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Computing coproducts of finitely presented Gödel algebras

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Abstract

We obtain an algorithm to compute finite coproducts of finitely generated Gödel algebras, i.e. Heyting algebras satisfying the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$. (Since Gödel algebras are locally finite, 'finitely generated', 'finitely presented', and 'finite' have identical meaning in this paper.) We achieve this result using ordered partitions of finite sets as a key tool to investigate the category opposite to finitely generated Gödel algebras (forests and open order-preserving maps). We give two applications of our main result. We prove that finitely presented Gödel algebras have free products with amalgamation; and we easily obtain a recursive formula for the cardinality of the free Gödel algebra over a finite number of generators first established by A. Horn.

Keywords: Heyting algebras; Gödel algebras; Coproducts; Open maps; Trees; Forests; Ordered partitions

1. Introduction

A *Gödel algebra* (a.k.a. an *L-algebra*, or *Gödel–Dummett algebra*) is a Heyting algebra satisfying the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$. For background and references on Gödel logic and algebras we refer to [4]; for background on Heyting algebras, we refer to [7,3].

Given a variety (i.e., equationally definable class) of algebras V, let V_{fp} denote the full subcategory of finitely presented objects. Informally, a Stone-type duality for V_{fp} amounts to an explicit description of some category C that is dually equivalent to V_{fp}. Evidently, not all such dualities are equally informative: taking $\mathsf{C} = \mathsf{V}_{\mathsf{fp}}^{\mathsf{op}}$, for instance, yields virtually no new insight. It has been authoritatively maintained [8,3] that a reasonable benchmark for the usefulness of a Stone-type duality is the degree to which it affords computation of limits, and thus of colimits in the original category V_{fp}.

The variety of Gödel algebras is locally finite and has finite signature, whence the finitely presentable algebras coincide with the finite and the finitely generated ones. Starting from the standard duality between finite posets and finite distributive lattices, it is a simple matter to develop an effective Stone-type duality for finite Gödel algebras; this we do in Section 2. We marshal enough information on the dual category to eventually establish, in Section 3, an algorithm to compute finite coproducts of finite Gödel algebras in terms of ordered partitions of finite sets. Our results also allow computation of fibred coproducts.

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In Section 4, we give two applications. It is known that Gödel algebras have the amalgamation property [9]. We prove that finitely presented Godel algebras in fact have free products with amalgamation (Corollary 4.1). Horn [5] established a recursive formula for the cardinality of the free Gödel algebra over n generators. We reobtain Horn's formula as an easy application of our main results (Corollary 4.2).

Notation. We let $\mathbb{N} = \{1, 2, \ldots\}$. We write $|A|$ for the cardinality of the set A, and $\setminus$ to denote set-theoretic difference. We write $f \restriction A$ to denote the restriction of the function f to A. We let G denote the category of Gödel algebras, and G_{fp} the full subcategory of finitely presented (equivalently, finite, or finitely generated) algebras. We let $\mathcal{G}_n$ denote the free Gödel algebra over n generators.

2. Preliminaries: Duals of finitely presented Gödel algebras

For general Stone-type dualities see [7]. On the basis of [5, Theorem 2.4], we provide an explicit description of $\mathsf{G}_{\mathsf{fp}}^{\mathsf{op}}$ in terms of forests and open order-preserving maps. As is standard, by a *chain* we throughout mean a totally ordered set.

Definition 2.1. A *forest* is a finite poset F such that for every $x \in F$, the set $\{y \in F \mid y \le x\}$ is a chain when endowed with the order inherited from F. If $S \subseteq F$, the *down-set* of S is[1] $\downarrow S = \{x \in F \mid x \le y, \text{ for some } y \in S\}$. A *tree* is a forest with a bottom element, called the *root* of the tree. An order-preserving map $f : F \to G$ between forests is *open* iff it carries down-sets to down-sets — for every $S \subseteq F$, $f(\downarrow S) = \downarrow T$ for some $T \subseteq G$. We let F denote the category of forests and open order-preserving maps, and T the full subcategory of trees.

Remark 1. It is easy to see that the condition $f(\downarrow S) = \downarrow T$ above may be replaced by $f(\downarrow x) = \downarrow f(x)$ for every $x \in F$.

The following is a version of the standard duality between finite posets and finite distributive lattices (see e.g. [1]) for finite Gödel algebras:

Proposition 2.2. *The categories G_{fp} and F are dually equivalent via the functor* **Spec** *that sends a finite Gödel algebra to the poset of its prime filters* (*ordered by reverse*[2] *set-theoretic inclusion*), *and a morphism $f : A \to B$ of algebras to the order-preserving map given by*

$$\mathbf{Spec}(f) : \mathfrak{p} \in \mathbf{Spec}(B) \;\mapsto\; \{a \in A \mid f(a) \in \mathfrak{p}\} \in \mathbf{Spec}(A)\,.$$

Proof. A straightforward verification. To check that duals of finite Gödel algebras are precisely forests, one uses the easily proved fact that the prime filters of a finite (in fact, by [5, Theorem 2.4], of any) Gödel algebra form a forest under reverse set-theoretic inclusion. Moreover, a Heyting algebra whose prime filters form a forest is necessarily a Gödel algebra. It is well-known that order-preserving maps preserve implication iff they are open — for details see for instance [3]. □

The terminal object in F (and T) is a tree consisting of the root only. Finite coproducts in F (and T) are disjoint unions. Finite (fibred) products in F exist because G is a variety; a moment's reflection shows that T has finite products, and a finite product computed in T coincides with the same product computed in F.

Notation. We use $\prod$ and $\coprod$ for products and coproducts, respectively; we also write $X \times Y$ for the product of two objects, and $X \times_Z Y$ for the product of X and Y fibred over Z. We identify without further warning a finite set $\{T_1, \ldots, T_n\}$ of trees with the forest F whose trees are precisely the T_i's — in other words, $F = \coprod_{i=1}^{n} T_i$. We let $\mathcal{C}_n$ denote (the isomorphism type in F of) a chain of cardinality $n + 1$, for $n \ge 0$. We let $\mathcal{S} = \{\mathcal{C}_0, \mathcal{C}_1\}$.

A trivial computation shows that $\mathcal{G}_1$ is the lattice-theoretic product of two chains of respective lengths 2 and 3, whence $\mathbf{Spec}(\mathcal{G}_1) = \mathcal{S}$. Thus, $\mathbf{Spec}(\mathcal{G}_n) = \prod_{i=1}^{n} \mathcal{S} = \mathcal{S}^n$ for any $n \in \mathbb{N}$. Fig. 1(a) shows $\mathcal{S}$. Fig. 1(b), as proved in Section 3, is a picture of $\mathcal{S}^2$.

