project_Spasiano_1889394

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1 Data Privacy and Security Project

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Goal : Primality Testing. The goal of this project is to better understand the task of primality testing. Write a program (using your favourite programming language) implementing the following primality testing algorithms: The Fermat's Test and the Miller-Rabin Test. Compare the performances of the two tests.

What I have done: I implemented Fermat's Test, Miller-Rabin Test, Solovay-Strassen test and a simple factoring algorithm in Python. Described the theory behind them, along with advantages and disadvantages. Then compared the performance of these tests in terms: of complexity, running time and accuracy. Set of numbers considered: generic primes and composites, Mersenne numbers and Fermat numbers.

Why do we care about prime numbers? Prime numbers always had an important role in the development of mathematics. In particular the problem of determining whether a given integer is prime or not is one of the better known and most easily understood problems of pure mathematic. In 300BC, Euclid was able to prove that there were infinitely many primes and the Fundamental Theorem of Arithmetic, states that every integer is defined as the unique product of powers of prime numbers. Later on, greek mathematicians discovered the Sieve of Eratosthenes, which represents a basic approach in counting primes smaller than a given number. It's a trivial primality testing that becomes inefficient and too expensive to implement when the magnitude of the number increases. For instance, McGregor-Dorsey explained in that for an input number of 20 digits, at a rate of discovering one prime number per second, it would take more than 14 years to determine the primality of the number.

The advent of cryptographic systems that use large primes was the main driving force for the development of fast and reliable methods for primality testing. Over the last decades primes became one of the main tools of many cryptography applications and many protocols are based on prime numbers,for example RSA, in which one needs to produce large prime numbers.

There are two ways to assert the primality of a number: using **factorization** algoritms or use **primality** tests. - Factorization algorithm might be seen as more interesting, because they give more information about a number, not only if it's a prime or not, but also its unique factorization. And being able to factorize a number for an attacker might be the way of breaking security of a cryptosystem. For example in RSA we perform the protocol in $\mathbb{Z}_n$ with $n = p \cdot q$ (product of two large primes), so if we are able to factorize n, we can compute $\phi(n)$ that is the number of natural numbers coprimes with n. Computing $\phi(n)$ allow us to find the multiplicative inverse of $e \mod \phi(n)$, that is the secret key d, so break the protocol. However factorization is an hard problem. - On the other side it's easier to test if a number is a prime or not, and even more easier to test if $m \mid n$.

Theorem: There are infite prime numbers.

PROOF: This proof was made by Euclid and is the most famous. It follows by absurd. We assume there are *finite* prime numbers, say $P = \{p_1, ..., p_k\}$. Given the number $n = p_1 \cdot ... \cdot p_k + 1$. Then since P is finite, n is not divisible by any of the primes in P and n is not factorizable, that is impossible. So there are an infinte numbers of primes. QED

Basic tests:

- We have cited Sieve of Eratosthenes that is an easy method to find all primes smaller that a value n. This is quite expensive, indeed the basic algorithm requires $\mathcal{O}(n)$ of memory.
- Another one we all know from primary school is the following: given any number n, we check all numbers less than or equal to $\sqrt{n}$ and see if any of them divides n. If at least one ($\neq 1$) of them does, the algorithm outputs composite, otherwise it outputs prime. This follows from the fact that if it's indeed composite, it must be possible for us to factor it into at least two factors and at least one of them must be less than or equal to $\sqrt{n}$. However, this test, that is also a **factoring algorithm** involving $\sqrt{n}$ operations, is $\mathcal{O}(\sqrt{n})$ and hence the algorithm becomes computationally infeasible for large numbers. I implemented a very basic version of this test just to compare it computationally speaking with the others.

A **primality test** does not give any information about the factorization, it only provides an answer to the question: is this number prime or not? The idea behind probabilitic primality tests is to levergae on prime properties without looking for factors. A probabilistic primality repatedly test tries to prove that a number behaves as a primes.

Primality tests can be classified based on the form of the given number which is going to be tested. There are tests for numbers of special forms, and there are tests for generic numbers such as Fermat, Miller-Rabin, Solovay-Strassen test. I implemented and evaluated the performance of these three algorithms on generic numbers and paricular ones.

Primality tests can be classified as **deterministic** or **probabilistic**. The main difference is that deterministic tests behave predictably and can establish exactly if a given number is prime or not, whereas the probabilistic tests can, with low error probability, classify a composite number as being prime. Even if probabilistic tests, cannot assure always max accuracy, they are more efficient to use instead of deterministic algorithms. Infact they may become impractical for large numbers.

A probablistic test T is a sequence of tests $\{T_m\}_{m \in \mathbb{N}}$ and a sequence of $\{\epsilon_m\}_{m \in \mathbb{N}}$ such that the test T_i: - If n doesn't pass test T_i than it's not prime and outputs composite, otherwise n goes to the next test T_{i+1}; - The probability that n passes tests $T_1, ..., T_m$ not being prime is $< \epsilon_m$.

Useful definitions and notations A **Fermat Number** is a number of the form: $F_n = 2^{2^m} + 1$. This sequence grows incredibly fast: $F_0 = 3, F_1 = 5, F_2 = 17, F_4 = 65537$. If $2^k + 1$ is prime and $k > 0$, then k must be a power of 2, so $2k + 1$ is a Fermat number; such primes are called Fermat primes of the form $2^{2^n}+1$. A Fermat conjecture says all Fermat numbers are prime, however it was found to be false, in fact $F_5 = 4294967297$ is composite ($F_5 = 641 \cdot 6700417$)

A **Mersenne Number** is a number of the form $M = 2^k - 1$, with k prime. Notice that if M is prime, then it is called a Mersenne prime number.

Notation:

- $(\frac{a}{p})$ with p prime, is the *Legendre* symbol;

- $(\frac{a}{n})$ with n composite is the *Jacobi* symbol. $n = p_1^{c_1} \cdot ... \cdot p_k^{c_k}$ and $(\frac{a}{n}) = \prod_{i=1}^{k} (\frac{a}{p_i})^{c_i}$;

- *Euler criterion*: Given p prime and a such that $(a, p) = 1$, then $\frac{a}{p}) \equiv a^{\frac{p-1}{2}} \mod p$;

- The *Square & Multiply* algorithm to compute modular exponentiation $g^a \mod n$ has complexity $\mathcal{O}(\log^3(n))$.

1.1 Fermat's Test

This test is based on **Fermat's Little Theorem**: If p is prime, then $\forall a \in \mathbb{Z}$, $a^p \equiv p \mod p$, Which means that is $p \nmid a$, then :

$$a^{p-1} \equiv 1 \mod p$$

Fermat Test: Given a number n and an $a \leq n-1$ and $p \nmid a$: 1. if $a^{n-1} \not\equiv 1 \mod n$, then N is not a prime. 2. If $a^{n-1} \equiv 1 \mod n$, then we can't say anything.

A composite number n that passes Fermat test for a particular base a is called **pseudoprime in base** a, and a is called **Fermat Liar**, $n = PSP(a)$. For example, $341 = 11 \cdot 31$, but $2^{340} \equiv 1 \mod 341$, so $341 = PSP(2)$.

A **Charmichael Number** is a composite number that passes Fermat Test for every possible base a. In other words, $\forall a$ such that $1 \leq a \leq n-1$ and $(a, n) = 1$, n is a composite number and $a^{n-1} \equiv 1 \mod n$. (an example is $561 = 3 \cdot 11 \cdot 17$). Carmichael numbers are fairly sparse, but infinite.

Probabilistic Fermat Test The probabilistic Fermat Test is based on Fermat test. It is indeed constructed in this way: - pick an appropriate value m, 1. repeat m times the following: - pick a random base a coprime with n, - do Fermat test: compute $a^{n-1} \mod n$. If $a^{n-1} \equiv 1 \mod n$ then n go to the next test 2. If the number succeds all the test, then we can say that with high probability it is a prime. If it is rejected by one test, we call it composite.

At every step i, the if n succeds the test i, then it's a *pseudoprime* in a particular base. The key point is: if a number is pseudoprime for a sufficiently large set of bases, than with high probability n is actually prime.

Complexity: Using the S&M algorithm for modular exponentiation, the time complexity of this test id $\mathcal{O}(m \cdot \log^3(n))$.

The Fermat Test is fast and easy to implement. The test does however have a serious flaw: Carmichael numbers completely fools it, the big problem is that there are infinitely many Carmichael numbers. Thus these numbers cause the test to wrongly claim them as prime.

The probability of error can be made arbitrary small by picking a large number of bases. But the probability of error is 1 for Carmichael numbers.

