Volume 2

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Chapter VI: Łukasiewicz Logic and MV-Algebras

ANTONIO DI NOLA AND IOANA LEUŞTEAN

1 Introduction

MV-algebras were defined by Chang [12] as algebraic structures corresponding to Łukasiewicz ∞-valued propositional logic Ł [50]. This logical system has the real unit interval $[0,1]$ as set of truth values and its basic logical connectives, $*$ (negation) and $\rightarrow$ (implication), can be expressed using the real algebraic operations:

$$a^* = 1 - a \quad \text{and} \quad a \rightarrow b = \min\{1, 1 - a + b\}.$$

The usual lattice operations on $[0,1]$ ($a \vee b = \max\{a,b\}$, $a \wedge b = \min\{a,b\}$) are defined by $a \vee b = (a \rightarrow b) \rightarrow b$ and $a \wedge b = (a^* \vee b^*)^*$. The negation $*$ is an involution, but the excluded middle principle is not satisfied. If one defines on $[0,1]$ the operations $\oplus$ and $\odot$ by

$$a \oplus b = \min\{a+b, 1\} \quad \text{and} \quad a \odot b = (a^* \oplus b^*)^* = \max\{a+b-1, 0\},$$

then $a \oplus a^* = 1$ and $a \odot a^* = 0$. Note that $\oplus$ is in fact the real numbers addition truncated to $[0,1]$ and $\odot$ is its dual with respect to involutive negation. Moreover, the lattice operations can be also expressed by:

$$a \vee b = a \oplus (b \odot a^*) = b \oplus (a \odot b^*) \quad \text{and} \quad a \wedge b = a \odot (b \oplus a^*) = b \odot (a \oplus b^*).$$

When he defined the structure of MV-*algebra*, Chang's main goal was to provide an algebraic proof for the completeness theorem of Łukasiewicz propositional calculus Ł (i.e., any formula of Ł is provable if and only if it holds in $[0,1]$) [13]. He chose $\oplus$, $\odot$ and * as primary operations. Many of his initial axioms are inspired by the properties that are satisfied on $[0,1]$ [12]. A main tool in Chang's proof of the completeness theorem was the bijective correspondence between the linearly ordered MV-algebras and the linearly ordered Abelian lattice ordered groups with strong unit.

The deep connection between MV-algebras and Abelian lattice ordered groups with strong unit was completely clarified by Mundici in [60]: the two categories are equivalent. Many important aspects in the theory of MV-algebras have their origin in this correspondence. The historical remarks at the end of this chapter aim to give an account on the various developments in MV-algebras, as well as suggestions for further reading. A standard reference for the elementary theory of MV-algebras is [18], while [71] approaches advanced topics.

This chapter presents some fundamental aspects of MV-algebra theory and it is intended to be self-contained. Some results are particular cases of results proved in other chapters. Some definitions and constructions are just specializations of general notions in logic and universal algebra. We included them in our presentation for the sake of completeness. Łukasiewicz logic, developed in Section 6 is a core fuzzy logic, so main theorems are instances of more general results from Chapters I and II. Note that Chapter III contains a proof theoretic approach to Łukasiewicz logic, while its computational complexity is analyzed in Chapter X. Since MV-algebras are BL-algebras with involutive negation, aspects of MV-algebra theory directly follow from Chapter V. The second section of Chapter IX presents two important issues concerning MV-algebras: McNaughton theorem and functional characterization of the free structures.

2 Definitions and basic properties

We give below a simplified definition of MV-algebras, which is due to Mangani [52]. We prove that the class of MV-algebras is polynomially equivalent with the class of *Wajsberg algebras*, structures that are obtained through a direct algebraization of Ł [33]. We provide few examples of MV-algebras, some of which will be frequently referred. We prove that an interval of an Abelian lattice ordered group can be endowed with an MV-algebra structure. The fundamental connection between MV-algebras and Abelian lattice ordered groups will be deeply analyzed in Section 5.

2.1 MV-algebras

DEFINITION 2.1.1. *An* MV-algebra *is a structure* $\langle A, \oplus, {}^*, 0\rangle$, *where* $\oplus$ *is a binary operation,* * *is a unary operation and* 0 *is a constant such that the following axioms are satisfied for any* a, $b \in A$*:*

$(\mathrm{MV1})$ $\langle A, \oplus, 0\rangle$ *is an Abelian monoid,*

$(\mathrm{MV2})$ $(a^*)^* = a$,

$(\mathrm{MV3})$ $0^* \oplus a = 0^*$,

$(\mathrm{MV4})$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

In order to simplify the notation, an MV-algebra $\langle A, \oplus, {}^*, 0\rangle$ will be referred by its support set, A. An MV-algebra is *trivial* if its support is a singleton. On an MV-algebra A we define the constant 1 and the auxiliary operation $\odot$ as follows:

$$1 := 0^* \qquad\qquad a \odot b := (a^* \oplus b^*)^*$$

for any a, $b \in A$. We will also use the notation $a^{**} := (a^*)^*$. Unless otherwise specified by parentheses, the order of evaluation of these operations is first *, then $\odot$, and finally $\oplus$. Using these notations, the axioms $(\mathrm{MV3})$ and $(\mathrm{MV4})$ have the equivalent forms $(\mathrm{MV3}')$ and respectively $(\mathrm{MV4}')$:

$(\mathrm{MV3}')$ $1 \oplus a = 1$,

$(\mathrm{MV4}')$ $(a \odot b^*) \oplus b = (b \odot a^*) \oplus a$,

for any $a, b \in A$. A list of direct consequences is given below.

PROPOSITION 2.1.2. *If A is an* MV*-algebra then the following identities hold for any a, b and $c \in A$:*

(a) $1^* = 0$,
(b) $a \oplus a^* = 1$,
(c) $a \odot 1 = a$,
(d) $a \odot 0 = 0$,
(e) $a \odot a^* = 0$,
(f) $(a \odot b)^* = a^* \oplus b^*$,
(g) $(a \oplus b)^* = a^* \odot b^*$,
(h) $a \odot (a^* \oplus b) = b \odot (b^* \oplus a)$,
(i) $a \odot (b \odot c) = (a \odot b) \odot c$,
(j) $a \odot b = b \odot a$.

PROPOSITION 2.1.3. *If A is an* MV*-algebra and $a \in A$, then $x = a^*$ is the unique solution of the system*

$$a \oplus x = 1,$$
$$a \odot x = 0.$$

Proof. If x is the solution of the above system, then $x^* \odot a^* = (x \oplus a)^* = 1^* = 0$ by Proposition 2.1.2 (g). It follows that $x = x \oplus 0 = x \oplus (x^* \odot a^*)$. Using $(\mathrm{MV4}')$ and $(\mathrm{MV2})$ we get $x = a^* \oplus (a \odot x) = a^* \oplus 0 = a^*$. □

PROPOSITION 2.1.4. *In any* MV*-algebra A there is at most one element $a \in A$ such that $a = a^*$.*

Proof. Let us suppose that there are a, $b \in A$ such that $a = a^*$ and $b = b^*$. Using $(\mathrm{MV3}')$, $(\mathrm{MV4}')$ and Proposition 2.1.2 (c), (b), (h), (g) we obtain: $a = a \odot 1 = a \odot (a^* \oplus a \oplus b) = (a \oplus b) \odot (a^* \odot b^* \oplus a) = (a \oplus b) \odot (a^* \odot b \oplus a) = (a \oplus b) \odot (b^* \odot a \oplus b) = (a \oplus b) \odot (b^* \odot a^* \oplus b) = b \odot (b^* \oplus a \oplus b) = b \odot 1 = b$.

Hence, if there is an element $a \in A$ such that $a = a^*$, then a is the unique element satisfying this property. □

NOTATION 2.1.5. Let A be an MV-algebra, $a \in A$ and $n \in \mathsf{N}$. We introduce the following notations:

$$0a = 0, \qquad na = a \oplus (n-1)a \quad \text{for any } n \geq 1,$$
$$a^0 = 1, \qquad a^n = a \odot (a^{n-1}) \quad \text{for any } n \geq 1.$$

We say that the element a *has order* n, and we write $ord(a) = n$, if n is the least natural number such that $na = 1$. We say that the element a *has a finite order*, and we write $ord(a) < \infty$, if a has order n for some $n \in \mathsf{N}$. If no such n exists, we say that a *has infinite order* and we write $ord(a) = \infty$.

DEFINITION 2.1.6. *Let $\langle A, \oplus, {}^*, 0\rangle$ be an* MV*-algebra and $B \subseteq A$ such that the following conditions are satisfied:*

(S1) $0 \in B$,
(S2) *if a, $b \in B$, then $a \oplus b \in B$,*
(S3) *if $a \in B$, then $a^* \in B$.*

Thus, if we consider the restriction of $\oplus$ *and* * *to* B*, we get an* MV*-algebra* $\langle B, \oplus, ^*, 0\rangle$ *which is an* MV-subalgebra *(or, simply,* subalgebra*) of the* MV*-algebra* A*.*

If $S \subseteq A$ *is a subset of* A*, then we will denote by* $\langle S\rangle$ *the least subalgebra of* A *which includes* S *and it will be called* the subalgebra generated by S in A. *We will say that* S *is a* system of generators *for* $\langle S\rangle$*.*

EXAMPLE 2.1.7. In any MV-algebra A the subalgebra generated by the empty set is $\langle\emptyset\rangle = \{0, 1\}$.

2.2 The lattice structure of an MV-algebra

In the sequel, A will be an MV-algebra.

PROPOSITION 2.2.1. *For any* a*,* $b \in A$ *the following are equivalent:*

(a) $a^* \oplus b = 1$,

(b) $a \odot b^* = 0$,

(c) $b = a \oplus b \odot a^*$,

(d) *there is* $c \in A$ *such that* $b = a \oplus c$,

(e) *there is* $d \in A$ *such that* $a = b \odot d$.

Proof. (a) ⇒ (b) $a \odot b^* = (a^* \oplus b^{**})^* = (a^* \oplus b)^* = 1^* = 0$.
(b) ⇒ (c) $a \oplus b \odot a^* = b \oplus a \odot b^* = b \oplus 0 = b$.
(c) ⇒ (d) We consider $c = b \odot a^*$.
(d) ⇒ (e) For $d = a \oplus b^*$ we get $b \odot d = a \odot (b \oplus a^*) = a \odot (a \oplus c \oplus a^*) = a \odot 1 = a$.
(e) ⇒ (a) $a^* \oplus b = b^* \oplus d^* \oplus b = 1 \oplus d^* = 1$. □

DEFINITION 2.2.2. *We define a binary relation* $\leq$ *on* A *by* $a \leq b$ *iff* a *and* b *satisfy one of the equivalent conditions from Proposition 2.2.1.*

PROPOSITION 2.2.3. $\leq$ *is an order relation on* A*.*

Proof. The relation $\leq$ is reflexive, since $a \oplus a^* = 1$. If $a \leq b$ and $b \leq a$, then $b = a \oplus b \odot a^*$ and $a = b \oplus a \odot b^*$. Using the axiom $(\mathrm{MV4}')$ we get $a = b$, so $\leq$ is an antisymmetric relation. In order to prove the transitivity, we consider $a \leq b$ and $b \leq c$. Thus, there are x and $y \in A$ such that $b = a \oplus x$ and $c = b \oplus y$. Hence $c = a \oplus (x \oplus y)$, so $a \leq c$. □

PROPOSITION 2.2.4. *For any* a*,* b *and* $c \in A$*, the following properties hold:*

(a) $0 \leq a \leq 1$,

(b) $a \leq b$ *iff* $b^* \leq a^*$,

(c) $a \odot b \leq c$ *iff* $a \leq b^* \oplus c$,

(d) $a \leq b$ *implies* $a \oplus c \leq b \oplus c$ *and* $a \odot c \leq b \odot c$,

(e) $a \odot c^* \leq a \odot b^* \oplus b \odot c^*$,

(f) $a \odot (b \oplus c) \leq b \oplus (a \odot c)$.

Proof. Let a, b and c be arbitrary elements of A.

(a) follows by $(\mathrm{MV3}')$ and Proposition 2.1.2 (d).

(b) $a \leq b$ iff $a^* \oplus b = 1$ iff $a^* \oplus b^{**} = 1$ iff $b^* \leq a^*$.

(c) $a \odot b \leq c$ iff $(a \odot b)^* \oplus c = 1$ iff $a^* \oplus b^* \oplus c = 1$ iff $a \leq b^* \oplus c$.

(d) We get $b \oplus c \oplus (a \oplus c)^* = b \oplus (c \oplus a^* \odot c^*) = b \oplus (a^* \oplus a \odot c) = (b \oplus a^*) \oplus a \odot c = 1 \oplus a \odot c = 1$, so $a \oplus c \leq b \oplus c$. The other inequality follows similarly.

(e) By (c), it suffices to prove that $a \leq c \oplus a \odot b^* \oplus b \odot c^*$. We have $c \oplus a \odot b^* \oplus b \odot c^* = (c \oplus b \odot c^*) \oplus a \odot b^* = (b \oplus c \odot b^*) \oplus a \odot b^* = (b \oplus a \odot b^*) \oplus c \odot b^* = a \oplus b \odot a^* \oplus c \odot b^*$. Thus, the desired inequality becomes $a \leq a \oplus b \odot a^* \oplus c \odot b^*$, which obviously holds by Definition 2.2.2 and Proposition 2.2.1 (a).

(f) We show that $(a \odot (b \oplus c))^* \oplus (b \oplus (a \odot c)) = 1$. Indeed we have: $(a \odot (b \oplus c))^* \oplus (b \oplus (a \odot c)) = a^* \oplus (b \oplus c)^* \oplus b \oplus (a \odot c) = (a^* \vee c) \oplus b \oplus (b^* \odot c^*) = (a^* \vee c) \oplus (b \vee c^*) = 1$. Hence, by Proposition 2.2.1 (a) we get the desired inequality. □

We introduce two auxiliary operations, $\vee$ and $\wedge$, by setting

$$a \vee b := a \oplus b \odot a^* = b \oplus a \odot b^* \quad \text{and} \quad a \wedge b := a \odot (b \oplus a^*) = b \odot (a \oplus b^*).$$

PROPOSITION 2.2.5. *If $a, b \in A$, then:*

(a) $a \odot b \leq a \wedge b \leq a \leq a \vee b \leq a \oplus b$,

(b) $(a \vee b)^* = a^* \wedge b^*$,

(c) $(a \wedge b)^* = a^* \vee b^*$,

(d) $a \oplus b = (a \vee b) \oplus (a \wedge b)$,

(e) $a \odot b = (a \vee b) \odot (a \wedge b)$,

(f) $a \wedge b = 0$ *implies* $a \oplus b = a \vee b$,

(g) $a \vee b = 1$ *implies* $a \odot b = a \wedge b$.

Proof. (a) By definition it is obvious that $a \wedge b \leq a \leq a \vee b$. In order to prove the other two inequalities, we will use Proposition 2.2.1 (a). Thus, $(a \wedge b) \oplus (a \odot b)^* = a \odot (a^* \oplus b) \oplus a^* \oplus b^* = (a \odot (a^* \oplus b) \oplus a^*) \oplus b^* = (a^* \oplus b \oplus (a^* \odot a \odot b)) \oplus b^* = 1$, and also $(a \oplus b) \oplus (a \vee b)^* = b \oplus (a \oplus a^* \odot (a \oplus b^*)) = b \oplus a \oplus b^* \oplus (a \odot a^* \odot b) = 1$.

(b) $(a \vee b)^* = (a \oplus a^* \odot b)^* = a^* \odot (a \oplus b^*) = a^* \odot (a^{**} \oplus b^*) = a^* \wedge b^*$.

(c) $(a \wedge b)^* = (a \odot (a^* \oplus b))^* = a^* \oplus a \odot b^* = a^* \oplus a^{**} \odot b^* = a^* \vee b^*$.

(d) $(a \vee b) \oplus (a \wedge b) = a \oplus (a^* \odot b) \oplus (b \odot (a \oplus b^*)) = a \oplus (a^* \odot b) \oplus b \odot (a^* \odot b)^* = a \oplus b \oplus (b^* \odot a^* \odot b) = a \oplus b \oplus 0 = a \oplus b$.

(e) $a \odot b = a^* \oplus b^{**} = ((a^* \vee b^*) \oplus (a^* \wedge b^*))^* = (a^* \vee b^*)^* \odot (a^* \wedge b^*)^* = (a^{**} \wedge b^{**}) \odot (a^{**} \vee b^{**}) = (a \wedge b) \odot (a \vee b)$.

(f) Follows by (d).

(g) Follows by (e). □

PROPOSITION 2.2.6. *The partially ordered set $\langle A, \leq\rangle$ is a bounded lattice such that* 0 *is the first element,* 1 *is the last element and*

$$l.u.b.\{a, b\} = a \vee b \qquad\qquad g.l.b.\{a, b\} = a \wedge b,$$

for any a, $b \in A$.

Proof. By Proposition 2.2.4 (a), it follows that 0 is the first element and 1 is the last element of A. In order to prove that $l.u.b.\{a, b\} = a \vee b$, note that $a \leq a \vee b$ and $b \leq a \vee b$ by Proposition 2.2.5 (a). If $c \in A$ such that $a \leq c$ and $b \leq c$, then $c = a \oplus a^* \odot c$ and $c \oplus b^* = 1$. We get $c \oplus (a \vee b)^* = a \oplus a^* \odot c \oplus (a^* \wedge b^*) = a^* \odot c \oplus (a \oplus a^* \odot (a \oplus b^*)) = a^* \odot c \oplus a \oplus b^* \oplus a \odot a^* \odot b = (a^* \odot c \oplus a) \oplus b^* \oplus 0 = a \odot c^* \oplus c \oplus b^* = a \odot c^* \oplus 1 = 1$, so $a \vee b \leq c$. Now we prove that $g.l.b.\{a, b\} = a \wedge b$. By Proposition 2.2.5 (a), we have $a \wedge b \leq a$ and $a \wedge b \leq b$. Let c be in A such that $c \leq a$ and $c \leq b$. By Proposition 2.2.4 (b), it follows that $a^* \leq c^*$ and $b^* \leq c^*$. Hence $l.u.b.\{a^*, b^*\} = a^* \vee b^* \leq c^*$. Using again Proposition 2.2.4 (b), we get $c \leq (a^* \vee b^*)^*$, so $c \leq a \wedge b$ by Proposition 2.2.5 (c). □

We will denote $L(A) = \langle A, \vee, \wedge, 0, 1\rangle$, the lattice structure of A. We call $L(A)$ *the lattice reduct of* A.

DEFINITION 2.2.7. *An* MV*-algebra A is* complete *(σ-complete) if the lattice reduct of A is a complete (σ- complete) lattice.*

If $\{a_i \mid i \in I\}$ is a family of elements from A, then we will write $\bigvee a_i$ instead of $\bigvee\{a_i \mid i \in I\}$ and $\bigwedge a_i$ instead of $\bigwedge\{a_i \mid i \in I\}$ if there are no possible confusions.

LEMMA 2.2.8. *Let $\{a_i \mid i \in I\}$ be a family of elements from A.*

(a) *If $\bigvee a_i$ exists, then $\bigwedge {a_i}^*$ exists and $\bigwedge {a_i}^* = (\bigvee a_i)^*$.*

(b) *If $\bigwedge a_i$ exists, then $\bigvee {a_i}^*$ exists and $\bigvee {a_i}^* = (\bigwedge a_i)^*$.*

Proof. We will use Proposition 2.2.4 (b) and Proposition 2.2.5 (b), (c).
(a) If $a = \bigvee a_i$, then $a_i \leq a$ for any $i \in I$. It follows that ${a_i}^* \geq a^*$ for any $i \in I$, so a^* is a lower bound of the family $\{{a_i}^* \mid i \in I\}$. Let $z \in A$ such that $z \leq {a_i}^*$ for any $i \in I$. We have $z^* \geq a_i$ for any $i \in I$, so $a \leq z^*$. Thus, $z \leq a^*$, so a^* is the greatest lower bound of the family $\{{a_i}^* \mid i \in I\}$ in A.
(b) follows similarly. □

PROPOSITION 2.2.9. *For any $a \in A$ and for any family of elements $\{b_i \mid i \in I\} \subseteq A$, the following properties hold whenever $\bigvee\{b_i \mid i \in I\}$ and $\bigwedge\{b_i \mid i \in I\}$ exist:*

(a) $a \odot \bigvee\{b_i \mid i \in I\} = \bigvee\{a \odot b_i \mid i \in I\}$,

(b) $a \wedge \bigvee\{b_i \mid i \in I\} = \bigvee\{a \wedge b_i \mid i \in I\}$,

(c) $a \oplus \bigvee\{b_i \mid i \in I\} = \bigvee\{a \oplus b_i \mid i \in I\}$,

(d) $a \oplus \bigwedge\{b_i \mid i \in I\} = \bigwedge\{a \oplus b_i \mid i \in I\}$,

(e) $a \vee \bigwedge\{b_i \mid i \in I\} = \bigwedge\{a \vee b_i \mid i \in I\}$,

(f) $a \odot \bigwedge\{b_i \mid i \in I\} = \bigwedge\{a \odot b_i \mid i \in I\}$.

Proof. (a) It is obvious that $a \odot b_i \leq a \odot \bigvee b_i$ for any $i \in I$. Let z be another upper bound of the family $\{a \odot b_i \mid i \in I\}$, so $a \odot b_i \leq z$ for any $i \in I$. We get

$$b_i \leq b_i \vee a^* = a^* \oplus a \odot b_i \leq a^* \oplus z$$

for any $i \in I$. It follows that $\bigvee b_i \leq a^* \oplus z$. Thus, we have $a \odot \bigvee b_i \leq a \odot (a^* \oplus z) = a \wedge z \leq z$, so $a \odot \bigvee b_i$ is the least upper bound of the family $\{a \odot b_i \mid i \in I\}$.
(b) We have $a \wedge \bigvee b_i \geq a \wedge b_i$ for any $i \in I$. We consider $z \in A$ such that $z \geq a \wedge b_i$ for any $i \in I$. Since ${b_i}^* \geq (\bigvee b_i)^*$, we have

$$z \geq a \wedge b_i = b_i \odot (a \oplus {b_i}^*) \geq b_i \odot \Big(a \oplus \Big(\bigvee b_i\Big)^*\Big),$$

for any $i \in I$. Using (a), it follows that

$$a \wedge \bigvee b_i = \Big(\bigvee b_i\Big) \odot \Big(a \oplus \Big(\bigvee b_i\Big)^*\Big) = \bigvee \Big(b_i \odot \Big(a \oplus \Big(\bigvee b_i\Big)^*\Big)\Big) \leq z.$$

Thus, $a \wedge \bigvee b_i$ is the least upper bound of the family $\{a \wedge b_i \mid i \in I\}$.
(c) We have $a \oplus \bigvee b_i \geq a \oplus b_i$ for any $i \in I$. Hence, $a \oplus \bigvee b_i$ is an upper bound of the family $\{a \oplus b_i \mid i \in I\}$. Let z be another upper bound, so $z \geq a \oplus b_i$ for any $i \in I$. We get $z \geq a$ and

$$z \odot a^* \geq (a \oplus b_i) \odot a^* = a^* \wedge b_i$$

for any $i \in I$. Using (b) it follows that

$$a^* \wedge \bigvee b_i = \bigvee (a^* \wedge b_i) \leq z \odot a^*.$$

We infer that

$$a \oplus \bigvee b_i = a \oplus \Big(a^* \wedge \bigvee b_i\Big) \leq a \oplus z \odot a^* = a \vee z = z,$$

so $a \oplus \bigvee b_i$ is the least upper bound of the family $\{a \oplus b_i \mid i \in I\}$.
(d), (e) and (f) Follows from (a), (b) and (c), respectively, using Lemma 2.2.8. We will only give the proof of (d):

$$a \oplus \bigwedge b_i = \Big(a^* \odot \bigvee {b_i}^*\Big)^* = \Big(\bigvee (a^* \odot {b_i}^*)\Big)^* = \bigwedge (a^* \odot {b_i}^*)^* = \bigwedge (a \oplus b_i). \quad \square$$

PROPOSITION 2.2.10. *If $a, x_1, \ldots, x_n \in A$ for some $n \geq 1$, then the following properties hold:*

(a) $a \vee (x_1 \oplus \cdots \oplus x_n) \leq (a \vee x_1) \oplus \cdots \oplus (a \vee x_n)$,
(b) $a \wedge (x_1 \oplus \cdots \oplus x_n) \leq (a \wedge x_1) \oplus \cdots \oplus (a \wedge x_n)$,
(c) $a \vee (x_1 \odot \cdots \odot x_n) \geq (a \vee x_1) \odot \cdots \odot (a \vee x_n)$,
(d) $a \wedge (x_1 \odot \cdots \odot x_n) \geq (a \wedge x_1) \odot \cdots \odot (a \wedge x_n)$.

Proof. (a) We prove by induction on $n \geq 1$. For $n = 2$ we use Proposition 2.2.9 (c):

$$\begin{aligned}(a \vee x_1) \oplus (a \vee x_2) &= (a \oplus a) \vee (a \oplus x_2) \vee (a \oplus x_1) \vee (x_1 \oplus x_2)\\ &\geq a \vee a \vee a \vee (x_1 \oplus x_2) = a \vee (x_1 \oplus x_2).\end{aligned}$$

Now, the induction step follows:

$$\begin{aligned} a \vee (x_1 \oplus \cdots \oplus x_{n+1}) &\leq (a \vee (x_1 \oplus \cdots \oplus x_n)) \oplus (a \vee x_{n+1}) \\ &\leq (a \vee (x_1) \oplus \cdots \oplus (a \vee x_n) \oplus (a \vee x_{n+1}). \end{aligned}$$

(b), (c) and (d) follows similarly. □

PROPOSITION 2.2.11. *For any a, $b \in A$ the following properties hold:*

(a) $(a \odot b^*) \wedge (a^* \odot b) = 0$,

(b) $(a \oplus b^*) \vee (a^* \oplus b) = 1$,

(c) $a \wedge a^* \leq b \vee b^*$,

(d) $a \oplus b = a \oplus b \oplus a \odot b$,

(e) $a \oplus b \leq 2a \vee 2b$.

