The Algebra of Geometry

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1 Improvements

Equation (3) is easier to understand in the original form

$$
\begin{aligned}
&\backslash e_{12}, e_{23}, \dots, e_{n-1,n}| \\
&= \backslash e_{12}, \dots, e_{(r-1)r}|\,\backslash e_{(r+1)(r+2)}, \dots, e_{(n-1)n}| \\
&- \sum_{\mu\in\mathcal{C}^r_{r-2}} \sum_{\nu\in\mathcal{C}^{n-r}_2(\mathbb{N}^n_{r+1})} (-1)^\sigma \,\backslash e_{\mu_1\mu_2}, \dots, e_{\mu_{r-3}\mu_{r-2}}||e_{\mu_{r-1},\nu_{r+1}}, e_{\mu_r,\nu_{r+2}}|\,\backslash e_{\nu_{r+3},\nu_{r+4}}, \dots, e_{\nu_{n-1}\nu_n}| \\
&+ \sum_{\mu\in\mathcal{C}^r_{r-4}} \sum_{\nu\in\mathcal{C}^{n-r}_4(\mathbb{N}^n_{r+1})} (-1)^\sigma \,\backslash e_{\mu_1\mu_2}, \dots, e_{\mu_{r-5}\mu_{r-4}}||e_{\mu_{r-3},\nu_{r+1}}, \dots, e_{\mu_r,\nu_{r+4}}|\,\backslash e_{\nu_{r+5},\nu_{r+6}}, \dots, e_{\nu_{n-1}\nu_n}| \\
&- \dots \\
&\pm \begin{cases}
|e_{1(r+1)}, e_{2(r+2)}, \dots, e_{rn}| & \text{if } r = \frac{n}{2} \\
\sum\limits_{\mu\in\mathcal{C}^r_{2r-n}} (-1)^\sigma \,\backslash e_{\mu_1\mu_2}, \dots, e_{\mu_{2r-n-1}\mu_{2r-n}}||e_{\mu_{2r-n+1},r+1}, \dots, e_{\mu_r n}| & \text{if } r > \frac{n}{2} \\
\sum\limits_{\nu\in\mathcal{C}^{n-r}_r(\mathbb{N}^n_{r+1})} (-1)^\sigma |e_{1,\nu_{r+1}}, \dots, e_{r,\nu_{2r}}|\,\backslash e_{\nu_{2r+1}\nu_{2r+2}}, \dots, e_{\mu_{n-1}\nu_n}| & \text{if } r < \frac{n}{2}
\end{cases}
\end{aligned}
$$

where $\mathcal{C}^n_r$ is the $\binom{n}{r}$ sets of indices partitioning $\mathbb{N}^n_1$ into r and $n-r$ parts and σ is the parity of the partition. $\mathcal{C}^n_r(\mathbb{N}^{n+r}_{r+1})$ is same combination but over indices $\{r+1, r+2, \dots, n+r\}$.

The equation for the determinant after Pythogora's rule (which should be Theorem) becomes the Pfaffian $\backslash \mathbf{a}\wedge\mathbf{b}, \mathbf{c}\wedge\mathbf{d}, \dots, \mathbf{m}\wedge\mathbf{n}|$ after it is reduced to the last term which is the volume. Then considering the 4-simplex is it obvious that any pair of opposing pairs provide the same volume and each alternate term cancels apart from one

$$\{(\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d})\}_4 = -\{(\mathbf{a}\wedge\mathbf{c})(\mathbf{b}\wedge\mathbf{d})\}_4 = \{(\mathbf{a}\wedge\mathbf{d})(\mathbf{b}\wedge\mathbf{c})\}_4$$

This argument extends to any degree simplex as $\mathcal{P}_{n,n} = (n/2)!\,\mathcal{P}'_{n,n}$ or

