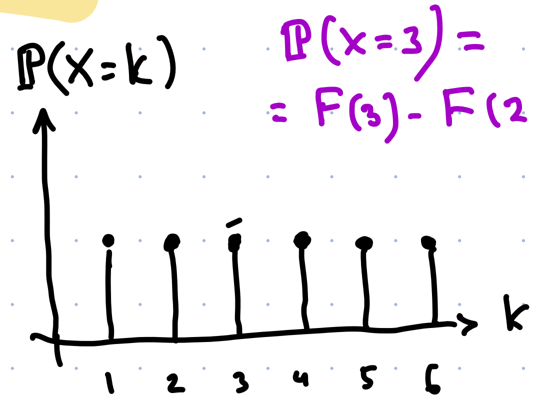
① Функция распределения

$X: \Omega \to \mathbb{R}$

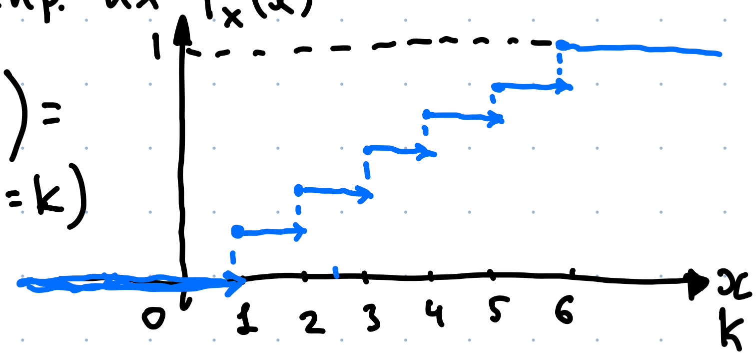
X	1	2	3	4	6
$\mathbb{P}(X=k)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$



$\mathbb{P}(X=3) =$
$= F(3) - F(2)$

<u>Опр.</u> функция распр.-ия $F_X(x)$

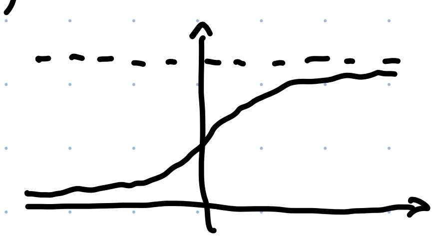
$$F_X(x) = \mathbb{P}(X \le x) = \sum_{k \le x} \mathbb{P}(X=k)$$



<u>Св-ва:</u>

1) $0 \le F_X(x) \le 1$
2) $\forall x_1 \le x_2 \quad F_X(x_1) \le F_X(x_2)$
3) $F_X(x)$ непр. справа
4) $\lim\limits_{x \to -\infty} F_X(x) = 0$

   $\lim\limits_{x \to +\infty} F_X(x) = 1$



② Непрерывная сл. вел.

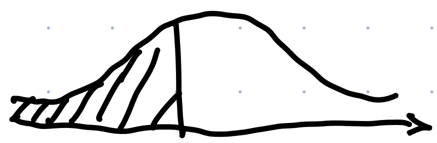
$$\mathbb{P}(X \in [t_0; t_0+\Delta]) = \Delta \cdot f(t_0) + o(\Delta)$$

$$\Delta \to 0$$

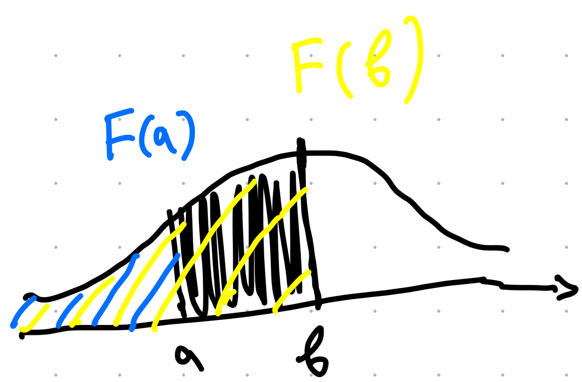
$f_X(t) \sim$ что-то типа вероятностей

<u>Опр.</u> Сл. вел. X наз. непрерывной, если

$$\exists \ f_X(t) \geq 0 \ \ \forall t : \ F_X(t) = \int_{-\infty}^{t} f_X(t)\, dt$$



$\mathbb{P}(X \leq t)$



<u>Св-ва:</u>

1) $f_X(t) \geq 0$

2) $\int_{-\infty}^{+\infty} f_X(t)\, dt = 1$

3) $\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(t)\, dt \,) = F_X(b) - F_X(a)$

4) $\mathbb{P}(X = a) = \mathbb{P}(a \leq X \leq a) = \int_a^a f_X(t)\, dt = 0$

Упражнение

$X \quad [0;2]$

$F_X(t) = c \cdot t^2 = \frac{t^2}{4}$

а) c - ?