Proof. (a) $0 = (a \wedge b) \odot (a \wedge b)^* = (a \wedge b) \odot (a^* \vee b^*) = (a \odot (a^* \vee b^*)) \wedge (b \odot (a^* \vee b^*)) = (a \odot a^* \vee a \odot b^*) \wedge (b \odot a^* \vee b \odot b^*) = (0 \vee a \odot b^*) \wedge (b \odot a^* \vee 0) = (a \odot b^*) \wedge (b \odot a^*)$.

(b) Using (a) we get $1 = 0^* = ((a \odot b^*) \wedge (b \odot a^*))^* = (a^* \oplus b) \vee (b^* \oplus a)$.

(c) We will use (a) and Proposition 2.2.1 (b). Thus, we have $(a \wedge a^*) \odot (b \vee b^*)^* = (a \wedge a^*) \odot (b \wedge b^*) = (a \odot b) \wedge 0 \wedge (a^* \odot b^*) = 0$, and so $a \wedge a^* \leq b \vee b^*$.

(d) From (b) we get $1 = (a \oplus b) \vee (a^* \oplus b^*) = (a \oplus b) \oplus (a \oplus b)^* \odot (a^* \oplus b^*) = (a \oplus b) \oplus a^* \odot b^* \odot (a \odot b)^* = (a \oplus b) \oplus (a \oplus b \oplus a \odot b)^*$. Using Proposition 2.2.1 (a), it follows that $a \oplus b \geq a \oplus b \oplus a \odot b$. Since the converse inequality is also true by Proposition 2.2.5 (a), the desired result holds.

(e) By Proposition 2.2.5 (d) we have $a \oplus b = (a \vee b) \oplus (a \wedge b) = (a \oplus (a \wedge b)) \vee (b \oplus (a \wedge b)) = (2a \wedge (a \oplus b)) \vee (2b \wedge (a \oplus b)) = (a \oplus b) \wedge (2a \vee 2b) \leq 2a \vee 2b$. □

PROPOSITION 2.2.12. *Let a and b be two elements of A.*

(a) If $a \wedge b = 0$, then $na \wedge nb = 0$ for any $n \in \mathsf{N}$.

(b) If $a \vee b = 1$, then $a^n \vee b^n = 1$ for any $n \in \mathsf{N}$.

Proof. (a) If $a \wedge b = 0$ we have $a = a \oplus (a \wedge b) = 2a \wedge (a \oplus b) \geq 2a \wedge b$. It follows that $2a \wedge b \leq a \wedge b = 0$, so $2a \wedge b = 0$. Moreover, $b = b \oplus (2a \wedge b) = (b \oplus 2a) \wedge 2b \geq 2a \wedge 2b$, so $2a \wedge 2b \leq 2a \wedge b = 0$. Thus we proved that $a \wedge b = 0$ implies $2a \wedge 2b = 0$. By induction, it follows that $2^k a \wedge 2^k b = 0$ for any $k \in \mathsf{N}$. For any $n \in \mathsf{N}$ there is $k \in \mathsf{N}$ such that $n \leq 2^k$. Hence $na \wedge nb \leq 2^k a \wedge 2^k b = 0$, so $na \wedge nb = 0$.

(b) If $a \vee b = 1$, then $a^* \wedge b^* = 0$. Using (a), we get $na^* \wedge nb^* = 0$ for any $n \in \mathsf{N}$. It follows that $1 = 0^* = (na^* \wedge nb^*)^* = a^n \vee b^n$ for any $n \in \mathsf{N}$. □

COROLLARY 2.2.13. *For any a, $b \in A$ and for any $n \in \mathsf{N}$ we have:*

(a) $n(a \odot b^*) \wedge n(b \odot a^*) = 0$,

(b) $(a \oplus b^*)^n \vee (b \oplus a^*)^n = 1$.

Proof. (a) Follows by Proposition 2.2.11 (a) and Proposition 2.2.12 (a).

(b) Follows by Proposition 2.2.11 (b) and Proposition 2.2.12 (b). □

COROLLARY 2.2.14. *For any a, $b \in A$, if $ord(a \odot b) < \infty$, then $a \oplus b = 1$.*

Proof. By hypothesis, for some $n \geq 1$, $n(a \odot b) = 1$. Hence, by Corollary 2.2.13 (a), we get $n(b^* \odot a^*) = 0$. It follows that $(a \oplus b)^n = 1$, so $a \oplus b = 1$. □

We recall that a *Kleene algebra* is a structure $\langle L, \vee, \wedge, ^*, 0, 1\rangle$ where $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and * is a unary operation such that the following properties hold for any $a, b \in L$:

$(\mathrm{K1})$ $(a^*)^* = a$,

$(\mathrm{K2})$ $(a \vee b)^* = a^* \wedge b^*$,

$(\mathrm{K3})$ $a \wedge a^* \leq b \vee b^*$.

PROPOSITION 2.2.15. *For any MV-algebra A, the structure $\langle L(A), \vee, \wedge, ^*, 0, 1\rangle$ is a Kleene algebra.*

Proof. By Proposition 2.2.6, the structure $\langle L(A), \vee, \wedge, 0, 1\rangle$ is a bounded lattice. The lattice $L(A)$ is distributive (Proposition 2.2.9 (b) and (e)). The property $(\mathrm{K1})$ is $(\mathrm{MV2})$, $(\mathrm{K2})$ follows from Proposition 2.2.5 (b) and $(\mathrm{K3})$ follows by Proposition 2.2.11 (c). □

2.3 The implication

In the sequel, $\langle A, \oplus, \odot, \vee, \wedge, 0, 1\rangle$ will be an MV-algebra. The *implication* is defined by

$$a \to b := a^* \oplus b \text{ for any } a, b \in A.$$

LEMMA 2.3.1. *The following properties hold for any a, $b \in A$:*

(a) $a^* = a \to 0$,

(b) $a \leq b$ *iff* $a \to b = 1$,

(c) $a \wedge b = a \odot (a \to b)$,

(d) $a \vee b = (a \to b) \to b$,

(e) $(a \to b)^n \vee (b \to a)^n = 1$ *for any* $n \in \mathsf{N}$.

Proof. (a), (b), (c) follows by definition.
(d) $(a \to b) \to b = (a \to b)^* \oplus b = (a^* \oplus b)^* \oplus b = (a \odot b^*) \oplus b = a \vee b$,
(e) is straightforward by Corollary 2.2.13 (b). □

LEMMA 2.3.2. *For any $a \in A$ and for any family of elements $\{b_i \mid i \in I\} \subseteq A$, the following properties hold whenever $\bigvee\{b_i \mid i \in I\}$ and $\bigwedge\{b_i \mid i \in I\}$ exist:*

(a) $a \to \bigvee\{b_i \mid i \in I\} = \bigvee\{a \to b_i \mid i \in I\}$,

(b) $a \to \bigwedge\{b_i \mid i \in I\} = \bigwedge\{a \to b_i \mid i \in I\}$,

(c) $(\bigvee\{b_i \mid i \in I\}) \to a = \bigwedge\{b_i \to a \mid i \in I\}$,

(d) $(\bigwedge\{b_i \mid i \in I\}) \to a = \bigvee\{b_i \to a \mid i \in I\}$.

Proof. We will only prove (a) and (c), using Proposition 2.2.9 and De Morgan laws.

(a) $a \to \bigvee\{b_i \mid i \in I\} = a^* \oplus \bigvee\{b_i \mid i \in I\} = \bigvee\{a^* \oplus b_i \mid i \in I\} =$
$= \bigvee\{a \to b_i \mid i \in I\}$.

(c) $(\bigvee\{b_i \mid i \in I\}) \to a = (\bigvee\{b_i \mid i \in I\})^* \oplus a = (\bigwedge\{{b_i}^* \mid i \in I\}) \oplus a =$
$= \bigwedge\{{b_i}^* \oplus a \mid i \in I\} = \bigwedge\{b_i \to a \mid i \in I\}$. □

LEMMA 2.3.3. *If a, b, $x \in A$, then:*

(a) $a \odot x \leq b$ *iff* $x \leq a \to b$,

(b) $a \to b = \max\{x \in A \mid a \odot x \leq b\}$.

Proof. (a) follows by Proposition 2.2.4 (c).
(b) is a direct consequence of (a). □

COROLLARY 2.3.4. *The structure $\langle L(A), \vee, \wedge, \odot, \to, 0, 1\rangle$ is a residuated lattice for any MV-algebra A.*

PROPOSITION 2.3.5. *In any MV-algebra A the following properties hold:*

(W1) $1 \to a = a$,

(W2) $(a \to b) \to ((b \to c) \to (a \to c)) = 1$,

(W3) $(a \to b) \to b = (b \to a) \to a$,

(W4) $(a^* \to b^*) \to (b \to a) = 1$.

Proof. (W1) $1 \to a = 0 \oplus a = a$.
(W2) Using consecutively Lemma 2.3.1 (b), Lemma 2.3.3 (a) and Proposition 2.2.1 (b), it follows that

$$\begin{aligned}(a \to b) \to ((b \to c) \to (a \to c)) = 1 &\text{ iff } (a \to b) \leq (b \to c) \to (a \to c)\\ &\text{ iff } (a \to b) \odot (b \to c) \leq a \to c\\ &\text{ iff } (a \to b) \odot (b \to c) \odot (a \to c)^* = 0.\end{aligned}$$

We prove the last equality: $(a \to b) \odot (b \to c) \odot (a \to c)^* = (a \to b) \odot (b \to c) \odot a \odot c^* = a \odot (a^* \oplus b) \odot c^* \odot (b^* \oplus c) = (a \wedge b) \odot (c^* \wedge b^*) \leq b \odot b^* = 0$, and so $(a \to b) \odot (b \to c) \odot (a \to c)^* = 0$.
(W3) By Proposition 2.3.1 (d), $(a \to b) \to b = a \vee b = b \vee a = (b \to a) \to a$.
(W4) $(a^* \to b^*) \to (b \to a) = (a \oplus b^*)^* \oplus b^* \oplus a = a^* \odot b \oplus b^* \oplus a = (a^* \vee b^*) \oplus a = (a^* \oplus a) \vee (b^* \oplus a) = 1 \vee (b^* \oplus a) = 1$. □

DEFINITION 2.3.6. *A* Wajsberg algebra *is a structure $\langle W, \to, *, 1\rangle$, where $\to$ is a binary operation, $*$ is a unary operation, and 1 is a constant and the identities* (W1)–(W4) *from Proposition 2.3.5 hold.*

COROLLARY 2.3.7. *If $\langle A, \oplus, {}^*, 0\rangle$ is an MV-algebra, then $W_A = \langle A, \to, {}^*, 1\rangle$ is a Wajsberg algebra, where $\to$ is the MV-algebra implication and $1 = 0^*$.*

In the sequel, we will prove that the converse is also true, i.e. any Wajsberg algebra has an MV-algebra structure. Moreover, the category of MV-algebras and the category of Wajsberg algebras are equivalent.

PROPOSITION 2.3.8 ([33]). *In any Wajsberg algebra $\langle W, \to, {}^*, 1\rangle$ the following properties hold for any a, b and $c \in W$:*

(a) $a \to a = 1$,

(b) *if* $a \to b = 1$ *and* $b \to a = 1$, *then* $a = b$,

(c) *if* $a \to b = 1$ *and* $b \to c = 1$, *then* $a \to c = 1$,

(d) $a \to 1 = 1$,

(e) $a \to (b \to a) = 1$,

(f) $a \to ((a \to b) \to b) = a \to ((b \to a) \to a) = 1$,

(g) $a \to (b \to c) = b \to (a \to c)$,

(h) $(a \to b) \to ((c \to a) \to (c \to b)) = 1$,

(i) $1^* \to a = 1$,

(j) $a = a^* \to 1^*$,

(k) $a^* = a \to 1^*$,

(l) $(a^*)^* = a$,

(m) $a^* \to b^* = b \to a$.

Proof. (a) Using $(\mathrm{W2})$ and $(\mathrm{W1})$ we get $(1 \to 1) \to ((1 \to a) \to (1 \to a)) = 1$, $1 \to (a \to a) = 1$, and so finally $a \to a = 1$.

(b) By $(\mathrm{W1})$ and $(\mathrm{W3})$ it follows that $a = 1 \to a = (b \to a) \to a = (a \to b) \to b = 1 \to b = b$.

(c) Straightforward by $(\mathrm{W2})$ and $(\mathrm{W1})$.

(d) From $(\mathrm{W3})$ and (a) we infer that $(a \to 1) \to 1 = (1 \to a) \to a = a \to a = 1$. Thus, by $(\mathrm{W2})$ and $(\mathrm{W1})$, it follows that

$$\begin{aligned}(1 \to a) \to ((a \to 1) \to (1 \to 1)) &= 1\\ a \to ((a \to 1) \to 1) &= 1\\ a \to 1 &= 1.\end{aligned}$$

(e) Using $(\mathrm{W2})$, $(\mathrm{W1})$ and (d) we get

$$\begin{aligned}(b \to 1) \to ((1 \to a) \to (b \to a)) &= 1\\ 1 \to (a \to (b \to a)) &= 1\\ a \to (b \to a) &= 1.\end{aligned}$$

(f) By (e), $a \to ((b \to a) \to a) = 1$. The other equality follows by $(\mathrm{W3})$.

(g) If we denote $x = ((b \to c) \to c) \to (a \to c) = ((c \to b) \to b) \to (a \to c)$ then $(a \to (b \to c)) \to x = 1$ by $(\mathrm{W2})$. Moreover, using also (f) we get

$$\begin{aligned}(b \to ((b \to c) \to c)) \to (x \to (b \to (a \to c))) &= 1\\ 1 \to (x \to (b \to (a \to c))) &= 1\\ (x \to (b \to (a \to c))) &= 1.\end{aligned}$$

By (c), we have $(a \to (b \to c)) \to (b \to (a \to c)) = 1$ for any $a, b \in W$. Hence, we also have $(b \to (a \to c)) \to (a \to (b \to c) = 1$, so the desired equality follows by (b).

(h) Straightforward by $(\mathrm{W2})$ and (g).

(i) Using $(\mathrm{W1})$ and $(\mathrm{W4})$, we get $(a^* \to 1^*) \to a = (a^* \to 1^*) \to (1 \to a) = 1$. Thus, by (h) we obtain

$$\begin{aligned} ((a^* \to 1^*) \to a) \to ((1^* \to (a^* \to 1^*)) \to (1^* \to a)) &= 1 \\ 1 \to ((1^* \to (a^* \to 1^*)) \to (1^* \to a)) &= 1 \\ (1^* \to (a^* \to 1^*)) \to (1^* \to a) &= 1. \end{aligned}$$

From (e) and $(\mathrm{W1})$, we get $1^* \to a = 1$.

(j) By (e) and $(\mathrm{W4})$, we have $a^* \to (1^{**} \to a^*) = 1$ and $(1^{**} \to a^*) \to (a \to 1^*) = 1$. Using (c), it follows that $a^* \to (a \to 1^*) = 1$ so, by (g), $a \to (a^* \to 1^*) = 1$. By $(\mathrm{W4})$, we also have $(a^* \to 1^*) \to a = (a^* \to 1^*) \to (1 \to a) = 1$. Hence, from (b), we deduce that $a = a^* \to 1^*$.

(k) By $(\mathrm{W1})$, (i), $(\mathrm{W3})$ and (j) we obtain that: $a^* = 1 \to a^* = (1^* \to a^*) \to a^* = (a^* \to 1^*) \to 1^* = a \to 1^*$.

(l) By (k) and (j) we get: $(a^*)^* = a^* \to 1^* = a$.

(m) By (l) and $(\mathrm{W4})$ we get $(b \to a) \to (a^* \to b^*) = ((b^*)^* \to (a^*)^*) \to (a^* \to b^*) = 1$. The desired equality follows by $(\mathrm{W4})$ and (b). $\square$

FACT 2.3.9. *If* $\langle W, \to, ^*, 1\rangle$ *is a Wajsberg algebra and we define*

$$a \leq b \quad \textit{iff} \quad a \to b = 1,$$

for any a, $b \in W$*, then* $\leq$ *is a partial order on* W *by Proposition 2.3.8 (a), (b), (c). By (d) and (i), it follows that* 1 *is the last element and* 1^* *is the first element. Moreover, one can prove that* $\langle W, \leq\rangle$ *is a lattice where for any* $a, b \in A$*:*

$$a \vee b = (a \to b) \to b \quad \textit{and} \quad a \wedge b = (a^* \vee b^*)^*.$$

PROPOSITION 2.3.10. *If in a Wajsberg algebra* $\langle W, \to, ^*, 1\rangle$ *we define*

$$a \oplus b = a^* \to b \quad \textit{and} \quad 0 = 1^*$$

for any a, $b \in W$*, then* $A_W = \langle W, \oplus, ^*, 0\rangle$ *is an* MV*-algebra.*

Proof. $(\mathrm{MV1})$ We have to prove that $\langle W, \oplus, 0\rangle$ is an Abelian monoid. The commutativity of $\oplus$ is straightforward using Proposition 2.3.8 (m) and (l):

$$a \oplus b = a^* \to b = b^* \to (a^*)^* = b^* \to a = b \oplus a.$$

From (j), we get $a \oplus 0 = a^* \to 1^* = a$. Now, using the commutativity of $\oplus$ and (g), we are able to prove the associativity: $a \oplus (b \oplus c) = a \oplus (c \oplus b) = a^* \to (c^* \to b) = c^* \to (a^* \to b) = c \oplus (a \oplus b) = (a \oplus b) \oplus c$.

$(\mathrm{MV2})$ Follows from Proposition 2.3.8 (l).

$(\mathrm{MV3})$ By Proposition 2.3.8 (l), we get $0^* = 1$. Thus, using (i) it follows that

$$0^* \oplus a = 1 \oplus a = 1^* \to a = 1 = 0^*.$$

$(\mathrm{MV4})$ Follows from $(\mathrm{W4})$:

$$(a \oplus b^*)^* \oplus a = (a \oplus b^*) \to a = (b \to a) \to a = (a \to b) \to b = (b \oplus a^*)^* \oplus b. \ \square$$

COROLLARY 2.3.11. *For any* MV*-algebra A and for any Wajsberg algebra W, the following properties hold:*

(a) $A_{W_A} = A$,

(b) $W_{A_W} = W$.

Proof. Note that the support sets of A_{W_A} and A coincide, as well as the * operation. Similarly, the support sets and the operation * on W_{A_W} and W coincide. Thus, we only have to prove that $\oplus_{A_{W_A}} = \oplus_A$ and $\rightarrow_{W_{A_W}} = \rightarrow_W$.
(a) If we denote $B = A_{W_A}$, then for any $a,\, b \in A$ we get $a \oplus_B b = a^* \rightarrow_{W_A} b = (a^*)^* \oplus_A b = a \oplus_A b$.

(b) If we denote by $V = W_{A_W}$, then for any $a, b \in W$ we get $a \rightarrow_V b = a^* \oplus_{A_W} b = (a^*)^* \rightarrow_W b = a \rightarrow_W b$. □

2.4 Examples of MV-algebras

We will describe some basic examples of MV-algebras.

EXAMPLE 2.4.1. Any Boolean algebra is an MV-algebra in which the operations $\oplus$ and $\vee$ coincide,

EXAMPLE 2.4.2 ($[0,1]$, $\mathsf{Z} \cap [0,1]$, $\mathbf{Ł}_{n+1}$). Let R denote the set of real numbers and let Z denote the set of rational numbers. For any $n \in \mathsf{N}$, $n \geq 1$ we define $L_{n+1} = \{0, 1/n, \dots, (n-1)/n, 1\}$. If a and b are real numbers we define

$$a \oplus b := \min\{a+b, 1\} \quad \text{and} \quad a^* := 1 - a.$$

One can easily see that the unit interval $[0,1]$, the set $\mathsf{Z} \cap [0,1]$ and the set L_{n+1} with $n \geq 1$ are closed under the above defined operations. Straightforward computations show that $\langle [0,1], \oplus, ^*, 0\rangle$, $\langle \mathsf{Z} \cap [0,1], \oplus, ^*, 0\rangle$, and $\langle L_{n+1}, \oplus, ^*, 0\rangle$ are MV-algebras, which will be simply denoted by $[0,1]$, $\mathsf{Z} \cap [0,1]$ and $\mathbf{Ł}_{n+1}$ respectively. If $n = 1$, then $L_{n+1} = L_2 = \{0,1\}$, the Boolean algebra with two elements. Moreover, the auxiliary operation $\odot$ is given by $a \odot b = \max\{a+b-1, 0\}$ and the order is the natural order of the real numbers.

EXAMPLE 2.4.3 (A^X, $[0,1]^X$). Let $\langle A, \oplus, ^*, 0\rangle$ be an MV-algebra and X a nonempty set. The set A^X of all the functions $f\colon X \to A$ becomes an MV-algebra with the pointwise operations, i.e., if $f, g \in A^X$, then $(f \oplus g)(x) := f(x) \oplus g(x)$, $f^*(x) := f(x)^*$ for any $x \in X$ and 0 is the constant function associated with $0 \in A$. A special significance has the MV-algebra $[0,1]^X$, where $[0,1]$ is the MV-algebra defined in Example 2.4.2. An element $f \in [0,1]^X$ is called *fuzzy subset of* X and, for any $x \in X$, $f(x)$ represents the *degree of membership* of x to f. The subalgebras of $[0,1]^X$ are called *bold algebras of fuzzy sets*.

EXAMPLE 2.4.4 ($C(X)$). Let X be a topological space and consider $[0,1]$ the unit real interval equipped with the natural topology. We consider

$$C(X) = \{f\colon X \to [0,1] \mid f \text{ is continuous}\}.$$

One can easily see that $C(X)$ is a subset of $[0,1]^X$ from Example 2.4.3 and $C(X)$ is closed under the MV-algebra operations defined pointwise. Thus, if $f, g \in C(X)$, then $f \oplus g$ and $f^* \in C(X)$ where $(f \oplus g)(x) = \min\{f(x)+g(x), 1\}$ and $f^*(x) = 1 - f(x)$ for any $x \in X$. We obtain the MV-algebra $\langle C(X), \oplus, {}^*, 0\rangle$, where 0 is the constant function associated with $0 \in [0,1]$.

EXAMPLE 2.4.5 (Chang's MV-algebra $\boldsymbol{C}$). Let $\{c, 0, 1, +, -\}$ be a set of formal symbols. For any $n \in \mathsf{N}$ we define the following abbreviations:

$$nc := \begin{cases} 0 & \text{if } n = 0, \\ c & \text{if } n = 1, \\ c + (n-1)c & \text{if } n > 1. \end{cases}$$

$$1 - nc := \begin{cases} 1 & \text{if } n = 0, \\ 1 - c & \text{if } n = 1, \\ 1 - (n-1)c - c & \text{if } n > 1. \end{cases}$$

We consider $C = \{nc \mid n \in \mathsf{N}\} \cup \{1 - nc \mid n \in \mathsf{N}\}$ and we define the MV-algebra operations as follows:

($\oplus$1) if $x = nc$ and $y = mc$, then $x \oplus y := (m+n)c$,

($\oplus$2) if $x = 1 - nc$ and $y = 1 - mc$, then $x \oplus y := 1$,

($\oplus$3) if $x = nc$ and $y = 1 - mc$ and $m \leq n$, then $x \oplus y := 1$,

($\oplus$4) if $x = nc$ and $y = 1 - mc$ and $n < m$, then $x \oplus y := 1 - (m-n)c$,

($\oplus$5) if $x = 1 - mc$ and $y = nc$ and $m \leq n$, then $x \oplus y := 1$,

($\oplus$6) if $x = 1 - mc$ and $y = nc$ and $n < m$, then $x \oplus y := 1 - (m-n)c$,

(*1) if $x = nc$, then $x^* := 1 - nc$,

(*2) if $x = 1 - nc$, then $x^* := nc$.

Hence, the structure $\boldsymbol{C} = \langle C, \oplus, {}^*, 0\rangle$ is an MV-algebra, which is called *Chang's algebra* since it was defined by Chang [12]. The order relation is defined by:

$$\begin{aligned} x \leq y \quad \text{iff} \quad & x = nc \ \text{ and } \ y = 1 - mc \ \text{ or} \\ & x = nc \ \text{ and } \ y = mc \text{ and } \ n \leq m \ \text{ or} \\ & x = 1 - nc \ \text{ and } \ y = 1 - mc \text{ and } \ m \leq n. \end{aligned}$$

Since $0 \leq c \leq \cdots \leq nc \leq \cdots \leq 1 - nc \leq \cdots \leq 1 - c \leq 1$, $\boldsymbol{C}$ is an MV-chain.

EXAMPLE 2.4.6 (The interval algebra $A(0,a)$). Let $\langle A, \oplus, {}^*, 0\rangle$ be an MV-algebra and assume $a > 0$ in A. Denote

$$A(0,a) = [0,a] = \{x \in A \mid 0 \leq x \leq a\}.$$

For any $x, y \in A(0,a)$ define

$$x \oplus_{[0,a]} y := (x \oplus y) \wedge a \quad \text{and} \quad x^{*[0,a]} := x^* \odot a.$$

The structure $\langle A(0,a), \oplus_{[0,a]}, {}^{*[0,a]}, 0\rangle$ is an MV-algebra.

EXAMPLE 2.4.7 (The Lindenbaum–Tarski algebra **Ł**). From the logical point of view, the most important example of an MV-algebra is the algebra **Ł** arising from the ∞-valued Łukasiewicz propositional logic, which will be developed in Section 6. The formulas in this logic are built up of denumerable many propositional variables with two operations $\neg$ and $\to$ inductively in the following manner:

(f1) every propositional variable is a formula,
(f2) if φ is a formula, then $\neg\varphi$ is a formula,
(f3) if φ and ψ are formulas, then $(\varphi \to \psi)$ is a formula,
(f4) a string of symbols is a formula of Ł iff it can be shown to be a formula by a finite number of applications of (f1), (f2), and (f3).

