Structure of the Sedenion Series

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Abstract *Viewing the Cayley-Dickson construction as graded exposes structure on the algebras such as the associative quaternion sub-algebra parts of the octonions. More generally, associativity is divided into three parts, one which specifies zero divisors and matches a Moufang loop identity. Sedenion zero dividors are summarised by seven of them with rules generating all 84 at level four. The 1428 zero divisors for the Cayley-Dickson process at level five are also partially derived from these primaries. The cardinality of zero dividors up to level ten are provided.*

Keywords. Octonions, sedenions, split-octonions, split-sedenions, zero divisors, graded algebras, Moufang loop.

1 Introduction

The octonions are generally not considered as a graded algebra because multiplication of elements do not require such a scheme. But the Cayley-Dickson process is graded with the following graded form for multiplication at level n

$$(a,b)(c,d) = (ac + \epsilon_n d^{*(n-1)}b, da + bc^{*(n-1)}). \tag{1}$$

where generally $\epsilon_i = -1$ for $i \in \mathbb{N}_1^n$, the set of natural numbers up to n and the conjugate $(a,b)^{*n} = (a^{*(n-1)}, -b)$ applies at level n only and is usually denoted without showing the level[1]. The internal products, ac for instance, are then expanded at level $(n-1)$ and the process repeated generating a pyramid easily represented by a computer program. This process stops at level 0 which represents the reals, $\mathbb{R}$.

For $n = 1$, $(a,b)(c,d) = (ac - db, da + bc)$ since $a, b, c, d \in \mathbb{R}$ and $c^{*(0)} = c$ and $d^{*(0)} = d$. For $n = 2$ with $a = (a_1, a_2)$ and similarly for b, c, d we have

$$\begin{aligned}
(a,b)(c,d) &= (((a_1,a_2),(b_1,b_2))((c_1,c_2),(d_1,d_2))) \\
&= ((a_1,a_2)(c_1,c_2) - (d_1,d_2)^{*(1)}(b_1,b_2), (d_1,d_2)(a_1,a_2) + (b_1,b_2)(c_1,c_2)^{*(1)}) \\
&= ((a_1c_1 - a_2c_2, a_1c_2 + a_2c_2) - (d_1b_1 + d_2b_2, d_1b_2 - d_2b_1), \\
&\quad (d_1a_1 - d_2a_2, d_1a_2 + d_2a_1) + (b_1c_1 + b_2c_2, -b_1c_2 + b_2c_1)) \\
&= ((a_1c_1 - a_2c_2 - d_1b_1 - d_2b_2, a_1c_2 + a_2c_1 - d_1b_2 + d_2b_1), \\
&\quad (d_1a_1 - d_2a_2 + b_1c_1 + b_2c_2, d_1a_2 + d_2a_1 - b_1c_2 + b_2c_1)).
\end{aligned} \tag{2}$$

Choose generators, o_i, such that $o_i^2 = \epsilon_i = -1$, for all $i \in \mathbb{N}_1^n$, and call the products of generators pure instead of imaginary to later cover the split case of $\epsilon_i = 1$. For $n = 1$ we have $(a,b)(c,d) = (a + bo_1)(c + do_1) = ac + bo_1do_1 + ado_1 + bo_1c$. For $n = 2$ with $a = a_1 + a_2o_1$ and similarly for b, c, d and for convenience assume product expansion from the left-hand side, ie $o_1o_2o_1 = (o_1o_2)o_1$. Then we have

$$
\begin{aligned}
(a+bo_2)(c+do_2) &= ac + bo_2do_2 + ado_2 + bo_2c \\
&= (a_1+a_2o_1)(c_1+c_2o_1) + (b_1+b_2o_1)o_2(d_1+d_2o_1)o_2 \\
&\quad +(a_1+a_2o_1)(d_1+d_2o_1)o_2 + (b_1+b_2o_1)o_2(c_1+c_2o_1) \\
&= a_1c_1 - a_2c_2 + (a_1c_2+a_2c_1)o_1 - b_1d_1 + b_2d_2o_1o_2o_1o_2 \\
&\quad +b_1d_2o_2o_1o_2 - b_2d_1o_1 + (a_1d_1 - a_2d_2)o_2 + (a_1d_2+a_2d_1)o_1o_2 \\
&\quad +b_1c_1o_2 + b_2c_2o_1o_2o_1 + b_1c_2o_2o_1 + b_2c_1o_1o_2 \\
&= a_1c_1 - a_2c_2 - b_1d_1 - b_2d_2 + (a_1c_2 + a_2c_1 - b_2d_1 + b_1d_2)o_1 \\
&\quad +(a_1d_1 - a_2d_2 + b_1c_1 + b_2c_2)o_2 + (a_2d_1 + a_1d_2 - b_1c_2 + b_2c_1)o_{12}.
\end{aligned}
$$

This equals the representation in (2) if $o_1o_2o_1o_2 = -1$, $o_2o_1o_2 = o_1$, $o_1o_2o_1 = o_2$ and $o_{12} = o_1o_2 = -o_2o_1$. These relations mean $o_{12}o_1 = o_2$ and $o_{12}o_2 = -o_1$ and o_1 and o_2 anticommute along with a rule seen in the corollary below.

Starting with $n = 1$ then $\{1, o_1\}$ is the basis of the complex numbers, $\mathbb{C}$. Next $\{1, o_1, o_2, o_{12}\}$ is the basis of the quaternions, $\mathbb{H}$. The triple of the pure basis elements generates the usual cyclic product structure that defines rotations, $o_1o_2 = o_{12}$, $o_2o_{12} = o_1$, $o_{12}o_1 = o_2$ and which implies these elements anticommute. It is easy to multiple graded elements, it is only the sign that is difficult.

Lemma

Extending o_i by multiplying with larger generators from the right is always positive, $o_io_j \ldots o_n = o_{ij\ldots n}$, for $i < j < n$, still using product expansion from the left-hand side. Secondly, that multiplying the last pair first turns this negative, $o_io_j \ldots (o_{n-1}o_n) = -o_{ij\ldots n}$.

