The Algebra of Geometry

G. P. Wilmot

1 Introduction

This book explores a fundamental connection between geometry and algebra that is so natural that it is an ideal introduction to the theory of algebra. It has simple rules that introduce imaginary numbers and half of the classical matrix groups using geometry rather than the complicated rules of matrix multiplication and vector identities. Instead it uses symmetry and antisymmetry inherent within geometry to define an algebra called the geometric algebra. Having a geometric viewpoint allows for terminology that is easy to comprehend and this understanding carries over into other algebras. The geometric approach simplifies and amalgamates concepts and focuses on the most important ones and from simple rules the most important equations in physics can be developed and appreciated when viewed using geometry. This algebra was discovered by William Kingdon Clifford so is also known as Clifford algebra and the two identifiers distinguish the two ways of viewing the algebra.

The geometric algebra defines rotations and reflections of objects and includes those objects. Matrices are usually used to represent rotations via either the orthogonal group or the spin group. These can have many representations of different sizes for the same dimension so is not as simple as geometric algebra. The matrices operate on themselves and column or row matrices in different ways requiring many rules to be remembered. The spin group is used to represent $\mathbb{G}$ but suffers from not distinguishing the geometry of the objects. Increasing dimensions means finding a representation of larger matrix sizes and embedding smaller dimensions inside. The notation quickly becomes unwieldy and requires computers to do the calculations which obscures the understanding of the operation. In Clifford algebra the operation of rotation is the same effort in nine dimensions as it is in two and can be done manually without requiring a computer. Although the algebra only provides rotations and reflections this approach naturally introduces imaginary numbers and covers parts of the remaining matrix groups. It also extends to algebras not covered by matrices without exotic multiplication schemes. I hope this book provides an easy introduction to many areas of mathematics and a pathway into its complicated terminology.

The first chapter introduces Pascal's Triangle as a table of objects and connections that define a simple geometric shape called a simplex. This simple start provides the basis for deriving geometric algebra. Although it is equivalent to Clifford algebra, the latter is viewed by mathematicians as imaginary numbers whereas the former starts with the much more geometric concept of arrows. This distinction is so important to the geometric understanding I will denote the geometric approach to Clifford algebra as $\mathbb{G}$, for geometric algebra. Imaginary numbers are derived from this with the advantage of having a geometric interpretation. Quaternions are introduced but using the simplex viewpoint and their connection to geometry completes the definition of $\mathbb{G}$. $\mathbb{G}$ actually denotes many Clifford algebras, $\mathbb{G}_n$, where n is the number of dimensions. The three dimensional version, $\mathbb{G}_3$, is the most fruitful algebra to learn and this understanding will follow through to higher dimensions. We will find that thinking in 5 dimensions is fairly easy and this can be extended to 7 dimensions with a bit more work.

The second chapter covers products of quaternions and the most general products in $\mathbb{G}$. The 3 dimensional algebra, $\mathbb{G}_3$, by itself could be called the geometric algebra. It merges arrows, quaternions and the first imaginary number. The chapter leads onto space-time and the single equation covering all eight of Maxwell's Equations which includes monopoles. The geometric interpretation merges physics and geometry and the extension to 5 dimensions introduces more physics as it describes electrons. Since the elements of $\mathbb{G}$ can be of any dimension, they are called forms which were introduced by Grassmann to define the basis of Euclidean spaces. These generate the exterior algebra which uses determinants to define the exterior products of forms. The equivalent expansion for $\mathbb{G}$ extends this definition to include the Pfaffian expansion which incorporates the metric tensor. Hence $\mathbb{G}$ is seen as the algebra that extends exterior space to a metric space.

Hamilton introduced the term quaternion along with their vector products and the third chapter uncovers how these were applied in an inferior fashion into the upcoming vector algebra. This led to what is known as the vector algebra wars with the proponents of quaternions saying that the new vector algebra had lost the elegance and power of Hamilton's algebra. Chapter three provides a history of this unfortunate episode in mathematics which could have been avoided if Clifford algebra, at the time, had been recognised as the algebra of geometry. The algebra was introduced at the start of these wars but due to the untimely death of W. K. Clifford, it was lost until the quantum mechanics of the electron was discovered 49 years later.

The fourth chapter deals with the relationship of $\mathbb{G}$ to the matrix algebras and to differential forms. Chapter five then shows the relationship to octonions and the series of sedenions and chapter six exposes a natural connection of $\mathbb{G}$ to the simplest Lie Exceptional algebra, G2. With this background, chapter seven proceeds to classify the products of all these algebras which exposes a wealth of non-associative algebras with only a few currently being studied. I hope the classification of these algebras provides incentive for further investigation.

2 The Algebra of Geometry

Geometry's intimate connection to the geometric algebra $\mathbb{G}$ starts with Pascal's Triangle which is usually introduced as the expansion of polynomials called the binomial theorem. At the end of this chapter the binomial theorem will be described from a geometric viewpoint and the expansion shown to justify the proof of Pascal's Triangle's connection to $\mathbb{G}$. For now this is presented as a table of numbers following rules similar to Fibonacci's Numbers. These are a sequence of numbers generated by a simple rule; each number is the sum of the two preceding numbers assuming the sequence starts with a zero and a one. This is called a recurrence relation which is expressed as

$$F_n = F_{n-1} + F_{n-2}, \text{ where } F_0 = 0 \text{ and } F_1 = 1.$$

and provides the series 0, 1, 1, 2, 3, 5, 8, ... This sequence has many applications since it describes the idealised birth rate of pairs of rabbits and is related to the golden ratio. The sequence generally does not start at 0 but it is important for Pascal's Triangle below and it makes sense if starting with one pair of rabbits. The relationship to the golden ratio allows Fibonacci's Numbers to describe patterns in nature such as the number of petals of a flower or the number of seeds in the spirals of a sunflower.

Pascal's Triangle extends Fibonacci's Numbers to two dimensions using a table of numbers governed by a similarly simple rule; each number is the sum of the two numbers immediately above it, starting with a row of zeros and with a single one. This rule assumes a symmetric or isosceles triangle but skewing it into a right triangle shape allows the columns to be labelled. This means with 1 at the top of the table and not showing zeros then for each row add the number above and the number above to the left. The resulting table, shown in Table 1, has Fibonacci's Numbers as the sum of each diagonal that slopes down to the left but this is not relevant here since we are only interested in each row separately.

Table 1: Pascal's Triangle

N	**V**	**E**	**F**	$\cdots$				
1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The rule for each cell, $P_{r,c}$, written as a recursion relationship from the previous row $(r-1)$ is

$$P_{r,c} = P_{r-1,c} + P_{r-1,c-1}, \text{ where } r \text{ and } c \text{ are row and column positions, respectively.}$$

The first column under the initial one is all 1's and labelled as N to mean nothing, which will be justified later. The next 3 columns are labelled **V**, **E** and **F** for Vertices, Edges and Faces, respectively. These labels describe the number of elements of n-simplices which are generalisations of triangles to n dimensions with $\mathbf{V} = n+1$. An n-simplex is defined as the simplest polytope in n dimensions. Polytopes generalise polyhedrons but here we are not interested in hypercubes or hyper octahedrons, etc, just hyper tetrahedrons.

Ignoring the first column in Pascal's Triangle and hence the first row for now, the second row, $\mathbf{V} = 1$, specifying zero dimensions has 1 vertex. This is the tip of a triangle, a single point, and that's the only possibility in a space with no dimensions. Next is 2 points and 1 edge or line segment. This is a 1-D triangle and a line segment is, at most, all that would be visible of a triangle in one dimension. The fourth row, $\mathbf{V} = 3$, is a triangle with 3 edges and 1 face. Next is a tetrahedron or 3-simplex with 4 faces. Each simplex matches the numbers in Table 1 and are shown in Figure 1.

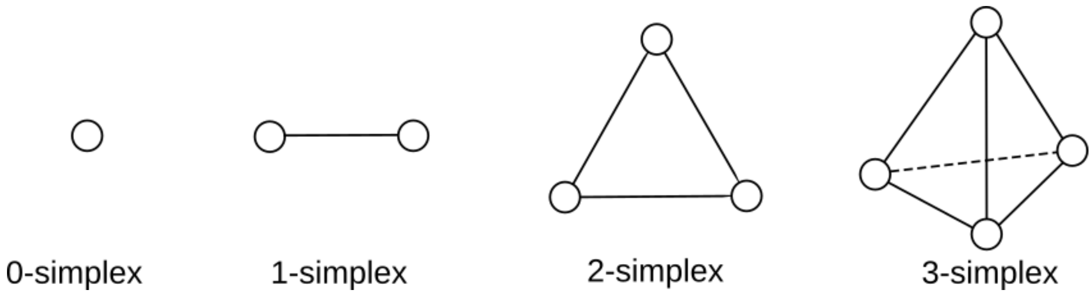

Figure 1: **The first four simplices**

The 3-simplex is drawn using Projective Geometry since it is a 3-D object displayed flat on the page in 2-D. We can go further by putting a point in the middle of the tetrahedron and connecting all vertices to this point. This is a 5-simplex shown in Figure 2 with the internal connections shown as dotted lines and again using projections now you are thinking in four dimensions. This is a four dimensional object displayed in the two dimensions of the page. The diagram shows 10 edges and 10 faces again matching the $\mathbf{V} = 5$ row of Pascal's Triangle. The diagram also shows that there are 5 tetrahedrons which is the next column in Table 1.

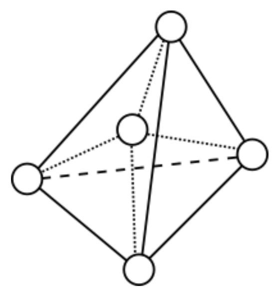

Figure 2: **The 4-simplex**

Names can be introduced for all these geometric elements in order to make it easier to proceed to higher dimensions. A single point is designed e_1 and two points as e_1 and e_2 with connecting edge labelled as e_{12}. With a third vertex, e_3, the extra edges are e_{23} and e_{31} and the face is e_{123}. This is shown in Figure 3 but it is also important to specify the ordering of points. The face e_{123} shown in Figure 3 designates a clockwise ordering of the points while e_{321} would show an anticlockwise order. Such an arrow defines a 3-cycle because starting from e_1 there are three steps to get back to e_1.

Starting with a triangle with face e_{123} we can add a point, e_4, in the middle and connect this to the starting three points. Figure 1 shows a tetrahedron looking from the top or a projection of the tetrahedron onto the plane of e_{123}. For clarity only the point labels are shown. The 3-simplex, by Pascal's Triangle, has another three edges labelled e_{14}, e_{24} and e_{34} and four faces, e_{123}, e_{124}, e_{134} and e_{234}. Later the last element on each row will be called the pseudo-scalar, e_{1234}, in this case, which corresponds to the whole tetrahedron. This is a good example to show pseudo symmetry. Removing each vertex from e_{1234} gives four triangles, e_{123}, e_{124}, e_{134} and e_{234}. These are called pseudo-vertices since they match the symmetry shown in Pascal's Triangle.

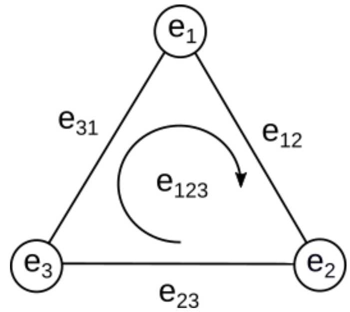

Figure 3: **Clockwise order**

Returning to the 3-simplex, it can't have four 3-cycles, one for each face, because then the labelling would be inconsistent. Only two faces can be consistent, so choose the clockwise one on the bottom, e_{123}, which is actually anticlockwise when looking from below, and, for example, another clockwise from the outside on side, e_{143}. The shared edge has the same direction for the arrows so both 3-cycles can be labelled using e_{31}. No other 3-cycle can be chosen and keep this consistency on the shared edges. But this scheme provides a constraint on the 3-cycles. Changing e_{123} to e_{321} means e_{143} must also change to e_{134}. In Chapter 5 we will find that keeping the sense of the 3-cycles independent is an important concept. In fact leaving them undefined generates $\mathbb{G}$ and constraining them generates different algebras such as octonians for the 6-simplex. So the tetrahedron can only have one independent 3-cycle. The 4-simplex in Figure 2 can have two, one outside and one inside and, using projections, we can prove this and extend it to further dimensions.

