Notes on ACM

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November 4, 2018

1 Introduction

Yields can be decomposed into two components: expectations of future short rates and term premium. The term premium is the compensation that investors require for bearing the risk that short rates might not evolve as expected. This decomposition cannot be inferred directly from market prices, so we need to impose some structure and some hypothesis in order to do this decompostion using econometric models.

Models typically have three assumptions:

- pricing kernel is exponentially affine in the shocks that drive the economy
- prices of risk are affine in the state variables
- shocks to state variables and log yields observation errors are normally distributed

These assumptions give rise to yields that are an affine function of state variables and coefficient restrictions across maturities. If you also impose a no-arbitrage restriction, the yield for bonds of different maturities can be seen as interlinked functions of the pricing factors. We can use these functions to consistently estimate the price-of-risk parameters. If we then set these parameters to zero, we could build a risk-neutral curve.

Usually, papers use maximum likelihood estimates to explore the distribution assumption. This paper uses a three-step OLS regression based approach:

1. Decompose pricing factors into predictable components and factor innovations by regressing factors on their lagged levels.
2. Estimate the exposures of treasury returns with respect to lagged variables of pricing factors and contemporaneous pricing factors innovations.
3. Obtain market price of risk parameters from a cross-sectional regression of the exposure of returns to the lagged pricing factors onto exposures to contemporaneous pricing factor innovations.

2 The Model

2.1 State variables and expected returns

$$X_{t+1} = \mu + \Phi X_t + \upsilon_{t+1} \tag{1}$$

where X_t is a $K \times 1$ vector of state variables and $\upsilon_t \sim N(0, \Sigma)$[1]. Denote $P_t^{(n)}$ as the price at time t of a zero coupon treasury bond with maturity in n periods and unitary face value. The assumption of no-arbitrage implies that exists a pricing kernel M_t such that:

$$P_t^{(n)} = E_t\left[M_{t+1} P_{t+1}^{(n-1)}\right] \tag{2}$$

We assume that the pricing kernel M_{t+1} is exponentially affine

$$M_{t+1} = \exp\left(-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right) \tag{3}$$

[1] Since Σ is a symmetric matrix it can be diagonalized as $\Sigma = VDV^{-1}$, where D is a diagonal matrix of eigenvalues of Σ and V is a matrix where each column is the corresponding eigenvector. So we can write $\Sigma^{-\frac{1}{2}} = VD^{-\frac{1}{2}}V^{-1}$

where $r_t = \ln\left(P_t^{(1)}\right)$ is the continously compounded short term risk-free rate and λ_t is the market price of risk at time t, which is assumed to be an affine function of the state variables following:

$$\lambda_t = \Sigma^{-\frac{1}{2}}\left(\lambda_0 + \lambda_1 X_t\right) \tag{4}$$

Where λ_0 and λ_1 are matrices of coeficients with dimensions$K \times 1$ and $K \times K$. We denote $rx_{t+1}^{(n)}$ is the one-period log excess holding return of a bond maturing in n periods, derived as follows:

$$rx_{t+1}^{(n-1)} = \frac{\ln\left(\frac{P_{t+1}^{(n-1)}}{P_t^{(n)}}\right)}{1} - r_t = \ln\left(P_{t+1}^{(n-1)}\right) - \ln\left(P_t^{(n)}\right) - r_t \tag{5}$$

Substituting equation (3) in (2) we get:

$$P_t^{(n)} = E_t\left[\exp\left(-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)P_{t+1}^{(n-1)}\right]$$

Isolating r_t in equation (5) and substituting in the previous equation, we get:

$$\begin{aligned}
P_t^{(n)} &= E_t\left[\exp\left(-\left(\ln\left(P_{t+1}^{(n-1)}\right) - \ln\left(P_t^{(n)}\right) - rx_{t+1}^{(n-1)}\right) - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)P_{t+1}^{(n-1)}\right] \\
P_t^{(n)} &= E_t\left[\exp\left(-\ln\left(P_{t+1}^{(n-1)}\right) + \ln\left(P_t^{(n)}\right) + rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)P_{t+1}^{(n-1)}\right] \\
P_t^{(n)} &= E_t\left[\frac{-1}{P_{t+1}^{(n-1)}}P_t^{(n)}\exp\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)P_{t+1}^{(n-1)}\right] \\
P_t^{(n)}\frac{1}{P_t^{(n)}} &= E_t\left[\frac{1}{P_{t+1}^{(n-1)}}\exp\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)P_{t+1}^{(n-1)}\right] \\
1 &= E_t\left[\exp\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)\right] \qquad (6)
\end{aligned}$$

