Recursive Formulas to Compute Coproducts of Finite Gödel Algebras and Related Structures

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***Abstract*—Gödel logic and its algebraic semantics, namely, the variety of Gödel algebras, play a major rôle in mathematical fuzzy logic. The category of finite Gödel algebras and their homomorphisms is dually equivalent to the category FF of finite forests and order-preserving open maps. The combinatorial nature of FF allows to reduce the usually difficult problem of computing coproducts of algebras and their cardinalities to the combinatorial problem of computing products of finite forests. In this paper we propose a neat, purely combinatorial, recursive formula to compute the product objects. Further, we formulate a dual equivalence between finite Gödel$_\triangle$-algebras and a category of finite multisets of finite chains, and we provide recursive formulas to compute coproducts, and their cardinalities, in the categories of finite Gödel hoops and of finite Gödel$_\triangle$-algebras.**

I. Introduction

Gödel logic and its algebraic semantics, the variety of Gödel algebras, play a major rôle in mathematical fuzzy logic, for it is the logic of one of the fundamental continuous t-norms.

Computing coproducts between algebras in a variety of interest for fuzzy logicians, bears an interest which is not only theoretical. As a matter of fact, assume that you have two distinct algebras A and B in a chosen variety capturing the behaviour of some observables: one set of observables for A and a disjoint set of observables for B. We may suppose that the interrelations between observables in these sets are formulated as identities between terms built with elements of the aforementioned algebras. Algebraically, this means that A and B are each *presented* by a distinct set of identities. Then, if one has to model the joint behaviour of these two sets of observables under the assumption of minimal interaction between the two sets, using the same kind of algebraic model, then one has to compute the algebra presented by the (disjoint) union of the identities presenting A and B. This amounts exactly to compute the coproduct of A and B in the chosen variety (of course, when the interaction between the two sets is not minimal, one has to resort to fibred coproducts).

In general, computing the coproduct object $A+B$ (that is, the algebraic structure of $A+B$) of two algebras A and B in a given variety is a difficult problem. For instance, consider the case that A and B are both meant to model set of observables which do not require the specification of any interrelation, that is, they are both presented by the empty set of identities. This means that A and B are free algebras (over some sets of generators) in their variety. Then $A+B$, presented again emptily over the disjoint union of the two sets of generators, is, by definition, a free algebra in the same variety, and the determination of the structure of free algebras in a given variety can notoriously be a really tough task to embark on.

For the case of Gödel algebras, the knowledge of a combinatorial category dually equivalent to the finite slice of the algebraic one greatly simplifies this task.

As is well known, the category of finite Gödel algebras and their homomorphisms is dually equivalent to the category FF of finite forests and order-preserving open maps. The combinatorial nature of FF allows to reduce the usually difficult problem of computing coproducts of algebras and their cardinalities to the combinatorial problem of computing products of finite forests. Notice however, that the computation of the product of finite forests is not a trivial task: as a matter of fact the underlying set of the product $F \times G$ of two finite forests F and G is in general not the cartesian product of the underlying sets of F and G.

While general procedures to compute the product in FF are available [12] [11], [16], they are rather indirect, relying on *intermediate steps* such as labelings, paths, and other devices, or being produced by unfolding much more general constructions.

In this paper we propose a neat, purely combinatorial, recursive formula to compute the product objects.

While this formula has been already applied in recent literature ([3], [5], [10, Chap. IX] to cite a few), there is no single place where it is fully proved. In this note we present a detailed proof of the validity of this recursive formula.

Further, we formulate a dual equivalence between finite Gödel$_\triangle$-algebras and a category of finite multisets of finite chains, and we provide recursive formulas to compute coproducts, and their cardinalities, in the categories of finite Gödel hoops and of finite Gödel$_\triangle$-algebras.

II. Gödel algebras and forests

In this section we define the two categories of Gödel algebras and homomorphisms, and of finite forests and order-preserving open maps, and we recall their well-known dual equivalence. Further we state basic properties of the category of finite forests.

A. Gödel algebras, MTL-algebras, residuated lattices

An *MTL-algebra* is a system $\mathcal{A} = (A, *, \to, \vee, \wedge, 0, 1)$ such that:

- $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice;
- $(A, *, 1)$ is a *commutative monoid* (notice, the property that 1 is both the top of the lattice and the unit of the monoid is referred to as the *integrality* of the lattice $\mathcal{A}$);
- $\mathcal{A}$ is *residuated*, that is, $x * z \leq y$ iff $z \leq x \to y$ for all $x, y, z \in A$;
- $\mathcal{A}$ is *prelinear*, that is, $(x \to y) \vee (y \to x) = 1$ for all $x, y \in A$.

Equivalently, let us call *prelinear semihoop* a prelinear, integral, distributive, commutative, residuated lattice. Then an MTL-algebra is a bounded prelinear semihoop.

MTL-algebras are major objects of study in mathematical fuzzy logic, as the associated logic, *Monoidal t-norm based logic*, introduced by Esteva and Godo in [14], was proved to be the logic of all left-continuous t-norms and their residua by Jenei and Montagna in [20]. Among them, a major rôle is played by Gödel algebras, and their associated logic, *Gödel-Dummett logic* (see [17] for background).

A *Gödel algebra* is an *idempotent* MTL-algebra, that is $x * x = x$ holds for all $x \in A$. (Equivalently, a Gödel algebra is a prelinear Heyting algebra). As in each Gödel algebra $\mathcal{A} = (A, *, \to, \vee, \wedge, 0, 1)$ it holds that $* = \wedge$, we display $\mathcal{A}$ as $(A, \vee, \wedge, \to, 0, 1)$.

Gödel algebras form a variety, denoted $\mathbb{G}$. The variety $\mathbb{G}$ can be considered a category, with objects the algebras in the variety, and with arrows the homomorphisms between them.

In this paper we shall focus on the full subcategory $\mathbb{G}_{fin}$ of $\mathbb{G}$, whose objects are the Gödel algebras of finite cardinality.

The following general notions and results on MTL-algebras, and on prelinear semihoops are well-established. We recall them here since we shall use them in the paper.

Let $\mathcal{A}$ be an MTL-algebra or a prelinear semihoop. Then $\mathfrak{p} \subseteq A$ is a *filter* of $\mathcal{A}$ if it is an upward closed, $*$-closed subset of A, that is for all $y \in A$, if there is $x \in \mathfrak{p}$ with $x \leq y$ then $y \in \mathfrak{p}$, and $x * y \in \mathfrak{p}$ for all $x, y \in \mathfrak{p}$.

Notice that if $\mathcal{A}$ is a Gödel algebra, then each filter $\mathfrak{p}$ is just an upward closed subset of A, that is $\mathfrak{p} = \{y \in A \mid \exists x \in \mathfrak{p}\colon x \leq y\}$.

Filters of $\mathcal{A}$ are in bijection with congruences over $\mathcal{A}$, via the following definitions: $x \theta_{\mathfrak{p}} y$ iff $(x \to y) \wedge (y \to x) \in \mathfrak{p}$, and $\mathfrak{p}_\theta = \{x \in A \mid x\theta 1\}$.

In the following, we shall write $\mathcal{A}/\mathfrak{p}$ to denote the quotient algebra $\mathcal{A}/\theta_{\mathfrak{p}}$.