We will denote by $Fm_{\text{Ł}}$ the set of all formulas of Ł. The particular four *axiom schemes* of this propositional calculus are:

(A1) $\varphi \to (\psi \to \varphi)$,
(A2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(A3) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$,
(A4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$.

The *deduction rule* is *modus ponens* (MP): from φ and $\varphi \to \psi$ infer ψ. The notation $\vdash \varphi$ means that the formula φ is provable from (A1)–(A4) using only *modus ponens*. On the set of all formulas we define the equivalence relation $\equiv$ by: $\varphi \equiv \psi$ iff $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$. If we denote by $[\varphi]$ the equivalence class of the formula φ determined by $\equiv$, then $\text{Ł} = \{[\varphi] \mid \varphi \text{ is a formula}\}$ is the set of all the equivalence classes. It is obvious that **Ł** is a Wajsberg algebra. Consequently, the MV-algebra operations on **Ł** are defined as follows:

$$[\varphi] \oplus [\psi] := [\neg\varphi \to \psi] \quad \text{and} \quad [\varphi]^* := [\neg\varphi].$$

If we also define $0 := [\varphi]$ for any φ such that $\vdash \neg\varphi$, then the structure $\mathbf{Ł} = \langle \text{Ł}, \oplus, {}^*, 0\rangle$ is an MV-algebra. One can easily prove that $[\varphi] \vee [\psi] = [(\varphi \to \psi) \to \psi]$ and $1 = [\varphi]$ iff $\vdash \varphi$.

2.5 The interval MV-algebra of an ℓu-group

A *lattice-ordered group* (ℓ*-group*) is a structure $\langle G, +, 0, \leq\rangle$ such that $\langle G, +, 0\rangle$ is a group, $\langle G, \leq\rangle$ is a lattice and any group translation is isotone. For a comprehensive study of ℓ-groups theory one can see [3, 7, 8, 34].

If G and H are ℓ-groups, then $h\colon G \to H$ is an ℓ*-group homomorphism* if h is both a group homomorphism and a lattice homomorphism. For G an ℓ-group we denote by G_+ the positive cone of G. If $g \in G$, then the positive and, respectively, the negative part of g are $g_+ = g \vee 0$ and $g_- = (-g) \vee 0$. We remind that $g = g_+ - g_-$ and $|g| = g \vee (-g) = g_+ + g_-$.

A positive element u in G is a *strong unit* if for every $g \in G_+$ there is $n \in \mathsf{N}$ such that $g \leq nu$. An Abelian ℓ-group with strong unit $\langle G, u\rangle$ will be simply called ℓu-group.

We begin with the simple, but very important observation that, given an Abelian ℓ-group $\langle G, +, 0, \leq\rangle$ and a positive element $u > 0$ in G, the interval $[0, u]$ can be endowed with an MV-algebra structure. Moreover, in Section 5 we will prove that any MV-algebra coincides with the interval $[0, u]$ of an ℓu-group $\langle G, u\rangle$.

LEMMA 2.5.1. *Let* $\langle G, +, 0, \leq\rangle$ *be an Abelian* ℓ*-group and* $u \in G$ *such that* $u > 0$*. If we define on the interval* $[0, u] = \{a \in G \mid 0 \leq a \leq u\}$ *the operations*

$$a \oplus b := (a + b) \wedge u \quad \textit{and} \quad a^* := u - a,$$

then the structure $\langle [0, u], \oplus, {}^*, 0\rangle$ *is an* MV*-algebra. Moreover, for any* $a, b \in [0, u]$ *we get* $a \odot b = (a + b - u) \vee 0$ *and the lattice operations on* $[0, u]$ *are the restriction of the lattice operations on* G.

Proof. We will only prove the associativity of $\oplus$ and the axiom $(\mathrm{MV4})$, since the other axioms are straightforward. For any a, b and $c \in [0, u]$, we have

$$\begin{aligned}
(a \oplus b) \oplus c \quad &= (((a + b) \wedge u) + c) \wedge u = (a + b + c) \wedge (u + c) \wedge u \\
&= (a + b + c) \wedge u = (a + b + c) \wedge (a + u) \wedge u \\
&= (a + ((b + c) \wedge u) \wedge u = a \oplus (b \oplus c). \\
a \oplus (b^* \oplus a)^* \quad &= (a + (u - ((u - b + a) \wedge u))) \wedge u \\
&= (a + ((b - a) \vee 0)) \wedge u = (b \vee a) \wedge u \\
&= b \vee a.
\end{aligned}$$

The axiom $(\mathrm{MV4})$ is a direct consequence of the last equality. □

If G is an Abelian ℓ-group and $u > 0$ in G, then the MV-algebra $\langle [0, u], \oplus, {}^*, 0\rangle$ from Lemma 2.5.1 will be denoted by $[0, u]_G$.

EXAMPLE 2.5.2. (1) If $G = \mathsf{R}$, the ℓ-group of the real numbers and $u = 1$, then $[0, u]_G$ is the MV-algebra $[0, 1]$ from Example 2.4.2.

(2) If $G = \mathsf{Z}$ and $u = n$, then $[0, u]_G$ is isomorphic with the MV-algebra $\mathbf{Ł}_{n+1}$ from Example 2.4.2.

EXAMPLE 2.5.3 (${}^*[0, 1]$). Let R be the set of the real numbers, $\mathcal{P}(\mathsf{N})$ the Boolean algebra of all the subsets of N and $\mathcal{F} \subseteq \mathcal{P}(\mathsf{N})$ the ultrafilter which contains all the cofinite subsets (i.e., the sets with finite complements). We denote ${}^*\mathsf{R} := \mathsf{R}^{\mathsf{N}}/\mathcal{F}$, the ultrapower of R in the class of ℓ-groups. The elements of ${}^*\mathsf{R}$ are called *nonstandard reals*. We will briefly describe the structure of ${}^*\mathsf{R}$. If f, $g\colon \mathsf{N} \to \mathsf{R}$ are two elements from R^{N} we define $f \sim g$ iff $\{n \in \mathsf{N} \mid f(n) = g(n)\} \in \mathcal{F}$. One can easily prove that $\sim$ is an equivalence, so we consider ${}^*\mathsf{R} = \mathsf{R}^{\mathsf{N}}/\mathcal{F} = \{[f] \mid f \in \mathsf{R}^{\mathsf{N}}\}$ the set of all the equivalence classes with respect to $\sim$. If we define $[f] + [g] := [f + g]$ and $[f] \leq [g]$ iff $\{n \in \mathsf{N} \mid f(n) \leq g(n)\} \in \mathcal{F}$ then ${}^*\mathsf{R}$ becomes an ℓ-group. Moreover, since $\mathcal{F}$ is an ultrafilter, ${}^*\mathsf{R}$ is linearly ordered. A *real* element of ${}^*\mathsf{R}$ is an element of the form $[r]$ where r is a constant function R^{N}. An *infinitesimal* is an element $\tau \in {}^*\mathsf{R}$ such that $|\tau| \leq [1/n]$ for any $n \in \mathsf{N}$, where $|\tau| = \max\{\tau, -\tau\}$ is the absolute value of τ. For example, if $t \mid \mathsf{N} \to \mathsf{R}$ is defined by $t(0) = 0$ and $t(n) = 1/n$ for $n > 0$, then $\tau = [t]$ is an infinitesimal in ${}^*\mathsf{R}$. Results from nonstandard analysis shows that any nonstandard real has one of the forms $[r] + \tau$ or $[r] - \tau$ where $[r]$ is a real and τ is an infinitesimal. Now, we consider the interval ${}^*[0, 1] = \{[f] \in {}^*\mathsf{R} \mid [0] \leq [f] \leq [1]\}$ and we define the operations

$$[f] \oplus [g] := \max\{[f] + [g], [1]\} \quad \text{and} \quad [f]^* := [1] - [f]$$

for any $[f]$, $[g] \in {}^*[0, 1]$. As in Example 2.4.2, $\langle {}^*[0, 1], \oplus, {}^*, [0]\rangle$ is an MV-algebra.

A standard construction in ℓ-group theory is the *lexicographic product:* if G_1 is a totally ordered group and G_2 is an ℓ-group, then their lexicographic product is the ℓ-group $G_1 \times_{lex} G_2$, whose support set is $G_1 \times G_2$, the group operations are defined on components but the order relation is lexicographic:

$$\langle x_1, x_2\rangle \leq \langle y_1, y_2\rangle \quad \text{iff} \quad x_1 < y_1 \quad \text{or} \quad x_1 = y_1 \text{ and } x_2 \leq y_2,$$

for any $x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$.

EXAMPLE 2.5.4 (Komori chains $\boldsymbol{K}_n$). If $G = Z \times_{lex} Z$, then the MV-algebras $\boldsymbol{K}_{n+1} = [\langle 0,0\rangle, \langle n,0\rangle]_G$ where $n > 0$ are called *Komori chains*. These structures were introduced by Komori [43] and they are used for characterizing the equational classes of MV-algebras (Section 7). Note that $\boldsymbol{K}_2 = [\langle 0,0\rangle, \langle 1,0\rangle]_G$ is isomorphic with Chang's algebra $\boldsymbol{C}$ from Example 2.4.5.

2.6 The distance function

In an MV-algebra $\langle A, \oplus, ^*, 0\rangle$ we define *the distance function* $d\colon A \times A \to A$ by

$$d(a,b) := (a \odot b^*) \oplus (b \odot a^*).$$

PROPOSITION 2.6.1. *For any a, b, x, $y \in A$ the following properties hold:*

(d1) $d(a,b) = a \odot b^* \vee b \odot a^*$,
(d2) $d(a,b) = 0$ *iff* $a = b$,
(d3) $d(a,0) = a$,
(d4) $d(a,1) = a^*$,
(d5) $d(a^*,b^*) = d(a,b)$,
(d6) $d(a,b) = d(b,a)$,
(d7) $d(a,c) \leq d(a,b) \oplus d(b,c)$,
(d8) $d(a \oplus c, b \oplus e) \leq d(a,b) \oplus d(c,e)$,
(d9) $d(a \odot c, b \odot e) \leq d(a,b) \oplus d(c,e)$.

Proof. (d1) Follows by Propositions 2.2.5 (d) and 2.2.11 (a).

(d2) If $a = b$, then it is obvious that $d(a,b) = 0$. Conversely, if $d(a,b) = 0$, then $a \odot b^* = b \odot a^* = 0$. We get $a \leq b$ and $b \leq a$, so $a = b$.

(d3), (d4), (d5), (d6) Follows by easy computations.

(d7) Using Proposition 2.2.4 (e) we have $d(a,c) = a \odot c^* \oplus c \odot a^* \leq (a \odot b^* \oplus b \odot c^*) \oplus (c \odot b^* \oplus b \odot a^*) = (a \odot b^* \oplus b \odot a^*) \oplus (b \odot c^* \oplus c \odot b^*) = d(a,b) \oplus d(b,c)$.

(d8) We first prove that inequality $(a \oplus c)^* \odot (b \oplus e) \leq b \odot a^* \oplus e \odot c^*$. We have $((a \oplus c)^* \odot (b \oplus e))^* \oplus b \odot a^* \oplus e \odot c^* = a \oplus c \oplus b^* \odot e^* \oplus b \odot a^* \oplus e \odot c^* = (a \oplus b \odot a^*) \oplus (c \oplus e \odot c^*) \oplus b^* \odot e^* = (b \oplus a \odot b^*) \oplus (c \vee e) \oplus (b^* \odot e^*) = (b \oplus b^* \odot e^*) \oplus (c \vee e) \oplus a \odot b^* = (b \vee e^*) \oplus (c \vee e) \oplus a \odot b^* \geq e^* \oplus e = 1$. The inequality then follows by Proposition 2.2.1 (a). Now we prove (d8) using the above inequality twice: $d(a \oplus c, b \oplus e) = (a \oplus c)^* \odot (b \oplus e) \oplus (b \oplus e)^* \odot (a \oplus c) \leq (b \odot a^* \oplus e \odot c^*) \oplus (a \odot b^* \oplus c \odot e^*) = (b \odot a^* \oplus a \odot b^*) \oplus (e \odot c^* \oplus c \odot e^*) = d(a,b) \oplus d(c,e)$.

(d9) Follows by (d5) and (d8): $d(a \odot c, b \odot e) = d(a^* \oplus c^*, b^* \oplus e^*) \leq d(a^*,b^*) \oplus d(c^*,e^*) = d(a,b) \oplus d(c,e)$. $\square$

EXAMPLE 2.6.2. (1) If A is a Boolean algebra, then the distance function is $d(a,b) = (a \wedge b^*) \vee (b \wedge a^*) = (a \leftrightarrow b)^*$.

(2) In the MV-algebras $[0,1]$, $\mathsf{Z} \cap [0,1]$ and $\mathbf{Ł}_n$ from Example 2.4.2 the distance function is $d(a,b) = |a-b|$, where $|r|$ denotes the absolute value of r for any real number r.

(3) In Chang's MV-algebra $\boldsymbol{C}$ the distance function is given by

$$d(nc, mc) = d(1-nc, 1-mc) = |n-m|c,$$
$$d(nc, 1-mc) = 1-(n+m)c \text{ for any } n, m \in \mathsf{N},$$

where $|n-m|$ is the absolute value of $n-m$.

(4) In the MV-algebra Łfrom Example 2.4.7, the distance function is $d([\varphi],[\psi]) = (\varphi \to \psi) \to \neg(\psi \to \varphi)$.

(5) If A is an MV-algebra and $a \in A$ such that $0 < a$, then the distance function in the interval algebra $A(0,a)$ is

$$\begin{aligned} d_{[0,a]}(x,y) &= (x^{*[0,a]} \odot_{[0,a]} y) \vee_{[0,a]} (y^{*[0,a]} \odot_{[0,a]} x) \\ &= (x^* \odot y) \vee (y^* \odot x) = d(x,y), \end{aligned}$$

where $x, y \in [0,a]$ and $d(x,y)$ is the distance function in A.

2.7 Ideals, filters, and homomorphisms

In the sequel A is an MV-algebra.

DEFINITION 2.7.1. *A nonempty set $I \subseteq A$ is an* ideal *if the following properties are satisfied:*

(I1) $a \leq b$ *and* $b \in I$ *implies* $a \in I$,

(I2) $a, b \in I$ *implies* $a \oplus b \in I$.

We will denote by $Id(A)$ the set of all the ideals of A. An ideal is proper *if it does not coincide with the entire algebra. An ideal I is* closed *if $X \subseteq I$ implies $\bigvee\{x \mid x \in X\} \in I$ whenever $\bigvee\{x \mid x \in X\}$ exists in A.*

FACT 2.7.2. *The following properties are straightforward:*

(i1) $\{0\}$ *and A are ideals,*

(i2) $0 \in I$ *for any ideal I of A,*

(i3) *an ideal I is proper iff $1 \notin I$,*

(i4) *if I is a proper ideal and $a \in I$, then $ord(a) = \infty$,*

(i5) *if I is an ideal and $a, b \in I$, then $a \wedge b$, $a \odot b$, $a \vee b$ and $a \oplus b \in I$,*

(i6) *an ideal of A is also an ideal of $L(A)$,*

(i7) *$Id(A)$ is partially ordered by set theoretical inclusion,*

(i8) *if $ord(x) = \infty$, then there exists a proper ideal I such that $x \in I$.*

The next result can be easily proved.

PROPOSITION 2.7.3. *Let F be a nonempty subset of A. Then the following conditions are equivalent:*

(a) $1 \in F$ *and for any* a, $b \in A$ *if* $a \in F$ *and* $a \to b \in F$, *then* $b \in F$,

(b) F *satisfies the conditions below (b1) and (b2):*

(b1) *for any* a, $b \in A$ *if* $a \in F$ *and* $a \leq b$, *then* $b \in F$,
(b2) *if* $a \in F$ *and* $b \in F$, *then* $a \odot b \in F$,

(c) *the set* $F^* = \{a^* \mid a \in F\}$ *is an ideal of* A.

DEFINITION 2.7.4. *A nonempty subset F of A is a* filter *if it satisfies one of the equivalent conditions from Proposition 2.7.3.*

LEMMA 2.7.5. *Let A be an* MV*-algebra and I an ideal of A. Then the* MV*-subalgebra generated by I in A is* $\langle I \rangle = I \cup I^*$.

Proof. $I \cup I^*$ is obviously closed to the unary operation * and $0 \in I$. We only have to prove that $I \cup I^*$ is also closed to $\oplus$. Since I is an ideal and I^* is a filter of A, both I and I^* are closed to $\oplus$. If $x \in I$ and $y \in I^*$, then $x \oplus y \geq y$, so $x \oplus y \in I^*$. Hence $I \cup I^*$ is an MV-subalgebra of A and it is obviously generated by I. $\square$

DEFINITION 2.7.6. *If A and B are two* MV*-algebras, then a* homomorphism *is a function* $f\colon A \to B$ *which satisfies the following conditions:*

(M1) $f(0) = 0$,

(M2) $f(a \oplus b) = f(a) \oplus f(b)$ *for any* a, $b \in A$,

(M3) $f(a^*) = f(a)^*$ *for any* $a \in A$.

An injective homomorphism is called embedding. *We say that the* MV*-algebra A is* embedded *in the* MV*-algebra B if there is an embedding* $f\colon A \to B$. *A homomorphism which is also a bijective function is called* isomorphism. *We say that the* MV*-algebras A and B are* isomorphic *if there is an isomorphism* $f\colon A \to B$. *We denote* $A \simeq B$ *whenever A and B are isomorphic* MV*-algebras.*

FACT 2.7.7. *Let* $f\colon A \to B$ *be an* MV*-algebra homomorphism. One can immediately prove that:*

(a) $f(1) = 1$,

(b) $f(a \odot b) = f(a) \odot f(b)$,

(c) $f(a \vee b) = f(a) \vee f(b)$,

(d) $f(a \wedge b) = f(a) \wedge f(b)$,

(e) $f(a \to b) = f(a) \to f(b)$,

for any a, $b \in A$. *Thus, f is also a lattices homomorphism from* $L(A)$ *to* $L(B)$. *In particular, f is an increasing function.*

EXAMPLE 2.7.8. Let ${}^*[0,1]$ be the MV-algebra from Example 2.5.3 and $\tau \in {}^*[0,1]$ an infinitesimal. One can easily prove that the set $\{n\tau \mid n \in \mathsf{N}\} \cup \{[1] - n\tau \mid n \in \mathsf{N}\}$ is an MV-subalgebra of ${}^*[0,1]$ which is isomorphic to Chang's MV-algebra $\boldsymbol{C}$ from Example 2.4.5.

FACT 2.7.9. *If $f\colon A \to B$ is an* MV*-algebra homomorphism then, by Remark 2.7.7 (e), f is also a Wajsberg algebra homomorphism between W_A and W_B. Moreover, a Wajsberg algebras homomorphism is also an* MV*-algebra homomorphism between the corresponding* MV*-algebras structures. These facts, together with Corollary 2.3.11, assert that there is a categorical equivalence between the category of* MV*-algebras and the category of Wajsberg algebras.*

If $f\colon A \to B$ is an MV-algebra homomorphism, then the *kernel* of f is $ker(f) = f^{-1}(0) = \{a \in A \mid f(a) = 0\}$.

PROPOSITION 2.7.10. *If $f\colon A \to B$ is an* MV*-algebra homomorphism, then the following assertions hold:*

- *(a)* $ker(f)$ *is an ideal of A,*
- *(b)* *f is injective iff* $ker(f) = \{0\}$,
- *(c)* *if $J \subseteq B$ is an ideal, then $f^{-1}(J)$ is an ideal of A and $ker(f) \subseteq f^{-1}(J)$,*
- *(d)* *if f is surjective and $I \subseteq A$ is an ideal such that $ker(f) \subseteq I$, then $f(I)$ is an ideal of B.*

Proof. (a) Since f is a homomorphism we have $f(0) = 0$, so $0 \in ker(f)$. If $a, b \in A$ such that $a \leq b$ and $b \in ker(f)$, then $f(a) \leq f(b)$ and $f(b) = 0$. We get $f(a) = 0$, so $a \in ker(f)$. If a, $b \in ker(f)$, then $f(a \oplus b) = f(a) \oplus f(b) = 0 \oplus 0 = 0$, so $a \oplus b \in ker(f)$. Hence, $ker(f)$ is an ideal.
(b) We suppose that f is injective and let $a \in ker(f)$. Then $f(a) = f(0) = 0$, so $a = 0$. Conversely, let $ker(f) = \{0\}$ and $a, b \in A$ such that $f(a) = f(b)$. It follows that $f(d(a,b)) = d(f(a), f(b)) = 0$. Since $ker(f) = \{0\}$ we get $d(a,b) = 0$, so $a = b$. Thus, f is an injective homomorphism.
(c) If J is an ideal of B, then $0 \in J$. Hence, $f(a) = 0 \in J$ for any $a \in ker(f)$, so $ker(f) \subseteq J$. Let a_1, $a_2 \in A$ such that $a_1 \leq a_2$ and $a_2 \in f^{-1}(J)$. It follows that $f(a_1) \leq f(a_2)$ and $f(a_2) \in f(f^{-1}(J)) \subseteq J$, so $f(a_2) \in J$. Thus, $a_2 \in f^{-1}(J)$. Similarly, if a_1, $a_2 \in f^{-1}(J)$, then $f(a_1)$, $f(a_2) \in f(f^{-1}(J)) \subseteq J$, so $f(a_1 \oplus a_2) = f(a_1) \oplus f(a_2) \in J$ and $a_1 \oplus a_2 \in f^{-1}(J)$. We proved that $f^{-1}(J)$ is an ideal of A.
(d) Since $0 \in I$, we get $0 = f(0) \in f(I)$, so $f(I)$ is not empty. Let $b_1 \leq b_2 \in B$ and $b_2 \in f(I)$. It follows that there is $a_2 \in I$ such that $f(a_2) = b_2$. Because f is surjective, we also find an element $a_1 \in A$ such that $f(a_1) = b_1$. Thus, $f(a_1) \leq f(a_2)$ which implies that $f(a_1 \odot a_2^*) = f(a_1) \odot f(a_2)^* = 0$. Hence $a_1 \odot a_2^* \in Ker(f) \subseteq I$ and $a_2 \in I$, so $a_2 \oplus a_1 \odot a_2^* = a_2 \vee a_1$ is in I. It follows that $a_1 \in I$, which means that $b_1 = f(a_1) \in f(I)$. Moreover, if $f(a_1)$, $f(a_2)$ are in $f(I)$ for some a_1, $a_2 \in I$, then $a_1 \oplus a_2 \in I$, so $f(a_1) \oplus f(a_2) = f(a_1 \oplus a_2)$ is also in $f(I)$. We proved that $f(I)$ is an ideal of B. $\square$

COROLLARY 2.7.11. *If $f\colon A \to B$ is a surjective* MV*-algebra homomorphism, then there is a bijective correspondence between $\{I \mid I \in Id(A),\, ker(f) \subseteq I\}$ and $Id(B)$.*

Proof. By Proposition 2.7.10 (c) and (d), the desired correspondence is given by

$$J \mapsto f^{-1}(J) \quad \text{and} \quad I \mapsto f(I),$$

for any $J \in Id(B)$ and $I \in Id(A)$ such that $ker(f) \subseteq I$. We have to prove that $f(f^{-1}(J)) = J$ and $f^{-1}(f(I)) = I$. The first relation is satisfied because f is surjective. Since $I \subseteq f^{-1}(f(I))$ always holds, we only have to prove the converse inclusion. If $a \in f^{-1}(f(I))$, then $f(a) \in f(I)$, so there is $x \in I$ such that $f(a) = f(x)$. It follows that $f(a \odot x^*) = f(a) \odot f(x)^* = 0$, so $a \odot x^* \in ker(f) \subseteq I$. Hence $x \in I$ and $a \odot x^* \in I$. We get $a \vee x = x \oplus a \odot x^* \in I$, which implies that $a \in I$. Our proof is now complete. □

LEMMA 2.7.12. *If A is an* MV*-algebra, $f\colon A \to A$ an* MV*-algebra homomorphism and $a \in A$ such that $a = a^*$, then $f(a) = a$.*

Proof. Since f is a homomorphism, we have $f(a)^* = f(a^*) = f(a)$. Thus, a and $f(a)$ are elements of A such that $a = a^*$ and $f(a) = f(a)^*$. Using Proposition 2.1.4 we infer that $f(a) = a$. □

EXAMPLE 2.7.13 (The ideals and the homomorphisms of $[0,1]$). Let $[0,1]$ be the MV-algebra from Example 2.4.2 and $I \subseteq [0,1]$ and ideal. Suppose that there is $a \in I$ such that $a \neq 0$. It follows that there is $n \in \mathsf{N}$ such that $a + \overset{n \text{ times}}{\cdots} + a \geq 1$, where $+$ denotes the real numbers addition. We get $na = 1$, so $ord(a) < \infty$. Since I is an ideal, then $na = 1 \in I$ and $I = [0,1]$. We conclude that $Id([0,1]) = \{\{0\}, [0,1]\}$. Now, let $f\colon [0,1] \to [0,1]$ be an MV-algebra homomorphism. By Proposition 2.7.10 (a), $ker(f)$ is a proper ideal of $[0,1]$, so $ker(f) = \{0\}$. Thus, by Proposition 2.7.10 (b), f is injective. Since $1/2^* = 1 - 1/2 = 1/2$ we get $f(1/2) = 1/2$ by Lemma 2.7.12. Let $m > 2$ be an even natural number and let $k = m/2$. We have $k(1/m) = 1/2$, so $kf(1/m) = f(1/2) = 1/2$, since f is a homomorphism. Recall that $a \oplus b = \min\{a + b, 1\}$ for any a, $b \in [0,1]$, i.e. $a \oplus b = a + b$ if $a + b \leq 1$ and $a \oplus b = 1$ otherwise. Hence, $kf(1/m) = 1/2$, so $f(1/m) = (1/2)/k = (1/2)/(m/2) = 1/m$. If $m > 2$ is an odd natural number, then $1/m = (1/2m) \oplus (1/2m)$, so $f(1/m) = f(1/2m) \oplus f(1/2m) = (1/2m) \oplus (1/2m) = 1/m$. We proved that $f(1/m) = 1/m$ for any $m \in \mathsf{N}$ such that $m \neq 0$. Consider n, $m \in \mathsf{N}$ such that $m \neq 0$ and $n \leq m$. It follows that $f(n/m) = f(n(1/m)) = nf(1/m) = n(1/m) = n/m$. Thus for any rational number $r \in [0,1]$ we get $f(r) = r$. Let $a \in [0,1]$ be an arbitrary real number. We know that there are two sequences of rational numbers $(r_n)_{n \in \mathsf{N}} \subseteq [0,1]$ and $(s_n)_{n \in \mathsf{N}} \subseteq [0,1]$ such that $r_n \leq a \leq s_n$, $r_n \nearrow a$ and $s_n \swarrow a$. Since f is increasing, we get $r_n = f(r_n) \leq f(a) \leq f(s_n) = s_n$, so $lim_{n \to \infty} r_n \leq f(a) \leq lim_{n \to \infty} s_n$. Thus, $a \leq f(a) \leq a$ and $f(a) = a$. We proved that $f(a) = a$ for any $a \in [0,1]$. In conclusion, the only MV-algebra homomorphism $f\colon [0,1] \to [0,1]$ is the identity. A similar conclusion can be obtained if we consider the MV-algebra $\mathsf{Z} \cap [0,1]$ or the MV-algebras $\mathbf{Ł}_n$ with $n \geq 2$.

