Spectra of Gödel Algebras

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Abstract. We exploit the duality between finite Gödel algebras and their homomorphisms, and the category of finite forests and open maps, to compute the duals of non-isomorphic k-element Gödel algebras, for $k \geq 1$. From this construction we obtain a recurrence formula to compute the *fine spectrum* of the variety of Gödel algebras $\mathbb{G}$. Using such a formula we easily compute the set of cardinalities of finite Gödel algebras (the *spectrum* of $\mathbb{G}$), that is equal to the set of positive integers $\mathbb{N}^+$. To complete the picture on spectra of Gödel algebras, we recall the well-known recurrence formula to compute the cardinality of every free k-generated Gödel algebra, that is the *free spectrum* of $\mathbb{G}$.

1 Introduction

Essential knowledge about the finite members of a class of structures $\mathcal{C}$ can be obtained by computing, for every natural number $k \geq 1$:

- *spectrum* of $\mathcal{C}$: $\mathit{Spec}(\mathcal{C}) = \{k \mid k = |C|, C \in \mathcal{C}\}$, that is the set of cardinalities of structures occurring in $\mathcal{C}$;

when $\mathcal{C}$ is a variety of algebras we can also define:

- *fine spectrum* of $\mathcal{C}$: $\mathit{Fine}_{\mathcal{C}}(k)$, that is the function counting non-isomorphic k-element structures in $\mathcal{C}$,
- *free spectrum* of $\mathcal{C}$: $\mathit{Free}_{\mathcal{C}}(k) = |\mathbf{F}_{\mathcal{C}}(k)|$, that is the function computing the sizes of the free k-generated algebra $\mathbf{F}_{\mathcal{C}}(k)$ in $\mathcal{C}$.

Fagin characterizes the sets of integers that are $\mathit{Spec}(\mathcal{C})$ when $\mathcal{C}$ is the class of all models of first-order formulas of a first-order language [24]. The fine spectrum has been introduced by Taylor in [32] when considering $\mathcal{C}$ a variety of algebras, and Quackenbush [31] states that "the fine spectrum problem is usually hopeless" when dealing with ordered structures. Given such a function, the computation of the spectrum of $\mathcal{C}$ is straightforward,

$$\mathit{Spec}(\mathcal{C}) = \{k \in \mathbb{N}^+ \mid \ \mathit{Fine}_{\mathcal{C}}(k) > 0\}. \tag{1}$$

One of the most famous enumeration problems is the *Dedekind's Problem* [20], that is the free spectrum problem when $\mathcal{C}$ is the class of distributive lattices.

The variety of Gödel algebras is obtained by the class of Heyting algebras (that are bounded residuated distributive lattices) adding the prelinearity equation. Gödel algebras are the algebraic semantics of Gödel logic, a non-classical logic whose studies date back to Gödel [28] and Dummett [21]. Indeed Gödel logic can be obtained by adding the prelinearity axiom to Intuitionistic logic. Furthermore, Gödel logic is one of the three major (many-valued) logics in Hajek's framework of Basic Logic, that is the logic all continuous t-norms and their residua [25].

Given a finite Gödel algebra $\mathbf{A}$, the set of prime filters of $\mathbf{A}$ ordered by reverse inclusion forms a finite *forest*. Viceversa, given a finite forest F, the collection of all subforests of F, equipped with properly defined operations, is a finite Gödel algebra. This construction is functorial, meaning that it can be extended to obtain a dual equivalence between the category of Gödel algebras and their homomorphisms, and the category of finite forests and open maps (see [16,2] for details and proofs).

The above duality is a special case of the Birkhoff duality, between the category of finite distributive lattices and complete lattice homomorphisms, and the category of finite posets and open maps. Dropping finiteness restrictions, it is possible to obtain the Priestley duality between the category of bounded distributive lattices and bounded lattices homomorphisms, and the category of Priestley spaces and continuous order-preserving maps (see [18] for both Birkhoff and Priestley dualities). In [19] was observed that open maps dually correspond to Heyting algebras homomorphisms. Hence, the category of finite posets and open maps is dually equivalent to the category of complete and completely join-generated Heyting algebras with complete Heyting algebra homomorphisms [9,10]. Finally, Esakia in [22] establishes a duality between Heyting algebras and so-called Esakia spaces.

In this paper we exploit the category of forests to solve the fine and free spectra problems when $\mathcal{C}$ is the variety of Gödel algebras $\mathbb{G}$. The spectrum problem for $\mathbb{G}$ is easily solved. Indeed $Spec(\mathbb{G}) = \mathbb{N}^+$, because every finite chain carries the structure of a Gödel algebra.

In literature one can find many solutions to the free spectrum problem for $\mathbb{G}$. Indeed, already Horn in 1969 has obtained a recurrence formula to compute the cardinalities of free k-generated Gödel algebras [26]. Another solution to this problem can be achieved by restating the Horn's recurrence in terms of finite forests [16]. Conversely to the best of our knowledge, the fine spectrum problem for $\mathbb{G}$ has never been considered before. In this paper we show how to obtain a set of forests S_k such that for every Gödel algebra $\mathbf{A}$ of cardinality k there exists $F \in S_k$ such that $\mathbf{A}$ is isomorphic to the downsets of F. That is, given a cardinal k we can build the set of finite Gödel algebras with k elements. As a corollary, we obtain a recurrence $Fine_{\mathbb{G}}(k)$ that computes the cardinality of S_k for any $k \geq 1$. Solving in this way the fine spectrum problem for $\mathbb{G}$.