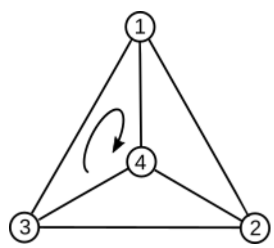

Figure 4: **The 3-simplex projection**

Starting with the tetrahedron with a base of e_{123} and apex e_4 and a 3-cycle chosen as e_{143}, as shown in Figure 4, then the tetrahedron can be tilted so the apex is above e_{13}. The 3-cycle will disappear so to avoid the loss of information an arrow is added to the line to indicate that e_{143} is a 3-cycle, as shown in Figure 5.

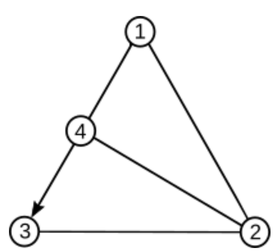

Figure 5: **The 3-simplex tilted projection**

This is now flattened by projecting it onto the plane of e_{123}. Similar to adding a point to the middle of triangle to get a 3-simplex, a point, e_5, can be added into the projected tilted 3-simplex and connected to all other points. The face e_{125} is independent so can be labelled as a 3-cycle and the simplex tilted towards e_{12}. The 3-cycle becomes an arrow and again projected down to the base plane. The resulting 4-simplex is shown in Figure 6 with the new internal connections shown as dotted lines. The figure shows that the 4-simplex has two independent 3-cycles.

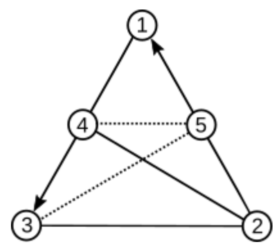

Figure 6: **The 4-simplex projection**

Another point, e_6, can be added and the process repeated for the edge e_{23} to get e_{236} which is also an independent 3-cycle. But we now have an extra independent 3-cycle, e_{456}, shown with dotted lines in the 5-simplex of Figure 7, to give 4 independent 3-cycles. The dotted lines are replaced with an arrowed circle before the final point, e_7, is added above the flattened 5-simplex. This automatically introduces another three independent 3-cycles for a total of seven for the 6-simplex. These are denoted e_{143}, e_{125}, e_{236}, e_{456}, e_{176}, e_{274}, e_{375} and show the remarkable coincidence that each vertex is listed three times and each edge is listed once. This will be exploited in Chapter 5 to define 7 dimensional algebras based on the example for quaternions introduced shortly.

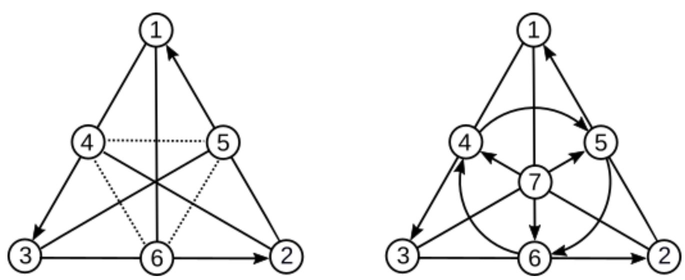

Figure 7: **The 5 and 6-simplex projections**

The resulting diagram for the 6-simplex projection with 7 independent 3-cycles is called the Fano Plane and is usually interpreted as the projection of the cube in dual Projective Geometry where the 3-cycles and points can swap places.

Row $\mathbf{V} = 7$ of Pascal's Triangle shows that the 6-simplex has 21 edges and 35 faces. The Fano Plane diagram shows 15 edges and seven 3-cycles and 28 other labelled triangles and so has hidden 6 edges but all faces can be observed. The missing edges are e_{12}, e_{23}, e_{13}, e_{16}, e_{24} and e_{35} while some of the observed faces, such as e_{156}, may be hard to spot. Chapter 5 shows that there are 30 such unique projections with different edges missing and that this covers all possible projections that can be represented by the Fano Plane.

You may have noticed that the sense of 3-cycles provides a sense for each edge. Each edge in any simplex is also a 1-simplex that connects two vertices so the connection e_{12} can be interpreted as not just connecting e_1 and e_2 but as the operation of changing e_1 into e_2, under the action of some geometric operation, $e_1 e_{12} \to e_2$.

This is inferred by the 3-cycle e_{123} but with the opposite sense, e_{321}, would mean the reverse operation, $e_2 e_{21} \to e_1$. This will be explored shortly.

Now continuous Perspective Geometry can be used to interpret points as lines, lines as areas, faces as volumes, etc. This is achieved by re-labelling the columns in Table 1 from **V**, **E**, **F** to Dimension, Area, Volume, respectively. The dimensions labelled as $e_1, e_2, \ldots$ represent arrows in space. Once this is done the 4-simplex allows you to think in 5 dimensions. The line segments connecting each dimension now describe two dimensional areas but are now easier to visualise as rotations between these dimensions. Furthermore, each face of a simplex describes a volume but also defines a 3-cycle of rotations. In this perspective space the simplex labels are called n-forms, arrows are 1-forms, areas are 2-forms, etc. Defining a geometric product between 1-forms and 2-forms allows the 3-cycle, e_{123}, to mean $e_1 \to e_2 \to e_3$, which can be represented as $e_1 e_{12} = e_2$, $e_2 e_{23} = e_3$, $e_3 e_{31} = e_1$. To understand e_{12}, e_{23}, e_{31} further the algebra of rotations needs to be considered and this is where quaternions are involved.

Hamilton discovered quaternions, now denoted $\mathbb{H}$ in his honour, while looking for the algebra of rotations in three dimensions. The complex number, i, provides rotations in two dimensions, as shown by Argand in 1806 and published in 1813. Hamilton spent years looking for the three dimensional equivalent. In 1843 he found the solution using 3 complex numbers which can be summarised as $ij = k$, where i is the rotation down one wall, j is the rotation across the floor and k is the rotation down the other wall. Each of these are special rotations of 90° so that if we imagine an upwards pointing arrow in the corner of the room then a 90° downward rotation will change it to point along the floor. Hence a 90° rotation down one wall and another across the floor will produce the same effect as rotating the arrow down the other wall. This is the meaning of $ij = k$. Quaternions also have cyclic rules for each triple of operations, $jk = i$ and $ki = j$.

Notice we have not specified which wall to use and quaternions are agnostic to the order of the three rotations. It is only in their interaction with arrows that the sense is defined. Here I have talked about arrows but technically I should have specified vectors. The difference is that arrows can move around and be anywhere in space, the so called Affine Space, but vectors are fixed to a single point, called the origin which defines a Vector Space. It is convenient for you to choose this origin to be in the corner of your room.

From this description we can see that i, j and k are not ordinary numbers but that they operate on vectors. We will see later when compared to $\mathbb{G}$ they also operate on themselves as 90° rotations so we could call them numbers with strange rules but operators is preferable. These operators all have the property that the square is negative meaning they are called imaginary,

$$i^2 = j^2 = k^2 = -1.$$

This is easy to see because i operating on an upwards pointing vector moves its direction down the wall to point along the floor. Operating with i again will rotate this vector so that it points downwards, opposite to the direction it started. So we now have four ways of getting an arrow to point in the opposite direction. Multiplying by -1 is the zero dimension approach. Reflecting by any point within the arrow is the 1-D solution. Rotating twice around any dimension not in the line is the 2-D way and now we can operate with i, then j, then k. This is the meaning of $ijk = -1$ which is derived by operating on $ij = k$ by k on both sides. The equation $i^2 = j^2 = k^2 = ijk = -1$ was famously inscribed by Hamilton into the Broom (originally Bougham) Bridge in Dublin when he first discovered the algebra of rotations. More of this in Chapter 3 but now with the visualisation of these operators we can uncover another property that distinguishes quaternions from normal numbers.

After rotating the upwards vector by i and j how do we go backwards. We need to operate with $-k$ or by $-j$ then $-i$ to get back to the start. Hence $-k = (-j)(-i) = ji$ but $-k = -ij$ so $ij = -ji$, the rotation operators anticommute. Geometrically, this is the statement that if a rotation plane is inverted then the rotation changes sign. A clockwise rotation becomes anticlockwise and vice versa. This is a natural consequence of geometry and can now be applied to the 2-forms, introduced above. We have defined these to be $90°$ rotations so should have the property $e_{12} = -e_{21}$ and $e_{12}^2 = -e_{12}e_{21} = -1$. The 2-forms, e_{32}, e_{21} and e_{31}, should be equivalent to i, j and k, given the previous geometric interpretation of i changing e_3 to e_2, j changing e_2 to e_1 and k changing e_3 to e_1. Hence the rule $ij = k$ becomes $e_{32}e_{21} = e_{31}$, and similarly for the other quaternion cycle rules. Also $ij = -ji$ simplifies to $e_{31} = -e_{13}$.

It is not hard to see that with an appropriate geometric product then

$$e_{12} = e_1e_2 \text{ and } e_1^2 = e_2^2 = e_3^2 = 1.$$

So an area is defined by multiplying two different 1-forms and multiplying a 1-from by itself gives a positive number under the geometric product. Changing the order of multiplication means turning the area over which changes the sign. This argument assumes distributivity of the 1-form product, as can be seen when e_{13} rotates e_{12},

$$e_{13}e_{12} = (e_1e_3)(e_1e_2) = (e_1e_3e_1)e_2 = -(e_1e_1e_3)e_2 = -e_3e_2 = e_{23}.$$

This must be taken as an axiom but since simplices can be built up from the binomial theorem, as seen soon, there is no inconsistency in this definition.

The forms e_{13}, e_{12}, e_{23} can easily be seen to generate a 3-cycle of 2-forms as each rotates the next in turn which is the same as quaternions. The standard way of diagrammatically showing quaternion multiplication is shown in Figure 8 which demonstrates the 3-cycle. The diagram for $-i, -k, -j$ is also shown to expose the ambiguity of the sense of quaternions. Chapter 2 shows that quaternions generate a negative cross product but this makes sense since these are "imaginary". Imaginary is not the appropriate term for these operators since they don't commute. Negative length is more appropriate to distinguish such operators and Chapter 2 shows that this also specifies time-like operations in space-time.

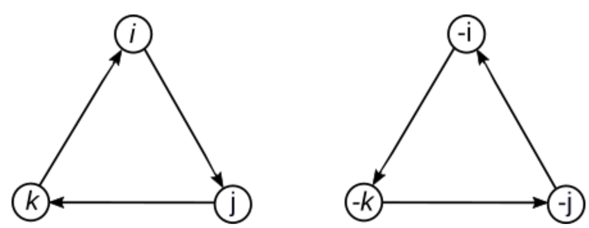

Figure 8: **Standard Quaternion Geometry**

But these diagrams do not correspond with the 2-simplex view described above. It only shows the operators as rotation planes and ignores vectors or dimensions which are shown as nodes above. The previous geometric interpretation of quaternion multiplication of vectors was given in terms of $ij = k$ operating on a vector pointing upwards. This description is best shown using $jk = i$ instead, as shown in Figure 9. Here the quaternions act as rotations on the vectors e_1, e_2, e_3, with e_3 pointing upwards. But the vectors are not part of the algebra, so generally, the following substitution is made, $i = e_1$, $j = e_2$, $k = e_3$. But this replacement in the figure breaks the multiplication rule unless $i = -e_1$ is used, as shown in the second diagram in Figure 9. Substituting i for $-i$ changes the direction of both arrows impinging on the i node thus providing a 3-cycle seen in the third diagram of Figure 9 but only if i represents $-e_1$.