[1] Following widespread usage, we write $\downarrow x$ for $\downarrow \{x\}$.

[2] A convention herein adopted to make trees grow upwards.

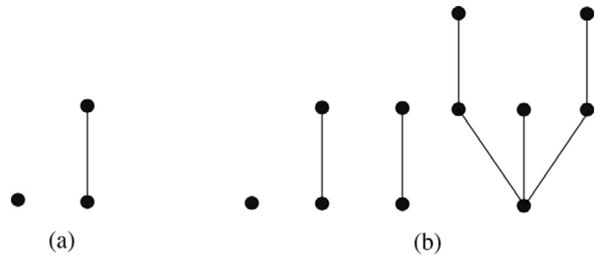

Fig. 1. (a) The trees $\mathcal{C}_0$, $\mathcal{C}_1$ of $\mathcal{S}$. (b) The forest $\mathcal{S}^2$.

The following is an explicit description of equalisers in T.

Proposition 2.3. T *has equalisers. Specifically, let* $f, g: T_1 \rightrightarrows T_2$ *be given. Let* $e: S \to T_1$ *be the subposet of* T_1 *that equalises* f, g *as set-theoretic functions. Then there exists a unique inclusion-maximal subtree* T *of* T_1 *whose underlying set is contained in* $e(S)$*. Then the inclusion map* $\iota: T \to T_1$ *is the equaliser of* f, g.

Proof. Observe that the set-theoretic union of two subobjects of T_1 (i.e. subtrees with the property that the down-set of each node is a branch of the subtree) is again a subtree. It follows that there does exist a unique inclusion-maximal subtree T of T_1 whose underlying set is contained in $e(S)$, whence ι is a morphism of trees. Since e is the set-theoretic equaliser of the functions f and g, and since $T \subseteq e(S)$, it follows that $\iota: T \to T_1$ equalises f and g. By the same argument, any morphism $\tilde{e}: \tilde{T} \to T_1$ in T that equalises f and g satisfies $\tilde{e}(\tilde{T}) \subseteq S$. Hence, by the maximality of T, $\tilde{e}(\tilde{T}) \subseteq T$, and the universal property of ι immediately follows. □

Caution: in the proposition above, the subposet $e(S)$ of T_1 with support the whole of S need not be a tree.

It follows that T has fibred finite products, and these coincide with the corresponding fibred products computed in F.

The problem of explicitly describing the forest $F \times G$ is easily reduced to that of describing its trees. Indeed, given forests $F = \{T_1, \ldots, T_r\}$, $G = \{U_1, \ldots, U_s\}$, we have $F \times G = \{W_{ij}\}$, where $W_{ij} = T_i \times U_j$, $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, s\}$; the projection maps $\pi_1: F \times G \to F$, $\pi_2: F \times G \to G$ are given by $\pi_1(x) = \pi_1^{ij}(x)$ and $\pi_2(x) = \pi_2^{ij}(x)$ for $x \in W_{ij}$, where π_1^{ij}, π_2^{ij}, are the projection maps of W_{ij}. The case of fibred products $F \times G$ is an obvious extension of the foregoing using equalisers. Equalisers in F are obtained as follows. Given $f, g: F_1 \rightrightarrows F_2$, first take the set-theoretic equaliser $e: S \to F_1$ of f and g; then replace each element $s \in S$ with the equaliser in T of $f \restriction e(s)\,,\ g \restriction e(s) :\ e(s) \rightrightarrows f(e(s)) = g(e(s))$.

We shall eventually need to use subobject classifiers in T and F. (For a definition of 'subobject classifier' see e.g. [3, p. 227].)

Proposition 2.4. *The monomorphism* $\mathsf{true}: \mathcal{C}_0 \rightarrowtail \mathcal{C}_1$ *that maps* $\mathcal{C}_0$ *to the root of* $\mathcal{C}_1$ *is the subobject classifier in* T. *The monomorphism* $\mathsf{true}: \{\mathcal{C}_0\} \rightarrowtail \mathcal{S}$ *that maps* $\mathcal{C}_0$ *to the root of* $\mathcal{C}_1 \in \mathcal{S}$ *is the subobject classifier in* F.

Proof. Fix a monomorphism $m: S \rightarrowtail T$ of trees. Define a map $\chi_m: T \to \mathcal{C}_1$ as follows. If $x \in T$ is such that x is not in the set-theoretical range of m, let $\chi_m(x)$ be the top element of $\mathcal{C}_1$; otherwise, let $\chi_m(x)$ be the root of $\mathcal{C}_1$. A moment's reflection shows χ_m is a morphism in T. The square

$$\begin{tikzcd} S \arrow[r, "!"] \arrow[d, "m"'] & \mathcal{C}_0 \arrow[d, "\mathsf{true}"] \\ T \arrow[r, "\chi_m"'] & \mathcal{C}_1 \end{tikzcd}$$

ie evidently commutative. Moreover, it is a pullback square: the product $T \times \mathcal{C}_0$ is isomorphic to T, and the equaliser $e: T \times_{\mathcal{C}_1} \mathcal{C}_0 \to T \times \mathcal{C}_0 \rightrightarrows \mathcal{C}_1$ selects in $T \cong T \times \mathcal{C}_1$ just the subobject S.

The proof of the second assertion is equally straightforward. □

Remark 2. With Propositions 2.2 and 2.4 available, one can recover a Gödel algebra from its spectrum as follows. Let F be a forest, $\mathcal{F}$ the set of all maps of forests $f: F \to \mathcal{S}$, and Sub F the set of subobjects of F in F. Then Sub F

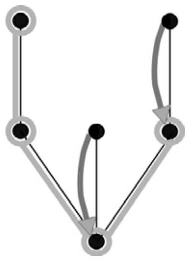

Fig. 2. The unique retraction of a tree ($\mathfrak{C}_1^2$, cf. Section 3) onto the subtree with larger hollow nodes.

is a finite distributive lattice under inclusion, and thus has a unique Heyting algebra structure. Up to an isomorphism, Sub F is the unique Gödel algebra whose spectrum is F. Equivalently, endow $\mathcal{F}$ with pointwise order ($f \leq g$ iff for every $x \in F$ one has $f(x) \leq g(x)$). Then again $\mathcal{F}$ is the Gödel algebra whose spectrum is F.