2 Finite Forests and Gödel Algebras

Let P and Q be two disjoint posets, their vertical sum $P \oplus Q$ is the poset over $P \cup Q$ obtained by taking the order relation defined in the following way: let x and y be two elements that belong to $P \cup Q$, then $x \leq y$ if the pair (x, y) fall in one of the following three mutually disjoint cases; $x \leq y$ if and only if $x, y \in P$ and $x \leq y$ in P, second $x, y \in Q$ and $x \leq y$ in Q and finally $x \in P$ and $y \in Q$. A special case of vertical sum is the *lifting*, $P_\perp := \{\perp\} \oplus P$, with $\perp \notin P$.

Given a poset $(P, \leq)$ and a subset $Q \subseteq P$, the *downset* of Q is $\downarrow Q = \{x \in P \mid x \leq q, \text{ for some } q \in Q\}$. By abuse of notation we write $\downarrow q$ for $\downarrow \{q\}$.

Let P and Q be two posets, an *open* map is an order-preserving map from P to Q that sends downsets of P to downsets of Q.

A poset F is a *forest* when for all $q \in F$ the downset $\downarrow q$ is a *chain*, that is $\downarrow q$ is totally ordered. A *tree* is a forest with a bottom element, called the *root* of the tree. A *subforest* of a forest F is the downset of some $Q \subseteq F$.

Finite forests and open maps form a category FF. Let F,F' and G be forests in FF. The coproduct of F and F' is the disjoint union $F \sqcup F'$ of F and F'. The product of $F \times F'$ is given by the following isomorphisms,

$$
\begin{aligned}
&F \times F' \cong F' \text{ when } |F| = 1;\\
&G \times (F \sqcup F') \cong (G \times F) \sqcup (G \times F');\\
&F_\perp \times F'_\perp \cong ((F_\perp \times F') \sqcup (F \times F') \sqcup (F \times F'_\perp))_\perp.
\end{aligned}
$$

These characterizations of products and coproducts in FF are fully proved in [5].

A Heyting algebra is a structure $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, \top, \perp\rangle$ such that $\langle A, \wedge, \vee, \top, \perp\rangle$ is a bounded distributive lattice, and $(\rightarrow, \wedge)$ forms a residuated couple, that is $x \wedge y \leq z$ if and only if $y \leq x \rightarrow z$. Negation operation is usually defined as $\neg x = x \rightarrow \perp$. A Gödel algebra is a Heyting algebra $\mathbf{A}$ that satisfies the prelinearity condition

$$(x \rightarrow y) \vee (y \rightarrow x) = \top.$$

The variety of Gödel algebras $\mathbb{G}$ is generated by the *standard* algebra $[\mathbf{0}, \mathbf{1}] = \langle [0, 1], \wedge, \vee, \rightarrow, 1, 0\rangle$ where the operations are defined as follows.

$$
x \wedge y = \min\{x, y\} \qquad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y,\\ y & \text{otherwise} \end{cases} \qquad \neg x = \begin{cases} 1 & \text{if } x = 0,\\ 0 & \text{otherwise.} \end{cases}
$$

Since the algebra $[\mathbf{0}, \mathbf{1}]$ singly generates the whole variety $\mathbb{G}$, from universal algebraic facts we have that the free k-generated algebra $\mathbf{F}_{\mathbb{G}}(k)$ is isomorphic with the subalgebra of the algebra of all functions $f \colon [0, 1]^k \rightarrow [0, 1]$ generated by the projection functions $x_i \colon (t_1, \ldots, t_k) \mapsto t_i$, for all $i \in \{1, 2, \ldots, k\}$. Functional and combinatorial representations of $\mathbf{F}_{\mathbb{G}}(k)$ can be found in [6].

Let $\mathbf{A}$ be a Gödel algebra, then a non-empty subset $\mathfrak{p}$ of A is a *filter* of $\mathbf{A}$ if for all $y \in A$, if there is x in $\mathfrak{p}$ such that $x \leq y$ then $y \in \mathfrak{p}$, and $x \wedge y \in \mathfrak{p}$ for all $x, y \in \mathfrak{p}$. We call *proper* the filters $\mathfrak{p}$ such that $\mathfrak{p} \neq A$.

A filter $\mathfrak{p}$ of $\mathbf{A}$ is *prime* if it is proper and for all $x, y \in A$, either $x \to y \in \mathfrak{p}$ or $y \to x \in \mathfrak{p}$. The set of all prime filters $\mathsf{Prime}(\mathbf{A})$ [1] of $\mathbf{A}$ ordered by reverse inclusion is called the *prime spectrum* of $\mathbf{A}$.