Proof At level $n = 1$, $1o_1 = (1, 0)(0, 1) = (0, da) = o_1$. At $n = 2$,

$$o_1o_2 = ((0, 1), (0, 0))((0, 0), (1, 0)) = ((0, 0), (0, d_1a_2)).$$

Level $n = 3$, $o_{12}o_3$ multiples as

$$(((0, 0), (0, 1)), ((0, 0), (0, 0)))(((0, 0), (0, 0)), ((1, 0), (0, 0))) = (((0, 0), (0, 0)), ((0, 0), (0, d_{11}a_{22}))).$$

It is easy to see that at each level of the pyramid $o_{12\ldots(n-1)}$ is the last position in the left hand half of the expansion, $a_{22} = 1$ in this case, and o_n is the first position in the right hand half of the expansion, $d_{11} = 1$ in this case. From (2) this provides the a_2d_1 term in the last position of the multiplied expansion, which is always positive. If any generator is skipped the d_1 becomes c_1 and the only term in the product expansion is a_2c_1 which is again positive and doesn't include o_j for level $n = j$.

For the second half of the lemma, starting at level 2 for association of the last two terms

$$1o_{12} = ((1, 0), (0, 0))((0, 0), (0, 1)) = ((0, 0), (0, d_2a_1)).$$

With an extra generator at level $n = 3$, o_1o_{23}, then

$$(((0, 1), (0, 0)), ((0, 0), (0, 0)))(((0, 0), (0, 0)), ((0, 0), (1, 0))) = (((0, 0), (0, 0)), ((0, 0), (0, d_{21}a_{12}))).$$

Because we are always dealing with the first and last pair at any level then we generalise the argument. In the first product at level n, parallel terms vanish and only the da cross term survives from (1). The next level product, again containing only cross terms, has a $bc^{*(n-2)}$ term but c has come from a_1 which represents o_1 so is negated by the conjugate. Further products and expansions follow the da scheme and do not change sign including with skipped generators. Thus $o_{12\ldots(n-1)n} = -o_{12\ldots(n-2)}o_{(n-1)n}$ and similarly for missing generators $\in \mathbb{N}_1^{n-2}$ apart from at least one.

Corollary

For x pure and $x^2 = -1$ then $xxy = -y$ and $yxx = -y$.

Proof First note that left-hand product expansion can be called left-association and allows the left most two elements to swap places by changing sign whereas multiplication without brackets can only be carried out from the right-hand side. It is easy to see the first result if $x^2 = -1$ but the second result requires the use of equation (2). Multiplying the $o_1 o_2$ product from the Lemma by o_2 gives

$$((0,0),(0,1))((0,0),(1,0)) = ((0,-d_1 b_2),(0,0)).$$

This is applicable for o_1 and o_2 at any level since these form a subalgebra of higher levels. Again the argument can be generalised to arbitrary pure grades including those with missing generators.

Definition The **associator** is defined, using left-association, as

$$[a,b,c] = (ab)c - a(bc) = abc - a(bc),$$

which is zero for $\mathbb{H}$ which means quaternions are associative. This is not the case for octonions, $\mathbb{O}$, with basis $\{1, o_1, o_2, o_3, o_{12}, o_{23}, o_{13}, o_{123}\}$. While quaternions have one triple, the octonions have $\binom{7}{3} = 35$. Of these 28 have a non-zero associator and 7 are zero. These are

$$(o_1,o_2,o_{12}),(o_2,o_3,o_{23}),(o_1,o_3,o_{13}),(o_1,o_{23},o_{123}),(o_2,o_{13},o_{123}),(o_3,o_{12},o_{123}),(o_{12},o_{13},o_{23}).$$

All of these have quaternion cyclic product rules and the first one is the quaternion subset from level 2. The next two have the same structure and would be called quaternions if the basis where reordered. The last triple provides rotations between the three generators which is a feature of quaternions if the generators represent three dimensional vector space, $\mathbb{R}^3$. The three remaining triples can not be interpreted in this way but it is easy to see that the product of the three numbers in all seven triples provides a scalar, either ± 1.

Definition There are N^3 triple products bcd and $\binom{N}{3}$ naturally ordered triples with pure $b < c < d$ where $N = 2^n - 1$ and n is the number of generators. This is the same as removing all pairs and triples of the same element from the N^3 set and dividing by the number of permutations, 6. A **triad** is defined as ordered triples containing unique, pure elements such that bcd is non-scalar and $b < c < d$. Hence $a = bcd$ can be defined for triads and a is unique and pure. Of course, there are 6 unordered triples for each triad.

Associativity Theorem

The 24 permutations of triads and their product generate three types of associativity, called types a, b and c. These are shown in Table 1 noting that $[a,b,c] = -[c,b,a]$ so only 12 permutations are required. The three types are not completely distinct sets but c-associativity triads have a-associative and b-associative distinct subsets.

Table 1: Triad Associativity Types

a-associativity	$[a,b,c] \approx [a,d,c] \approx [b,a,d] \approx [b,c,d]$
b-associativity	$[b,a,c] \approx [a,b,d] \approx [a,c,d] \approx [b,d,c]$
c-associativity	$[c,b,d] \approx [a,c,b] \approx [a,d,b] \approx [c,a,d]$

Proof Relation $a = bcd$ means $ad^{-1} = bc$ or $d = -a(bc)^{-1} = -bca/b^2/c^2$, noting that $(bc)^2 = -b^2c^2$. Assuming a-associativity $[a,b,c] = abc - a(bc) = 0$ or $[c,b,a] = cba - c(ba) = 0$, then substituting d into the other three associators in the first row, gives

$$\begin{aligned}
[a,d,c](bc)^2 &= a(bca)c - a(bcac) = -(bca)ac + a(c(ba)c) = -bcca^2 + (bacc)a = 0, \\
[b,a,d](bc)^2 &= ba(bca) - b(a(bca)) = -ba(c(ba)) + b(bc)a^2 = c(ba)(ba) + cbba^2 = 0 \;\text{ and} \\
[b,c,d](bc)^2 &= bc(bca) - b(c(bca)) = -(bca)(bc) - (bca)cb = -(bc)a(bc) + c(ba)cb \\
&= a(bc)(bc) - baccb = a(bc)^2 + abbc^2 = -ab^2c^2 + ab^2c^2 = 0.
\end{aligned}$$

Similarly for b-associativity and c-associativity. More important are the associators that don't involve a which have c, d and b as the middle term but these are not as memorable.

Substituting c-associativity $[c, b, d] = cbd - c(bd) = 0$ into the a-associator and b-associator that don't involve a gives

$$[b, c, d] = bcd - b(cd) = -c(bd) + (cd)b = cdb + c(db) \text{ and}$$
$$[b, d, c] = bdc - b(dc) = -c(bd) + dcb = -cbd + dcb = bcd + b(cd).$$

The first equation results in the reversed b-associator but with positive sign and the second has the a-associator with positive sign. Thus a-associativity and c-associativity can not be b-associative and similarly bc-associativity can not be a-associative.