While the foregoing is all we shall need to prove our main results, we add a note. A simple consequence of Proposition 2.2 is that any finitely generated Gödel algebra is projective. This is also observed in [2], but within a broader context. For general Gödel algebras, nothing seems to be known. (See [2] and references therein for the crucial rôle finitely presented projective objects play in Automated Deduction and related areas.)

Proposition 2.5. *Any epimorphism in* G_{fp} *is a retraction. Consequently, any finitely generated Gödel algebra is projective.*

Proof. (Cf. Fig. 2.) By duality, it suffices to show that any monomorphism in F is a section, and this in turn is immediately seen to hold iff monomorphisms in T are sections. Let $s: A \to B$ be a monomorphism of trees. There exists a *unique* retraction $r: B \to A$, constructed as follows. Let $p \in B$ be given. The set

$$M_p = \{b \in B \mid s^{-1}(b) \neq \emptyset \text{ and } b \leq p\}$$

is nonempty, because it contains the root of B. Since $M_p \subseteq \downarrow p$, M_p is a chain (with the order inherited from B). Hence M_p has a maximum m, and its inverse image $s^{-1}(m) = \{p_m\}$ is a singleton. We set

$$r: p \in B \mapsto p_m \in A\ .$$

It is clear that r is order-preserving: if $p \geq q$ then $M_p \supseteq M_q$, and thus $p_m \geq q_m$. To see that r is open, just note that by construction $r(\downarrow p) = \downarrow p_m = \downarrow r(p)$. It is an exercise to show that r is unique. □

Evidently, uniqueness of retractions fails for forests. It is clear that not every monomorphism in G_{fp} is a section.

3. Ordered partitions, foliages, and trees

Throughout, *set* means *finite set*.

Definition 3.1 (*Ordered Partition, Foliage, Tree*). An *ordered partition* σ is a sequence of pairwise disjoint nonempty sets. We write $\sigma = \{A_1, \dots, A_m\}$, where the A_i's are the *blocks* of σ, and it is understood that A_i precedes A_j iff $i \leq j$. Given ordered partitions $\sigma = \{A_1, \dots, A_m\}$, $\tau = \{B_1, \dots, B_n\}$ with $m \leq n$, we write $\sigma \leq \tau$ iff $A_i = B_i$ for every $i \in \{1, \dots, m\}$. A *foliage* is a set of mutually incomparable (according to $\leq$) ordered partitions (i.e., an antichain). Given a foliage T, we set Tree $T = \{\sigma \mid \sigma$ is an ordered partition such that $\sigma \leq \tau \in T\}$. Given an ordered partition σ, its *support* is supp $\sigma = \bigcup \sigma$. Similarly, if T is a foliage, we set supp $T = \bigcup_{\sigma \in T}$ supp σ.

Clearly, $\leq$ is a partial order on ordered partitions. Note that $\sigma = \{A_1, \dots, A_m\}$ and $\tau = \{B_1, \dots, B_n\}$ with $m \leq n$ are incomparable iff there exists $i \in \{1, \dots, m\}$ with $A_i \neq B_i$. Also note that $\emptyset$ is the only ordered partition of $\emptyset$, and that $\emptyset \leq \sigma$ for every ordered partition σ. The collection of ordered partitions below $\sigma = \{A_1, \dots, A_m\}$ is the chain $\emptyset$, $\{A_1\}$, $\{A_1, A_2\}$, $\dots$, $\{A_1, \dots, A_m\}$. Thus, Tree T is a tree whose leaves (i.e. maximal elements) are precisely the elements of T, and whose root is $\emptyset$. If $\sigma, \tau \in$ Tree T, σ covers τ (i.e. $\sigma > \tau$, and there is no $\theta \in$ Tree T satisfying $\sigma > \theta > \tau$) iff σ can be obtained from τ appending to it one trailing block.

The next two definitions are of key importance for the contents of this paper.

Definition 3.2 (*Merged Shuffle*). Let σ and τ be ordered partitions with disjoint supports. An ordered partition θ is a *shuffle* of σ and τ iff σ and τ are subsequences of θ, and $\operatorname{supp}\theta = \operatorname{supp}\sigma \cup \operatorname{supp}\tau$. We stipulate that the unique shuffle of σ and $\{\emptyset\}$ is σ. We inductively define a *merged shuffle* (*of order* $t \geq 0$) of σ and τ as follows. A merged shuffle of order 0 of σ and τ is just a shuffle of σ and τ. If $t > 0$, θ is a merged shuffle of order t of σ and τ iff there exists a merged shuffle $\lambda = \{B_1, \ldots, B_k\}$ of order $t-1$ of σ and τ, together with an index $i \in \{1, \ldots, k-1\}$, such that $\theta = (\lambda \setminus \{B_i, B_{i+1}\}) \cup \{B_i \cup B_{i+1}\}$ with $B_i \in \sigma$ and $B_{i+1} \in \tau$, or $B_i \in \tau$ and $B_{i+1} \in \sigma$, where the block $B_i \cup B_{i+1}$ of θ comes just after B_{i-1} (or first, if $i = 1$) and just before B_{i+2} (or last, if $i = k-1$).

Example 1. Consider $\sigma = \{\{a\}, \{b\}\}$ and $\tau = \{\{x\}\}$. In this and subsequent examples, we omit inner braces in displaying ordered partitions: σ is $\{a|b\}$ and τ is $\{x\}$. There are 3 shuffles of σ and τ, namely $\{a|b|x\}$, $\{a|x|b\}$, $\{x|a|b\}$. Besides these merged shuffles of order 0, there are 2 merged shuffles of σ and τ, namely $\{a|bx\}$, $\{ax|b\}$. They are both of order 1.

Definition 3.3 (*Product*). Let S and T be foliages with disjoint supports. We call

$$S \times T = \{\theta \mid \theta \text{ is a merged shuffle of some } \sigma \in S, \tau \in T\}$$

the *product* of S and T.

Notation. Given an ordered partition $\theta = \{B_1, \ldots, B_n\}$ and a set X, we let $\theta - X$ denote the ordered partition $\{B_1 \setminus X, \ldots, B_n \setminus X\} \setminus \{\emptyset\}$.

Remark 3. We record three remarks to be used repeatedly in the following.