Assuming that $rx_{t+1}^{(n-1)}$ and υ_{t+1} are jointly normally distributed, we find that the term inside the expectations operator is a log-normal variable, so that

$$\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right) \sim N\left(\mu_{r,t}, \Sigma_{r,t}\right)$$

$$\begin{aligned}
\mu_{r,t} &= E_t\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right) \\
&= E_t\left(rx_{t+1}^{(n-1)}\right) - \frac{1}{2}\lambda_t'\lambda_t
\end{aligned}$$

$$\begin{aligned}
\Sigma_{r,t} &= Var_t\left(rx_{t+1}^{(n-1)} - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right) \\
&= Var_t\left(rx_{t+1}^{(n-1)} - \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right) \\
&= Var_t\left(rx_{t+1}^{(n-1)}\right) + \lambda_t'\Sigma^{-\frac{1}{2}}Var_t\left(\upsilon_{t+1}\right)\Sigma^{-\frac{1}{2}}\lambda_t + 2Cov_t\left(rx_{t+1}^{(n-1)}, -\lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right) \\
&= Var_t\left(rx_{t+1}^{(n-1)}\right) + \lambda_t'\lambda_t - 2Cov_t\left(rx_{t+1}^{(n-1)}, \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)
\end{aligned}$$

Now that we know these moments, we can write the expected value of the log-normal distribution from equation (6) as:

$$1 = \exp\left(\mu_{r,t} + \frac{1}{2}\Sigma_{r,t}\right)$$

$$1 = \exp\left(E_t\left(rx_{t+1}^{(n-1)}\right) - \frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}\left(Var_t\left(rx_{t+1}^{(n-1)}\right) + \lambda_t'\lambda_t - 2Cov_t\left(rx_{t+1}^{(n-1)}, \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)\right)\right)$$

$$1 = \exp\left(E_t\left(rx_{t+1}^{(n-1)}\right) - \frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right) + \frac{1}{2}\lambda_t'\lambda_t - Cov_t\left(rx_{t+1}^{(n-1)}, \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)\right)$$

$$1 = \exp\left(E_t\left(rx_{t+1}^{(n-1)}\right) + \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right) - Cov_t\left(rx_{t+1}^{(n-1)}, \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)\right)$$

Apply the natural logarithm on both sides to obtain:

$$0 = E_t\left(rx_{t+1}^{(n-1)}\right) + \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right) - Cov_t\left(rx_{t+1}^{(n-1)}, \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right)$$

$$E_t\left(rx_{t+1}^{(n-1)}\right) = Cov_t\left(rx_{t+1}^{(n-1)}, \lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}\right) - \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right)$$

Since the term $\lambda_t'\Sigma^{-\frac{1}{2}}\upsilon_{t+1}$ is a scalar, we can work with its transpose:

$$E_t\left(rx_{t+1}^{(n-1)}\right) = Cov_t\left(rx_{t+1}^{(n-1)}, \upsilon_{t+1}'\Sigma^{-\frac{1}{2}}\lambda_t\right) - \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right)$$

$$E_t\left(rx_{t+1}^{(n-1)}\right) = Cov_t\left(rx_{t+1}^{(n-1)}, \upsilon_{t+1}'\right)\Sigma^{-\frac{1}{2}}\lambda_t - \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right)$$

Using equation (4) we can write:

$$E_t\left(rx_{t+1}^{(n-1)}\right) = Cov_t\left(rx_{t+1}^{(n-1)}, \upsilon_{t+1}'\right)\Sigma^{-\frac{1}{2}}\Sigma^{-\frac{1}{2}}\left(\lambda_0 + \lambda_1 X_t\right) - \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right)$$

$$E_t\left(rx_{t+1}^{(n-1)}\right) = Cov_t\left(rx_{t+1}^{(n-1)}, \upsilon_{t+1}'\right)\Sigma^{-1}\left(\lambda_0 + \lambda_1 X_t\right) - \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right)$$