2.8 The Boolean center

Let A be an MV-algebra.

PROPOSITION 2.8.1. *If $a \in A$, then the following are equivalent:*

(a) $a \oplus a = a$,

(b) $a \wedge a^* = 0$,

(c) $a \vee a^* = 1$,

(d) $a \odot a = a$.

Proof. (a) ⇒ (b) $\quad a \wedge a^* = a^* \odot (a^{**} \oplus a) = a^* \odot (a \oplus a) = a^* \odot a = 0$.
(b) ⇒ (c) $\quad 1 = 0^* = a \wedge a^{**} = a^* \vee a$.
(c) ⇒ (d) $\quad a = a \odot 1 = a \odot (a \vee a^*) = (a \odot a) \vee (a \odot a^*) = (a \odot a) \vee 0 = a \odot a$.
(d) ⇒ (a) By hypothesis, $a^* \oplus a^* = a^*$. It follows that $a = a \oplus 0 = a \oplus a \odot a^* = a \oplus a \odot (a^* \oplus a^*) = a \oplus (a \wedge a^*) = (a \oplus a) \wedge (a \oplus a^*) = (a \oplus a) \wedge 1 = a \oplus a$. □

DEFINITION 2.8.2. *We denote $B(A) = \{a \in A \mid a \oplus a = a\}$. By Proposition 2.8.1, $B(A)$ is the set of all the complemented elements with respect to the lattice structure of A. We will call $B(A)$ the* Boolean center *of A. Since, in any bounded distributive lattice, the sublattice of all the complemented elements is a Boolean algebra, it follows that $\langle B(A), \vee, \wedge, {}^*, 0, 1\rangle$ is a Boolean algebra.*

LEMMA 2.8.3. *If $a \in B(A)$, then $a \oplus b = a \vee b$ and $a \odot b = a \wedge b$ for any $b \in A$.*

Proof. We will prove that $a \oplus b \leq a \vee b$ and $a \wedge b \leq a \odot b$, since the converse inequalities hold by Proposition 2.2.5 (a). We have

$$\begin{aligned}(a \oplus b) \odot (a \vee b)^* &= (a \oplus b) \odot (a^* \wedge b^*) = ((a \oplus b) \odot a^*) \wedge ((a \oplus b) \odot b^*)\\ &= b \wedge a^* \wedge a \wedge b^* = 0,\\ (a \wedge b) \odot (a \odot b)^* &= (a \wedge b) \odot (a^* \oplus b^*) = (a \odot (a^* \oplus b^*)) \wedge (b \odot (a^* \oplus b^*))\\ &= a \wedge b^* \wedge b \wedge a^* = 0.\end{aligned}$$

The desired inequalities follows by Proposition 2.2.1 (b). □

FACT 2.8.4. *If $a \in B(A)$, then in the interval MV-algebra $A(0, a)$ the MV-algebra operations are defined by:*

$$x \oplus_{[0,a]} y = x \oplus y, \quad x \odot_{[0,a]} y = x \odot y, \quad \textit{and} \quad x^{*[0,a]} = a \wedge x^*,$$

for any x and $y \in A(0, a)$.

LEMMA 2.8.5. *If $a \in B(A)$, then the function $f\colon A \to A(0, a)$ defined as $f(x) = x \wedge a$ is an MV-algebra homomorphism.*

Proof. We have $f(0) = 0 \wedge a = 0$. If $x, y \in A$, then $f(x^*) = x^* \wedge a = x^{*[0,a]}$ and $f(x \oplus y) = (x \oplus y) \wedge a = (x \oplus y) \oplus a = (x \oplus y) \oplus a \oplus a = (x \oplus a) \oplus (y \oplus a) = f(x) \oplus f(y) = f(x) \oplus_{[0,a]} f(y)$, so f is an MV-algebra homomorphism. □

PROPOSITION 2.8.6. *Let* $n \in \mathsf{N}$ *and* $a_1, \ldots, a_n \in B(A)$ *such that* $a_i \wedge a_j = 0$ *for any* $i \neq j$ *and* $a_1 \vee \cdots \vee a_n = 1$. *Then* A *is isomorphic with the direct product of the family* $\{A(0, a_i) \mid i \in \{1, \ldots, n\}\}$ *via the isomorphism* f *defined as*

$$f(x) = \langle x \wedge a_1, \ldots, x \wedge a_n \rangle.$$

Proof. We recall that the operations of the direct product are defined on components. The function f is an MV-algebra homomorphism since, by Lemma 2.8.5, the morphism conditions are satisfied on each component. If $x, y \in A$ such that $f(x) = f(y)$, then $x \wedge a_i = y \wedge a_i$ for any $i \in \{1, \ldots, n\}$. We get $x = x \wedge 1 = x \wedge (a_1 \vee \cdots \vee a_n) = (x \wedge a_1) \vee \cdots \vee (x \wedge a_n) = (y \wedge a_1) \vee \cdots \vee (y \wedge a_n) = y \wedge (a_1 \vee \cdots \vee a_n) = y$, so f is injective. In order to prove the surjectivity, we consider $\langle x_1, \ldots, x_n \rangle \in \prod\{A(0, a_i) \mid i \in \{1, \ldots, n\}\}$. Note that $x_i \wedge a_i = x_i$ for any $i \in \{1, \ldots, n\}$ and $x_i \wedge a_j \leq a_i \wedge a_j = 0$, so $x_i \wedge a_j = 0$ for any $i \neq j$. If $x = x_1 \vee \cdots \vee x_n$, then $x \wedge a_i = (x_1 \wedge a_i) \vee \cdots \vee (x_n \wedge a_n) = x_i$. Thus, $f(x) = \langle x_1, \ldots, x_n \rangle$ and f is surjective. We have proved that f is an MV-algebra isomorphism. $\square$

An MV-algebra A is *indecomposable* if A is nontrivial and $A \simeq A_1 \times A_2$ implies either A_1 or A_2 is trivial.

PROPOSITION 2.8.7. *An* MV*-algebra* A *is indecomposable iff* $B(A) = \{0, 1\}$.

Proof. Let A be an indecomposable MV-algebra and suppose that there is $a \in B(A) \setminus \{0, 1\}$. We have $a \vee a^* = 1$ and $a \wedge a^* = 0$ so, by Proposition 2.8.6, A is isomorphic to the direct product $A(0, a) \times A(0, a^*)$. Since $a \neq 0$ and $a^* \neq 0$, neither $A(a)$ or $A(a^*)$ are trivial so we get a contradiction with the hypothesis that A is indecomposable. Conversely, let A be an MV-algebra such that $B(A) = \{0, 1\}$ and suppose that $A \simeq A_1 \times A_2$ where A_1 and A_2 are nontrivial MV-algebras. It follows that the element $\langle 0, 1 \rangle \in A_1 \times A_2$ is in the Boolean center. Thus, there is a Boolean element in $B(A) \setminus \{0, 1\}$, which is a contradiction. We proved that A is a indecomposable MV-algebra. $\square$

EXAMPLE 2.8.8. The MV-algebra $[0, 1]$ and Chang's MV-algebra $\boldsymbol{C}$ are indecomposable.

2.9 The Riesz decomposition property

Let $\langle A, \oplus, {}^*, 0 \rangle$ be an MV-algebra. We will define a partial binary operation $+$ on A as follows:

for any $a, b \in A$, $a + b$ is defined iff $a \leq b^*$ and, in this case, $a + b := a \oplus b$.

One can easily see that $a \leq b^*$ iff $b \leq a^*$, so $a + b$ is defined iff $b + a$ is defined and, in this case, $a + b = b + a$.

EXAMPLE 2.9.1. If A is one of the MV-algebras from Example 2.4.2 ($[0, 1]$, $\mathsf{Z} \cap [0, 1]$, or $\mathbf{Ł}_n$), then the partial addition $+$ on A is just the real numbers addition.

NOTATION 2.9.2.
For any $a \in A$ we will denote $(0\, a)^+ = 0$. For any $n \geq 1$, if $((n-1)a)^+$ is defined and $((n-1)a)^+ + a$ is defined, then we will denote $(na)^+ = ((n-1)a)^+ + a$.

PROPOSITION 2.9.3. *For any a, b, $c \in A$ the following properties hold:*

(a) $a + 0 = a$,

(b) $a + a^* = 1$,

(c) $a \vee b = a + (a^* \odot b)$,

(d) *if* $a + b$ *and* $(a + b) + c$ *are defined, then* $b + c$ *and* $a + (b + c)$ *are defined and* $(a + b) + c = a + (b + c)$,

(e) *if* $a + b = 1$, *then* $b = a^*$,

(f) *if* $a + b = c$, *then* $b = a^* \odot c$.

Proof. (a) Since $a \leq 1 = 0^*$ we have $a + 0 = a \oplus 0 = a$.
(b) Since $a \leq a = (a^*)^*$ we have $a + a^*$ is defined and $a + a^* = a \oplus a^* = 1$.
(c) Since $a \leq a \oplus b^* = (a^* \odot b)^*$ we know that $a + (a^* \odot b)$ is defined and $a + (a^* \odot b) = a \oplus (a^* \odot b) = a \vee b$.
(d) Since $(a + b) + c$ is defined, we have $a + b \leq c^*$ which is equivalent to $c \leq a^* \odot b^*$. It follows that $b \leq a \oplus b = a + b \leq c^*$, so $b + c$ is defined. Moreover, $b + c = b \oplus c \leq b \oplus (a^* \odot b^*) = b \vee a^*$. Because $a + b$ is defined, we get $b \leq a^*$. Thus, $b + c \leq a^*$ and $a + (b + c)$ is also defined. Finally, we have $(a + b) + c = a + (b + c) = a \oplus b \oplus c$.
(e) If $a + b$ is defined and $a + b = 1$, then $b \leq a^*$ and $a \oplus b = 1$. We get $a^* \leq b$ using Proposition 2.2.1 (a), so $b = a^*$.
(f) If $a + b$ is defined and $a + b = c$, then $b \leq a^*$ and $a \oplus b = c$. It follows that $b = b \wedge a^* = a^* \odot (a \oplus b) = a^* \odot c$. □

LEMMA 2.9.4. *If $a, b \in A$, then $a \leq b$ iff there is $c \in A$ such that $a + c = b$. Moreover, if $a \leq b$, then $c = a^* \odot b$ is the unique element of A with the property that $a + c = b$.*

Proof. If $a + c$ is defined and $a + c = b$, then $a \leq a \oplus c = a + c = b$. Conversely, let us suppose that $a \leq b$ and define $c = a^* \odot b$. It follows that $a \leq a \oplus b^* = c^*$, so $a + c$ is defined and $a + c = a \oplus c = a \vee b = b$. The uniqueness of c follows by Proposition 2.9.3 (f). □

PROPOSITION 2.9.5. *For any a, b, x, $y \in A$, the following cancellation properties hold:*

(a) *if* $a + x = a + y$, *then* $x = y$,

(b) *if* $a + x \leq a + y$, *then* $x \leq y$,

(c) *if* $a \leq x$, $b \leq y$ *and* $a + b = x + y$, *then* $a = x$ *and* $b = y$.

Proof. (a) Since $a + x$ and $a + y$ exist, we get $x \leq a^*$ and $y \leq a^*$. Then $x = x \wedge a^* = a^* \odot (a \oplus x) = a^* \odot (a + x) = a^* \odot (a + y) = a^* \odot (a \oplus y) = a^* \wedge y = y$.
(b) If $a + x \leq a + y$ then, by Lemma 2.9.4, there is $b \in A$ such that $a + x + b = a + y$. Using (a), we get $x + b = y$. Our conclusion follows by Lemma 2.9.4.
(c) By Lemma 2.9.4, there are v, $w \in A$ such that $x = a + v$ and $y = b + w$. It follows that $a + b = x + y = a + b + v + w$ and, by (a), we get $v + w = 0$. Hence $v = w = 0$, which proves that $a = x$ and $b = y$. □

PROPOSITION 2.9.6. *In any* MV*-algebra* A*, the following are equivalent:*

(a) *if* $c \leq a+b$*, then there are* $a_1, b_1 \in A$ *such that* $a_1 \leq a$*,* $b_1 \leq b$ *and* $c = a_1 + b_1$*,*

(b) *if* $x + y = a + b$*, then there are* $z_{11}, z_{12}, z_{21}, z_{22} \in A$ *such that*

$$x = z_{11} + z_{12} \qquad a = z_{11} + z_{21} \qquad y = z_{21} + z_{22} \qquad b = z_{12} + z_{22}.$$

Proof. (a) ⇒ (b) If $x + y = a + b$, then $x \leq a + b$. By hypothesis, we infer that there are $z_{11}, z_{12} \in A$ such that $z_{11} \leq a$, $z_{12} \leq b$ and $x = z_{11} + z_{12}$. We define $z_{21} = {z_{11}}^* \odot a$ and $z_{22} = {z_{12}}^* \odot b$. By Lemma 2.9.4, z_{21} is the unique element of A such that $z_{11} + z_{21} = a$ and z_{22} is the unique element of A such that $z_{12} + z_{22} = b$. Since $x + y = a + b$, it follows that $z_{11} + z_{12} + y = z_{11} + z_{21} + z_{12} + z_{22}$. Using Proposition 2.9.5 (b) we get $y = z_{21} + z_{22}$ and the desired conclusion is proved.
(b) ⇒ (a) If $c \leq a + b$, then, by Lemma 2.9.4, there is $d \in A$ such that $c + d = a + b$. By hypothesis there are $z_{11}, z_{12}, z_{21}, z_{22} \in A$ such that $c = z_{11} + z_{12}$, $a = z_{11} + z_{21}$ and $b = z_{12} + z_{22}$. If we consider $a_1 = z_{11}$ and $b_1 = z_{12}$, then the desired result is straightforward. □

DEFINITION 2.9.7 (Riesz decomposition property). *We say that an* MV*-algebra* A *has the* Riesz decomposition property *if one of the equivalent conditions from Proposition 2.9.6 is satisfied.*

PROPOSITION 2.9.8. *Any* MV*-algebra* A *has the Riesz decomposition property.*

Proof. We will prove that any MV-algebra A satisfies the condition (a) from Proposition 2.9.6. Let $c \leq a + b$ and consider $a_1 = a \wedge (c \odot b^*)$. Then $a_1 \leq a$. Moreover, $a_1 \odot c^* = (a \odot c^*) \wedge (c \odot b^* \odot c^*) = (a \odot c^*) \wedge 0 = 0$, so $a_1 \leq c$. By Lemma 2.9.4, we infer that $a_2 = {a_1}^* \odot c$ is the unique element such that $c = a_1 + a_2$. To complete our proof, we must show that $a_2 \leq b$. Note that $c \leq a+b$ implies $c \odot a^* \odot b^* = 0$. Thus, we get $a_2 \odot b^* = c \odot b^* \odot {a_1}^* = c \odot b^* \odot (a^* \vee (c^* \oplus b)) = (c \odot b^* \odot a^*) \vee ((c \odot b^*) \odot (c^* \oplus b)) = 0 \vee ((c \odot b^*) \odot (c \odot b^*)^*) = 0 \vee 0 = 0$, so $a_2 \leq b$. □

3 Ideals in MV-algebras

Throughout this section, A will be an MV-algebra. We will investigate the set $Id(A)$, of all the ideals of A, as well as *Spec*(A) (the set of all prime ideals of A) and $Max(A)$ (the set of all maximal ideals of A). It is proved that $Id(A)$ is a Brouwerian lattice and that the ideals of A are in bijective correspondence with the congruences of A. Some classical results (the first and the second isomorphism theorem, the Chinese remainder theorem, a subdirect representation theorem, the prime ideal theorem) are proved in the context of MV-algebras. Section 3.3 is concerned with the analysis of *Spec*(A), while Section 3.4 deals with $Max(A)$. We also introduce the *primary* ideals, i.e. those ideals that can be embedded in an unique maximal ideal, and we prove that any prime ideal is primary. The *radical* of an MV-algebra, defined as usual as the intersection of all the maximal ideals, is investigated in Section 3.5, as well as the finite and infinite elements of an MV-algebra. In Section 3.6 we define in classical manner the spectral topology on *Spec*(A) and $Max(A)$.

3.1 The lattice of the ideals of *A*

We recall we have denoted by $Id(A)$ the set of all the ideals of A. In the following we will describe the ideal generated by a given set of elements, we will further analyze the principal ideals of an MV-algebra A and we will prove that $Id(A)$ is a Brouwerian lattice.

LEMMA 3.1.1. *If $\{I_k \mid k \in K\}$ is a family of ideals from A, then the intersection $\bigcap\{I_k \mid k \in K\}$ is also an ideal of A. Hence, $Id(A)$ is closed under arbitrary intersections.*

Proof. We denote $I = \bigcap\{I_k \mid k \in K\}$. Obviously, I is not empty because $0 \in I$. If $a \leq b$ and $b \in I$, then $b \in I_k$ for any $k \in K$. We get $a \in I_k$ for any $k \in K$ and, thus, $a \in I$. Similarly, if $a, b \in I$, then $a, b \in I_k$ for any $k \in K$, so $a \oplus b \in I_k$ for any $k \in K$. It follows $a \oplus b \in I$. We proved that I is an ideal. □

DEFINITION 3.1.2. *Let S be a subset of A. We will denote by $(S\,]$ the* ideal generated *by S, i.e. the smallest ideal that includes S. If $a \in A$, then the ideal generated by $\{a\}$ will be simply denoted $(a]$. An ideal I is called* principal *if there is $a \in A$ such that $I = (a]$. For any ideal I of A we define*

$$W(I) = \{a \in I \mid I = (a]\}.$$

Obviously, an ideal I is principal iff $W(I) \neq \emptyset$.

FACT 3.1.3. *By Lemma 3.1.1, the ideal $(S\,]$ exists for any $S \subseteq A$. Moreover, $(\emptyset] = \{0\}$.*

PROPOSITION 3.1.4. *If S is a nonempty set of A, then*

$$(S\,] = \{a \in A \mid a \leq x_1 \oplus \cdots \oplus x_n \textit{ for some } n \in \mathsf{N} \textit{ and } x_1, \ldots, x_n \in S\}.$$

Proof. If $I = \{a \in A \mid a \leq x_1 \oplus \cdots \oplus x_n$ for some $n \in \mathsf{N}$ and $x_1, \ldots, x_n \in S\}$, then we will prove that I is the smallest ideal containing S. Note that I is not empty because $S \subseteq I$. Let $a \leq b$ and $b \in I$, so there are $n \in \mathsf{N}$ and $x_1, \ldots, x_n \in S$ such that $a \leq b \leq x_1 \oplus \cdots \oplus x_n$. It follows that $a \in I$. If $a, b \in I$, then $a \leq x_1 \oplus \cdots \oplus x_n$ and $b \leq y_1 \oplus \cdots \oplus y_m$ for some $x_1, \ldots, x_n, y_1, \ldots, y_m \in S$. We get $a \oplus b \leq x_1 \oplus \cdots \oplus x_n \oplus y_1 \oplus \cdots \oplus y_m$, so $a \oplus b \in I$. Thus I is an ideal containing S. Let J be another ideal of A that contains S and let a be an arbitrary element from I. Hence $a \leq x_1 \oplus \cdots \oplus x_n$ and $x_1, \ldots, x_n \in S \subseteq J$. Because J is an ideal, it follows that $x_1 \oplus \cdots \oplus x_n \in J$, so $a \in J$ and $I \subseteq J$. We proved that $I = (S\,]$. □

COROLLARY 3.1.5. *The following assertions hold:*

(a) *if $a \in A$, then $(a] = \{x \in A \mid x \leq na$ for some $n \in \mathsf{N}\}$,*

(b) *if $a \in A$, then $(a]$ is proper iff $ord(a) = \infty$,*

(c) $W(A) = \{a \in A \mid ord(a) < \infty\}$,

(d) *if $a \in A$, $I \in Id(A)$ and $a \notin I$, then $(I \cup \{a\}] = \{x \in A \mid x \leq b \oplus na$ for some $n \in \mathsf{N}$ and $b \in I\}$,*

(e) *if $I, J \in Id(A)$, then $(I \cup J\,] = \{x \in A \mid x \leq a \oplus b$ for some $a \in I$ and $b \in J\}$.*

Proof. (a) The proof is straightforward by Proposition 3.1.4.
(b) The ideal $(a]$ is proper iff $1 \notin (a]$ iff $1 \neq na$ for any $n \in \mathsf{N}$ iff $ord(a) = \infty$.
(c) Follows directly from (b).
(d) If $x \in (I \cup \{a\}]$ then, by Proposition 3.1.4, there are $m, n \in \mathsf{N}$ and $x_1, \ldots, x_m \in I$ such that $x \leq x_1 \oplus \cdots \oplus x_m \oplus na$. If we denote $b = x_1 \oplus \cdots \oplus x_m$, then $b \in I$ since I is an ideal. The desired result is now obvious.
(e) Follows by Proposition 3.1.4 and the fact that any ideal is closed under finite sums. $\square$

LEMMA 3.1.6. *If I is an ideal of A, then*

$$W(I) = \{a \in I \mid \textit{for any } b \in I \textit{ there is } n \in \mathsf{N} \textit{ such that } b \leq na\}.$$

Proof. Let $a \in W(I)$ and $b \in I$. It follows that $b \in I = (a]$ so, by Corollary 3.1.5 (a), there is $n \in \mathsf{N}$ such that $b \leq na$. Conversely, suppose $a \in I$ with the property that for any $b \in I$ there is $n \in \mathsf{N}$ such that $b \leq na$. Since $a \in I$, we get $(a] \subseteq I$. If $b \in I$ we know that $b \leq na$ for some natural number n, so $b \in (a]$. Thus $I \subseteq (a]$, so $I = (a]$. $\square$

FACT 3.1.7. *If we denote $I \vee J := (I \cup J\,]$ and $I \wedge J := I \cap J$ for any two ideals I and J, then $\langle Id(A), \vee, \wedge \rangle$ is a bounded lattice in which the first element is $\{0\}$ and the last element is A.*

We recall that a lattice $\langle L, \vee, \wedge \rangle$ is called *Brouwerian* if

$$x \wedge \bigvee \{y_k \mid k \in K\} = \bigvee \{x \wedge y_k \mid k \in K\} \text{ for any } x \in L,$$

whenever $\bigvee \{y_k \mid k \in K\}$ exists in L. Obviously, a Brouwerian lattice is distributive. An element x of L is called *compact* if, whenever $\bigvee S$ exists and $x \leq \bigvee S$ for $S \subseteq L$, then $x \leq \bigvee T$ for some finite $T \subseteq S$. The lattice L is *compactly generated* if every element of L is supremum of compact elements. The lattice L is *algebraic* if it is complete and compactly generated.

PROPOSITION 3.1.8. *$\langle Id(A), \vee, \wedge \rangle$ is a complete Brouwerian algebraic lattice.*

Proof. If $\{I_k \mid k \in K\}$ is a family of ideals from A, then the infimum and the supremum of the family are

$$\bigwedge \{I_k \mid k \in K\} = \cap \{I_k \mid k \in K\} \quad \text{and} \quad \bigvee \{I_k \mid k \in K\} = (\bigcup \{I_k \mid k \in K\}].$$

Thus, $Id(A)$ is a complete lattice. Obviously, the compact elements of $Id(A)$ are the finite generated ideals, i.e. the ideals generated by finite sets. For any ideal I we have $I = \bigvee \{(a] \mid a \in I\}$, so the lattice $Id(A)$ is algebraic. In order to prove that $Id(A)$ is Brouwerian we must show that $J \cap (\bigcup \{I_k \mid k \in K\}] = (\bigcup \{J \cap I_k \mid k \in K\}]$. In fact, we prove that $J \cap (\bigcup \{I_k \mid k \in K\}]$ is the smallest ideal which contains $\bigcup \{J \cap I_k \mid k \in K\}$. Let U be an ideal of A containing $\bigcup \{J \cap I_k \mid k \in K\} = J \cap \bigcup \{I_k \mid k \in K\}$. Thus, $J \subseteq U$ and $\bigcup \{I_k \mid k \in K\} \subseteq U$, so $J \subseteq U$ and $(\bigcup \{I_k \mid k \in K\}] \subseteq u$, so $J \cap (\bigcup \{I_k \mid k \in K\}] \subseteq U$. Our proof is finished. $\square$

3.2 Congruences and quotient MV-algebras

In the general study of the algebraic structures, the notion of *congruence* is fundamental. The congruences are in strong connection with the construction of the quotient structures and many representation theorems are based on the characterization of some congruence classes. Using the distance function of an MV-algebra, we prove that there is a bijective correspondence between the lattice of all the congruences defined on the MV-algebra A and the lattice of all the ideals of A. In this section we also prove some classical results of universal algebra, like the first and the second isomorphism theorem, the Chinese remainder theorem, a subdirect representation theorem. Due to the fact that a congruence uniquely determines an ideal (and vice-versa), all this results are expressed using ideals instead of congruences.