This shows that the quaternions when operating on vectors do not form a 3-cycle because the triple product is $ijk = -1$. To return a vector back to the start via a 3-cycle, $i'jk = 1$, needs

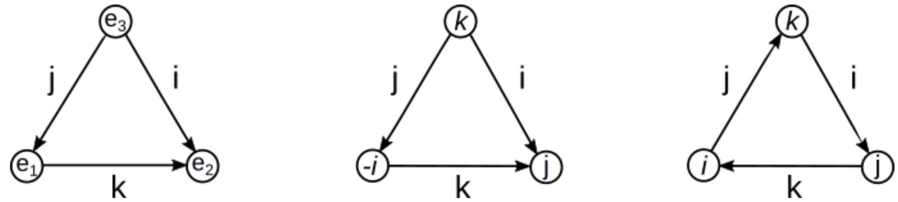

Figure 9: **Quaternion 2-simplex Geometry**

$$i'j = -k, \text{ where } i' = -i.$$

Hence the interpretation of vectors and rotation multiplication in quaternions is different. The standard geometry in Figure 8 uses just the vertices as edges of simplices which is confusing and requires the $\mathbb{G}$ notation to separate the two concepts. In $\mathbb{G}$ the arrows are unconstrained because the edges are labelled and the label can change sense. However, moving away from $\mathbb{G}$ then constraining the arrows defines a new algebra. With labels $p_1, p_2, p_3, \ldots$ of just the vertices of the simplices then for the 2-simplex or triangle there are 6 combinations for the directions of the three connecting arrows. Of course, these map to quaternions or the negative quaternions, $-i, -j, -k$. For higher dimension simplices we need independent 3-cycles to avoid multiplication conflicts but Figure 2 shows that not all vertices are connected by arrows. The multiplication table would have holes for p_2p_3, for example. The 6-simplex in Figure 7 shows the 7 independent arrows connect each node and this provides a complete multiplication table and depending on the direction of the arrows can define octonions $\mathbb{O}$. The same thing happens in 15 and 31 dimensions and Chapter 5 investigates all the associated algebras defined by Clifford algebra.

The 1-forms defined by the nodes of simplices are called vectors and are usually denoted with arrows $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$, blackboard notation, $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$, or bold face $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Since we are dealing with multiple dimensional n-forms it is convenient to avoid such notation since this distinguishes n-forms from the usual matrix notation.

Another important concept of the algebra is that n-forms play dual roles. They represent $(n+1)$-simplices as objects in space with signs that change the orientation. A negative arrow points in the opposite direction, a negative plane is one that is turned upside down, a negative volume defines a left-hand screw and numbers, which we will see define the lengths or sizes of these objects, have negatives that indicate other properties. Each of these n-forms also represent an operation. We have seen 2-forms act as rotations and this will become more explicit in Chapter 2. The geometric product of a 3-form swaps between 1 and 2-forms. For example

$$e_{123}e_3 = e_{12}e_3^2 = e_{12} \text{ and } e_{123}e_{12} = -e_1^2e_2^2e_3 = -e_3.$$

So the geometric action of a 3-form is to swap planes and lines. Chapter 2 shows that a 1-form acts as a reflection and also shows the role that the pseudo-scalar 4-form plays in the physics of space-time. This geometric concept includes numbers, denoted $\mathbb{R}$, that operate by magnifying or shrinking the size of objects and also changing the sense of other operations. Simplices do not include numbers because these would be denoted as a dimension of -1 and obviously raise the question of further negative dimensions. But $\mathbb{G}$ does include numbers, denoting them as 1's in the $\mathbf{N}$ column of Table 1 and just like e_1 is the basis of all vectors pointing in the e_1 direction, 1 is the basis of $\mathbb{R}$, the set of real numbers. The sole 1 in the first row of the table was interpreted as none above, meaning there is no simplex. But this doesn't make sense for subsequent rows. So Pascal's Triangle more closely represents the algebra than simplices and this leads to an interesting interpretation of the proof of Pascal's Triangle's recursion relationship. The proof uses the binomial theorem, which represents each element of Pascal's Triangle, $P_{r,c}$, as coefficients of the following polynomial expansion.

$$(1+x)^n = 1 + P_{n,1}x + P_{n,2}x^2 + \cdots + P_{n,n}x^n.$$

This is proved using the recursion

$$B_n = (1+x)B_{n-1}, \text{ where } B_n = (1+x)^n \text{ so that } B_0 = 1.$$

The dimensional approach provides a geometric re-interpretation of this relationship. Using Projective Geometry where points become dimensions allows the single 1 in the first row of Table 1 to represent emptiness without space. For want of a better name I shall call this the void and every possible number of dimensions given by subsequent rows includes the void. The x in the recurrence relationship for B_n above indicates a single dimension and x^2 implies two dimensions, etc. The coefficient of x^n indicates the cardinality of each bundle of n grouped dimensions which is the number of inside $(n-1)$-simplices or n-forms. Multiplying $(1+x)$ by B_{n-1} as $(1 +$ "x terms in B_{n-1}") gives four parts for B_n,

- void squared ($1^2 = 1$) —- meaning there is only one void,
- void times the "x terms in B_{n-1}" — replicates all the existing simplices of B_{n-1},
- x times void — adds the new dimension to the new space, and
- x times "x terms in B_{n-1}" — connects the new dimension to all existing simplices.

The first and thirds items in this list correspond to the first two columns of Table 1. The other two items correspond to the two parts of the Pascal's Triangle's recursion relationship applied to all the other columns. The existing simplices of the same size are in the same column and a connection to simplices one size smaller are in the column to the left, both from the row above. Hence the $P_{n,c}$ factors indicate how many inner c-simplices exist in n dimensions plus the void. Each simplex connects pairs of dimensions and this is exactly what is expected for the definition of a metric and this is where the most important definition of the Pfaffian is involved at the heart of $\mathbb{G}$. The Pfaffian is a rule applied to all the edges of a simplex that generates a pseudo-scalar. This rule is easier to explain for even number of vertices or odd simplices. All the edges are listed in the form of half a matrix. The rule for a 1-simplex is trivial, $\backslash e_{12}| = e_{12}$ where the vertical lines represents the Pfaffian. For a 3-simplex the expansion is

$$\left| \begin{matrix} e_{12} & e_{13} & e_{14} \\ & e_{23} & e_{24} \\ & & e_{34} \end{matrix} \right| = e_{12}e_{34} - e_{13}e_{24} + e_{14}e_{23}$$

The Pfaffian can be defined using a recurrence rule. It is easy to see that if we add another column then it would include all four 2-forms in the 4-simplex. As noted this expansion needs an odd simplex so the Pfaffian for the 5-simplex has another five 2-forms and a shorthand notation is introduced showing just the lower diagonal

$$\left| \begin{matrix} e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ & e_{23} & e_{24} & e_{25} & e_{26} \\ & & e_{34} & e_{35} & e_{36} \\ & & & e_{45} & e_{46} \\ & & & & e_{56} \end{matrix} \right| = \backslash e_{12}, e_{23}, e_{34}, e_{45}, e_{56}|.$$

The recurrence relation here expands each signed term of the newly added column with the smaller Pfaffian without these indices. This is called the co-factor expansion, for n even,

$$
\begin{aligned}
\backslash e_{12}, e_{23}, \dots e_{(n-1)n}| \quad = \quad & e_{1n} \backslash e_{23}, e_{34}, \dots, e_{(n-2)(n-1)}| \\
+ \quad & \sum_{j=2}^{n-2} (-1)^{(j-1)} e_{jn} \backslash \dots, e_{(j-1)(j+1)}, \dots| \ . \\
+ \quad & e_{(n-1)n} \backslash e_{12}, e_{23}, \dots, e_{(n-3)(n-2)}|
\end{aligned}
\tag{1}
$$

The first and last rows cover j missing in the Pfaffian as the first or last index otherwise j is missing from the middle of the pairs of indicies. So the criteria is that as j sequences over 1 though $(n-1)$ then j and n indices are removed from the original Pfaffian by removing the last column, the row with j as the first index as well as the column with j as the second index, if it exists. For the 5-simplex above this is

$$
e_{16} \backslash e_{23}, e_{34}, e_{45}| - e_{26} \backslash e_{13}, e_{34}, e_{45}| + e_{36} \backslash e_{12}, e_{24}, e_{45}| - e_{46} \backslash e_{12}, e_{23}, e_{35}| + e_{56} \backslash e_{12}, e_{23}, e_{34}|.
$$

The last row of the co-factor expansion applied repeatedly gives the last term of the Pfaffian expansion as

$$
e_{12}e_{34} \dots e_{(n-1)n} = \ \backslash e_{12}, 0, e_{34}, 0, \dots, 0, e_{(n-1)n}| \tag{2}
$$

where all the other terms are zeros. This is an important result when extended to antisymmetric matrices and can be used to show that the determinant applied to simplices defines the square of the hyper-volume defined by an $(n-1)$-simplex, as demonstrated in the next chapter.

Now that the Pfaffians for odd simplices are known there is a trick due to de Bruijn to derive the Pfaffian for even simplices. This just sets the last column to 1's. So the Pfaffian for a 2-simplex is $|e_{12}| - |e_{13}| + |e_{23}|$. This still connects all vertices and is inherent in the structure of $\mathbb{G}$ but is not applicable to this structure because the algebra has fundamentally different properties in the even and odd cases.

The important property of the Pfaffian is that the square of the Pfaffian equals the determinant of a skew-symmetric matrix which is what a simplex describes. So the Pfaffian gives the hyper-volume of the Perspective simplex edges as 2-forms. This seems strange that the edges of a simplex defined as connections between vectors in $(n-1)$ different dimensions can give the hyper-volume of the shape with vectors in n dimensions as edges. For example, the 2-simplex has a triangular area which the Pfaffian maps to the volume of the cube with edges e_1, e_2 and e_3. The explination of this needs to be left to the next chapter where the connection of Pfaffians to $\mathbb{G}$ is shown. It is well known that the determinant defines the hyper-volume for n vectors and this is normally shown as a matrix calculation. But is is more fundamentally a sum of all permutations of the n vectors in what is called an n-form of the exterior algebra, also known as Grassmann algebra. This algebra does not include a metric and products of forms generate higher dimensional forms or, if the forms overlap, zero. As seen above forms can be generated in $\mathbb{G}$ without reverting to determinants or permutations making working with them easier.

There is an easy notation to make this obvious in terms of permutations. All $n!$ permutations of the set of numbers $\mathbb{N}_1^n = \{1, 2, \dots, n\}$ are represented as $[1, 2, 3, \dots, n]$. If we introduce normal, round brackets to restrict all permutations to those which have increasing order within the brackets then we can, for example, specify the $n(n-1)/2$ combinations $\binom{n}{2}$ as $[(1, 2), 3, 4, \dots, n]$, if just the numbers within the normal brackets are taken and the rest ignored. Partitions are represented with multiple brackets using all permutated numbers ordered within brackets. For $\mathbb{N}_1^n$ divided into two parts of sizes $r+s = n$ then $[(1, 2, \dots, r)(r+1, \dots, r+s)]$ provides the unique identifiers for each set. The Pfaffian takes one more step. The another set of normal brackets surround brackets whose first element must be in increasing order $[((1, 2), (3, 4), \dots, (n-1, n))]$ for n even. From this it is easy to see that if n is removed then this will effect each pair so $j \in \mathbb{N}_1^{n-1}$ needs to also be removed giving the e_{jn} terms of the co-factor expansion (1).