Now let us define

$$\beta_t^{(n-1)\prime} = Cov_t\left(rx_{t+1}^{(n-1)}, \upsilon_{t+1}'\right)\Sigma^{-1} \tag{7}$$

so we can rewrite

$$E_t\left(rx_{t+1}^{(n-1)}\right) = \beta_t^{(n-1)\prime}\left(\lambda_0 + \lambda_1 X_t\right) - \frac{1}{2}Var_t\left(rx_{t+1}^{(n-1)}\right) \tag{8}$$

Now that we have the expected value of the excess returns of bonds, we can decompose the unexpected excess returns into two components, the first one is correlated with the innovations and the second one is conditionally orthogonal to the first one. Se we can write:

$$rx_{t+1}^{(n-1)} - E_t\left(rx_{t+1}^{(n-1)}\right) = \gamma_t^{(n-1)\prime}\upsilon_{t+1} + e_{t+1}^{(n-1)}$$

Since the term $\gamma_t^{(n-1)\prime}\upsilon_{t+1}$ represents the orthogonal projection of $rx_{t+1}^{(n-1)}$ on υ_{t+1} then it is straightforward to show that $\gamma_t^{(n-1)} = \beta_t^{(n-1)}$. We assume that the pricing errors $e_{t+1}^{(n-1)}$ are independent and identically distributed with variance σ^2. Since the variables in X_t are linear combinations of log yields and the model parameters are estimated using holding period returns, then by construction (functional form of X_t), $\beta_t^{(n-1)}$ is constant $\beta^{(n-1)}$. So we now have:

$$rx_{t+1}^{(n-1)} - E_t\left(rx_{t+1}^{(n-1)}\right) = \beta^{(n-1)\prime}\upsilon_{t+1} + e_{t+1}^{(n-1)} \tag{9}$$

From this last equation we can show that:

$$\begin{aligned}
Var_t\left(rx_{t+1}^{(n-1)}\right) &= Var_t\left(E_t\left(rx_{t+1}^{(n-1)}\right) + \beta^{(n-1)\prime}\upsilon_{t+1} + e_{t+1}^{(n-1)}\right) && (10)\\
&= Var_t\left(\beta^{(n-1)\prime}\upsilon_{t+1} + e_{t+1}^{(n-1)}\right) && (11)\\
&= \beta^{(n-1)\prime}Var_t\left(\upsilon_{t+1}\right)\beta^{(n-1)} + Var_t\left(e_{t+1}^{(n-1)}\right) && (12)\\
&= \beta^{(n-1)\prime}\Sigma\beta^{(n-1)} + \sigma^2 && (13)
\end{aligned}$$

Using equations (9), (8) and (13) we can write the return generating process for log excess returns as:

$$rx_{t+1}^{(n-1)} = \underbrace{\beta^{(n-1)\prime}\left(\lambda_0 + \lambda_1 X_t\right)}_{Expected\ Return} - \underbrace{\frac{1}{2}\left(\beta^{(n-1)\prime}\Sigma\beta^{(n-1)} + \sigma^2\right)}_{Convexity\ Adjustment} + \underbrace{\beta^{(n-1)\prime}\upsilon_{t+1}}_{Priced\ Return\ Innovation} + \underbrace{e_{t+1}^{(n-1)}}_{Return\ Pricing\ Error} \tag{14}$$

Stacking this last equation in a system across maturities and time periods, we can write all the equations of all the observations as:

$$rx = \beta' \left(\lambda_0 \iota_T' + \lambda_1 X\right) - \frac{1}{2}\left(B^\star vec\left(\Sigma\right) + \sigma^2 \iota_N\right)\iota_T' + \beta' V + E \tag{15}$$

where ι_T and ι_N are $T \times 1$ and $N \times 1$ vectors of ones, λ_0 and λ_1 have already been defined in equation (4) and:

$$rx = \begin{pmatrix} rx_1^{(1)} & rx_2^{(1)} & \cdots & rx_T^{(1)} \\ rx_1^{(2)} & rx_2^{(2)} & \cdots & rx_T^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ rx_1^{(N)} & rx_2^{(N)} & \cdots & rx_T^{(N)} \end{pmatrix}_{N \times T} \qquad \beta = \begin{pmatrix} \beta_1^{(1)} & \beta_1^{(2)} & \cdots & \beta_1^{(N)} \\ \beta_2^{(1)} & \beta_2^{(2)} & \cdots & \beta_2^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_K^{(1)} & \beta_K^{(2)} & \cdots & \beta_K^{(N)} \end{pmatrix}_{K \times N}$$

$$X = \begin{pmatrix} x_{1,0} & x_{1,1} & \cdots & x_{1,T-1} \\ x_{2,0} & x_{2,1} & \cdots & x_{2,T-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{K,0} & x_{K,1} & \cdots & x_{K,T-1} \end{pmatrix}_{K \times T} \qquad B^\star = \begin{pmatrix} vec\left(\beta^{(1)}\beta^{(1)\prime}\right)' \\ vec\left(\beta^{(2)}\beta^{(2)\prime}\right)' \\ \vdots \\ vec\left(\beta^{(N)}\beta^{(N)\prime}\right)' \end{pmatrix}_{N \times K^2}$$

$$V = \begin{pmatrix} \upsilon_{1,1} & \upsilon_{1,2} & \cdots & \upsilon_{1,T} \\ \upsilon_{2,1} & \upsilon_{2,2} & \cdots & \upsilon_{2,T} \\ \vdots & \vdots & \ddots & \vdots \\ \upsilon_{K,1} & \upsilon_{K,2} & \cdots & \upsilon_{K,T} \end{pmatrix}_{K \times T} \qquad E = \begin{pmatrix} e_1^{(1)} & e_2^{(1)} & \cdots & e_T^{(1)} \\ e_1^{(2)} & e_2^{(2)} & \cdots & e_T^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ e_1^{(N)} & e_2^{(N)} & \cdots & e_T^{(N)} \end{pmatrix}_{N \times T}$$

2.2 Estimation

We want to estimate all the parameters from equation (15). The authors do this with a four-step method:

1. Estimate equation (1) using OLS (basically a VAR(1)). This allows to decompose X_{t+1} into a predictable component and an estimate of the inovation $\hat{\upsilon}_{t+1}$. We can stack these estimated innovations into matrix $\hat{V}$ and construct an estimator of the state variable covariance matrix:

$$B = (\mu, \Phi)$$

$$\hat{B} = YZ' \left(ZZ'\right)^{-1}$$

$$\hat{\Sigma} = \frac{\hat{V}\hat{V}'}{T} \qquad or \qquad \hat{\Sigma} = \frac{\hat{V}\hat{V}'}{T-K-1}$$

2. Regress the excess returns on a constant and lagged pricing factors and contemporaneuos pricing factor innovations according to

$$rx = a\iota_T' + \beta'\hat{V} + cX + E$$

We can organize the regressors in a $(2K+1) \times T$ matrix $Z = \begin{pmatrix} \iota_T & \hat{V}' & X' \end{pmatrix}'$ so that the estimator becames:

$$\begin{pmatrix} \hat{a} & \hat{\beta}' & \hat{c} \end{pmatrix} = rxZ' \left(ZZ'\right)^{-1}$$

We collect the residuals from this regression into de $N \times T$ matrix $\hat{E}$ to estimate:

$$\hat{\sigma}^2 = \frac{trace\left(\hat{E}\hat{E}'\right)}{NT}$$

We can construct $\hat{B}^\star$ from $\hat{\beta}$.

3. The prices of risk cofficients are estimated via cross-sectional regression. We know from equation (15) that:

$$a = \beta'\lambda_0 - \frac{1}{2}\left(B^\star vec\left(\Sigma\right) + \sigma^2 \iota_N\right) \qquad c = \beta'\lambda_1$$

