Construction of Exceptional Lie Algebra G2 and Non-Associative Algebras using Clifford Algebra

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Abstract *This article uses Clifford algebra to derive octonions and the Lie Exceptional algebra G2 from* $\mathrm{Spin}(7)$ *calibrations. This is simpler than the usual exterior algebra derivation and uncovers an invertible element using the 4-form calibration that is used to classify 6 other algebras which are found to be related to the symmetries of G2. The calibration terms provide a direct construction of G2 for each of the 480 representations of the octonions. This result is extended to* $\mathrm{Spin}(15)$*, deriving another 12 algebras including the sedenions. Also the 15 terms of the 8-form calibration are a commuting sub-algebra analogous to the* $\mathrm{Spin}(7)$ *case.*
Keywords. Clifford algebra, octonions, sedenions, Lie algebra G2, $\mathrm{Spin}(7)$.

1 Introduction

The Clifford algebra is generally denoted using matrices which hides its graded structure and geometric interpretation. It is can also be expressed using a negative signature which again obfuscates the geometric interpretation. The even subsets of Clifford algebras regain the negative signature parts without loss of generality and using a positive signature exposes the underling geometry inherent in this algebra. Here $\mathrm{Cl}(n)$ denotes the vector space $\mathbb{R}_n$ for vectors with squares of positive quadratic signature. The even part subset of Clifford algebra, Cl^+, is the same for both positive and negative signatures and is isomorphic to the spinor algebra,

$$\mathrm{Spin}(n) \approx \mathrm{Cl}^+(n).$$

The positive signature equates with the standard use of vectors and the space of products of 1-forms or vectors which covers the Grassmann forms of the same dimension. Marcel Riesz, [1], exposed the relationship between Clifford algebra and Grassmann forms by deriving $\mathrm{Cl}(n)$ from the external algebra and vise versa, as constructed by P. Lounesto in [2]. The fundamental structure equation for multi-vector products provides the mapping between the two [3, 4, 5],

$$\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\ldots\mathbf{a}_n = \sum_{i=0}^{[\frac{n}{2}]}\sum_{\mu\in\mathcal{C}}(-1)^k\mathrm{pf}(\mathbf{a}_{\mu_1}\cdot\mathbf{a}_{\mu_2},\ldots,\mathbf{a}_{\mu_{2i-1}}\cdot\mathbf{a}_{\mu_{2i}})\mathbf{a}_{\mu_{2i+1}}\wedge\ldots\wedge\mathbf{a}_{\mu_n},$$

where $\mathrm{pf}(A)$ is the Pffafian of A and $\mathcal{C} = \binom{n}{2i}$ provides combinations, μ, of n indicies divided into $2i$ and $n-2i$ parts and k is the parity of the combination. In the following a p-form is a p dimensional subspace of n-dimensional exterior space. The wedge product of Grassmann algebra, $\wedge$, can only be represented in Clifford algebra as a selection of the exterior part of the Clifford product, which, in general, is a semi-graded product as shown in [4]. The wedge product is used to signify the mapping between the two algebras except when applied to vectors where the definition is the same. The Pffafian expansion above can be derived from the $\mathrm{Cl}(n)$ product of any two orthonormal basis vectors:

$$e_ie_j = \delta_{ij} + e_i\wedge e_j, \quad 1\le i,j\le n.$$

Since $\mathrm{Cl}(n)$ is graded, representations using matricies, which aren't graded, lose the geometric information provided by forms. In three dimensions, $\mathrm{Cl}(3)$ has 3 vectors, e_1, e_2, e_3, the quaternions 1, e_{12}, e_{23}, e_{13} and the pseudo-scalar, e_{123}. With positive signature $e_{123}^2 = -1$. It also commutes with $\mathrm{Cl}(3)$ so is the commuting imaginary normally used in complex $\mathbb{R}_3$ vector space. Calling it "i" would be confusing later for higher dimensions and would make the cross product confusing. With a positive signature metric the cross product for a right hand screw rule is,

$$\mathbf{a} \times \mathbf{b} = e_{321} \mathbf{a} \wedge \mathbf{b}.$$

The 2-form, $\mathbf{a} \wedge \mathbf{b}$, is a pure quaternion and these provide the basis for rotation operators. They are called versors and generalise to arbitrary dimensions in $\mathrm{Cl}(n)$, $R_{ij}(\theta), i \neq j$ with angle θ and any multivector $A \in \mathrm{Cl}(n)$ as

$$\tilde{A} = R_{ij}(\theta) A R_{ji}(\theta),$$
$$R_{ij}(\theta) = cos(\tfrac{\theta}{2}) + e_{ij} sin(\tfrac{\theta}{2}).$$

An important operation is the rotation of $90°$ which sends any e_j component in A to e_i and e_i to $-e_j$. This shall be denoted as simply R_{ij}. Using this swapping trick rather than applying the rotation $R_{ij}(\pi/2)$ avoids rounding errors being accumulated. In dimensions greater than 3 the product of 2 versors may involve 4-forms. For example,

$$R_{12} R_{34} = \frac{1}{2}(1 + e_{12} + e_{34} + e_{1234}).$$

This same 4-form occurs in $R_{13}R_{24}$ and $R_{14}R_{23}$ but is part of $\mathrm{Spin}(4)$ and adds to the rotation angle in all cases. A product of three versors in dimensions greater than 5 may involve 6-forms but in the following only 4-forms need to be considered. Hence, the notation R_{ijkl} is shorthand for all three $90°$ versor pairs; $R_{ij}R_{kl}$, $R_{ik}R_{jl}$ and $R_{il}R_{jk}$.

2 Spin(7)

In 3-D, the psuedo-vector of $\mathrm{Cl}(3)$, e_{123}, represents a triangle with verticies labelled $1, 2, 3$ and edges defining the rotations e_{12}, e_{23}, e_{31}. The order here is important since the rotations define a 3-cycle, taking the vector e_1 to itself through e_2 and e_3. This order is denotated with a circular clockwise arrow inside the triangle, as shown in the Fano Plane diagram later (Figure 1). The alternate is $e_{321} = -e_{123}$ representing an anti-clockwise arrow. Both triangles define 3-cycles specifying a right-hand or a left-hand screw rule in the cross product and the arrow defines these product rules.

In general $\mathrm{Cl}(n)$ can be "visualised" as an n-D simplex or n-simplex which extends the two and three dimensional triangle and tetrahedron, respectively, into higher dimensions. In 7-D, $\mathrm{Cl}(7)$ has grade sizes given by Pascal's Triangle as $(1, 7, 21, 35, 35, 21, 7, 1)$. So the 7-simplex has 7 vertices or basis vectors, 21 edges or rotation planes and 35 faces or 3-forms. Also, since n is odd and $\sum_{i=1}^{n-1} i$ is odd, the pseudo-scalar is again a commuting imaginary, $e_{1234567}^2 = -1$.

Since the number of edges and faces in $\mathrm{Cl}(7)$ are both divisible by 3 and 7, then 7 primary faces can be selected so that each edge is selected once and all dimensions are covered 3 times:

$$\Phi_1 = e_{123} + e_{145} + e_{167} + e_{246} + e_{257} + e_{347} + e_{356}.$$

This is called the form of the associator calibration by Harvey and Lawson, [6], and has the well known property that a cross product for the orthogonal group with 7-vectors can be defined analogous to 3-D as

$$\mathbf{a} \times \mathbf{b} = \Phi_1 \, \mathbf{a} \wedge \mathbf{b}.$$

There are 30 ways to select 7 primary faces in this way which can be found by enumerating all combinations of 7 of the 35 triples of $\mathbb{N}_1^7$, eliminating those with duplicate edges. Alternatively, applying $90°$

rotations and reflections to Φ_1 can be used, as will be done later. The $\Phi_i, 1 \leq i \leq 30$, with positive terms, will be called primary 3-forms and this provides 30 possibilities for defining the cross product, apart from signs of each term. This works because the cross product selects a single term from Φ_1 for any pair of bases and contracts to a unique vector. For the moment we consider the $\mathrm{SL}(3, \mathbb{Z}_2)$ component of Φ_1 but first note that any single rotation R_{ij} will take Φ_1 to another 3-form in the primary list, apart from signs. Thinking geometrically, each vertex has 3 touching primary faces and swapping the vertices of any edge changes the triple of one face and swaps the vertex indices between the pair of touching faces at one end with the pair at the other end. But these pairs can not touch the remote end, guaranteeing each triple still has unique edges.