б) $f_X(t)$ - ?

в) $\mathbb{P}(X < 1)$

г) $\mathbb{P}(X \in [1, 1.5])$

д) $F_X(-5) \quad F_X(10)$

е) $\mathbb{E}(X) \quad Var(X)$

$$f_X(t) = F_X'(t) = 2 \cdot c \cdot t$$

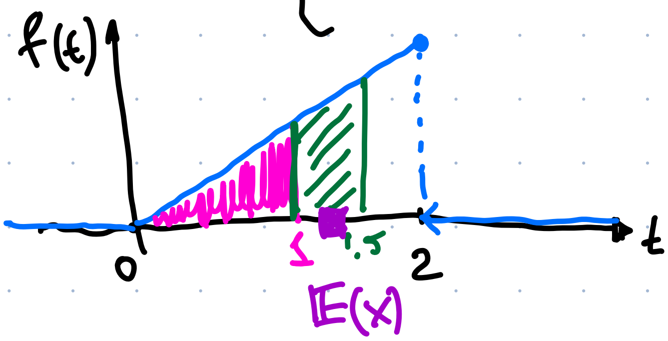
$$\int_{-\infty}^{+\infty} f_X(t)\,dt = 1$$

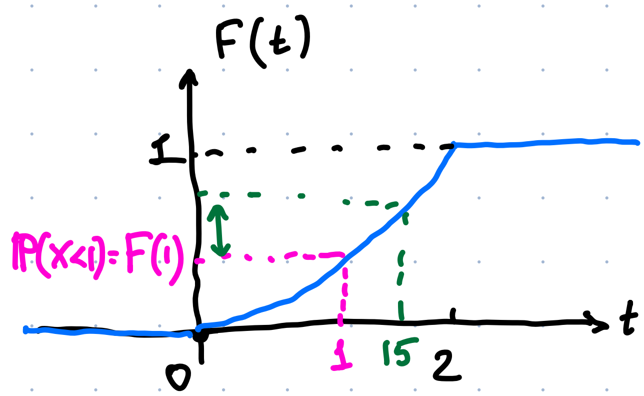
$$\int_0^2 2 \cdot c \cdot t \, dt = 1$$

$$2 \cdot c \cdot \frac{t^2}{2} \Big|_0^2 = 1$$

$$c \cdot (2^2 - 0^2) = 1 \Rightarrow c = 1/4$$

$$f_X(t) = \begin{cases} t/2, & t \in [0;2] \\ 0, & t \notin [0;2] \end{cases}$$





$$\mathbb{P}(X < 1) = \mathbb{P}(X \le 1) = F(1) = \int_0^1 f(t)\,dt = \frac{1}{4}$$

$$\mathbb{P}(1 \le X \le 1.5) = F(1.5) - F(1) = \frac{1.5^2}{4} - \frac{1^2}{4}$$

$F(-5) = 0 \qquad F(10) = 1$

$\mathbb{E}(Y) = \sum_{k} P_k \cdot k$

Дискретные сл. велич.

$Var(Y) = \mathbb{E}(Y - \mathbb{E}(Y))^2 =$

$= \sum_{k} P_k \cdot (k - \mathbb{E}(Y))^2$

$Var(Y) = \mathbb{E}(Y^2) - \mathbb{E}^2(Y)$

В непр. сл. мы меняем $\sum$ на $\int$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(t) \cdot t \, dt = \int_0^2 t/2 \cdot t \, dt =$$

$$= \frac{1}{2} \cdot \frac{t^3}{3} \Big|_0^2 = \frac{1}{6} \cdot (2^3 - 0^3) = \frac{8}{6} = \frac{4}{3}$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(t) \cdot t^2 \, dt = \int_0^2 \frac{t}{2} \cdot t^2 \, dt =$$

$$= \frac{1}{2} \cdot \frac{t^4}{4} \Big|_0^2 = \frac{1}{8} \cdot (2^4 - 0^4) = 2$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}^2(X) = 2 - \frac{16}{9} = \frac{18-16}{9} =$$

$$= 2/9$$

③ Преобразование сл. вел.

Опр.