(1) If θ is a merged shuffle of σ and τ, then $\tau = \theta - \operatorname{supp}\sigma$, and $\sigma = \theta - \operatorname{supp}\tau$.
(2) If $\theta_1 \leq \theta_2$, then $\theta_1 - X \leq \theta_2 - X$ for any set X.
(3) If θ_1 and θ_2 are distinct merged shuffles of σ and τ, then they are incomparable. Indeed, if $\theta_1 < \theta_2$ and $\theta_1 = \{B_1, \ldots, B_n\}$, then $\theta_2 = \{B_1, \ldots, B_n, C_1, \ldots, C_m\}$, which is impossible because $\operatorname{supp}\theta_1 = \operatorname{supp}\theta_2 = \operatorname{supp}\sigma \cup \operatorname{supp}\tau$.

Lemma 3.4. *For any two foliages S and T with disjoint supports, $S \times T$ is a foliage satisfying*

$$\operatorname{Tree}(S \times T) = \{\theta' \mid \theta' \text{ is a merged shuffle of some } \sigma' \in \operatorname{Tree} S, \tau' \in \operatorname{Tree} T\}.$$

Proof. We first show $S \times T$ is a foliage. To avoid trivialities, assume $S \neq \{\emptyset\} \neq T$. Fix $\sigma_1, \sigma_2 \in S$ and $\tau_1, \tau_2 \in T$. Let θ_1 be a merged shuffle of σ_1, τ_1, and θ_2 be a merged shuffle of σ_2, τ_2. Suppose $\theta_1 \leq \theta_2$, and let $\theta_1 = \{B_1, \ldots, B_n\}$, $\theta_2 = \{B_1, \ldots, B_n, C_1, \ldots, C_m\}$. By Remark 3(1), it follows that $\sigma_1 \leq \sigma_2$, $\tau_1 \leq \tau_2$. Since S and T are foliages, we have $\sigma_1 = \sigma_2$, $\tau_1 = \tau_2$. By Remark 3(3), we infer $\theta_1 = \theta_2$. In other words, $S \times T$ is a foliage.

To prove the remaining part of the lemma, let $\theta' \in \operatorname{Tree}(S \times T)$. Then there exists $\theta \in S \times T$ such that $\theta' \leq \theta$. It follows that there exist $\sigma \in S$, $\tau \in T$ such that θ is a merged shuffle of σ and τ. Let $\sigma' = \theta' - \operatorname{supp}\tau$ and $\tau' = \theta' - \operatorname{supp}\sigma$. Then $\operatorname{supp}\theta' = \operatorname{supp}\sigma' \cup \operatorname{supp}\tau'$, because $\operatorname{supp}\theta' \subseteq \operatorname{supp}\theta$, and the latter is the disjoint union of $\operatorname{supp}\sigma$ and $\operatorname{supp}\tau$. Since $\theta' \leq \theta$, by Remark 3(2) we have $\theta' - \operatorname{supp}\tau \leq \theta - \operatorname{supp}\tau$, that is $\sigma' \leq \sigma$. Similarly, we deduce $\tau' \leq \tau$. Therefore, $\sigma' \in \operatorname{Tree} S$ and $\tau' \in \operatorname{Tree} T$. It now suffices to show that θ' is a merged shuffle of σ' and τ'. If $B \in \theta'$, then $B \in \theta$, whence either $B \in \sigma$ or $B \in \tau$ or $B = B_1 \cup B_2$ for some $B_1 \in \sigma$ and $B_2 \in \tau$. Say for definiteness the latter is the case. Then, since $\operatorname{supp}\theta' = \operatorname{supp}\sigma' \cup \operatorname{supp}\tau'$ and the union is disjoint, we have $B_1 \in \sigma'$ and $B_2 \in \tau'$, which implies θ' is a merged shuffle of σ' and τ'.

Conversely, let θ' be a merged shuffle of $\sigma' = \{A_1, \ldots, A_h\} \in \operatorname{Tree} S$ and $\tau' = \{B_1, \ldots, B_k\} \in \operatorname{Tree} T$. Choose $\sigma = \{A_1, \ldots, A_h, A_{h+1}, \ldots, A_{h+m}\} \in S$ and $\tau = \{B_1, \ldots, B_k, B_{k+1}, \ldots, B_{k+n}\} \in T$. Consider $\theta = \theta' \cup \{A_{h+1}, \ldots, A_{h+m}\} \cup \{B_{k+1}, \ldots, B_{k+n}\}$, where θ' is the bottom, $\{A_i\}$ the middle, and $\{B_j\}$ the top segment of θ. Now $\theta \in S \times T$, because θ evidently is a merged shuffle of σ and τ. Moreover, $\theta' \leq \theta$ by construction, and thus $\theta' \in \operatorname{Tree}(S \times T)$, as was to be shown. □

Definition 3.5 (*Projections*). Let S and T be foliages, and set $X = \operatorname{supp} S$, $Y = \operatorname{supp} T$. Assume $X \cap Y = \emptyset$. Let $A = \operatorname{Tree} S$, $B = \operatorname{Tree} T$, and $C = \operatorname{Tree}(S \times T)$. We define a function $\pi_S : C \to A$ by $\theta \mapsto \theta - Y$, and similarly $\pi_T : C \to B$ by $\theta \mapsto \theta - X$. We call π_S and π_T the *projections induced by* $S \times T$.

We prepare a second lemma.

Lemma 3.6. *Let S and T be foliages with disjoint supports. Then the projections π_S, π_T induced by $S \times T$ are epimorphisms in* T. *Moreover, for every $\sigma \in \mathrm{Tree}\, S$ and $\tau \in \mathrm{Tree}\, T$, the intersection of the set-theoretic fibres $\pi_S^{-1}(\sigma) \cap \pi_T^{-1}(\tau)$ is precisely the set of merged shuffles of σ and τ.*

Proof. By Remark 3(2), projections are order-preserving. As to openness, let $\theta = \{B_1, \ldots, B_k\} \in \mathrm{Tree}\,(S \times T)$, and let $\lambda = \pi_S(\theta) = \{B_1 \setminus Y, \ldots, B_k \setminus Y\} \setminus \{\emptyset\}$, where $Y = \mathrm{supp}\, T$. If $\lambda' \le \lambda$, then $\lambda' = \{B_1 \setminus Y, \ldots, B_h \setminus Y\} \setminus \{\emptyset\}$ for some $h \le k$. But then $\theta' = \{B_1, \ldots, B_h\}$ satisfies $\theta' \le \theta$, whence $\theta' \in \mathrm{Tree}\,(S \times T)$ by Lemma 3.4. By construction, $\pi_S(\theta') = \lambda'$. This proves projections are open maps.
In T, epimorphisms are just surjections. To check that projections are epic, let $\sigma \in \mathrm{Tree}\, S$ be given. The only merged shuffle of σ and $\emptyset$ is σ itself, and since $\emptyset \in \mathrm{Tree}\, T$, by Lemma 3.4 we conclude $\sigma \in \mathrm{Tree}\, S \times T$. Evidently, $\pi_S(\sigma) = \sigma$.

