of unstructured grid data.For structured and unstructured grids the tool to render the isosurface is the *Contour Filter* tool. Unfortunately the documentation does not present which algorithm the tool uses. It can be inferred that it is an extended *Marching Cubes* algorithm. The *Contour Filtering* seemed to work fine but the visualization of our data was still not possible and an intermediate step was needed. We used the *Implicit Modelling* tool which is a filter that

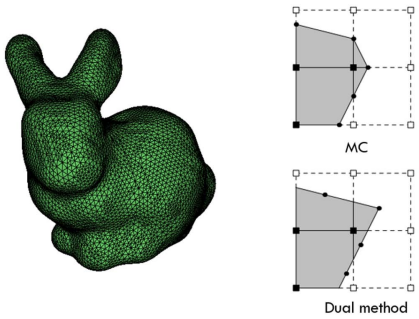


Figure 2.7.: *Left:* The famous Stanford Bunny. *Right:* Main difference between MC and Dual methods

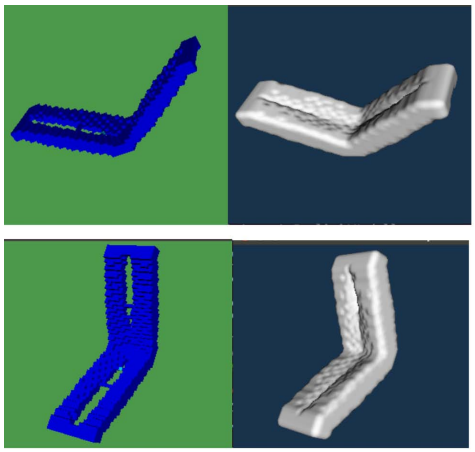


Figure 2.8.: Contour Filtering tool after Implicit Modelling

computes the distance from the input geometry to the points of an output structured point set. This distance function can then be "contoured" to generate new, offset surfaces from the original geometry. Although this approach allowed the visualisation, some crucial information was lost. In particular, holes are not represented in the final model.

A further idea to solve this problem is to convert ~~at the first step~~ the volume data into point data and only then present it to the *Contour Filtering* tool (Figure 2.8). ~~This will be implemented by the next milestone.~~

In order to reduce computational costs of the following *NURBS fitting* process, presented in the next section, we need to create a coarser mesh from the fine one. The number of triangles that represent the isosurface can be reduced with the *Decimation* tool. A smoothing step is necessary in between to get the new connections right. The top part of figure 2.9 shows a 50 % reduction of the triangles, a noticeable difference can not be perceived. On the lower part a 90 % reduction is obtained, it is nevertheless still difficult to see a difference. Triangle meshes can be easily coarsened since there are many open source algorithms that simplify the triangles. VTK has the decimation tool which works for 3D triangle data.

Figure 2.9.: Decimation of triangles. *Top:* 50% *Lower:* 90%

2.4.3. Long Road to NURBS

There are two possible roads to go from the voxel data to the CAD representation (in our case NURBS based representation).

Quad Contouring

This approach uses the dual contouring algorithm as first step in order to obtain a quad mesh representation from the voxel data. The first challenge is to implement correctly the algorithm with the ideas presented in [??1]. The original marching cubes algorithm is implemented in VTK but the source code is not public, therefore not only an extension of it is needed, but a full implementation. Once this first step is done, the quads will be chosen for the NURBS parametrization. A second step considers multiple smaller quads which have to be combined into one larger patch. This is another challenge, since the remeshing of quad meshes is not as straightforward as with the triangles. Different approaches have been taken in order to achieve this coarsening. In [??2] an incremental and greedy approach, which is based on local operations only, is presented. It depicts an iterative process which performs local optimizing, coarsening and smoothing operations. Another approaches, like the one presented in [??3] uses smooth harmonic scalar fields to simplify the mesh.

Multiresolution Analysis of Arbitrary Meshes

With *Multiresolution Analysis of Arbitrary Meshes* approach there is no need to apply a Dual Contouring algorithm, since it takes as beginning data the triangles from the Marching Cubes. The main concepts are shown in the paper [3]. It mainly takes a series of intermediate steps which permits a parametrization of data. It includes a partitioning scheme based on the ideas of the Voronoi Diagrams and Delaunay triangulations. Large patches or quads are obtained with this method.

Both approaches have not been implemented in open source documentation, therefore there is a need to implement it from scratch. Up to now, the second approach has been chosen.

In this project, we decided ot proceed with the second approach, since ...

2.5. From a surface representation to NURBS

Parametrised geometries are often given in terms of NURBS descriptions (e.g. in FreeCAD software). To define NURBS from a mathematical standpoint, we first define so-called *Bezier curves* which we will use later for the definition of NURBS.

2.5.1. Bezier Curves

Bezier curve is a *parametric* curve, which often used for producing a smooth approximation of a given set of data points.

~~**Analytical expression**~~ No need for subtitles when the section is so short.