DEFINITION 3.2.1. *An equivalence relation $\sim$ on A is a* congruence *if the following properties are satisfied:*

(C1) *if $a \sim b$, then $a^* \sim b^*$,*

(C2) *if $a \sim b$ and $x \sim y$, then $(a \oplus x) \sim (b \oplus y)$,*

for any a, b, x, $y \in A$.

We will denote by $Con(A)$ the set of all the congruence relations on A. Obviously, $Con(A)$ is partially ordered by set theoretical inclusion.

LEMMA 3.2.2. *For any congruence $\sim$ on A and a, $b \in A$ we have:*

(a) *if $a \sim b$ and $x \sim y$, then $a \odot x \sim b \odot y$, $a \vee x \sim b \vee y$ and $a \wedge x \sim b \wedge y$,*

(b) *if $a \oplus b \sim 0$, then $a \sim 0$,*

(c) *$a \sim b$ iff $d(a, b) \sim 0$.*

Proof. (a) $a \odot x = (a^* \oplus x^*)^* \sim (b^* \oplus y^*)^* = b \odot y$. The desired relations for $\vee$ and $\wedge$ follows similarly.
(b) If $a \oplus b \sim 0$, then $a^* \oplus a \oplus b \sim a^*$, so $1 \sim a^*$. It follows that $a \sim 0$.
(c) If $a \sim b$, then $a \odot b^* \sim 0$ and $b \odot a^* \sim 0$, so $d(a, b) \sim 0$. Conversely, if $d(a, b) \sim 0$, by (b), we get $a \odot b^* \sim 0$ and $a^* \odot b \sim 0$. Thus, $a \vee b = b \oplus a \odot b^* \sim b$ and $a \vee b \sim a \oplus b \odot a^* \sim a$. By transitivity, we infer that $a \sim b$. □

LEMMA 3.2.3. *If I is an ideal, then the relation $\sim_I$ defined by*

$$a \sim_I b \quad \textit{iff} \quad d(a, b) \in I$$

is a congruence on A.

Proof. Firstly we prove that $\sim_I$ is an equivalence on A. The relation $\sim_I$ is obviously symmetric. The reflexivity follows by the fact that $d(a, a) = 0 \in I$ for any $a \in A$. In order to prove the transitivity, we suppose that $a \sim_I b$ and $b \sim_I c$, i.e. $d(a, b)$ and $d(b, c)$ are in I. Thus, $d(a, b) \oplus d(b, c)$ is in I. By Proposition 2.6.1 (d7), we have $d(a, c) \leq d(a, b) \oplus d(b, c)$, so $d(a, c) \in I$ and $a \sim_I c$. Now we have to prove the congruence properties. If $a \sim_I b$, then $d(a, b) = d(a^*, b^*) \in I$, so $a^* \sim_I b^*$. Suppose $a \sim_I b$ and $x \sim_I y$, i.e. $d(a, b)$ and $d(x, y)$ are in I. By Proposition 2.6.1 (d8), $d(a \oplus x, b \oplus y) \leq d(a, b) \oplus d(x, y)$ so $d(a \oplus x, b \oplus y) \in I$. Hence, $a \oplus x \sim_I b \oplus y$ and we have proved that $\sim_I$ is a congruence relation on A. □

LEMMA 3.2.4. *If* $\sim$ *is a congruence on* A*, then the set* $I_{\sim} = \{a \in A \mid a \sim 0\}$ *is an ideal.*

Proof. Because $\sim$ is reflexive we get $0 \in I$, so I is nonempty. If $a \leq b$ and $b \in I$, then $a = a \wedge b \sim a \wedge 0 = 0$, so $a \in I$. If a and b are in I, then $a \oplus b \sim 0 \oplus 0 = 0$, so $a \oplus b \in I$. Hence I is an ideal. □

PROPOSITION 3.2.5. *The partially ordered sets* $Id(A)$ *and* $Con(A)$ *are isomorphic via isomorphism* θ *defined as* $\theta(I) = \sim_I$.

Proof. Let I and J be two ideals such that $\theta I = \theta J$. If $a \in A$ we get $a = d(a, 0) \in I$ iff $a \sim_I 0$ iff $a \sim_J 0$ iff $a = d(a, 0) \in J$, so $I = J$. Thus, θ is injective. The map θ is also surjective since, for any $\sim\, \in Con(A)$, we have $\theta(I_{\sim}) = \sim$. Indeed, for any a and b in A, we have $a\theta(I_{\sim})b$ iff $d(a, b) \in I_{\sim}$ iff $d(a, b) \sim 0$ iff $a \sim b$. We finish our proof showing that $I \subseteq J$ iff $\sim_I \subseteq \sim_J$ for any two ideals I and J. If $I \subseteq J$ and $a \sim_I b$, then $d(a, b) \in I \subseteq J$, so $a \sim_J b$. Conversely, if $\sim_I \subseteq \sim_J$ and $a \in I$, then $a = d(a, 0) \in I$, so $a \sim_I 0$. It follows that $a \sim_J 0$. Thus $a \in J$ and our proof is now complete. □

If I is an ideal of A and $a \in A$ we will denote by $[a]_I$ the congruence class of a with respect to $\sim_I$, i.e. $[a]_I = \{b \in A \mid a \sim_I b\}$. One can easily see that $a \in I$ iff $[a]_I = [0]_I$. We will denote by $A/I = \{[a]_I \mid a \in A\}$ the set of all the congruence classes determined by $\sim_I$. Since $\sim_I$ is a congruence relation, the MV-algebra operations on A/I given by

$$[a]_I \oplus [b]_I := [a \oplus b]_I \quad \text{and} \quad ([a]_I)^* := [a^*]_I,$$

are well defined. Hence, $\langle A/I, \oplus, ^*, [0]_I \rangle$ is an MV-algebra which is called *the quotient of* A *by* I. The function $\pi_I \colon A \to A/I$ defined by $\pi_I(a) = [a]_I$ for any $a \in A$ is a surjective homomorphism, which is called *the canonical projection from* A *to* A/I. One can easily prove that $Ker(\pi_I) = I$.

LEMMA 3.2.6. *If* I *is an ideal of* A *and* $a, b \in A$*, then the following are equivalent:*

(a) $[a]_I = [b]_I$,

(b) $a = (b \oplus x) \odot y^*$ *for some* $x, y \in I$.

Proof. (a) ⇒ (b) We denote $x = a \odot b^*$ and $y = b \odot a^*$. By hypothesis, x and y are in I. Note that $b \oplus x = a \vee b = a \oplus y$, so $(b \oplus x) \odot y^* = a \wedge y^*$. Since $a \leq a \oplus b^* = y^*$, the desired equality follows.
(b) ⇒ (a) $[a]_I = ([b]_I \oplus [x]_I) \odot {[y]_I}^* = ([b]_I \oplus [0]_I) \odot [1]_I = [b]_I$. □

PROPOSITION 3.2.7. *If* I *is an ideal of* A*, then:*

(a) $\pi_I(J)$ *is an ideal of* A/I*, where* J *is an ideal of* A *containing* I*,*

(b) the correspondence $J \mapsto \pi_I(J)$ *is a bijection between the set of the ideals of* A *containing* I *and the set of the ideals of* A/I.

Proof. (a) Follows by Proposition 2.7.10 (d).
(b) Straightforward by Corollary 2.7.11. □

The well-known isomorphism theorems have corresponding versions for MV-algebras. We only enunciate the first and the second isomorphism theorem, since their proof follows directly from the classical ones.

THEOREM 3.2.8 (The first isomorphism theorem). *If A and B are two* MV*-algebras and $h\colon A \to B$ is a homomorphism, then $A/Ker(h)$ and $h(A)$ are isomorphic* MV*-algebras.*

THEOREM 3.2.9 (The second isomorphism theorem). *If A is an* MV*-algebra and I, J are two ideals such that $I \subseteq J$, then $(A/I)/\pi_I(J)$ and A/J are isomorphic* MV*-algebras.*

THEOREM 3.2.10 (The Chinese remainder theorem). *Let $n \geq 1$ and let $I_1, \ldots, I_n$ be ideals of an* MV*-algebra A such that $I_i \vee I_j = A$ for any i, $j \in \{1, \ldots, n\}$ such that $i \neq j$. Then for every $x_1, \ldots, x_n \in A$ there is $x \in A$ such that $[x]_{I_i} = [x_i]_{I_i}$ for any $i \in \{1, \ldots, n\}$.*

Proof. If $n = 1$ the desired result obviously follows, since any two elements are equivalent with respect to $I_1 = A$. We prove the theorem for an arbitrary $n \geq 2$. Let i, $j \in \{1, \ldots, n\}$ such that $i \neq j$. Since $1 \in A = I_i \vee I_j$, by Remark 3.1.7 and Corollary 3.1.5 (e), there are $a_{ij} \in I_i$ and $a_{ji} \in I_j$ such that $a_{ij} \oplus a_{ji} = 1$. Thus, $[a_{ij}]_{I_i} = [0]_{I_i}$ and $[a_{ji}]_{I_i} = [1]_{I_i}$ for any i, $j \in \{1, \ldots, n\}$ such that $i \neq j$. Let $x_1, \ldots, x_n \in A$ and denote

$$y_i = x_i \odot a_{1i} \odot \cdots \odot a_{(i-1)i} \odot a_{(i+1)i} \odot \cdots \odot a_{ni}$$

for any $i \in \{1, \ldots, n\}$. Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. Note that

$$\begin{aligned}
[y_i]_{I_i} &= [x_i]_{I_i} \odot [a_{1i}]_{I_i} \odot \cdots \odot [a_{(i-1)i}]_{I_i} \odot [a_{(i+1)i}]_{I_i} \odot \cdots \odot [a_{ni}]_{I_i} \\
&= [x_i]_{I_i} \odot [1]_{I_i} \odot \cdots \odot [1]_{I_i} \odot [1]_{I_i} \odot \cdots \odot [1]_{I_i} \\
&= [x_i]_{I_i},
\end{aligned}$$

$$\begin{aligned}
[y_i]_{I_j} &= [x_i]_{I_j} \odot [a_{1i}]_{I_j} \odot \cdots \odot [a_{(i-1)i}]_{I_j} \odot [a_{(i+1)i}]_{I_j} \odot \cdots \odot [a_{ni}]_{I_j} \\
&= [x_i]_{I_j} \odot [a_{1i}]_{I_j} \odot \cdots \odot [a_{ji}]_{I_j} \odot \cdots \odot [a_{ni}]_{I_j} \\
&= [x_i]_{I_j} \odot [a_{1i}]_{I_j} \odot \cdots \odot [0]_{I_j} \odot \cdots \odot [a_{ni}]_{I_j} \\
&= [0]_{I_j},
\end{aligned}$$

for any $j \neq i$. Our thesis follows by taking $x = y_1 \oplus \cdots \oplus y_n$. We have

$$[x]_{I_i} = [y_i]_{I_i} = [x_i]_{I_i} \text{ for any } i \in \{1, \ldots, n\}. \qquad \square$$

The following result is a version of a well known theorem due to Birkhoff.

PROPOSITION 3.2.11. *Let A be an* MV*-algebra and $\{I_k \mid k \in K\}$ a family of ideals of A such that $\bigcap\{I_k \mid k \in K\} = \{0\}$. Then A is a subdirect product of the family $\{A/I_k \mid k \in K\}$.*

Proof. It is straightforward to prove that the function

$$f\colon A \to \prod\{A/I_k \mid k \in K\} \quad \text{defined as} \quad f(a) = ([a]_{I_k})_{k\in K}$$

is obviously a homomorphism of MV-algebras. If $f(a) = f(b)$, then $f(d(a,b)) = d(f(a), f(b)) = 0$, so $d(a,b) \in I_k$ for any $k \in K$. By hypothesis, $d(a,b) = 0$ so $a = b$. Thus f is an injective homomorphism. For $k \in K$ we denote by π_k the corresponding projection. We have to prove that $\pi_k \circ f$ is a surjective homomorphism. This is obvious since $(\pi_k \circ f)(a) = \pi_k(f(a)) = [a]_{I_k}$ for every $a \in A$. We conclude that A is a subdirect product of the family $\{A/I_k \mid k \in K\}$. □

COROLLARY 3.2.12. *If A is an* MV*-algebra and $I_1, \ldots, I_n$ are ideals of A such that $I_1 \cap \ldots \cap I_n = \{0\}$ and $I_i \vee I_j = A$ for any $i \neq j$, then A is isomorphic to the direct product $(A/I_1) \times \cdots \times (A/I_n)$.*

Proof. We define $f\colon A \to (A/I_1) \times \cdots \times (A/I_n)$ by $f(a) = \langle [a]_{I_i} \rangle_{i\in\{1,\ldots,n\}}$. Using Proposition 3.2.11, f is an injective homomorphism. We only have to prove that f is surjective. For any element x $= \langle [x_i]_{I_i} \rangle_{i\in\{1,\ldots,n\}} \in I_1 \times \cdots \times I_n$, by Theorem 3.2.10, there is an element $x \in A$ such that $[x]_{I_i} = [x_i]_{I_i}$ for any $i \in \{1, \ldots, n\}$. Thus $f(x) =$ x, so f is surjective. We proved that f is an isomorphism. □

FACT 3.2.13. *Let A be an* MV*-algebra and $F \subseteq A$ a filter in A. We define*

$$a \sim_F b \quad \textit{iff} \quad (a \to b) \wedge (b \to a) \in F,$$

for any a, $b \in F$. One can easily see that $a \sim_F b$ iff $a \to b \in F$ and $b \to a \in F$. By Proposition 2.7.3 (c), $I = F^* = \{a^* \mid a \in F\}$ is an ideal of A. Note that $((a \to b) \wedge (b \to a))^* = d(a^*, b^*)$. It follows that $a \sim_F b$ iff $a \sim_I b$. Thus, the congruence relations corresponding to F and I coincide and, consequently, the quotient MV*-algebras A/F and A/I coincide.*

3.3 Prime ideals

In the structures belonging to the algebra of logic (i.e., algebraic structures that correspond to some logical system), the prime ideals are involved at least in three important matters, in algebra, topology and logic. They are extensively used for proving the algebraic representation theorems, as well as the topological duality results. The duals of the prime ideals (i.e. the prime filters) models the deduction in the corresponding logical system and they are frequently used in the algebraic proofs of the completeness theorems.

We further investigate the prime ideals of an MV-algebra and we will prove two important results: the prime extension property and the prime ideal theorem. Finally, we get a subdirect representation theorem for MV-algebras in terms of prime ideals.

PROPOSITION 3.3.1. *The following properties are equivalent for any ideal P of A:*

(a) *for any a, $b \in A$, $a \odot b^* \in P$ or $a^* \odot b \in P$,*

(b) *for any a, $b \in A$, if $a \wedge b \in P$, then $a \in P$ or $b \in P$,*

(c) *for any I, $J \in Id(A)$, if $I \cap J \subseteq P$, then $I \subseteq P$ or $J \subseteq P$.*

Proof. Let $a, b \in A$.
(a) ⇒ (b) Suppose $a \wedge b \in P$ and $a \odot b^* \in P$. It follows that $c = (a \wedge b) \oplus (a \odot b^*) = (a \oplus a \odot b^*) \wedge (b \vee a) \in P$. Since $a \leq c$, we get $a \in P$. Similarly, if $a^* \odot b \in P$ we infer that $b \in P$.
(b) ⇒ (c) Let I and J be two ideals of A such that $I \cap J \subseteq P$. If we suppose that $I \nsubseteq P$ and $J \nsubseteq P$, then there are $a \in I \setminus P$ and $b \in J \setminus P$. We get $a \wedge b \in I \cap J \subseteq P$ and, by hypothesis, $a \in P$ or $b \in P$ which is a contradiction. Thus, $I \subseteq P$ or $J \subseteq P$.
(c) ⇒ (a) Let a and b be two arbitrary elements of A. If we consider $I = (a^* \odot b]$ and $J = (b^* \odot a]$, then $I \cap J = \{0\}$ by Propositions 2.2.11 (a) and 2.2.12 (a). It follows that $I \cap J \subseteq P$, so $I \subseteq P$ or $J \subseteq P$. Hence, $a^* \odot b \in P$ or $b^* \odot a \in P$. □

DEFINITION 3.3.2. *An ideal of A is* prime *if it is proper and it satisfies one of the equivalent conditions from Proposition 3.3.1. We will denote by $Spec(A)$ the set of all the prime ideals of A.*

FACT 3.3.3. *If I and P are ideals of A such that $I \subseteq P$, then one can easily prove that*

$$P \in Spec(A) \quad \textit{iff} \quad \pi_I(P) \in Spec(A/I).$$

Thus, there is a bijective correspondence between the prime ideals of A containing I and the prime ideals of A/I.

PROPOSITION 3.3.4 (Prime extension property). *If P and I are proper ideals of A such that $P \subseteq I$ and P is prime, then I is also prime.*

Proof. Let $a, b \in A$. Since P is prime it follows that $a \odot b^* \in P$ or $a^* \odot b \in P$. Because $P \subseteq I$ we get $a \odot b^* \in I$ or $a^* \odot b \in I$, so I is a prime ideal of A. □

LEMMA 3.3.5. *If $\{P_t \mid t \in T\} \subseteq Spec(A)$ is totally ordered by inclusion, then $\bigcap\{P_t \mid t \in T\}$ is also a prime ideal.*

Proof. By Lemma 3.1.1, $P := \bigcap\{P_t \mid t \in T\}$ is an ideal of A. Let $a, b \in A$ such that $a \wedge b \in P$ and suppose that $a \notin P$ and $b \notin P$. Hence there are t_1 and $t_2 \in T$ such that $a \notin P_{t_1}$ and $b \notin P_{t_2}$. Since P_{t_1} and P_{t_2} are prime ideals, it follows that $b \in P_{t_1}$ and $a \in P_{t_2}$. The family $P_t \mid t \in T$ is totally ordered so $P_{t_1} \subseteq P_{t_2}$ or $P_{t_2} \subseteq P_{t_1}$. If $P_{t_1} \subseteq P_{t_2}$, then $b \notin P_{t_1}$, which is a contradiction. Similarly, we get a contradiction if $P_{t_2} \subseteq P_{t_1}$. It follows that $a \in P$ or $b \in P$, so P is a prime ideal of A. □

PROPOSITION 3.3.6. *Let I be a prime ideal of A. Then the set*

$$\mathcal{I} = \{J \mid I \subseteq J \textit{ and } J \textit{ is a proper ideal of } A\}$$

is linearly ordered with respect to set-theoretical inclusion.

Proof. We consider $J, K \in \mathcal{I}$ and we suppose that $J \nsubseteq K$ and $K \nsubseteq J$. Thus, there are $a \in J \setminus K$ and $b \in K \setminus J$. Since I is prime, we get $a \odot b^* \in I \subseteq K$ or $a^* \odot b \in I \subseteq J$. It follows that $a \vee b = b \oplus a \odot b^* \in K$ or $a \vee b = a \oplus b \odot a^* \in J$, so $a \in K$ or $b \in J$ which is a contradiction. Thus, $J \subseteq K$ or $K \subseteq J$ and $\mathcal{I}$ is linearly ordered. □

COROLLARY 3.3.7. *$Spec(A)$, partially ordered by set inclusion, is a* root system*, i.e. a partially ordered set in which the upper bounds of any element form a chain.*

THEOREM 3.3.8 (Prime ideal theorem). *Let I be a proper ideal of A and let S be a nonempty $\wedge$-closed subset of A (i.e. if $a, b \in S$, then $a \wedge b \in S$) such that $I \cap S = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap S = \emptyset$.*

Proof. We define $\mathcal{J} = \{J \mid J \text{ is a proper ideal}, I \subseteq J \text{ and } S \cap J = \emptyset\}$. A routine application of Zorn's Lemma shows that $\mathcal{J}$ has a maximal element P. We will prove that P is prime. Let $a, b \in A$ and suppose $a \odot b^* \notin P$ and $a^* \odot b \notin P$. Thus, the ideals $J = (P \cup \{a \odot b^*\}]$ and $K = (P \cup \{a^* \odot b\}]$ are not in $\mathcal{J}$, so there are $c \in J \cap S$ and $d \in K \cap S$. If we denote $u = a \odot b^*$ and $v = a^* \odot b$, then $c \leq p_1 \oplus n_1 u$ and $d \leq p_2 \oplus n_2 v$ for some $p_1, p_2 \in P$ and $n_1, n_2 \in \mathsf{N}$. Note that $nu \wedge nv = 0$ for any $n \in \mathsf{N}$. Let $p = p_1 \vee p_2 \in P$ and $n = \max\{n_1, n_2\}$. It follows $c \wedge d \leq (p \oplus nu) \wedge (p \oplus nv) = p \oplus (nu \wedge nv) = p \in P$, so $c \wedge d \in P$. Since S is $\wedge$-closed we also have $c \wedge d \in S$ which contradicts the fact that $S \cap P = \emptyset$. Thus $a \odot b^* \in P$ or $a^* \odot b \in P$ and P is the desired prime ideal. □

COROLLARY 3.3.9. *Any proper ideal I of A can be extended to a prime ideal.*

Proof. Apply Theorem 3.3.8 for I and $S = \{1\}$. □

COROLLARY 3.3.10. *If $a \in A$, then $ord(a) < \infty$ iff $a \notin P$ for any $P \in Spec(A)$.*

Proof. Assume that $ord(a) < \infty$ and there exists $P \in Spec(A)$ such that $a \in P$. Then for some $n \in \mathsf{N}$, $na = 1 \in P$, so P is not a proper ideal, which is impossible. Conversely, assume that $a \notin P$ for any P in $Spec(A)$ and $ord(a) = \infty$. Hence $(a]$ is a proper ideal and we get a contradiction using Corollary 3.3.9. □

PROPOSITION 3.3.11. *If $a \neq 0$ there is a prime ideal P such that $a \notin P$.*

Proof. Apply Theorem 3.3.8 for $I = \{0\}$ and $S = \{a\}$. □

COROLLARY 3.3.12. *In every* MV*-algebra A the intersection of all the prime ideals of A is $\{0\}$.*

Proof. By Proposition 3.3.11. □

PROPOSITION 3.3.13. *Any* MV*-algebra A is a subdirect product of the family $\{A/P \mid P \in Spec(A)\}$.*

Proof. By Proposition 3.2.11 and Corollary 3.3.12. □

3.4 Maximal ideals

In this section we characterize the maximal ideals of an MV-algebra and we prove that any prime ideal can be embedded in an unique maximal ideal (Proposition 3.4.5).

DEFINITION 3.4.1. *An ideal M of A is* maximal *if it is a maximal element in the partially ordered set of all the proper ideals of A. This means that M is proper and, for any proper ideal I, if $M \subseteq I$, then $M = I$. We will denote by $Max(A)$ the set of all the maximal ideals of A.*

PROPOSITION 3.4.2. *If M is a proper ideal of A, then the following are equivalent:*

(a) M *is maximal,*

(b) *for any $a \in A$ if $a \notin M$, then there is $n \in \mathsf{N}$ such that $(a^*)^n \in M$.*

Proof. (a) ⇒ (b) Let a be an element of A such that $a \notin M$. Since M is maximal, the ideal $(M \cup \{a\}]$ coincide with A. Hence, there is $b \in M$ and $n \in \mathsf{N}$ such that $1 = b \oplus na$, so $(a^*)^n = (na)^* \leq b$. We get $(a^*)^n \in M$.
(b) ⇒ (a) Suppose that I is an ideal of A such that $M \subseteq I$ and $M \neq I$. Then there exists an element $a \in I \setminus M$. By hypothesis, $(a^*)^n \in M$ for some $n \in \mathsf{N}$. It follows that $na \in I$ and $(na)^* = (a^*)^n \in M \subseteq I$, so $I = A$. □

FACT 3.4.3. *If I and M are ideals of A such that $I \subseteq M$, then one can easily prove:*

$$M \in Max(A) \quad \textit{iff} \quad \pi_I(M) \in Max(A/I).$$

Thus, there is a bijective correspondence between the maximal ideals of A containing I and the maximal ideals of A/I.

LEMMA 3.4.4. *Any maximal ideal of an* MV*-algebra is a prime ideal.*

Proof. Let M be a maximal ideal of A. Because M is a proper ideal of A, there is a prime ideal P of A such that $M \subseteq P$. Since P is proper, it follows that $M = P$. Hence, M is prime. □

PROPOSITION 3.4.5. *Any proper ideal of A can be extended to a maximal ideal. Moreover, for any prime ideal of A there is a unique maximal ideal containing it.*

Proof. Let I be a proper ideal of A. By Corollary 3.3.9, there is a prime ideal P of A such that $I \subseteq P$. Let $\mathcal{P} = \{J \mid P \subseteq J \text{ and } J \text{ is a proper ideal of} A\}$. By Proposition 3.3.6, $\mathcal{P}$ is linearly ordered, so $U = \bigcup\{J \mid J \in \mathcal{P}\}$ is a proper ideal and $I \subseteq U$. We will prove that U is a maximal ideal of A. Let H be a proper ideal of A such that $U \subseteq H$. It follows that $H \in \mathcal{P}$, hence $H \subseteq U$. We get $H = U$, so U is a maximal ideal which includes I. Let P be a prime ideal and suppose that there are two maximal ideals M and U such that $P \subseteq M$ and $P \subseteq U$. It follows that $M, U \in \mathcal{P}$ which is a linearly ordered set, so $M \subseteq U$ or $U \subseteq M$. Since both U and M are maximal ideal we infer that $U = M$. Thus, any prime ideal is contained in a unique maximal ideal. □

3.5 The radical

In the following, the radical $Rad(A)$ of the MV-algebra A is defined as the intersection of its maximal ideals. This notion is found in algebra especially in ring theory. Since the MV-algebra operation $\oplus$ is cancellative on $Rad(A)$, it coincides with the partial operation $+$ from Section 2.9. The nonzero elements of $Rad(A)$ are called *infinitesimals*, since they are characterized by a property that defines the infinitesimals in nonstandard analysis (see Example 2.5.3).