When $\mathbf{A}$ is finite, each prime filter $\mathfrak{p}$ of $\mathbf{A}$ is generated by a join-irreducible element a as $\mathfrak{p} = \{b \in \mathbf{A} \mid a \leq b\}$. On the other hand, each join-irreducible element of $\mathbf{A}$ singly generates a prime filter of $\mathbf{A}$. Hence, $\mathsf{Prime}(\mathbf{A})$ is isomorphic with the poset of the join-irreducible elements of $\mathbf{A}$. See Figure 1 as an example where $\mathbf{A} = \mathbf{F}_{\mathbb{G}}(1)$.

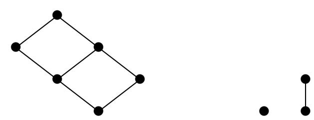

Fig. 1. The free Gödel Algebra on one generator $\mathbf{F}_{\mathbb{G}}(1)$ and its prime spectrum.

In [27], Horn established that the prime spectrum of a finite Gödel algebra forms a forest. Conversely, given $F \in \mathsf{FF}$ we can equip the finite set of subforests $\mathsf{Sub}(F)$ with the structure of a Gödel algebra $\langle \mathbf{Sub}(F), \cap, \cup, \to, \emptyset, F\rangle$, where $F' \to F'' = F \setminus \uparrow (F' \setminus F'')$, for all $F', F'' \in \mathbf{Sub}(F)$. Hence, we obtain the following isomorphisms

$$\mathsf{Prime}(\mathsf{Sub}(F)) \cong F \quad \text{and} \quad \mathsf{Sub}(\mathsf{Prime}(\mathbf{A})) \cong \mathbf{A}.$$

The above equivalence can be extended to a full duality between FF and the category of finite Gödel algebras and their homomorphisms, by making Sub and Prime functors acting also on open maps and Gödel algebra homomorphisms, respectively. These constructions go beyond the scope of the present paper, we refer the interested reader to [16,2,5].

3 Fine Spectrum

To compute the fine spectrum of $\mathbb{G}$ we need to recall some definition on factorizations of natural numbers and to collect some results on subforests.

Let k be a natural number greater than 1. By the fundamental theorem of arithmetic [17], there exists a unique expansion of k into a product of prime numbers $k = p_1 \times \ldots \times p_m$, such that $p_1 \leq \cdots \leq p_m$.

For our purpose we are interested in a slightly different problem. Given a composite natural number k, we want to find all the possible factorizations $n_1 \times \cdots \times n_t$ with $t > 1$, such that $n_1 \times \cdots \times n_t = k$ and $n_1 \leq \cdots \leq n_t$. That is, we need the following set of t-uples,

$$\mathsf{fact}(k) := \{(n_1, \ldots, n_t) \mid k = n_1 \times \cdots \times n_t, n_1 \leq \cdots \leq n_t, t > 1\}. \tag{2}$$

[1] In [16,2,5] the set of prime filters of a Gödel algebra $\mathbf{A}$ is usually denoted as $\mathsf{Spec}(\mathbf{A})$, here we adopt $\mathsf{Prime}(\mathbf{A})$ to avoid confusion with $\mathit{Spec}(\mathbb{G})$.

The set $\mathsf{fact}(k) \cup (k)$ is known as the set of *unordered factorizations* (or *multiplicative partitions*) of k. Indeed, since the order of the factors does not matter, elements of $\mathsf{fact}(k)$ are exactly t-uples composed of natural numbers whose product is equal to k. For details and further information on multiplicative partitions see [29]. Finally, we define a function such that

$$pr(k) = \begin{cases} 0 & \text{if } k \text{ is prime;} \\ 1 & \text{otherwise.} \end{cases}$$

From now on, bold faced integers $\mathbf{k}$ will be used to denote the k-element totally ordered tree.

Let F be a finite forest.

Lemma 1. $|\mathsf{Sub}(F)| = k$ *if and only if* $|\mathsf{Sub}(F_\perp)| = k + 1$.

Proof. By definition of lifting, we notice that $F_\perp = F \cup \{\perp\}$, and that $\perp$ has to belong to every subforest of $F_\perp$. Hence, for every subforest F' of F there exists a subforest $F' \cup \{\perp\}$ of $F_\perp$. Now, it is easy to check that the map $F' \mapsto F' \cup \{\perp\}$ is a bijection from $\mathsf{Sub}(F)$ to $\mathsf{Sub}(F_\perp) \setminus \emptyset$. Hence, $|\mathsf{Sub}(F_\perp)| = |\mathsf{Sub}(F)| + |\mathsf{Sub}(\emptyset)|$, that is $|\mathsf{Sub}(F_\perp)| = k + 1$. □

Let F be a forest such that $F = F' \sqcup F''$. Since $\sqcup$ is a disjoint union, then for every forest E' in $\mathsf{Sub}(F')$ there exists a forest E in $\mathsf{Sub}(F' \sqcup F'')$ such that $E = E' \sqcup E''$, for each $E'' \in \mathsf{Sub}(F')$. Then,

$$\mid \mathsf{Sub}(F') \mid = n \text{ and } \mid \mathsf{Sub}(F'') \mid = m \text{ if and only if } \mid \mathsf{Sub}(F) \mid = n \times m. \tag{3}$$

By iterating (3) we derive

Lemma 2. *Let F be a forest such that $F = F_1 \sqcup \cdots \sqcup F_t$, for $t > 1$. Then,* $\mid \mathsf{Sub}(F_i) \mid = n_i$ *with* $1 \leq i \leq t$*, if and only if* $\mid \mathsf{Sub}(F) \mid = n_1 \times \cdots \times n_t$.