For $\sigma \in \mathrm{Tree}\, S$ and $\tau \in \mathrm{Tree}\, T$, set $I = \pi_S^{-1}(\sigma) \cap \pi_T^{-1}(\tau)$. If θ is a merged shuffle of σ and τ, then $\theta \in I$ by Remark 3(1). Conversely, suppose $\theta \in I$, i.e. $\pi_S(\theta) = \sigma$ and $\pi_T(\theta) = \tau$. By Lemma 3.4, there are $\sigma' \in \mathrm{Tree}\, S$ and $\tau' \in \mathrm{Tree}\, T$ such that θ is a merged shuffle of σ' and τ'. Remark 3(1), $\pi_S(\theta) = \sigma'$ and $\pi_T(\theta) = \tau'$, whence $\sigma' = \sigma$ and $\tau' = \tau$, and the proof is complete. $\square$

We are now ready to prove the main theorem of our paper.

Theorem 3.7. *Let S and T be foliages with disjoint supports. Then*

$$\mathrm{Tree}\, S \overset{\pi_S}{\leftarrow} \mathrm{Tree}\,(S \times T) \overset{\pi_T}{\rightarrow} \mathrm{Tree}\, T$$

is the product of $\mathrm{Tree}\, S$ *and* $\mathrm{Tree}\, T$ *in* T.

Proof. Let P be a tree, and suppose morphisms of trees $p_S : P \to \mathrm{Tree}\, S$ and $p_T : P \to \mathrm{Tree}\, T$ be given. By induction on the number of nodes $|P| = n$ of P. The case $n = 1$ is clear — the only element of $\mathrm{Tree}\, S \times T$ that projects both onto the root of $\mathrm{Tree}\, S$ and onto the root of $\mathrm{Tree}\, T$ is its root. Assume the theorem holds up to $n - 1$. Choose any leaf $l \in P$, and consider the tree $P^* = P \setminus \{l\}$ ($\neq \emptyset$ by $n > 1$) with maps $p_S^* : P^* \to \mathrm{Tree}\, S$ and $p_T^* : P^* \to \mathrm{Tree}\, T$ defined by restriction of p_S and p_T, respectively; they are clearly morphisms in T. By the induction hypothesis, there exists a unique map $f^* : P^* \to \mathrm{Tree}\,(S \times T)$ factoring p_S^* and p_T^* through π_S and π_T, respectively. Since P is a tree, there exists a unique element $c \in P$ that is covered by l. If $p_S(l) = p_S^*(c) = p_S(c)$, we say p_S *folds* l. We apply analogous terminology to p_T. Observe that any map of trees $f : P \to \mathrm{Tree}\,(S \times T)$ factoring p_S and p_T through π_S and π_T, respectively, must extend f^*, by the uniqueness of f^*. Let

$$E = \{e \in \mathrm{Tree}\,(S \times T) \mid \text{ either } e = f^*(c) \text{ or } e \text{ covers } f^*(c)\},$$

whence, in particular, $E \neq \emptyset$. Then any $f : P \to \mathrm{Tree}\,(S \times T)$ extending f^* must satisfy $f(l) \in E$, because morphisms in T are open maps. We distinguish three cases.

Case 1. Both p_S and p_T fold l.

We claim there is a unique $e \in E$ such that both $\pi_S(e) = p_S(c)$ and $\pi_T(e) = p_T(c)$, namely $e = f^*(c)$. Indeed, $f^*(c)$ does satisfy these properties, because both p_S and p_T fold l. Now let $e' \in E$, and suppose $f^*(c) = \{B_1, \ldots, B_k\}$. If e' covers $f^*(c)$, then $e' = \{B_1, \ldots, B_k, B_{k+1}\}$ for some $B_{k+1} \neq \emptyset$. Thus, either $B_{k+1} \setminus \mathrm{supp}\, S$ or $B_{k+1} \setminus \mathrm{supp}\, T$ is nonempty, whence either $\pi_S(e') \neq \pi_S(f^*(c))$ or $\pi_T(e') \neq \pi_T(f^*(c))$. It follows that the map of trees $f : P \to \mathrm{Tree}\,(S \times T)$ extending f^* by $f(l) = f^*(c)$ satisfies the universal property of products, and the theorem is proved if Case 1 holds.

Case 2. Neither p_S nor p_T fold l.

Let us display $\sigma = p_S(l) \in \mathrm{Tree}\, S$ and $\tau = p_T(l) \in \mathrm{Tree}\, T$ as $\sigma = \{A_1, \ldots, A_{h+1}\}$, $\tau = \{B_1, \ldots, B_{k+1}\}$. Then $\sigma^* = p_S^*(c) = p_S(c) \in \mathrm{Tree}\, S$ is necessarily given by $\sigma^* = \{A_1, \ldots, A_h\}$. Similarly, $\tau^* = \{B_1, \ldots, B_k\}$. Since $\pi_S(f^*(c)) = p_S^*(c) = \sigma^*$ and $\pi_T(f^*(c)) = p_T^*(c) = \tau^*$, by Lemma 3.6 $f^*(c)$ is a merged shuffle of σ^* and τ^*. Therefore, $e = f^*(c) \cup \{A_{h+1} \cup B_{k+1}\}$ (with $A_{h+1} \cup B_{k+1}$ as last block) is a merged shuffle of σ and τ, and thus $e \in \mathrm{Tree}\, S \times T$ by Lemma 3.4. By construction, we have $e \in E$, $\pi_S(e) = \sigma = p_S(l)$, and $\pi_T(e) = \tau = p_T(l)$. We claim these properties uniquely determine $e \in E$. For suppose $e' \in E$ satisfies $\pi_S(e') = \sigma$ and $\pi_T(e') = \tau$. Then $e' = f^*(c) \cup \{L\}$ (with L as last block), and $L \setminus \mathrm{supp}\, T = A_{h+1}$, $L \setminus \mathrm{supp}\, S = B_{k+1}$. Thus, $L = A_{h+1} \cup B_{k+1}$, and therefore $e' = e$. We conclude that the map $f : P \to \text{Tree}\,(S \times T)$ extending f^* by $f(l) = e$ satisfies the universal property of products, and the theorem holds in Case 2.