Definition Associativity for triads are paired, so that a-associativity means either b- or c-associativity as well, which are called **ab-associativity** and **ac-associativity**, respectively. The alternatives are **bc-associativity** or non-associativity which also has sub-cases. If a triad is ab-associative then it is c-non-associative and similarly for the other cases. A complete non-associative triad is **abc-non-associative**.

Octonians are completely non-associative while sedenions have 420 triads of which 84 are ab-associative, 84 are ac-associative and 252 are abc-non-associative. Hence there are 336 c-non-associative triads but this overlaps with the 252 abc-non-associative and the 84 ab-associative triads. Similarly for the 336 b-non-associative triads. So the non-associative structure is complimentary to the associative structure.

Moufang Associativity Theorem

A Moufang loop [2] is a non-associative algebraic structure that satisfies four identities. Calling these types 1 to 4, the first three are not equivalent and when types 1 and 2 show associativity this is the same as a-associativity and c-associativity, respectively. Moufang non-associativity of types 1 and 2 are defined as

$$d(b(dc)) - dbdc = 0 \text{ and } b(d(cd)) - bdcd = 0.$$

Proof Inequality of these equations therefore defines associativity. The equations, multiplied by d and assuming $d^2 = -1$, become

$$d(b(dc))d - dbdcd = -b(dc)dd + bddcd = dcbdd - bcd = -dcb - bcd = -bcd - b(cd)$$
$$b(d(cd))d - bdcdd = -d(cd)bd + dbcdd = -cddbd + dbcdd = cbd - dbc = cdb + c(db).$$

The first is a-associativity with a positive sign instead of negative and the second is c-associativity with positive sign. Hence a-associativity means that Moufang type 1 is greater than 0 and thus associative and similarly for c-associativity and type 2.

Definition Zero divisors have the property $(a + b)(c + d) = 0$ for a, b, c, d all single pure and unique and $a^2 = b^2$ and $c^2 = d^2$.

Zero Divisors Theorem

The parallel terms and cross terms of the zero divisors equation separate into

$$ac + bd = 0 \text{ and } ad + bc = 0. \tag{3}$$

with the second equation being an identity thus reducing the definition to just $ac = -bd$. Further, this is equivalent to c-associativity.

Proof Uniqueness of a, b, c and d ensures ac does not have the same grade as either ad or bc, so the two equations have unique grades and are separable. Since b, c, d are all single, non-scalar elements, they anti-commute, so the equations can be re-arranged as

$$a = -(bd)c^{-1} = c(bd)/c^2 \text{ and } a = -(bc)d^{-1} = (cb)d/d^2,$$

which can only be equal if b, c, d are c-associative, $[c, b, d] = 0$, since $c^2 = d^2$. Also the product $a \approx bcd$ means a is not a scalar because (3) would become $c \approx db$ and $d \approx -bc$ and by substitution, $c \approx cbb = cb^2$. This is a contradiction because $b^2 < 0$. In the other direction

$$[c, b, d] = cbd - c(bd) = cbd + c(ac) = cbd - bcdcc/d^2 = cbd - cbd = 0.$$

Finally, substituting for a, $ad = bcdd = -bc$ so that $ad + bc = 0$ is an identity.

Dual and Extended Theorem

Calling (3) the prime zero divisor then if the triad for any prime satisfies ac-associativity then $(-d + b)(c + a) = 0$, which is called the dual of the prime. If the prime satisfies b-associativity then either $(a' + b)(c + db) = 0$, which is called the extended zero divisor, with $a' = bc(db)$, or its dual is satisfied. Some a-associative primes may also have extended and its dual solutions.

Proof C-associativity for the dual equals a-associativity for the prime, by symmetry, $[c, b, a] = -[a, b, c] = 0$, hence is a zero divisor. And vise versa, a-associativity for the dual equals c-associativity for the prime, $[-d, b, c] = [c, b, d] = 0$.

The extended zero divisor $(a' + b)(c + db) = 0$ separates into $a'c + b(db) = 0$ and $bc + a'(db) = 0$. The extended dual is $(-db + b)(c + a')$ which separates into $dbc - ba' = 0$ and $bc - dba' = 0$. The second equation for the extended case and its dual are the same, since $dba' = -a'(db)$, and is the identity $bc + bc(db)(db) = 0$ assuming $d^2b^2 = 1$. The definition of zero divisors required that $a^2 = b^2$ and $c^2 = d^2$. For the extended equation to hold these become, $a'^2 = b^2$ and $c^2 = (db)^2$ which implies $d^2 = a'^2 = 1$. Hence for all subsequent work we assume $a^2 = b^2 = c^2 = d^2 = -1$.

The two cases for b-associativity are separated by whether the new d satisfies c-associativity, $[db, b, c] = 0$, or not. If the db triad is c-associative then the extended solution holds, $a'c + b(db) = 0$. Multiplying by c^{-1} and using $[db, b, c] = 0$ gives

$$a' = (db)bc/c^2 = (db)(bc)/c^2 = -bc(db)/c^2,$$

which holds since $c^2 = -1$. If db is c-non-associative then it is either ab-associative or completely non-associative. Trying a-associativity, $[b, c, db] = 0$, then we find the extended dual solution holds, $dbc - ba' = 0$. That is

$$ba' = b(bc(db)) = -b(c(db))b = c(db)b^2 = -c(db) = dbc.$$

Definition Naturally ordered basis pairs form **3-cycles**, $(b, c), (b, bc)$ and (c, bc) which cover all ordered pairs. This is because there are $(2^n - 1)(2^n - 2)/2$ combinations of pairs which have factors one and two numbers less than a power of 2 so one must have a factor of 3. These pairs are ordered since $b < c < bc$ and are called the first, second and third cycle, respectively.

Cycle Theorem

All triads with first pair bc-associativity form bc-associative second and ac-associative third cycles. All triads with first pair ac-associativity either have the ac-associativity for all cycle pairs or don't have second and third cycle solutions. The former case occurs when the first cycle triad has an extended solution and the later when it does not. The missing cycles are not prime solutions and, as a comment, some would already have been found if $d < bc$.

Proof The second pair triad has c-associativity $[bc, b, d] = bcbd - bc(bd)$ and the third pair $[bc, c, d] = bccd - bc(cd)$, which we need to prove.