Let T be a tree, and denote by r the root of T. Then, the bottom of $\mathsf{Sub}(T)$ is $\emptyset$ and its cover is $\{r\}$. In words, $\mathsf{Sub}(T)$ has a unique atom and it is the root of T. By (3) and the above discussion we conclude,

Proposition 1. *If* $|\mathsf{Sub}(F)|$ *is prime then* F *is a tree.*

The converse is not true. For instance, take the tree $\mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{1} \sqcup \mathbf{1})$. It has 6 subforests, and 6 is not a prime number.

Let k be a positive integer. We denote by $A(k)$ the set of non-isomorphic Gödel algebras with k elements. That is $A(k) = \{[\mathbf{A}] \in \mathbb{G} \mid k = |A|\}$, where $[\mathbf{A}]$ is the class of finite Gödel algebras isomorphic with $\mathbf{A}$.

Define the set of of forests corresponding to non-isomorphic k-elements Gödel algebras as:

$$S_k = \{\mathsf{Prime}(\mathbf{A}) \mid [\mathbf{A}] \in A(k)\},$$

then the fine spectrum of $\mathbb{G}$ is

$$Fine_{\mathbb{G}}(k) = |A(k)| = |S_k|. \tag{4}$$

Now define the following set of forests,

$$H_1 = \{\emptyset\} \tag{H_1}$$

$$H_k = P_k \cup Z_k \tag{H_k}$$

$$P_k = \{F_\perp | F \in H_{k-1}\} \tag{P_k}$$

$$Z_k = \{F_1 \sqcup \cdots \sqcup F_t \mid F_1 \in P_{n_1}, \ldots, F_t \in P_{n_t}, (n_1, \ldots, n_t) \in \mathsf{fact}(k)\} \tag{Z_k}$$

Theorem 1. $S_k = H_k$.

Proof. We proceed by strong induction over k. To settle the base case $k = 1$ notice that $S_1 = \emptyset$ because $A(1)$ contains only the trivial Gödel algebra, and hence $H_1 = \{\emptyset\} = S_1$. Notice also that $A(2)$ contains only the two element Gödel chain, whose spectrum is the tree $\mathbf{1}$, then $H_2 = S_2$ by definition of H_2.

For the inductive step, we assume that $H_n = S_n$ for every $1 \leq n < k$ and we prove that $H_k = S_k$.

Take a forest $F \in S_k$, either F is a tree, or F can be decomposed in a disjoint union of a family of subforests. Suppose F is a tree $F = (F')_\perp$. By the definition of S_k we have $|\mathsf{Sub}(F)| = k$, and hence by Lemma 1 $|\mathsf{Sub}(F')| = k - 1$, that is $F' \in S_{k-1}$. By induction $F' \in H_{k-1}$, and hence we conclude $F \in H_k$ by (P_k). Now suppose F can be decomposed in $F_1 \sqcup \cdots \sqcup F_n$. Then, $k = n_1 \times \cdots \times n_t$ for $(n_1, \ldots, n_t) \in \mathsf{fact}(k)$, and by Lemma 2 $|\mathsf{Sub}(F_j)| = n_j$, that is $F_j \in S_j$, for $1 \leq j \leq t$. By inductive hypothesis $F_j \in H_j$, for $1 \leq j \leq t$, hence $F \in H_k$ by (Z_k).

We have shown $S_k \subseteq H_K$. Now we prove the opposite relation.

Take a forest $F \in H_k$. By definition, either $F \in P_k$ or $F \in Z_k$. Suppose $F \in P_k$, then by definition $F = (F')_\perp$ with $F' \in H_{k-1}$. By induction $F' \in H_{k-1} = S_{k-1}$, and hence by Lemma 1 $|\mathsf{Sub}(F')| = k-1$ if and only if $|\mathsf{Sub}(F)| = k$, that is $F \in S_k$. Now suppose that $F \in Z_k$. Then, by definition $F = F_1 \sqcup \cdots \sqcup F_t$ such that $F_j \in P_j \subseteq H_j$, for $1 \leq j \leq t$. By induction $F_j \in S_j$ for every $1 \leq j \leq t$, and by Lemma 2 we have $|\mathsf{Sub}(F)| = \prod_{j=1}^{t} |\mathsf{Sub}(F_j)|$ that is $F \in S_k$.

This concludes the proof. □

In the Appendix the sets S_k are depicted for $1 \leq k \leq 12$.

