The Algebra of Geometry

G. P. Wilmot – Copyright (c) August 8, 2023

1 Improvements

Equation (3) is easier to understand in the original form

$$\begin{aligned}
&\backslash e_{12}, e_{23}, \ldots, e_{n-1,n}| \\
&= \backslash e_{12}, \ldots, e_{(r-1)r}| \backslash e_{(r+1)(r+2)}, \ldots, e_{(n-1)n}| \\
&- \sum_{\mu \in \mathcal{C}^r_{r-2}} \sum_{\nu \in \mathcal{C}^{n-r}_2(\mathbb{N}^n_{r+1})} (-1)^\sigma \backslash e_{\mu_1\mu_2}, \ldots, e_{\mu_{r-3}\mu_{r-2}}||e_{\mu_{r-1},\nu_{r+1}}, e_{\mu_r,\nu_{r+2}}| \backslash e_{\nu_{r+3},\nu_{r+4}}, \ldots, e_{\nu_{n-1}\nu_n}| \\
&+ \sum_{\mu \in \mathcal{C}^r_{r-4}} \sum_{\nu \in \mathcal{C}^{n-r}_4(\mathbb{N}^n_{r+1})} (-1)^\sigma \backslash e_{\mu_1\mu_2}, \ldots, e_{\mu_{r-5}\mu_{r-4}}||e_{\mu_{r-3},\nu_{r+1}}, \ldots, e_{\mu_r,\nu_{r+4}}| \backslash e_{\nu_{r+5},\nu_{r+6}}, \ldots, e_{\nu_{n-1}\nu_n}| \\
&- \ldots \\
&\pm \begin{cases} |e_{1(r+1)}, e_{2(r+2)}, \ldots, e_{rn}| & \text{if } r = \frac{n}{2} \\ \sum\limits_{\mu \in \mathcal{C}^r_{2r-n}} (-1)^\sigma \backslash e_{\mu_1\mu_2}, \ldots, e_{\mu_{2r-n-1}\mu_{2r-n}}||e_{\mu_{2r-n+1},r+1}, \ldots, e_{\mu_r n}| & \text{if } r > \frac{n}{2} \\ \sum\limits_{\nu \in \mathcal{C}^{n-r}_r(\mathbb{N}^n_{r+1})} (-1)^\sigma |e_{1,\nu_{r+1}}, \ldots, e_{r,\nu_{2r}}| \backslash e_{\nu_{2r+1}\nu_{2r+2}}, \ldots, e_{\mu_{n-1}\nu_n}| & \text{if } r < \frac{n}{2} \end{cases}
\end{aligned}$$

where $\mathcal{C}^n_r$ is the $\binom{n}{r}$ sets of indices partitioning $\mathbb{N}^n_1$ into r and $n-r$ parts and σ is the parity of the partition. $\mathcal{C}^n_r(\mathbb{N}^{n+r}_{r+1})$ is same combination but over indices $\{r+1, r+2, \ldots, n+r\}$.

The equation for the determinant after Pythogora's rule (which should be Theorem) becomes the Pfaffian $\backslash \mathbf{a} \wedge \mathbf{b}, \mathbf{c} \wedge \mathbf{d}, \ldots, \mathbf{m} \wedge \mathbf{n}|$ after it is reduced to the last term which is the volume. Then considering the 4-simplex is it obvious that any pair of opposing pairs provide the same volume and each alternate term cancels apart from one

$$\{(\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \wedge \mathbf{d})\}_4 = -\{(\mathbf{a} \wedge \mathbf{c})(\mathbf{b} \wedge \mathbf{d})\}_4 = \{(\mathbf{a} \wedge \mathbf{d})(\mathbf{b} \wedge \mathbf{c})\}_4$$

This argument extends to any degree simplex as $\mathcal{P}_{n,n} = (n/2)!\, \mathcal{P}'_{n,n}$ or

$$[(1,2),(3,4),\ldots,(n-1,n)] = (n/2)!\; [((1,2),(3,4),\ldots,(n-1,n))].$$

So the determinant equation becomes

$$\begin{aligned}
|\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \ldots, \mathbf{m}, \mathbf{n}| &= \mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} \wedge \cdots \wedge \mathbf{m} \wedge \mathbf{n} \\
&= \tfrac{1}{n!} \sum_{\mu \in \mathcal{P}_n} (-1)^\sigma a_{\mu_1} e_{\mu_1} b_{\mu_2} e_{\mu_2} c_{\mu_3} e_{\mu_3} d_{\mu_4} e_{\mu_4} \ldots m_{\mu_m} e_{\mu_m} n_{\mu_n} e_{\mu_n} \\
&= \tfrac{1}{(n/2)!} \sum_{\mu \in \mathcal{P}'_{n,n}} (-1)^\sigma (a_{\mu_1} e_{\mu_1} \wedge b_{\mu_2} e_{\mu_2})(c_{\mu_3} e_{\mu_3} \wedge d_{\mu_4} e_{\mu_4}) \ldots (m_{\mu_m} e_{\mu_m} \wedge n_{\mu_n} e_{\mu_n}) \\
&= \left\{ \sum_{\mu \in \mathcal{P}_{n,n}} (-1)^\sigma (a_{\mu_1} e_{\mu_1} \wedge b_{\mu_2} e_{\mu_2})(c_{\mu_3} e_{\mu_3} \wedge d_{\mu_4} e_{\mu_4}) \ldots (m_{\mu_m} e_{\mu_m} \wedge n_{\mu_n} e_{\mu_n}) \right\}_n \\
&= \{ \backslash (\mathbf{a} \wedge \mathbf{b}), (\mathbf{c} \wedge \mathbf{d}), \ldots, (\mathbf{m} \wedge \mathbf{n})| \}_n \\
&= \pm V e_{12\ldots n}
\end{aligned}$$