Defining the compliment to Φ_i, Φ'_1 using the pseudo-scalar, generates a 4-form, called the form of the coassociative calibration [6],

$$\Phi'_i = -e_{1234567}\ \Phi_i, \quad 1 \leq i \leq 30.$$

This form is related to the Hodge operator acting on Φ as a contraction operator of the Grassmann algebra. In Clifford algebra this operation is just multiplication by the psuedo-vector. A psuedo-versor such as $R'_{12} = (e_{1234567} + e_{34567})/\sqrt{2}$ can be defined but this has the same action as the versor. The most general versor in $\mathrm{Cl}(7)$ is

$$\frac{1}{\sqrt{2}}(1 \pm e_{1234567}) \prod_{\mu,\nu \in \mathcal{C}} R_{\mu,\nu}(\theta_{\mu,\nu}),$$

where the combinatorial expansion $\mathcal{C} = \binom{7}{2}$ provides all combinations of pairs, $1 \leq \mu, \nu \leq 7, \mu < \nu$. The pseudo-scalar term is an idempotent that splits $\mathrm{Cl}(7)$ into two simple algebras but it has no effect on rotations or reflections so psuedo-versors are not considered. Also the combinations of pairs $\mu, \nu \in \mathbb{N}_1^7$ provide products of up to three rotations (containing 6-forms) but only two rotations, such as those provided by the Φ'_i, are necessary below.

Lemma

The terms of $\{1 + \Phi'_i\}$ and $\{1 + \Phi_i + \Phi'_i + e_{1234567}\}$, $1 \leq i \leq 30$, form commuting sub-algebras of Spin(7) and Pin(7), respectively.

Proof By the definition of Φ_i a selected term, e_{jkl}, has each index occurring uniquely in a pair of other terms of Φ_i and no other shared index with e_{jkl}. Multiplying the selected term by the pair that shares j, $e_{jop} + e_{jqr}$ (j, l, o, p, q, r unique), then j will anti-commute twice (with o and p or q and r) and k and l will each anti-commute three times. Thus giving even parity so each term of Φ_i commutes with Φ_i itself. Since $e_{1234567}$ commutes with $\mathrm{Cl}(7)$ then Φ'_i is a 4-form with terms considered as two orthogonal rotations of 90° so is a subalgebra of $\mathrm{Spin}(7)$.

Single basis terms, e_i, under conjugation provide mirror reflections in $\mathrm{Cl}(n)$ of the other dimensions. The generalised reflection for any form reflects all dimensions not included, as

$$\tilde{A} = (-1)^{\sigma} r A r^{-1}, \tag{1}$$

where σ is the rank of the form r. This is 3 for the 3-form Φ_i which is a series of reflections that are part of $\mathrm{Pin}(7)$. Hence the lemma holds. It is interesting to note that the seven 6-forms refer to 3 rotations of 90° so applying twice is the same as negating or reflecting the missing basis from the 6-form.

The closure of the Φ'_i terms can be expressed as $\Phi_i\ \Phi'_i = \langle \Phi_i\ terms \rangle + 7e_{1234567}$. This is equivalent to the exterior form $\Phi_i \wedge \Phi'_i = 7e_{1234567}$ except that Clifford algebra products inherently contain contractions which leads to a Classification Theorem for all signed terms of Φ_i.

Since the terms of Φ_i along with the pseudo-scalar are closed, the multiplication of the terms of Φ_1^2 contain only the terms of the 4-form

$$\Phi'_1 = e_{1247} + e_{1256} + e_{1346} + e_{1357} + e_{2345} + e_{2367} + e_{4567}. \tag{2}$$

Permutations of the indices of each term of Φ_1' follows the Symmetric group S_4 that has order 24. Each permutation defines rotations that leave Φ_1 invariant, apart from signs, and since there are 7 terms this is $\mathrm{SL}(3,\mathbb{Z}_2)$ isomorphic to the projective group of the cube $\mathrm{PSL}(2,\mathbb{Z}_7)$ which has order $168 = 7 * 24$. The permutations of the 7 vertices of the 7-simplex is S_7 which has order $7! = 30 * 168$. Thus the primary faces provide complete coverage of S_7. The 24 permutations of 4 indices surject to only three rotations in $\mathrm{Cl}(7)$, apart from sign, and this is represented by Φ_i'. Of course, each Φ_i has $2^7 = 128$ combinations of signs which are now considered.

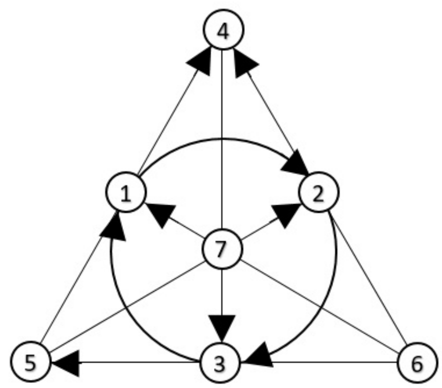

Figure 1: **Fano Plane Diagram**

The Fano Plane diagram in Figure 1 shows a projection of the 7-simplex with arrows defined by Φ_1. Like the arrow in $\mathrm{Cl}(3)$, the arrows define the elements and product rules but this diagram is not the usual representation that would define the product rules of octonions. Instead it generates a natural cross product whereby the basis index of each 3-cycle only decreases once. This represents another non-associative algebra that will be exposed shortly. Changing the sign of e_{146} means reversing the arrow on the right hand side of the diagram and this is now a representation of the octonion multiplication rule. There are 16 representations of octonions in each primary and denoting any of these as Φ_O then the Clifford algrebra statement of the well know exterior formulation, $\Phi \wedge \Phi'$, [6], where the product here generates a 7-form, is

$$\Phi_O^2 + 7 = (-1)^\sigma 6 e_{1234567} \Phi_O. \tag{3}$$

The σ factor in (3) defines a sign parity for the octonions and classifies 8 $\mathbb{O}^+$ and 8 $\mathbb{O}^-$ algebras with sign $(-1)^\sigma$, for σ even or odd respectively. There are 2^7 or 128 combinations of signs for the 7 terms of Φ_i and σ_j is the number of minus signs in these combinations. Defining $\Phi_{i,j}$, for $1 \leq i \leq 30$ and $1 \leq j \leq 128$ to be all signed combinations of the 7 terms of Φ_i, then the sequence starts with all positive terms, followed by 7 single minus terms, etc. Thus $\Phi_{1,5}$ denotes the $\mathbb{O}^-$ algebra with e_{246} negated in Φ_1, as discussed above. It is only necessary to consider the first half terms because the second half is just the negation of the first half which swaps the $\mathbb{O}^+$ and $\mathbb{O}^-$ classes, $\Phi_{i,j+64} = -\Phi_{i,j}$ and $\sigma_{j+64} = 7 - \sigma_j$ for $1 \leq j \leq 64$.

This has lain the groundwork for classifying the seven non-associative algebras found in the $30 * 128$ representations provided by Clifford algebra. The σ parity can be discovered by considering (3) and replacing Φ_O with any $\Phi_{i,j}$. If the equality is satisfied or there are 6 excess terms, then the parity is correct. Otherwise, there are seven terms and the opposite parity is needed. A better way to represent (3) is to define an invertible multivector,

$$\rho_{i,j} = \frac{1}{4}(3e_{1234567} - (-1)^\sigma \Phi_{i,j}),$$

where, if $\Phi_{i,j} = \Phi_O$ then $\rho_O\, \rho_O = -1$, so $\rho_O^{-1} = -\rho_O$. In general,

$$2e_{1234567}(\rho_{i,j}^2+1)=\begin{cases}0, & \text{or}\\ \langle\Phi_{i,j}\ term\rangle+(-1)^{\sigma}\Phi_{i,j}.\end{cases} \tag{4}$$

Classification Theorem

For octonions, $\rho_{i,j}^2 = 0$, otherwise $\mathrm{abs}(\langle\Phi_{i,j}\ term\rangle)$ in (4) defines a non-associative algebra with 4, 8, 10, 12, 14 or 16 non-associative products which shall be designated $\mathbb{O}_2, \mathbb{O}_4, \mathbb{O}_5, \mathbb{O}_6, \mathbb{O}_7$ or $\mathbb{O}_8$, respectively. Hence, the number of non-associative triplets in the algebra $\mathbb{O}_k$ is $2k$. Note that the octonion representations have 28 non-associative triples and 7 associative triples. The sign parity, σ, can be derived as above but it is easy to count the number of minus signs in $\Phi_{i,j}$, compared to the primary Φ_i. It is convenient to define both 0 and the $\langle\Phi_{i,j}\ term\rangle$ as the remainder, as shown in Table 1. Each of the signed 3-forms, $\Phi_{i,j}$, has a remainder that defines an algebra $\mathbb{O}$, if 0, or $\mathbb{O}_k$ provided by the Classes column in Table 1 for each Φ_i term as remainder, in order. For example, the classes for Φ_1 show that a remainder of e_{123} means $\mathbb{O}_2$, the next two terms as remainders are $\mathbb{O}_6$ and the remaining four are $\mathbb{O}_7$. The third column in Table 1 shows the remainder for the primary so that Φ_1 represents $\mathbb{O}_7$. Note that Φ_{11} and Φ_{20} have no remainder so these primaries generate $\mathbb{O}$.