From the Dual and Extended Theorem, b-associativity means either the extended or its dual is a solution. The extended solution's c-associativity is $[c, b, db] = 0$ so that $cb(db) = cbdb = c(bd)b$ by c-associativity. Using b-associativity as $[c, d, b] = 0$ gives $c(bd)b = -c(db)b = -cdb^2 = cd$. Hence c-associativity for the second cycle in this case is

$$[bc, b, d] = bcbd - bc(bd) = -cdb^2 - cb(db) = cd - cd = 0.$$

Also for the second cycle, b-associativity is

$$[b, d, bc] = bd(bc) - b(d(bc)) = -cb(db) - bcdb = -cbdb + cbdb = 0,$$

where c-associativity of the extended solution has been used $[c, b, db] = 0$. Hence b-associativity means ac-associativity for the second cycle. For the third cycle, assume extended c-associativity, $[db, b, c] = dbbc - db(bc) = 0$. Substitute $cd = db(bc)$ into c-associativity for the third cycle

$$[bc, c, d] = bccd - bc(cd) = -bd - bc(db(bc)) = db + db(bc)(bc) = 0.$$

Also, using c-associativity $[c, b, d] = -[d, b, c] = 0$ in b-associativity of the third cycle is

$$[c, d, bc] = cdbc - c(d(bc)) = c(db)c - c(dbc) = db + dbc^2 = 0,$$

so the third cycle is ac-associative if the first cycle is bc-associative.

For the first cycle having ac-associativity and the extended solution then the second and third cycles have already been shown to satify c-associativity above, without using b-associativity. The a-associative second and third cycles are $[b, bc, d] = bbcd - b(bcd) = -cd + b(cd)b = 0$, using $[b, c, d] = 0$ and $[c, bc, d] = cbcd - c((bc)d) = -bdc^2 - dbcc = 0$, using c-associativity $[d, b, c] = 0$. Thus ac-associative first cycles give ac-associative second and third cycles if they have extended solutions. If not, then $cb(db) \neq cbdb$ and so both second and third cycles are c-non-associative giving first cycle single solutions for extended non-associativity which completes the theorem.

Octonians, $\mathbb{O}$, have 7 pure elements and thus $7 \times 6 \times 5/6 = 35$ ordered triples split into 28 non-associative and 7 associative. But the candidates for zero divisors, the 7 associative triples all have scalar products. For example the quaternions o_1, o_2, o_{12} are a subset of $\mathbb{O}$ and are well known to have the product -1 and cyclic rotation structure. The remaining associative triads are quaternion-like, with rotation structure. Hence octonions do not have zero divisors. To achieve this we need another generator, o_4, for example, which using the lemma gives $o_{23}o_4 = -o_2o_{34}$ and hence $[o_2, o_3, o_4] = 0$ and $a = o_{234}$.

2 Sedenion Series

Cawagas and the Wikipedia, [3][4], label the pure sedenions as e_i, for $i \in \mathbb{N}_1^{16}$. The graded single, pure elements, called the basis of $\mathbb{S}$, have a natural order generated by the Cayley-Dickson process, which match these in order as

$$(e_1, e_2, \ldots e_{15}) = (o_1, o_2, o_{12}, o_3, o_{13}, o_{23}, o_{123}, o_4, o_{14}, o_{24}, o_{124}, o_{34}, o_{134}, o_{234}, o_{1234}).$$

The first three elements are seen to be the quaternions and the first seven match the octonions[5]. Sedenions are power associative which means the associator with repeated terms is always zero, $[a, a, a] = 0$. This is obvious for an algebra where the single terms square to -1 and other than the zero divisors all other terms have inverses. Octonians are alternate associative because $[a, a, b] = [a, b, a] = 0$ for all $a, b \in \mathbb{O}$. But sedenions are not alternate associative. For example

$$[o_1, o_2, o_{34}] = -o_{12}o_{34} - o_1o_{234} = -o_{1234} + o_{1234} = 0 \text{ and } [o_1, o_{34}, o_2] = o_{134}o_2 + o_1o_{234} = -2o_{1234}.$$

This is the reason the formula for zero divisors above contained the fourth generator in the last term. This example produces the zero divisor $(o_1 - o_{1234})(o_2 + o_{34})$ which provides an example that non-associativity is not always anti-associativity. Note that $(o_2 + o_{34})$ is invertible so that $(o_2 + o_{34})^2 = -2$. Then $((o_1 - o_{1234})(o_2 + o_{34}))(o_2 + o_{34}) = 0$ whereas $(o_1 - o_{1234})((o_2 + o_{34})(o_2 + o_{34})) = 2(o_{1234} - o_1)$. For $\mathbb{S}$, we found above that zero divisors need $d = o_{i4}$ or $d = o_{ij4}$ or $d = o_{1234}$ but b and c do not contain o_4 in order to satisfy $[c, b, d] = 0$. This means a also contains the o_4 generator, satisfying the symmetry required for the dual and, indeed, all $\mathbb{S}$ zero divisors are ac-associative.

Sedenions contain 455 ordered triples split into 252 non-associate and 203 a-associative. Of these 35 are quaternion-like and the 168 remaining are split into 84 b-associatives and 84 c-associatives that are distinct. Thus the b-associative half are not c-associative so there are no extended only zero divisors in $\mathbb{S}$. Half of the c-associatives are provided in Table 2 in prime and extended forms and duals can be obtained by swapping a with $-d$ and ab with $-db$ which matches the 84 zero divisors listed in [3]. These triads also allow for 3-cycles and the primes are shown in the d column of Table 3. The bc 3-cycles cover all octonian pairs and as mentioned above only the d term contains o_4. Swapping d with a b or c term just provides an existing solution so the ordering $b < c < d$ is appropriate.