Including a permutation section at the end allows determinants to be declared as well

$$\mathcal{P}_{n,r} = [((1,2),(3,4),\ldots,(r-1,r)), r+1,\ldots,n]$$

For index numbers over $\mathbb{N}_s^{n+s}$ the Pfaffian of degree r is represented as $\mathcal{P}_{n,r}^{s,n+s}$. The general Pfaffian expansion, split into r and $n-r$ sized parts, for $n > r$ both even, can be expressed as

$$\begin{aligned}
&\sum_{\mu\in\mathcal{P}_{n,n}} e_{\mu_1\mu_2}\ldots e_{\mu_{n-1}\mu_n} \\
&\qquad = \sum_{\mu\in\mathcal{P}_{r,r}} \sum_{\nu\in\mathcal{P}^{r+1,n}_{n-r,n-r}} (-1)^\sigma e_{\mu_1\mu_2}\ldots e_{\mu_{r-1}\mu_r} e_{\nu_1\nu_2}\ldots e_{\nu_(n-r-1)\nu_{n-r}} \\
&\qquad - \sum_{\mu\in\mathcal{P}_{r,r-2}} \sum_{\nu\in\mathcal{P}^{r+1,n}_{n-r,n-r-2}} \frac{(-1)^\sigma}{2!} e_{\mu_1\mu_2}\ldots e_{\mu_{r-3}\mu_{r-2}} e_{\nu_1\nu_2}\ldots e_{\nu_{n-r-3}\nu_{n-r-2}} e_{\mu_{r-1}\nu_{n-r-1}} e_{\mu_r\nu_{n-r}} \\
&\qquad + \ldots
\end{aligned} \tag{3}$$

which continues until there is a single determinant if $n = 2r$

$$\pm \sum_{\mu\in\mathcal{P}_{r,0}} \sum_{\nu\in\mathcal{P}^{r+1,n}_{r,0}} \frac{(-1)^\sigma}{r!} e_{\mu_1\nu_{r+1}}\ldots e_{\mu_r\nu_n}$$

or a Pfaffian one side and the remaining terms as a determinant. The σ term gives the parity of μ and ν. For the 5-simplex above with $r = 2$ ($n = 6$) and noting that [((3,4)),5,6] = [3,4,((5,6))] then

$$[((1,2),(3,4),(5,6))] = [((1,2))][((3,4),(5,6))] - [1,2][3,4,((5,6))]/2!$$

where $[1,2][3,4]/2!$ indicates the determinant $e_{13}e_{24} - e_{14}e_{23}$. Of course $-[3,4,((5,6))]$ has the six Pfaffians $\backslash -e_{56}|$, $\backslash e_{46}|$, $\backslash -e_{36}|$, $\backslash e_{35}|$, $\backslash -e_{45}|$ and $\backslash -e_{34}|$ and associated determinants. The first expansion term with two Pfaffians is

$$\backslash e_{12}|\, \backslash e_{34}, e_{45}, e_{56}| = e_{12}(e_{34}e_{56} - e_{35}e_{46} + e_{36}e_{45})$$

Identifying the e_{36}, e_{46} and e_{56} terms here and adding to the first three of the six Pfaffians, along with the determinant part, gives the last three terms in the 5-simplex co-factor expansion from (1). The remaining terms are

$$(e_{14}e_{26} - e_{16}e_{24})e_{35} - (e_{13}e_{26} - e_{16}e_{23})e_{45} - (e_{15}e_{26} - e_{16}e_{25})e_{34}$$

which can be rearranged to give the remaining e_{16} and e_{26} terms of the 5-simplex expansion. The ifirst line ion the right hand side of (3) is the product of two Pfaffians so all the other terms ($[1,2][3,4,((5,6))]/2!$ in the 5-simplex case) cancel terms from the original Pfaffian.

Pfaffians expose the mathematics behind the geometry of all connections between vertices of a simplex. When the edges are replaced with the metric and added to the exterior algebra then geometry naturally leads to Clifford algebra. Pfaffians describe simplices and so do not include the first column of Pascal's Triangle described as the void or empty space. It is only when the metric is added that we get scalars and can explain the first column. Pascal's Triangle tells us that in every dimensional space the void always exists and above 1 in the table are only zeros. This can be interpreted as saying nothing does not exist; once the void existed other dimension can be added. Of course this is not proof of the non-existence of nothing but it is interesting that mathematics distinguishes between the concepts of void and nothingness. Alternatively, if the algebra describes reality then nothingness is outside of reality and only exists as a mathematical concept or, logically, be unobservable to objects within dimensional space. As we will see over and over again, there is nothing new in the mathematics here, it is the geometric insights that make it easier to comprehend.

3 Geometric Algebra

The geometric algebra, $\mathbb{G}$, has many imaginary numbers and not just multiples of i or addition with real numbers to form complex numbers. Here it is important to recognise geometric operations and distinguish these from numbers. The distinction is that numbers commute and operators do not commute. Commutation for numbers means multiplication is independant of order. So $ab = ba$, for real numbers $a, b \in \mathbb{R}$. Of course numbers act as operations of magnification, shrinking or a change of sense but an operator that doesn't commute means the geometry has changed. The first chapter introduces the quaternions as operators of rotations with the property $ij = -ji$ which is called anticommutation. Chapter 1 has also introduced psuedo-scalars acting as operators but if these have negative length and commute then the symmetry of Pascal's Triangle means half of the algebra is defined as psuedo-forms using a imaginary number unique to the algebra.

In $\mathbb{G}$, negative length means the square is negative, $p^2 < 0$. If p commutes with all the elements of the algebra it will be called imaginary, otherwise, it is an operator. Here we will start with the historical definition of imaginary numbers. There is really only one that commutes with all of $\mathbb{G}$, the square root of -1, $\sqrt{-1}$, which is imaginary $\sqrt{-1}^2 = \sqrt{-1}\sqrt{-1} = -1$. Scaling and adding any real number, $x + y\sqrt{-1}, x, y \in \mathbb{R}$ introduces complex numbers, $\mathbb{C}$. Such numbers confused mathematicians for hundreds of years and they tried to work around them. It was only when they were accepted as normal numbers that the beginnings of modern mathematics could finally solve all quadratic equations.

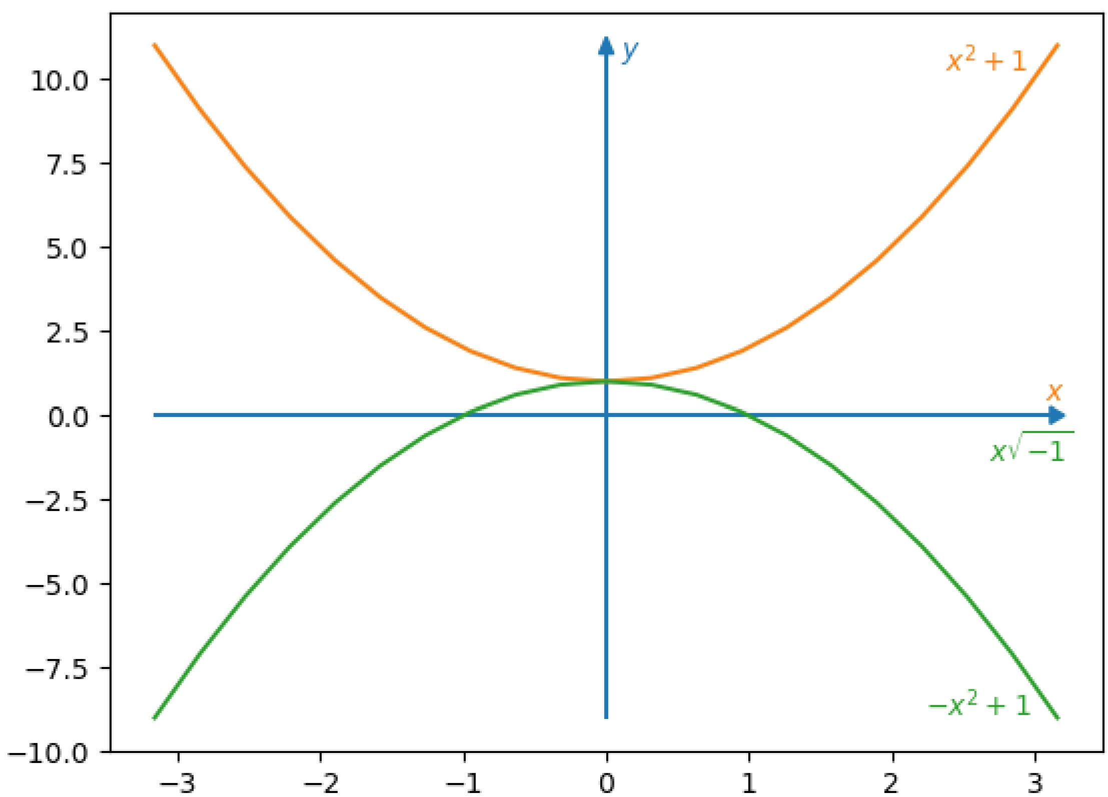

Figure 10: **The first imaginary number**

The problem can be demonstrated with the simple quadratic equation $y = x^2 + 1$ as shown in Figure 10. There is a well known equation that provides the solutions to the location where any quadratic equation crosses the horizontal x axis. But, as can be seen in the figure, the top curve does not cross the x axis so the solutions to the quadratic equation formula are imaginary, $\sqrt{-1}$ and $-\sqrt{-1}$. That's why they are called imaginary which is better than calling them absurd or impossible numbers which was fortunately corrected by Descartes.

Applying the transformation $x \to x\sqrt{-1}$ changes the quatratic equation to $y = -x^2 + 1$ with the crossing points $x = 1$ and $x = -1$ as shown in the lower curve in Figure 10. This transformation is a rotation of $180°$. More generally for $y = x^2 + bx + c$ the transformation $x \to x\sqrt{-1} - \frac{b}{2}(1 + \sqrt{-1})$ can be used.

This is not the same as the rotation for complex numbers in the 2-dimensional complex plane $\{1, i\}$ seen in the Argand diagram shown in Figure 11. The real numbers, $\mathbb{R}$, lie on the horizontal dimension and imaginary numbers on the vertical line with basis vectors shown bold. Multiplying a real number, say 1, by i gives $\boldsymbol{i}$, which is a rotation of $90°$. Multiplying by i again gives -1 because $i^2 = ii = -1$ (note that algebraic multiplication does not have a symbol so that $i + i = 2i$). This is another rotation of $90°$ and demonstrates the relationship between imaginary numbers and reflections. Another two multiplcations by i is another reflection and returns to the starting point. These two numbers share the imaginary property and so have been seen as equivalent. But geometrically these are different rotations and one can be seen as an extension of the real numbers, $\mathbb{R}$, and the other as an operator that changes the geometry.

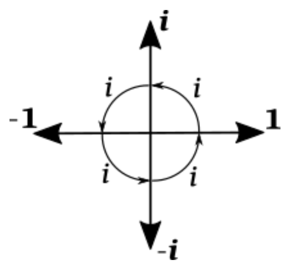

Figure 11: **The second imaginary number**

After Argent's success, and others, with rotations in 2-dimensions using i, Hamilton decided to introduce another imaginary number, j, $(j^2 = -1)$. He was looking for rotations in 3-dimensions. Of course, this produces another number, ij, which Hamilton called k. The set i, j, k defines the rotations in 3-dimensions if they satisfy the following cyclic multiplication property: $ij = k$, $jk = i$, $ki = j$. This is possible only if $k^2 = -1$, another imaginary number. The set $\{1, i, j, k\}$ is the basis of quaternions, $\mathbb{H}$, and these can be identified as operators rather than numbers. Even the number 1 is an operator which generates no rotation and it's inclusion extends i, j and k to all rotations, not just $90°$.

Hamilton then introduced quaternion vectors using a different notation to the modern one used here. A quaternion vector, $\mathbf{p}$, is $\mathbf{p} = p_1 i + p_2 j + p_3 k$ where (p_1, p_2, p_3) are the coordinates of a point in i, j, k space. These could be called directed numbers but could equally be called directed areas so the term shall be avoided. The fundamental equation for the product of $\mathbf{p}$ and another quaternion vector, $\mathbf{q}$ is

$$\mathbf{pq} = \mathbf{p} \times \mathbf{q} - \mathbf{p} \cdot \mathbf{q}.$$

Note that general multiplication is shown without a symbol while $\times$ is the cross product and $\cdot$ the dot product. The dot product denotes the length for vectors and as can be seen, is negative. The length is usually understood to be positive so this equation did not make sense at the time. Furthermore, in order to satisfy the cyclic multiplication rotations rules they are anticommutative, as shown in Chapter 1. This was another strange property for a vector algebra. The progression followed by Hamilton of adding an extra imaginary operator continued to octonions and more recently sedenions and these algebras have even stranger properties that will be explored in Chapter 6. Octonions were discovered at the same time as matrix algebra was being developed and the two approaches to understanding rotations split the mathematics world. The modern vector algebra aquired the products and triplet notation of quaternions and incorporated them into matrix notation. This split escalated into what is known as the vector algebra war which is discussed in the next chapter. It is unfortunate that Clifford developed his algebra during this time but it was not recognised and was lost until matrix spinors were discovered 45 years later.