Case 3. Just one among p_S and p_T folds l.

Without loss of generality, say just p_S folds l. Let us display $\sigma = p_S(l) \in \text{Tree}\,S$ and $\tau = p_T(l) \in \text{Tree}\,T$ as $\sigma = \{A_1, \ldots, A_h\}$, $\tau = \{B_1, \ldots, B_{k+1}\}$. Then $\sigma^* = p_S^*(c) = p_S(c) = \sigma$ and $\tau^* = p_T^*(c) = p_T(c) = \{B_1, \ldots, B_k\}$. By Lemma 3.6, $f^*(c)$ is a merged shuffle of σ and τ^*, and thus $e = f^*(c) \cup \{B_{k+1}\}$ (with B_{k+1} as last block) is a merged shuffle of σ and τ. By Lemma 3.4, then, $e \in \text{Tree}\,(S \times T)$. By construction, $e \in E$, $\pi_S(e) = \sigma = p_S(l)$, and $\pi_T(e) = \tau = p_T(l)$. Moreover, suppose $e' \in E$ satisfies $\pi_S(e') = \sigma$ and $\pi_T(e') = \tau$. Then $e' = f^*(c) \cup \{L\}$ (with L as last block), and $L \setminus \text{supp}\,T = \emptyset$, $L \setminus \text{supp}\,S = B_{k+1}$. Thus, $e' = e$. It follows that the map $f : P \to \text{Tree}\,(S \times T)$ extending f^* by $f(l) = e$ satisfies the universal property of products, and the theorem holds in Case 3.

Since the three cases above provide an exhaustive classification of possible actions of p_S and p_T, the theorem is proved. $\square$

To apply the preceding result to arbitrary trees, we need one more proposition. Let us write $\cong_o$ to denote isomorphism of posets.

Proposition 3.8. *There is an algorithm that, when input a tree $\hat{T}$, outputs a foliage T with* Tree $(T) \cong_o \hat{T}$.

Proof. Let $a_1, \ldots, a_k$ be the atoms of $\hat{T}$, and let $\hat{T}_i$ be the filter generated by a_i. Each $\hat{T}_i$ is obviously a tree. By induction on the height of $\hat{T}$, there exist foliages T_i, $i \in \{1, \ldots, k\}$, with Tree $T_i \cong_o \hat{T}_i$. Without loss of generality, we may assume $\text{supp}\,T_i \cap \text{supp}\,T_j = \emptyset$ whenever $i \neq j$. Let $\{x_i\}_{i=1}^k$ be any set such that $\{x_i\}_{i=1}^k \cap \bigcup_{i=1}^k \text{supp}\,T_i = \emptyset$. Set $\tilde{T}_i = \{\{\{x_i\}\} \cup \sigma \mid \sigma \in T_i\}$, where it is understood that each $\{x_i\}$ is the first element of the ordered partition $\{\{x_i\}\} \cup \sigma$, and the order on σ is unchanged. Note that each $\tilde{T}_i$ is a foliage such that Tree $\tilde{T}_i$ is isomorphic to $\hat{T}_i$ with a new root r_i added, and any such isomorphism sends r_i to the ordered partition $\emptyset$. Set

$$T = \bigcup_{i=1}^{k} \tilde{T}_i\ ,$$

Then T is a foliage: two ordered partitions $\{\{x_i\}\} \cup \sigma_1$ and $\{\{x_j\}\} \cup \sigma_2$, with $\sigma_1 \in T_i$, $\sigma_2 \in T_j$ and $i \neq j$, are necessarily incomparable, because $x_i \neq x_j$. It immediately follows that Tree $T \cong_o \hat{T}$. $\square$

In our construction of T above, supp T is far larger than it need be. For example, consider a tree $\hat{T}$ of height 1 with k atoms (thus, each atom is a leaf). The construction above yields $|\,\text{supp}\,T\,| = k$, whereas it is easy to see that there exists a foliage S with $|\,\text{supp}\,S| = 1 + \lfloor \log_2 k \rfloor$ such that Tree $S \cong_o \hat{T}$. The problem of minimizing $|\,\text{supp}\,T|$ has nontrivial logical meaning. Let $\Theta(X_1, \ldots, X_n)$ be a finite set of axioms in Gödel logic in the variables $X_1, \ldots, X_n$. Suppose the corresponding Tarski–Lindenbaum algebra G has a unique maximal filter, i.e. **Spec** (G) is a tree.[3] Let T be a foliage with $|\,\text{supp}\,T\,| = m$. Then one can effectively compute a finite set of axioms $\Lambda(X_1, \ldots, X_m)$ equivalent (in the sense of mutual interpretability) to $\Theta(X_1, \ldots, X_n)$. Thus, minimizing supp T amounts to minimize the number of variables of a set of axioms Λ into which one can (effectively) rewrite the given finite set of axioms Θ.

Proposition 3.8 and Theorem 3.7 provide an algorithm to compute products of trees, or coproducts of Gödel algebras with a unique maximal filter. The extension to forests and finite Gödel algebras is straightforward, cf. Section 2, *passim*.

Example 2. Consider two Gödel algebras L and M, with L a chain of 4 elements and M a chain of 3 elements. The trees **Spec** (M), **Spec** (N) dual to them are just chains of 3 and 2 elements, respectively. The foliages $\{\sigma\}$ and $\{\tau\}$, with σ and τ as in Example 1, are such that Tree $\{\sigma\}$, Tree $\{\tau\}$ are isomorphic (in T or F) to **Spec** (M), **Spec** (N), respectively. Again in Example 1, we computed $\{\sigma\} \times \{\tau\} = \{\{a|b|x\}, \{a|x|b\}, \{x|a|b\}, \{a|bx\}, \{ax|b\}\}$. Fig. 3 displays $T = \text{Tree}\,(\{\sigma\} \times \{\tau\})$. The Gödel algebra whose underlying lattice is the collection of subforests of T (ordered by inclusion) is thus the coproduct $L \coprod M$. It has 229 elements, as can be seen counting the maps of forests from T to $\mathbb{S}$. The reader will spare us the task of drawing the Hasse diagram of $L \coprod M$.