A *filter* $\mathfrak{p}$ of $\mathcal{A}$ is *prime* if it is proper, that is $\mathfrak{p} \neq A$, and for all $x, y \in A$, either $x \to y \in \mathfrak{p}$ or $y \to x \in \mathfrak{p}$.

The set of all prime filters of $\mathcal{A}$, partially ordered by *reverse inclusion*, is called the *prime spectrum* of $\mathcal{A}$, and it is denoted $Spec(\mathcal{A})$. The inclusion-maximal elements of $Spec(\mathcal{A})$ are the *maximal filters* of $\mathcal{A}$, and they form its *maximal spectrum* $Max(\mathcal{A}) \subseteq Spec(\mathcal{A})$.

Recall that an algebra is *simple* if its lattice of congruences contains only the total relation and the diagonal one. If $\mathfrak{p} \in Max(\mathcal{A})$, then $A/\mathfrak{p}$ is simple.

The following theorem collects some well-known results about decompositions of MTL-algebras and prelinear semihoops. See, *e.g.* [14], [15].

Theorem 2.1: Let $\mathcal{A}$ be a prelinear semihoop. Then:

1) For each $\mathfrak{p} \in Spec(\mathcal{A})$, the quotient $\mathcal{A}/\mathfrak{p}$ is a *chain*, that is, its lattice reduct is a totally ordered set.
2) The subdirectly irreducible MTL-algebras are chains.
3) $\mathcal{A}$ is isomorphic with a subdirect product of the family $\{\mathcal{A}/\mathfrak{p} \mid \mathfrak{p} \in Spec(\mathcal{A})\}$.
4) If $\mathcal{A}$ has finite cardinality, then $\mathcal{A} \cong \prod_{\mathfrak{p} \in Max(\mathcal{A})} \mathcal{A}/\mathfrak{U}_{\mathfrak{p}}$, where $\mathfrak{U}_{\mathfrak{p}}$ is the intersection of all prime filters contained in $\mathfrak{p}$.

Notice that $Max(\mathcal{A}) = Spec(\mathcal{A})$ iff $\mathcal{A}$ is *semisimple*, that is, $\mathcal{A}$ is a subdirect product of simple algebras.

Theorem 2.1 applies evidently to all extensions of MTL-algebras and of prelinear semihoops. For extensions/expansions of MTL-algebras, one needs to adapt the definition of filter to the new operations, in order to maintain the bijection filters-congruences.

In each finite Gödel algebra, each filter is *principal*, that is, it is the set of all elements $\geq$ than some fixed element. In particular every prime filter $\mathfrak{p}$ is of the form $\{y \in A \mid y \geq x_{\mathfrak{p}}\}$ for some join irreducible element $x_{\mathfrak{p}}$. We recall that $x \in A$ is *join irreducible* if $x = y \vee z$ implies $x = y$ or $x = z$.

B. Finite forests

Given a subposet Q of a poset P, we call *downset* of Q the downward closed poset $\downarrow Q = \{y \leq x \mid x \in Q\}$. Analogously, we call *upset* of Q the upward closed poset $\uparrow Q = \{y \geq x \mid x \in Q\}$.

A *forest* is a poset F such that, for each $x \in F$, the downset $\downarrow\{x\}$ is totally ordered by restriction of the order of F.

A forest is *finite* if its underlying set is of finite cardinality.

A map $f\colon F \to G$ between two finite forests is *order-preserving* if $x \leq y$ implies $f(x) \leq f(y)$ for each $x, y \in F$; f is *open* if it carries downsets to downsets, or, equivalently for each $x \in F$ and $y \leq f(x)$ there is $z \in F$, with $z \leq x$ such that $f(z) = y$.

The category FF of finite forests has finite forests as objects, and order-preserving open maps between them as morphisms.

Let $\mathbf{1}$ denote the singleton forest $\{*\}$, and $\mathbf{0}$ the empty forest $\emptyset$.

Lemma 2.2: $\mathbf{1}$ and $\mathbf{0}$ are respectively the terminal and the initial objects in FF.

Proof: Clearly, for each $F \in \mathsf{FF}$ there are a unique map $F \to \mathbf{1}$, and a unique map $\mathbf{0} \to F$. ∎

Lemma 2.3: The coproduct $F+G$ of two finite forests F and G in FF is given by the the disjoint union of F with G (in the sense that the partial order relation of $F+G$ is the disjoint union of the partial order relations of F and G).

Proof: Write the disjoint union of the underlying sets of F and G as $F+G = \{(x,0) \mid x \in F\} \cup \{(x,1) \mid x \in G\}$. Then the inclusion maps $\iota_F \colon F \to F+G$ and $\iota_G \colon G \to F+G$ are given by $\iota_F(x) = (x,0)$ and $\iota_G(x) = (x,1)$, which are clearly order-preserving and open. Take now any $H \in \mathsf{FF}$, and let $f \colon F \to H$ and $g \colon G \to H$ be two arrows in FF. Define $f+g \colon F+G \to H$ as the map $(f+g)(x,0) = f(x)$ if $x \in F$, while $(f+g)(x,1) = g(x)$ if $x \in G$. It is straightforward to check that $f+g$ is the unique map $h \colon F+G \to H$ in FF such that $h \circ \iota_F = f$ and $h \circ \iota_G = g$. ∎

For each forest $F \in \mathsf{FF}$, let $F_\perp$ be the poset obtained by adding a fresh bottom element to F, that is, once we fix an element $\perp \notin F$, then $F_\perp = F \cup \{\perp\}$, with $\perp \leq x$ for all $x \in F$.

A *tree* is a forest with minimum, which is called the *root* of the tree. The following lemma is straightforward.

Lemma 2.4: For each forest $F \in \mathsf{FF}$, the poset $F_\perp$ is a tree in FF. Moreover, each tree in FF has the form $F_\perp$ for some $F \in \mathsf{FF}$.

C. Categorical equivalence

As is well known, the category $\mathbb{G}_{fin}$ of finite Gödel algebras is dually equivalent to the category FF of finite forests. This result was left implicit in the works by Horn [18], [19] (see also [1]), and made recently explicit in, for instance, [12] (see also [10, Chap. IX]). It can also be obtained specialising Esakia duality [13] between Heyting algebras and Esakia spaces to the case of finite and prelinear Heyting algebras.

The functors implementing the equivalence are:

$$\mathrm{Spec} \colon \mathbb{G}_{fin} \to \mathsf{FF}\,,$$

defined on objects as $\mathrm{Spec}\,\mathcal{A} = \mathit{Spec}(\mathcal{A})$, and on arrows by taking preimages, that is, if $h \colon \mathcal{A} \to \mathcal{B}$ then $\mathrm{Spec}\,h \colon \mathrm{Spec}\,\mathcal{B} \to \mathcal{A}$ is given by $(\mathrm{Spec}\,h)(\mathfrak{p}) = h^{-1}[\mathfrak{p}]$ for each $\mathfrak{p} \in \mathit{Spec}(\mathcal{B})$. (Notice that $\mathrm{Spec}\,\mathcal{A}$ is indeed a forest, as a consequence of $\mathcal{A}$ being prelinear.) And

$$\mathrm{Sub} \colon \mathsf{FF} \to \mathbb{G}_{fin}\,,$$

defined on objects as

$$\mathrm{Sub}\,F = (\{G \subseteq F \mid G = \downarrow G\}, \cup, \cap, \to, \emptyset, F)\,,$$

where, for all downward closed $G, H \subseteq F$:

$$G \to H = F \setminus \uparrow (G \setminus H)\,,$$

and on arrows by taking preimages, that is, if $f \colon F \to G$, then $\mathrm{Sub}\,f \colon \mathrm{Sub}\,G \to \mathrm{Sub}\,F$ is given by $(\mathrm{Sub}\,f)(H) = f^{-1}[H]$ for each downward closed $H \subseteq G$.