Hamilton also introduced the versor and the expression of this object in $\mathbb{G}$ is fundamental to understanding rotations. Instead of adding more imaginary operators, $\mathbb{G}$ introduces them along with geometric properties. The simplex approach of Chapter 1 introduced this geometry and here we will follow the same path introducing $\mathbb{G}$ for each dimension. Note that if the quaternion multiplication was changed to obey the rules for a 3-cycle with i replaced with $i' = -i$ then the cross product in the above product of quaternions would become negative, the same as the dot product. This is just an example where the properties of quaternions are better explained by Clifford's notation. The notation presented here avoids vector notation for basis n-forms and this is applicable for arbitrary forms apart from combined 1-forms where it is convenient to use vector notation in n dimensional space

$$\mathbf{a} = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$

with $(a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, known as the Vector Space, and $\{e_1, e_2, \dots, e_n\}$ are orthogonal and of unit length, called orthonormal. Vector addition and scalar multiplication is the same as for vectors, $e_i + e_i = 2e_i$ and $(a+b)e_i = ae_i + be_i$ for $i = 1, \dots, n$ and $a, b \in \mathbb{R}$.

It was Descartes who realised the value of a line segment that has unit length which allows a point to be described by coordinates (a_1, a_2) in a two dimensional space. A vector turns this point into an arrow as the sum of a set of orthonormal basis vectors, e_1 and e_2, in this example. Unfortunately, both (a_1, a_2) and $a_1e_1 + a_2e_2$ are called vectors but here Hamiltons original definition is required. In n dimensions, vector or 1-form addition follows as for vectors but it is the geometric product of 1-forms that is new

$$e_i e_j = \delta_{ij} + e_i \wedge e_j, \quad \text{where } i, j \in \mathbb{N}_1^n.$$

$\mathbb{N}_1^n$ is shorthand for the set of natural numbers $\{1, 2, \dots, n\}$ and delta has the propery $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j,\ i, j \in \mathbb{N}_1^n$. The square of a vector is a scalar because the cross terms have basis terms $e_ie_j + e_je_i$, $i \neq j$, which all cancel to give

$$\mathbf{a}^2 = \mathbf{aa} = a_1^2 + a_2^2 + \cdots + a_n^2.$$

This is the Pythagorean theorem for n dimensions which is usually expressed in two dimensions. It allows a length to be defined as $|\mathbf{a}| = \sqrt{\mathbf{a}^2}$. It also means every vector has an inverse $\mathbf{a}^{-1} = \frac{1}{\mathbf{a}} = \frac{\mathbf{a}}{\mathbf{a}^2}$.

So the product of two 1-forms, called a bivector, contains both a 0-form and a 2-form. These are called the contraction and expansion components which extends to 1-forms multiplied by any r-form so give, in general, an $(r-1)$-form and a $(r+1)$-form. The product of arbitrary s and t-forms is not well defined since it can involve forms of degree $s+t, s+t-2, s+t-4, \dots, 1$ or 0. Such products are best analysed in terms of multivectors, $\mathbf{abc}\dots\mathbf{n}$, where each vector can be expanded into basis components which provides a very complicated expansion over all forms of degrees 0 through n. To arrive at this point it is best to step through each Clifford algebra, $\mathbb{G}_n$, starting with $n = 0$.

The first Geometric algebra $\mathbb{G}_0 = \mathbb{R}$ has no dimensions and matches the first row of Pascal's Triangle in Table 1 even though this does not corrrespond to a simplex. The basis of $\mathbb{G}_0$ is $\{1\}$, and all real numbers are multiples of this basis. The 0-simplex consisting of one point or dimension is $\mathbb{G}_1$ with basis $\{1, e_1\}$. This provides the first example of a spinor, actually two, $s_\pm = \frac{1}{2}(1 \pm e_1)$. The important property of spinors is that they are projection operators $s_\pm^2 = s_\pm$ also called idempotents. This is important in physics as the objects that represent electron spin and this formula in physics indicates a constructor as opposed to the destructor operation $s_+s_- = 0$. Every dimension of $\mathbb{G}$ after this has spinors and these are explored at the end of the chapter and for now the focus is on rotations and reflections.

The 1-simplex represents $\mathbb{G}_2$ with basis $\{1, e_1, e_2, e_{12}\}$ with the simplex edge of the 2-form now defined as the product of two 1-forms, $e_{12} = e_1e_2$. It operates as the $90°$ rotation $e_1e_{12} = e_2$ and $e_2e_{12} = -e_1$. An important property to demonstrate here is that the set of even forms is another Clifford algebra and $\{1, e_{12}\}$ can be mapped to $\{1, i\}$ so the subalgebra is isomorphic to $\mathbb{C}$. This introduces imaginary numbers and could be called i but not for $\mathbb{G}_2$ since it is an operator of rotation. Put another way, it anticommutes with both vectors $e_ie_{12} = -e_{12}e_i$, for $i = 1, 2$. In higher dimensions any vector outside the plane of e_{12} will anticommute with both indices so commute with the plane which means e_{12} can only operate within it's plane. This is the first example of the symmetry inherent in the algebra and the reason why an analysis of rotations and reflections in $\mathbb{G}_2$ is applicable in any dimension.

Vectors as arrows are drawn starting at an origin point and stretch to arbitrary coordinates. Two arbitrary vectors in two dimensions, $\mathbf{a} = a_1e_1 + a_2e_2$ and $\mathbf{b} = b_1e_1 + b_2e_2$, as shown in Figure 12, can be added forming a parallelogram to give $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} = (a_1 + b_1)e_1 + (a_2 + b_2)e_2$. This is the well-known head-to-tail addition rule probably known to Euclid. In Figure 12 $a_1 = 0$, arbitrary, to aid later analysis.

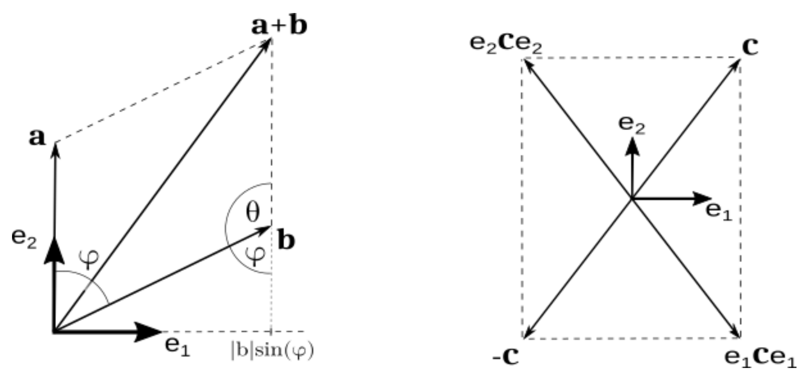

Figure 12: **Vector addition and reflection**

Addition of vectors is well defined whereas multiplication of arbitrary 1-forms requires Clifford algebra. For the vector $\mathbf{c} = c_1e_1 + c_2e_2$, place a basis either side and move one side through $\mathbf{c}$. This basis will commute with the parallel part of $\mathbf{c}$ and anticommute with the other part changing its sign before cancelling with the other side. This produces the reflections seen in the second diagram in Figure 12

$$e_1\mathbf{c}e_1 = c_1e_1 - c_2e_2 \quad \text{and} \quad e_2\mathbf{c}e_2 = -c_1e_1 + c_2e_2.$$

Applying the two consequtively gives a reflection of both axes or a rotation of $180°$

$$e_{12}\mathbf{c}e_{21} = e_1e_2\mathbf{c}e_2e_1 = -c_1e_1 - c_2e_2 = -\mathbf{c}.$$

If this 2-form is applied from one side only then Figure 13 shows half of this rotation or $90°$ in the direction from e_2 to e_1.

$$e_{12}\mathbf{c} = e_{12}(c_1e_1 + c_2e_2) = -c_1e_2 + c_2e_1.$$

If the plane of the 2-form was turned upside-down then multiplying by e_{21} from the left would rotate anticlockwise instead. Multiply from the right gives the opposite sense thus giving the $180°$ rotation above. This is a demonstration of an important fact that two reflections make a rotation and later we will show that a rotation is always composed of two relections. For the moment it is important to recognise the power of

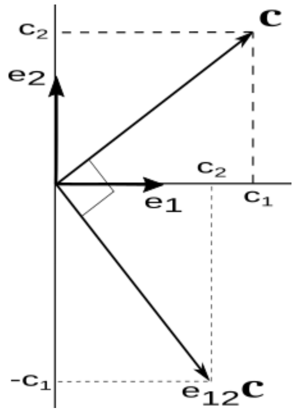

Figure 13: **Vector rotation**

the two sided operation. Both rotations and reflections for any vector can be expressed as $\tilde{\mathbf{a}} = r\mathbf{a}r^{-1}$ and this generalises to the fundamental operation of any multivector $A = \mathbf{abc}\ldots\mathbf{n}$ as

$$\tilde{A} = rAr^{-1} = r\mathbf{a}(r^{-1}r)\mathbf{b}(r^{-1}r)\mathbf{c}(r^{-1}\ldots r)\mathbf{n}r^{-1}. \tag{4}$$

In general, $e_i\mathbf{a}e_i$ reflects all components perpendicular to e_i, a_je_j for $i \neq j$ and commutes with a_ie_i leaving it unchanged. The general form of a reflection for any s-form is

$$\tilde{\mathbf{a}} = (-1)^s r\mathbf{a}r^{-1} \tag{5}$$

where r is an s-form and the parity changes the reflection so that all components in $\mathbf{a}$ with basis in r are reflected and the perpendicular components are untouched. For a multivector with n vectors this becomes

$$\tilde{A} = (-1)^{sn} rAr^{-1}$$

which can be shown using fundamental operation (4). Note that if $r = e_{ijk}$ then $r^{-1} = e_{kji}$ so another intepretation is that as each basis, e_k of r passes through each vector of A from right to left it acts symmetrically for parallel vectors and changes the sign for perpendicular parts. It then disappears because $e_ke_k = 1$. We could replace r with an arbitrary vector and $\tilde{A}$ would be the sum of each component acting on A but r can't be replaced with a multivector due to the general product shown later.

Returning to the progression through each algebra, in 3 dimensions, $\mathbb{G}_3$ has 8 basis elements

$$\{1, e_1, e_2, e_3, e_{12}, e_{23}, e_{13}, e_{123}\},$$

where $e_{123} = e_1e_2e_3$. Chapter 1 shows that the three bivectors or planes have negative length and the same cyclic products as the quaternions, $e_{12}e_{23} = e_{13}$, $e_{23}e_{13} = e_{12}$, $e_{13}e_{12} = e_{23}$, and since these along with the scalar 1, form the even subalgebra, they are isomorphic to the quaternions. The 2-forms rotate the vectors so the interpretation of quaternions is that they rotate themselves because they are themselves composed of 1-forms. The final element, e_{123} is called the psuedo-scalar of $\mathbb{G}_3$. The octonians mentioned in Chapter 1 also contain the quaternions and have 8 basis elements including 1. The difference is that in $\mathbb{G}_3$ the psuedo-scalar is a commuting, imaginary operator. It is easy to check that $e_{123}^2 = -1$ and that any vector component in $\mathbb{G}_3$ commutes with one basis vector of e_{123}, and anticommutes with the other two. Operating on vectors turns them into 2-forms, which can be called psuedo-vectors in $\mathbb{G}_3$. The operation in reverse generates axial vectors because the cross product in $\mathbb{G}_3$ is defined as

$$\mathbf{a} \times \mathbf{b} = e_{321}\mathbf{a} \wedge \mathbf{b}.$$

The 3-form here defines a right-hand screw rule for the vector product whereas $e_{123} = -e_{321}$ would define a left-hand screw rule. Chapter 1 touched on the fact that all forms in $\mathbb{G}$ are objects, operators and have a sense. The axial vector given by the cross product defines the volume of the parallelopiped which is an extension of the parallelogram in Figure 12 to 3 dimensions. This is the voume $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ where $\mathbf{c} = \mathbf{a} \times \mathbf{b}$. The action of the 3-form is to convert between vectors and planes. Planes themselves represent an area and act as rotations, clockwise or anticlockwise. Vectors are arrows in space pointing in one of two directions and operate as reflections. Scalars themselves magnify and specify the size, including negative size which now has a geometric interpretation.