Table 2: $\mathbb{S}$ Zero Divisors

	a	***b***	***c***	***d***	***ab***	***db***
1	$-o_{1234}$	o_1	o_2	o_{34}	o_{234}	o_{134}
2	o_{234}	o_1	o_{12}	o_{34}	o_{1234}	o_{134}
3	o_{1234}	o_1	o_3	o_{24}	$-o_{234}$	o_{124}
4	$-o_{234}$	o_1	o_{13}	o_{24}	$-o_{1234}$	o_{124}
5	o_{134}	o_1	o_{23}	o_{24}	o_{34}	o_{124}
6	$-o_{34}$	o_1	o_{123}	o_{24}	o_{134}	o_{124}
7	$-o_{134}$	o_2	o_{12}	o_{34}	o_{1234}	o_{234}
8	$-o_{1234}$	o_2	o_3	o_{14}	$-o_{134}$	$-o_{124}$
9	o_{234}	o_2	o_{13}	o_{14}	o_{34}	$-o_{124}$
10	$-o_{134}$	o_2	o_{23}	o_{14}	o_{1234}	$-o_{124}$
11	o_{34}	o_2	o_{123}	o_{14}	$-o_{234}$	$-o_{124}$
12	o_{234}	o_{12}	o_3	o_{14}	$-o_{134}$	o_{24}
13	o_{1234}	o_{12}	o_{13}	o_{14}	o_{34}	o_{24}
14	$-o_{34}$	o_{12}	o_{23}	o_{14}	o_{1234}	o_{24}
15	$-o_{134}$	o_{12}	o_{123}	o_{14}	$-o_{234}$	o_{24}
16	$-o_{124}$	o_3	o_{13}	o_{24}	$-o_{1234}$	$-o_{234}$
17	o_{124}	o_3	o_{23}	o_{14}	o_{1234}	$-o_{134}$
18	$-o_{24}$	o_3	o_{123}	o_{14}	$-o_{234}$	$-o_{134}$
19	o_{24}	o_{13}	o_{23}	o_{14}	o_{1234}	o_{34}
20	o_{124}	o_{13}	o_{123}	o_{14}	$-o_{234}$	o_{34}
21	o_{124}	o_{23}	o_{123}	o_{24}	o_{134}	o_{34}

Definition The **sedenion series** will be denoted with the number of generators, n, as $\mathbb{S}_n$. Thus octonions are $\mathbb{S}_3 \equiv \mathbb{O}$ and sedonions are $\mathbb{S}_4 \equiv \mathbb{S}$ with the 15 basis terms above. In general the number of pure basis terms is $N = 2^n - 1$ so that $\mathbb{S}_5$ has 31 pure basis terms.

The sedenions with five generators, $\mathbb{S}_5$, has 4495 ordered triples split into 2156 non-associative and 2339 a-associative. Of these 155 are quaternion-like with scalar a but unlike $\mathbb{S}_4$ some of the c-associatives are b-associative. There are 1428 c-associatives split into 1092 ac-associatives and 336 bc-associatives. Table 3 shows solutions derived from $\mathbb{S}$ which used the 4$^{\text{th}}$ generator in the d term. It uses the Cycle Theorem to simplify the 84 zero divisors of $\mathbb{S}$ to just 7 primes with, since they are ac-associative, dual, extended and dual extended solutions derived from column d. This is extended by $\mathbb{S}_5$ to include the 5$^{\text{th}}$ generator and both together, o_{45}, as shown in the d_1 and d_2 columns of the table. All of these triads are ac-associative with 3-cycle solutions thus giving giving three lots of 84 zero divisors.

Table 3: Zero Divisors in $\mathbb{S}_5$ derived from $\mathbb{O}$

bc			***d***	$\boldsymbol{d}_1$	$\boldsymbol{d}_2$
o_1o_2	o_1o_{12}	o_2o_{12}	o_{34}	o_{35}	o_{45}
o_1o_3	o_1o_{13}	o_3o_{13}	o_{24}	o_{25}	o_{45}
o_1o_{23}	o_1o_{123}	$o_{23}o_{123}$	o_{24}	o_{25}	o_{45}
o_2o_3	o_2o_{23}	o_3o_{23}	o_{14}	o_{15}	o_{45}
o_2o_{13}	o_2o_{123}	$o_{13}o_{123}$	o_{14}	o_{15}	o_{45}
$o_{12}o_3$	$o_{12}o_{123}$	o_3o_{123}	o_{14}	o_{15}	o_{45}
$o_{12}o_{13}$	$o_{12}o_{23}$	$o_{13}o_{23}$	o_{14}	o_{15}	o_{45}

Five of the rows in this table can be extended by rotating the bc indices positively and one negatively, wrapping $1 \rightarrow 4$. The other two rows, $o_2o_{13} \rightarrow o_{13}o_4$ and $o_{12}o_{13} \rightarrow o_{123}o_4$, do not follow a pattern nor do the d columns when compaired with Table 4. Again all triads in Table 4 are ac-associative for each 3-cycle thus giving another $21 \times 4 \times 3 = 252$ zero divisors.

Table 4: Zero Divisors in $\mathbb{S}_5$ independent of $\mathbb{O}$

bc			$\boldsymbol{d}_1$	$\boldsymbol{d}_2$	$\boldsymbol{d}_3$
o_1o_4	o_1o_{14}	o_4o_{14}	o_{25}	o_{35}	o_{235}
o_2o_4	o_2o_{24}	o_4o_{24}	o_{15}	o_{35}	o_{135}
$o_{12}o_4$	$o_{12}o_{124}$	o_4o_{124}	o_{15}	o_{35}	o_{135}
o_3o_4	o_3o_{34}	o_4o_{34}	o_{15}	o_{25}	o_{125}
$o_{123}o_4$	$o_{123}o_{1234}$	o_4o_{1234}	o_{15}	o_{25}	o_{125}
$o_{23}o_4$	$o_{23}o_{234}$	o_4o_{234}	o_{15}	o_{25}	o_{125}
$o_{13}o_4$	$o_{13}o_{134}$	o_4o_{134}	o_{15}	o_{25}	o_{125}

We can now cover all other bc pairs that don't include o_5. These 21 sets of 3-cycles are listed in Table 5 but this time only the first d_1 column is a-associative for all 3-cycles with both prime and extended with dual solutions giving $21 \times 3 \times 4 = 252$ zero divisors. The other two columns are b-associative for the first and second cycles, with only prime and extended solutions while the third cycle has a-associativity with only prime and dual solutions. Thus the d_2 and d_3 columns provide $21 \times 3 \times 2 = 126$ solutions each and the table provides $252 + 126 \times 2 = 504$ zero divisors.