DEFINITION 3.5.1. *The intersection of the maximal ideals of A is called the* radical *of A. It will be denoted by $Rad(A)$. It is obvious that $Rad(A)$ is an ideal, since an intersection of ideals is also an ideal.*

LEMMA 3.5.2. *For any $a, b \in Rad(A)$, the following identities hold:*

(a) $a \odot b = 0$,

(b) $a \oplus b = a + b$,

(c) $a \leq b^*$.

Proof. (a) Let a, $b \in Rad(A)$ and suppose $a \odot b \neq 0$. By Proposition 3.3.11 there is a prime ideal P such that $a \odot b \notin P$. Since P is prime it follows that $a^* \odot b^* \in P$. By Proposition 3.4.5 there is a maximal ideal M such that $P \subseteq M$. Thus $a \oplus b = (a^* \odot b^*)^* \notin M$ so $(M \cup \{a \oplus b\}] = A$ and $1 = x \oplus n(a \oplus b)$ for some $x \in M$ and $n \in \mathsf{N}$. We get $(n(a \oplus b))^* \leq x$. If we denote $c = n(a \oplus b)$, then we infer that c and c^* are in M which contradicts the fact that M is a proper ideal.
(b) By (a), $a \leq b^*$, so $a + b$ is defined and $a + b = a \oplus b$.
(c) Equivalent with (a) by Proposition 2.2.1. □

COROLLARY 3.5.3. $\langle Rad(A), \oplus, 0\rangle$ *is a naturally ordered cancellative monoid.*

DEFINITION 3.5.4. *An element a of A is called* infinitesimal *if $a \neq 0$ and $na \leq a^*$ for any $n \in \mathsf{N}$. An element x of A is called* finite *if $ord(x) < \infty$ and $ord(x^*) < \infty$.*

FACT 3.5.5. *Using Notation 2.9.2, we can equivalently define an infinitesimal as an element $a \in A$ such that $(na)^+$ is defined for any $n \in \mathsf{N}$.*

PROPOSITION 3.5.6. *For any $a \in A$, $a \neq 0$, the following are equivalent:*

(a) *a is infinitesimal,*

(b) $a \in Rad(A)$,

(c) $(na)^2 = 0$ *for every* $n \in \mathsf{N}$.

Proof. (a) ⇒ (b) Let a be an infinitesimal and suppose $a \notin Rad(A)$. Thus, there is a maximal ideal M of A such that $a \notin M$ so $(M \cup \{a\}] = A$. We get $1 = na \oplus b$ for some $n \in \mathsf{N}$ and $b \in M$. It follows $(na)^* \leq b \in M$ so $(na)^* \in M$. By hypothesis $na \leq a^*$, so $a \leq (na)^*$. We conclude that $a \in M$ which is a contradiction. Thus $a \in Rad(A)$.
(b) ⇒ (c) Since $Rad(A)$ is an ideal it follows that $na \in Rad(A)$ for any $n \in \mathsf{N}$. The desired result follows by Lemma 3.5.2.
(c) ⇒ (a) If $n = 0$, then $na = 0 \leq a^*$. If $n \geq 1$, then $(na) \odot a \leq (na)^2 = 0$ so $(na) \odot a = 0$. We get $na \leq a^*$ for any $n \in \mathsf{N}$. Thus a is an infinitesimal. □

COROLLARY 3.5.7. *If B is an* MV*-subalgebra of A, then $Rad(B) = Rad(A) \cap B$.*

COROLLARY 3.5.8.
Let $f\colon A \to B$ be an MV*-algebra homomorphism. Then $f(Rad(A)) \subseteq Rad(f(A))$.*

COROLLARY 3.5.9. *If B is* MV*-algebra, then $Rad(A \times B) = Rad(A) \times Rad(B)$.*

PROPOSITION 3.5.10. *If a is an infinitesimal of A, then the set*

$$C(a) = \{na \mid n \in \mathsf{N}\} \cup \{(na)^* \mid n \in \mathsf{N}\}$$

is a subalgebra of A, isomorphic to Chang's algebra $\boldsymbol{C}$.

Proof. By Lemma 3.5.2 we get $(na) \odot (ma) = 0$, so $na \leq (ma)^*$ for any $n, m \in \mathsf{N}$. In order to prove that $C(a)$ is a subalgebra, it is sufficient to show that $na \oplus (ma)^* \in C(a)$ for any $n, m \in \mathsf{N}$. If $n \leq m$, then $na \oplus (ma)^* = na \oplus (na \oplus (m-n)a)^* = na \oplus (na)^* \odot ((m-n)a)^* = na \vee ((m-n)a)^* = ((m-n)a)^* \in C(a)$.

If $n > m$, then $na \oplus (ma)^* = (n-m)a \oplus ma \oplus (ma)^* = 1 \in C(a)$. Thus, $C(a)$ is a subalgebra of A. An isomorphism f between $C(a)$ and C is defined by: $f(a) = c$, $f(a^*) = 1 - c$, $f(na) = nc$, $f((na)^*) = 1 - nc$ for any $n \in \mathsf{N}$. □

LEMMA 3.5.11. *If $x \in A$ is a finite element, then the following properties hold for any $a, b, c \in Rad(A)$:*

(a) $a < x < a^*$,

(b) $x \odot a^*$ *is a finite element,*

(c) $a < b^* \odot (x \oplus c)$,

(d) $(x \oplus a) \odot b^* = x \odot b^* \oplus a$,

(e) $a^* \odot b^* = x \odot a^* \oplus x^* \odot b^*$.

Proof. Set $ord(x) = n$.
(a) Note that, for every $a \in Rad(A)$, $n(a \wedge x) = na$ implies $a \wedge x = a$, i.e. $a < x$. Similarly one proves that $a < x^*$, so $a < x < a^*$.
(b) From Lemma 3.2.6 we obtain $[x]_{Rad(A)} = [x \odot a^*]_{Rad(A)}$. Hence $n(x \odot a^*) \in Rad(A)^*$, i.e. $ord(x \odot a^*) \leq 2n < \infty$. Since $ord(x^* \oplus a) \leq ord(x^*) < \infty$, $x \odot a^*$ is a finite element of A.
(c) Follows by (a), (b) and Lemma 3.2.6.
(d) Using (a) and (b) we get $(x \oplus a) \odot b^* = (x \odot b^* \oplus b \oplus a) \odot b^* = (x \odot b^* \oplus a) \wedge b^* = x \odot b^* \oplus a$.
(e) Using (d) we get the following:

$$\begin{aligned} (x \oplus a) \odot b^* &= x \odot b^* \oplus a \\ (x \oplus a)^* \oplus (x \oplus a) \odot b^* \odot a^* &= (x \oplus a)^* \oplus (x \odot b^* \oplus a) \odot a^* \\ (x \oplus a)^* \vee (b^* \odot a^*) &= (x^* \odot a^*) \oplus (x \odot b^* \wedge a^*) \\ b^* \odot a^* &= (x^* \odot a^*) \oplus (x \odot b^*). \end{aligned}$$ □

LEMMA 3.5.12. *The following are equivalent for any finite element $x \in A$ and for any $a, b, c, d \in Rad(A)$:*

(a) $x \odot a^* \oplus b = x \odot c^* \oplus d$,

(b) $b \oplus c = a \oplus d$.

Proof. (a) ⇒ (b) We have $x \odot a^* \oplus b \oplus a \oplus c = x \odot c^* \oplus d \oplus a \oplus c$. By Lemma 3.5.11 (a) follow $x \oplus b \oplus c = x \oplus a \oplus d$ and $b \oplus c = a \oplus d$.
(b) ⇒ (a) Since $x \oplus b \oplus c = x \oplus a \oplus d$, by Lemma 3.5.11 (a), one gets $x \odot a^* \oplus a \oplus b \oplus c = x \odot c^* \oplus c \oplus a \oplus d$, i.e. $x \odot a^* \oplus b = x \odot c^* \oplus d$. □

LEMMA 3.5.13. *If $a, b, c, d \in Rad(A)$ and x, y are finite elements such that $x \oplus y$ is a finite element, then*

$$(x \oplus a) \odot b^* \oplus (y \oplus c) \odot d^* = (x \oplus y \oplus a \oplus c) \odot (b \oplus d)^*.$$

Proof. Using Lemmas 3.5.11 and 3.5.12 we get:

$$\begin{aligned}(x \vee b) \oplus (y \vee d) &= (x \oplus y) \vee (b \oplus d)\\ x \odot b^* \oplus b \oplus y \odot d^* \oplus d &= (x \oplus y) \odot b^* \odot d^* \oplus b \oplus d\\ (b \oplus d)^* \odot (b \oplus d \oplus x \odot b^* \oplus y \odot d^*) &= (b \oplus d)^* \odot (b \oplus d \oplus (x \oplus y) \odot b^* \odot d^*)\\ (b \oplus d)^* \wedge (x \odot b^* \oplus y \odot d^*) &= (b \oplus d)^* \wedge (x \oplus y) \odot b^* \odot d^*\\ x \odot b^* \oplus y \odot d^* &= (x \oplus y) \odot b^* \odot d^*\\ a \oplus c \oplus x \odot b^* \oplus y \odot d^* &= a \oplus c \oplus (x \oplus y) \odot b^* \odot d^*\\ (x \oplus a) \odot b^* \oplus (y \oplus c) \odot d^* &= (x \oplus y \oplus a \oplus c) \odot (b \oplus d)^*. \qquad \square\end{aligned}$$

3.6 The spectral topology

Let $\langle A, \oplus, ^*, 0\rangle$ be an MV-algebra. In this section we define a topology on $Spec(A)$ and we display some properties of $Spec(A)$ and $Max(A)$. For any $I \in Id(A)$ we define

$$r(I) := \{P \in Spec(A) \mid I \not\subseteq P\}.$$

PROPOSITION 3.6.1. *The following properties hold:*

(a) $r(\{0\}) = \emptyset$,

(b) $r(A) = Spec(A)$,

(c) $r(I \wedge J) = r(I) \cap r(J)$ *for any* $I, J \in Id(A)$,

(d) $r(\bigvee\{I_k \mid k \in K\}) = \bigcup\{r(I_k) \mid k \in K\}$, *for any set* $\{I_k \mid k \in K\} \subseteq Id(A)$,

(e) $I \subseteq J$ *iff* $r(I) \subseteq r(J)$ *for any* $I, J \in Id(A)$,

(f) *if* $X \subseteq A$, *then* $\{P \in Spec(A) \mid X \not\subseteq P\} = r((X])$.

Proof. (a) and (b) Obvious.
(c) Let I and J be ideals of A. Recall that $I \wedge J = I \cap J$. For any prime ideal P, by Proposition 3.3.1, we have $I \cap J \subseteq P$ iff $I \subseteq P$ or $J \subseteq P$. It follows that $I \cap J \not\subseteq P$ iff $I \not\subseteq P$ and $J \not\subseteq P$, so $r(I \wedge J) = r(I) \cap r(J)$.
(d) If $\{I_k \mid k \in K\}$ is a set of ideals, then $\bigvee\{I_k \mid k \in K\} = (\bigcup\{I_k \mid k \in K\}]$. For any prime ideal P we have $P \in r(\bigvee\{I_k \mid k \in K\})$ iff $(\bigcup\{I_k \mid k \in K\}] \not\subseteq P$ iff there is $k \in K$ such that $I_k \not\subseteq P$ iff there is $k \in K$ such that $P \in r(I_k)$ iff $P \in \bigcup\{r(I_k) \mid k \in K\}$.
(e) If $I \subseteq J$ and $P \in r(I)$, then $I \not\subseteq P$. It follows that $J \not\subseteq P$, so $P \in r(J)$. Conversely, we consider $r(I) \subseteq r(J)$. If $J = A$, then the desired result is obvious. Thus, we can assume that J is a proper ideal. We suppose that $I \not\subseteq J$. It follows that there is an element $a \in I \setminus J$. Using Theorem 3.3.8 for J and $S = \{a\}$, we infer that there exists a prime ideal P such that $J \subseteq P$ and $a \notin P$. We get $J \subseteq P$ and $I \not\subseteq P$, so $P \notin r(J)$ and $P \in r(I)$, which is a contradiction. We proved that $I \subseteq J$.
(f) A consequence of the fact that $X \subseteq P$ iff $(X] \subseteq P$ for any $P \in Spec(A)$. $\square$

If we define $\tau := \{r(I) \mid I \in Id(A)\}$, then $\langle Spec A, \tau\rangle$ becomes a topological space by Proposition 3.6.1 (a), (b), (c) and (d). In the sequel τ will be referred as the *spectral topology*.

PROPOSITION 3.6.2. *Let* A *be an* MV-*algebra. Then ideals of* A *are in one to one correspondence with open sets in* $Spec(A)$.

Proof. Let $\mathbf{r}$ be the mapping from ideals of A to open sets in $Spec(A)$ defined as follows:

$$\mathbf{r}\colon I \in Id(A) \to r(I).$$

Thus, trivially, $\mathbf{r}$ is surjective. The injectivity follows by Proposition 3.6.1 (e). □

For any $a \in A$ we define $r(a) := \{P \in Spec(A) \mid a \notin P\}$.

LEMMA 3.6.3. *The following properties hold:*

(a) $r(a) = r((a])$ *for any* $a \in A$,

(b) $r(0) = \emptyset$,

(c) $r(1) = Spec(A)$,

(d) $r(a \vee b) = r(a \oplus b) = r(a) \cup r(b)$ *for any* $a, b \in A$,

(e) $r(a \wedge b) = r(a) \cap r(b)$ *for any* $a, b \in A$,

(f) $r(I) = \bigcup\{r(a) \mid a \in I\}$ *for any* $I \in Id(A)$.

Proof. (a) Straightforward, since for any element $a \in A$ and for any ideal P we have $a \in P$ iff $(a] \subseteq P$.
(b), (c) Obvious.
(d) Let $a, b \in A$. We have $P \in r(a \vee b)$ iff $a \vee b \notin P$ iff $a \notin P$ or $b \notin P$ iff $P \in r(a) \cup r(b)$. The other equality follows similarly.
(e) For any prime ideal P, we have $a \wedge b \in P$ iff $a \in P$ or $b \in P$.
(f) Let P be any prime ideal of A. We have $I \nsubseteq P$ iff there exists $a \in I$ such that $a \notin P$. Hence $P \in r(I)$ iff there exists $a \in I$ such that $P \in r(a)$. The desired equality is now obvious. □

PROPOSITION 3.6.4. *The family* $\{r(a) \mid a \in A\}$ *is a basis for the topology* τ *on* $Spec(A)$. *The compact open subsets of* $Spec(A)$ *are exactly the sets of the form* $r(a)$ *for some* $a \in A$.

Proof. By Lemma 3.6.3 (a), $r(a)$ is an open subset of $Spec(A)$ for any $a \in A$. If $I \in Id(A)$ then, by Proposition 3.6.3 (f), any open subset of $Spec(A)$ is a union of elements $r(a)$ with $a \in A$. This means that the family $\{r(a) \mid a \in A\}$ is a basis for the topology τ. Now we will prove that $r(a)$ is a compact element for any $a \in A$. Let $a \in A$ and $\{I_k \mid k \in K\} \subseteq Id(A)$ such that $r(a) \subseteq \bigcup\{r(I_k) \mid k \in K\}$. Hence $r(a) \subseteq r(\bigvee\{I_k \mid k \in K\})$ and, by Proposition 3.6.1 (e), $a \in \bigvee\{I_k \mid k \in K\}$. It follows that there are $n \in \mathsf{N}$ and $a_{k1} \in I_{k1}, \ldots, a_{kn} \in I_{kn}$ such that $a \leq a_{k1} \oplus \cdots \oplus a_{kn}$. We get $r(a) \subseteq r(a_{k1}) \cup \cdots \cup r(a_{kn}) \subseteq r(I_{k1}) \cup \cdots \cup r(I_{kn})$, so $r(a)$ is a compact subset of $Spec(A)$. Conversely, let $r(I)$ be a compact open subset of $Spec(A)$. Since $r(I) = \bigcup\{r(a) \mid a \in I\}$ and $r(I)$ is compact, it follows that there are $n \in \mathsf{N}$ and $a_1, \ldots, a_n \in I$ such that $r(I) = r(a_1) \cup \cdots \cup r(a_n)$. By Lemma 3.6.3 (d), we infer that $r(I) = r(a_1 \vee \cdots \vee a_n)$. Our proof is finished. □

THEOREM 3.6.5. *For any* MV*-algebra* A*, the prime ideal space* $Spec(A)$ *is a compact* T_0 *space with respect to the spectral topology.*

Proof. By Proposition 3.6.4 it follows that $Spec(A) = r(1)$ is compact. Now we have to prove that $Spec(A)$ is a T_0 space, which means that for any two distinct prime ideals $P \neq Q \in Spec(A)$ at least one of them has an open neighbourhood not containing the other. If $P \neq Q$, then $P \setminus Q \neq \emptyset$ or $Q \setminus P \neq \emptyset$. If $P \setminus Q \neq \emptyset$, then there exists $a \in P \setminus Q$, so $Q \in r(a)$ and $P \notin r(a)$. If $Q \setminus P \neq \emptyset$, then there exists $b \in Q \setminus P$, so $P \in r(b)$ and $Q \notin r(b)$. □

THEOREM 3.6.6. *If A is an* MV*-algebra, then the following are equivalent:*

(a) $Spec(A)$ *is connected,*

(b) $B(A) = \{0, 1\}$.

Proof. (a) ⇒ (b) Assume that $B(A) \neq \{0, 1\}$. Then there exists $a \in B(A) \setminus \{0, 1\}$. Since $a \wedge a^* = 0$, we have $a \in P$ iff $a^* \notin P$, for any $P \in Spec(A)$. It follows that $Spec(A) = r(a) \cup r(a^*)$, $r(a) \cap r(a^*) = \emptyset$, $r(a) \neq \emptyset$ and $r(a^*) \neq \emptyset$, which contradicts our hypothesis. Hence $B(A)$ is the two element algebra.
(b) ⇒ (a) If $Spec(A)$ is not connected, then $Spec(A) = V_1 \cup V_2$, where V_1 and V_2 are non-empty disjoint open subsets of $Spec(A)$. Thus there are X, $Y \subseteq A$ such that $V_1 = \bigcup_{x \in X} r(x)$ and $V_2 = \bigcup_{y \in Y} r(y)$. Since $Spec(A)$ is compact, it follows that there are $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_m \in Y$ with $V_1 = \bigcup_{i=1}^{n} r(x_i)$ and $V_2 = \bigcup_{j=1}^{m} r(y_j)$. If we set $x = x_1 \oplus \cdots \oplus x_n$ and $y = y_1 \oplus \cdots \oplus y_m$ then, by Lemma 3.6.3 (d), we get $V_1 = r(x)$, $V_2 = r(y)$ and $Spec(A) = r(x) \cup r(y)$. Then, for any $P \in Spec(A)$, $x \in P$ iff $y \notin P$. Thus $x \wedge y \in P$ and $x \oplus y \notin P$ for any $P \in Spec(A)$. By Corollaries 3.3.12 and 3.3.10 it follows that $x \wedge y = 0$ and $ord(x \oplus y) < \infty$, so $nx \oplus ny = 1$ for some $n \in \mathsf{N}$. By Proposition 2.2.12 (a), $nx \wedge ny = 0$ so, by Proposition 2.2.5 (f), we also get $nx \vee ny = 1$. It follows that nx and ny are Boolean elements of A and $nx = (ny)^*$. If $nx = 0$, then $x = 0$ and $V_1 = r(0) = \emptyset$, which is impossible. Hence $rx = 1$, so $ry = 0$ and $V_2 = r(0) = \emptyset$ which is also impossible. We proved that $Spec(A)$ is connected. □

PROPOSITION 3.6.7. *Let A be an* MV*-algebra and $\mathcal{S}$ a subset of* $Spec(A)$*. Then the following are equivalent:*

(a) $\mathcal{S}$ *is closed in* $Spec(A)$*,*

(b) for every $P \in Spec(A)$, $P \in \mathcal{S}$ *iff* $\bigcap_{H \in \mathcal{S}} H \subseteq P$.

Proof. (a) ⇒ (b) Assume $\mathcal{S}$ is closed, $P \in Spec(A)$ and $P \supseteq \bigcap_{H \in \mathcal{S}} H$. If $\mathcal{V}$ is any open set of $Spec(A)$ containing P, $P \in r(a) \subseteq V$ for some $a \in A$. Since $a \notin P$, we get $a \notin \bigcap_{H \in \mathcal{S}} H$. Then, there exists a prime ideal $K \in \mathcal{S}$ such that $a \notin K$. Hence $K \in r(a) \cap \mathcal{S} \subseteq (\mathcal{V} \cap \mathcal{S})$, so every neighborhood of P contains a point of $\mathcal{S}$. It follows that P is a limit point of $\mathcal{S}$. Since $\mathcal{S}$ is closed, we have $P \in \mathcal{S}$ and (b) is proved.
(b) ⇒ (a) Let $P \in Spec(A)$ be a limit point of $\mathcal{S}$ and $P \notin \mathcal{S}$. Then $\bigcap_{H \in \mathcal{S}} H \nsubseteq P$, so there exists an $a \in (\bigcap_{H \in \mathcal{S}} H) \setminus P$. It follows that $P \in r(a)$. But $r(a)$ is a neighborhood of P, so $r(a) \cap \mathcal{S} \neq \emptyset$. If $Q \in (r(a) \cap \mathcal{S})$, then $Q \in \mathcal{S}$ and $a \notin Q$. But we have $a \in (\bigcap_{H \in \mathcal{S}} H) \subseteq Q$, so $a \in Q$ which is a contradiction. Thus $P \in \mathcal{S}$ and $\mathcal{S}$ is closed. □

Let $\mathbb{MV}$ be the category of MV-algebras with MV-homomorphisms and $\mathbb{T}\mathrm{op}$ the category of topological spaces with continuous functions. Then we can prove:

THEOREM 3.6.8. *The mapping* $\mathbf{Spec}\colon A \in \mathbb{MV} \to \mathit{Spec}(A) \in \mathbb{T}\mathrm{op}$ *defines a contravariant functor.*

Proof. Indeed, let $\varphi\colon A \to B$ be an MV-algebra homomorphism and define $\mathbf{Spec}(\varphi) = \varphi^{-1}$. It is easy to check that for any prime ideal Q of B, $\varphi^{-1}(Q)$ is a prime ideal of A. So $\mathbf{Spec}(\varphi)$ maps $\mathit{Spec}(B)$ to $\mathit{Spec}(A)$. Let us show that $\mathbf{Spec}(\varphi)$ is continuous. Indeed, for any open set $r(I)$ of $\mathit{Spec}(A)$, with I an ideal of A, we have:

$$\begin{aligned}(\mathbf{Spec}(\varphi)^{-1}(r(I)) &= \{Q \in \mathit{Spec}(B) \mid \mathbf{Spec}(\varphi)(Q) \in r(I)\}\\ &= \{Q \in \mathit{Spec}(B) \mid \varphi^{-1}(Q) \in r(I)\}\\ &= \{Q \in \mathit{Spec}(B) \mid I \nsubseteq \varphi^{-1}(Q)\}\\ &= \{Q \in \mathit{Spec}(B) \mid \varphi(I) \nsubseteq Q\}.\end{aligned}$$

By Proposition 3.6.1 (f), $\{Q \in \mathit{Spec}(B) \mid \varphi(I) \nsubseteq Q\} = r((\varphi(I)])$, which is an open set of $\mathit{Spec}(B)$. So, $\mathbf{Spec}(\varphi)$ is continuous. Moreover, we have:

$$\mathbf{Spec}(\psi \circ \varphi) = \mathbf{Spec}(\psi) \circ \mathbf{Spec}(\varphi)$$

with φ and ψ MV-algebra homomorphisms. Hence $\mathbf{Spec}$ is contravariant. □

In the sequel, we will characterize the maximal ideal space $\mathit{Max}(A)$ of an MV-algebra A. Since $\mathit{Max}(A) \subseteq \mathit{Spec}(A)$, we endow $\mathit{Max}(A)$ with the topology induced by the spectral topology τ on $\mathit{Spec}(A)$. This means that the open sets of $\mathit{Max}(A)$ are

$$s(I) = r(I) \cap \mathit{Max}(A) = \{M \in \mathit{Max}(A) \mid I \nsubseteq M\}.$$

In consequence, for any $a \in A$ and $I \in \mathit{Id}(A)$,

$$s(a) = r(a) \cap \mathit{Max}(A) = \{M \in \mathit{Max}(A) \mid a \notin M\} \quad \text{and} \quad s(I) = \bigcup\{s(a) \mid a \in I\}.$$

Hence the family $\{s(a) \mid a \in A\}$ is a basis for the induced topology on $\mathit{Max}(A)$.

LEMMA 3.6.9. *The following properties hold:*

(a) $s(0) = \emptyset$,

(b) $s(A) = s(1) = \mathit{Max}(A)$,

(c) $s(I \cap J) = s(I) \cap s(J)$ *for any* $I, J \in \mathit{Id}(A)$,

(d) $s(\bigvee\{I_k \mid k \in K\}) = \bigcup\{s(I_k) \mid k \in K\}$, *for any set* $\{I_k \mid k \in K\} \subseteq \mathit{Id}(A)$,

(e) $I \subseteq J$ *implies* $s(I) \subseteq s(J)$ *for any* $I, J \in \mathit{Id}(A)$,

(f) *if* $X \subseteq A$, *then* $\{P \in \mathit{Max}(A) \mid X \nsubseteq P\} = r((X])$,

(g) $s(I) = \mathit{Max}(A)$ *iff* $I = A$ *for any* $I \in \mathit{Id}(A)$.