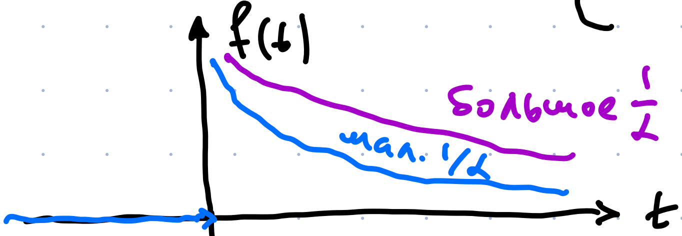
$X \sim Exp(\lambda)$ Экспоненциальное р.-е

$$F_X(t) = \begin{cases} 1 - e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Упражнение

$f(t)$ - ? $\mathbb{E}(X)$ $Var(X)$

$$f_X(t) = F_X'(t) = \begin{cases} \lambda \cdot e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



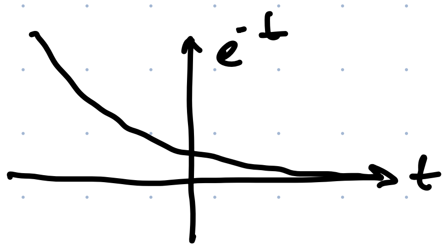
$$\int_a^b u(t)\,dv(t) = u(t)\cdot v(t)\Big|_a^b - \int_a^b v\,du$$

$$\mathbb{E}(X) = \int_0^{+\infty} t \cdot \lambda \cdot e^{-\lambda t}\,dt \quad \cdot -\frac{1}{\lambda} \quad = \int_0^{+\infty} -t \cdot \underbrace{d e^{-\lambda t}}_{-\lambda \cdot e^{-\lambda t} dt} =$$

$$= -t e^{-\lambda t}\Big|_0^{+\infty} + \int_0^{+\infty} e^{-\lambda t}\,dt =$$

$$= -\left[\cancel{\lim_{t\to+\infty} \frac{t}{e^{\lambda t}}}^{\,0} - \cancel{\lim_{t\to 0} \frac{t}{e^{\lambda t}}}^{\,0}\right] - \frac{1}{\lambda} e^{-\lambda t}\Big|_0^{+\infty} =$$

$$= -\frac{1}{\lambda}\left(e^{-\infty} - e^{0}\right) = \frac{1}{\lambda}$$



$$\lim_{t \to +\infty} e^{-t} = 0$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Найдите её сами

$\mathbb{E}(X^2)$ - два интеграла по частям

Упражнение

$X \sim \mathrm{Exp}(1)$

$Y = X^2 \quad f_Y(t) - ?$

$Z = \sqrt{X} \quad f_Z(t) - ?$

функция случайного аргумента

Везде $t > 0$

$$F_Z(t) = \mathbb{P}(Z \le t) = \mathbb{P}(\sqrt{X} \le t) =$$

$$= \mathbb{P}(X \le t^2) = F_X(t^2) = 1 - e^{-t^2}$$

$$f_Z(t) = F_Z'(t) = \left[F_X(t^2)\right]' = 2t \cdot F_X'(t^2) =$$

$$= 2t \cdot f_X(t^2) = 2t\, e^{-t^2}$$

Дальше я могу сделать с Z что угодно!

$$F_Y(t) = \mathbb{P}(Y \le t) = \mathbb{P}(X^2 \le t) =$$

$$= \mathbb{P}(-\sqrt{t} \le X \le \sqrt{t}) = F_X(\sqrt{t}) - F_X(-\sqrt{t}) =$$

$X \quad t \in [0; +\infty)$ $\qquad$ $\sqrt{X} \quad t \in [0; +\infty)$

$X^2 \quad t \in [0; +\infty)$

$$= 1 - e^{-\sqrt{t}}$$

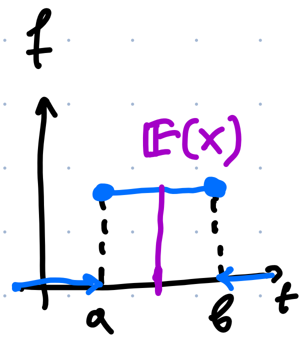
$$f_Y(t) = \frac{1}{2\sqrt{t}} e^{-\sqrt{t}} \qquad \int_0^{+\infty} \frac{1}{2\sqrt{t}} e^{-\sqrt{t}} \, dt =$$