Test	Pros	Cons	Computational Complexity	Probability of Error
Fermat	Fast	Carmichael numbers It cannot determine the number of Fermat liars for a pseudoprime	$\mathcal{O}(m \cdot \log^3(n))$	Probability of failure for Carmichael numbers is 1

1.2 The Miller Rabin Primality Test

The Miller-Rabin Test improves on the weaknesses of the Fermat Test. It consists in a sequence of m test $\{T_m\}_{m\in\mathbb{N}}$ and $\{\epsilon_m\}_{m\in\mathbb{N}}$. A test T_i is composed by the following steps:

1. Pick a random b_i ($\neq b_1 ... \neq b_{i-1}$), write $n-1 = 2^s \cdot t$ with t odd.
2. compute (b_i, n).
   - If it is > 1 than we can conclude the test and say n is not prime (and we also have information about the factorization).
   - If it is $= 1$, then we can start the test i.
3. Since $(b_i, n) = 1$,
   - Do
     - Compute b_i^t and if $b_i^t \not\equiv \pm 1$
     - then compute $b_i^{2^1 t}$ and if $b_i^{s^1 t} \not\equiv \pm 1$
     - then compute $b_i^{2^2 t}$ and if $b_i^{s^2 t} \not\equiv \pm 1$
     - ...
     - compute $b_i^{2^{s-1} t}$ and if $b_i^{2^{s-1} t} \not\equiv \pm 1$, then n is composite.
   - otherwise if:
     - $b_i^t \equiv \pm 1$ or
     - or $\exists 0 \leq r < s$ such that $b_i^{2^r t} \not\equiv -1$, then n pass the test and text test is performed.

The probability that n succedd until test i without being prime is $\leq \frac{1}{4^i}$. So with a large enough m (say ≈ 30) we can conclude n is prime.

The test relies on theorems and results in **number theory** that I can summarize in:

1. If p is prime, then $p-1 = 2^s t$, while computing $b^t, b^{2t}, ..., b^{2^{s-s} t}$ we will find that one of these values is equivalent to a -1 and then the next iteration we'll get a 1. This is because $b^{p-1} \equiv 1 \mod p$ (PTF) and due to the fact that $\mathbb{Z}_p$ is a field and in this field the equation $x^2 - 1 \equiv 0 \mod p$ has **only** two **trivial** solutions $x = \pm 1$.
2. On the other hand if n is composite, then $n-1 = 2^s t$, and computing $b^t, b^{2t}, ..., b^{2^{s-1} t}$ it might happen to find $b^{2^k t} \equiv 1 \mod n$ but its square root $b^{2^{k-1} t} \not\equiv \pm 1 \mod n$, which means that we obtained a non trivial solution, so n fails the test.

Composite numbers that pass one iteration of Miller-Rabin test are called **strong-pseudoprimes** and they are much fewer than the pseudoprimes.

Complexity: Using the S&M algorithm, Miller-Rabin tests the primality of any odd number n in time $\mathcal{O}(m \cdot log^3(n))$ (thanks to FFT-based multiplication it's possible to push the running time down to $\mathcal{O}(m \cdot \log^2(n) \cdot \log(\log(n)) \cdot \log(\log(\log(n)))) = \tilde{\mathcal{O}}(m \cdot \log^2(n)))$ [5]).

Test	Pros	Cons	Computational Complexity	Probability of Error
Fermat	Fast	Carmichael numbers It cannot determine the number of Fermat liars for a pseudoprime	$\mathcal{O}(m \cdot \log^3(n))$	Probability of failure for Carmichael numbers is 1
Miller-Rabin	Very fastVery small probability of errorStrong-pseudoprimes are fewer than pseudoprimes	Strong-pseudoprimes fools it	$\tilde{\mathcal{O}}(m \cdot \log^2(n))$	$< 4^{-m}$

1.3 Solovay Strassen Primality Test

The test is based on **Euler's criterion**: Let p be an prime and b be an integer coprime to p. Then:

$$b^{\frac{p-1}{2}} \equiv \left(\frac{b}{p}\right) \mod p$$

The Solovay Strassen test consists in a sequence of m test $\{T_m\}_{m \in \mathbb{N}}$ and $\{\epsilon_m\}_{m \in \mathbb{N}}$. A test T_i is composed by the following steps: 1. Pick a random b_i ($\neq b_1 ... \neq b_{i-1}$). 2. compute (b_i, n). - If it is > 1 than we can conclude the test and say n is not prime (and we also have information about the factorization). - If it is $= 1$, then we can start the test i. 3. Since $(b_i, n) = 1$, compute $(\frac{b_i}{n})$ and $b_i^{\frac{n-1}{2}}$. - If $(\frac{b_i}{n}) \not\equiv b_i^{\frac{n-1}{2}} \mod n$ then we conclude n is not prime. - If $(\frac{b_i}{n}) \equiv b_i^{\frac{n-1}{2}} \mod n$ then n is potentially prime. The probability that n succedd the test m without being prime is $\leq \frac{1}{2^i}$. So with a large enough m (say ≈ 30) we can conclude n is prime.

The Solovay Stassen test relies on Euler's Criterion and the following theorem that assure: that if n is compsite, then a particular equivalence is not satisfied for a given base b.

Theorem: If n is an odd composite number, there exists a b, $(b, n) = 1$ such that $(\frac{b}{n}) \not\equiv b^{\frac{n-1}{2}} \mod n$.

Note that the theorem is not a iif condition, in fact if $(\frac{b}{n}) \equiv b^{\frac{n-1}{2}} \mod n$ it does not means that n is not composite (prime). But if we test the condition with a large set of bases, then n is a prime with high probability.

A composite number n that passes the Euler test for a particular base b is called *Euler pseudoprime in base* b, so $n = PSPE(b)$.

The key point of the probabilistic Solovay Strassen test is that at if a number is EPSP for a sufficiently large set of bases, than with high probability n is actually prime.

Complexity: Computing the Jacobi symbol $(\frac{b}{n})$, that adds more computation, is isomorphic to computing (b, n); this can be done in $\mathcal{O}(log^2(n))$, and thus the cost of whole algorithm the sum of: finding gcd, computing of Jacobi symbol, and computing powers of b. The complexity is $\mathcal{O}(m \cdot \log^2(n)) + \mathcal{O}(m \cdot \log^2(n)) + \mathcal{O}(m \cdot \log^3(n)) \sim \mathcal{O}(m \cdot \log^3(n))$.

Test	Pros	Cons	Computational Complexity	Probability of Error
Fermat	Fast	Carmichael numbers It cannot determine the number of Fermat liars for a pseudoprime	$\mathcal{O}(m \cdot \log^3(n))$	Probability of failure for Carmichael numbers is 1
Miller-Rabin	Very fastVery small probability of errorStrong-pseudoprimes are fewer than Pseudoprimes	Strong-pseudoprimes fools it	$\tilde{\mathcal{O}}(m \cdot \log^2(n))$	$< 4^{-m}$
Solovay-Strassen	Fast Euler pseudoprimes are much less dense than pseudoprimes	Euler Pseudoprime fools it	$\mathcal{O}(m \cdot \log^3(n))$	$< 2^{-m}$

In conclusion theoretically: - Fermat Test is very fast and quite accurate, but the problem of Carmichael number constitutes a big issue in using this test. Solovay-Strassen can be more expansive and on a average its failure probability is twice that of Miller-Rabin's, so normally Miller-Rabin's test is used to decide whether an odd number is prime or not in time $\tilde{\mathcal{O}}(m \log^2(n))$; the probability that it erroneously calls 'prime' a composite number is less than 4^{-m}.

1.3.1 Let's see now my implementation and practical results

[1]:
```
#number of iterations
m = 100

Prime = True
Composite = False

# imporr the module where all the functions are
import functions_1889394 as functions
```

1.4 Generic numbers that are going to be tested

The input are the numbers of various sizes, starting from 2 digits to 100 digit. There are 2 representative numbers for each of the considered number of digits, a prime numbers and composite numbers.

Number	#digits	Composition
13	2	Prime
34	2	Compsite
991	3	Prime
169	3	Compsite
3793	4	Prime
7621	4	Composite
11621	5	Prime
10036	5	Composite
5915587277	10	Prime
2860486317	10	Composite
29497513910652490397	20	Prime
12764787846358441471	20	Composite
590872612825179551336102196593	30	Prime
280829369862134719390036617061	30	Composite
2425967623052370772757633156976982469681	40	Prime
2425967623052370772757633156976982469682	40	Composite
29927402397991286489627837735179186385188296382227	50	Prime
54987541231012456465487987655132654568741532457817	50	Composite
470287785858076441566723507866751092927015824834881906763507	60	Prime
574876541556778984542315454987666541234557898465415657796415	60	Composite
5850725702766829291491370712706286009948642125131436113342815786444567	70	Prime
4237080979868607742750808600846638318022863593147774739556642794329	70	Composite
34263233064835421125264776608016344053792570599796234659697780346203384105962872 3	80	Prime
1475998436180202124541047592801669395348791811705709117374129427051861355011151	80	Composite
463199005416013829210323411591132845972525641604435693287586851332821637442813833942427923	90	Prime
370332600450952648802345609998335058273399487356359263038584017827194636172568988257769603	90	Composite
20747222467734852078216952221076085874809964747211172927529925899121966847505496583100844167325 50	100	Prime
21939929932186043108844618646108001945131790925282531768679169054889241527895222169476723691605898	100	Composite

As you can see there are also more numbers, in particular 20 primes and 20 composite to even more test the algorithms.