An analytical expression for the Bezier curve is given by:

$$\vec{B}(t) = \sum_{i=0}^{n} b_i^n(t)\vec{P}_i$$

where $\vec{P}_i$ is the i–th control point (we have $n+1$ control points). And

$$b_i^n(t) = \begin{pmatrix} n \\ i \end{pmatrix} (1-t)^{(n-i)} t^i$$

is the i–th Bernstein polynomial of degree n.

Additionally to the expression with the Bernstein polynomials, one can use a recursion formula (so-called **de Casteljau Algorithm**) for the construction of the Bezier curve, which we will not cover here.

~~**Surfaces**~~

Analogically to Bezier curves, but with $n \cdot m$ Points $\vec{P}_{i,j}$, one can define a *Bezier surface*, given by the analytical expression

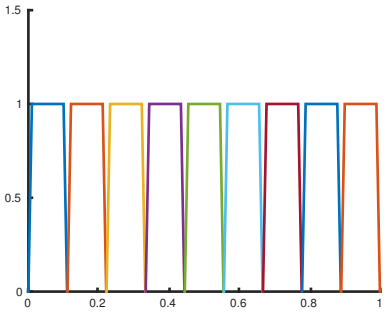
$$\vec{S}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_i^n(u) b_j^m(v) \vec{P}_{i,j}$$

Note, that Bezier curves and surfaces may be unstable – minor changes in control points might lead to major global changes.

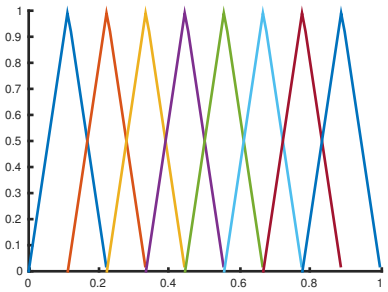
2.5.2. NURBS basis functions

Extending the idea, described in previous section, one could use NURBS basis functions instead of simple Bezier curves.

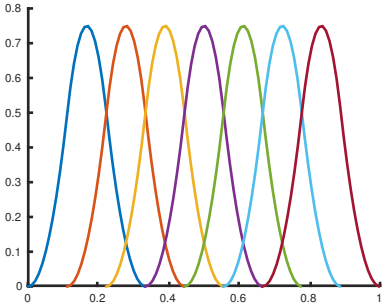
Unlike Bezier curves, for the B–spline basis a parameter domain is subdivided with so-called *knots*. In particular, given the parameter domain $[u_0, u_m]$ (in 1D), *knot vector* is given by $u_0 \leq u_1 \leq ... \leq u_m$. In most cases $u_0 = 0, u_m = 1$, so we get unit interval for our parameter values.



(a) B-spline basis for $p = 0$



(b) B-spline basis for $p = 1$



(c) B-spline basis for $p = 2$

Figure 2.10.: B-spline basis functions

Recall, that, N in NURBS stands for *non-uniform*. This means, that our knots $u_0, ..., u_m$ are not equidistant.

Given *knot vector* $[u_0, u_m]$ and a degree of B-spline p one can find i-th B-spline basis function recursively as follows:

$$N_{i,0}(u) = \begin{cases} 1, & \text{if } u_i \leq u < u_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \tag{2.2}$$

For $p = 0$ we get just step functions (see fig. 2.5.2), for $p = 1$ we get familiar hat functions (see fig. 2.5.2). Quadratic basis looks a bit more complicated, but also quite intuitive (fig. 2.5.2).

Given these basis functions, Non-Uniform Rational B-Spline (NURBS) curve is given by:

$$C(u) = \frac{\sum_{i=1}^{k} N_{i,n} \omega_i P_i}{\sum_{i=1}^{k} N_{i,n} \omega_i}, \tag{2.3}$$

where k is number of points, $\{P_i\}$ are given control points.

B–splines are have the following properties, which are useful for our problem:

- Degree n and number of control points $\vec{P}_{i...m}$ are independent.

- B–Splines only change locally (depends on the degree n) when a control point is changed.

Analogically, one can define B–spline surfaces.

2.5.3. Minimization problem

Now, once we defined all necessary tools, we proceed to the fitting problem.
The goal is to fit in a parametric curve to the set of given data points. In our case our given set of points is a mesh, obtained from surface contouring.

2.5.4. Minimization Problem: Bezier curve

First we want to find a Bezier–curve $\vec{B}_n(t)$ of degree n which is approximating a given spline $\vec{s}_m(t)$ defined by m points in a optimal way. For this purpose we want to minimize the L2–error. This leads to the minimization problem:

$$\text{find} \min_{\vec{B}_n \in \mathbb{B}_n} \left\| \vec{B}_n - \vec{s}_m \right\|_{L_2} \tag{2.4}$$

Minimizing the L2–norm is equal to minimizing the functional:

$$F(\vec{B}_n) = \int_{t=0}^{t=1} \left(\vec{B}_n - \vec{s}_m \right)^2 \mathrm{dt} \tag{2.5}$$

Using the variational principle we get the the system of linear equations

$$Aa = b \tag{2.6}$$

where A is the stiffness matrix, b is the vector of scalar products of the given spline s_m and basis functions (Bernstein polynomials) and a is a required vector of coefficients for Bezier curve representation.