We can use these expressions to obtain the estimators for λ_0 and λ_1:

$$\hat{a} = \hat{\beta}'\hat{\lambda}_0 - \frac{1}{2}\left(\hat{B}^{\star}vec\left(\hat{\Sigma}\right) + \hat{\sigma}^2\iota_N\right) \qquad \hat{c} = \hat{\beta}'\hat{\lambda}_1$$

$$\hat{\beta}'\hat{\lambda}_0 = \hat{a} + \frac{1}{2}\left(\hat{B}^{\star}vec\left(\hat{\Sigma}\right) + \hat{\sigma}^2\iota_N\right) \qquad \hat{c} = \hat{\beta}'\hat{\lambda}_1$$

$$\hat{\beta}\hat{\beta}'\hat{\lambda}_0 = \hat{\beta}\left[\hat{a} + \frac{1}{2}\left(\hat{B}^{\star}vec\left(\hat{\Sigma}\right) + \hat{\sigma}^2\iota_N\right)\right] \qquad \hat{\beta}\hat{c} = \hat{\beta}\hat{\beta}'\hat{\lambda}_1$$

$$\hat{\lambda}_0 = \left(\hat{\beta}\hat{\beta}'\right)^{-1}\hat{\beta}\left[\hat{a} + \frac{1}{2}\left(\hat{B}^{\star}vec\left(\hat{\Sigma}\right) + \hat{\sigma}^2\iota_N\right)\right] \qquad \hat{\lambda}_1 = \left(\hat{\beta}\hat{\beta}'\right)^{-1}\hat{\beta}\hat{c}$$

4. To find the values of δ_0 and δ_1 we need to regress the yield of the first maturity on the factors. This can be done with the traditional OLS estimator for a multiple linear regression.

$$r_t^{(1)} = \delta_0 + \delta_1' X_t + \xi_t$$

$$\begin{pmatrix}\hat{\delta}_0\\ \hat{\delta}_1\end{pmatrix} = \left(X^{\star\prime}X^{\star}\right)^{-1} X^{\star\prime} r$$

where the matrices are given by

$$X^{\star} = \begin{pmatrix} 1 & x_{1,1} & x_{2,1} & \cdots & x_{K,1}\\ 1 & x_{1,2} & x_{2,2} & \cdots & x_{K,2}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_{1,T} & x_{2,T} & \cdots & x_{K,T}\end{pmatrix}_{T\times(K+1)} \qquad r = \begin{pmatrix} r_1^{(1)}\\ r_2^{(1)}\\ \vdots\\ r_T^{(1)}\end{pmatrix}_{T\times 1}$$

2.3 Inference

Soon...

2.4 Affine yields

From the estimated model parameters, we can generate a zero coupon yield curve. Under the assumptions we have made so far, we can show that bond prices are exponentially affine in the vector of state variables:

$$\ln\left(P_t^{(n)}\right) = A_n + B_n' X_t + u_t^{(n)}$$

Substituting this in equation 5, we get:

$$\begin{aligned} rx_{t+1}^{(n-1)} &= \ln\left(P_{t+1}^{(n-1)}\right) - \ln\left(P_t^{(n)}\right) - r_t\\ &= A_{n-1} + B_{n-1}' X_{t+1} + u_{t+1}^{(n-1)} - A_n - B_n' X_t - u_t^{(n)} + A_1 + B_1' X_t + u_t^{(1)}\end{aligned}$$

Setting this last equation equal to equation (14), we get:

$$A_{n-1} + B_{n-1}' X_{t+1} + u_{t+1}^{(n-1)} - A_n - B_n' X_t - u_t^{(n)} + A_1 + B_1' X_t + u_t^{(1)} = \beta^{(n-1)\prime}\left(\lambda_0 + \lambda_1 X_t\right) - \frac{1}{2}\left(\beta^{(n-1)\prime}\Sigma\beta^{(n-1)} + \sigma^2\right) + \beta^{(n-1)\prime}\upsilon_{t+1} + e_{t+1}^{(n-1)}$$

Using the state variable dynamics defined in (1):

$$A_{n-1} + B_{n-1}'\left(\mu + \Phi X_t + \upsilon_{t+1}\right) + u_{t+1}^{(n-1)} - A_n - B_n' X_t - u_t^{(n)} + A_1 + B_1' X_t + u_t^{(1)} = \beta^{(n-1)\prime}\left(\lambda_0 + \lambda_1 X_t\right) - \frac{1}{2}\left(\beta^{(n-1)\prime}\Sigma\beta^{(n-1)} + \sigma^2\right) + \beta^{(n-1)\prime}\upsilon_{t+1} + e_{t+1}^{(n-1)}$$