Table 1: Primary Table and Classification Map

i	$\Phi_\mathbf{i}$	Classes	Remainder
1	$e_{123}+e_{145}+e_{167}+e_{246}+e_{257}+e_{347}+e_{356}$	$(\mathbb{O}_2, 2\mathbb{O}_6, 4\mathbb{O}_7)$	$-e_{246}$
2	$e_{123}+e_{145}+e_{167}+e_{247}+e_{256}+e_{346}+e_{357}$	$(\mathbb{O}_2, 2\mathbb{O}_6, 4\mathbb{O}_7)$	$-e_{357}$
3	$e_{123}+e_{146}+e_{157}+e_{245}+e_{267}+e_{347}+e_{356}$	$(\mathbb{O}_2, 2\mathbb{O}_7, 2\mathbb{O}_6, 2\mathbb{O}_7)$	$-e_{157}$
4	$e_{123}+e_{146}+e_{157}+e_{247}+e_{256}+e_{345}+e_{367}$	$(\mathbb{O}_2, 4\mathbb{O}_7, 2\mathbb{O}_6)$	$-e_{146}$
5	$e_{123}+e_{147}+e_{156}+e_{245}+e_{267}+e_{346}+e_{357}$	$(\mathbb{O}_2, 2\mathbb{O}_7, 2\mathbb{O}_6, 2\mathbb{O}_7)$	$-e_{346}$
6	$e_{123}+e_{147}+e_{156}+e_{246}+e_{257}+e_{345}+e_{367}$	$(\mathbb{O}_2, 4\mathbb{O}_7, 2\mathbb{O}_6)$	$-e_{257}$
7	$e_{124}+e_{135}+e_{167}+e_{236}+e_{257}+e_{347}+e_{456}$	$(\mathbb{O}_4, 2\mathbb{O}_6, 3\mathbb{O}_7, \mathbb{O}_5)$	$-e_{135}$
8	$e_{124}+e_{135}+e_{167}+e_{237}+e_{256}+e_{346}+e_{457}$	$(\mathbb{O}_4, 2\mathbb{O}_6, 3\mathbb{O}_7, \mathbb{O}_5)$	$-e_{124}$
9	$e_{124}+e_{136}+e_{157}+e_{235}+e_{267}+e_{347}+e_{456}$	$(\mathbb{O}_4, 2\mathbb{O}_7, 2\mathbb{O}_6, \mathbb{O}_7, \mathbb{O}_5)$	$-e_{267}$
10	$e_{124}+e_{136}+e_{157}+e_{237}+e_{256}+e_{345}+e_{467}$	$(\mathbb{O}_4, 4\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_4)$	$-e_{237}$
11	$e_{124}+e_{137}+e_{156}+e_{235}+e_{267}+e_{346}+e_{457}$	$(\mathbb{O}_4, 2\mathbb{O}_7, 2\mathbb{O}_6, \mathbb{O}_7, \mathbb{O}_5)$	0
12	$e_{124}+e_{137}+e_{156}+e_{236}+e_{257}+e_{345}+e_{467}$	$(\mathbb{O}_4, 4\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_4)$	$-e_{156}$
13	$e_{125}+e_{134}+e_{167}+e_{236}+e_{247}+e_{357}+e_{456}$	$(2\mathbb{O}_5, \mathbb{O}_6, \mathbb{O}_7, \mathbb{O}_8, \mathbb{O}_6, \mathbb{O}_5)$	$-e_{247}$
14	$e_{125}+e_{134}+e_{167}+e_{237}+e_{246}+e_{356}+e_{457}$	$(2\mathbb{O}_5, \mathbb{O}_6, \mathbb{O}_7, \mathbb{O}_8, \mathbb{O}_6, \mathbb{O}_5)$	$-e_{356}$
15	$e_{125}+e_{136}+e_{147}+e_{234}+e_{267}+e_{357}+e_{456}$	$(\mathbb{O}_5, \mathbb{O}_7, \mathbb{O}_8, \mathbb{O}_5, 2\mathbb{O}_6, \mathbb{O}_5)$	$-e_{136}$
16	$e_{125}+e_{136}+e_{147}+e_{237}+e_{246}+e_{345}+e_{567}$	$(\mathbb{O}_6, 4\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_2)$	$-e_{136}$
17	$e_{125}+e_{137}+e_{146}+e_{234}+e_{267}+e_{356}+e_{457}$	$(\mathbb{O}_5, \mathbb{O}_7, \mathbb{O}_8, \mathbb{O}_5, 2\mathbb{O}_6, \mathbb{O}_5)$	$-e_{457}$
18	$e_{125}+e_{137}+e_{146}+e_{236}+e_{247}+e_{345}+e_{567}$	$(\mathbb{O}_6, 4\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_2)$	$-e_{247}$
19	$e_{126}+e_{134}+e_{157}+e_{235}+e_{247}+e_{367}+e_{456}$	$(\mathbb{O}_6, \mathbb{O}_5, \mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_8, 2\mathbb{O}_5)$	$-e_{134}$
20	$e_{126}+e_{134}+e_{157}+e_{237}+e_{245}+e_{356}+e_{467}$	$(\mathbb{O}_6, \mathbb{O}_5, 3\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_4)$	0
21	$e_{126}+e_{135}+e_{147}+e_{234}+e_{257}+e_{367}+e_{456}$	$(2\mathbb{O}_6, \mathbb{O}_8, \mathbb{O}_5, \mathbb{O}_7, 2\mathbb{O}_5)$	$-e_{257}$
22	$e_{126}+e_{135}+e_{147}+e_{237}+e_{245}+e_{346}+e_{567}$	$(\mathbb{O}_6, 4\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_2)$	$-e_{245}$
23	$e_{126}+e_{137}+e_{145}+e_{234}+e_{257}+e_{356}+e_{467}$	$(\mathbb{O}_6, 2\mathbb{O}_7, \mathbb{O}_5, \mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_4)$	$-e_{126}$
24	$e_{126}+e_{137}+e_{145}+e_{235}+e_{247}+e_{346}+e_{567}$	$(\mathbb{O}_6, 4\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_2)$	$-e_{137}$
25	$e_{127}+e_{134}+e_{156}+e_{235}+e_{246}+e_{367}+e_{457}$	$(\mathbb{O}_6, \mathbb{O}_5, \mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_8, 2\mathbb{O}_5)$	$-e_{235}$
26	$e_{127}+e_{134}+e_{156}+e_{236}+e_{245}+e_{357}+e_{467}$	$(\mathbb{O}_6, \mathbb{O}_5, 3\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_4)$	$-e_{467}$
27	$e_{127}+e_{135}+e_{146}+e_{234}+e_{256}+e_{367}+e_{457}$	$(2\mathbb{O}_6, \mathbb{O}_8, \mathbb{O}_5, \mathbb{O}_7, 2\mathbb{O}_5)$	$-e_{146}$
28	$e_{127}+e_{135}+e_{146}+e_{236}+e_{245}+e_{347}+e_{567}$	$(\mathbb{O}_4, 4\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_2)$	$-e_{135}$
29	$e_{127}+e_{136}+e_{145}+e_{234}+e_{256}+e_{357}+e_{467}$	$(\mathbb{O}_6, 2\mathbb{O}_7, \mathbb{O}_5, \mathbb{O}_6, \mathbb{O}_4)$	$-e_{357}$
30	$e_{127}+e_{136}+e_{145}+e_{235}+e_{246}+e_{347}+e_{56}$	$(\mathbb{O}_6, 4\mathbb{O}_7, \mathbb{O}_6, \mathbb{O}_2)$	$-e_{246}$

Proof The proof is split into the first 8 signed combinations with all positive terms ($\Phi_{i,1}, 1 \leq i \leq 30$) or a single negative term ($\Phi_{i,j}, 2 \leq j \leq 8$). It will be seen that these cover all remainders and can be extended to all 128 signs using automorphisms, reflections and negation. The proof starts with the observation that the remainder for each primary, $\Phi_{i,1}$, is the term that is negated to provide the first element of the octonions, apart from Φ_{11} and Φ_{20}. Considering Φ_1, the permutations of the basis indices needed to proceed to the negation of the first term, e_{123}, can be found by forming the multiplication table for both $\Phi_{1,1}$ and $\Phi_{1,2}$, and searching for an isomorphism. Table 2 shows the continuing permutations from $\Phi_{1,1}$ to $\Phi_{1,2}$, etc up to $\Phi_{1,8}$, with the final permutation wrapping back to $\Phi_{1,1}$. Negative permutations change the term's sign if it contains the negatated index. The first permutation changes the sign of all terms in $\Phi_{1,1}$ that contain e_1 and needs to also swap e_4 to e_5 and e_6 to e_7 plus in the opposite direction with a sign change. The last permutation which wraps to the first row also shows a single signed permutation because Φ is changing its signed parity, σ. Notice that the remainder is also modified to match the next row in each case.

