The Algebra of Geometry

G. Wilmot (Copyright, 2023)

Introduction

This book explores a fundamental connection between geometry and algebra that is so natural that it is an ideal introduction to the theory of algebra. It has simple rules that introduce imaginary numbers and half of the matrix groups using geometry rather than unjustified and complicated rules. It is so immersed in geometry it is called the geometric algebra. Having a geometric viewpoint allows for terminology that is easy to comprehend and this understanding carries over into other algebras. The geometric approach simplifies and amalgamates concepts and focuses on the most important ones. This algebra is called Clifford algebra and from simple rules the most important equations in physics can be developed and appreciated when viewed using geometry.

The first chapter introduces Pascal's Triangle as a table of objects and connections that define a simple geometric shape called a simplex. This simple start provides the basis for deriving Clifford algebra. Although it is equivalent to geometric algebra, the former is viewed by mathematicians as imaginary numbers whereas the later starts with the much more geometric concept of arrows. This distinction is so important to the geometric understanding I will denote the geometric approach to Clifford algebra as $\mathbb{G}$, for geometric algebra. Imaginary numbers are derived from this with the advantage of having a geometric interpretation. Quaternions are then introduced but again from a geometric viewpoint and their connection to geometry completes the definition of $\mathbb{G}$. $\mathbb{G}$ actually denotes many Clifford algebras, $\mathbb{G}_n$, where n is the number of dimensions. The three dimensional version, $\mathbb{G}_3$, will be the easiest algebra to understand and this understanding will follow through to higher dimensions. We will find that thinking in 5 dimensions is fairly easy and this can be extended to 7 dimensions with a bit more work.

The second chapter covers products of quaternions and the most general products in $\mathbb{G}$. The 3 dimensional algebra, $\mathbb{G}_3$, by itself could be called the geometric algebra. It merges arrows, quaternions and the first imaginary number. The chapter leads onto space-time and the single equation covering all eight of Maxwell's Equations which includes monopoles. The geometric interpretation merges physics and geometry and the extension to 5 dimensions introduces more physics as it describes electrons. Since the elements of $\mathbb{G}$ can be of any dimension, they are called forms which were introduced by Grassmann to define the basis of Euclidean spaces. These generate the exterior algebra which uses determinants to define the exterior products of forms. The equivalent expansion for $\mathbb{G}$ extends this definition to include the Pfaffian expansion which incorporates the metric tensor. Hence $\mathbb{G}$ is seen as the algebra that extends exterior space to a metric space.

Hamilton introduced the term quaternion along with their vector products and the third chapter uncovers how these were applied in an inferior fashion into the upcoming vector algebra. This led to what is known as the vector algebra wars with the proponents of quaternions saying that the new vector algebra had lost the elegance and power of Hamilton's algebra. Chapter three provides a history of this unfortunate episode in mathematics which could have been avoided if Clifford algebra, at the time, had been recognised as the algebra of geometry. The algebra was introduced at the start of these wars but due to the untimely death of William Clifford, it was lost until the quantum mechanics of the electron was discovered 45 years later.

The fourth chapter deals with the relationship of Ꞡ to the matrix algebras and to differential forms. Chapter five then shows the relationship to octonions and the series of sedenions and chapter six exposes a natural connection of Ꞡ to the simplest Lie Exceptional algebra, G2. With this background, chapter seven proceeds to classify the products of all these algebras which exposes a wealth of non-associative algebras with only a few currently being studied. I hope the classification of these algebras provides incentive for further investigation.