Hence we have complex numbers, quaternions, vectors and scalars all within one algebra that naturally incorporates rotations and the imaginary number with geometric meaning. It is tragedy that this algebra has not been recognised as playing the centrol role in the split between octonians, differential geometry and matrices for over 130 years. As an introduction to mathematics it would entice students into operations in higher dimensional spaces and lead into the study of matrices since rotations and reflections are derived from the more fundamental permutation group, as we shall soon see.

For 3-forms the sense of direction is called parity. Any swapping of the indices is a change of parity which changes the sign of the 3-form. In the theory of exterior algebra an n-form is defined in terms of the determinant which is just a procedure for enumerating all n components of each of n vectors taking into account the parity

$$\det(\mathbf{a}, \mathbf{b}, \dots, \mathbf{n}) = \frac{1}{n!} \sum_{\mu \in \mathcal{P}_n} (-1)^{\sigma} a_{\mu_1} b_{\mu_2} \dots n_{\mu_n}.$$

where $\mathcal{P}_n$ provides all $n!$ permutations of the indicies $\mu_1, \mu_2, \dots, \mu_n$ over $\mathbb{N}_1^n$ and σ is the sign of the permutation. The exterior algebra is defined by $\mathbf{a} \wedge \mathbf{b} \wedge \cdots \wedge \mathbf{n} = \det(\mathbf{a}, \mathbf{b}, \dots, \mathbf{n}) e_{12\dots n}$. For a 3-form in terms of the determinant this is

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \frac{1}{6}(a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1)e_{123}.$$

This is not a valid Clifford product which is only defined in terms of pairs of vectors. Stretching the determinant definition above to cover 2-forms in 3 dimensions, the triple multivector product $\mathbf{abc}$ contains

$$(\mathbf{a} \wedge \mathbf{b})\mathbf{c} = ((a_1b_2 - a_2b_1)e_{12} + (a_2b_3 - a_3b_2)e_{23} + (a_1b_3 - a_3b_1)e_{13})\mathbf{c}$$

which gives the same terms as above for the 3-form e_{123}. The 1-form part of $\mathbf{abc}$ involving the Pfaffian will be analysed shortly.

Products in $\mathbb{G}$ reduce to symmetry for pairs of vectors. Assuming only distributivity of vectors, the bivector $\mathbf{ab}$ can be divided into half and $\mathbf{b}(\mathbf{a} - \mathbf{a}) = 0$ added. Then rearranging terms gives the fundamental equation for two vectors

$$\begin{aligned} \mathbf{ab} &= \frac{1}{2}(\mathbf{ab} + \mathbf{ab}) + \frac{1}{2}(\mathbf{ba} - \mathbf{ba}) \\ &= \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) + \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}. \end{aligned} \tag{6}$$

The first term is the symmetric product since $\mathbf{aa} = \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, the length of $\mathbf{a}$ squared. The second antisymmetric term uses the Grassmann wedge ($\wedge$) operator because this is the definition of a 2-form using the determinate in the exterior algebra in the plane of $\mathbf{a}$ and $\mathbf{b}$. This is another example of symmetry that allows rotations to be analysed without resorting to a basis, as shown by D. Hestenes. Applying this to quaternions shows a negative dot product and a fixed sense for the cross product.

The multiplication of two vectors can be interpreted geometrically using symmetry alone. Splitting $\mathbf{b}$ into parts parallel and perpendicular to $\mathbf{a}$, $\mathbf{b} = \mathbf{b}_{||}$ and $\mathbf{b}_{\perp}$, respectively. Figure 14 shows via trigonometry that $|\mathbf{b}_{||}| = |\mathbf{b}|\cos(\varphi)$ and $|\mathbf{b}_{\perp}| = |\mathbf{b}|\sin(\varphi)$. Defining the 2-form

$$I_{\mathbf{ab}} = \frac{\mathbf{a}\mathbf{b}_{\perp}}{|\mathbf{a}||\mathbf{b}_{\perp}|}, \tag{7}$$

this can be substituted into $\mathbf{ab}$ in (6) and expanded using Euler's formula for the complex exponential function since $I_{\mathbf{ab}}^2 = -1$, as

$$\begin{aligned}\mathbf{ab} &= \mathbf{a}(\mathbf{b}_{||} + \mathbf{b}_{\perp})\\ &= |\mathbf{a}|(|\mathbf{b}_{||}| + |\mathbf{b}_{\perp}|I_{\mathbf{ab}})\\ &= |\mathbf{a}||\mathbf{b}|(\cos(\varphi) + I_{\mathbf{ab}}\sin(\varphi))\\ &= |\mathbf{a}||\mathbf{b}|\exp(I_{\mathbf{ab}}\varphi).\end{aligned} \tag{8}$$

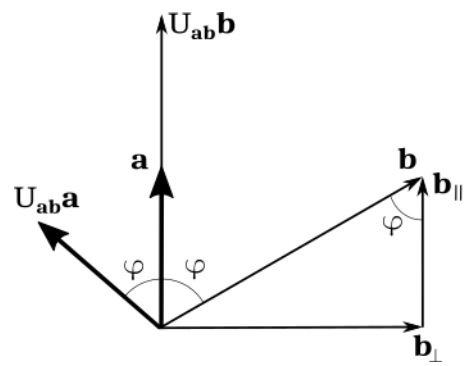

Figure 14: **Full rotation**

This defines a rotation of angle φ and is called a versor if $|\mathbf{ab}| = 1$, defined as

$$U_{\mathbf{ab}} = \frac{\mathbf{ab}}{|\mathbf{a}||\mathbf{b}|} = \exp(I_{\mathbf{ab}}\varphi)$$

and this, operating from the left, rotates vectors in the plane of $\mathbf{a} \wedge \mathbf{b}$ by angle φ in the direction of $\mathbf{b}$ towards $\mathbf{a}$. This is shown for $\mathbf{a}$ and $\mathbf{b}$ but first note that $\mathbf{ab}_{\perp}\mathbf{a}$ is a reflection

$$U_{\mathbf{ab}}\mathbf{b} = \frac{|\mathbf{b}|}{|\mathbf{a}|}\mathbf{a} \quad \text{and}$$

$$U_{\mathbf{ab}}\mathbf{a} = \frac{\mathbf{a}(\mathbf{b}_{||} + \mathbf{b}_{\perp})\mathbf{a}}{|\mathbf{a}||\mathbf{b}|} = \frac{|\mathbf{a}|}{|\mathbf{b}|}(\mathbf{b}_{||} - \mathbf{b}_{\perp}).$$

An arbitrary vector with components along $\mathbf{a}$ and $\mathbf{b}$ will be rotated by the same amount and any components outside the plane $\mathbf{a} \wedge \mathbf{b}$ will not be changed since the versor commutes with these components. The inverse is $U_{\mathbf{ba}} = \exp(-I_{\mathbf{ab}}\varphi)$ so the general rotation for any number of vectors is

$$\tilde{M} = \exp(\frac{1}{2}I_{\mathbf{ab}}\varphi)M\exp(-\frac{1}{2}I_{\mathbf{ab}}\varphi).$$

Given that this has the form of reflections using vectors, $\mathbf{ab}M\mathbf{ba}$, it immediately shows that two reflections make a rotation. The fundamental operation of multivectors shows how each vector in M is rotated in the same way. Another way of looking at it is that if a vector in M is not in the plane then the rotation from one side commutes and cancels the versor on the other side. But for any vector in M within the plane $\mathbf{a} \wedge \mathbf{b}$ then the versor is inverted as it passes through and doubles the one sided rotation on the other side.

The utility of the fundamental product of two vectors is shown using polarisation to derive the cosine rule from symmetry alone without resorting to a basis and co-ordinates.

$$\begin{aligned}(\mathbf{a}+\mathbf{b})^2 &= \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{ab} + \mathbf{ba} \\ &= \mathbf{a}^2 + \mathbf{b}^2 + 2\mathbf{a} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta).\end{aligned}$$

The first line is Pythagoras's rule if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal which is a very simple statement of symmetry. Note that in the final line θ has changed to the angle opposite $\mathbf{a}+\mathbf{b}$, shown in Figure 12 as $\theta = 180° - \varphi$, which introduces the minus sign.

The versor $\exp(e_{12}\varphi)$ is 1 for $\varphi = 0$ and e_{12} for $\varphi = \pi/2$ which is a rotation of 90°. The product of two external rotations of 45° is

$$\exp(e_{12}\pi/4)\exp(e_{34}\pi/4) = \frac{1}{2}(1 + e_{12} + e_{34} + e_{1234}).$$

The 4-form also occurs in the product of $\exp(e_{14}\pi/4)\exp(e_{23}\pi/4)$ and $-\exp(e_{13}\pi/4)\exp(e_{24}\pi/4)$. These are the elements of the Pfaffian for a 3-simplex and also correspond to the 3 pairs of opposite edges of the 3-simplex.

The anaylsis of Pfaffians used basis vectors which were assumed orthonormal. This is extended to arbitrary vectors by returning to the notation of Chapter 1 and representing a determinant of n vectors as $|\mathbf{a}, \mathbf{b}, \ldots, \mathbf{n}|$ where n is even and each vector is represented as a column of vector components and basis elements. Then

$$\begin{aligned}|\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \ldots, \mathbf{m}, \mathbf{n}| &= \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} \wedge \cdots \wedge \mathbf{m} \wedge \mathbf{n} \\ &= \tfrac{1}{n!} \sum_{\mu \in \mathcal{P}_n} (-1)^\sigma a_{\mu_1} e_{\mu_1} b_{\mu_2} e_{\mu_2} c_{\mu_3} e_{\mu_3} d_{\mu_4} e_{\mu_4} \ldots m_{\mu_m} e_{\mu_m} n_{\mu_n} e_{\mu_n} \\ &= \tfrac{n}{n!} \sum_{\mu \in \mathcal{P}'_{n,n}} (-1)^\sigma (a_{\mu_1} e_{\mu_1} \wedge b_{\mu_2} e_{\mu_2})(c_{\mu_3} e_{\mu_3} \wedge d_{\mu_4} e_{\mu_4}) \ldots (m_{\mu_m} e_{\mu_m} \wedge n_{\mu_n} e_{\mu_n}) \\ &= \{(\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \wedge \mathbf{d}) \ldots (\mathbf{m} \wedge \mathbf{n})\}_n \\ &= \pm V e_{12\ldots n}\end{aligned}$$

where $\mathcal{P}_n = [1, 2, \ldots, n]$, $\mathcal{P}'_{n,n} = [(1,2), (3,4), \ldots, (m,n)]$, $\{A\}_n$ means take the n-form part and $V = \det(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{n})$ is the volume of the hyper-parallelogram with sides $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{n}$. The hyper-parallelogram extends the parallelogram and parallolopided in two and three dimensions, respectively, to n dimensions. The wedge product of two vectors gives the area, $\mathbf{a} \wedge \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin(\varphi_{\mathbf{ab}}) = |\mathbf{a}||\mathbf{b}_\perp|$, of the parallelogram as shown in Figure 12 where $\varphi = \varphi_{\mathbf{ab}}$. The wedge product provides the $\sin$ term and it does that in each dimension. The pairs of vectors in $\mathcal{P}_{n,n}$ each provide $\binom{n}{2}$ factors which is the same number of cross terms in $\mathbf{ab}$ so this can be replaced with $\mathbf{a} \wedge \mathbf{b}$, etc. Removing the permutation (and $1/(n-1)!$ factor) over the 2-forms may introduce cross terms between the 2-forms hence it is necessary to take the n-form part. This guarantees the $\sin$ term is applied between each 2-form and will give $V = 0$ if any pair of vectors are linearly dependent but is non-zero otherwise. This is a statement that the determinant of n vectors in $\mathbb{G}_n$ is the same as incrementally including the area of each parallelogram and it's exterior separation from the hyper-parallelogram that contructs the hyper-volume V.