#Digits	Prime Numbers	Composites Numbers
10	3267000013	6198669063
10	5915587277	4765202170
20	48112959837082048697	76263224987803629328
20	54673257461630679457	17423384515502858017
30	671998030559713968361666935759	598766443293888611474166490469
30	282174488599599500573849980904	048604051580451278912136018920
40	2425967623052370772757633156976982469681	9715834861814413114504570241353393569583

#Digits	Prime Numbers	Composites Numbers
40	1451730470513778492236629598[illegible]	[illegible]8278688510306641702775894463
50	229536868677196912300027078[illegible]	[illegible]7059387406395854249
50	307625422503012706920514605[illegible]	[illegible]5576966561642372607
60	6222880974989264961410958692[illegible]	[illegible]1172465083
60	610692533270508750441931226[illegible]	[illegible]733422065
70	5885903965180586669073549360[illegible]	[illegible]
70	377318081621938460678418953[illegible]	[illegible]
80	1853239550094717445070938338[illegible]	[illegible]
80	4482248151160106609871348145[illegible]	[illegible]
90	2827554835337072870547521843[illegible]	[illegible]
90	4631990054160138292103234115[illegible]	[illegible]
100	1814159566819970307982681716[illegible]	[illegible]
100	6513516734600035718300327211[illegible]	[illegible]

[2]:
```
# 28 normal numbers
numbers = [13, 34, 991 ,169, 3793, 7621, 11621, 10036, 5915587277, 2860486317, 29497513910652490397,
           12764787846358441471, 590872612825179551336102196593,280829369862134719390036617061,
           2425967623052370772757633156976982469681 ,2425967623052370772757633156976982469682,
           2992740239799128648962783773417918638518829638227, 54987541231012456465487987654132654568741532457817,
           470287785858076441566723507866751092927015824834881906763507,
           574876541556778984542315454987666541234557898465415657795415 ,
           5850725702766829291491370712136286009948642125131436113342815786444567,
           423708097986860774275080860084663831802286359314777473955642794329,
           34263233064835421125264776608163440537925705997962346596977803462033841059628723,
           14759984361802021245410475928101669395348791811705709117374129427051861355011151,
           463199005416013829210323411514132845972525641604435693287586851332821637442813833942427923,
           370332600450952648802345609908335058273399487356359263038584017827194636172568988257769603,
           207472224677348520782169522210760858748099647472111729275299258991219668475054965831008441673
           219399299321860431088446186461800194513179092528253176867916905438924152789522216947672369160

# Other 20 primes
other_primes = [3267000013, 5915587277, #2 x 10 digits
                48112959837082048697, 54673257461630679457, #2 x 20 digits
```

```
               671998030559713968361666935769, 282174488599599500573849980909,␣
↪#2 x 30
               2425967623052370772757633156976982469681,␣
↪1451730470513778492236629598992166035067, #2 x 40
               22953686867719691230002707821868552601124472329079,␣
↪30762542250301270692051460539586166927291732754961, #2 x 50
               622288097498926496141095869268883999563096063592498055290461,␣
↪610692533270508750441931226384209856405876657993997547171387, # 2 x 60
               ␣
↪5885903965180586669073549360644800583458138238012033647539649735017287,␣
↪3773180816219384606784189538899553110499442295782576702222280384917551, # 2 x␣
↪70
               ␣
↪18532395500947174450709383384936679868383424444311405679463280782405796233163977,␣
↪44822481511601066098713481453161748979849764719554039096395688045048053310178487,␣
↪# 2 x 80
               ␣
↪282755483533707287054752184321121345766861480697448703443857012153264407439766013042402571,␣
↪463199005416013829210323411514132845972525641604435693287586851332821637442813833942427923,␣
↪# 2 x 90
               ␣
↪181415956681997030798268171682210701603892017050439145746256348519812691673516726021561952342
↪651351673460003571830032721125092823717828175849441735756008682841686392927045143712602194985
↪# 2 x 100

# Other 20 composites
other_composites= [6198669063,4765202170, #2 x 10 digits
                   76263224987803629328, 17423384515502858017, #2 x 20 digits
                   578766443293888611474166490469,448604051580451278912136018920,␣
↪#2 x 30 digits
                   ␣
↪7615353861114413114504570241353393569583,1947696553298278688510306641702775894463,␣
↪#2 x 40 digits
                   75128567454332200997487059387406395854249942560639,␣
↪25512976467887390219455576966561642372607061829171, #2 x 50 digits
                   350724107805662340204391321697711724650837028619404411998100,␣
↪283853317129073568596730410492507334220655195793712288798718, #2 x 60 digits
                   ␣
↪4448822320341298002327767469322167218785631231602699883049759451980034,␣
↪2524252130882273102040850065646873717114946297391270905937770144764029, #2 x␣
↪70 digits
                   ␣
↪35164229577514178181168622408091604002805565338433768558910859481454836351654907,␣
↪57841046017887403629651561187922554682614984840584319258001793299928867489493677,␣
↪#2 x 80 digits
```

```
                  ␣
↪569538061237235290108472366481505284888394210328372559896442803412779067923942016876408369,␣
↪891042121690333452972999043011726585530165321972219423184476655890889415661665353083911937,␣
↪#2 x 90 digits
                  ␣
↪218553086503759256433772472098446819928625546500479638095719165282445084008898387125878356745
↪301499158287606277482132327275170081500375446475640391993616442531317227865253497564603227055
↪#2 x 100 digits
                  ]
#number of digits
d =␣
↪[2,2,3,3,4,4,5,5,10,10,20,20,30,30,40,40,50,50,60,60,70,70,80,80,90,90,100,100]
d_others = [10,10,20,20,30,30,40,40,50,50,60,60,70,70,80,80,90,90,100,100]

#composition of "numbers"
composition = [Prime, Composite,Prime,Composite,Prime,Composite,␣
↪Prime,Composite,Prime, Composite,Prime, Composite,  Prime,Composite,Prime,
               Composite,Prime, Composite,Prime,Composite,Prime,␣
↪Composite,Prime, Composite, Prime, Composite,Prime, Composite]
composition_primes = [Prime for i in range(len(other_primes))]
composition_composite = [Composite for i in range(len(other_composites))]

#how many numbers to test
print("Generic numbers that will be tested: ",len(numbers))

print("Others prime numbers that will be tested: ",len(other_primes))
print("Others composite numbers that will be tested: ",len(other_composites))
```

```
Generic numbers that will be tested:  28
Others prime numbers that will be tested:  20
Others composite numbers that will be tested:  20
```

1.5 Special number that are going to be tested: Mersenne Numbers and Fermat numbers

As before the input are the numbers of various sizes, starting from 2 digits to 39 digits. These are classified according to being prime/composite and to the form and the number of digits.

Number	#digits	Composition	Special form
5	1	Prime	Fermat
9	1	Compsite	Fermat
7	1	Prime	Mersenne
17	2	Prime	Fermat
33	2	Composite	Fermat
31	2	Prime	Mersenne
257	3	Prime	Fermat
129	3	Composite	Fermat

Number	#digits	Composition	Special form
127	3	Prime	Mersenne
2047	4	Composite	Mersenne
65537	5	Prime	Fermat
16385	5	Composite	Fermat
131071	6	Prime	Mersenne
524287	6	Composite	Mersenne
4294967297	10	Composite	Fermat
2147483647	10	Prime	Mersenne
137438953471	12	Composite	Mersenne
576460752303423487	18	Composite	Mersenne
2305843009213693951	19	Prime	Mersenne
18446744073709551617	20	Composite	Fermat
618970019642690137449562111	27	Prime	Mersenne
162259276829213363391578010288127	33	Prime	Mersenne
340282366920938463463374607431768211457	39	Composite	Fermat
170141183460469231731687303715884105727	39	Prime	Mersenne

[3]:

```
special_numbers=[5,9,7,17,33,31,257,129,127,2047, 65537, 16385,131071,
                 524287, 4294967297, 2147483647, 137438953471,
                  576460752303423487, 2305843009213693951,
                
 ↪18446744073709551617,618970019642690137449562111,162259276829213363391578010288127,
                 
 ↪340282366920938463463374607431768211457,170141183460469231731687303715884105727]
#number of digits
d_special = [1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 5, 5, 6, 6, 10, 10, 12, 18, 19, 20,
 ↪27, 33, 39, 39]

form = ['fermat', 'fermat', 'mersenne',
        'fermat', 'fermat', 'mersenne',
        'fermat', 'fermat', 'mersenne',
        'mersenne', 'fermat', 'fermat',
        'mersenne', 'mersenne', 'fermat',
        'mersenne', 'mersenne', 'mersenne',
        'mersenne', 'fermat', 'mersenne',
        'mersenne', 'fermat', 'mersenne']

special_form = [Prime,Composite, Prime, Prime, Composite,  Prime, Prime,
 ↪Composite,
               Prime, Composite, Prime, Composite,Prime, Composite, Composite,
 ↪Prime,
                Composite, Composite,Prime,Composite, Prime, Prime, Composite,
 ↪Prime ]
```

```
#how many special numbers to test
print("Special numbers that will be tested: ",len(special_numbers))
```

```
Special numbers that will be tested:  24
```