---

1. The Algebra of Geometry

Geometry's intimate connection to the geometric algebra Ꞡ starts with Pascal's Triangle which is usually introduced as the expansion of polynomials called the binomial theorem. At the end of this chapter the binomial theorem will be described from a geometric viewpoint and the expansion shown to justify the proof of Pascal's Triangle's connection to Ꞡ. For now this is presented as a table of numbers following rules similar to Fibonacci's Numbers. The latter is a sequence of numbers generated by a simple rule; each number is the sum of the two preceding numbers assuming the sequence starts with a zero and a one. This is called a recurrence relation which is expressed as:

$$F_n = F_{n-1} + F_{n-2}, \quad \text{where } F_0=0 \text{ and } F_1=1,$$

and provides the series 0, 1, 1, 2, 3, 5, 8, ... This sequence has many applications since it describes the idealised birth rate of pairs of rabbits and is related to the golden ratio. The sequence generally does not start at 0 but it is important for Pascal's Triangle below and it makes sense if starting with one pair of rabbits. The relationship to the golden ratio allows Fibonacci's Numbers to describe patterns in nature such as the number of petals of a flower or the number of seeds in the spirals of a sunflower.

Pascal's Triangle extends Fibonacci's Numbers to two dimensions using a table of numbers governed by a similarly simple rule; each number is the sum of the two numbers immediately above it, starting with a row of zeros and with a single one. This rule assumes a symmetric or isosceles triangle but skewing it into a right triangle shape allows the columns to be labelled. This means with 1 at the top of the table and not showing zeros then for each row add the number above and the number above to the left. The resulting table, shown in Table 1, has Fibonacci's Numbers as the sum of each diagonal that slopes down to the left but this is not relevant here since we are only interested in each row separately.

Table 1 Pascal's Triangle

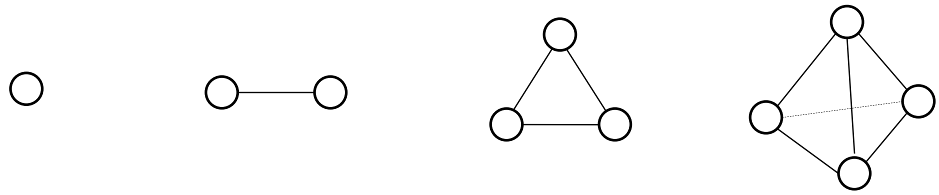
N	**V**	**E**	**F**	**…**			
1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1
			…				

The rule for each cell, Pr,c, written as a recursion relationship from the previous row (r-1) is

$$P_{r,c} = P_{r-1,c} + P_{r-1,c-1}, \quad \text{where r, c are row and column positions.}$$

The first column under the initial one is all 1's and labelled as **N** to mean nothing, which will be justified later. The next 3 columns are labelled **V**, **E** and **F** for Vertices, Edges and Faces, respectively. These labels describe the elements of n-simplices which are generalisations of triangles to n-1 dimensions with V=n. An n-simplex is defined as the simplest polytope in (n-1) dimensions. Polytopes generalise polyhedrons but here we are not interested in hypercubes or hyper octohedrons, etc, just hyper tetrahedrons.

Ignoring the first column in Pascal’s Triangle and hence the first row for now, the second row, V=1, specifying zero dimensions has 1 vertex. This is the tip of a triangle, a single point, and that’s the only possibility in a space with no dimensions. Next is 2 points and 1 edge or line segment. This is a 1-D triangle and a line segment is, at most, all that would be visible of a triangle in one dimension. The fourth row, V=3, is a triangle with 3 edges and 1 face. Next is a tetrahedron or 4-simplex with 4 faces. Each simplex matches the numbers in Table 1 and are shown in Figure 1.



1-simplex 2-simplex 3-simplex 4-simplex

Figure 1 The first 4 simplices

The 4-simplex is drawn using Projective Geometry since it is a 3-D object displayed flat on the page in 2-D. We can go further by putting a point in the middle of the tetrahedron and connecting all vertices to this point. This is a 5-simplex shown in Figure 2 with the internal connections shown as dotted lines and again using projections now you are thinking in four dimensions. This is a four dimensional object displayed in the two dimensions of the page. The diagram shows 10 edges and 10 faces again matching the V=5 row of Pascal’s Triangle. The diagram also shows that there are 5 tetrahedrons which is the next column in Table 1.