$$[(1,2),(3,4),\dots,(n-1,n)] = (n/2)!\,[((1,2),(3,4),\dots,(n-1,n))].$$

So the determinant equation becomes

$$
\begin{aligned}
|\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\dots,\mathbf{m},\mathbf{n}| &= \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}\wedge\dots\wedge\mathbf{m}\wedge\mathbf{n} \\
&= \frac{1}{n!}\sum_{\mu\in\mathcal{P}_n} (-1)^\sigma a_{\mu_1}e_{\mu_1}b_{\mu_2}e_{\mu_2}c_{\mu_3}e_{\mu_3}d_{\mu_4}e_{\mu_4}\dots m_{\mu_m}e_{\mu_m}n_{\mu_n}e_{\mu_n} \\
&= \frac{1}{(n/2)!}\sum_{\mu\in\mathcal{P}'_{n,n}} (-1)^\sigma (a_{\mu_1}e_{\mu_1}\wedge b_{\mu_2}e_{\mu_2})(c_{\mu_3}e_{\mu_3}\wedge d_{\mu_4}e_{\mu_4})\dots(m_{\mu_m}e_{\mu_m}\wedge n_{\mu_n}e_{\mu_n}) \\
&= \left\{\sum_{\mu\in\mathcal{P}_{n,n}} (-1)^\sigma (a_{\mu_1}e_{\mu_1}\wedge b_{\mu_2}e_{\mu_2})(c_{\mu_3}e_{\mu_3}\wedge d_{\mu_4}e_{\mu_4})\dots(m_{\mu_m}e_{\mu_m}\wedge n_{\mu_n}e_{\mu_n})\right\}_n \\
&= \{\,\backslash(\mathbf{a}\wedge\mathbf{b}),(\mathbf{c}\wedge\mathbf{d}),\dots,(\mathbf{m}\wedge\mathbf{n})|\}_n \\
&= \pm V e_{12\dots n}
\end{aligned}
$$

The derivation equation using the grad operator on page 22 is misleading since it has not shown derivations which puts it at odds with the usual vector notation. The Pfaffian three vector expansion is

$$\mathbf{abc} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) + (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}. \tag{1}$$

This can be separated as $\mathbf{abc} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) + \mathbf{a}(\mathbf{b}\wedge\mathbf{c})$. The wedge product from exterior algeba is well defined as being the exterior part of any multivector product including for scalars so that $\mathbf{a}(\mathbf{b}\cdot\mathbf{c}) = \mathbf{a}\wedge(\mathbf{b}\cdot\mathbf{c})$. The dot product has no equivalent since the multivector product is semi-graded for symmetric and anti-symmetric products and the smallest term is scalar for even products and a vector for odd multi-vector products. We can only uniquely define $\mathbf{aM}$ having a contracted part and an exterior part, $\mathbf{a}\wedge\mathbf{M} = \frac{1}{2}(\mathbf{aM} + (-1)^m\mathbf{Ma})$ if $\mathbf{M}$ has is the product of m vectors. The contracted part has the opposite sign. This means

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \mathbf{a}\wedge(e_{123}\mathbf{b}\times\mathbf{c}) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})e_{123}.$$

When the e_{123} is removed from the wedge product it changes from a 2-form to a 1-form thus changing the symmetry to generate a dot product. Another vector identity is $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ so the 3 multivector becomes

$$\mathbf{abc} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) - \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})e_{123}.$$

Now we can proceed to the gradient operator applied to a versor

$$\nabla\mathbf{ab} = \nabla(\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}) = \nabla(|\mathbf{a}||\mathbf{b}|(\cos(\phi) + I_{\mathbf{ab}}\sin(\phi))). \tag{2}$$

This can be compared to the Pfaffian expansion in terms of changes applicable within the plane and those exterior to it that move the plane by operating on $I_{\mathbf{ab}}$. Taking the last term then differential geometry has $\nabla\wedge\mathbf{a}\wedge\mathbf{b} = (\nabla\wedge\mathbf{a})\wedge\mathbf{b} - \mathbf{a}\wedge(\nabla\wedge\mathbf{b})$. This is called an anti-derivation because the operator must apply to both components but under the exterior product the sign changes when it is applied to $\mathbf{b}$. This is the same as the psuedo scalar (or complex) vector calculus identity $\nabla\cdot(\mathbf{a}\times\mathbf{b}) = (\nabla\times\mathbf{a})\cdot\mathbf{b} - \mathbf{a}\cdot(\nabla\times\mathbf{b})$.

The middle two terms of (1) again use an anti-derivation which is the same as the vector calculus identity

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a}.$$

Hence the Pffafian expansion in (1) as a derivation provides

$$\nabla\mathbf{ab} = \nabla(\mathbf{a}\cdot\mathbf{b}) - \nabla\times(\mathbf{a}\times\mathbf{b}) + \nabla\cdot(\mathbf{a}\times\mathbf{b})e_{123},$$

where the first term is the change of the $\cos$ term in (2), the middle term is the change to the plane of $\mathbf{a}\wedge\mathbf{b}$ and the last term is the change of the $\sin$ part within the plane $\mathbf{a}\wedge\mathbf{b}$.