[62]:
```
import importlib
importlib.reload(functions)
```

[62]:
```
<module 'functions_1889394' from '/home/flaminia/Documents/Sapienza/DPS_(Data
Privacy and Security)/Project/functions_1889394.py'>
```

2 Tests performances on regular numbers

[4]:
```
# TRIVIAL FACTORING
times_factoring, results_factoring = functions.
 ↪eval_factoring(numbers,composition)

# (In this particular case what I wanted to show was the fact that subsequent␣
 ↪divisions ar really expansive and all computations were aborted)
```

```
Accuracy of: Trivial Division is  0.0
Computation aborted: 28  times
```

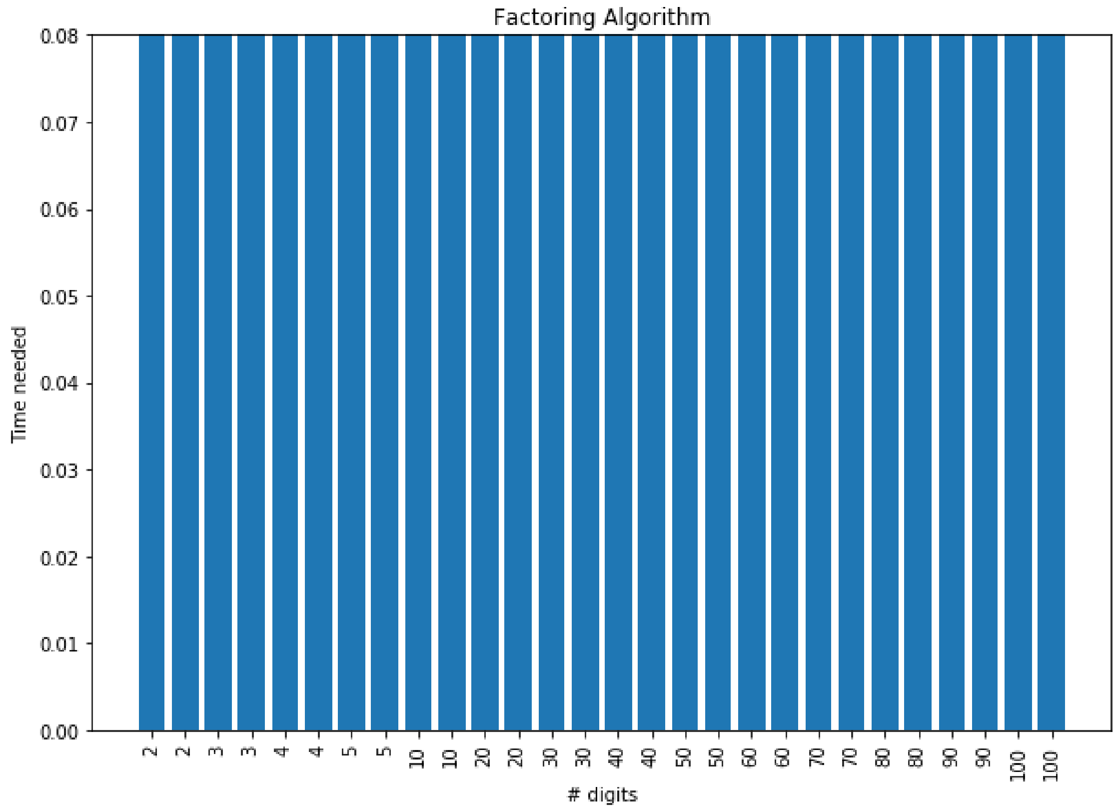
[5]:
```
# FERMAT
times_fermat, results_fermat = functions.evaluate_test('fermat',␣
 ↪numbers,composition, m)

# MILLER-RABIN
times_rabin, results_rabin = functions.evaluate_test('m-r', numbers,composition,␣
 ↪m)

# SOLOVAY-STRASSEN
times_solovay, results_solovay = functions.evaluate_test('s-s',␣
 ↪numbers,composition, m)
```

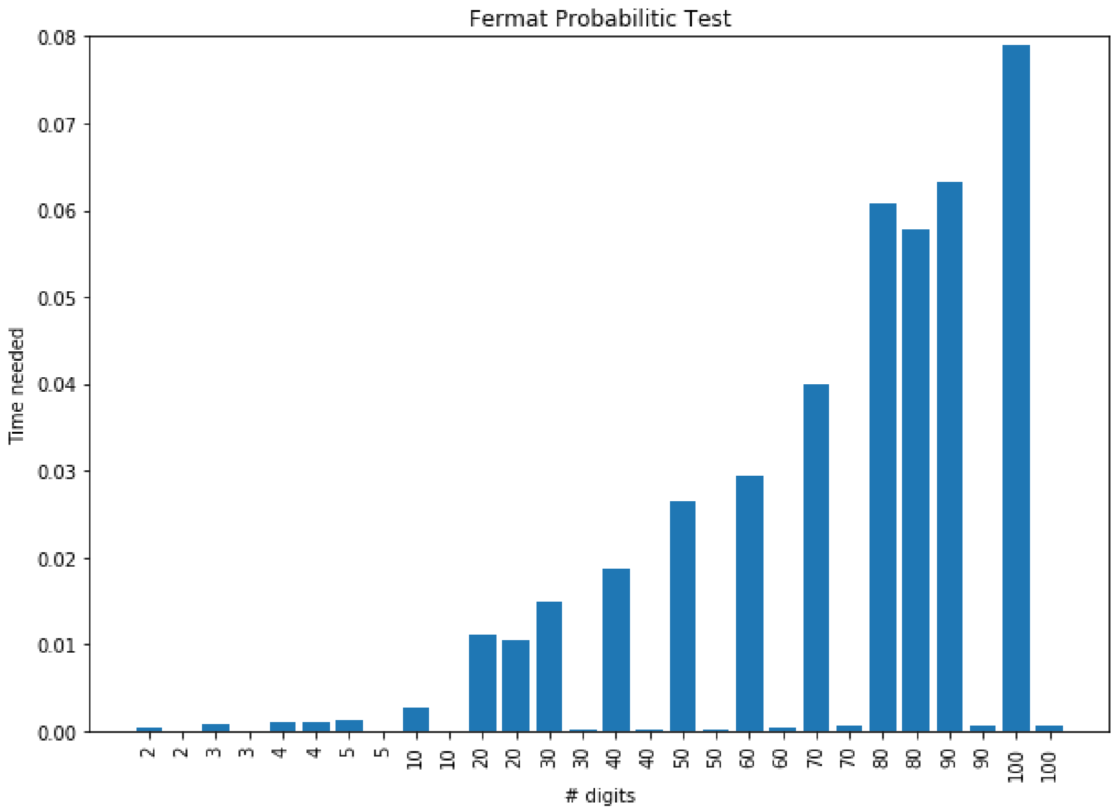
```
Accuracy of:  Fermat Probabilitic Test is  0.8928571428571429
Accuracy of:  Miller Rabin Probabilitic Test is  0.75
Accuracy of:  solovay Strassen Probabilitic Test is  0.6428571428571429
```

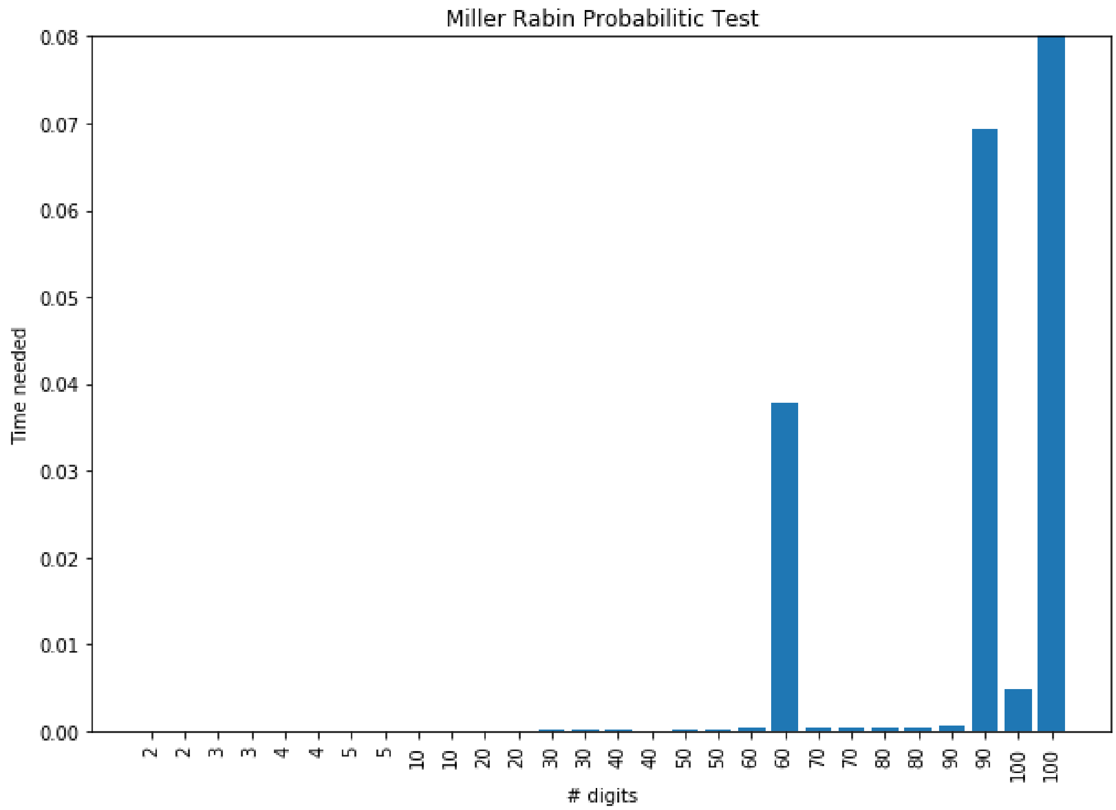
Plotting time needed for every #digit number