Table 5: Zero Divisors in $\mathbb{S}_5$ with mainly a-associativity

bc			***d***	$\boldsymbol{d}_1$	$\boldsymbol{d}_2$	***bc***			***d***	$\boldsymbol{d}_1$	$\boldsymbol{d}_2$
o_1o_{24}	o_1o_{124}	$o_{24}o_{124}$	o_{25}	o_{345}	o_{2345}	o_1o_{34}	o_1o_{134}	$o_{34}o_{134}$	o_{35}	o_{245}	o_{2345}
o_1o_{234}	o_1o_{1234}	$o_{234}o_{1234}$	o_{235}	o_{245}	o_{345}	o_2o_{14}	o_2o_{124}	$o_{14}o_{124}$	o_{15}	o_{345}	o_{1345}
o_2o_{34}	o_2o_{234}	$o_{34}o_{234}$	o_{35}	o_{145}	o_{1345}	o_2o_{134}	o_2o_{1234}	$o_{134}o_{1234}$	o_{135}	o_{145}	o_{345}
$o_{12}o_{14}$	$o_{12}o_{24}$	$o_{14}o_{24}$	o_{15}	o_{345}	o_{1345}	$o_{12}o_{34}$	$o_{12}o_{1234}$	$o_{34}o_{1234}$	o_{35}	o_{145}	o_{1345}
$o_{12}o_{134}$	$o_{12}o_{234}$	$o_{134}o_{234}$	o_{135}	o_{1245}	o_{345}	o_3o_{14}	o_3o_{134}	$o_{14}o_{134}$	o_{15}	o_{245}	o_{2345}
o_3o_{24}	o_3o_{234}	$o_{24}o_{234}$	o_{25}	o_{145}	o_{1245}	o_3o_{124}	o_3o_{1234}	$o_{124}o_{1234}$	o_{125}	o_{145}	o_{245}
$o_{13}o_{14}$	$o_{13}o_{34}$	$o_{14}o_{34}$	o_{15}	o_{245}	o_{1245}	$o_{13}o_{24}$	$o_{13}o_{1234}$	$o_{24}o_{1234}$	o_{25}	o_{145}	o_{1245}
$o_{13}o_{124}$	$o_{13}o_{234}$	$o_{124}o_{234}$	o_{125}	o_{145}	o_{245}	$o_{23}o_{14}$	$o_{23}o_{1234}$	$o_{14}o_{1234}$	o_{15}	o_{245}	o_{1245}
$o_{23}o_{24}$	$o_{23}o_{34}$	$o_{24}o_{34}$	o_{25}	o_{145}	o_{1345}	$o_{23}o_{124}$	$o_{23}o_{134}$	$o_{124}o_{134}$	o_{125}	o_{145}	o_{245}
$o_{123}o_{14}$	$o_{123}o_{234}$	$o_{14}o_{234}$	o_{15}	o_{245}	o_{1245}	$o_{123}o_{24}$	$o_{123}o_{134}$	$o_{24}o_{134}$	o_{25}	o_{145}	o_{1245}
$o_{123}o_{124}$	$o_{123}o_{34}$	$o_{124}o_{34}$	o_{125}	o_{145}	o_{245}						

Table 5 can be replicated with o_4 replaced with o_5 in the bc pairs, apart from the first d column. The last two columns, d_1 and d_2, are replicated identically in Table 6 and have the same properties. Hence this represents $21 \times 3 \times 4 = 252$ solutions. Tables 5 and 6 provide the only b-associative triads in two columns for two cycles giving $21 \times 2 \times 2 \times 2 = 336$ b-associatives, as expected.

Table 6: Zero Divisors in $\mathbb{S}_5$ with mainly b-associativity

bc			$\boldsymbol{d}_1$	$\boldsymbol{d}_2$	***bc***			$\boldsymbol{d}_1$	$\boldsymbol{d}_2$
o_1o_{25}	o_1o_{125}	$o_{25}o_{125}$	o_{345}	o_{2345}	o_1o_{35}	o_1o_{135}	$o_{35}o_{135}$	o_{245}	o_{2345}
o_1o_{235}	o_1o_{1235}	$o_{235}o_{1235}$	o_{245}	o_{345}	o_2o_{15}	o_2o_{125}	$o_{15}o_{125}$	o_{345}	o_{1345}
o_2o_{35}	o_2o_{235}	$o_{35}o_{235}$	o_{145}	o_{1345}	o_2o_{135}	o_2o_{1235}	$o_{135}o_{1235}$	o_{145}	o_{345}
$o_{12}o_{15}$	$o_{12}o_{25}$	$o_{15}o_{25}$	o_{345}	o_{1345}	$o_{12}o_{35}$	$o_{12}o_{1235}$	$o_{35}o_{1235}$	o_{145}	o_{1345}
$o_{12}o_{135}$	$o_{12}o_{235}$	$o_{135}o_{235}$	o_{145}	o_{345}	o_3o_{15}	o_3o_{135}	$o_{15}o_{135}$	o_{245}	o_{2345}
o_3o_{25}	o_3o_{235}	$o_{25}o_{235}$	o_{145}	o_{1245}	o_3o_{125}	o_3o_{1235}	$o_{125}o_{1235}$	o_{145}	o_{245}
$o_{13}o_{15}$	$o_{13}o_{35}$	$o_{15}o_{35}$	o_{245}	o_{1245}	$o_{13}o_{25}$	$o_{13}o_{1235}$	$o_{25}o_{1235}$	o_{145}	o_{245}
$o_{13}o_{125}$	$o_{13}o_{235}$	$o_{125}o_{235}$	o_{145}	o_{245}	$o_{23}o_{15}$	$o_{23}o_{1235}$	$o_{15}o_{1235}$	o_{245}	o_{1245}
$o_{23}o_{25}$	$o_{23}o_{35}$	$o_{25}o_{35}$	o_{145}	o_{1245}	$o_{23}o_{125}$	$o_{23}o_{135}$	$o_{125}o_{135}$	o_{145}	o_{245}
$o_{123}o_{15}$	$o_{123}o_{235}$	$o_{15}o_{235}$	o_{245}	o_{1245}	$o_{123}o_{25}$	$o_{123}o_{135}$	$o_{25}o_{135}$	o_{145}	o_{1245}
$o_{123}o_{125}$	$o_{123}o_{35}$	$o_{125}o_{35}$	o_{145}	o_{245}					

Finally, Table 7 shows the remaining c-associative triads containing o_5. These are all a-associative with only prime and dual solutions for the first cycles while the second and third cycles are not c-associative. Only the first cycle pairs need be displayed in this table which thus provides $42 \times 4 = 168$ solutions. Adding all solutions from Tables 3 through 7 gives $252 + 252 + 504 + 252 + 168 = 1428$ which matches the total number of c-associatives.