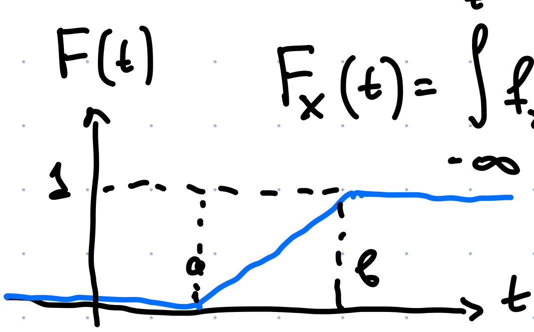
$$= \int_0^{+\infty} e^{-\sqrt{t}} \, d(\sqrt{t}) = \int_0^{+\infty} e^{-v} \, dv =$$ Проверка

$$= -e^{-v} \Big|_0^{+\infty} = 0 - (-1) = 1$$

Опр. $X \sim U[a; b]$

$$f_X(t) = \begin{cases} \frac{1}{b-a}, & t \in [a; b] \\ 0, & t \notin [a; b] \end{cases}$$





$$F_X(t) = \int_{-\infty}^{t} f_X(t) \, dt = \begin{cases} \frac{t}{b-a}, & t \in [a; b] \\ 0, & t \le a \\ 1, & t > b \end{cases}$$

$$\mathbb{E}(X) = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12}$$

Задание: самим найти эти величины

Т. (Квантильное преобразование)

$$X \sim Exp(\lambda)$$

$$Y = F_X(X) = 1 - e^{-\lambda X} \sim ?$$

$$F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(1 - e^{-\lambda X} \leq t) =$$

$$= \mathbb{P}(e^{-\lambda X} \geq 1-t) = \mathbb{P}(-\lambda X \geq \ln(1-t)) =$$

$$= \mathbb{P}\left(X \leq -\frac{1}{\lambda}\ln(1-t)\right) = F_X\left(-\frac{1}{\lambda}\ln(1-t)\right) =$$

$$= 1 - e^{\ln(1-t)} = 1 - 1 + t = t$$

$$X \quad t \in [0; +\infty)$$

$$Y \quad t \in [1 - e^{-0}; 1 - e^{-\infty}] = [0; 1]$$

$$Y \sim U[0; 1]$$

$X \sim \mathcal{F}_\theta$ Квантильное пр.-е

$Y = F_X(X) \sim U[0;1]$

$F_Y(t) = \mathbb{P}(Y \le t) = \mathbb{P}(F_X(X) \le t) =$

$= \mathbb{P}(X \le F_X^{-1}(t)) = F_X(F_X^{-1}(t)) = t$

Компьютер хорошо генерирует равномерное распределение

$Y \quad F_Y(t)$

1. $X \sim U[0;1]$
2. $F_Y^{-1}(X)$

$x_1, \dots, x_n \sim iid\ U[0;1]$

$y_i = -\frac{1}{\lambda} \ln(1 - x_i)$

$y_1, \dots, y_n \sim iid\ Exp(\lambda)$

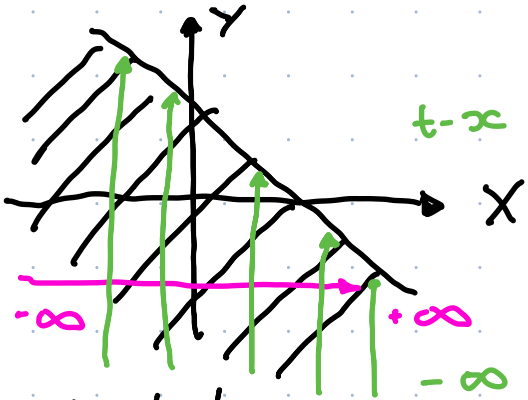
Для нормального такой прием не поможет!!

4 Формула для суммы сл. вел. (формула свёртки)

X, Y - независимые $\quad f_X(t_1) \quad f_Y(t_2)$

$Z = X + Y \sim ? \quad f_Z(t_2) - ?$

$$F_Z(t) = \mathbb{P}(Z \le t) = \mathbb{P}(X + Y \le t) = *$$



$$Y \le t - X$$

$$* = \int_{-\infty}^{+\infty}\int_{-\infty}^{t-t_1} \underbrace{f(t_1, t_2)}_{f_X(t_1)\cdot f_Y(t_2)} \, dt_1 \, dt_2 = \int_{-\infty}^{+\infty} f_X(t_1)\, dt_1 \int_{-\infty}^{t-t_1} f_Y(t_2)\, dt_2$$

$$F_Z(t) = \int_{-\infty}^{+\infty} f_X(t_1)\, dt_1 \int_{-\infty}^{t-t_1} f_Y(t_2)\, dt_2$$

$$f_Z(t) = F_Z'(t) = \int_{-\infty}^{+\infty} f_X(t_1) \left(\int_{-\infty}^{t-t_1} f_Y(t_2)\, dt_2 \right)_t' dt_1 =$$