With some simplifications:

$$A_{n-1} + B_{n-1}'\mu + B_{n-1}'\Phi X_t + B_{n-1}'\upsilon_{t+1} + u_{t+1}^{(n-1)} - A_n - B_n' X_t - u_t^{(n)} + A_1 + B_1' X_t + u_t^{(1)} = \beta^{(n-1)\prime}\lambda_0 + \beta^{(n-1)\prime}\lambda_1 X_t - \frac{1}{2}\left(\beta^{(n-1)\prime}\Sigma\beta^{(n-1)} + \sigma^2\right) + \beta^{(n-1)\prime}\upsilon_{t+1} + e_{t+1}^{(n-1)}$$

Now we apply the undetermined coeficients method to solve the difference equation system. The first step is to group each side as coeficients that are constants and the ones that multiply the same variables, X_t, υ_{t+1} and error terms, to get:

$$\begin{aligned}&\left(A_{n-1}+B'_{n-1}\mu-A_n+A_1\right)+\left(B'_{n-1}\Phi-B'_n+B'_1\right)X_t+\left(B'_{n-1}\right)\upsilon_{t+1}+\left(u_{t+1}^{(n-1)}-u_t^{(n)}+u_t^{(1)}\right)\\ =&\left(\beta^{(n-1)\prime}\lambda_0-\frac{1}{2}\left[\beta^{(n-1)\prime}\Sigma\beta^{(n-1)}+\sigma^2\right]\right)+\left(\beta^{(n-1)\prime}\lambda_1\right)X_t+\left(\beta^{(n-1)\prime}\right)\upsilon_{t+1}+\left(e_{t+1}^{(n-1)}\right)\end{aligned}$$

The second step is to solve for the coefficients to be the same:

$$\begin{aligned}A_{n-1}+B'_{n-1}\mu-A_n+A_1 &= \beta^{(n-1)\prime}\lambda_0-\tfrac{1}{2}\left(\beta^{(n-1)\prime}\Sigma\beta^{(n-1)}+\sigma^2\right)\\ B'_{n-1}\Phi-B'_n+B'_1 &= \beta^{(n-1)\prime}\lambda_1\\ B'_{n-1} &= \beta^{(n-1)\prime}\\ u_{t+1}^{(n-1)}-u_t^{(n)}+u_t^{(1)} &= e_{t+1}^{(n-1)}\end{aligned}$$

These equations must hold at all periods in order to generate an arbitrage free yield curve. By definition, $A_0=0$ and $B_0=0$. By setting $A_1=-\delta_0$ and $B_1=-\delta_1$, we can use these conditions to derive:

$$\begin{aligned}A_n &= A_{n-1}+B'_{n-1}\left(\mu-\lambda_0\right)+\frac{1}{2}\left(B'_{n-1}\Sigma B_{n-1}+\sigma^2\right)-\delta_0\\ B'_n &= B'_{n-1}\left(\Phi-\lambda_1\right)-\delta'_1\\ \beta^{(n)\prime} &= B'_n\end{aligned}$$

2.5 Term Premium / Bond Risk Premium

Setting price risk parameters λ_0 and λ_1 to zero in the affine yield recursions generates de risk-neutral pricing parameters A_n^{RF} and B_n^{RF}, so that $-\frac{1}{n}\left(A_n^{RF}+B_n^{RF\prime}X_t\right)$ equals the time t expectation of average future short rates over the next n periods. These risk-neutral yields are independent of economic interest, so the risk premium φ_t can be calculated as the difference between the model fitted yields (or the observed yields) and the risk-neutral yields.

$$\varphi_t^{(n)}=\frac{1+y_t^{(n)}}{1+y_{RN,t}^{(n)}}-1$$

or simply

$$\varphi_t^{(n)}=y_t^{(n)}-y_{RN,t}^{(n)}$$

3 Discussion

- This model has become the most popular for term premium estimates for financial markets, due to its ease of implementation.
- Term premium estimates can change a lot depending on the time window of your sample and on the holding period that is used to calculate the excess returns.
- The model lacks exogenous inputs that might shift the dynamics away from the long-run average and mitigate mean-reversion behavior.
- The expected short rate (risk-neutral) generated by the model is usually very different from survey estimates. A possible extension of the model is to include survey data as a variable to be fitted, and not as an explanatory variable.