Table 2: Φ_1 Simple Sign Combinations and Tranformation Constructions

j	$\mathbf{\Phi_{1,j}}$	Remainder	Permutation	Rotation	Reflection
1	$e_{123}+e_{145}+e_{167}+e_{246}$ $+e_{257}+e_{347}+e_{356}$	$-e_{246}$	$(-1)(45)(67)$	$R_{45}R_{67}$	e_{157}
2	$-e_{123}+e_{145}+e_{167}+e_{246}$ $+e_{257}+e_{347}+e_{356}$	e_{257}	$(23)(45)$	$R_{23}R_{45}$	e_{12467}
3	$e_{123}-e_{145}+e_{167}+e_{246}$ $+e_{257}+e_{347}+e_{356}$	e_{347}	$(45)(67)$	$R_{45}R_{67}$	e_{12346}
4	$e_{123}+e_{145}-e_{167}+e_{246}$ $+e_{257}+e_{347}+e_{356}$	e_{356}	$(1463)(25)$	$R_{14}R_{16}R_{13}R_{25}$	e_{127}
6	$e_{123}+e_{145}+e_{167}+e_{246}$ $-e_{257}+e_{347}+e_{356}$	e_{123}	$(2\ {-4})(3\ {-5})$	$R_{24}R_{35}$	1
7	$e_{123}+e_{145}+e_{167}+e_{246}$ $+e_{257}-e_{347}+e_{356}$	e_{145}	$(-2)(-3)(46)(57)$	$R_{46}R_{57}$	e_{145}
8	$e_{123}+e_{145}+e_{167}+e_{246}$ $+e_{257}+e_{347}-e_{356}$	e_{167}	$(-12)(4576)$	$R_{12}R_{45}R_{47}R_{46}$	e_{34}

Table 2 shows permutations as cycles, with (i, j, k) meaning $(i \to j, j \to k, k \to i)$ and rotations with reflections to change signs. Permutation indices with minus signs mean the term changes sign. Equivalently, 2-form rotations and reflections to change signs can be used to replace any permutation. For example, the permutation $(j, k) \to (k, j)$ is R_{jk} and a reflection of basis k. Similarly, $(i, j, k, l) \to (j, k, l, i)$ can be expressed as rotations $R_{ij}R_{ik}R_{il}$ followed by sign changes for j, k and l, which, from (1), is the reflection 4-form with j, k and l indices missing. Since these rotations involve the 4-forms from Φ_1', which commute with Φ_1, then equation (4) is invariant apart from sign changes of Φ_1. Finally, reflections are used to correct the rotation signs as well as address permutation sign changes. Rotations and reflections matching the permutations to change from $\Phi_{1,j}$ to $\Phi_{1,j+1}, 1 \leq j \leq 8$, are provided in Table 2. A no reflection case is shown for $\Phi_{1,6}$ which is 1 according to (1). Table 2 shows the transitions to move the minus sign to each term in succession and to transform remainders correctly as defined by (4).

This process has been verified for all primary simple sign combinations for the first 8 progressions for each $\Phi_i, 1 \leq i \leq 30, i \neq 11, 20$, and also for $\Phi_{11,2}$ and $\Phi_{20,2}$. Note that the octonions are not isomorphic to the $\mathbb{O}_k$ algebras and the construction skips such cases to avoid the remainder becoming 0. The construction has progressed through each term of Φ_i and its remainder. The reflections can only change the sign of these remainders in (4), so while the rotations may cycle through the $\mathbb{O}_k$ algebras, the reflections only move to equivalent signed $\mathbb{O}_k$ or $\mathbb{O}$ representations. Reflections are now used to generate all 128 combinations for each Φ_i.

Table 3 shows the reflections of the first 8 signed combinations of $\Phi_{1,k}, 1 \leq k \leq 8$, with a single basis. Along with the $\Phi_{1,k}$ row, this gives 64 elements which are all unique as shall now be proved. By the definition

Table 3: Single Basis Reflections

Reflection	$\mathbf{\Phi_{1,1}}$	$\mathbf{\Phi_{1,2}}$	$\mathbf{\Phi_{1,3}}$	$\mathbf{\Phi_{1,4}}$	$\mathbf{\Phi_{1,5}}$	$\mathbf{\Phi_{1,6}}$	$\mathbf{\Phi_{1,7}}$	$\mathbf{\Phi_{1,8}}$
e_1	$\Phi_{1,99}$	$\Phi_{1,114}$	$\Phi_{1,119}$	$\Phi_{1,120}$	$\Phi_{1,64}$	$\Phi_{1,63}$	$\Phi_{1,62}$	$\Phi_{1,61}$
e_2	$\Phi_{1,90}$	$\Phi_{1,105}$	$\Phi_{1,60}$	$\Phi_{1,54}$	$\Phi_{1,117}$	$\Phi_{1,118}$	$\Phi_{1,48}$	$\Phi_{1,47}$
e_3	$\Phi_{1,85}$	$\Phi_{1,100}$	$\Phi_{1,55}$	$\Phi_{1,49}$	$\Phi_{1,46}$	$\Phi_{1,45}$	$\Phi_{1,115}$	$\Phi_{1,116}$
e_4	$\Phi_{1,79}$	$\Phi_{1,59}$	$\Phi_{1,104}$	$\Phi_{1,43}$	$\Phi_{1,111}$	$\Phi_{1,38}$	$\Phi_{1,113}$	$\Phi_{1,36}$
e_5	$\Phi_{1,76}$	$\Phi_{1,56}$	$\Phi_{1,101}$	$\Phi_{1,40}$	$\Phi_{1,37}$	$\Phi_{1,110}$	$\Phi_{1,35}$	$\Phi_{1,112}$
e_6	$\Phi_{1,72}$	$\Phi_{1,52}$	$\Phi_{1,42}$	$\Phi_{1,103}$	$\Phi_{1,106}$	$\Phi_{1,33}$	$\Phi_{1,32}$	$\Phi_{1,109}$
e_7	$\Phi_{1,71}$	$\Phi_{1,51}$	$\Phi_{1,41}$	$\Phi_{1,102}$	$\Phi_{1,34}$	$\Phi_{1,107}$	$\Phi_{1,108}$	$\Phi_{1,31}$

of the primary 3-forms each $\Phi_{1,k}$ contains the reflection e_i three times so each row reflection changes the sign of the remaining 4 terms. These terms are unique for each row and have different non-overlapping sign differences for each $\Phi_{1,k}$ so each term is unique. The first column has 4 negative terms for each row after $\Phi_{1,1}$ in the header. Other columns may have 3 or 5 negative terms or 1 in the header. Now consider the complimentary table starting with header $\Phi_{1,128}, \Phi_{1,127}, \ldots, \Phi_{1,121}$. The first column has 7 negatives in the header followed by 3 negatives for each row. The remaining columns have 2 or 4 negative terms and 6 in the header. So only the cases of 3 and 4 negative terms need be considered since these may overlap.

But these terms are mixed from the first column in one table and the other columns from the other table. The reflections in the first column start with all terms having the same sign so each reflection changes terms that don't contain the reflection. In order for the remaining columns in the alternate table to have the same number of terms then the single term from the header, e_{ijk}, that has a different sign must be negated. This means the reflection can't involve any of the basis indicies, i, j, k, and the three terms that contain one of these basis indicies and the reflection basis will not change sign. That leaves three remaining terms, each containing one of these indicies that change sign and thus differ to e_{ijk}. This does not match the pattern from the first column where the three same sign terms share a basis index. Thus all terms of Table 3 and its compliment are unique and this covers all possible $2^7 = 128$ sign combinations of Φ_1. This argument can be extended to be applicable to all primary 3-forms. Finally, the count of associative or non-associative triples must remain intact to keep the same remainder in (4) due to $\Phi_{i,j}$ being squared. This completes the proof of the Classification Theorem.

The primary 3-forms can be divided up into groups of six as seen in Table 1. All Φ_i rows contain $\mathbb{O}_6$ and $\mathbb{O}_7$. For each group of six rows the following extra classes can be found, $\{\mathbb{O}_2\}, \{\mathbb{O}_4, \mathbb{O}_5\}, \{\mathbb{O}_2, \mathbb{O}_5, \mathbb{O}_8\}$, $\{\mathbb{O}_2, \mathbb{O}_4, \mathbb{O}_5, \mathbb{O}_8\}, \{\mathbb{O}_2, \mathbb{O}_4, \mathbb{O}_5, \mathbb{O}_8\}$. Thus applying rotations from Φ_i' to $\Phi_{i,j}$ not including the octonions will only generate classes within these groups. Applying a term from Φ_i' as a rotation to an octonion primary, $\Phi_{1,5}$ say, will transform to another octonion representation because the remainder in (4) is zero and remains so under the transform. Hence the $\Phi_{1,5}$ column represents the eight $\mathbb{O}^-$ octonians in primary Φ_1. The remaining eight of the complimentary set, $\mathbb{O}^+$, are obtained by negating the $\mathbb{O}^-$ cases. The same applies for all 30 primary representations giving 480 representations of the octonions. Applying the 3 sets of rotations from the terms of Φ_1' to $\Phi_{1,5}$ can also be used to generate the eight members of $\mathbb{O}^-$ with some redundancy.