III. Using the dual equivalence to compute coproducts of Gödel algebras

In order to speak of computing the coproduct of two finite given algebras A and B, we shall agree on how these algebras are effectively *given* in input. There may be several distinct points of views on this matter. For instance, we may be given the (finite) presentations of these two algebras. Then, as specified in the introduction, we can readily obtain the presentation of the coproduct algebra as the disjoint union of the two given presentations. Notice that this does not solve automatically the problem of determining the concrete algebraic structure of the coproduct algebra. For instance, consider again the case A and B being free algebras in their variety. Then they are both presented by the empty set of identities over distinct sets of generators. Then $A+B$ is readily presented by the empty set of identities over the disjoint union of the two sets of generators, but this fact alone in general does not cast enough light on the structure of $A+B$: it may hard or impossible to compute, just to cite a major parameter, its cardinality.

The problem becomes generally harder and more interesting if the input algebras A and B are not specified through their presentations. Dealing in this paper only with finite algebras, we may suppose that generally we are given A and B *extensively*, that is, we are explicitly given the tables of all the operations of A and B.

With this general assumption in place, we point out that all the information we shall extract from A and B, which for all the kind of algebras considered in this paper amounts to the identification of the join irreducible elements of A and B, can be effectively obtained algorithmically from A and B, albeit not so efficiently.

We finally observe that another very advantageous way to specify concretely A and B, is to provide directly their dual objects (the posets $\mathrm{Spec}\,A$ and $\mathrm{Spec}\,B$, in the case of finite Gödel algebras), since these objects, as the content of the paper and its examples should show, are in general combinatorially much less complex than the original algebras themselves.

Summing up, we assume from now on that the dual objects of any algebra involved in the computation of a coproduct are available to us, either as the original input, or as the output of some pre-processing. We then reduce the problem of computing the coproducts of algebras A and B to the computation of the product of the duals of A and B in the dually equivalent category.

Before moving to the next section, we remark here that knowledge of a dual category to a variety of algebras provides us with a very powerful tool to study several features of these algebras (and related structures) beyond computing coproducts and their cardinalities. For the case of Gödel algebras, the dual approach has provided, just to mention a few, results about the structure and cardinality of free algebras ([12], [6]) (as a matter of fact, computation of free algebras are particular cases of computation of coproducts, since the k-generated free algebra in a variety is the kth copower of the 1-generated one); the computation of fibred coproducts and amalgamation [12]; the determination of categorical equivalences between $\mathbb{G}$ and other varieties of algebras [5]; the classification of subvarieties of $\mathbb{G}$ and related structures [5]; the computation of the structure and cardinality of the automorphisms groups of finite Gödel algebras [8]; the computation of minimal axiomatisations of theories in Gödel-Dummett propositional logic [4], the computation of the structure of the free distributive lattices over a finite Gödel algebra [7].

IV. Recursive computation of products of finite forests

Lemma 4.1: For all $F \in \mathsf{FF}$,

$$F \times \mathbf{1} \cong F\,.$$

Proof: By Lemma 2.2, $\mathbf{1}$ is the terminal object in FF. ∎

Lemma 4.2: For all $F, G, H \in \mathsf{FF}$,

$$F \times (G + H) \cong (F \times G) + (F \times H)\,.$$

Proof: We will prove that $(F \times G) + (F \times H)$ is the product object in FF of F with $G + H$. We shall then show that there are morphisms $\pi_F\colon (F \times G) + (F \times H) \to F$ and $\pi_{G+H}\colon (F\times G)+(F\times H) \to G+H$ such that for each $K \in \mathsf{FF}$, and every pair of morphisms $f_F\colon K \to F$, $f_{G+H}\colon K \to G+H$, there is a unique morphism $f\colon K \to (F \times G) + (F \times H)$ such that $\pi_F \circ f = f_F$ and $\pi_{G+H} \circ f = f_{G+H}$.

To this purpose, let us first consider f_{G+H}, and let $K_G = \{k \in K \mid f_{G+H}(k) \in G\}$, and $K_H = \{k \in K \mid f_{G+H}(k) \in H\}$ (here, not to add notational burden, we have identified G and H with their isomorphic images in the coproduct $G+H$). Since the coproduct in FF is just the disjoint union, we have that $K = K_G + K_H$, that is there are maps $\iota_G\colon K_G \to K$ and $\iota_H\colon K_H \to K$ such that, for each map $g_G\colon K_G \to K'$, and $g_H\colon K_H \to K'$, there is a unique map $g\colon K \to K'$ such that $g \circ \iota_G = g_G$ and $g \circ \iota_H = g_H$.

Moreover, there is a unique map $f_{F\times G}\colon K_G \to F\times G$ such that $(f_F \restriction K_G) = \pi'_F \circ f_{F\times G}$ and $(f_{G+H} \restriction K_G) = \pi'_G \circ f_{F\times G}$, for $\pi'_F\colon F \times G \to F$ and $\pi'_G\colon F \times G \to G$ being the projection maps. Analogously, there is a unique map $f_{F\times H}\colon K_H \to F \times H$ such that $(f_F \restriction K_H) = \pi''_F \circ f_{F\times H}$ and $(f_{G+H} \restriction K_H) = \pi''_H \circ f_{F\times H}$, for $\pi''_F\colon F \times H \to F$ and $\pi''_H\colon F \times H \to H$ being the projection maps.

Using again the fact that coproduct is disjoint union, we then infer that there is a unique map $g\colon K \cong K_G + K_H \to (F\times G)+(F\times H)$ such that $g \circ \iota_G = f_{F\times G}$ and $g \circ \iota_H = f_{F\times H}$.

Let us now define the maps $\pi_F\colon (F \times G) + (F \times H) \to F$ by $\pi_F = \pi'_F + \pi''_F$ and $\pi_{G+H}\colon (F \times G) + (F \times H) \to G + H$ by $\pi_{G+H} = (\iota'_G \circ \pi'_G) + (\iota'_H \circ \pi''_H)$, for $\iota'_G\colon G \to G + H$ and $\iota'_H\colon H \to G + H$ being the injection maps of the coproduct $G + H$. We are left to show that $\pi_F \circ g = f_F$ and $\pi_{G+H} \circ g = f_{G+H}$.

Now, let us pick $k \in K$. First suppose $k \in K_G$. Then $f_F(k) = (f_F \restriction K_G)(k) = (\pi'_F \circ f_{F\times G})(k) = (\pi'_F \circ g \circ \iota_G)(k)$. By definition of g, $(g \circ \iota_G)(k) = g(k) \in F \times G$. Noticing that π_F coincides by definition with π'_F over $F \times G$, we conclude $(\pi'_F \circ g \circ \iota_G)(k) = (\pi_F \circ g)(k)$, that is $f_F(k) = (\pi_F \circ g)(k)$. In a completely analogous manner, one can show that $k \in K_H$ implies $f_F(k) = (\pi''_F \circ g \circ \iota_H)(k) = (\pi_F \circ g)(k)$. Whence, $\pi_F \circ g = f_F$.