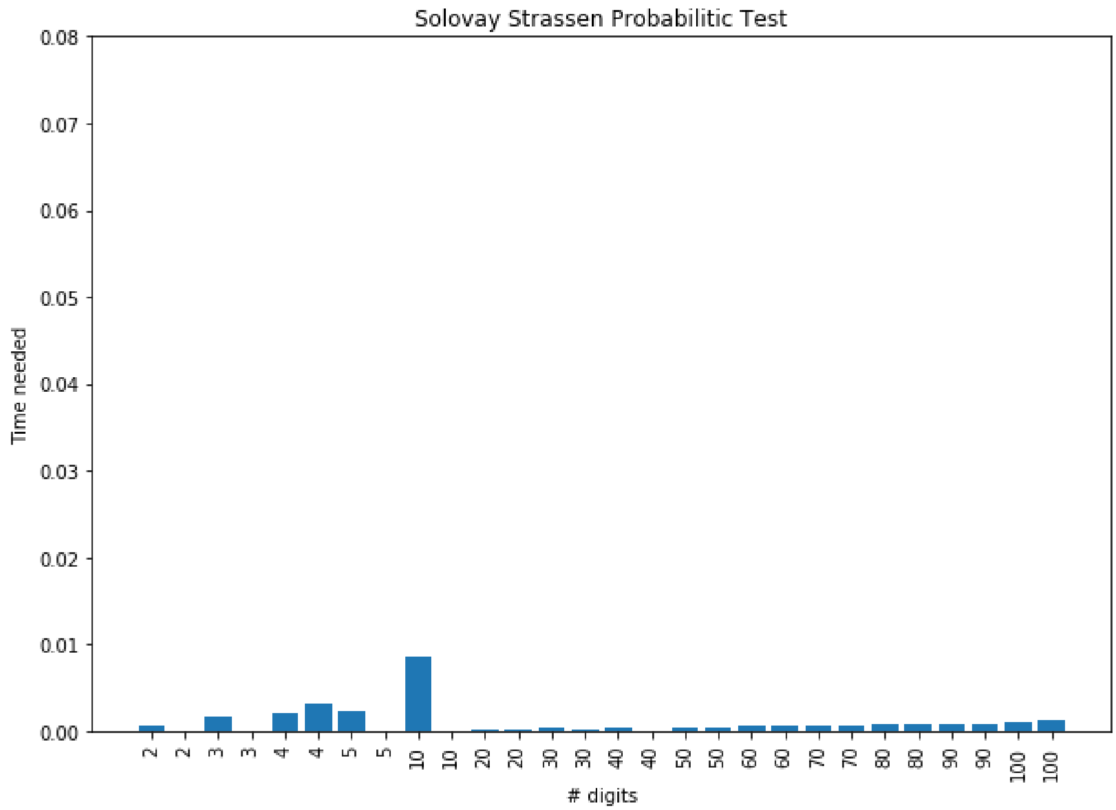
[6]:
```
functions.barplot(d, times_factoring, 'factoring')

functions.barplot(d, times_fermat, 'fermat')

functions.barplot(d, times_rabin,'rabin')

```

```
functions.barplot(d, times_solovay, 'solovay')
```




Fermat Probabilitic Test
Time needed
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
2
2
3
3
4
4
5
5
10
10
20
20
30
30
40
40
50
50
60
60
70
70
80
80
90
90
100
100
digits


Miller Rabin Probabilitic Test
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
Time needed
2 2 3 3 4 4 5 5 10 10 20 20 30 30 40 40 50 50 60 60 70 70 80 80 90 90 100 100
digits



2.0.1 Now let's look at tests performance on other primes of high magnitude

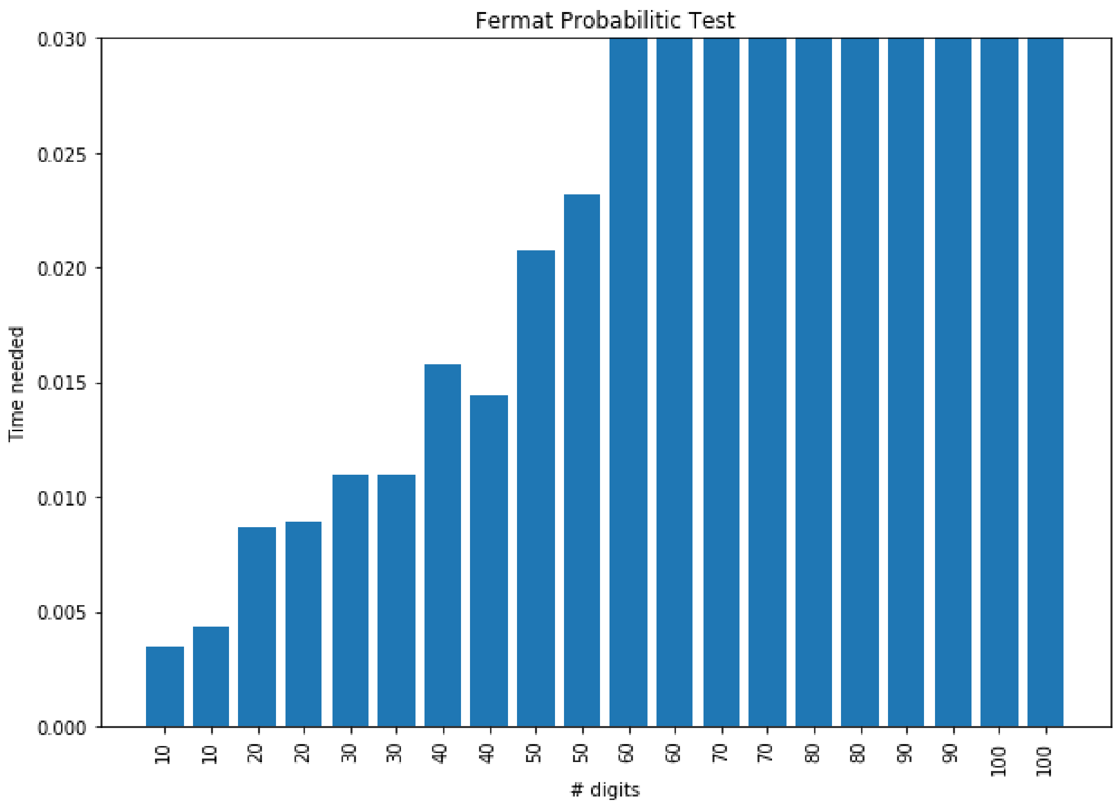
We consider two prime numbers, that have a number of digits from 10 to 100