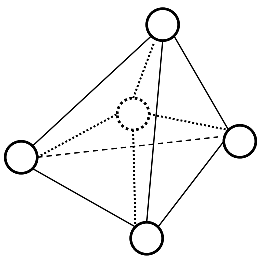


Figure 2 The 5-simplex

Names can be introduced for all these geometric elements in order to make it easier to proceed to higher dimensions. A single point is designed e_1 and two points as e_1 and e_2 with connecting edge labelled as e_{12}. With a third vertex, e_3, the extra edges are e_{23} and e_{31} and the face is e_{123}. This is shown in Figure 3 but it is also important to specify the ordering of points.

The face e_{123} shown in Figure 3 designates a clockwise ordering of the points while e_{321} would show an anti-clockwise order. Such an arrow defines a 3-cycle because starting from e_1 there are three steps to get back to e_1.

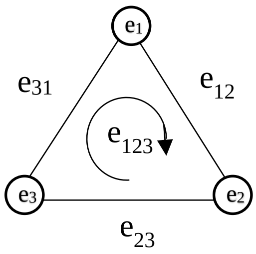


Figure 3   Clockwise order

Starting with a triangle with face e_{123} we can add a point, e_4, in the middle and connect this to the starting three points. Figure 4 shows a tetrahedron looking from the top or a projection of the tetrahedron onto the plane of e_{123}. For clarity only the point labels are shown. Now the 4-simplex can have more than one 3-cycles but can only have two consistent 3-cycles, the clock-wise one on the bottom face e_{123}, which is actually anti-clockwise when looking from below, and another clock-wise from the outside on side, e_{143}, for example. The shared edge has the same direction for the arrows so both 3-cycles can be labelled using e_{31}. No other 3-cycle can be chosen and keep this consistency on the shared edges. But this scheme provides a constraint on the 3-cycles. Changing e_{123} to e_{321} means e_{143} must also change to e_{134}. In Chapter 5 we will find that keeping the sense of the 3-cycles independent is an important concept. In fact leaving them undefined generates Ǥ and constraining them generates different algebras such as the octonians. So the tetrahedron can only have one independent 3-cycle. The 5-simplex in Figure 2 can have two, one outside and one inside and, using projections, we can prove this and extend it to further dimensions.

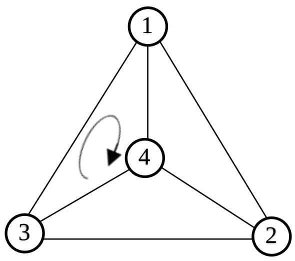


Figure 4   The 4-simplex projection

Starting with the tetrahedron with a base of e_{123} and apex e_4 and a 3-cycle chosen as e134, as shown in Figure 4, then the tetrahedron can be tilted so the apex is above e_{13}. The 3-cycle will disappear so to avoid the loss of information an arrow is added to the line to indicate that e_{143} is a 3-cycle, as shown in Figure 5.

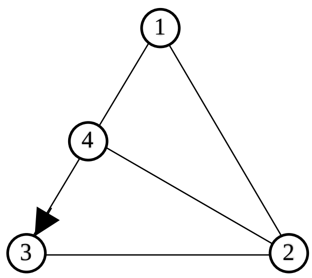


Figure 5   The 4-simplex tilted projection

This is now flattened by projecting it onto the plane of e_{123}. Similar to adding a point to the middle of triangle to get a 4-simplex, a point, e_5, can be added into the projected tilted 4-simplex and connected to all other points. The face e_{125} is independent so can be labelled as a 3-cycle and the simplex tilted towards e_{12}. The 3-cycle becomes an arrow and we again project down to the base plane. The resulting 5-simplex is shown in Figure 6 with the new internal connections shown as dotted lines. The figure shows that the 5-simplex has two independent 3-cycles.