The column for $\Phi_{1,5}$ in Table 3 shows eight $\mathbb{O}^-$ since $\Phi_{1,5}$ has 1 negative term and each reflection changes the sign of four terms keeping the parity negative. Notice that four 3-forms have index above 64 and four are below this. The complimentatary table changes these to $\mathbb{O}^+$ and swaps the top half indicies with the bottom half. So the distribution is three $\mathbb{O}^+$ and five $\mathbb{O}^-$ for indicies above 64 and five $\mathbb{O}^+$ and three $\mathbb{O}^-$ in the bottom half of indices. All primaries share this distribution apart from $\Phi_{11,1}$ and $\Phi_{20,1}$ which have a distribution of seven $\mathbb{O}^-$ and one $\mathbb{O}^+$ in the lower half and the opposite above. The representation of octonions selected by Baez, [7], corresponds to $\Phi_{11,1}$ which has this less common distribution.

An example of classification is $\Phi_{1,6}$ which has remainder e_{123} and, from Table 1, represents $\mathbb{O}_2^-$. The multiplication table for $\mathbb{O}_2$, as generated from the product rules dictated by $\Phi_{1,6}$, is shown in Table 4.

Table 4: Multiplication Table for $\mathbb{O}_2$

$\mathbb{O}_2^2$	$\mathbf{p_1}$	$\mathbf{p_2}$	$\mathbf{p_3}$	$\mathbf{p_4}$	$\mathbf{p_5}$	$\mathbf{p_6}$	$\mathbf{p_7}$
$\mathbf{p_1}$	-1	p_3	$-p_2$	p_5	$-p_4$	p_7	$-p_6$
$\mathbf{p_2}$	$-p_3$	-1	p_1	p_6	$-p_7$	$-p_4$	p_5
$\mathbf{p_3}$	p_2	$-p_1$	-1	p_7	p_6	$-p_5$	$-p_4$
$\mathbf{p_4}$	$-p_5$	$-p_6$	$-p_7$	-1	p_1	p_2	p_3
$\mathbf{p_5}$	p_4	p_7	$-p_6$	$-p_1$	-1	p_3	$-p_2$
$\mathbf{p_6}$	$-p_7$	p_4	p_5	$-p_2$	$-p_3$	-1	p_1
$\mathbf{p_7}$	p_6	$-p_5$	p_4	$-p_3$	p_2	$-p_1$	-1

The non-associative triples of this algebra refer to faces of the 7-simplex not included in the primary faces.

$$(p_4, p_5, p_6), (p_4, p_5, p_7), (p_4, p_6, p_7), (p_5, p_6, p_7).$$

Defining a graded product for the octonions exposes a representation independent form of the algebra,

$$\Phi_{1,5} \to \mathbb{O} = \{o_1, o_2, o_{12}, o_3, o_{13}, o_{23}, o_{123}\}.$$

The associative parts only involve quaternion product triples giving scalar results,

$$\begin{gathered}(o_1, o_2, o_{12}), (o_1, o_3, o_{13}), (o_2, o_3, o_{23}),\\ (o_1, o_{23}, o_{123}), (o_2, o_{13}, o_{123}), (o_{12}, o_3, o_{123})\\ (o_{12}, o_{13}, o_{23})\end{gathered}$$

The other 28 triples give non-associative products so the graded definition exposes the structure of $\mathbb{O}$. Another advantage of this approach is that it is easy to implement a recursive Cayley-Dickson product and extend to basis indices 4, 5, etc, which define a series of sedenion algebras all having power associativity. The algebra $\mathbb{O}_2$ has four non-associative triples. Using the same labelling as $\mathbb{O}$ above, these are

$$(o_3, o_{13}, o_{23}), (o_3, o_{13}, o_{123}), (o_3, o_{23}, o_{123}), (o_{13}, o_{23}, o_{123})$$

These don't correspond to the associative products above so $\mathbb{O}_2$ can be considered to be a subset of the non-associative products of $\mathbb{O}$. This is true for all $\mathbb{O}_k$ except $\mathbb{O}_4$ and $\mathbb{O}_5$.

3 Construction of G2

The Wikipedia article [8] defines G2 as the Lie algebra with conjugations leaving the 3-form $\Phi_{1,64}$ invariant. This follows the associated calibration used by Bryant, [9], whereas [6] used $\Phi_{1,23}$ which both represent octonions. Bryant then provides the same G2 generators as the Wikipedia with different labels. He stated that he could not find an elegant proof and resorted to an explicit case. The generators in [8] are provided as matrices which can be converted to Bryant's form and represented in Spin(7) as

$$\begin{array}{lll}
A = (e_{23} - e_{45})/2 & H = (e_{45} - e_{67})/2 & A + H = (e_{23} - e_{67})/2\\
B = (-e_{13} - e_{46})/2 & I = (e_{46} + e_{57})/2 & B + I = (-e_{13} + e_{57})/2\\
C = (e_{12} + e_{47})/2 & J = (-e_{47} + e_{56})/2 & C + J = (e_{12} + e_{56})/2\\
D = (e_{15} + e_{26})/2 & K = (-e_{26} - e_{37})/2 & D + K = (-e_{15} - e_{37})/2\\
E = (e_{14} - e_{27})/2 & L = (-e_{27} + e_{36})/2 & E - L = (e_{14} - e_{36})/2\\
F = (e_{17} + e_{24}/2 & M = (-e_{17} + e_{35})/2 & F + M = (e_{24} + e_{35})/2\\
G = (-e_{16} - e_{25})/2 & N = (e_{25} - e_{34})/2 & G + N = (-e_{16} - e_{34})/2
\end{array} \tag{5}$$

The first two columns contain repeated terms which provides the additional seven elements of G2 in column three. Applying the pairs of terms in this representation as 90° rotations to $\Phi_{1,64}$ leaves it invariant.

Table 5: Anti-symmetric Product Table for G2

[,]	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**	**N**
A	$-(C+J)$	$B+I$	$-L-E$	$-(D+K)$	N	$F+M$	0	J	$-I$	L	$-K$	N	$-2(F+M)$
B	0	$-(A+H)$	$-(F+M)$	$-G+N$	$-K$	$-(L-E)$	$-J$	0	H	F	$G+N$	$D+K$	$-(L-E)$
C	.	0	$-G$	$-2F$	$2E$	D	I	$-H$	0	$G+N$	$-F$	$-E$	$-(D+K)$
D	.	.	0	$-H$	I	$-2(C+J)$	E	$-F$	G	0	$-(A+H)$	$-(B+I)$	$C+J$
E	.	.	.	0	$-2C$	I	$-D$	$-G$	$-F$	$-(A+H)$	0	C	$-(B+I)$
F	.	.	.	.	0	H	$-G$	D	E	$-B$	C	0	A
G	.	.	.	.	.	0	F	E	$-D$	$-(C+J)$	$-(B+I)$	$A+H$	0
H	.	.	.	.	.	.	0	$-2J$	$2I$	L	$-K$	$-(G+N)$	$F+M$
I	.	.	.	.	.	.	.	0	$-2H$	$-(F+M)$	$-N$	$D+K$	L
J	.	.	.	.	.	.	.	.	0	$-N$	$F+M$	$-(L-E)$	K
K	.	.	.	.	.	.	.	.	.	0	$2(A+H)$	$-B+I$	$-J$
L	.	.	.	.	.	.	.	.	.	.	0	$C+J$	$-I$
M	.	.	.	.	.	.	.	.	.	.	.	0	A
N	.	.	.	.	.	.	.	.	.	.	.	.	0

This representation has the Lie Product Table shown in Table 5 for the first 14 elements and with only the upper half shown since the table is anti-symmetric. But other octonion representations are not invariant and G2 must be transformed using the same transforms for $\Phi_{1,64}$.

Firstly consider transformations that convert Φ_1 to $\Phi_{i,1}, 1 \leq i \leq 30, i \neq 11, 20$, and Φ_1 to $\Phi_{11,2}$ and $\Phi_{20,2}$. These are provided in Table 6 which was generated by considering the permutations of the isomorphism between the multiplication tables generated by Φ_1 and Φ_i, etc. This process follows the same rotation and reflection scheme as for the signed variations of Table 2 but here some extra rotations are needed because reflections alone don't necessarily return $\tilde{\Phi}_i$ to a primary. These cases are shown in Table 6 with either no reflection, a single basis or as a 6-form.