Finally, we show that $\pi_{G+H} \circ g = f_{G+H}$. Pick $k \in K_G$. Then $f_{G+H}(k) = (f_{G+H} \restriction K_G)(k) = (\pi'_G \circ f_{F\times G})(k) = (\pi'_G \circ g \circ \iota_G)(k)$. As before, $(g \circ \iota_G)(k) = g(k) \in F \times G$. But by definition π_{G+H} coincides with π'_G over $F \times G$, whence $f_{G+H}(k) = (\pi'_G \circ g \circ \iota_G)(k) = (\pi_{G+H} \circ g)(k)$. The case $k \in K_H$ is identical, *mutatis mutandis*. ∎

The following lemma states the recursive formula to compute products of forests.

Lemma 4.3:

$$F_\perp \times G_\perp \cong ((F_\perp \times G) + (F \times G) + (F \times G_\perp))_\perp$$

for all $F, G \in \mathsf{FF}$.

Proof: Let us display $F_\perp$ isomorphically as $(F_1 + F_2 + \cdots + F_u)_\perp$ for a uniquely determined family $\{F_i\}_{i=1}^u$ of trees. Analogously $G_\perp \cong (G_1 + G_2 + \cdots + G_v)_\perp$, for a uniquely determined family $\{G_j\}_{j=1}^v$ of trees. By Lemma 4.2, we can rewrite the forest $(F \times G_\perp) + (F \times G) + (F_\perp \times G)$ isomorphically as the following coproducts of trees computed in FF:

$$\sum_{i=1}^{u}(F_i \times G_\perp) + \sum_{i=1}^{u}\sum_{j=1}^{v}(F_i \times G_j) + \sum_{j=1}^{v}(F_\perp \times G_j)\,.$$

We denote r_0 and s_0 the roots of $F_\perp$ and $G_\perp$, respectively. Further, we denote t_0 the root of the tree $((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp$.

We next define the maps $\pi_{F_\perp}\colon ((F\times G_\perp)+(F\times G)+(F_\perp \times G))_\perp \to F_\perp$ and $\pi_{G_\perp}\colon ((F\times G_\perp)+(F\times G)+(F_\perp\times G))_\perp \to G_\perp$ as follows. First we set $\pi_{F_\perp}(t_0) = r_0$ and $\pi_{G_\perp}(t_0) = s_0$. Now, each $x \in (F\times G_\perp)+(F\times G)+(F_\perp\times G)$, must belong to a unique tree either of the form $F_i \times G_\perp$ or $F_i \times G_j$ or $F_\perp \times G_j$.

Writing F_0 and G_0 for $F_\perp$ and $G_\perp$, respectively, then $x \in F_i \times G_j$, for a uniquely determined pair (i, j) with $i + j > 0$. We define $\pi_{F_\perp}(x) = \iota_{F_i}(\pi_{F_i}(x))$, where $\pi_{F_i}\colon F_i \times G_j \to F_i$ is the projection function, while $\iota_{F_i}\colon F_i \to F_\perp$ is the set-theoretic inclusion of the support of F_i into $F_\perp$. Similarly, we define $\pi_{G_\perp}(x) = \iota_{G_j}(\pi_{G_j}(x))$. It is clear that both $\pi_{F_\perp}$ and $\pi_{G_\perp}$ are well-defined morphisms of finite forests.

Now, let us take a forest H and two morphisms $f\colon H \to F_\perp$ and $g\colon H \to G_\perp$. We shall construct a map $(f, g)\colon H \to ((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp$ such that $\pi_{F_\perp} \circ (f, g) = f$ and $\pi_{G_\perp} \circ (f, g) = g$. To accomplish this task we partition the set H as follows. Let $R_0 = f^{-1}(r_0)$ and $R_1 = H \setminus R_0$. Analogously, let $S_0 = g^{-1}(s_0)$ and $S_1 = H \setminus S_0$. Then the following is a partition of the set H:

$$\{R_0 \cap S_0, R_0 \cap S_1, R_1 \cap S_0, R_1 \cap S_1\}\,.$$

We refine this partition by further subdividing $R_1 \cap S_1$, as follows. Let R_2 be the set of all $x \in R_1 \cap S_1$ such that there is $y < x$ in H with $f(y) = r_0$ and $g(y) \neq s_0$. Notice that, since g is an order-preserving open map, if G_j is the unique tree in $\{G_h\}_{h=1}^v$ such that $g(x) \in G_j$, then also $g(y) \in G_j$. Similarly, let S_2 be the set of all $x \in R_1 \cap S_1$ such that there is $y < x$ in H with $f(y) \neq r_0$ and $g(y) = s_0$. Finally let $T_2 = (R_1 \cap S_1) \setminus (R_2 \cup S_2)$. Then

$$\{R_0 \cap S_0, R_0 \cap S_1, R_1 \cap S_0, R_2, S_2, T_2\}$$

is a partition of the set H. For each $x \in R_0 \cap S_0$ we let $(f,g)(x) = t_0$; for each $x \in R_0 \cap S_1$ we note that there is a unique tree G_j such that $g(x) \in G_j$: we then let $(f,g)(x)$ be the uniquely determined element t of $F_\perp \times G_j$ such that $\pi_{F_\perp}(t) = f(x) = r_0$ and $\pi_{G_j}(t) = g(x)$, where these two maps are the projections of the product $F_\perp \times G_j$; for each $x \in R_1 \cap S_0$ we reason analogously, letting $(f,g)(x)$ be the uniquely determined element t of $F_i \times G_\perp$ such that $\pi_{F_i}(t) = f(x)$ and $\pi_{G_\perp}(t) = g(x) = s_0$.

Some additional care is needed to deal with the remaining cases. For each $x \in R_2$ we note that there are uniquely determined trees F_i and G_j such that $f(x) \in F_i$ and $g(x) \in G_j$. Since $x \in R_2$, there is $y < x$ in H such that $f(x) = r_0$ and $g(x) \neq s_0$. As morphisms of finite forests must carry downsets to downsets, $(f,g)(x)$ must belong to $F_\perp \times G_j$, but now, reasoning as in the preceding cases we let $(f,g)(x)$ be the uniquely determined element t of $F_\perp \times G_j$ such that $\pi_{F_\perp}(t) = f(x)$ and $\pi_{G_j}(t) = g(x)$. For each $x \in S_2$ we reason analogously, letting $(f,g)(x)$ be the uniquely determined element t of $F_i \times G_\perp$ such that $\pi_{F_i}(t) = f(x)$ and $\pi_{G_\perp}(t) = g(x)$.

Finally, the last case $x \in T_2$ is again similarly dealt with, as we let $(f,g)(x)$ be the uniquely determined element t of $F_i \times G_j$ such that $\pi_{F_i}(t) = f(x)$ and $\pi_{G_j}(t) = g(x)$. A simple check now shows $\pi_{F_\perp} \circ (f,g) = f$ and $\pi_{G_\perp} \circ (f,g) = g$ as desired.

There remains to show that (f,g) is the only map with this property. Let $h\colon H \to ((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp$ be a morphism in FF such that $\pi_{F_\perp} \circ h = f$ and $\pi_{G_\perp} \circ h = g$. Clearly h must coincide with (f,g) over $R_0 \cap S_0$. Notice that if $x \in R_0 \cap S_1$ then $f(x) = r_0$ and $g(x) \neq s_0$, whence $h(x)$ must belong to $F_\perp \times G_j$ for a uniquely determined G_j. But then $h(x) = (f,g)(x)$ as otherwise $F_\perp \times G_j$, together with its projections, would not be the product in F of $F_\perp$ and G_j.