Table 7: Zero Divisors in $\mathbb{S}_5$ with 1-cycle only

bc	$\boldsymbol{d}_1$	$\boldsymbol{d}_2$	$\boldsymbol{d}_3$	$\boldsymbol{d}_4$	***bc***	$\boldsymbol{d}_1$	$\boldsymbol{d}_2$	$\boldsymbol{d}_3$	$\boldsymbol{d}_4$
$o_{14}o_{25}$	o_{345}	o_{1345}	o_{2345}	o_{12345}	$o_{14}o_{125}$	o_{345}	o_{1345}	o_{2345}	o_{12345}
$o_{14}o_{35}$	o_{245}	o_{1245}	o_{2345}	o_{12345}	$o_{14}o_{135}$	o_{245}	o_{1245}	o_{2345}	o_{12345}
$o_{14}o_{235}$	o_{245}	o_{1245}	o_{345}	o_{1345}	$o_{14}o_{1235}$	o_{245}	o_{1245}	o_{345}	o_{1345}
$o_{24}o_{15}$	o_{345}	o_{1345}	o_{2345}	o_{12345}	$o_{24}o_{125}$	o_{345}	o_{1345}	o_{2345}	o_{12345}
$o_{24}o_{35}$	o_{145}	o_{1245}	o_{1345}	o_{12345}	$o_{24}o_{135}$	o_{145}	o_{1245}	o_{345}	o_{2345}
$o_{24}o_{235}$	o_{145}	o_{1245}	o_{1345}	o_{12345}	$o_{24}o_{1235}$	o_{145}	o_{1245}	o_{345}	o_{2345}
$o_{124}o_{15}$	o_{345}	o_{1345}	o_{2345}	o_{12345}	$o_{124}o_{25}$	o_{345}	o_{1345}	o_{2345}	o_{12345}
$o_{124}o_{35}$	o_{145}	o_{245}	o_{1345}	o_{2345}	$o_{124}o_{135}$	o_{145}	o_{245}	o_{345}	o_{12345}
$o_{124}o_{235}$	o_{145}	o_{245}	o_{345}	o_{12345}	$o_{124}o_{1235}$	o_{145}	o_{245}	o_{1345}	o_{2345}
$o_{34}o_{15}$	o_{245}	o_{1245}	o_{2345}	o_{12345}	$o_{34}o_{25}$	o_{145}	o_{1245}	o_{1345}	o_{12345}
$o_{34}o_{125}$	o_{145}	o_{245}	o_{1345}	o_{2345}	$o_{34}o_{135}$	o_{245}	o_{1245}	o_{2345}	o_{12345}
$o_{34}o_{235}$	o_{145}	o_{1245}	o_{1345}	o_{12345}	$o_{34}o_{1235}$	o_{145}	o_{245}	o_{1345}	o_{2345}
$o_{134}o_{15}$	o_{245}	o_{1245}	o_{2345}	o_{12345}	$o_{134}o_{25}$	o_{145}	o_{1245}	o_{345}	o_{2345}
$o_{134}o_{125}$	o_{145}	o_{245}	o_{345}	o_{12345}	$o_{134}o_{35}$	o_{245}	o_{1245}	o_{2345}	o_{12345}
$o_{134}o_{235}$	o_{145}	o_{245}	o_{345}	o_{12345}	$o_{134}o_{1235}$	o_{145}	o_{1245}	o_{345}	o_{2345}
$o_{234}o_{15}$	o_{245}	o_{1245}	o_{345}	o_{1345}	$o_{234}o_{25}$	o_{145}	o_{1245}	o_{1345}	o_{12345}
$o_{234}o_{125}$	o_{145}	o_{245}	o_{345}	o_{12345}	$o_{234}o_{35}$	o_{145}	o_{1245}	o_{1345}	o_{12345}
$o_{234}o_{135}$	o_{145}	o_{245}	o_{345}	o_{12345}	$o_{234}o_{1235}$	o_{245}	o_{1245}	o_{345}	o_{1345}
$o_{1234}o_{15}$	o_{245}	o_{1245}	o_{345}	o_{1345}	$o_{1234}o_{25}$	o_{145}	o_{1245}	o_{345}	o_{2345}
$o_{1234}o_{125}$	o_{145}	o_{245}	o_{1345}	o_{2345}	$o_{1234}o_{35}$	o_{145}	o_{245}	o_{1345}	o_{2345}
$o_{1234}o_{135}$	o_{145}	o_{1245}	o_{345}	o_{2345}	$o_{1234}o_{235}$	o_{245}	o_{1245}	o_{345}	o_{1345}

There are 57 3-cycle bc pairs not mentioned above that don't form c-associative triads and all have o_5 in the c term,

$$o_1o_5, o_1o_{45}, o_1o_{245}, o_1o_{345}, o_1o_{2345}, o_2o_5, o_2o_{45}, o_2o_{145}, o_2o_{345}, o_2o_{1345}, o_{12}o_5, o_{12}o_{45}, o_{12}o_{145}, o_{12}o_{345}, o_{12}o_{1345}, o_3o_5, o_3o_{45}, o_3o_{145}, o_3o_{245}, o_3o_{1245}, o_{13}o_5, o_{13}o_{45}, o_{13}o_{145}, o_{13}o_{245}, o_{13}o_{1245}, o_{23}o_5, o_{23}o_{45}, o_{23}o_{145}, o_{23}o_{245}, o_{23}o_{1245}, o_{123}o_5, o_{123}o_{45}, o_{123}o_{145}, o_{123}o_{245}, o_{123}o_{1245}, o_4o_5, o_4o_{15}, o_4o_{25}, o_4o_{125}, o_4o_{35}, o_4o_{135}, o_4o_{235}, o_4o_{1235}, o_{14}o_5, o_{14}o_{15}, o_{24}o_5, o_{24}o_{25}, o_{124}o_5, o_{124}o_{125}, o_{34}o_5, o_{34}o_{35}, o_{134}o_5, o_{134}o_{135}, o_{234}o_5, o_{234}o_{235}, o_{1234}o_5, o_{1234}o_{1235}.$$

This provides complete coverage of the $\binom{31}{2} = 465$ ordered pairs of $\mathbb{S}_5$ and demonstrates the structure found in higher levels of the sedenion series. The only complication is that at higher levels a-associative triads without cycle solutions like those found in Table 7 may be mixed with 3-cycle a-associative triads for sedenions with levels greater than 5.