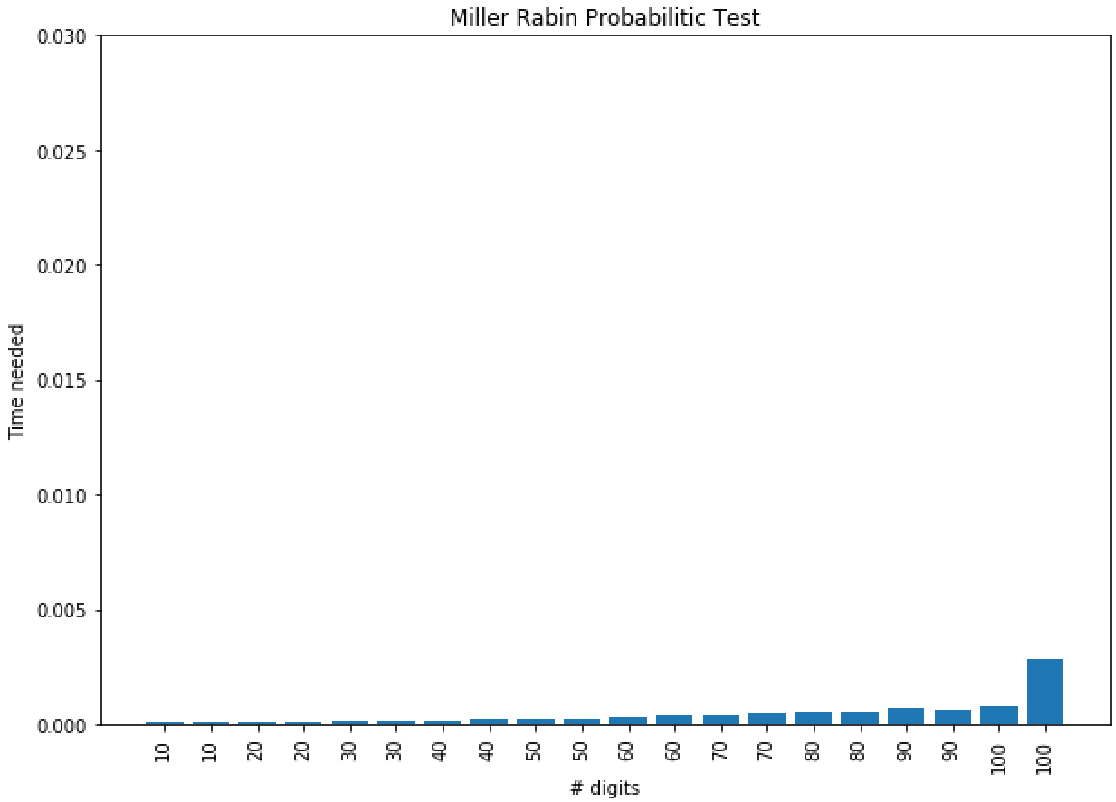
[7]:
```
# FERMAT
times_fermat, results_fermat = functions.evaluate_test('fermat', other_primes,␣
↪composition_primes, m)

# MILLER-RABIN
times_rabin, results_rabin = functions.evaluate_test('m-r',␣
↪other_primes,composition_primes, m)

# SOLOVAY-STRASSEN
times_solovay, results_solovay = functions.evaluate_test('s-s', other_primes,␣
↪composition_primes, m)
```

```
Accuracy of:  Fermat Probabilitic Test is  1.0
Accuracy of:  Miller Rabin Probabilitic Test is  0.75
Accuracy of:  solovay Strassen Probabilitic Test is  0.1
```

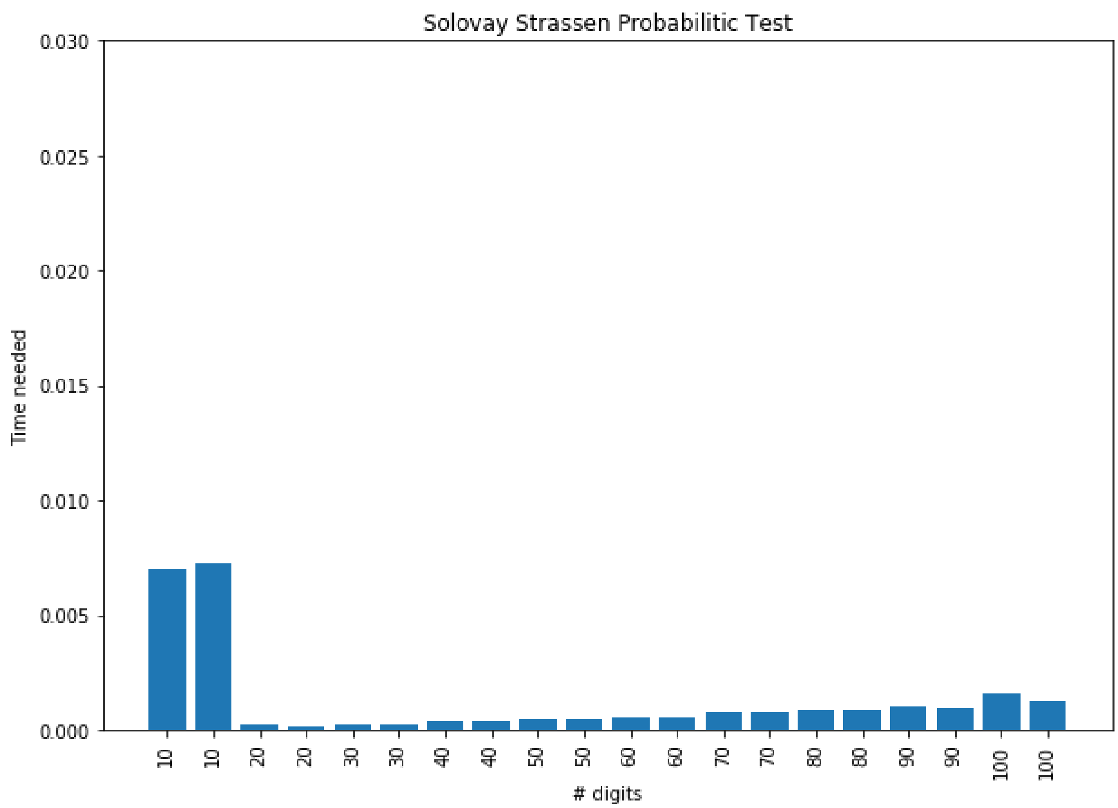
Plotting time needed for every #digit number

[8]:
```
functions.barplot(d_others, times_fermat, 'fermat')

functions.barplot(d_others, times_rabin,'rabin')

functions.barplot(d_others, times_solovay, 'solovay')
```





Miller Rabin Probabilitic Test
0.030
0.025
0.020
0.015
0.010
0.005
0.000
Time needed
10
10
20
20
30
30
40
40
50
50
60
60
70
70
80
80
90
90
100
100
digits



2.0.2 Now let's look at tests performance on other composites of high magnitude

We consider two composite numbers, that have a number of digits from 10 to 100

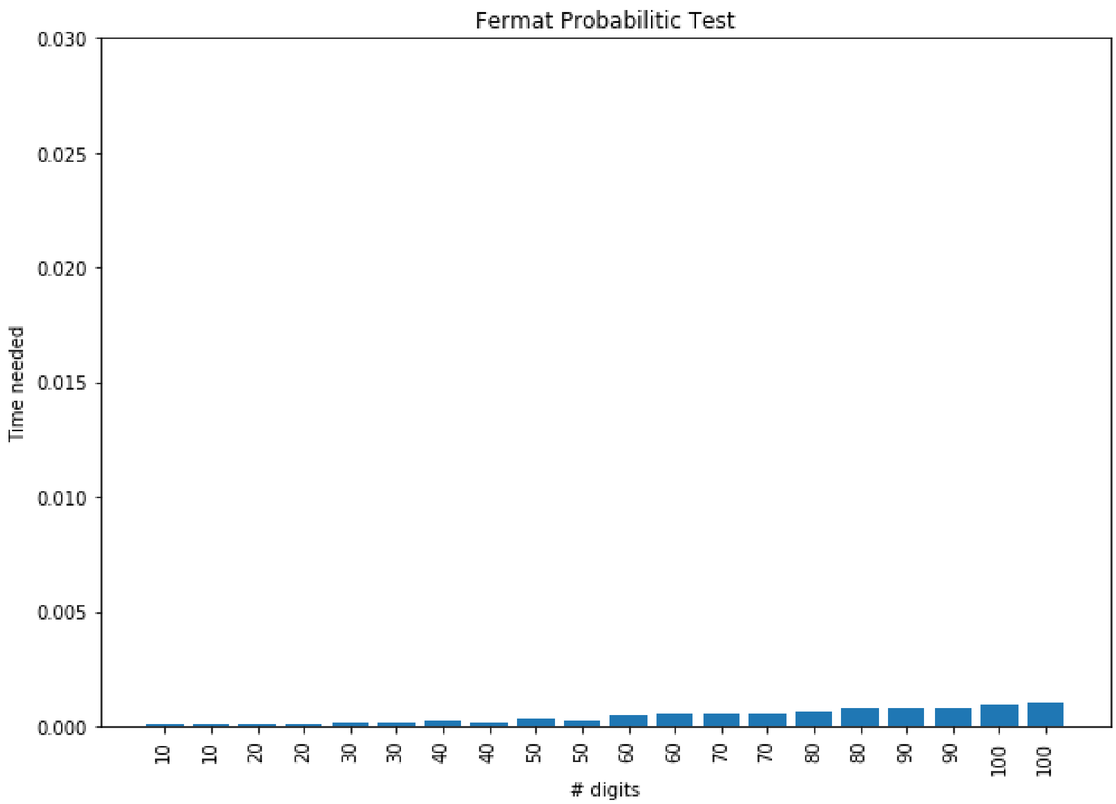
[9]:
```
# FERMAT
times_fermat, results_fermat = functions.evaluate_test('fermat',␣
 ↪other_composites,composition_composite, m)

# MILLER-RABIN
times_rabin, results_rabin = functions.evaluate_test('m-r',␣
 ↪other_composites,composition_composite, m)

# SOLOVAY-STRASSEN
times_solovay, results_solovay = functions.evaluate_test('s-s',␣
 ↪other_composites,composition_composite, m)
```