The same result could be acheived with $Ve_{123\ldots n} = \pm\{\mathbf{abc}\ldots\mathbf{n}\}_n$ remembering that all forms in $\mathbb{G}$ have a sense of operation so may be negative. But the approach can be seen to be the last term of the Pfaffian (2) introduced in Chapter 1,

$$V = \pm \setminus \mathbf{a}\wedge\mathbf{b}, 0, \mathbf{c}\wedge\mathbf{d}, 0, \ldots, 0, \mathbf{m}\wedge\mathbf{n}|.$$

This shows that the Pfaffian in terms of basis elements $\setminus e_{12}, 0, e_{34}, 0, \ldots, 0, e_{(n-1)n}| = e_{123\ldots n}$ provides volume $V = 1$, as expected for an orthonormal basis. When separating the determinate into pairs, although the notation would not allow for it, we would have acheived the same result by starting with any pair of vectors, $\mathbf{a}\wedge\mathbf{c}$, for instance. There are $n-1$ ways to do this for $\mathbf{a}$, etc, and these correspond to all the terms in the expansion of the following Pfaffian.

$$\left| \begin{matrix} \mathbf{a}\wedge\mathbf{b}, & \mathbf{a}\wedge\mathbf{c}, & \mathbf{a}\wedge\mathbf{d}, & \ldots, & \mathbf{a}\wedge\mathbf{n} \\ & \mathbf{b}\wedge\mathbf{c}, & \mathbf{b}\wedge\mathbf{d}, & \ldots, & \mathbf{b}\wedge\mathbf{n} \\ & & \mathbf{c}\wedge\mathbf{d}, & \ldots, & \mathbf{c}\wedge\mathbf{n} \\ & & & \ddots & \vdots \\ & & & & \mathbf{m}\wedge\mathbf{n} \end{matrix} \right|$$

Fortunately, each term negates the previous volume leaving the last term providing $\pm V e_{12\ldots n}$. There are other forms in this expansion so the n-form part needs to be taken but interestingly the scalar part vanishes. It is in this form that an antisymmetric matrix can be created with the order of each wedge product reversed below the diagonal. It is well known the determinant of an antisymmetric matrix is the square of the equivalent Pfaffian, V^2 in the case. Such a determinant involves each term of the Pfaffian squared plus all cross terms which means a massive amount of redundancy. But is it also well known that reflections applied to the antisymmetric matrix can transform it to the equivalent of the first Pfaffian term as an antisymmetric matrix. This means all intermediate wedge products become zero, such as $\mathbf{b}\wedge\mathbf{c} = 0$, by finding rotations that align the last term edges, for this example $\mathbf{b}$ and $\mathbf{c}$ are parallel, etc.

The derivation of Pascal's triangle in Chapter 1 and it's relationship to simplices and Pfaffians has shown each additional dimension connects to all previous ones. This has led to Pfaffians with the exterior product generating the volume of the hyper-parallelogram which is one for the hyper-cubes shown in Chapter 1 using basis forms. But the fundamental product of vectors, $\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}$, shows that a Pfaffian of contractions would provide a measure of parallelism connecting all dimensions. Replacing the external products with contractions introduces a metric on the structure of space which introduces the fundamental expansion of Clifford multivectors

$$\mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3\ldots\mathbf{a}_n = \sum_{i=0}^{[\frac{n}{2}]}\sum_{\mu\in\mathcal{C}}(-1)^\sigma \setminus \mathbf{a}_{\mu_1}\cdot\mathbf{a}_{\mu_2}, \ldots, \mathbf{a}_{\mu_{2i-1}}\cdot\mathbf{a}_{\mu_{2i}}|\;\mathbf{a}_{\mu_{2i+1}}\wedge\cdots\wedge\mathbf{a}_{\mu_n} \tag{9}$$

where $\mathcal{C} = \binom{n}{2i}$ provides combinations, μ, of n indicies divided into $2i$ and $n-2i$ parts and σ is the parity of the combination. Notice that for even simplices the Pfaffian trick used by de Bruijn becomes an expansion over vectors instead. For 3 vectors the contracted part of the multivector is

$$\begin{aligned} \mathbf{a}_1\,\mathbf{a}_2\,\mathbf{a}_3 - \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} &= \textstyle\sum_{\mu\in[(1,2),3]}(-1)^\sigma \setminus \mathbf{a}_{\mu_1}\cdot\mathbf{a}_{\mu_2}|\mathbf{a}_{\mu_3} \\ &= (\mathbf{a}_1\cdot\mathbf{a}_2)\mathbf{a}_3 - (\mathbf{a}_1\cdot\mathbf{a}_3)\mathbf{a}_2 + (\mathbf{a}_2\cdot\mathbf{a}_3)\mathbf{a}_1 \\ &= \left| \begin{matrix} \mathbf{a}_1\cdot\mathbf{a}_2 & \mathbf{a}_1\cdot\mathbf{a}_3 & \mathbf{a}_1 \\ & \mathbf{a}_2\cdot\mathbf{a}_3 & \mathbf{a}_2 \\ & & \mathbf{a}_3 \end{matrix} \right| \end{aligned}$$

This notation doesn't simplify the fundamental expansion because it involves various Pfaffian sizes so combinations of the last column would still be needed.

More importantly, it can be proved using the general Pfaffian expansion (3) of Chapter 1 that the product of two multivectors expands to the same expansion of the whole multivector. This result proves that the fundamental operation of multivectors, $\tilde{A} = rMr^{-1}$, expanded in (4), operates in the same way on each vector separately

$$\mathbf{a}_1\,\mathbf{a}_2\ldots\mathbf{a}_{r+s} = (\mathbf{a}_1\,\mathbf{a}_2\ldots\mathbf{a}_r)(\mathbf{a}_{r+1}\,\mathbf{a}_{r+2}\ldots\mathbf{a}_{r+s}). \tag{10}$$

The Pfaffian expansion also works for the differential operator called the grad vector,

$$\nabla = e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + \cdots + e_n\frac{\partial}{\partial x_n}.$$

For a function this derives the slope or gradient. When applied to a two dimensional surface $f(x_1, x_2)$, which can be interpreted as the height above some substratum, it provides a field of arrows pointing uphill, $\nabla f = e_1\frac{\partial f}{\partial x_1} + e_2\frac{\partial f}{\partial x_2}$. For a vector field, such as the velocity arrows at every point in a river at a single point of time, it defines the divergence and 2-form curl, $\nabla\mathbf{a} = \nabla\cdot\mathbf{a} + \nabla\wedge\mathbf{a}$. In three dimensions the usual curl vector is recovered as $\nabla\times\mathbf{a} = e_{321}\nabla\wedge\mathbf{a}$. In general, the 2-form curl is all the vector product cross terms $\nabla\wedge\mathbf{a} = e_{12}(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}) + \cdots + e_{(n-1)n}(\frac{\partial a_n}{\partial x_{n-1}} - \frac{\partial a_{n-1}}{\partial x_n})$. The wedge notation is superior because it is applicable in n dimensions and is easily seen to be the rotation of the velocity as $\mathbf{x}$ moves away from the origin. The divergence, $\nabla\cdot\mathbf{a} = \frac{\partial a_1}{\partial x_1} + \cdots + \frac{\partial a_n}{\partial x_n}$, is the change of length of the velocity in space. Applying this to a vector in the field, $\mathbf{a}'$, reproduces the vector

$$\mathbf{a}' = (\nabla\mathbf{a}')\mathbf{a}.$$

The symmetric and antisymmetric forms of the vector product in this case are not instructive as they require the introduction of a right-handed differential operator, $\overleftarrow{\nabla}$, which can be avoided by going straight to components. In general, the grad operator, $\nabla = \overrightarrow{\nabla}$, operating from the left as a normal vector in the Pfaffian expansion (9) will apply the operator to each part in turn. Operating on a bivector gives

$$\nabla\mathbf{ab} = \nabla(\mathbf{a}\cdot\mathbf{b}) + (\nabla\cdot\mathbf{a})\mathbf{b} - (\nabla\cdot\mathbf{b})\mathbf{a} + \nabla\wedge(\mathbf{a}\wedge\mathbf{b}).$$

Each of these terms is zero for static vectors and are only applicable to vector fields. The first three terms have been discussed above and the 3-form needs differential forms in Chapter 4 to explain. The notation was introduced here to show that the Pfaffian expansion naturally applies ∇ to each vector.

Now that we have seen the roles played by odd and even forms and shown that the fundamental expansion (9) extends this structure to arbitrary dimensions there is no reason to continue to higher dimensions for rotations. It is more important to introduce hyperbolic geometry.

Imaginary operators and numbers describe important actions of the algebra but have another role to play. Clifford allowed for negative length basis elements so a basis $e_1, e_2, \ldots, e_n$ can be extended with m negative signature elements denoted $\mathbb{G}_{n,m}$ where $e_{n+i}^2 = -1$ for $i \in \mathbb{N}_1^m$. Mathematicians generally deal with $\mathbb{G}_{0,m}$ as an extension of complex numbers thus matching octonions but the even subalgebras are isomorphic to the even subalgebra of $\mathbb{G}_m$ so the Spin algebras are the same. In mathematical physics dealing with space-time, $\mathbb{G}_{3,1}$, it is convenient to instead introduce the basis vector e_0 with $e_0^2 = -1$. Then, since $(e_0e_i)^2 = 1$ for $i \in \mathbb{N}_1^3$, the hyperbolic Euler equation in (8) is

$$e_0\mathbf{b} = |\mathbf{b}|(\cosh(\varphi) + e_0\frac{\mathbf{b}}{|\mathbf{b}|}\sinh(\varphi)).$$

This is called a boost because it instantaniously changes the velocity, just like a rotation instantaneously changes orientation. So negative length is important in hyperbolic geometry too. Of course a boost will not cross the light cone and increase the velocity past the speed of light, c. Note that the light cone is very flat due to $c \approx 300$ thousand kilometers per second. Drawing a ray of light would mean that the x-axis would almost reach the moon before the y-axis reached 1 second. Changing to units of light seconds instead of meters then $c = 1$ and the light cone has an angle of $45°$ to all axes. This most general picture is commonly encountered and the scale should really be provided. In space-time it is time that is usually scaled as ct so setting $c = 1$ also simplifies Maxwell's equations. In space-time the change of time is included in the grad operator as $\nabla_4 = e_0 \frac{\partial}{\partial t} + \nabla$. The index here should be $3, 1$ but space-time vectors are generally just called 4-vectors. For convenience, 4-vectors are shown without vector notation to distinguish them from vectors that can have curl applied. The 4-current has the charge density as the time component, $q = q_0 e_0 + \mathbf{J}$, where $\mathbf{J}$ is the charge current. With $\mathbf{E}$ being the electric field, $\mathbf{B}$ the magnetic field and μ the monopole 4-current then all eight Maxwell's equations, are

$$\nabla_4(e_0\mathbf{E} - e_{321}\mathbf{B}) = q + e_{0321}\mu.$$

This looks like $\mathbf{E}$ driving boosts and $\mathbf{B}$ driving rotations of the current. The grad function complicates things but the time derivative of the electric field and the curl of the magnetic field can be interpreted as boosting the 3-current. Setting $\mu = 0$ and applying the grad operator to a bivector, as given above, then separating the time and space components of the above equation uncovers the four electro-magnetic Maxwell equations in the usual vector form. Including μ and setting $q = 0$ provides another four equations, separated out by the space-time psuedo-scalar, e_{0321}, that describe interactions for magnetic monopoles.

The electro-magnetic field is usually written as $F = \mathbf{E}+i\mathbf{B}$ but this does not expose the underlying geometry that F is really a plane in space-time with a time-like 2-form $e_0\mathbf{E}$ and a space 2-form $e_{321}\mathbf{B}$. The field for electro-magnetic waves, or light and radio waves, is obtained by setting both charges and currents to zero, $q = \mu = 0$, and then applying the grad function again to get the 4-Laplacian

$$\nabla_4^2(e_0\mathbf{E} - e_{321}\mathbf{B}) = e_0(-\frac{\partial \mathbf{E}^2}{\partial t^2} + \nabla^2\mathbf{E}) + e_{321}(-\frac{\partial \mathbf{B}^2}{\partial t^2} + \nabla^2\mathbf{B}),$$

which are wave equations with suggested solution

$$F_0 = (e_0\mathbf{E_0} - e_{321}\mathbf{B_0})\exp(\omega I_{\mathbf{EB}}(\mathbf{v}\cdot\mathbf{x} - t)),$$

where, $I_{\mathbf{EB}}$ from (7) is a unit 2-form in the plane of $\mathbf{E}$ and $\mathbf{B}$, $\mathbf{E_0}$ and $\mathbf{B_0}$ are constant vectors and $\mathbf{v}$ is a unit vector in the direction of the velocity. Usually the wave equation has exponential component $i(k\mathbf{x} \pm \omega t)$, but the wave number k is $k = \frac{\omega}{\mathbf{v}}$ where ω is the angular frequency and v is the velocity. Since this is expected to be the speed of light which has been scaled to $c = 1$ then $\mathbf{v}^{-1} = \mathbf{v}$.