Table 6: Transformations from Φ_1 for the Primary 3-forms

Result	**Rotation**	**Reflection**	**Result**	**Rotation**	**Reflection**
Φ_1	1	1	Φ_{16}	$e_{35}+e_{46}+e_{12}+e_{34}$	e_6
Φ_2	$e_{23}+e_{45}+e_{67}$	e_{246}	Φ_{17}	$e_{35}+e_{47}+e_{24}$ $+e_{23}+e_{56}+e_{57}$	e_2
Φ_3	$e_{12}+e_{45}+e_{67}$	e_{256}	Φ_{18}	$e_{35}+e_{47}+e_{46}$	e_{145}
Φ_4	$e_{12}+e_{57}$	e_{345}	Φ_{19}	$e_{12}+e_{36}+e_{34}$	e_{146}
Φ_5	$e_{23}+e_{57}$	e_{157}	$\Phi_{20,2}$	$e_{36}+e_{47}+e_{13}+e_{12}$ $+e_{47}+e_{46}$	e_{57}
Φ_6	$e_{45}+e_{46}+e_{47}$	e_{123}	Φ_{21}	$e_{36}+e_{57}+e_{35}+e_{47}$	e_3
Φ_7	$e_{12}+e_{34}+e_{56}$	e_{146}	Φ_{22}	$e_{36}+e_{34}+e_{35}$	e_{256}
Φ_8	$e_{34}+e_{67}+e_{13}$ $+e_{12}+e_{45}+e_{47}$	1	Φ_{23}	$e_{36}+e_{13}+e_{12}+e_{46}+e_{47}$	e_7
Φ_9	$e_{34}+e_{56}+e_{36}+e_{57}$	e_7	Φ_{24}	$e_{36}+e_{45}+e_{12}+e_{57}$	e_4
Φ_{10}	$e_{34}+e_{56}+e_{57}$	e_{367}	Φ_{25}	$e_{37}+e_{46}+e_{34}+e_{56}$	e_5
$\Phi_{11,2}$	$e_{34}+e_{56}+e_{67}$ $+e_{12}+e_{23}+e_{47}+e_{45}$	e_{234567}	Φ_{26}	$e_{23}+e_{25}+e_{27}+$ $e_{34}+e_{36}+e_{37}+e_{35}$	1
Φ_{12}	$e_{34}+e_{57}+e_{12}+e_{35}$	e_3	Φ_{27}	$e_{12}+e_{37}+e_{35}$	e_{356}
Φ_{13}	$e_{35}+e_{67}$	e_{257}	Φ_{28}	$e_{37}+e_{56}+e_{12}+e_{34}$	e_3
Φ_{14}	$e_{23}+e_{24}+e_{25}$	e_{167}	Φ_{29}	$e_{37}+e_{45}+e_{23}+e_{67}$	e_1
Φ_{15}	$e_{35}+e_{46}+e_{47}$ $+e_{12}+e_{34}+e_{37}+e_{36}$	e_7	Φ_{30}	e_{37}	e_{123}

Applying the Table 6 transformation to $\Phi_{1,64}$ will produce octonion representations for each primary and it is a simple matter to use reflections to negate some terms to uncover the 16 octonion representations for each primary. Table 7 shows the reflections required after Table 6 is applied to $\Phi_{1,64}$ to obtain the first half octonions in each primary. The Φ_{11} and Φ_{20} rows have translated to $\Phi_{11,79}$ and $\Phi_{20,79}$ which is in the top half of signed 3-forms and become $\Phi_{11,50}$ and $\Phi_{20,50}$ under negation, which is shown in Table 7 as an extra reflection e_{123567} and e_{123457}, respectively. The 240 representations of the G2 algebra generated from Tables 5 and 6 all have the same multiplication table, shown in Table 5 and leave invariant the 3-form generated by applying the same transformations to $\Phi_{1,64}$. By the definition (1), the terms of G2 act independently of the sign so all 16 representations of any primary act the same. So we can introduce $G2_i, 1 \leq i \leq 30$ that keeps all $\mathbb{O}$ and $\mathbb{O}_k$ representations derived from $\Phi_{i,j}, 1 \leq j \leq 128$, invariant.

Table 7: Octonion Reflections within each Primary

i	Reflection for each Octonion	Extra
Φ_1	$e_{14567}, e_{124567}, e_{123467}, e_{123456}, e_{23456}, e_{23467}, e_{24567}$	1
Φ_2	$e_{14567}, e_{134567}, e_{123567}, e_{123457}, e_{23457}, e_{23567}, e_{34567}$	1
Φ_3	$e_{24567}, e_{124567}, e_{123567}, e_{123457}, e_{23467}, e_{23456}, e_{14567}$	1
Φ_4	$e_{24567}, e_{124567}, e_{123456}, e_{123467}, e_{23567}, e_{23457}, e_{14567}$	1
Φ_5	$e_{14567}, e_{134567}, e_{123456}, e_{123467}, e_{23467}, e_{23456}, e_{34567}$	1
Φ_6	$e_{14567}, e_{124567}, e_{123567}, e_{123457}, e_{23457}, e_{23567}, e_{24567}$	1
Φ_7	$e_{23567}, e_{123567}, e_{123457}, e_{123456}, e_{24567}, e_{23467}, e_{13567}$	1
Φ_8	$e_{23467}, e_{123467}, e_{123456}, e_{123457}, e_{34567}, e_{23567}, e_{13567}$	1
Φ_9	$e_{13567}, e_{123567}, e_{124567}, e_{123467}, e_{23467}, e_{24567}, e_{23567}$	1
Φ_{10}	$e_{13567}, e_{123567}, e_{123457}, e_{123467}, e_{23467}, e_{23457}, e_{23567}$	1
Φ_{11}	$e_{23456}, e_{123456}, e_{123467}, e_{123457}, e_{3567}, e_{234567}, e_{134567}$	e_{123567}
Φ_{12}	$e_{23567}, e_{123567}, e_{124567}, e_{123456}, e_{23457}, e_{23467}, e_{13567}$	1
Φ_{13}	$e_{13467}, e_{123467}, e_{124567}, e_{123457}, e_{23457}, e_{24567}, e_{23467}$	1
Φ_{14}	$e_{13467}, e_{134567}, e_{123567}, e_{123456}, e_{23456}, e_{23567}, e_{34567}$	1
Φ_{15}	$e_{23467}, e_{123467}, e_{123567}, e_{123456}, e_{24567}, e_{23457}, e_{13467}$	1
Φ_{16}	$e_{23467}, e_{123467}, e_{123456}, e_{123567}, e_{24567}, e_{23457}, e_{13467}$	1
Φ_{17}	$e_{13467}, e_{134567}, e_{124567}, e_{123457}, e_{23457}, e_{24567}, e_{34567}$	1
Φ_{18}	$e_{13467}, e_{123467}, e_{124567}, e_{123457}, e_{23457}, e_{24567}, e_{23467}$	1
Φ_{19}	$e_{23457}, e_{123457}, e_{123467}, e_{123456}, e_{24567}, e_{23567}, e_{13457}$	1
Φ_{20}	$e_{23567}, e_{123567}, e_{123456}, e_{123467}, e_{3457}, e_{234567}, e_{134567}$	e_{123457}
Φ_{21}	$e_{13457}, e_{123457}, e_{124567}, e_{123567}, e_{23567}, e_{24567}, e_{23457}$	1
Φ_{22}	$e_{13457}, e_{123457}, e_{124567}, e_{123456}, e_{23456}, e_{24567}, e_{23457}$	1
Φ_{23}	$e_{23456}, e_{123456}, e_{123567}, e_{123467}, e_{34567}, e_{23457}, e_{13457}$	1
Φ_{24}	$e_{23457}, e_{123457}, e_{123467}, e_{123567}, e_{24567}, e_{23456}, e_{13457}$	1
Φ_{25}	$e_{13456}, e_{123456}, e_{123567}, e_{123457}, e_{23457}, e_{23567}, e_{23456}$	1
Φ_{26}	$e_{13456}, e_{134567}, e_{124567}, e_{123467}, e_{23467}, e_{24567}, e_{34567}$	1
Φ_{27}	$e_{23456}, e_{123456}, e_{124567}, e_{123467}, e_{23567}, e_{23457}, e_{13456}$	1
Φ_{28}	$e_{23456}, e_{123456}, e_{123457}, e_{123567}, e_{24567}, e_{23467}, e_{13456}$	1
Φ_{29}	$e_{13456}, e_{134567}, e_{123457}, e_{123567}, e_{23567}, e_{23457}, e_{34567}$	1
Φ_{30}	$e_{13456}, e_{123456}, e_{124567}, e_{123467}, e_{23467}, e_{24567}, e_{23456}$	1

The G2 representations provide a relationship with the three 2-form rotations generated from the Φ'_i 4-forms. The representation of G2 in (5) corresponds to the rotations provided by the terms of (2). In general, the three 90° rotation pairs derived from the primary 4-forms, Φ'_i, provide all 21 candidates for each $\Phi_{i,j} = \Phi_O$. For $\Phi'_{1,64}$, this is shown in Table 9, alongside the representation from (5). The relative signs for the G2 elements, whether the two terms have the same sign or opposite signs, are provided by the following. The Mixed Rotations column has odd parity since $e_{14}e_{27} = -e_{1247}$ while the other two columns have even parity. The relative sign parity also includes the sign of the row given by whether the 4-form, Φ'_O, has a negative term.