A completely analogous argument shows that h must coincide with (f,g) over $R_1 \cap S_0$. If $x \in R_1 \cap S_1$ then there are uniquely determined F_i and G_j, $i \neq 0 \neq j$, such that $\pi_{F_i}(h(x)) = f(x)$ and $\pi_{G_j}(h(x)) = g(x)$, where projections are those of the product $F_i \times G_j$. If $x \in R_2 \subseteq R_1 \cap S_1$ then there is $y < x$ in H such that $\pi_{F_\perp}(h(y)) = r_0$ while $\pi_{G_\perp}(h(y)) \neq s_0$, that is $y \in R_0 \cap S_1$, and hence $h(y) \in F_\perp \times G_j$. Since h is order-preserving and open, x must belong to the isomorphic copy of F_i included as a set in $F_\perp$, but then $h(x) = (f,g)(x)$ as otherwise $F_\perp \times G_j$, together with its projections, would not be the product in F of $F_\perp$ and G_j. Similar arguments hold for $x \in S_2$ or $x \in T_2$. Hence $h = (f,g)$ and the proof is complete. ∎

Notice that the proof of Lemma 4.3 is taken from [10, Chap. IX], while Lemma 4.2 is never proved explicitly. This is the first location where all the proofs needed to implement the following algorithm are fully detailed.

Theorem 4.4: The following recursive algorithm computes the product of two forests $F, G \in \mathsf{FF}$.

1) Apply repeatedly Lemma 4.2 to reduce the computation of $F \times G$ to the computation of $(F_1 \times G_1) + (F_2 \times G_2) + \cdots + (F_u \times G_u)$, where $F_1, F_2, \ldots, F_u, G_1, G_2, \ldots, G_u$ are all trees.
2) For $i \in \{1, 2, \ldots, u\}$, if (either $F_i \cong \mathbf{1}$ or $G_i \cong \mathbf{1}$) then apply Lemma 4.1, if not, then apply Lemma 4.3 to rewrite recursively $F_i \times G_i$.

Proof: The correctness and termination of the algorithm are immediate by Lemma 4.1, Lemma 4.2 and Lemma 4.3. ∎

Example 4.5: Let us compute the product of $\mathbf{1} + \mathbf{1}_\perp$ with $\mathbf{1}_\perp$. By Lemma 4.2 we have that

$$(\mathbf{1} + \mathbf{1}_\perp) \times \mathbf{1}_\perp \cong (\mathbf{1} \times \mathbf{1}_\perp) + (\mathbf{1}_\perp \times \mathbf{1}_\perp).$$

Now, by Lemma 4.1, $\mathbf{1} \times \mathbf{1}_\perp \cong \mathbf{1}_\perp$, while, applying Lemma 4.3, we get

$$\mathbf{1}_\perp \times \mathbf{1}_\perp \cong ((\mathbf{1}_\perp \times \mathbf{1}) + (\mathbf{1} \times \mathbf{1}) + (\mathbf{1} \times \mathbf{1}_\perp))_\perp.$$

Proceeding recursively, we get

$$(\mathbf{1}_\perp \times \mathbf{1}) + (\mathbf{1} \times \mathbf{1}) + (\mathbf{1} \times \mathbf{1}_\perp) \cong \mathbf{1}_\perp + \mathbf{1} + \mathbf{1}_\perp,$$

so that, the desired product turns out to be

$$(\mathbf{1} + \mathbf{1}_\perp) \times \mathbf{1}_\perp \cong \mathbf{1}_\perp + (\mathbf{1}_\perp + \mathbf{1} + \mathbf{1}_\perp)_\perp.$$

Notice that $\mathbf{1} + \mathbf{1}_\perp$ is the spectrum of the free 1-generated Gödel algebra, so it can be presented as $(\{x\}, \emptyset)$, meaning that it is generated by a single element x, without imposing any identity. On the other hand $\mathbf{1}_\perp$ is the spectrum of a 1-generated Gödel algebra that can be presented as $(\{y\}, y \to 0 = 0)$. Whence, the coproduct $\mathrm{Sub}\,(\mathbf{1}_\perp + (\mathbf{1}_\perp + \mathbf{1} + \mathbf{1}_\perp)_\perp)$ is the algebra presented as $(\{x, y\}, y \to 0 = 0)$.

V. CARDINALITY

Lemma 5.1: Each finite forest can be built from finitely many copies of the terminal object $\mathbf{1}$, just using the operations of (binary) coproduct $(F, G) \mapsto F + G$ and of lifting $F \mapsto F_\perp$.

Proof: By induction on the cardinality $|F|$ of the underlying set of a forest $F \in \mathsf{FF}$. If $|F| \leq 1$, then either $F \cong 0$ or $F \cong 1$, and then F is built using 0 or 1 copies of the terminal object. If $|F| > 1$ then, by finiteness, it has the form $F \cong G_\perp + H$ for $G \in \mathsf{FF}$ being such $G_\perp$ is a subposet of F, and the possibly empty forest $H \in \mathsf{FF}$ being uniquely determined by the pair (F, G). By inductive hypothesis, G and H are built from copies of the terminal object using only (binary) coproducts and liftings. ∎

For any finite forest $F \in \mathsf{FF}$, let us write $||F||$ for the cardinality of the algebra $\mathrm{Sub}\, F$.

Theorem 5.2: The following recursive algorithm computes the cardinality of any finite Gödel algebra A, having in input its prime spectrum $F \in \mathsf{FF}$.

1) If $F \cong 0$ then $||F|| = 1$.
2) If $F \cong 1$ then $||F|| = 2$.
3) If $F \cong G_\perp + H$ then $||F|| = (||G|| + 1)||H||$.

Proof: The correctness and termination of the algorithm follows immediately by Lemma 5.1. ∎

Example 5.3: Consider the two forests of Example 4.5. Then $||\mathbf{1}_\perp|| = 2 + 1 = 3$ and $||\mathbf{1} + \mathbf{1}_\perp|| = 2 \cdot (2 + 1) = 6$. Their coproduct has cardinality $||\mathbf{1}_\perp + (\mathbf{1}_\perp + \mathbf{1} + \mathbf{1}_\perp)_\perp|| = 3 \cdot (||\mathbf{1}_\perp + \mathbf{1} + \mathbf{1}_\perp|| + 1) = 3 \cdot ((3 \cdot 2 \cdot 3) + 1) = 57$.

VI. Gödel hoops and trees

Gödel hoops are exactly the 0-free subreducts of Gödel algebras, that is they are systems $\mathcal{A} = (A, \wedge, \vee, \to, 1)$ such that $\mathcal{A}$ is a topped, distributive, integral, residuated, prelinear lattice $(A, *, \wedge, \vee, \to, 1)$ that is idemponent, *i.e.* $x * x = x$, whence $* = \wedge$. Notice that Gödel hoops may lack a least element. Moreover, Gödel hoops are clearly prelinear semihoops, whence Theorem 2.1 applies.