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)\, dx = \underbrace{\int_{\alpha(t)}^{\beta(t)} \frac{df(x,t)}{dt}\, dx}_{0} +$$

$$+ f(\beta(t), t) \cdot \frac{d\beta(t)}{dt} - \underbrace{f(\alpha(t), t) \cdot \frac{d\alpha(t)}{dt}}_{0}$$

$$= \int_{-\infty}^{+\infty} f_X(t_1)\, f(t-t_1) \cdot 1 \, dt_1$$

$X \quad x \qquad Y \quad y$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_Y(y)\, f_X(z-y)\, dy$$

Упражнение:

$X \sim N(\mu, \sigma^2) \qquad f_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(t-\mu)^2}{2\sigma^2}}$

$\mathbb{E}(X) = \mu$

$Var(X) = \sigma^2$

$X \sim N(\mu_1, \sqrt{\tfrac{1}{2}}) \qquad \frac{1}{\sqrt{\pi}} e^{-(t-\mu_1)^2}$

$Y \sim N(\mu_2, \sqrt{1/2})$

$Z = X + Y \sim N(\mu_1 + \mu_2, \sqrt{1/2} + \sqrt{1/2})$

$$f_Z(t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-(y-\mu_1)^2} \cdot \frac{1}{\sqrt{\pi}} e^{-(z-y+\mu_2)^2} dy =$$

$= \boxed{\text{Решайте сами!}}$

Интеграл Гаусса-Пуасона:

$$\int_0^{+\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

Упражнение

$X \quad [0; 2] \qquad F_X(x) = cx^3 \qquad \{X \le 0.3\}$

а) c - ?

б) $\mathbb{P}(X \le 0.3 \mid X \le 0.6) = \dfrac{\mathbb{P}(\{X \le 0.3\} \cap \{X \le 0.6\})}{\mathbb{P}(X \le 0.6)} = \dfrac{F(0.3)}{F(0.6)} = \dfrac{0.3^3}{0.6^3} = \left(\dfrac{3}{6}\right)^3 = \dfrac{1}{8}$

в) $\mathrm{Cov}\left(X+1, \dfrac{1}{X}\right)$

$$F(2) - F(0) = 1$$

$$c \cdot 2^3 - c \cdot 0^3 = 1$$

$$c = \frac{1}{8}$$

б) $\mathbb{P}(A \mid B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

б) $\operatorname{Cov}\left(X+1, \frac{1}{X}\right) = \operatorname{Cov}\left(X, \frac{1}{X}\right) + \underbrace{\operatorname{Cov}\left(1, \frac{1}{X}\right)}_{0} =$

$$\underbrace{\mathbb{E}\left(X \cdot \frac{1}{X}\right)}_{1} - \mathbb{E}(X) \cdot \mathbb{E}\left(\frac{1}{X}\right)$$

$$\mathbb{E}(X) = \int_0^2 x \cdot f(x)\, dx = \int_0^2 x \, dF(x) =$$

$$= \int_0^2 x \, d\frac{x^3}{8} = \int_0^2 \frac{3x^3}{8} dx = \frac{3}{8} \cdot \frac{x^4}{4} \Big|_0^2 = \frac{3}{8} \cdot \frac{1}{4} \cdot 16 = \frac{3}{2}$$

$$\mathbb{E}\left(\frac{1}{X}\right) = \int_0^2 \frac{1}{x} f(x)\, dx = \int_0^2 \frac{1}{x} \cdot 3x^2 \cdot \frac{1}{8} dx = (*)$$

$$\mathbb{E}(\varphi(X)) = \int_{-\infty}^{+\infty} \varphi(x) \cdot f(x)\, dx$$

$$(*) = \int_0^2 3x \cdot \frac{1}{8} dx = \frac{3}{8} \cdot \frac{x^2}{2} \Big|_0^2 = \frac{3}{8} \cdot 2 = \frac{3}{4}$$

$$\operatorname{Cov}\left(X+1, \frac{1}{X}\right) = 1 - \frac{3}{2} \cdot \frac{3}{4} = 1 - \frac{9}{8} = -\frac{1}{8} < 0$$

$X \uparrow \quad \frac{1}{X} \downarrow$ интуитивно понятно