Notice that during the recursive building of H_k no forest is created more than once. This fact can be proven by noticing first that to generates two times a forest F in Z_k we need two factorizations $(n_1, \ldots, n_t) \neq (n_{1'}, \ldots, n_{t'})$ in $\mathsf{fact}(k)$, such that $F = F_1 \sqcup \cdots \sqcup F_t$ and $F = F_{1'} \sqcup \cdots \sqcup F_{t'}$. By (Z_k), every F_i for $1 \leq i \leq t$ is a tree belonging to P_{n_i}, and every $F_{i'}$ for $1' \leq i' \leq t'$ is a tree belonging to $P_{n_{i'}}$, we conclude that $t = t'$ and each tree F_i is equal to its corresponding $F_{i'}$. This means that there are at least two $n_i \neq n_{i'}$ such that $P_{n_i} = P_{n_{i'}}$, but this is in contradiction with (P_k). Indeed assuming without loss of generality that $n_i < n_{i'}$, then for each tree T in P_{n_i} there exists a tree $T' \in P_{n_{i'}}$ obtained by lifting T for $n_{i'} - n_i$ times. Therefore, $P_{n_i} \neq P_{n_{i'}}$.

From (4), and the above considerations on the construction of $H_k = S_k$, directly follows a formula to compute the cardinality of H_k, that is the fine spectrum of $\mathbb{G}$.

Corollary 1. $\mathit{Fine}_{\mathbb{G}}(k) = f(k) + pr(k) \times g(k)$ *with,*

$$f(2) = 1 \tag{f_1}$$

$$f(k) = \mathit{Fine}_{\mathbb{G}}(k-1) \tag{f_k}$$

$$g(k) = \sum_{(n_1,\ldots,n_t)\in\mathsf{fact}(k)} f(n_1) \times \cdots \times f(n_t) \tag{g_k}$$

Table 1 shows the number of non-isomorphic finite Gödel algebras for cardinalities $1 \le k \le 150$ computed using Corollary 1.

k	$\mathit{Fine}_{\mathbb{G}}(k)$	k	$\mathit{Fine}_{\mathbb{G}}(k)$	k	$\mathit{Fine}_{\mathbb{G}}(k)$	k	$\mathit{Fine}_{\mathbb{G}}(k)$	k	$\mathit{Fine}_{\mathbb{G}}(k)$
1	1	31	136	61	1484	91	7390	121	25519
2	1	32	162	62	1620	92	7987	122	27003
3	1	33	170	63	1679	93	8123	123	27347
4	2	34	193	64	1868	94	8668	124	29103
5	2	35	199	65	1892	95	8730	125	29249
6	3	36	248	66	2122	96	9627	126	31501
7	3	37	248	67	2122	97	9627	127	31501
8	5	38	279	68	2338	98	10318	128	33559
9	6	39	291	69	2390	99	10528	129	33965
10	8	40	344	70	2631	100	11439	130	36075
11	8	41	344	71	2631	101	11439	131	36075
12	12	42	406	72	2990	102	12418	132	38925
13	12	43	406	73	2990	103	12418	133	39018
14	15	44	466	74	3238	104	13387	134	41140
15	17	45	493	75	3341	105	13713	135	41878
16	23	46	545	76	3651	106	14573	136	44455
17	23	47	545	77	3675	107	14573	137	44455
18	31	48	646	78	4063	108	15947	138	47442
19	31	49	655	79	4063	109	15947	139	47442
20	41	50	740	80	4492	110	17085	140	50619
21	44	51	763	81	4608	111	17333	141	51164
22	52	52	860	82	4952	112	18646	142	53795
23	52	53	860	83	4952	113	18646	143	53891
24	69	54	986	84	5541	114	20119	144	57988
25	73	55	1002	85	5587	115	20223	145	58206
26	85	56	1132	86	5993	116	21604	146	61196
27	91	57	1163	87	6102	117	21955	147	61974
28	109	58	1272	88	6636	118	23227	148	65460
29	109	59	1272	89	6636	119	23296	149	65460
30	136	60	1484	90	7354	120	25455	150	69922

Table 1. The number of non-isomorphic k-element Gödel algebras with $1 \le k \le 150$.

By applying (1), we can use the above recurrence also to compute the spectrum of $\mathbb{G}$,

$$\mathrm{Spec}(\mathbb{G}) = \{k \in \mathbb{N}^+ \mid \mathrm{Fine}_{\mathbb{G}}(k) > 0\}.$$

Proposition 2. $Spec(\mathbb{G}) = \mathbb{N}^+$.

Proof. The result follows from the fact that every finite chain can be equipped with the structure of a Gödel algebra.

3.1 Sums of Oterms

The sequence of integer numbers reported in Table 1 and produced by $Fine_{\mathbb{G}}$, appears to be sequence A130841 of the *On-Line Encyclopedia of Integer Sequences*. Such a sequence gives the number of ways to express an integer as a sum of so-called *oterms* [2]. In this section, we see how a sum of oterms could be used to syntactically describe a finite forest.

Let w be a word over the language $\{(,),1,+,\times\}$. We denote $\|w\|$, the number of occurrences of 1 appearing in w. An *oterm* is a word over the language $\{(,),1,+,\times\}$, inductively defined by the following rules:

(O1) 1 is an oterm;
(O2) $(1+o_1)\times_1\cdots\times_{i-1}(1+o_i)$ is an oterm, when $o_1,\ldots,o_i$ are oterms and $i \geq 1$.

A *sum of oterms* is a string $o_1+o_2+\cdots+o_i$, where $o_1,o_2,\ldots,o_i$ are oterms such that $i \geq 1$ and $\|o_1\| \leq \cdots \leq \|o_i\|$. We call SO the class of all sum of oterms.