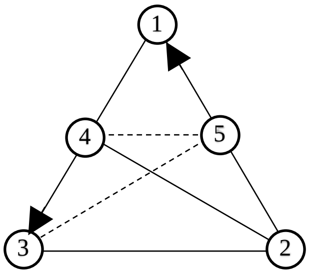


Figure 6 The 5-simplex projection

Another point, e_6, is can be added and the process repeated for the edge e_{23} to get e_{263} which is also an independent 3-cycle. But we now have an extra independent 3-cycle, e_{456}, shown with dotted lines in the 6-simplex of Figure 7, to give 4 independent 3-cycles. The dotted lines are replaced with an arrowed circle before the final point, e_7, is added above the flattened 6-simplex. This automatically introduces another three independent 3-cycles for a total of seven for the 7-simplex. These are denoted e_{143}, e_{125}, e_{236}, e_{456}, e_{176}, e_{274}, e_{375} and show the remarkable coincidence that each vertex is listed three times and each edge is listed once. The resulting diagram for the 7-simplex projection with 7 independent 3-cycles is called the Fano Plane and is usually interpreted as the projection of the cube in dual Projective Geometry where the 3-cycles and points can swap places.

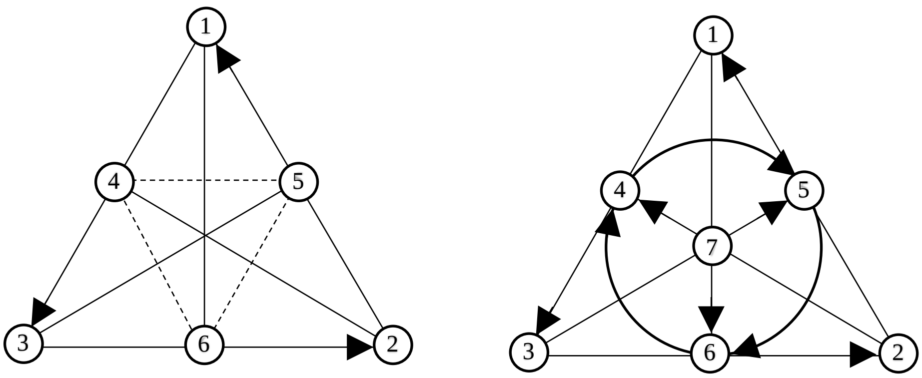


Figure 7 The 6 and 7-simplex projections

+

Row V=7 of Pascal's Triangle shows that the 7-simplex has 21 edges and 35 faces. The Fano Plane diagram shows 15 edges and seven 3-cycles and 28 other labelled triangles and so has hidden 6 edges but all faces can be observed. The missing edges are e_{12}, e_{23}, e_{13}, e_{16}, e_{24} and e_{35} while some of the observed faces, such as e_{156}, may be hard to spot. Chapter 5 shows that there are 30 such unique projections with different edges missing and that this covers all possible projections that can be represented by the Fano Plane.

You may have noticed that the sense of 3-cycles provides a sense for each edge. Each edge in any simplex is also a 2-simplex that connects two vertices so the connection e_{12} can be interpreted as not just connecting e_1 and e_2 but as the operation of changing e_1 into e_2, under the action of some geometric operation, $e_1\, e_{12} \rightarrow e_2$. This is inferred by the 3-cycle e_{123} but with the opposite sense, e_{321}, would mean the reverse operation, $e_2\, e_{21} \rightarrow e_1$. This will be explored shortly.

Now continuous Perspective Geometry can be used to interpret points as lines, lines as areas, faces as volumes, etc. This is achieved by re-labelling the columns in Table 1 as **V**,**E**,**F** for Dimension, Area, Volume, respectively. Once this is done the 5-simplex allows you to think in 5 dimensions. The line segments connecting each dimension describe 2-D areas but are now easier to visualise as rotations between these dimensions. Furthermore, each face of a simplex describes a volume but also defines a 3-cycle of rotations. In this perspective space the simplex labels are called n-forms. Defining a geometric product between 1-forms and 2-forms then the 3-cycle, e_{123}, meaning $e_1 \rightarrow e_2 \rightarrow e_3$, can be represented as

$$e_1\, e_{12} = e_2, \quad e_2\, e_{23} = e_3, \quad e_3\, e_{31} = e_1.$$

To understand e_{12}, e_{23}, e_{31} further the algebra of rotations needs to be considered and this is where quaternions are involved.