Table 8 provides the number of non-associative elements for the sedenion series up to level 10. These are labelled $\mathbb{S}_n$ where n is the number of generators. Each generator level defines an algebra with $N = 2^n - 1$ basis elements and $\binom{N}{3}$ ordered product triples. Some of these triples form triads with non-scalar products and some of these are associative which is shown as the sum of a-associatives and b-associatives, since they are disjoint. Some of these associatives are c-associative and these define the zero divisors for the algebra.

Table 8: Sedenion Series Cardinality

Label	**Pure Basis Size**	**Triads**	**Associatives**	**Zero Divisors**	**Factor**
$\mathbb{O} = \mathbb{S}_3$	7	28	7	0	0
$\mathbb{S} = \mathbb{S}_4$	15	420	252	84	1
$\mathbb{S}_5$	31	4340	3612	1428	17
$\mathbb{S}_6$	63	39060	36204	15876	567
$\mathbb{S}_7$	127	330708	319788	148932	1773
$\mathbb{S}_8$	255	2720340	2678508	1289988	15357
$\mathbb{S}_9$	513	22064980	21903084	10740996	127869
$\mathbb{S}_{10}$	1023	177736020	177102828	87676932	1043773

3 Split Sedenion Series

Equation (1) can also be defined with $\epsilon_i = 1$ for any combination of $i \in \mathbb{N}_1^n$. Such unitary generators are represented as u_i with $u_i^2 = 1$ for $i \in \mathbb{N}_1^n$. This provides mixed sign basis elements such as $o_i u_j, i \neq j$. Squaring such elements always involves a bd term in the scalar part and invokes $\epsilon_i = 1$ in the square of any unit generator, u_i. Thus the sign of the square changes for pure elements with a single unit generator from the octonion square. This is now applied to each level of the graded algebra.

Definition The **grade** of a term is defined as its number of generators and the number of terms for any grade k is given by the combination $\binom{n}{k}$ because this gives the unique selections of k grades from n generators. This includes $k = 0$, the zero grade term with no generators which corresponds to the number one.

By the binomial theorem, the number of simple elements is the sum of over all grades containing pure elements and the number one

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

The proof of the binomial theorem is expressed in Pascal's triangle which starts with

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

The recurrence relation for Pascal's triangle is that each position is the sum of the element above and the one to the above left. For graded algebras this is the statement that the number of terms of grade k is given by the number of terms at grade k without the generator o_n plus those that include this generator. Pascal's triangle is horizontally symmetric because for each term of grade k, the complimentary term with grade $n - k$ contains these k generators removed from the n generators. For n odd there are an equal number of terms with o_n as without in the complimentary terms. For n even the same applies apart from the middle grade $k = n/2$. But $\binom{n}{n/2}$ is always even for n even and by the definition of combinations it distributes the n generators equally into an even set, half of which contain o_n and the others exclude it. Of course, the left-hand column has grade $k = 0$ which specifies the basis number 1 while the right-hand side, the compliment of 1, has $k = n$ and always contains o_n.

Definition The pure trace of the multiplication table for the algebra with n generators is the sum of the squares of the $2^n - 1$ pure basis terms which is $-(2^n - 1)$ for sedenions $\mathbb{S}_n$ and thus is -7 for octonians as seen from the quadratic form without the first component[4].

Trace Theorem

The pure trace for all of the split sedenion series is one, as it is for split octonions, split quaternions and Lorentz numbers.

Proof As seen above, changing a single generator to be unitary with $\epsilon_i = 1$ changes the square for all terms involving with generator, u_i, $i \in \mathbb{N}_1^n$. Ignoring the first and last grades in the binomial theorem gives equal number of terms containing this generator and those without which provides a zero sum of squares. The remaining pure term is grade $k = n$ which contains u_i which has a positive square giving a pure trace of 1. Using Pascal's recursion relation and adding another unitary generator at any level will keep all existing terms plus multiply all terms by the new generator which swaps terms with an odd number of unitary generators with those with an even number which is the same apart from the compliment of 1. The square of this last element cancels with the existing term without the new generator so all squares cancel leaving the square of the new generator itself giving a pure trace of one. This argument can be applied recursively for all Cayley-Dickson construction levels. Considering equation (1) again, it is easy to see that a second unitary generator will change the sign of half the square terms. This operation can be repeated until there are n unitary generators keeping the balance of even and odd unitary grades at any level. The following corollary will complete the proof of the trace theorem.

Corollary

The pure trace for the case with all unitary generators is, of course, one. This is easy to see for odd n as each term in the binomial expansion has a compliment of opposite parity. For even n the complimentary terms have the same parity and the middle grade with $\binom{n}{n/2}$ terms is the largest number with even parity if $n/2$ is even and odd otherwise. This leads to the following identity for even n

$$\binom{n}{\frac{n}{2}} = -\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\frac{n}{2}-k}\binom{n}{k},$$

or more generally for any n,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0.$$

Similarly to the pure trace, the multiplication table for the pure basis has the same sign as for the non-split algebra if there are an even number of unitary generators in the product of any two elements and the opposite sign for an odd number of unitary generators. Since the pure elements can be considered identical for the same square sign if the grading is ignored, and they have the same signature and action when multiplied, then the basis can be permutated so the diagonals of two multiplication tables match and this gives the same table for any positive number of unitary generators. Hence these algebras are isomorphic and the sedenion series at any level contains only the sedenion, $\mathbb{S}_n$, and split-sedenion, $\tilde{\mathbb{S}}_n$, algebras, $n > 0$. For example, the split octonions can have nine combinations of one, two of three unity generators which are all isomorphic[1] with pure trace one.

The unitary a, b and c-associativity scheme does not change from the non-split case because the same sign changes apply to each associativity term so the zero result is the same which is not-applicable for zero divisors. These start appearing for unitary generators in the split-octonions, $\tilde{\mathbb{O}}$, which have the following four prime triads. The first three have dual solutions and the last one has an extended solution thus giving eight zero divisors

$$(u_1, u_2, u_{13}), (u_1, u_3, u_{23}), (u_2, u_3, u_{13}), (u_{12}, u_3, u_{13}).$$

Split-sedenions, $\tilde{\mathbb{S}}$, have 170 zero divisors derived from 82 primes that have all the combinations of extended and dual solutions but no cycles. All of this work was verified with the use of an octonion/sedenion calculator written in Python. The github URL for the calculators is
https://github.com/GPWilmot/geoalg.

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