Example 1. Let $o_1 = (1+1)$ and $o_2 = (1+1+1)$ be two words over the language $\{(,),1,+,\times\}$. The words o_1, o_2, and $o_1 \times o_2$ are oterms, $1+(o_1 \times o_2)$ is a sum of oterms, while $(o_1 \times o_2)+1 \notin \mathsf{SO}$ because $\|(o_1 \times o_2)\| \not\leq \|1\|$.

Now we show that sum of oterms are just another way to express finite forests (and viceversa), by defining a bijection between these two classes of objects. Without loss of generality we establish that the disjoint union decomposition of a forest $F \in \mathsf{FF}$ into a finite set of trees $T_1 \sqcup T_2 \sqcup \cdots \sqcup T_m$, is such that $|T_1| < |T_2| < \cdots < |T_m|$ and $m > 1$.

Let us define the functions $h : \mathsf{FF} \to \mathsf{SO}$ and $l : \mathsf{SO} \to \mathsf{FF}$ by the following prescriptions,

$$\begin{aligned}
&h(\emptyset) \mapsto 1 && l(1) \mapsto \emptyset\\
&h(\mathbf{1}) \mapsto (1+1) && l(1+1) \mapsto \mathbf{1}\\
&h(\mathbf{1} \oplus F) \mapsto (1+h(F')) && l(1+o) \mapsto (\mathbf{1} \oplus l(o))\\
&h(F \sqcup F') \mapsto h(F) \times h(F') && l(o \times o') \mapsto l(o) \sqcup l(o')
\end{aligned}$$

for $F, F' \in \mathsf{FF}$ non-empty forests, and $o, o' \in \mathsf{SO}$ with $o \neq 1 \neq o'$.

Notice that, a decomposition of a forest into a disjoint union of forests is unique up to associativity, and $h(F \sqcup (F' \sqcup F'')) = h((F \sqcup F') \sqcup F'')$. Analogously, $l(o_1 \times (o_2 \times o_3)) = l((o_1 \times o_2) \times o_3)$.

It is easy to see that l is the inverse function of h, and viceversa. Hence, we have established a bijection between FF and SO.

[2] The author has not been able to find references to oterms in the scientific literature. The only occurrence of such expressions is in the On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org [14].

Example 2. Let F be the spectrum of the free Gödel Algebra on one generator $\mathbf{F}_{\mathbb{G}}(1)$ depicted in Figure 1. Then, $h(F) = (1+1) \times (1 + (1+1))$. Viceversa, the oterm $o = 1 + ((1+1) \times (1+1))$ corresponds to the tree $l(o) = \mathbf{1} \oplus (\mathbf{1} \sqcup \mathbf{1})$.

Given integers $k, i \geq 1$, we let $s_k = \{s \in \mathsf{SO} \mid s = o_1 + o_2 + \cdots + o_i \text{ and } o_1 + o_2 + \cdots + o_i = k\}$. In words, s_k is the set of all possible ways to express k as a sum of oterms (when $i = 1$ we have unary sums). We define $a(k)$ as the cardinality of s_k, that is number of ways to write k as a sum of oterms

Example 3. Let $k = 12$. Then, $a(k) = 12$ because there are 12 ways to write k as a sum of oterms. That is,

$$
\begin{array}{ll}
(1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}1), & (1+1+((1+1)\times(1+1+1+1+1))),\\
(1{+}1{+}1{+}1{+}1{+}1{+}1{+}1{+}((1{+}1){\times}(1{+}1))), & (1{+}1{+}((1{+}1)\times(1{+}(1{+}1)\times(1{+}1))),\\
(1{+}1{+}1{+}1{+}1{+}1{+}((1{+}1){\times}(1{+}1{+}1))), & (1+1)\times(1+1+1+1+1),\\
(1{+}1{+}1{+}1{+}1{+}((1{+}1)\times(1{+}1{+}1{+}1))), & (1+1)\times(1+1+((1+1)\times(1+1))),\\
(1{+}1{+}1{+}1{+}((1{+}1){\times}(1{+}1){\times}(1{+}1))), & (1+1+1)\times(1+1+1+1),\\
(1+1+1+((1+1+1)\times(1+1+1))), & (1+1+1)\times(1+1)\times(1+1).
\end{array}
$$

Let S be a set of finite forests. We define the set of sum of oterms generated by the forests in S through the function h as,

$$H(S) = \{h(F) \mid F \in S\}.$$

Lemma 3. $H(S_k) = s_k$.

Proof. We proceed by induction over k. To settle the base case $k = 1$ it is sufficient to notice that $s_1 = 1$ and $S_1 = \{\emptyset\}$.

For the inductive step, we assume that $H(S_n) = s_n$ for every $1 \leq n < k$ and we prove that $H(S_k) = s_k$.

Let F be a forest in S_k. We have two cases. The first $F = \mathbf{1} \oplus F'$ and $F' \in S_{k-1}$. Hence, by induction $h(F') = s \in s_{k-1}$ and by definition of h we have $h(F) = (1 + h(F')) = (1 + s)$. That is, an oterm in s_k.