Proof. (a),(b),(c),(d),(f) Follows as in Proposition 3.6.1.
(e) Straightforward.
(g) One implication follows by (b) and (e). For the other one, suppose that $s(I) = \mathit{Max}(A)$. Then $I \nsubseteq M$ for any $M \in \mathit{Max}(A)$. By Proposition 3.4.5, I is not a proper ideal so $I = A$. □

THEOREM 3.6.10. *In any* MV*-algebra A, the maximal ideal space $Max(A)$ is a compact Hausdorff topological space with respect to the topology induced by the spectral topology on $Spec(A)$.*

Proof. We firstly prove that $Max(A)$ is compact. If $Max(A) = \bigcup\{s(I_k) \mid k \in K\}$ for some family $\{I_k \mid k \in K\} \subseteq Id(A)$ then, by Lemma 3.6.9 (d), $Max(A) = s(\bigvee\{I_k \mid k \in K\})$. by Lemma 3.6.9 (g) it follows that $A = \bigvee\{I_k \mid k \in K\}$. Since $1 \in \bigvee\{I_k \mid k \in K\}$, there are $n \in \mathsf{N}$ and $a_{k1} \in I_{k1}, \ldots, a_{kn} \in I_{kn}$ such that $1 = a_{k1} \oplus \cdots \oplus a_{kn}$. Using Lemma 3.6.9 (d), we get $Max(A) = s(1) \subseteq s(I_{k1} \vee \cdots \vee I_{kn}) = s(I_{k1}) \cup \cdots \cup s(I_{kn})$. Hence $Max(A)$ is a compact topological space. Let M and N be two distinct maximal ideals. We will prove that there are a, $b \in A$ such that $M \in s(a)$, $N \in s(b)$ and $s(a) \cap s(b) = \emptyset$. Since $M \neq N$, then $M \nsubseteq N$ and $N \nsubseteq M$, so there are $x \in M \setminus N$ and $y \in N \setminus M$. If $a = x^* \odot y$, then $a \notin M$ since, otherwise $x \oplus a = y \vee x \in M$, so $y \in M$ which is a contradiction. Similarly, we infer that $b = y^* \odot x \notin N$. Hence, we found $a, b \in A$ such that $M \in s(a)$ and $N \in s(b)$. Moreover, $s(a) \cap s(b) = s(a \wedge b) = s(0) = \emptyset$ by Proposition 2.2.11 (a). It follows that $Max(A)$ is Hausdorff. □

By Proposition 3.4.5, for any prime ideal $P \in Spec(A)$ there is a unique maximal ideal $M_P \in Max(A)$ such that $P \subseteq M_P$. It is straightforward to define an application

$$\mathcal{M}\colon Spec(A) \to Max(A) \quad \text{as} \quad \mathcal{M}(P) = M_P$$

for any $P \in Spec(A)$.

PROPOSITION 3.6.11. *$\mathcal{M}$ is a continuous function from $Spec(A)$ to $Max(A)$.*

Proof. Let U be a closed set of $Spec(A)$. We will prove that $\mathcal{M}(U)$ is a closed set of $Max(A)$ and this means that $\mathcal{M}$ is a continuous function. If U is closed in $Spec(A)$, then $U = Spec(A) \setminus r(I)$ for some ideal I of A. Thus, $U = \{P \in Spec(A) \mid I \subseteq P\}$ and $\mathcal{M}(U) = \{\mathcal{M}(P) \mid P \in Spec(A), I \subseteq P\}$. We prove that $\mathcal{M}(U) = Max(A) \setminus s(I)$. If $M \in \mathcal{M}(U)$, then $M = M_P$ for some $P \in Spec(A)$ such that $I \subseteq P$. We get $I \subseteq P \subseteq M$, so $M \in Max(A) \setminus s(I)$. Conversely, if $M \in Max(A) \setminus s(I)$, then $I \subseteq M$. Since $\mathcal{M}(M) = M$, we get $M \in \mathcal{M}(U)$. We know that $Max(A) \setminus s(I)$ is a closed set of $Max(A)$, so the desired conclusion is straightforward. □

PROPOSITION 3.6.12. *If A is an* MV*-algebra, the following are equivalent:*

(a) $Spec(A)$ *is Hausdorff,*

(b) $Spec(A) = Max(A)$.

Proof. (a) ⇒ (b) Let P, Q be prime ideals such that $P \subseteq Q$ and suppose that $P \neq Q$. By hypothesis, there are two ideals I and J such that $P \subseteq r(I)$, $Q \subseteq r(J)$ and $r(I) \cap r(J) = \emptyset$. It follows that $I \nsubseteq P$, $J \nsubseteq Q$ and $r(I \cap J) = r(0)$. Since $P \subseteq Q$ we infer that $J \nsubseteq P$ and, by Proposition 3.6.1 (e), $I \cap J = \{0\}$. Because $\{0\} \subseteq P$ and P is prime we get $I \subseteq P$ or $J \subseteq P$, which is a contradiction. Hence, $P = Q$ and P is a maximal ideal.
(b) ⇒ (a) Follows by Theorem 3.6.10. □

PROPOSITION 3.6.13. *If A is an* MV*-algebra such that $Spec(A) = Max(A)$, then $Spec(A)$ is a Boolean space.*

Proof. Recall that any compact set of a Hausdorff space is closed and any closed set of a compact space is compact. Since $Spec(A)$ is a compact Hausdorff space by Theorem 3.6.10, the compact open sets coincide with its clopen sets. By Proposition 3.6.4, the clopen sets $\{r(a) \mid a \in A\}$ form a basis so $Spec(A)$ is a Boolean space. □

4 Classes of MV-algebras

This section deals with some important classes of MV-algebras.

The MV-*chains* are characterized as quotients of MV-algebras with respect to prime ideals. Chang's representation theorem (Theorem 4.1.4) asserts that any MV-algebra is a subdirect product of MV-chains. Thus the algebraic calculus in arbitrary MV-algebras is reduced to calculus in MV-chains. The subalgebras of the MV-algebra $[0, 1]$ are characterized: they are either isomorphic with $\mathbf{Ł}_n$ for some natural number n, or dense in $[0, 1]$. We also investigate the *simple* MV-algebras (according to the classical definitions from universal algebra, these are structures with only two congruences), which are proved to be exactly the quotients with respect to maximal ideals. The *semisimple* structures have the property that the intersection of their maximal ideals is $\{0\}$, while the *Archimedean* MV-algebras are structures without infinitesimals (according to the definition of Archimedean Abelian lattice ordered groups). Hence, the semisimple and Archimedean MV-algebras coincide. *Local* MV-algebras are structures with only one maximal ideal. We further characterize and classify these algebras. The proper subclass of *perfect* MV-algebras, which are generated by their radical, is proved to be categorical equivalent with the Abelian ℓ-groups in Section 5.5.

4.1 MV-chains

In this section we prove that any MV-algebra can be represented as a subdirect product of MV-chains. An important property says that taking the quotient of an MV-algebra with respect to a prime ideal one gets an MV-chain. Hence, the algebraic calculus can be reduced to linearly ordered structures. An identity holds in any MV-algebra if and only if it holds considering all the possible orderings for the variables involved. We also classify the subalgebras of $[0, 1]$, which are either isomorphic with $\mathbf{Ł}_n$ for some natural number n, or dense in $[0, 1]$.

DEFINITION 4.1.1. *We say that an* MV*-algebra is* MV-chain *if it is linearly ordered.*

PROPOSITION 4.1.2. *For a nontrivial* MV*-algebra A the following are equivalent:*

(a) *A is an* MV*-chain,*

(b) *any proper ideal of A is prime,*

(c) *$\{0\}$ is a prime ideal,*

(d) *$Spec(A)$ is linearly ordered.*

Proof. (a) ⇒ (b) Let I be a proper ideal of A and $x, y \in A$. By hypothesis, it follows that $x \leq y$ or $y \leq x$, so $x \odot y^* = 0 \in I$ or $y \odot x^* = 0 \in I$. Thus, I is a prime ideal of A.

(b) ⇒ (c) Straightforward.
(c) ⇒ (d) By Proposition 3.3.4 and the fact that $\{0\}$ is a prime ideal, we infer that any proper ideal is prime, so

$$SpecA = \{I \subseteq A \mid I \text{ is a proper ideal and } \{0\} \subseteq I\}.$$

Thus $Spec(A)$ is linearly ordered under the set theoretical inclusion (Proposition 3.3.6).
(d) ⇒ (a) Let $x, y \in A$ and suppose $x \not\leq y$ and $y \not\leq x$, so $x \odot y^* \neq 0$ and $y \odot x^* \neq 0$. By Proposition 3.3.11, there are P and Q prime ideals such that $x \odot y^* \notin P$ and $y \odot x^* \notin Q$. Hence, using the definition of prime ideals, we get $y \odot x^* \in P$ and $x \odot y^* \in Q$. By hypothesis, $Spec(A)$ is linearly ordered, so $P \subseteq Q$ or $Q \subseteq P$. Thus, $y \odot x^* \in Q$ or $x \odot y^* \in P$ which is a contradiction. We have $x \leq y$ or $y \leq x$, so A is an MV-chain. □

PROPOSITION 4.1.3. *If A is an MV-algebra and I is a proper ideal of A, then the following are equivalent:*

(a) I is a prime ideal,

(b) A/I is an MV-chain.

Proof. (a) ⇒ (b) If $a, b \in A$, then $a \odot b^* \in I$ or $b \odot a^* \in I$, so $[a]_I \odot {[b]_I}^* = [0]_I$ or $[b]_I \odot {[a]_I}^* = [0]_I$. Thus, $[a]_I \leq [b]_I$ or $[b]_I \leq [a]_I$ in A/I, i.e. A/I is an MV-chain.
(b) ⇒ (a) If A/I is an MV-chain, then $[a]_I \leq [b]_I$ or $[b]_I \leq [a]_I$ for any $a, b \in A$. It follows that $a \odot b^* \in I$ or $b \odot a^* \in I$ for any $a, b \in A$, so I is a prime ideal. □

THEOREM 4.1.4 (Chang's representation theorem). *Every MV-algebra is a subdirect product of MV-chains.*

Proof. By Propositions 3.3.13 and 4.1.3. □

COROLLARY 4.1.5. *If A is an MV-algebra, $a, b \in A$ and $n \in \mathsf{N}$, then the following identities hold:*

(a) $n(a \vee b) = na \vee nb$,

(b) $n(a \wedge b) = na \wedge nb$,

(c) $b \oplus c = c$ implies $a \odot b \oplus a \odot c \oplus b = a \odot c \oplus b$.

Proof. (a) and (b) These identities obviously hold if A is an MV-chain. Hence, by Theorem 4.1.4, the identities hold in any MV-algebra.
(c) Let $b \oplus c = c$. By Theorem 4.1.4 we can assume that A is an MV-chain. Then we get $b = 0$ or $c = 1$. If $b = 0$ the equality $a \odot b \oplus a \odot c \oplus b = a \odot c \oplus b$ is trivial. If $c = 1$, by Proposition 2.2.11 (d) the equality holds too. □

PROPOSITION 4.1.6. *Every MV-chain is indecomposable.*

Proof. Let A be an MV-chain and $a \in B(A)$. If $a \leq a^*$, then $a = a \wedge a^* = 0$. If $a \geq a^*$, then $a = a \vee a^* = 1$. Thus, $B(A) = \{0, 1\}$ and the desired conclusion follows by Proposition 2.8.7. □

PROPOSITION 4.1.7. *If A is an MV-chain, then $M = \{a \in A \mid ord(a) = \infty\}$ is the unique maximal ideal of A.*

Proof. Obviously, M is not empty, since $0 \in M$. Suppose $a \leq b$ and $b \in M$. If a has a finite order, then so has b, which contradicts the fact that $b \in M$. Thus, $ord(a) = \infty$, so $a \in M$. Consider $a, b \in M$ and suppose $a \oplus b \notin M$. Therefore there is $n \in \mathsf{N}$ such that $n(a \oplus b) = 1$. Since A is an MV-chain we have $a \leq b$ or $b \leq a$. Hence $n(a \oplus b) \leq 2na$ or $n(a \oplus b) \leq 2nb$, so $2na = 1$ or $2nb = 1$. This contradicts the fact that a and b are in M. We get $a \oplus b \in M$, so M is a proper ideal of A. In order to prove that M is the unique maximal ideal of A, it is enough to prove that any proper ideal of A is included in M. Indeed, if I is a proper ideal of A and $a \in I$, then $ord(a) = \infty$, so $a \in M$. $\square$

COROLLARY 4.1.8. *If A is an* MV*-chain, then $Spec(A) = Id(A)$ is a bounded chain with first element $\{0\}$ and last element $M = \{a \in A \mid ord(a) = \infty\}$.*

Proof. By Propositions 4.1.2 and 4.1.7. $\square$

We recall some basic notions from the theory of partially ordered sets. If A is an MV-algebra, then an element $a \in A$ is an *atom* if $a > 0$ and there is no $x \in L$ such that $0 < x < a$. We say that A is *atomless* if it has no atoms. An MV-algebra A is *densely ordered* if the order relation of A is dense, i.e. $\{x \in A \mid a < x < b\} \neq \emptyset$ whenever $a < b$ in A. A subset $X \subseteq A$ is *dense* in A if $X \cap \{x \in A \mid a < x < b\} \neq \emptyset$ for any $a < b$ in A.

PROPOSITION 4.1.9. *Let A be an atomless* MV*-algebra. Then A is densely ordered.*

Proof. Assume that for some $a, b \in A$, $\{x \in A \mid a < x < b\} = \emptyset$. Let $c = b \odot a^* > 0$. A is atomless, so for some $x \in A$, $0 < x < c$. Then $b \odot x^* \geq b \odot c^* = b \odot (b^* \oplus a) = a$. Therefore, $a \oplus x \leq b$. This yields $a \oplus x = a$ or $a \oplus x = b$. The former implies $x = 0$, in contradiction with $x > 0$. If $a \oplus x = b$, then $c = a^* \wedge x \leq x$, absurd. $\square$

PROPOSITION 4.1.10. *Any* MV*-chain with an atom of order n is isomorphic to $\mathbf{Ł}_{n+1}$.*

Proof. Let A be an MV-chain and let $a \in A$ be an atom such that $ord(a) = n$. It follows that $0 < a < 2a < \cdots < (n-1)a < na = 1$. If $b \in A$, then there is $k < n$ such that $ka \leq b \leq (k+1)a$. It follows that $b \odot (ka)^* \leq (ka)^* \wedge a \leq a$. Since a is an atom we get $b \odot (ka)^* = 0$ or $b \odot (ka)^* = a$. If $b \odot (ka)^* = 0$, then $ka \leq b \leq ka$, so $b = ka$. If $b \odot (ka)^* = a$, then $b = b \vee ka = b \odot (ka)^* \oplus ka = a \oplus ka = (k+1)a$. We proved that for any element b of A there is $0 \leq k \leq n$ such that $b = ka$, i.e. $A = \{ka \mid 0 \leq k \leq n\}$. Now, we will prove that $(ka)^* = (n-k)a$ for any $0 \leq k \leq n$. Because $ka \oplus (n-k)a = na = 1$ we get $(ka)^* \leq (n-k)a$. If we suppose that $(ka)^* \leq (n-k-1)a$ we have $(n-1)a = ka \oplus (n-k-1)a = 1$, which contradicts the fact that $ord(a) = n$. Thus $(ka)^* = (n-k)a$ for any $0 \leq k \leq n$. Since we also have $ka \oplus la = (k+l)a$ for any $0 \leq k, l \leq n$, it is obvious that the function $f\colon A \to L_{n+1}$ defined by $f(ka) = k/n$ is an MV-algebra isomorphism. $\square$

COROLLARY 4.1.11. *Every finite subalgebra of $[0,1]$ is isomorphic to $\mathbf{Ł}_n$ for some $n \geq 2$.*

Proof. Let A be a finite subalgebra of $[0,1]$ and let $a = \min\{x \in A \mid x > 0\}$. It is obvious that a is an atom. Since A is a subalgebra of $[0,1]$, there exists $n \in \mathsf{N}$ such that $ord(a) = n$. If $n = 1$, then $A = L_2$. Otherwise, A has an atom of order n for some $n \geq 2$. By Proposition 4.1.10, A is isomorphic to $\mathbf{Ł}_{n+1}$. $\square$

COROLLARY 4.1.12. *Every infinite subalgebra of* $[0, 1]$ *is dense in* $[0, 1]$.

Proof. By Corollary 4.1.11 and Proposition 4.1.9. $\square$

4.2 Simple and semisimple MV-algebras

The simplicity and the semisimplicity are general notions of universal algebra. An algebra is *simple* if it has only two congruences and it is *semisimple* if the intersection of its maximal congruences contains only the identity. The MV-algebras of $[0, 1]$-valued functions are semisimple. In fact, any semisimple MV-algebra is isomorphic with a subalgebra of $[0, 1]^X$ for some nonempty set X, as we prove in Theorem 5.4.6.

DEFINITION 4.2.1. *An* MV*-algebra* A *is called* semisimple *if* $Rad(A) = \{0\}$*, which means that the intersection of the maximal ideals of* A *is* $\{0\}$.

EXAMPLE 4.2.2. For any nonempty set X the MV-algebra $[0, 1]^X$ is semisimple. Indeed, let $f \in [0, 1]^X$ and $n \in \mathsf{N}$ such that $nf \leq f^*$. It follows that $nf(x) \leq 1 - f(x)$, so $(n + 1)f(x) \leq 1$ for any natural number n and for any $x \in X$. Thus, $f(x) = 0$ for any $x \in X$. We infer that $[0, 1]^X$ has no infinitesimals, so it is a semisimple MV-algebra.

We define the *Archimedean* MV-algebras and we prove that they coincide with the semisimple ones.

LEMMA 4.2.3. *In any* MV*-algebra* A *the following are equivalent:*

(a) *for every* $a \in A$*,* $na \leq a^*$ *for any* $n \in \mathsf{N}$ *implies* $a = 0$*,*

(b) *for every* $a, b \in A$*,* $na \leq b$ *for any* $n \in \mathsf{N}$ *implies* $a \odot b = a$*.*

Proof. (a) $\Rightarrow$ (b) Let $a, b \in A$ such that $na \leq b$ for any $n \in \mathsf{N}$. We get $n(a \wedge b^*) \leq na \leq b \vee a^*$, so $n(a \wedge b^*) \leq (a \wedge b^*)^*$ for $n \in \mathsf{N}$. By hypothesis, $a \wedge b^* = 0$ so $a^* \vee b = 1$. Thus $a = a \odot (a^* \vee b) = (a \odot a^*) \vee (a \odot b) = 0 \vee (a \odot b) = a \odot b$.
(b) $\Rightarrow$ (a) Let $a \in A$ such that $na \leq a^*$ for any $n \in \mathsf{N}$. By hypothesis we get $a \odot a^* = a$, so $a = 0$. $\square$

DEFINITION 4.2.4. *An* MV*-algebra* A *is called* Archimedean *if the equivalent conditions from Lemma 4.2.3 are satisfied. One can easily see that an* MV*-algebra is Archimedean iff it has no infinitesimals.*

PROPOSITION 4.2.5. *An* MV*-algebra* A *is semisimple iff it is Archimedean.*

Proof. An MV-algebra A is semisimple iff it contains no infinitesimals. Thus, if $na \leq a^*$ for any $n \in N$, then $a = 0$. This means that A is Archimedean. $\square$

PROPOSITION 4.2.6. *Any* σ*-complete (complete)* MV*-algebra is semisimple.*

Proof. Let A be a σ-complete MV-algebra and $a \in A$ such that $na \leq a^*$ for any $n \in \mathsf{N}$. If $b = \bigvee\{na \mid n \in \mathsf{N}\}$, then $b \leq a^*$ and $b \oplus a = b$. Hence $a \leq b^*$ and $a \wedge b^* = b^* \odot (b \oplus a) = b^* \odot b = 0$, so $a = 0$. Thus, A is Archimedean, so it is semisimple. $\square$

PROPOSITION 4.2.7. *Any* MV*-algebra* A *which is not semisimple contains a subalgebra isomorphic to Chang's algebra* $\boldsymbol{C}$.

Proof. By Proposition 3.5.6, every MV-algebra A which is not semisimple contains an infinitesimal. The desired result follows by Proposition 3.5.10. □

DEFINITION 4.2.8. *An* MV*-algebra* A *is called* simple *if the only ideals are* $\{0\}$ *and* A*.*

PROPOSITION 4.2.9. *For an* MV*-algebra* A *the following are equivalent:*

(a) A *is simple,*

(b) $ord(a) < \infty$ *for any* $a \in A \setminus \{0\}$*,*

(c) $\{0\}$ *is a maximal ideal,*

(d) A *is linearly ordered and semisimple.*

Proof. (a) ⇒ (b) Let $a \in A$, $a \neq 0$ and suppose $ord(a) = \infty$. Then the ideal $(a]$ is proper and $(a] \neq \{0\}$ which contradicts the hypothesis.
(b) ⇒ (c) Let I be a proper ideal of A and $a \in I$. If $a \neq 0$, then $na = 1$ for some $n \in \mathsf{N}$. Because $na \in I$ we get a contradiction. Thus $a = 0$ so $\{0\}$ is the only proper ideal of A. Obviously, $\{0\}$ is the only maximal ideal.
(c) ⇒ (a) Obvious.
(c) ⇒ (d) Since $\{0\}$ is a maximal ideal it is also prime so, by Proposition 4.1.2, A is an MV-chain. It is obvious that the intersection of the maximal ideals of A is $\{0\}$. Hence, A is semisimple.
(d) ⇒ (b) Let a be a nonzero element from A. Since A is Archimedean, there is $n \in N$ such that $na \not\leq a^*$. It follows that $a^* \leq na$ so $a^* \oplus a \leq (n+1)a$, hence $ord(a) = n + 1 < \infty$. □

PROPOSITION 4.2.10. *The following are equivalent for any proper ideal* M *of* A*:*

(a) M *is maximal,*

(b) A/M *is simple.*

Proof. (a) ⇒ (b) Let a be in A such that $[a]_M \neq [0]_M$, so $a \notin M$. By Proposition 3.4.2, there is $n \in \mathsf{N}$ such that $(a^*)^n \in M$. Hence $([a]_M{}^*)^n = [0]_M$ and we get $n[a]_M = [1]_M$. We proved that every nonzero element from A/M has a finite order so, by Proposition 4.2.9, A/M is a simple MV-algebra.
(b) ⇒ (a) Let I be an ideal of A such that $M \subseteq I$ and consider $a \in I \setminus M$. Because $[a]_M \neq [0]_M$, by Proposition 4.2.9, there is $n \in \mathsf{N}$ such that $n[a]_M = [1]_M$, so $[na]_M = [1]_M$. Thus, $(na)^* = d(na, 1) \in M \subseteq I$. We get na, $(na)^* \in I$, so $I = A$. Hence M is maximal. □

PROPOSITION 4.2.11.

(a) *Any simple* MV*-algebra is an* MV*-chain.*

(b) *Any* MV*-chain which is not simple has a subalgebra isomorphic to Chang's algebra* $\boldsymbol{C}$.

Proof. (a) Let $a, b \in A$ such that $a \not\leq b$, i.e. $a \odot b^* \neq 0$. Then there is a natural number n such that $n(a \odot b^*) = 1$. By Propositions 2.2.11 and 2.2.12 we get $n(a^* \odot b) = 0$ so $a^* \odot b = 0$, i.e. $b \leq a$. Thus A is linearly ordered.
(b) Suppose A is an MV-chain which is not simple. Then there is a nonzero element $a \in A$ such that $ord(a) = \infty$. We will prove that a is an infinitesimal. Suppose that there is $n \in \mathsf{N}$ such that $na \not\leq a^*$. Since A is linearly ordered, we get $a^* \leq na$, so $(n+1)a = 1$ which is a contradiction. Thus, $na \leq a^*$ for any $n \in \mathsf{N}$, so a is an infinitesimal. The desired result follows by Proposition 3.5.10. □

THEOREM 4.2.12. *Every semisimple* MV*-algebra is a subdirect product of simple* MV*-algebras.*

Proof. By Proposition 3.2.11, 3.4.2, and 4.2.9. □

FACT 4.2.13. *In Section 5 we will prove that the simple* MV*-algebras are, up to isomorphism, the subalgebras of* $[0,1]$*. Thus, by Proposition 4.2.12 any semisimple* MV*-algebra is a subdirect product of subalgebras of* $[0,1]$*. In other words, for any semisimple* MV*-algebra A there is a set X such that A is isomorphic to a subalgebra of* $[0,1]^X$ *(the set X is in fact $Max(A)$). Using the terminology from [4], the subalgebras of* $[0,1]$*-valued functions are called* bold algebras of fuzzy sets*. Hence, by Example 4.2.2 and Proposition 4.2.12, we conclude that the semisimple* MV*-algebras are, up to isomorphism, the bold algebras of fuzzy sets.*

4.3 Local and perfect MV-algebras

PROPOSITION 4.3.1. *For any* MV*-algebra A, the following are equivalent:*

(a) *for any $a \in A$, $ord(a) < \infty$ or $ord(a^*) < \infty$,*

(b) *for any $a, b \in A$, $a \odot b = 0$ implies $a^n = 0$ or $b^n = 0$ for some $n \in \mathsf{N}$,*

(c) *for any $a, b \in A$, $ord(a \oplus b) < \infty$ implies $ord(a) < \infty$ or $ord(b) < \infty$,*

(d) *$\{a \in A \mid ord(a) = \infty\}$ is a proper ideal of A,*

(e) *A has only one maximal ideal.*

Proof. (a) ⇒ (b) Let $a, b \in A$ such that $a \odot b = 0$ and $a^k \neq 0$ for any $k \in \mathsf{N}$. Then $k(a^*) \neq 1$ for any $k \in \mathsf{N}$, so $ord(a^*) = \infty$. By (a) it follows that $ord(a) < \infty$, so there exists $n \in \mathsf{N}$ such that $na = 1$. But $a \odot b = 0$ implies that $a < b^*$. Hence $n(b^*) = 1$, which means that $b^n = 0$.
(b) ⇒ (c) If $a, b \in A$ such that $ord(a \oplus b) < \infty$, then there exists some $n \in \mathsf{N}$ such that $n(a \oplus b) = 1$. It follows that $(a^* \odot b^*)^n = 0$ so, by hypothesis, $(a^*)^n = 0$ or $(b^*)^n = 0$. Hence $na = 1$ or $nb = 1$ and the desired conclusion is obvious.
(c) ⇒ (d) Let $M = \{a \in A \mid ord(a) = \infty\}$ and $a < b \in A$ with $b \in M$. If we assume that $ord(a) < \infty$, then $na = 1$ for some $n \in \mathsf{N}$, so $nb = 1$ which is a contradiction. Hence $ord(a) = \infty$ and $a \in M$. Let $a, b \in M$, so $ord(a) = ord(b) = \infty$. Assume that $ord(x \oplus y) < \infty$. By hypothesis it follows that $ord(a) < \infty$ or $ord(b) = \infty$, so $a \notin M$ or $b \notin M$. But this is a contradiction. We proved that $a \oplus b \in M$, so M is an ideal of A. It is obvious that I is a proper ideal since $1 \notin M$.