Hamilton discovered quaternions while looking for the algebra of rotations in three dimensions. The complex number, i, provides rotations in two dimensions, as shown by Argand in 1806 and published publicly in 1813. Hamilton spent years looking for the three dimensional equivalent. In 1843 he found the solution using 3 complex numbers which can be summarised as $i\, j = k$, where i is the rotation down one wall, j is the rotation across the floor and k is the rotation down the other wall. Each of these are special rotations of 90° so that if we imagine an upwards pointing arrow in the corner of the room then a 90° rotation will change it to point along the floor. Hence a 90° rotation down one wall and across the floor will produce the same effect as rotating the arrow down the other wall. This is the meaning of $i\, j = k$. Quaternions also have cyclic rules for each triple of operations, $j\, k = i$ and $k\, i = j$.

Notice we have not specified which wall to use and quaternions are agnostic to the order of the three rotations. It is only in their interaction with arrows that the sense is defined. Here I have talked about arrows but technically I should have specified vectors. The difference is that arrows can move around and be anywhere in space, the so called Affine Space, but vectors are fixed to a single point, called the origin which defines a Vector Space. It is convenient for you to choose this origin to be in the corner of your room.

From this description we can see that i, j and k are not ordinary numbers but that they operate on vectors. We will see later when compared to Ȼ they also operate on themselves as 90° rotations so we could call them numbers with strange rules but operators is preferable. These operators all have the property that the square is negative meaning they are imaginary,

$$i^2 = j^2 = k^2 = -1.$$

This is easy to see because i operating on an upwards pointing vector moves its direction down the wall to point along the floor. Operating with i again will rotate this vector so that it points downwards, opposite to the direction it started. So we now have 4 ways of getting an arrow to point in the opposite direction. Multiplying by -1 is the zero dimension approach. Reflecting by any point within the arrow is the 1-D solution. Rotating twice around any dimension not in the line is the 2-D way and now we can operate with i, then j, then k. This is the meaning of i j k = -1 which is derived by operating on i j = k by k on both sides. The equation $i^2 = j^2 = k^2 = i\, j\, k = -1$ was famously inscribed by Hamilton into the Broom (originally Bougham) Bridge in Dublin when he first discovered the algebra of rotations. More of this in the Chapter 3 but now with the visualisation of these operators we can uncover another property that distinguishes quaternions from normal numbers.

After rotating the upwards vector by i and j how do we go backwards. We need to operate with -k or by -j then -i to get back to the start. Hence -k = j i = -i j, so that the rotation operators anti-commute. Geometrically, this is the statement that if a rotation plane is inverted then the rotation changes sign. A clockwise rotation becomes anti-clockwise and vise versa. This is a natural consequence of geometry and can now be applied to the 2-forms, introduced above. We have defined these to be 90° rotations so should have the property $e_{12} = -e_{21}$ and $e_{12}^2 = -e_{12}\, e_{21} = -1$. The 2-forms, e_{13}, e_{12} and e_{32}, should be equivalent to i, j and k, given the previous geometric interpretation of i changing e_3 to e_2, j changing e_2 to e_1 and k changing e_3 to e_1. Hence the rule i j = k becomes $e_{13}\, e_{12} = e_{32}$, and similarly for the other quaternion cycle rules. Also i j = -j i simplifies to the $e_{32} = -e_{23}$.

It is not hard to see that with an appropriate geometric product then

$$e_{12} = e_1\, e_2 \quad \text{and} \quad e_1^2 = e_2^2 = e_3^2 = 1.$$

So an area is defined by multiplying two different 1-forms and multiplying a 1-from by itself gives a positive number under the geometric product. Changing the order of multiplication means turning the area over which changes the sign. This argument assumes distributivity of the 1-form product, as can be seen when e_{13} rotates e_{12},

$$e_{13}\, e_{12} = (e_1\, e_3)\, (e_1\, e_2) = -(e_1\, e_1\, e_3)\, e_2 = -e_3\, e_2 = e_{23}.$$

This must be taken as an axiom but since simplices can be built up from the binomial theorem, as seen soon, there is no inconsistency in this definition.