Given any Gödel hoop $\mathcal{A} = (A, \vee, \wedge, \to, 1)$ and an element $\perp \notin A$, the structure $\mathcal{A}_\perp = (A \cup \{\perp\}, \vee_\perp, \wedge_\perp, \to_\perp, \perp, 1)$, where the restrictions of $\vee_\perp, \wedge_\perp, \to_\perp$ to A coincide respectively with $\vee, \wedge, \to$, while $\perp \vee_\perp x = x$, $\perp \wedge_\perp x = \perp$, $\perp \to_\perp x = 1$, $x \to_\perp \perp = \perp$, is a directly indecomposable Gödel algebra, called the *lift* of $\mathcal{A}$.

A. Finite trees

The category FT of *finite trees* is the full subcategory of FF whose objects are trees, that is, forests with a minimum element, called root. Equivalently, by Lemma 2.4, every object in FT is a finite forest of the form $F_\perp$.

Lemma 6.1: The following hold in FT.

1) The initial and terminal object in FT is the singleton forest $\mathbf{1}$.
2) The product $F_\perp \times G_\perp$ of two trees in FT is computed as in FF.
3) The coproduct $F_\perp + G_\perp$ of two trees in FT is the forest $(F + G)_\perp$, where the coproduct $F + G$ is computed in FF.

Proof: 1) Clearly, for each $F_\perp \in \mathsf{FT}$ there is a unique map $F_\perp \to \mathbf{1}$ and a unique map $\mathbf{1} \to F_\perp$.

2) Use the proof of Lemma 4.3, and notice that everything carries through from FF to FT.

3) Let r, s, t denote the roots of $F_\perp$, $G_\perp$ and $(F + G)_\perp$, respectively. Define the maps $\iota_F \colon F_\perp \to (F + G)_\perp$ and $\iota_G \colon G_\perp \to (F+G)_\perp$ as $\iota_F(r) = \iota_G(s) = t$, and $\iota_F(x) = (x, 0)$ for all $r \neq x \in F$, and $\iota_G(x) = (x, 1)$ for all $s \neq x \in G$. The claim now follows easily from Lemma 2.3, upon noticing that an order-preserving open maps between trees must map root to root. ∎

B. Categorical equivalence

The dual equivalence between the category $\mathbb{GH}_{fin}$ of finite Gödel hoops and their homomorphisms, and the category FT of finite trees and order-preserving open maps, is sketched in [10, Chap. IX] and fully proved in [5].

The functors implementing the equivalence are

$$\mathrm{Spec}^* \colon \mathbb{GH}_{fin} \to \mathsf{FT}\,,$$

and

$$\mathrm{Sub}^* \colon \mathsf{FT} \to \mathbb{GH}_{fin}\,,$$

that are defined on objects as

$$\mathrm{Spec}^* \mathcal{A} = \mathrm{Spec}\, \mathcal{A}_\perp\,, \quad \mathrm{Sub}^* F = \mathrm{Sub}\, F \setminus \{\emptyset\}\,,$$

while on arrows, they are defined as before by taking preimages.

C. Computing coproducts

It is immediate to check that one can use the algorithm of Theorem 4.4 to compute coproducts of finite Gödel hoops.

Example 6.2: The product of the tree $\mathbf{1}_\perp$ with itself is given by the tree $(\mathbf{1}_\perp + \mathbf{1} + \mathbf{1}_\perp)_\perp$.

To compute the cardinality of such a coproduct we provide a slight adaptation of Theorem 5.2. Let $||F||^* = |\mathrm{Sub}^* F|$.

Theorem 6.3: The following recursive algorithm computes the cardinality of any finite Gödel hoop A, having in input $F_\perp = \mathrm{Spec}^* A$.

1) If $F_\perp \cong 1$ then $||F_\perp||^* = 1$.
2) If $F_\perp \cong (G + H)_\perp$ then $||F||^* = ||G|| \cdot ||H||$.

Proof: The correctness and termination of the algorithm follows immediately by 5.2, upon recalling that $|\mathrm{Sub}^* F_\perp| = |\mathrm{Sub}\, F| - 1$. ∎

Example 6.4: The cardinality of $\mathrm{Sub}^* \mathbf{1}_\perp \cong \mathrm{Sub}^* (\mathbf{1}+\mathbf{0})_\perp$ is $2 \cdot 1 = 2$. The cardinality of $\mathrm{Sub}^* (\mathbf{1}_\perp \times \mathbf{1}_\perp)$ is $3 \cdot 2 \cdot 3 = 18$.

VII. Gödel$_\triangle$-algebras and multisets of chains

A *Gödel$_\triangle$-algebra* is a system $\mathcal{A} = (A, \vee, \wedge, \to, \Delta, 0, 1)$, where $(A, \vee, \wedge, \to, 0, 1)$ is a Gödel algebra, called the *Gödel reduct* of $\mathcal{A}$, while $\Delta \colon A \to A$ is the projection operator, satisfying the following properties.

$$\Delta\varphi \vee \neg\Delta\varphi = 1, \quad \Delta\varphi \to \varphi = 1, \quad \Delta\varphi = \Delta\Delta\varphi\,,$$

$$\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi) = 1, \quad \Delta(\varphi \vee \psi) = (\Delta\varphi \vee \Delta\psi)\,.$$

Notice that on each totally ordered Gödel$_\triangle$-algebra the projection operator is such that $\Delta x = 1$ if $x = 1$, $\Delta x = 0$ if $x \neq 1$.

A *filter* of a Gödel$_\triangle$-algebra $\mathcal{A}$ is an upward closed subset $\mathfrak{p}$ of A such that $x \in \mathfrak{p}$ implies $\Delta x \in \mathfrak{p}$. With this notion in place, filters are in bijection with congruences via the same correspondence holding for MTL-algebras, and Theorem 2.1 applies.

The following lemma is well-known. See for instance [2, Prop. 3].

Lemma 7.1: The variety $\mathbb{G}_\triangle$ of Gödel$_\triangle$-algebras is semisimple.

Proof: Let $\mathcal{A}$ be a Gödel$_\triangle$-chain, and let $x < y$ be two distinct elements in A. Notice that $y \to x < 1$. Assume a congruence θ is such that $x\theta y$. Then $(y \to x)\theta 1$, whence $\Delta(y \to x)\theta\Delta 1$, that is $0\theta 1$, which, by convexity of lattice congruences, means that θ is the total congruence: that is, $\mathcal{A}$ is simple. ∎

Lemma 7.1 and Theorem 2.1 imply the following.

Lemma 7.2: Each finite Gödel$_\triangle$-algebra $\mathcal{A}$ is a direct product of chains. Actually, $\mathcal{A} \cong \prod_{\mathfrak{p} \in Max(\mathcal{A})} \mathcal{A}/\mathfrak{p}$, and $Max(\mathcal{A}) = Spec(\mathcal{A})$.

Lemma 7.2 tells us that the spectrum of a Gödel$_\triangle$ algebra does not contain sufficient info to provide a dual equivalence: for instance, consider two Gödel$_\triangle$ chains of different cardinality: they are clearly not isomorphic, but their spectra coincide, being the singleton $\mathbf{1}$ in both cases.

Let $\mathcal{A}^-$ be the Gödel reduct of a Gödel$_\Delta$-algebra $\mathcal{A}$. Then we define

$$\operatorname{Spec}^\Delta \mathcal{A} := \operatorname{Spec} \mathcal{A}^- .$$

Notice that $\operatorname{Spec}^\Delta \mathcal{A}$ does not coincide in general with the spectrum $\operatorname{Spec} \mathcal{A} = Max(\mathcal{A})$ of $\mathcal{A}$. The reader may check that this coincidence happens iff $\mathcal{A}^-$ is a Boolean algebra. We write $(\mathbb{G}_\Delta)_{fin}$ for the full subcategory of $\mathbb{G}_\Delta$ whose objects have finite cardinality.