2.5.5. Minimization problem (least squares): NURBS

Although the approach used above showed good results, it appears to be very computationally intensive. Analogically, one could reduce the original problem to the *linear regression problem*, which allows us to reduce computational costs. Also, to improve the locality of our solution, from now on we are going to use NURBS basis functions with weights $\omega_j = 1$ instead of Bezier curves. For this purpose, we adopted an algorithm, provided in [1].

Let X^0 be the $n \times 2$ matrix of the given set of points, N^p - the basis functions of degree p ($n \times (n+p)$ matrix, where n - number of points), P^0 - the control points ($(n+p) \times 2$ matrix).

The original problem can be written as:

$$X_i^0 = \sum_{j=1}^{n+p} P_j^0 N_{i,j}^p, \quad i \in \{1, .., n\} \tag{2.7}$$

Or, in short:

$$X^0 = N^p P^0 \tag{2.8}$$

The above system needs to be solved for the unknown P^0. One of the ways to solve it is to use SVD decomposition. For prototyping, the built-in MATLAB solver can be used.

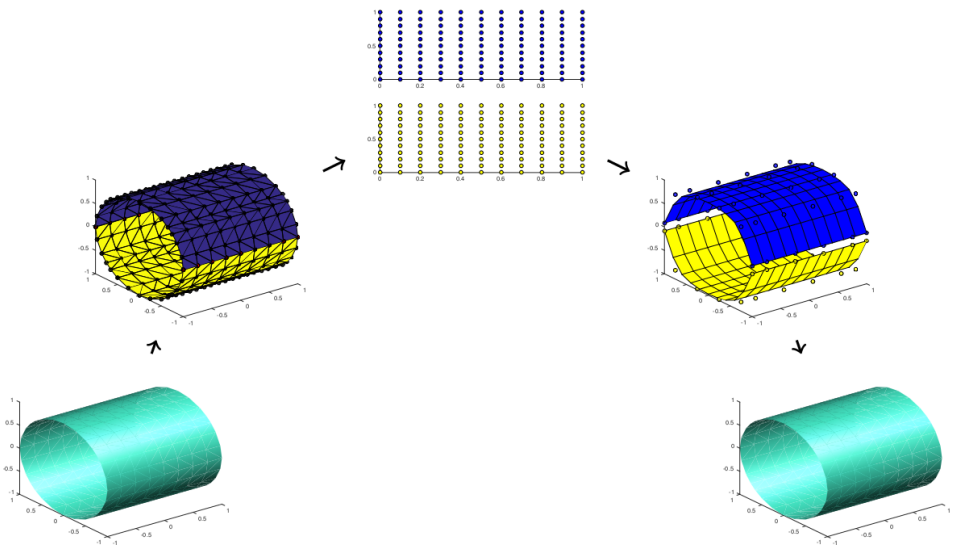


Figure 2.11.: NURBS fitting pipeline

2.5.6. Fitting pipeline

Since the geometry obtained after topology optimization can be arbitrary complex, we might not be able to find a good fit using only one patch. We seek a multi step algorithm, allowing us to break the overall big problem into smaller problems, which can be handled relatively easy. Based on the algorithm described in [3], our overall fitting pipeline looks as follows (see fig. 2.11):

- Patch selection (breaking our problem in small pieces which can be solved using least squares)
- Parametrization of obtained patches
- B–spline fitting using least squares
- Smooth connection of patches
- Conversion back to CAD

The pipeline given above, once implemented, will provide us with a flexible algorithm for converting an arbitrary complex mesh based geometry into NURBS and, hence, CAD–representation.

Bibliography

[1] Gerrit Becker, Michael Schäfer, and Antony Jameson. *An advanced NURBS fitting procedure for post-processing of grid-based shape optimizations*. 2011.

[2] Martin Philip Bendsøe and Ole Sigmund. *Topology Optimization: Theory, Methods and Applications*. Springer Science & Business Media, 2003.

[3] Matthias Eck and Hugues Hoppe. *Automatic reconstruction of B-spline surfaces of arbitrary topological type*. 1996.

[4] William Hunter. Predominantly solid-void three-dimensional topology optimisation using open source software. Master's thesis, Stellenbosch University, 2009.

[5] William Hunter. Topy - 2d and 3d topology optimization using python. `https://code.google.com/p/topy/`, 2009. lastly accessed on 21/08/2015.

[6] Ole Sigmund. A 99 line topology optimization code written in matlab. *Structural and Multidisciplinary Optimization*, 21(2):120–127, 2001.