The forms e_{13}, e_{12}, e_{23} can easily be seen to generate a 3-cycle of 2-forms as each rotates the next in turn. The quaternions do not form a 3-cycle unless one term is negated, k, say. Then

$$i\, j = -k', \quad \text{where } k' = -k.$$

Hence i j k' = 1 forms a 3-cycle. Chapter 2 shows that unlike quaternions this generates a negative cross product but this makes sense since these are “imaginary”. Imaginary is not the appropriate term for these operators since they don’t commute. Negative length is more appropriate to distinguish such operators and Chapter 2 shows that this also specifies time-like operations in space-time.

Another important concept here is that n-forms play dual roles. They represent n-simplices as objects in space with signs that change the orientation. A negative arrow points in the opposite direction, a negative plane is one that is turned upside down, a negative volume defines a left-hand screw and finally numbers, which we will see define the lengths or sizes of these objects, have negatives that indicate other properties. Each of these n-forms also represent an operation. We have seen 2-forms act as rotations. Chapter 2 shows 1-forms acting as reflections and 3-forms swapping these roles and the space-time 4-form playing a role in physics. But it is the role of numbers that operate by magnifying or shrinking the size of objects and also change the sense of all these operations. 1-forms are called unit vectors and are usually denoted with bold face $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ or blackboard notation $\underline{e}_1$, $\underline{e}_2$, $\underline{e}_3$. Since we are dealing with multiple dimensional n-forms it is convenient to avoid such notation since this distinguishes n-forms from the usual vector notation. The geometric interpretation of 1-form products will be shown in Chapter 2.

To finish this chapter Pascal's Triangle is reinterpreted in the dimensional view provided by G̸. The equation above, $P_{r,c} = P_{r-1,c} + P_{r-1,c-1}$, is usually interpreted as the coefficients of the binomial expansion,

$$(1+x)^n = 1 + P_{n,1}\, x + P_{n,2}\, x^2 + \ldots + P_{n,n}\, x^n.$$

This is proved using the recursion

$$B_n = (1 + x)\, B_{n-1}, \quad \text{where } B_n = (1+x)^n \text{ so that } B_0 = 1.$$

The dimensional approach provides an interesting re-interpretation of this relationship. Since Pascal's Triangle starts with 1 in the **N** column, without any dimensions, this was interpreted as none, meaning there is no simplex in zero dimensions. Using Projective Geometry where points become dimensions allows a new interpretation of this as the void and every possible number of dimensions given by subsequent rows includes the void. So Pascal's Triangle tells us that in every dimensional space the void always exists and above 1 in Table 1 are only zeros so nothing does not exist.

The x in the recurrence relationship for B_n above indicates a single dimension and x^2 implies two dimensions, etc. Multiplying (1+x) by B_{n-1} as (1 + "x terms in B_{n-1}") gives four parts,

- void squared ($1^2 = 1$) – meaning there is only one void,
- void times the "x terms in B_{n-1}" – replicates all the existing dimensions of B_{n-1},
- x times void – adds the new dimension to the result, and
- x times "x terms in B_{n-1}" – connects the new dimension to all existing dimensions.

The first and thirds items in this list correspond to the first two columns of Table 1. The other two items correspond to the two parts of the Pascal Triangle rules applied to all the other columns. The $P_{n,c}$ factors indicate how many connections exist between dimensions and the void for any collection of dimensions. This is exactly what is expected for the definition of a metric and is also the definition of the Pfaffian which is at the heart of G̸. As we will see over and over again, there is nothing new in the mathematics here, it is the geometric insights that make it easier to comprehend and will provide a consistent terminology.