Lemma 7.3: For each $\mathcal{A} \in (\mathbb{G}_\Delta)_{fin}$, the poset $\operatorname{Spec}^\Delta \mathcal{A}$ is a forest of chains, that is, it has the form $\operatorname{Spec}^\Delta \mathcal{A} \cong \sum_{i\in I} C_i$, where each tree C_i is a chain.

Proof: By Lemma 7.2 it is sufficient to notice that for each finite Gödel$_\Delta$-chain $\mathcal{C}$ we have that $\operatorname{Spec} \mathcal{C}^-$ is a chain. But this is obvious, as $\mathcal{C}^-$ itself is by assumption a chain. ∎

Lemma 7.4: Let $\mathcal{C}, \mathcal{D}$ be Gödel$_\Delta$ chains, and let $h\colon \mathcal{C} \to \mathcal{D}$ be a homomorphism. Then h is injective.

Proof: By way of contradiction, assume $h(x) = h(y)$ for some distinct elements $x, y \in C$, with $x > y$. Then $h(\Delta(x \to y)) = h(\Delta y) = h(0) = 0$, while $\Delta(h(x) \to h(y)) = \Delta 1 = 1$. Whence, $0 = 1$, which is impossible as $0 \leq y < x \leq 1$. Then h must be injective. ∎

Let $h\colon \mathcal{A} \to \prod_{i\in I} \mathcal{C}$ be a homomorphism of Gödel-algebras, where each $\mathcal{C}_i$ is a Gödel chain. Then h is called *chain-injective* if, given each projection $\pi_j\colon \prod_{i\in I} \mathcal{C}_i \to \mathcal{C}_j$, the homomorphism $\pi_j \circ h\colon \mathcal{A} \to \mathcal{C}_j$ is injective.

By Lemma 7.4 each homomorphism h of Gödel$_\Delta$-algebras is a chain-injective homomorphism between their Gödel reducts. We are ready to prove the following.

Theorem 7.5: The category $(\mathbb{G}_\Delta)_{fin}$ of finite Gödel$_\Delta$-algebras and their homomorphisms is equivalent with the non-full subcategory $\mathbb{G}^{c.i.}_{fin}$ of $\mathbb{G}_{fin}$ whose objects are direct products of finite Gödel chains, and whose arrows are the chain-injective homomorphisms.

Proof: The functor $R\colon (\mathbb{G}_\Delta)_{fin} \to \mathbb{G}^{c.i.}_{fin}$ defined on objects by $R(\mathcal{A}) = \mathcal{A}^-$ and on arrows by the identity $R(h) = h$ is easily seen to be 1) essentially surjective: let $\mathcal{A} \cong \prod_{i=1}^k \mathcal{C}_i$ be any direct product of finite Gödel chains. Take its expansion $\mathcal{A}_\Delta$ defined by adding to $\mathcal{A}$ the operation $\Delta\colon A \to A$ defined by $\Delta(c_1, \ldots, c_k) = (b_1, \ldots, b_k)$, for $b_i = 1$ iff $c_i = 1$ and $b_i = 0$ otherwise. Then $R(\mathcal{A}_\Delta) = (\mathcal{A}_\Delta)^- \cong \mathcal{A}$. 2) full and faithful: this follows promptly by R being the identity over morphisms, and by definition of morphism in $\mathbb{G}^{c.i.}_{fin}$. Whence R is an equivalence of categories. ∎

A. Categorical equivalence

Let MC be the category whose objects are finite multisets of (nonempty) finite chains, and whose morphisms $h\colon C \to D$, are defined as follows. Display C as $\{C_1, \ldots, C_m\}$ and D as $\{D_1, \ldots, D_n\}$. Then $h = \{h_i\}_{i=1}^m$, where each h_i is an order preserving surjection $h_i\colon C_i \twoheadrightarrow D_j$ for some $j = 1, 2, \ldots, n$.

By Theorem 7.5, the category $(\mathbb{G}_\Delta)_{fin}$ is equivalent with the subcategory $\mathbb{G}^{c.i.}_{fin}$ of $\mathbb{G}_{fin}$. Then the following lemma is an immediate consequence.

Lemma 7.6: $(\mathbb{G}_\Delta)_{fin}$ is dually equivalent with the non-full subcategory category FF^c of FF, whose objects are finite forests of chains $F = \sum_{i=1}^k C_i$, and whose arrows $f\colon \sum_{i=1}^k C_i \to \sum_{j=1}^h D_j$ are *chain-surjective*, that is, the restriction of f to each C_i maps surjectively to some chain D_l for $l \in \{1, \ldots, h\}$.

Proof: It follows promptly by specialising to the subcategory $\mathbb{G}^{c.i.}_{fin}$ the duality between $\mathbb{G}_{fin}$ and FF, and by composing with the functor R of Theorem 7.5. ∎

It is now easy to turn the dual equivalence of lemma 7.6 into a dual equivalence between $(\mathbb{G}_\Delta)_{fin}$ and MC. Notice that the proof of the following theorem amounts to prove the rather obvious fact that MC is equivalent to FF^c, which, however, are formally distinct categories.

Theorem 7.7: The categories $(\mathbb{G}_\Delta)_{fin}$ and MC are dually equivalent.

Proof: Let $T\colon \mathsf{FF}^c \to \mathsf{MC}$ be defined on objects as $T(\sum_{i=1}^k C_i) = \{C_i \mid i = 1, \ldots, k\}$, and on arrows $f\colon \sum_{i=1}^k C_i \to \sum_{j=1}^l D_j$ as $T(f)$ being the collection $\{f_i\colon C_i \to D_l \mid i = 1, \ldots, k\}$, where f_i is the restriction of f to C_i, and D_l is the image of f_i. It is straightforward to check that T realises an equivalence between the categories FF^c and MC. The rest follows from Lemma 7.6. ∎

Notice that the functor implementing the $(\mathbb{G}_\Delta)_{fin} \to \mathsf{MC}$ side of the dual equivalence is given by the composition

$$T \circ \operatorname{Spec} \circ R .$$

The other side of the equivalence is given by the functor $\operatorname{Sub}^\Delta\colon \mathsf{MC} \to (\mathbb{G}_\Delta)_{fin}$ defined on objects by

$$\operatorname{Sub}^\Delta \{C_i \mid i = 1, \ldots, k\} = \prod_{i=1}^k \operatorname{Sub} C_i ,$$

and on arrows as $\operatorname{Sub} \circ T^{-1}$.

B. Coproducts of G_Δ-algebras

For each finite forest F, let $\max F$ be the set of maximal elements of F. Then with each finite forest F one can associate the finite multiset of chains $C(F)$ defined as follows:

$$C(F) = \{\downarrow \{x\} \mid x \in \max F\} .$$

For each $x \in \max F$ and each $y \leq x$, we let $C(F)_x(y)$ be the copy of y in $\downarrow \{x\} \in C(F)$.

Let C be an object in MC and let $C^\top$ denote the object obtained adding to each chain in C a fresh maximum (being C a multiset of chains, the effect of adding a new maximum to each chain in C is the same as that of adding a new minimum: we have chosen this version to differentiate the notation from the lifting construction transforming a forest F into a tree $F_\perp$).