This works because Φ'_O is a commuting algebra which leads to all rotation terms leaving Φ_O invariant.

Table 8: G2 Multiplication Structure in Cartan representation

$[,]$	$\mathbf{H2}$	$\mathbf{X_1}$	$\mathbf{Y_1}$	$\mathbf{X_2}$	$\mathbf{Y_2}$	$\mathbf{X_3}$	$\mathbf{Y_3}$	$\mathbf{X_4}$	$\mathbf{Y_4}$	$\mathbf{X_5}$	$\mathbf{Y_5}$	$\mathbf{X_6}$	$\mathbf{Y_6}$
H1	0	$2X_1$	$-2Y_1$	$-3X_2$	$3Y_2$	$-X_3$	Y_3	X_4	$-Y_4$	$3X_5$	$-3Y_5$	0	0
H2	0	$-X_1$	Y_1	$2X_2$	$-2Y_2$	X_3	$-Y_3$	0	0	$-X_5$	Y_5	X_6	$-Y_6$
X1	.	0	$H1$	X_3	0	$2X_4$	$-3Y_2$	$-3X_5$	$-2Y_3$	0	Y_4	0	0
Y1	.	.	0	0	$-Y_3$	$3X_2$	$-2Y_4$	$2X_3$	$3Y_5$	$-X_4$	0	0	0
X2	.	.	.	0	$H2$	0	Y_1	0	0	$-X_6$	0	0	Y_5
Y2	.	.	.	.	0	$-X_1$	0	0	0	0	Y_6	$-X_5$	0
X3	.	.	.	.	.	0	$H1{+}3H2$	$-3X_6$	$2Y_1$	0	0	0	Y_4
Y3	.	.	.	.	.	.	0	$-2X_1$	$3Y_6$	0	0	$-X_4$	0
X4	.	.	.	.	.	.	.	0	$H1{+}2H2$	0	$-Y_1$	0	$-Y_3$
Y4	.	.	.	.	.	.	.	.	0	X_1	0	X_3	0
X5	.	.	.	.	.	.	.	.	.	0	$H1{+}H2$	0	$-Y_2$
Y5	.	.	.	.	.	.	.	.	.	.	0	X_2	0
X6	.	.	.	.	.	.	.	.	.	.	.	0	$H1{+}2H2$
Y6	.	.	.	.	.	.	.	.	.	.	.	.	0

The construction of G2 is a bonus. But Table 9 for any $\Phi_{i,j}$ has at least one invariant row. The remainder from the Classification Theorem leaves one row invariant because (4) is invariant for the three rotations involving the remainder's compliment. It is only when the remainder is zero that all rows are invariant. So the six algebras, $\mathbb{O}_2, \mathbb{O}_4, \mathbb{O}_5, \mathbb{O}_6, \mathbb{O}_7, \mathbb{O}_8$, provide three automorphisms each that are subsets of G2 and leave Φ invariant. This can be partially related to the 6 fold symmetry of the Cartan root system diagram for G2, [10], shown in Figure 2 using construction and destruction matrices, X_i and $Y_i, 1 < i < 6$, respectively, derived from the g matrices in Humphreys [11]. The mapping is

$$X_1 = g_2^T,\, X_2 = g_{1,-2},\, X_3 = -g_1^T,\, X_4 = g_3,\, X_5 = -g_{2,-3},\, X_6 = g_{1,-3},\, H_1 = [X_1, Y_1]$$
$$Y_1 = g_2,\;\; Y_2 = g_{1,-2}^T,\; Y_3 = -g_1,\;\; Y_4 = g_3^T,\, Y_5 = -g_{2,-3}^T,\;\; Y_6 = g_{1,-3}^T,\;\; H_2 = [X_2, Y_2].$$

These generate the G2 multiplication table in [10] replicated in Table 8. Matching these $90°$ rotations to Φ_{14} leads to the following

$$E - L = X_1 - Y_1, D + K = X_2 - Y_2, H = Y_3 - X_3.$$

Other terms can not be represented but these three terms match half of the symmetry provided by the G2 Cartan root diagram.

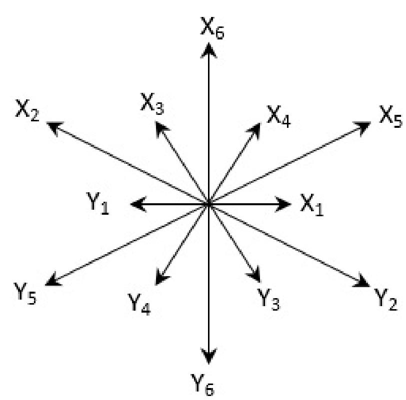

Figure 2: **Cartan Root Diagram for G2**

Table 9 generalises to relationships for other octonions with a similar pattern except the order of the rows and terms may differ. The one invariant is that the row that contains M always occurs in the row identified

Table 9: G2 Relationship to Φ_1'

$\boldsymbol{\Phi'_{1,64}}$ term	**Normal Rotations**	**Mixed Rotations**	**Outer Rotations**
e_{1247}	$C = (e_{12} + e_{47})/2$	$E = (e_{14} - e_{27})/2$	$F = (e_{17} + e_{24})/2$
e_{1256}	$C + J = (e_{12} + e_{56})/2$	$-D = (e_{15} - e_{26})/2$	$-G = (e_{16} + e_{25})/2$
e_{1346}	$-B = (e_{13} + e_{46})/2$	$E - L = (e_{14} - e_{36})/2$	$-G - N = (e_{16} + e_{34})/2$
$-e_{1357}$	$-B - I = (e_{13} - e_{57})/2$	$-D - K = (e_{15} + e_{37})/2$	$-M = (e_{17} - e_{35})/2$
$-e_{2345}$	$A = (e_{23} - e_{45})/2$	$F + M = (e_{24} + e_{35})/2$	$N = (e_{25} - e_{34})/2$
$-e_{2367}$	$A + H = (e_{23} - e_{67})/2$	$-K = (e_{26} + e_{37})/2$	$-L = (e_{27} - e_{36})/2$
$-e_{4567}$	$H = (e_{45} - e_{67})/2$	$I = (e_{46} + e_{57})/2$	$-J = (e_{47} - e_{56})/2$

Table 10: Construction of G2

$\Phi'_{1,64}$ **term**	**Normal Rotations**	**M Products**	**Cross Products**
e_{1247}	C, E, F	$-E, C, 0$	$-2F, 0, 0$
e_{1256}	$C + J, -D, -G$	$-L, B + I, -A - H$	$G, 2K, -C - J$
e_{1346}	$-B, E - L, -G - N$	$-D - K, -J, +H$	$G + N, E - L, -2I$
$-e_{1357}$	$-B - I, -D - K, -M$	$-2(D + K), 2(B + I), 0$	$-8M, 0, 0$
$-e_{2345}$	$A, F + M, N$	$N, 0, A$	$0, -2(F + M), 0$
$-e_{2367}$	$A + H, -K, -L$	$-G, B + I, -C - J$	$-L, -2D, A + H$
$-e_{4567}$	$H, I, -J$	$-G - N, D + K, L - E$	$-J, 2B, H$

from the Classification Theorem as the term which changes sign when starting from the primary (Φ_{11} and Φ_{20} are dealt with separately below). For the other algebras it is the remainder that identifies the M row. This observation allows for the construction of the representation of the G2 algebra, up to the sign of each element. For Φ_1, the negated term is e_{246} and $-e_{1234567}e_{246} = e_{1357}$ and this is consistent across all octonion representations in Φ_1.

The construction of G2 for any Φ starts with the three $90°$ rotations for Φ', as seen in Table 9. The M row is identified as above but it is uncertain which of the three rotations provides M so three candidates are trialled. First take the Lie product with all other terms then take cross products of these results. Using $Z_i, i \in \mathbb{N}_1^9$ to designate G2 terms in the columns of Table 10, then for rotation elements Z_1, Z_2 and Z_3, the Lie M products are $[M, Z_i] = Z_{i+3}, 1 \leq i \leq 3$, and cross terms are $[Z_4, Z_5] = Z_7, [Z_4, Z_6] = Z_8, [Z_5, Z_6] = Z_9$. These are shown for $\Phi_{1,64}$ in Table 10 and a pattern is obvious. Three rows have two zeros in the cross products, one which is the M row, and the other four rows have repeated M Product pairs, $B + I$ and $D + K$. These are the other elements in the M row which indicates a symmetry in G2 that could be hard to separate. Fortunately, of the two pairs, the $C + J$ and $E - L$ terms always occur first in the term ordering shown in the table.