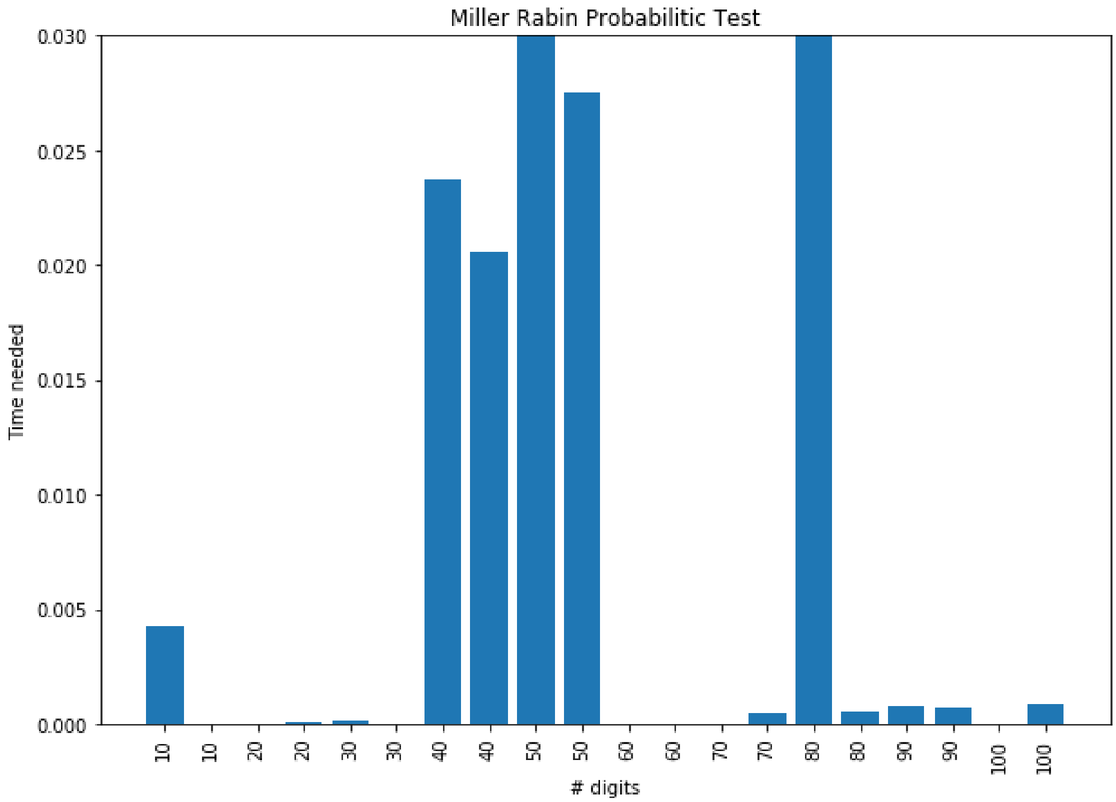
```
Accuracy of:  Fermat Probabilitic Test is  1.0
Accuracy of:  Miller Rabin Probabilitic Test is  0.7
Accuracy of:  solovay Strassen Probabilitic Test is  1.0
```

Plotting time needed for every #digit number

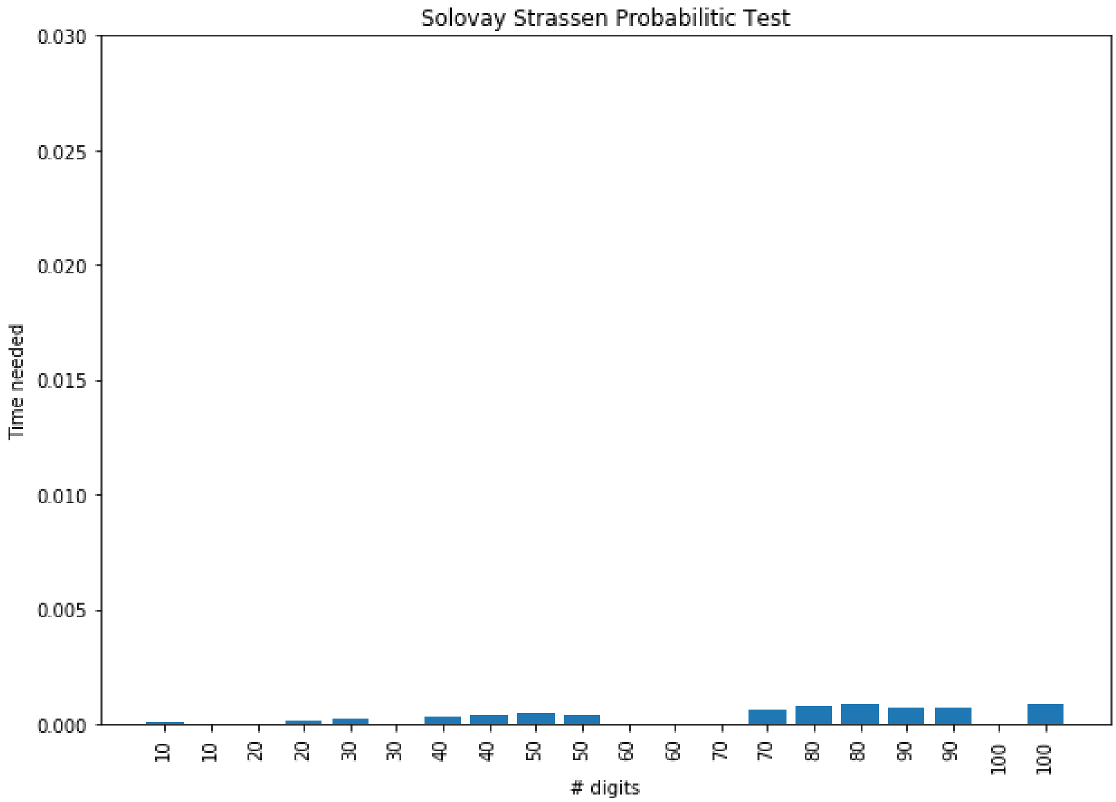
[10]:

```
functions.barplot(d_others, times_fermat, 'fermat')

functions.barplot(d_others, times_rabin,'rabin')

functions.barplot(d_others, times_solovay, 'solovay')
```





Miller Rabin Probabilitic Test
0.030
0.025
0.020
0.015
0.010
0.005
0.000
Time needed
10 10 20 20 30 30 40 40 50 50 60 60 70 70 80 80 90 90 100 100
digits



3 Tests's performances on special numbers

An example of what I was saying before, 561 is Carmichael number that fools Fermat Test

[11]:
```
print(functions.fermat_test(561))
```

```
True
```

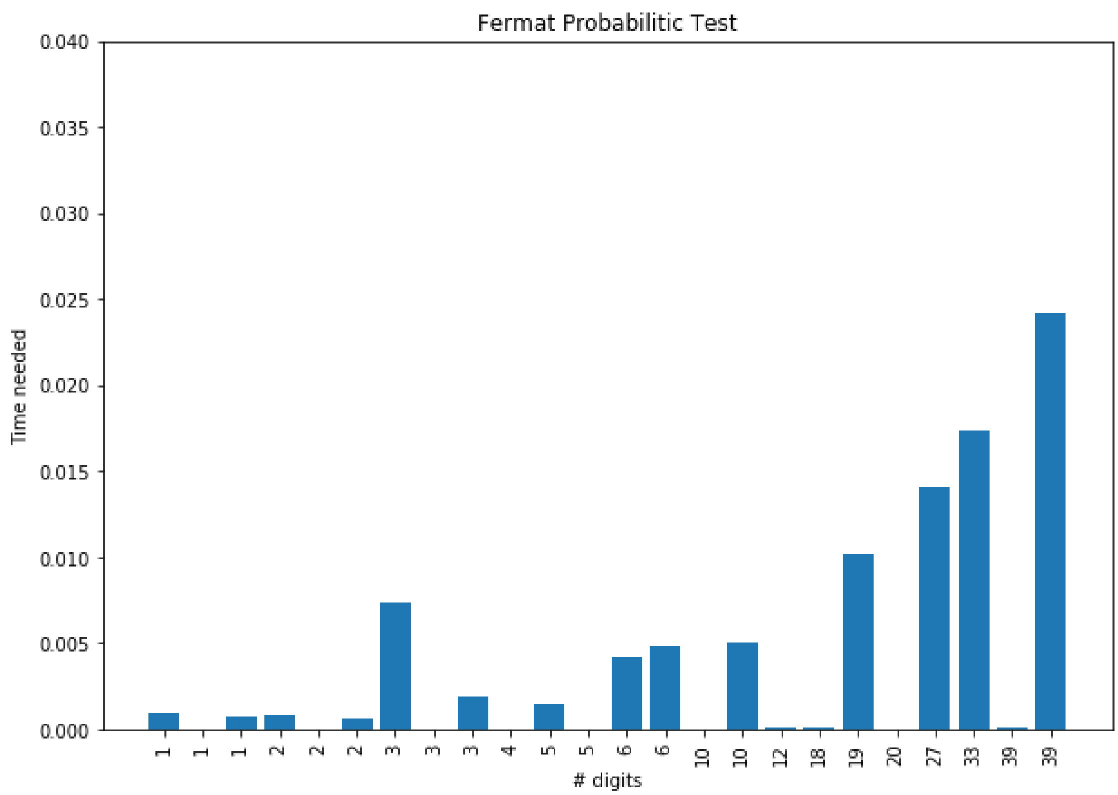
Now let's see more concrete examples on special numbers