For each integer $k > 0$ let $\mathbf{k}$ denote the k-element chain.

Lemma 7.8: In MC the following hold.

1) The terminal object is the multiset $\{\mathbf{1}\}$, and the initial object is the empty multiset $\emptyset$.
2) The coproduct $C + D$ of two multisets C and D is the disjoint union of C and D.

3) Products distribute over coproducts: $C \times (D+E) \cong (C \times D) + (C \times E)$.

4) The following recursive formula holds:

$$\{\mathbf{i+1}\} \times \{\mathbf{j+1}\} \cong$$
$$((\{\mathbf{i+1}\} \times \{\mathbf{j}\}) + (\{\mathbf{i}\} \times \{\mathbf{j}\}) + (\{\mathbf{i}\} \times \{\mathbf{j+1}\}))^{\top}.$$

Proof: 1) and 2) are straightforward by the definitions of terminal object, initial object and coproduct. The proof of 3) is completely analogous to the proof of Lemma 4.2, *mutatis mutandis*, as that proof essentially relies only on the fact that the coproduct is the disjoint union. The careful reader can go through that proof and check that everything still works. To prove 4), first we construct the forest $F := \{\mathbf{i+1}\} \times \{\mathbf{j+1}\} \cong (\{\mathbf{i+1}\} \times \{\mathbf{j}\}) + (\{\mathbf{i}\} \times \{\mathbf{j}\}) + (\{\mathbf{i}\} \times \{\mathbf{j+1}\}))_{\perp}$ computed in FF according to Lemma 4.3. Then we apply the map C to F, producing the multiset $C(F) = ((\{\mathbf{i+1}\} \times \{\mathbf{j}\}) + (\{\mathbf{i}\} \times \{\mathbf{j}\}) + (\{\mathbf{i}\} \times \{\mathbf{j+1}\}))^{\top}$. We are left to show that $C(F)$ is indeed the product $\{\mathbf{i+1}\} \times \{\mathbf{j+1}\}$ computed in MC. Take any $D = \{D_1, D_2, \ldots, D_u\} \in \mathsf{MC}$ and MC-maps $h_i\colon D \to \{\mathbf{i+1}\}$, and $h_j\colon D \to \{\mathbf{j+1}\}$. Consider the uniquely determined forest $D' \in \mathsf{FF}^c$ such that $T(D') = D$. We identify elements of D and D' through the bijection T. Then there is a unique order-preserving open map $g\colon D' \to F$ such that $\pi_i \circ g = h_i$ and $\pi_j \circ g = h_j$. Notice that, for each $i = 1, \ldots, u$, the image of $g \restriction D_i$ is of the form $\downarrow \{y_i\}$ for some $y_i \in \max F$. We define $g' = \{g_i\}_{i=1}^{u}\colon D \to C(F)$ as $g_i(z) = C(F)_{y_i}(g(z))$ for each $z \in D_i$. Observe that each $g_i\colon D_i \to \downarrow \{y_i\}$ is an order-preserving surjection, so g' is an MC-map, uniquely determined by g. For each $x \in \downarrow \{y\} \in C(F)$ let z_x be the uniquely determined element of F such that $C(F)_y(z_x) = x$. Let $\rho_i\colon C(F) \to \{\mathbf{i+1}\}$ and $\rho_j\colon C(F) \to \{\mathbf{j+1}\}$ be defined by $\rho_i(x) = \pi_i(z_x)$ and $\rho_j(x) = \pi_j(z_x)$. It is clear that $\rho_i \circ g' = h_i$ and $\rho_j \circ g' = h_j$. Uniqueness of g' is granted by uniqueness of g. Then $C(F) \cong \{\mathbf{i+1}\} \times \{\mathbf{j+1}\}$. ∎

It is clear that the four items of Lemma 7.8 are sufficient to compute all coproducts of finite Gödel$_\Delta$-algebras.

Example 7.9: Let us compute the product of the multiset $\{\mathbf{2}, \mathbf{2}\}$ with itself. Then $\{\mathbf{2}, \mathbf{2}\} \times \{\mathbf{2}, \mathbf{2}\} \cong 4 \cdot (\{\mathbf{2}\} \times \{\mathbf{2}\})$. Now, $\{\mathbf{2}\} \times \{\mathbf{2}\} \cong ((\{\mathbf{2}\} \times \{\mathbf{1}\}) + (\{\mathbf{1}\} \times \{\mathbf{1}\}) + (\{\mathbf{1}\} \times \{\mathbf{2}\}))^{\top}$. Whence, $\{\mathbf{2}, \mathbf{2}\} \times \{\mathbf{2}, \mathbf{2}\} \cong 4 \cdot \{\mathbf{3}, \mathbf{2}, \mathbf{3}\}$.

Computing the cardinality of a finite Gödel$_\Delta$-algebra given its dual multiset is very simple. Since $\mathrm{Sub}^{\Delta}\{C_i \mid i = 1, \ldots, k\} = \prod_{i=1}^{k} \mathrm{Sub}\, C_i$, its cardinality is just $\prod_{i=1}^{k}(|C_i| + 1)$.

Example 7.10: Let us consider the coproduct of $\mathrm{Sub}^{\Delta}\{\mathbf{2}, \mathbf{2}\}$ with itself. By Example 7.9 its cardinality is $(4 \cdot 3 \cdot 4)^4 = 5308416$.

C. DP-algebras

Drastic product algebras (DP-algebras, for short; also called $\mathrm{S_3MTL}$-algebras in [21]), are MTL-algebras satisfying $x \vee ((x * x) \to 0) = 1$. The paper [2] studies DP-algebras, and introduces, adapting a result of [9], a duality for finite DP-algebras which is based on a non-full subcategory of MC.

Proposition 7.11: Let $\mathsf{MC}^{\top}$ be the non-full subcategory of MC whose morphisms $h\colon C \to D$ satisfy the following additional constraint: for each $i = 1, 2, \ldots, m$, if the target D_j of h_i is not isomorphic with $\mathbf{1}$, then $h_i^{-1}(\max D_j) = \{\max C_i\}$. Then $\mathsf{MC}^{\top}$ is dually equivalent to the category of finite DP-algebras and their homomorphisms.

Proposition 7.11 implies that the category of finite DP-algebras and their homomorphisms is equivalent with a non-full subcategory of the category of finite Gödel$_\Delta$-algebras and their homomorphisms. In the same paper, a recursive formula for computing the product in $\mathsf{MC}^{\top}$ is given, which, as expected, turns out to be a slight variant of the formula of Lemma 7.8.4). For the sake of completeness of exposition, we report this formula here.

$$\{\mathbf{i}\} \times \{\mathbf{1}\} \cong \{\mathbf{i}\}, \qquad \{\mathbf{i+1}\} \times \{\mathbf{2}\} \cong \{\mathbf{i+1}\},$$
$$\{\mathbf{i+2}\} \times \{\mathbf{j+2}\} \cong$$
$$((\{\mathbf{i+2}\} \times \{\mathbf{j+1}\}) + (\{\mathbf{i+1}\} \times \{\mathbf{j+1}\}) + (\{\mathbf{i+1}\} \times \{\mathbf{j+2}\}))^{\top}$$

To compute cardinalities of finite DP-algebras, the same considerations made for the case of finite $\mathbb{G}_\Delta$-algebras apply.

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