Second case $F = F' \sqcup F''$ with $F' \in S_n$ and $F'' \in S_m$, for $n \times m = k$. By inductive hypothesis $h(F') = s' \in s_n$ and $h(F'') = s'' \in s_m$. Then, $h(F) = h(F') \sqcup h(F'') = s' \times s''$. Since $n \times m = k$ then $s' \times s'' \in s_k$.

We have shown that $H(S_k) \subseteq s_k$. To prove the opposite relation $s_k \subseteq H(S_k)$, it is now sufficient to notice that, since h is a bijective function, then for every $s \in s_k$ there exists a unique $F \in S_k$ such that $h^{-1}(s) = F$ and $s = h(F) \in H(S_k)$. □

Hence, we conclude that the number of non-isomorphic k-element Gödel algebras is equal to the number of way to express k as a sum of oterms.

Proposition 3. $Fine_{\mathbb{G}}(k) = |S_k| = |s_k| = a(k)$.

Proof. The first equality is given by (4), the second follows from Lemma 3, and the third is the definition of $a(k)$. □

4 Free Spectrum

In this section we show a recurrence formula to compute $\mathit{Free}_{\mathbb{G}}(k)$. Such formula has been obtained for the first time by Horn in [26], and it has been restated in dual terms by D'Antona and Marra in [16]. Here we recall the combinatorial approach given in [2].

In any variety $\mathbb{V}$, the free k-generated algebra is the coproduct of k copies of the free 1-generated algebra $\mathbf{F}_{\mathbb{V}}(1)$. Hence, we can dually describe the prime spectrum of $\mathbf{F}_{\mathbb{G}}(k)$ with

$$\mathsf{Prime}(\mathbf{F}_{\mathbb{G}}(k)) = \prod^{n} \mathsf{Prime}(\mathbf{F}_{\mathbb{G}}(1)),$$

by knowing that the prime spectrum of $\mathbf{F}_{\mathbb{G}}(1)$ is $\mathbf{1} \sqcup \mathbf{1}_{\perp}$, see Figure 1.

Thanks to the nice characterization of products and coproducts in FF recalled in Section 2, in the case of Gödel algebras we can do something better. Given $F \in \mathsf{FF}$, we write mF to denote $F \sqcup F \sqcup \cdots \sqcup F$ m-times, that is the disjoint union of m copies of F. Then, defining the following sets of forests,

$$D_0 = \emptyset \qquad D_n = \bigsqcup_{i=0}^{n-1} \binom{n}{i} (D_i)_{\perp},$$

it has been proved that,

Theorem 2 ([2]). Prime$(\mathbf{F}_{\mathbb{G}}(k)) = D_k + (D_k)_{\perp}$

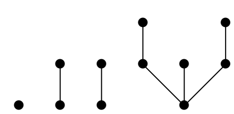

Fig. 2. The prime spectrum $\mathsf{Prime}(\mathbf{F}_{\mathbb{G}}(2))$ of the free Gödel Algebra on two generators, obtained as $D_2 + (D_2)_{\perp}$ where $D_2 = \bigsqcup_{i=0}^{1} \binom{n}{i} (D_i)_{\perp} = \binom{2}{0}(D_0)_{\perp} \sqcup \binom{2}{1}(D_1)_{\perp}$, that is $D_2 = \binom{2}{0}\mathbf{1} \sqcup \binom{2}{1}\mathbf{2} = \mathbf{1} \sqcup \mathbf{2} \sqcup \mathbf{2}$ and $\mathbf{1} = D_1 = (D_0)_{\perp}$.

From the above theorem directly follows a formula to compute $\mathit{Free}_{\mathbb{G}}(k)$,

Corollary 2 ([2]). *$\mathit{Free}_{\mathbb{G}}(k) = (d(k))^2 + d(k)$ with,*

$$d(0) = 1 \qquad d(n) = \prod_{i=0}^{n-1} (d(i) + 1)^{\binom{n}{i}}.$$

k	1	2	3	4
$Free_{\mathbb{G}}(k)$	6	342	137186159382	2.05740183252e+64

Table 2. The cardinalities of free k-generated Gödel algebras with $1 \leq k \leq 4$. The exact values for $k \geq 4$ can be computed with any software for unbounded length arithmetic.

5 Conclusions

In [7], authors used a brute-force algorithm to compute the number of non-isomorphic k-element residuated structures, including Gödel algebras, for $1 \leq k \leq 12$. In this paper we have tackled spectra problems by looking at duals of finite Gödel algebras, obtaining a recurrence formula to compute the number of non-isomorphic k-element Gödel algebras for any $k \geq 1$.

From this investigation, it appears that spectra problems are easier to handle when restated in dual terms. This seems to be confirmed by the preliminary results on the fine spectrum problem for the class of involutive bisemilattices (the algebraic counterpart of the Weak Kleene Logic) obtained in [12]. Hence, this duality-based approach is worth to be generalized to other varieties related to non-classical logics. A good starting point would be to investigate logics inside Monoidal T-norm Logic hierarchy [23]. Indeed, dual representations for corresponding varieties are already available, and have been used to compute free spectrum for Nilpotent Minimum logic [15,4] and Revised Drastic Product logic (RDP) [13]. The duality of [13] has been used in [33] to obtain subforests representations for varieties corresponding to RDP, DP and EMTL logic [34,1,11]. These logics, together with Gödel logic, are schematic extensions of the Weak Nilpotent Minimum logic [30], whose free spectrum problem has been solved in [3] through subdirect representation. We remark that the varieties corresponding to all the above listed logics, are locally finite.