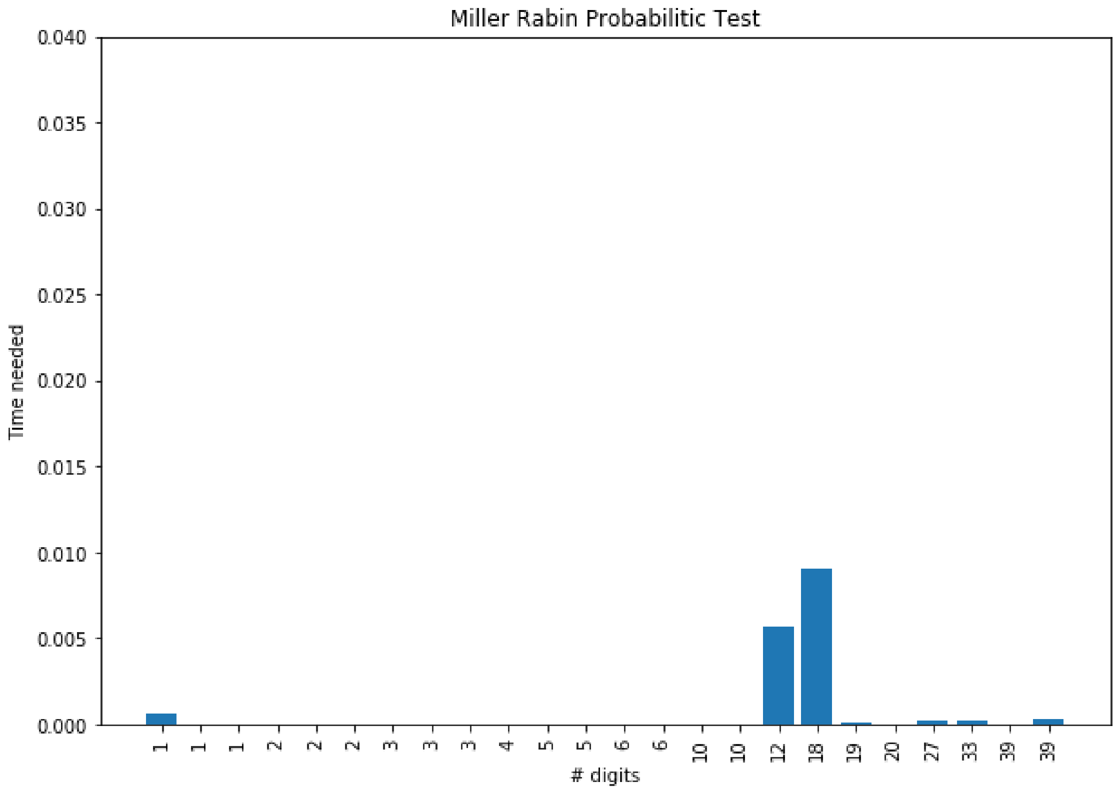
[12]:
```
# FERMAT
times_fermat_special, results_sp_fermat = functions.evaluate_test('fermat',␣
↪special_numbers,special_form, m)

# MILLER-RABIN
times_rabin_special, results_sp_rabin = functions.
↪evaluate_test('m-r',special_numbers,special_form, m)

# SOLOVAY-STRASSEN
times_solovay_special, results_sp_solovay = functions.evaluate_test('s-s',␣
↪special_numbers,special_form, m)
```

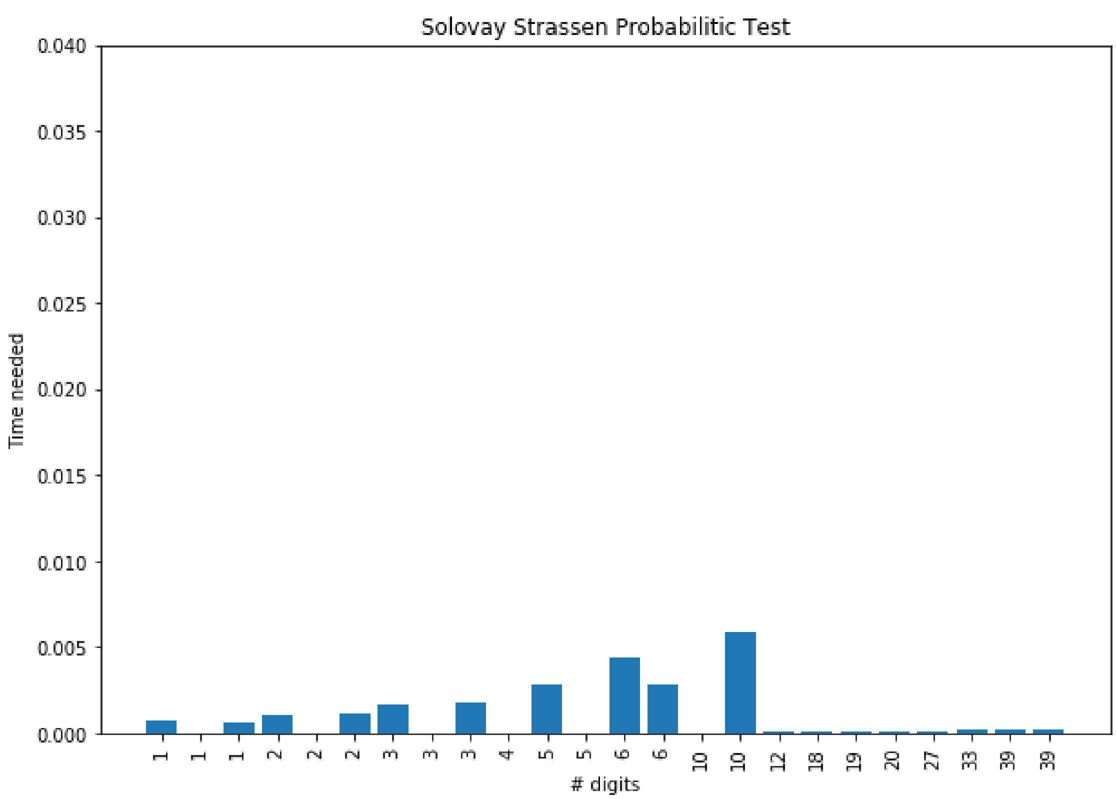
```
Accuracy of:  Fermat Probabilitic Test is  0.9583333333333334
Accuracy of:  Miller Rabin Probabilitic Test is  0.7083333333333334
Accuracy of:  solovay Strassen Probabilitic Test is  0.7916666666666666
```

Plotting time needed for every #digit number

[13]:
```
functions.barplot(d_special, times_fermat_special, 'fermat')

functions.barplot(d_special, times_rabin_special, 'rabin')

functions.barplot(d_special, times_solovay_special,  'solovay')
```




Miller Rabin Probabilitic Test
Time needed
digits
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
1 1 1 2 2 2 3 3 3 4 5 5 6 6 10 10 12 18 19 20 27 33 39 39


Solovay Strassen Probabilitic Test
Time needed
digits
0.040
0.035
0.030
0.025
0.020
0.015
0.010
0.005
0.000
1 1 1 2 2 2 3 3 3 4 5 5 6 6 10 10 12 18 19 20 27 33 39 39

4 In conclusion

In theory Miller Rabin is the best among the three probabilistic primality tests, in terms of running time and probability error. We have now seen that this statement is true also in practice, even if both Solovay Strassen and Fermat test performed really well.

Test	Pros	Cons	Computational Complexity	Probability of Error
Fermat	Fast	Carmichael numbers It cannot determine the number of Fermat liars for a pseudoprime	$\mathcal{O}(m \cdot \log^3(n))$	Probability of failure for Carmichael numbers is 1
Miller-Rabin	Very fastVery small probability of errorStrong-pseudoprimes are fewer than Pseudoprimes	Strong-pseudoprimes fools it	$\tilde{\mathcal{O}}(m \cdot \log^2(n)))$	$< 4^{-m}$
Solovay-Strassen	Fast Euler pseudoprimes are much less dense than pseudoprimes	Euler Pseudoprime fools it	$\mathcal{O}(m \cdot \log^3(n))$	$< 2^{-m}$

References: - [1] Framework for Evaluation and Comparison of Primality Testing Algorithms - [2] Schönhage, A., Strassen, V. Schnelle Multiplikation großer Zahlen. Computing 7, 281–292 (1971). - [3] Louis Monier, Evaluation and comparison of two efficient probabilistic primality testing algorithms,Theoretical Computer Science, Volume 12, Issue 1, 1980 - [4] Riley Worthington, Primality Testing: Theory, Complexity, and Applications - [5] Miller-Rabin Test, wikipedia - [6] Prime numbers with a number of digits - [7] Random numbers with a number of digits