Given any Φ_i', the relationship table can be formed and the M row for each trial group identified. Of the three trials trying to find which column represents M, the three rows with two zeros in the cross terms can be distinguished from the other four containing the M Product pairs. Excluding the $8M$ row from the two zeros rows leaves the C/E and A/N rows. The M Products here are asymmetric with $[M, C] = -E$ while all other products are positive. Hence C and E can be identified as well as F from the Cross Products. Because the A and N products are symmetric, these are left ambiguous but $F + M$ can be identified from the Cross Products.

The 2-form terms within C and E can now isolate the first rows with pairs $B + I$ and $D + K$ within the M Products, respectively. The other row of the pair has these same 2-form terms in J and L but as previously mentioned the C and E terms always appear first in the relationship table. With the location of the paired elements and the C and E terms identified then D, G, L and $A + H$ can be isolated from the first row of the $B + I$ pair and B, J, H from the first $D + K$ pair. The paired rows provide K and I using these paired locations. Finally, $A+H$ and H allows the ambiguity of A and N to be settled.

This process generates one of three solutions with M sometimes replaced by $D+K$ or $B+I$ in two trials. But one of each trial of the three selections of M generates the same solution as for the translation from $\Phi_{1,63}$ in Table 9. The same column for the choice of M applies to all octonion representations for the same primary. Most primaries use the third column and the first and second columns have 8 and 9 primaries each. Also the signs of each element are arbitrary and Table 5 can be used to correct this, up to an overall sign of all elements.

4 Summary

The $\mathbb{O}$ representation selected by Baez, Φ_{11}, is interesting in that it has positive terms which correspond to a "natural" ordering of triples

$$\Phi_{11,1} = e_{124} + e_{137} + e_{156} + e_{235} + e_{267} + e_{346} + e_{457}.$$

The 16 representations of G2 for any primary only differ by sign changes since they are derived from 4-forms with only sign changes. We have introduced $G2_i, 1 \leq i \leq 30$, that leaves Φ_i invariant so keeps all octonion and $\mathbb{O}_k$ representations derived from this primary, invariant. So any $\Phi_{i,j}$ can define a consistent cross product and the requirement of a "natural" ordering of triples with only one decreasing index in the cycle is purely asthetic. The connection between $\mathrm{Spin}(7)$, octonions and G2 is usually summarised as the dimension count relationship $21 = 7 + 14$. The definition of G2 can be expressed more precisely as $G2_i$ consists of the automorphisms of Spin(7) that preserve the related 3-form Φ_i. This can be extended to $G2_{i,j}$ as a subalgebra of $G2_i$ provided by the remainder row in Table 9 which are the automorphisms that preserve $\Phi_{i,j}$. Of course, a remainder of zero specifies $G2_i$. This means the octonions in the dimensional relationship can be replaced by any of the six $\mathbb{O}_k$ algebras and $\mathbb{O}$ can be interpreted as the seventh algebra in the series and designated as $\mathbb{O}_{14}$. The sedenion series under this scheme start with $\mathbb{O}_{126}, \mathbb{O}_{1078}, \mathbb{O}_{9366}, \mathbb{O}_{79926}$, and $\mathbb{O}_{665910}$, which are all power-associative algebras.

Clifford algebra has derived these relationships by identifying Φ'_i as a subalgebra of Spin(7). The Φ'_i automorphisms keep $\rho^2_{i,j}$ from (4) invariant for the remainder term's compliment, or for all terms if the remainder is zero. All these cases identify elements of G2 and the algebra can be constructed in a systematic way using Lie bracket operations. In this respect the connection between octonions and G2 is a more general connection to Spin(7). It is interesting to note that $\mathrm{Cl}(15)$ and $\mathrm{Cl}(31)$ also have commuting imaginary psuedo-vectors which may involve analogous relationships to the sedenion series.

In $\mathrm{Pin}(15)$, with commuting imaginary psuedo-vector, $e_{123456789ABCDEF}$, a 3-form can be generated that covers all edges of the 15-simplex once and all dimensions are listed 7 times. This should be the form for an associative calibration in 15 dimensions,

$$\begin{aligned}\Phi = \;& e_{123} + e_{145} + e_{167} + e_{189} + e_{1Ab} + e_{1CD} + e_{1EF} + e_{246} + e_{257} \\ & + e_{28A} + e_{29B} + e_{2CE} + e_{2DF} + e_{347} + e_{356} + e_{38B} + e_{39A} + e_{3CF} \\ & + e_{3CF} + e_{3DE} + e_{48C} + e_{49D} + e_{4AE} + e_{4BF} + e_{58D} + e_{59C} + e_{5AF} \\ & + e_{5BE} + e_{68E} + e_{69F} + e_{6AC} + e_{6BD} + e_{78F} + e_{79E} + e_{7AD} + e_{7BC}.\end{aligned}$$

For each algebra generated from this using the triple product rules and each term being negated in turn then counting the number of non-associative products gives the following association

$$\begin{aligned}&210, 210, 210, 210, 210, 210, 210, 210, 224, 200, 224, 200, 224, 200, 208, 208, 208, 208, \\ &208, 208, 252, 228, 204, 180, 236, 236, 188, 188, 220, 196, 220, 196, 204, 204, 204, 204.\end{aligned}$$

This represents 12 unique, non-associative algebras. Notice that the 20th term, -e_{48C}, when negated gives 252 non-associative triples. This should correspond to sedenions, called $\mathbb{O}_{126}$ above, especially since it is the middle term in both Spin(7) and Spin(15) that generates the largest non-associative algebra for both series. Mirror reflections of Φ generate 1024 algebras with the same properties so there are 1024 sedenion representations for each primary. The primary above includes each basis 7 times. So once 6 bivectors are selected for any basis then the rest are fixed. This is $14*13*12*11*10*9 = 2,162,160$ primaries. So there are $2,214,051,840$ representations of the first sedenion algebra, $\mathbb{O}_{126}$. I can not find an algebra equivalent to G2 for Spin(15) but there is a commuting subalgebra of 15 8-forms which is not coassociative to Φ but should be a calibration,

$$\begin{aligned} P_{15} = \quad & e_{12478BDE} + e_{12479ACF} + e_{12568BCF} + e_{12569ADE} + e_{13468ADF} + e_{13469BCE} \\ & + e_{13578ACE} + e_{13579BDF} + e_{234589EF} + e_{2345ABCD} + e_{236789CD} + e_{2367ABEF} \\ & + e_{456789AB} + e_{4567CDEF} + e_{89ABCDEF}. \end{aligned}$$

Another result from Clifford algebra is that for any octonion representation, Φ_O, we can define Ψ_O to be the 21 3-forms of $\mathrm{Cl}(7)$ not included in the terms of Φ_O, then with the correct signs we find

$$\Phi'_O \Psi_O = \Psi_O.$$

For $\Phi_{1,5}$ we have

$$\begin{aligned} \Psi_O = \quad & -e_{124} + e_{125} + e_{126} + e_{127} + e_{134} + e_{135} - e_{136} \\ & + e_{137} + e_{146} + e_{147} + e_{156} + e_{157} - e_{234} + e_{235} \\ & + e_{236} + e_{237} + e_{245} + e_{247} + e_{256} - e_{267} + e_{345} \\ & - e_{346} + e_{357} + e_{367} + e_{456} + e_{457} + e_{467} - e_{567}. \end{aligned}$$

A similar result appears in $\mathrm{Cl}(15)$.

This work was verified with the use of Clifford algebra, quaternion and octonion/sedenion calculators written in Python. Quaternions were used to verify the Clifford calculator using Euler angles and these generalised to arbitrary dimensions and compared to matrix expansions of rotations for up to seven dimensions. The octonion calculator also has 15 "dimensions" but can practially only support up to 8, giving sedenions up to five levels, called $\mathbb{O}_{665910}$ above. It also allows for unitary basis elements, $u_i^2 = +1$, which generates split octonions. I plan to extend the Clifford calculator to 31 dimensions to look for a connection to the second sedenion algebra, $\mathbb{O}_{1078}$. The github URL for the calculators is https://github.com/GPWilmot/geoalg.

An advantage of Clifford algebra is the conciseness of the notation. Matrix representations for Rij were not needed and many of the calculations can be done on the fly using the simple product and swapping rules. Another advantage is the geometric interpretation of each operation and the geometric understanding provided. For example, the subalgebra that Φ' spans does not involve 6-forms because a set of three rotations is inconsistent with octonion multiplication or for any of the six related algebras.

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