Free and fine spectra are closely related. Indeed, every k-generated algebra in $\mathbb{V}$ is a quotient of $\mathbf{F}_{\mathbb{V}}(k)$. However, different congruences may generates isomorphic quotients. Hence, studies on spectra can be also used to investigate congruence in varieties. Another interesting topic stemming out from research on spectra is the so-called *generative complexity* [8], that is the function that counts the number of k-generated algebras.

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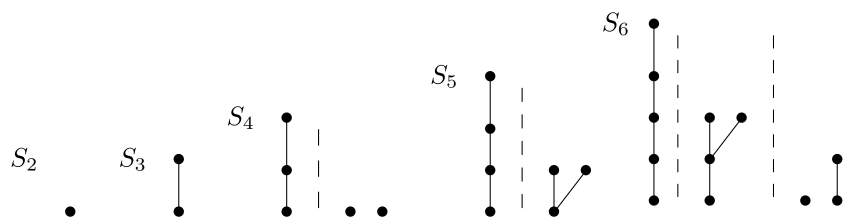
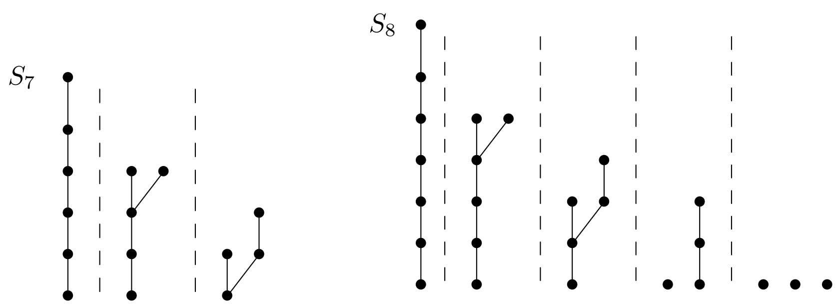
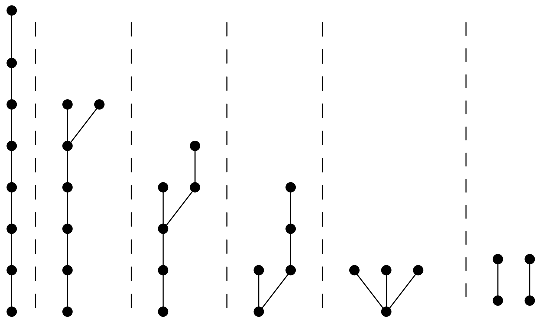
Appendix

We list here the sets S_k for $2 \leq k \leq 12$. Forests in each S_k are separated by dashed lines.

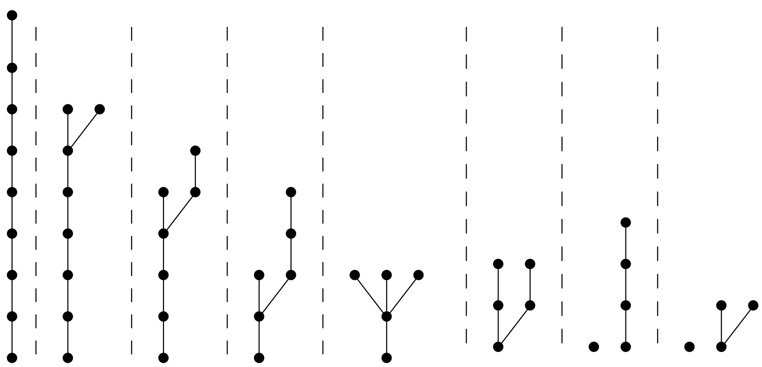




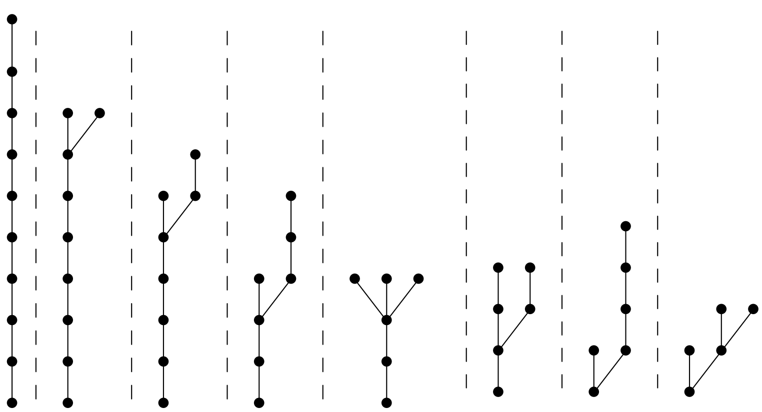
S_9



S_{10}



S_{11}



S_{12}