[3] We make this assumption merely for the sake of brevity. Extension of these considerations to general Gödel algebras is straightforward.

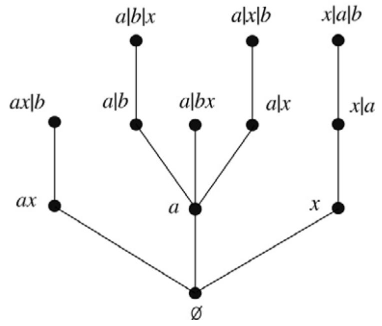

Fig. 3. The product Tree $(\{a|b\} \times \{x\})$ computed in Examples 1 and 2.

To compute fibred coproducts of finitely presented Gödel algebras, one may proceed as in the foregoing, using as an additional step the explicit description of equalisers in T and F provided by Proposition 2.3. Clearly, this procedure is hardly efficient, because it forces one to compute a whole product even when the given morphisms $S \to Q, T \to Q$ lead to a small equalising subobject $S \times_Q T \hookrightarrow S \times T$. It is an open problem whether it is possible to generalise our combinatorial treatment to yield a more direct algorithm to compute fibred products of trees.

4. Two applications

4.1. Free products with amalgamation

One says a category C has the *amalgamation property* iff for any two objects A and B in C, and for any common subobject $C \xrightarrow{m_1} A$, $C \xrightarrow{m_2} B$ of A and B, there exists a third object D with monomorphisms $A \xrightarrow{f} D$, $B \xrightarrow{g} D$ such that $f \circ m_1 = g \circ m_2$. The logical significance of amalgams is well-known. Specifically, if $\mathcal{L}$ is an intermediate propositional logic, and if $\mathsf{H}_{\mathcal{L}}$ is the corresponding variety of Heyting algebras, then $\mathcal{L}$ has Craig interpolation iff $\mathsf{H}_{\mathcal{L}}$ has the amalgamation property [9]. Subvarieties $\mathsf{H}_{\mathcal{L}}$ having the amalgamation property are classified in [9], where it is shown among other things that G enjoys amalgamation (i.e., Gödel logic has Craig interpolation). It is also proved in [9] that $\mathsf{H}_{\mathcal{L}}$ has the amalgamation property iff the full subcategory of finite algebras has the amalgamation property.

Now recall that a *free product of A and B with amalgamated subobject C* is just a fibred coproduct $A \coprod_C B$, that is, a push-out square

$$\begin{tikzcd} C \arrow[r, "m_1"] \arrow[d, "m_2"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"] & A \coprod_C B \end{tikzcd}$$

with *monic* push-out maps $f: A \to A \coprod_C B$ and $g: B \to A \coprod_C B$. (In certain contexts it may be necessary to require that m_1, m_2, f, g satisfy stronger conditions, as e.g. regularity; in our case such distinctions are immaterial.) Thus, if C is a variety, and therefore has fibred coproducts, a canonical form of the amalgamation property follows from the fact that monomorphisms are stable under push-outs.

Let us prove that finitely presented Gödel algebras have free products with amalgamation using the tools developed in Section 3. Dually, and restricting attention without further ado to trees, the question is whether a fibred product of trees

$$\begin{tikzcd} A \times_C B \arrow[r, "\pi_A^C"] \arrow[d, "\pi_B^C"'] & A \arrow[d, "s_A"] \\ B \arrow[r, "s_B"] & C \end{tikzcd}$$

is always such that π_A^C and π_B^C be epimorphisms whenever s_A, s_B are epimorphisms. Equivalently, we need to show that $\pi_A \circ eq$ and $\pi_B \circ eq$ are epic whenever s_A, s_B are epimorphisms, where π_A, π_B are the projections of the product $A \times B$, and $eq: A \times_C B \to A \times B$ is the equaliser yielding the fibred product $A \times_C B$.

Claim. *Given $a \in A$, $b \in B$, $c \in C$ such that $s_A(a) = s_B(b) = c$, there exists $z \in A \times_C B$ such that $\pi_A \circ eq(z) = a$, and $\pi_B \circ eq(z) = b$.*

Proof of Claim. We let $\rhd$ denote the covering relation. Let $a = a_0 \rhd \cdots \rhd a_r$, $b = b_0 \rhd \cdots \rhd b_s$, and $c = c_0 \rhd \cdots \rhd c_t$ be the unique chains from a, b, and c to the roots of A, B, and C, respectively. We induct on $r + s$. If $r + s = 0$, then $t = 0$, and the claim holds taking z to be the root of $A \times_C B$. For the induction step, suppose without loss of generality that $r > 0$. Let us identify A, B and $A \times B$ with posets of ordered partitions on the basis of Theorem 3.7 and Proposition 3.8.

First suppose $s_A(a_1) = c_0$. Then, by induction, there exists $z' \in A \times_C B$ with $\pi_A \circ eq(z') = a_1$, and $\pi_B \circ eq(z') = b_0$. Write $a_0 = a_1 \cup \{H\}$ for a trailing block H; then $z'' = eq(z') \cup \{H\} \in A \times B$, as in the proof of Theorem 3.7, and thus $\pi_A(z'') = a_0$, $\pi_B(z'') = b_0$, and $z'' \rhd eq(z')$. By Proposition 2.3, there is $z \in A \times_C B$ such that $eq(z) = z''$, and the claim is settled.

Suppose on the other hand that $s_A(a_1) = c_1$. Then, since s_B is open, there must be a smallest $u \in \{1, \ldots, s\}$ such that $s_B(b_u) = c_1$. If $u > 1$, we can apply induction to the triplet a_0, b_1, c_0, because $s_B(b_1) = c_0$, and prove the claim arguing as in the previous case (with the rôles of A and B interchanged). If, finally, $u = 1$, write $a_0 = a_1 \cup \{H\}$ and $b_0 = b_1 \cup \{K\}$, for trailing blocks H and K, respectively. By induction (applied to the triplet a_1, b_1, c_1), there exists $z' \in A \times_C B$ with $\pi_A \circ eq(z') = a_1$, and $\pi_B \circ eq(z') = b_1$. Then $z'' = eq(z') \cup \{H \cup K\} \in A \times B$, as in the proof of Theorem 3.7, and we have $\pi_A(z'') = a_0$, $\pi_B(z'') = b_0$, and $z'' \rhd eq(z')$; the claim is settled by Proposition 2.3. $\square$

We have proved:

Corollary 4.1. G_{fp} *has free products with amalgamation.* $\square$

4.2. Horn's result on the number of elements of $\mathcal{G}_n$

Recall that $\mathcal{S} = \{\mathcal{C}_0, \mathcal{C}_1\}$, and $\mathbf{Spec}\,(\mathcal{G}_n) = \mathcal{S}^n$. Thus,

$$\mathcal{S}^n = \{\mathcal{C}_1^0, \ldots, \underbrace{\mathcal{C}_1^k, \ldots, \mathcal{C}_1^k}_{\binom{n}{k} \text{ times}}, \ldots, \mathcal{C}_1^n\}\,,$$

where $\mathcal{C}_1^0 = \mathcal{C}_0$. By Proposition 2.4 and Remark 2, we know $|\mathcal{G}_n|$ is the number of morphisms in F from $\mathcal{S}^n$ to $\mathcal{S}$, written $|\mathsf{Mor}\,(\mathcal{S}^n, \mathcal{S})|$. Evidently, $|\mathsf{Mor}\,(\mathcal{C}_1^k, \mathcal{S})| = 1 + |\mathsf{Mor}\,(\mathcal{C}_1^k, \mathcal{C}_1)|$, and

$$|\mathsf{Mor}\,(\mathcal{S}^n, \mathcal{S})| = \prod_{k=0}^{n} (1 + |\mathsf{Mor}\,(\mathcal{C}_1^k, \mathcal{C}_1)|)^{\binom{n}{k}}\,.$$

Thus it remains to determine $|\mathsf{Mor}\,(\mathcal{C}_1^k, \mathcal{C}_1)|$.

Let $\mathcal{O}_k$ denote the set of all ordered partitions of the k-element set $\{x_1, \ldots, x_k\}$. It is clear that $\mathcal{O}_k$ is a foliage. Set $T_k = \mathrm{Tree}\,\mathcal{O}_k$.

Claim. $T_k \cong_o \mathcal{C}_1^k$ *for all $k \geq 0$.*

Proof of Claim. By induction on k. For $k = 0$, there is nothing to prove. Assume $k > 0$, and suppose the claim holds up to $k - 1$. Let $\{\chi\}$ be the foliage whose only ordered partition is $\chi = \{\{x_k\}\}$, whence $\mathrm{Tree}\,\{\chi\} \cong_o \mathcal{C}_1$. Since $\mathcal{C}_1^k = \mathcal{C}_1^{k-1} \times \mathcal{C}_1$, by Theorem 3.7 and the induction hypothesis we have $\mathrm{Tree}\,(\mathcal{O}_{k-1} \times \{\chi\}) \cong_o \mathcal{C}_1^k$. Thus it suffices to prove $\mathcal{O}_{k-1} \times \{\chi\} = \mathcal{O}_k$. Now, $\mathcal{O}_k \supseteq \mathcal{O}_{k-1} \times \{\chi\}$ holds by Definitions 3.2 and 3.3. To see that the converse inclusion also holds, let $\sigma \in \mathcal{O}_k$, and let $\tau = \sigma - \{x_k\}$ (cf. Notation 3). Suppose $\sigma = \{B_1, \ldots, B_h\}$, and assume $x_k \in B_i$. Then $\sigma' = \{B_1, \ldots, B_{i-1}, \{x_k\}, B_i \setminus \{x_k\}, B_{i+1}, \ldots, B_h\} \in \mathcal{O}_k$ is a merged shuffle of order 0 of $\tau \in \mathcal{O}_{k-1}$ and $\chi = \{\{x_k\}\}$; moreover, σ is a merged shuffle of order 1 of τ and χ, because σ is obtained from σ' by $B_i = \{x_k\} \cup (B_i \setminus \{x_k\})$. Hence, $\sigma \in \mathcal{O}_{k-1} \times \{\chi\}$, and the claim is settled. $\square$

It follows that $|\mathsf{Mor}\,(\mathcal{C}_1^k, \mathcal{C}_1)|$ equals the number of subobjects of $T_k = \mathrm{Tree}\,\mathcal{O}_k$. Now, the atoms of T_k are precisely the $2^k - 1$ nonempty ordered partitions σ into a single block with $\mathrm{supp}\,\sigma \subseteq \{x_1, \ldots, x_k\}$. Say $\sigma = \{B\}$ with $|B| = m$, for $1 \leq m \leq k$. Let $\langle\sigma\rangle$ denote the filter of T_k generated by σ. Then we have $\langle\sigma\rangle \cong_o T_{k-m}$, because the ordered partitions above σ in T_k are precisely those in $\mathcal{O}_k$ having B as a first block. Consider $f: T_k \to \mathcal{C}_1$. If $f(\sigma)$ is the top element of $\mathcal{C}_1$, f is the unique map sending every $\tau \in \langle\sigma\rangle$ to the top element of $\mathcal{C}_1$. If, on the other hand, $f(\sigma)$ is the bottom element of $\mathcal{C}_1$, then f uniquely determines a family of maps $f_m^j: T_{k-m} \to \mathcal{C}_1$, for $1 \le m \le k$ and $1 \le j \le \binom{k}{m}$; and, conversely, any such family of maps determines a unique $f: T_k \to \mathcal{C}_1$ having the property that $f(\sigma)$ is the bottom element of $\mathcal{C}_1$. Thus we obtain

$$|\mathsf{Mor}\,(\mathcal{C}_1^k, \mathcal{C}_1)| = \prod_{m=1}^{k} (1 + |\mathsf{Mor}\,(\mathcal{C}_1^{k-m}, \mathcal{C}_1)|)^{\binom{k}{m}},$$

and we have proved:

Corollary 4.2 (*Horn [5]*). *The cardinality of the free Gödel algebra $\mathcal{G}_n$ on $n \in \mathbb{N}$ generators satisfies*

$$|\mathcal{G}_n| = \prod_{k=0}^{n} (1 + |\mathsf{Mor}\,(\mathcal{C}_1^k, \mathcal{C}_1)|)^{\binom{n}{k}},$$

where

$$|\mathsf{Mor}\,(\mathcal{C}_1^k, \mathcal{C}_1)| = \prod_{m=1}^{k} (1 + |\mathsf{Mor}\,(\mathcal{C}_1^{k-m}, \mathcal{C}_1)|)^{\binom{k}{m}},$$

and

$$|\mathsf{Mor}\,(\mathcal{C}_1^0, \mathcal{C}_1)| = 1. \quad \square$$

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