Maxwell's equation then becomes $\nabla_4 F_0 = 0$ and expands as

$$\begin{aligned}
\nabla_4 F_0 &= \omega(-e_0 F_0 I_{\mathbf{EB}} + F_0 I_{\mathbf{EB}}\mathbf{v}) \\
&= \omega(\mathbf{E}' - e_{0321}\mathbf{B}' + e_0\mathbf{E}'\mathbf{v} - e_{321}\mathbf{B}'\mathbf{v})e^{\omega I_{\mathbf{EB}}(\mathbf{v}\cdot\mathbf{x}-t)} \\
&= \omega[\mathbf{E}' - \mathbf{B}'\times\mathbf{v} + e_0(\mathbf{E}' - \mathbf{B}'\times\mathbf{v})\mathbf{v} + e_0\mathbf{E}'\cdot\mathbf{v} - e_{321}\mathbf{B}'\cdot\mathbf{v}]e^{\omega I_{\mathbf{EB}}(\mathbf{v}\cdot\mathbf{x}-t)} \\
&= 0
\end{aligned}$$

where $\mathbf{E}' = \mathbf{E_0} I_{\mathbf{EB}}$ and $\mathbf{B}' = \mathbf{B_0} I_{\mathbf{EB}}$.

Taking space and time, 1 and 3-form parts gives $\mathbf{E}' = \mathbf{B}'\times\mathbf{v}$, twice, $\mathbf{E}'\cdot\mathbf{v} = 0$ and $\mathbf{B}'\cdot\mathbf{v} = 0$. Hence $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{v}$ are all perpendicular, so that $I_{\mathbf{EB}}$ commutes with $\mathbf{v}$. Using i in F_0 generates a polarised plane wave solution whereas by using $I_{\mathbf{EB}}$ we have a circularly polarised wave rotating the electric and magnetic fields around the direction of travel, $\mathbf{v}$. When t equals the period of oscilation for the frequency, $f^{-1} = \frac{2\pi}{\omega}$, then the electro-magnetic fields have done a complete rotation so the frequency determines the spin of the light wave.

But what is the meaning of the 4-form, e_{0321} which has property $e_{0321}^2 = -1$ but is not cummutative? Multiplying Maxwell's equation by e_{0321} swaps the interpretation of the charge and monopole 4-vectors and the electric and magnetic fields. So the 4-form acts to swap electrons and monopoles and the electric and magnetic fields. This is an alternate way to view physics but since there is no evidence of magnetic monopoles then μ is usually set to 0. It is interesting that the interpretation of e_{0321} has extended geometry into physics.

The psuedo-scalar in space-time is not commuting but adding another positive length basis generates a commuting imaginary pseudo-scalar. Extending space-time to $\mathbb{G}_{4,1}$ gives the psuedo-scalar e_{01234}, $e_{01234}^2 = -1$, which is used to define the Dirac equation. The Dirac Equation is the relativistic extension of the Schrodenger Equation and is usually represented by Pauli spin matricies. The matrix solutions are called spinors which are vectors for the spin matrices but in Clifford Algebra the matricies are replaced with 1-forms and solutions are ideals of the algebra. Ideals are the projection operators introduced by $\mathbb{G}_1$ above.

The Dirac equation for an electron of rest mass m_0 in free space is

$$(\hbar\nabla_4 - m_0)\phi = 0,$$

with solution, ϕ, for a free electron of 4-momentum p and spin s, $s^2 = 1$,

$$\phi_\pm = \frac{1}{4m_0}(m_0 \pm e_{01234}p)(1 + e_{0123}s)\exp\left(\frac{e_{01234}p \cdot x}{\hbar}\right)$$

where $\hbar$ is related to Planck's constant and x is a 4-vector. The first two terms in ϕ are projection operators due to Einstein's Equation, $E = mc^2$. Here m is the moving mass rather than the rest mass, m_0, and the hyperbolic form with $c = 1$ is $E^2 = \mathbf{P}^2 + m_0^2$. With definition $p = Ee_0 + \mathbf{P}$ then

$$p^2 = -E^2 + \mathbf{P}^2 + Ee_0\mathbf{P} + E\mathbf{P}e_0 = \mathbf{P}^2 - E^2 = -m_0^2$$

because $e_0 \cdot \mathbf{P} = 0$. This means that p is time-like, as expected for the electron travelling slower than the speed of light. Thus $(e_{01234}p)^2 = m_0^2$ and defining $\Phi_\pm = \frac{1}{2m_0}(m_0 \pm e_{01234}p)$ then

$$\Phi_\pm^2 = \frac{1}{4m_0}(m_0^2 + m_0^2 \pm 2m_0e_{01234}p) = \Phi_\pm.$$

Similarly, the spin vector s defines a projection operator because $(e_{0123}s)^2 = 1$ and the two projection operators commute with each other if $p \cdot s = 0$, (since p also anticommutes with e_{0123}) making ϕ into a projection operator. Note that $-s$ satisfies these conditions so spin-up and spin-down solutions are described here. The exponential term in ϕ provides the solution because it turns the Dirac Equation into a projection operator because p and s are fixed so ∇_4 only operates on x and $\nabla_4 p \cdot x = p$. Thus $\nabla_4 \exp(e_{01234}p \cdot x) = e_{01234}p\exp(e_{01234}p \cdot x)$. So in Dirac's equation $\hbar\nabla_4\phi_+ = m_0$ because $\Phi_\pm$ is interpreted as projecting positive or negative energy states for the electron and positron, respectively, as

$$e_{01234}p\Phi_\pm = \frac{1}{2m_0}(m_0e_{01234}p \pm m_0^2) = \pm m_0\Phi_\pm.$$

This shows that the Dirac Equation is a re-statement of Einstein's Equation but in a form that exposes positive and negative energy and dual spin solutions. The tensor solution provides the same information but, as Hestenes has pointed out, spinnors do not provide the geometric interpretation and brevity of the mathematics provided by an algebraic ϕ.

With the extra dimension, e_4, the psuedo-scalar becomes commutative and with the 4-momentum, p, becomes a 4-form providing boosts and rotations while the spin, s, acts as a reflection. Changing the sign of e_4 effects the psuedo-scalar which changes between electron and positrons in the Dirac equation. The interpretation in this case is that the 5-form psuedo-scalar transforms between positive and negative energy solutions, again extending geometry to physics.

Negative lengths are necessary for rotations, the hyperbolic geometry of space-time and for the three psuedo-scalars in $\mathbb{G}_2$, $\mathbb{G}_3$, $\mathbb{G}_{3,1}$ and $\mathbb{G}_{4,1}$. Starting with $\mathbb{G}_3$ each addition of two basis elements, one positive length and one negative to get $\mathbb{G}_{3+n,n}$ provides a psuedo-scalar which is a commuting imaginary since $e^2_{(n+3)(2n+3)} = 1$ and in the new psuedo-scalar all the other bases commute with the new 2-form. The product of two imaginary basis elements is just a normal rotation, but inside the hyperbolic space. It is hard to imagine the meaning of rotating time away to some other time-like dimension so if such axes exist then they must have very little influence. The psuedo-scalars are commuting for all odd dimensional spaces because there are an even number of other bases to anticommute thus giving even parity.

As the simplices in Chapter 1 increased in size, each extra dimension added $n-1$ extra edges. This is the sequence seen in Pascal's Triangle and the summation for the simplex with n vertices gives

$$1 + 2 + \cdots + (n-1) = \frac{1}{2}n(n-1).$$

Coincidently, this is the same as the number of basis swaps to make $e_{123\ldots(n-1)n}$ into $e_{n(n-1)\ldots321}$. This defines the inverse because $e_{123\ldots(n-1)n}e_{n(n-1)\ldots321} = 1$ so the psuedo-scalar has negative length if the number of swaps is odd. Adding each dimension n provided $n-1$ extra edges thus changing the parity when n is even then keeping the same parity on the next step since $n+1$ is odd. So the parity is organised in pairs, two odd then two even. Hence in $\mathbb{G}_n$ we have a negative length psuedo-scalar for $n \in \{2, 3, 6, 7, 10, 11, 14, 15, \ldots\}$ and since only odd dimensions are commuting we have imaginary psuedo-scalars, $e^2_{123\ldots n} = -1$, for $n \in \{3, 7, 11, 15, \ldots\}$. We have found that this sequence can be extended to negative signatures keeping the psuedo-scalars as (commuting) imaginaries by adding dimensions $e_{0(n+1)}$ and this can be repeated with different negative basis labels.

Hence there are an infinite number of imaginary numbers in different dimension algebras and not just by scaling or making them complex. In this chapter only $\mathbb{G}_3$ was considered. It is obvious from the 6-simplex shown in the Fano plane (Figure 7) that $\mathbb{G}_7$ is worth analysing. We will skip $\mathbb{G}_{11}$ and go straight to $\mathbb{G}_{15}$ because this has similar properties to $\mathbb{G}_7$. In Chapter 4, $\mathbb{G}_7$ and $\mathbb{G}_{15}$ are shown to generate the octonians and sedonions and many other non-associative algebras. Again the psuedo-scalars as imaginary numbers and operators play an important geometric role. Returning to 3-D vector space it is convenient to use $i = e_{123}$ but remember that the complex vector, $i\mathbf{p}$, is a quaternion providing axial rotations and this may help in interpreting an equation.

4 The Vector Algebra War

Clifford algebra was published by W. K. Clifford in 1878 at the start of the vector algebra war but due to his untimely death from tuberculosis the following year his method was lost for almost 50 years. In 1927, Pauli discovered matricies that identified the quantum mechanics of electron spin that were later recognised as satifying Clifford's equation. Matricies themselves are inadequate to describe geometry and needed to be replaced by tensors, as was done by Einstein in order to describe relativity. The path that led to the profusion of algebras and inconsistant terminology makes for an interesting history lesson.

Instead of repeating what is already well documented I will refer the reader to *A History of Vector Analysis*, Michael J. Crowe, University of Louisville, 2002 (https://www.researchgate.net/Home/Vectorization) and the first chapter of *Quaternion algebras*, John Voight, Springer, 2023 (https://math.dartmouth.edu/~jvoight/quat-book.pdf). Also see *The vector algebra war: a historical perspective*, J. M. Chappell, A. Iqbal, J. Hartnell and D. Abbott (https://arxiv.org/pdf/1509.00501.pdf). Unfortunately, these do not reference the part played by René Descartes and Pierre de Fermat for introducing what is now called Cartesian coordinates in the 17th century. J. C. Maxwell used quaternion notation in 1873 to group three Cartisean coordinates into a single vector to simplify his presentation of the equations of electro-magnetism but he didn't apply quaternion multiplication. This is one example of quaternion concepts being reused in the vector algebra which the quaternionists saw as an inferior algebra. Indeed the four Maxwell vector equations can be represented as two quaternion equations as is also true for differential algebra. Inventing another algebra because quaternions were hard to understand which had to be changed for Minkowski's 4-space vectors then tensors and even bra-ket notation shows that the quaternionists may have had a point.

It is also important to emphasise that the algebra, $\mathbb{G}_3$, which integrates reals, vectors, complex numbers and quaternions into a geometric framework, had it been recognised at the time, would have stopped the vector algebra war because both sides would have recognised that they could work together. The ambiguous definition of a vector as both a contravariant and covariant tensor as well as just one of these would have been avoided. Differential forms would also be included as they are a subalgebra of $\mathbb{G}$ whereby the metric Pfaffian is zero. This geometric algebra sits at the intersection of geometry, matricies, differential algebra and octonions so can be used as a springboard to introduce these fields. The following chapters tie $\mathbb{G}$ to various matrix algebras, differential forms, octonions and a series of power-associative algebras starting with sedenions.