(d) ⇒ (e) If J is a proper ideal of A, then $ord(x) = \infty$ for every $x \in J$. Hence $J \subseteq M = \{a \in A \mid ord(a) = \infty\}$ for any proper ideal J of A. It is straightforward that M is the only maximal ideal of A.
(e) ⇒ (a) Let M be the only maximal ideal of A. By Remark 2.7.2 (i8) and Proposition 3.4.5 it follows that, for any $a \in A$, $ord(a) = \infty$ implies that $a \in M$. Hence if we assume that $ord(a) = ord(a^*) = \infty$ for some $a \in A$, then $a, a^* \in M$ and M is not a proper ideal. In consequence, $ord(a) < \infty$ or $ord(a^*) < \infty$ for any $a \in A$. □

DEFINITION 4.3.2. *An* MV*-algebra A is called* local *if the equivalent conditions from Proposition 4.3.1 are satisfied.*

FACT 4.3.3. *If A is a local* MV*-algebra, then $Rad(A) = \{a \in A \mid ord(A) = \infty\}$ is the only maximal ideal of A.*

PROPOSITION 4.3.4. *For an* MV*-algebra A and a proper ideal $P \subseteq A$, the following are equivalent:*

- *(a) there is a unique maximal ideal containing P,*
- *(b) A/P is a local* MV*-algebra,*
- *(c) $a \odot b \in P$ implies $a^n \in P$ or $b^n \in P$ for some integer n,*
- *(d) for any $a \in A$ there exists some $n \in \mathsf{N}$ such that $a^n \in P$ or $(a^*)^n \in P$.*

Proof. (a) ⇒ (b) By Remark 3.4.3, the MV-algebra A/P has only one maximal ideal, so A/P is a local MV-algebra.
(b) ⇒ (c) Recall that for any $a \in A$, $a \in P$ iff $[a]_P = [0]_P$. Our conclusion follows by Proposition 4.3.1 (b).
(c) ⇒ (d) Obvious, since $a \odot a^* = 0 \in P$.
(d) ⇒ (a) Using Remark 3.4.3 we only have to prove that the MV-algebra A/P has a unique maximal ideal. But this follows by Proposition 4.3.1. □

DEFINITION 4.3.5. *An ideal P of an* MV*-algebra A is called* primary *if P is proper and there is a unique maximal ideal containing it.*

COROLLARY 4.3.6. *In an* MV*-algebra any prime ideal is a primary ideal.*

Proof. It follows by Proposition 3.4.5. □

EXAMPLE 4.3.7. If A is an MV-algebra and P a prime ideal of A, then

$$\mathcal{O}_P := \bigcap\{Q \mid Q \subseteq P, Q \in Spec(A)\}$$

is a primary ideal of A. It is trivial to check that $\mathcal{O}_P$ is an ideal of A. Let M_P be the unique maximal ideal of A including P. Assume there is a maximal ideal M such that $M \neq M_P$ and $\mathcal{O}_P \subseteq M$. Then there are elements a and b such that $a \in M$, $b \in M_P$ and $a \oplus b = 1$. Hence we have $a^* \odot b^* = 0$ and $(a^*)^2 \wedge (b^*)^2 = 0$. So for every prime ideal $Q \subseteq P$ we must have either $(a^*)^2 \in Q$ or $(b^*)^2 \in Q$. If $(b^*)^2 \in Q$, then $(b^*)^2 \in M_P$. Hence $b \vee b^* = b \oplus (b^*)^2 \in M_P$, so $b^* \in M_P$ which is absurd. So $(a^*)^2 \in Q$ for any prime ideal $Q \subseteq P$. It follows that $(a^*)^2 \in \mathcal{O}_P \subseteq M$. Similarly as above, we get $a \vee a^* = a \oplus (a^*)^2 \in M$ which is absurd. So, the ideal $\mathcal{O}_P$ is included in only one maximal ideal. We proved that $\mathcal{O}_P$ is a primary ideal.

PROPOSITION 4.3.8. *For an* MV*-algebra A, the following are equivalent:*

(a) *A is local,*

(b) *any proper ideal of A is primary,*

(c) *$\{0\}$ is a primary ideal,*

(d) *$Rad(A)$ contains a primary ideal.*

Proof. (a) ⇒ (b) Let P be a proper ideal of A. By Proposition 4.3.1, A has only one maximal ideal. Hence there exists a unique maximal ideal containing P. By Proposition 4.3.4, P is a primary ideal.
(b) ⇒ (c) Straightforward.
(c) ⇒ (d) Obvious, since $\{0\} \subseteq Rad(A)$.
(d) ⇒ (a) Let P be a primary ideal such that $P \subseteq Rad(A)$. By Proposition 4.3.4, there exists a unique maximal ideal containing P. Hence $Rad(A)$ is the only maximal ideal of A and, by Proposition 4.3.1, A is a local MV-algebra. □

COROLLARY 4.3.9. *Let A be a local* MV*-algebra. Then every homomorphic image of A is local.*

Proof. By Theorem 3.2.8, we only have to prove that A/I is local for any proper ideal I of A. Since A is local then any proper ideal I is primary by Proposition 4.3.8. Hence, by Proposition 4.3.4, A/I is a local MV-algebra for any proper ideal I of A. □

PROPOSITION 4.3.10. *For a local* MV*-algebra A the following properties hold:*

(a) *$B(A) = \{0, 1\}$,*

(b) *A is indecomposable,*

(c) *$Spec(A)$ is connected.*

Proof. (a) Let a be a Boolean element of A. By Proposition 4.3.1 (b), there exists some $n \in \mathsf{N}$ such that $a^n = 0$ or $(a^*)^n = 0$. But this implies that $a = 0$ or $a^* = 0$, so $B(A) = \{0, 1\}$.
(b) Follows by (a) and Proposition 2.8.7.
(c) Follows by (a) and Theorem 3.6.6. □

PROPOSITION 4.3.11. *Let A be an* MV*-algebra, P a prime ideal of A and $a \odot b \in P$. Then $a^2 \in P$ or $b^2 \in P$.*

Proof. P is prime so $a \odot b^* \in P$ or $a^* \odot b \in P$. We can assume $a \odot b^* \in P$. Thus $((a \odot b) \oplus (a \odot b^*)) \in P$. By Proposition 2.2.4 (g) we have $a \odot (b \oplus (a \odot b^*)) \in P$. Since $a \odot (b \oplus (a \odot b^*)) = a \odot (a \vee b) = a^2 \vee a \odot b$, it follows that $a^2 \vee a \odot b \in P$, so $a^2 \in P$. If $a^* \odot b \in P$ one can similarly prove that $b^2 \in P$. □

COROLLARY 4.3.12. *Any* MV*-chain is a local* MV*-algebra.*

Proof. If A is an MV-chain then, by Proposition 4.1.2, $\{0\}$ is a prime ideal. Using Corollary 4.3.6 and Proposition 4.3.4 it follows that $A = A/\{0\}$ is a local MV-algebra. □

The converse of the above proposition is false. To see this, let $\boldsymbol{C}$ be the Chang algebra and let A the subalgebra of $C \times C$ having as support the following set:

$$\{\langle nc, mc\rangle \mid n, m > 0\} \cup \{\langle 1 - nc, 1 - mc\rangle \mid n, m > 0\}.$$

Note the MV-algebra defined above is an example of a local MV-algebra that is not an MV-chain. Moreover the ideal $\{\langle 0, 0\rangle\} = 0$ satisfies $a \odot b = 0$ implies $a^2 = 0$ or $b^2 = 0$, but it is not a prime ideal.

The following proposition tells us which local MV-algebras are semisimple.

PROPOSITION 4.3.13. *Let A be a local MV-algebra. If A is semisimple, then A is simple.*

Proof. If A is local then, by Remark 4.3.3, $Rad(A)$ is the only maximal ideal of A. Since A is also semisimple, $Rad(A) = \{0\}$, so $\{0\}$ is a maximal ideal. By Proposition 4.2.9 it follows that A is a simple MV-algebra. □

LEMMA 4.3.14. *Let A be a local MV-algebra and $\rho\colon A \to [0, 1]$ a homomorphism. Then $ker(\rho) = Rad(A)$.*

Proof. By Proposition 4.2.10, $\rho^{-1}(0)$ is a maximal ideal of A. Since A is local it has only one maximal ideal, then $ker(\rho) = \rho^{-1}(0) = Rad(A)$. □

The following result is a topological characterization for local MV-algebras.

THEOREM 4.3.15. *For an MV-algebra A the following are equivalent:*

(a) A is local,

(b) $Spec(A)$ is connected and $Max(A)$ is closed in $Spec(A)$.

Proof. (a) ⇒ (b) Assume that A is a local MV-algebra. Then, by Proposition 4.3.10, $Spec(A)$ is connected. Let M be the unique maximal ideal of A. Then $Max(A) = \{M\} = Spec(A) \setminus r(M)$, so $Max(A)$ is closed.
(b) ⇒ (a) Since $Max(A)$ is closed, by Proposition 3.6.7 we have

$$Rad(A) \subseteq P \quad \text{iff} \quad P \in Max(A)$$

for any $P \in Spec(A)$. For every $Q' \in Spec(A/Rad(A))$ there is $Q \in Spec(A)$ such that $Rad(A) \subseteq Q$ and $Q' = Q/Rad(A)$. It follows that $Q \in Max(A)$ and $Q' \in Max(A/Rad(A))$. So we get $Spec(A/Rad(A)) = Max(A/Rad(A))$. Thus, by Proposition 3.6.13 $Spec(A/Rad(A))$ is a Boolean space. Assume that A is not local. Hence $Spec(A/Rad(A))$ is a non trivial Boolean space, hence is not connected. Now, each prime ideal $P \in Spec(A)$ is contained in a unique maximal ideal, thus we have a retraction from $Spec(A)$ to $Max(A)$ given by $\mathcal{M}\colon P \to M_P$. The map $\mathcal{M}$ is continuous. Since $Spec(A)$ is connected, then also $Max(A)$ is connected. We also have a bijection $\tau\colon M \to M/Rad(A)$ from $Max(A)$ to $Spec(A/Rad(A))$. It is straightforward to check that τ is continuous. Since $Max(A)$ is compact and $Spec(A/Rad(A))$ is Hausdorff, we can infer that $Max(A)$ is homeomorphic to $Spec(A/Rad(A))$. Thus $Spec(A/Rad(A))$ is connected. This contradiction shows that A must be local. □

Let A be an MV-algebra. We recall that an element a of A is called *finite* iff $ord(a) < \infty$ and $ord(a^*) < \infty$. Denote $Fin(A)$ the set of all finite elements of A.

PROPOSITION 4.3.16. *For an* MV*-algebra A the following are equivalent:*

(a) *A is local,*

(b) $A = \langle Rad(A)\rangle \cup Fin(A) = Rad(A) \cup Rad(A)^* \cup Fin(A)$.

Proof. Note that $\langle Rad(A)\rangle = Rad(A) \cup Rad(A)^*$ by Proposition 2.7.5.
(a) ⇒ (b) Assume A to be local and $x \in A \setminus \langle Rad(A)\rangle$. Then $[a]_{Rad(A)} > 0$ and $[a^*]_{Rad(A)} > 0$. Since $A/Rad(A)$ is simple, then there are integers n, m such that $[na]_{Rad(A)} = 1$ and $[ma^*]_{Rad(A)} = 1$. Hence $(na)^* \in Rad(A)$ and $(ma^*)^* \in Rad(A)$, i.e., $na \in (Rad(A))^*$ and $ma^* \in (Rad(A))^*$. So, $2na = 1$ and $2ma^* = 1$. That is $ord(a) < \infty$ and $ord(a^*) < \infty$. Hence a is finite.
(b) ⇒ (a) For $a \in \langle Rad(A)\rangle$ we have either $ord(a) < \infty$ or $ord(a^*) < \infty$. Of course, if a is finite we still have either $ord(a) < \infty$ or $ord(x^*) < \infty$. □

EXAMPLE 4.3.17. Let R be the Abelian ℓ-group of real numbers and G an Abelian ℓ-group. Consider $r_0 > 0$ in R and $g_0 \geq 0$ in G. If $H = R \times_{lex} G$ is the lexicographic product of R and G, then $\langle r_0, g_0\rangle > 0$ in H. One can easily see that

$$[\langle 0,0\rangle, \langle r_0, g_0\rangle] = \{\langle 0,g\rangle \mid g \in G, g \geq 0\} \cup \{\langle r_0, g\rangle \mid g \in G, g \leq g_0\} \cup \\ \{\langle r,g\rangle \mid r \in R, 0 < r < r_0, g \in G\}.$$

Consider now the MV-algebra $A = [\langle 0,0\rangle, \langle r_0, g_0\rangle]_H$ defined as in Lemma 2.5.1. Note that for every real number $r > 0$ there exists $n \in \mathsf{N}$ such that $nr > r_0$. Hence, for any $\langle r, g\rangle \in A$, $ord(\langle r,g\rangle) = \infty$ iff $r = 0$. One can prove that

$$\begin{aligned} Rad(A) &= \{\langle 0,g\rangle \mid g \in G, g \geq 0\}, \\ Rad(A)^* &= \{\langle r_0, g\rangle \mid g \in G, g \leq g_0\}, \\ Fin(A) &= \{(r,g) \mid r \in R, 0 < r < r_0, g \in G\}. \end{aligned}$$

Therefore A is a local MV-algebra. Moreover, A is an MV-chain iff G is a totally ordered group.

PROPOSITION 4.3.18. *Let A be a local* MV*-algebra. Then for every $a, b \in A$ such that $[a]_{Rad(A)} \neq [b]_{Rad(A)}$, either $a < b$ or $b < a$.*

Proof. By hypothesis we either have $a^* \odot b \notin Rad(A)$ or $b^* \odot a \notin Rad(A)$. In the first case, since A is local, then we have $ord(a^* \odot b) < \infty$ and then by Corollary 2.2.14, $a^* \oplus b = 1$, i.e., $a < b$. In the second case, from $b^* \odot a \notin Rad(A)$ we similarly obtain $b < a$. □

PROPOSITION 4.3.19. *For an* MV*-algebra A the following are equivalent:*

(a) *A is local,*

(b) *For every $x \in A$, $x \leq x^*$ or $x^* \leq x$ or $(d(x,x^*))^2 = 0$.*

Proof. (a) ⇒ (b) For any $x \in A$, if $x \equiv x^*$ does not hold, we have either $x < x^*$ or $x^* < x$. In the case that $x \equiv x^*$ we have $d(x, x^*) \in Rad(A)$, and then $d(x, x^*)^2 = 0$.
(b) ⇒ (a) If $x \leq x^*$, then $ord(x^*) < \infty$. Analogously, if $x^* \leq x$, then $ord(x) < \infty$. If $d(x, x^*)^2 = 0$, i.e., $(x^2 \oplus (x^*)^2)^2 = 0$, then for every prime ideal P of A we have the following cases:

(i) $[x]_P \leq [x^*]_P$,

(ii) $[x^*]_P \leq [x]_P$.

Assuming (i) we get $[x]_P \odot [x]_P = 0$, and then $([x^*]_P)^4) = 0$. Hence $ord([x]_P) \leq 4$. While, assuming (ii) holds we get $ord(x) \leq 2$. Hence, for every prime ideal P of A, $ord([x]_P) \leq 4$. This implies that $ord(x) < \infty$. Hence A is local. □

COROLLARY 4.3.20. *The class of local* MV*-algebras is a universal class.*

Proof. By Proposition 4.3.19. □

Let us describe an example of local MV-algebra that will result as a kind of prototypical local MV-algebra.

EXAMPLE 4.3.21. Let X be an arbitrary nonempty set, U a ultrapower of the MV-algebra $[0, 1]$, and $\mathbf{K}(U^X)$ the subset of the MV-algebra U^X as follows:

$$\mathbf{K}(U^X) = \{f \in U^X \mid f(X) \subseteq [a]_{Rad(U)} \quad \text{for some} \quad a \in U\}.$$

The algebra $\mathbf{K}(U^X)$ will be called the *the full* MV*-algebra of quasi constant functions* from X to U. Of course any element f from $\mathbf{K}(U^X)$ will be said *quasi constant* function from X to U. Any subalgebra of $\mathbf{K}(U^X)$ will be called an *algebra of quasi constant functions*.

PROPOSITION 4.3.22. $\mathbf{K}(U^X)$ *is a local* MV*-algebra.*

Proof. Let us show that $\mathbf{K}(U^X)$ is a subalgebra of U^X. The zero constant function f_0 belongs to $\mathbf{K}(U^X)$ because $f_0(X) \subseteq Rad(U) = [0]_{Rad(U)}$. Similarly, from $f_1(X) \subseteq (Rad(U))^* = [1]_{Rad(U)}$ we have $f_1 \in \mathbf{K}(U^X)$. Assume f satisfies $f(X) \subseteq [a]_{Rad(U)}$ for some $a \in U$. Then we have $f^*(X) \subseteq [a^*]_{Rad(U)}$. Finally, let $f, g \in \mathbf{K}(U^X)$ be such that $f(X) \subseteq [a]_{Rad(U)}$ for $a \in U$ and $g(X) \subseteq [b]_{Rad(U)}$ for $b \in U$. Then $(f \oplus g)(X) \subseteq [a \oplus b]_{Rad(U)}$. Hence $\mathbf{K}(U^X)$ is a subalgebra of U^X. To show that $\mathbf{K}(U^X)$ is local take $f \in \mathbf{K}(U^X)$. If $f(X) \subseteq [0]_{Rad(U)}$, then $f^*(X) \subseteq [1]_{Rad(U)}$ and $ord(f^*) < \infty$. If $f(X) \subseteq [1]_{Rad(U)}$ we have $ord(f) < \infty$. Assume now that $f(X) \subseteq [a]_{Rad(U)} \neq [0]_{Rad(U)} \neq [1]_{Rad(U)}$. Then, for every $x \in X$, $f(x) \sim_{Rad(U)} a$ and $a \notin Rad(U)$. Then $ord(f) < \infty$ and $ord(f^*) < \infty$. □

Given a local MV-algebra A we know the quotient algebra $A/Rad(A)$ to be simple.

DEFINITION 4.3.23. *Let n be a positive integer. Then a local* MV*-algebra A is said to be of* rank n *iff* $A/Rad(A) \simeq \text{Ł}_{n+1}$. *An* MV*-algebra A is said to be of* finite rank *iff A is a local* MV*-algebra of rank n for some integer n.*

PROPOSITION 4.3.24. *If A is a local* MV*-algebra of rank n, then there exists an element b in A such that the following hold:*

(a) $\{0, b, 2b, \ldots, nb\} \cap [x]_{Rad(A)}$ *is a singleton for every $x \in A$,*

(b) $(rb)^* = sb \oplus ((r+s)b)^*$ *and* $rb \oplus sb = (r+s)b$ *for every $r, s = 1, \ldots, n$ with $r + s \leq n$.*

Proof. (a) Let ψ be the isomorphism from $A/Rad(A)$ onto $\mathbf{Ł}_{n+1}$ and take any $a \in \psi^{-1}(1/n)$. Hence $na \in \psi^{-1}(1) = Rad^*(A)$ and $1 \neq (n-1)a \leq na \leq 1$. If $na < 1$, then $a > (na)^*$ and hence $(n+1)a = na \oplus a > na \oplus (na)^* = 1$, and so $ord(a) = n+1$. Note that both a and $((n-1)a)^* \in \psi^{-1}(1/n)$. Take $b = a \wedge ((n-1)a)^* \in \psi^{-1}(1/n)$. Hence (a) is satisfied.
(b) One can prove, by induction on $r = 1, \ldots, n$, that $(n-r)b \leq (rb)^*$. Hence, we have:

$$sb \oplus ((r+s)b)^* = sb \oplus ((rb \oplus sb))^* = sb \oplus ((rb)^* \odot (sb)^*) = sb \vee (rb)^*$$

where we applied the definition $x \vee y = x \oplus (x^* \odot y)$. Since $sb \leq (n-r)b \leq (rb)^*$ we get $(rb)^* = sb \oplus ((r+s)b)^*$. Hence $rb = (sb)^* \odot (r+s)b$ and so $rb \oplus sb = ((sb)^* \odot (r+s)b) \oplus sb = sb \vee (r+s)b$ and since $sb \leq (r+s)b$ we get the claim. ☐

MV-algebras of finite rank are used in Section 7 and they are characterized in Theorem 5.5.6.

In the following we will classify the local MV-algebras.

DEFINITION 4.3.25. *An* MV*-algebra A is called* perfect *if for any $a \in A$, $ord(a) = \infty$ iff $ord(a^*) < \infty$.*

It is straightforward that any perfect MV-algebra is local.

EXAMPLE 4.3.26. Chang's MV-algebra $\boldsymbol{C}$ from Example 2.4.5 is a perfect MV-algebra.

DEFINITION 4.3.27. *An* MV*-algebra A is called* singular *if A is local and there exist $a, b \in A$ such that $ord(a) < \infty$, $ord(b) < \infty$ and $a \odot b \in Rad(A) \setminus \{0\}$.*

EXAMPLE 4.3.28. The MV-algebra $^*[0,1]$ from Example 2.5.3 is a singular MV-algebra. Since $^*[0,1]$ is an MV-chain it follows that it is also local. Let $a = [1/2] + \tau$, where τ is an infinitesimal. Then $ord(a) = 2 < \infty$ and $a \odot a = 2\tau \in Rad(^*[0,1]) \setminus \{0\}$.

LEMMA 4.3.29. *Let A be a local* MV*-algebra which is not singular. If $S = (A \setminus \langle Rad(A) \rangle) \cup \{0, 1\}$, then:*

(a) *S is an* MV*-subalgebra of A,*

(b) *S is simple,*

(c) $S \simeq A/Rad(A)$.

Proof. (a) It is obvious that $a \in S$ implies $a^* \in S$. Since $0, 1 \in S$ it suffices to prove that S is closed with respect to the $\odot$ operation. If $a, b \in S \setminus \{0\}$ then, by Remark 4.3.4, $ord(a) < \infty$ and $ord(b) < \infty$. Assume that $a \odot b \notin S$. It follows that $a \odot b \in Rad(A) \setminus \{0\}$, so A is singular. But this fact contradicts our hypothesis, so $a \odot b \in S$.
(b) Follows by Remark 4.3.3.
(c) Let $h \colon S \to A/Rad(A)$ be defined by $h(s) = [s]_{Rad(A)}$ for any $s \in S$. It is obvious that h is an MV-algebra homomorphism. If $h(s) = [0]_{Rad(A)}$, then $s \in Rad(A) \cap S$, so $s = 0$. Thus h is an injective homomorphism. Let $[x]_{Rad(A)} \in A/Rad(A)$ such that $[x]_{Rad(A)} \neq [0]_{Rad(A)}$ and $[x]_{Rad(A)} \neq [1]_{Rad(A)}$. Hence $x \notin \langle Rad(A) \rangle$, so $x \in S$ and $h(x) = [x]_{Rad(A)}$. We proved that h is surjective, so h is an MV-algebra isomorphism. □

The following theorem provides a classification of local MV-algebras. Note that the only local MV-algebra which is a Boolean algebra is $\mathbf{Ł}_2 = \{0, 1\}$.

THEOREM 4.3.30. *For any local* MV*-algebra A if $A \neq \mathbf{Ł}_2$, then exactly one of the following holds:*

(a) *A is perfect,*

(b) *A is singular,*

(c) *A is simple.*

Proof. Assume that A is a local MV-algebra and $A \neq \mathbf{Ł}_2$. It is obvious that A cannot be perfect and simple simultaneously and A cannot be singular and simple simultaneously. If A is singular, there are $a, b \in A$ such that $ord(a) < \infty$, $ord(b) < \infty$ and $a \odot b \in Rad(A) \setminus \{0\}$. If A is also perfect, then $ord(a^*) = ord(b^*) = \infty$, so $a^*, b^* \in Rad(A)$. Hence $(a \odot b)^* = a^* \oplus b^* \in Rad(A)$, which is impossible because $Rad(A)$ is a proper ideal of A. We proved that A cannot be singular and perfect simultaneously. Assume now that A it neither perfect, nor singular and let $S = (A \setminus \langle Rad(A) \rangle) \cup \{0, 1\}$. We have to prove that A is simple, i.e. $Rad(A) = \{0\}$. Let $a \in Rad(A)$ be an arbitrary element. Since A is not perfect, it follows that there exists $s \in S$ such that $s \neq 0$ and $s \neq 1$. Then $s^* \in S$ and $ord(s^*) < \infty$, so $ord(a \oplus s^*) < \infty$. Hence $a \oplus s^* \notin Rad(A)$. If we assume that $a \oplus s^* \in Rad(A)^*$, then $s \odot a^* \in Rad(A)$ and $a \vee s = a \oplus s \odot a^* \in Rad(A)$. It follows that $s \in Rad(A)$, which is impossible. We proved that $a \oplus s^* \notin \langle Rad(A) \rangle$, so $a \oplus s^* \in S$. Hence $s \wedge a = s \odot (a \oplus s^*) \in S$. But $a \in Rad(A)$, so $s \wedge a \in Rad(A) \cap S = \{0\}$. We get $s \wedge a = 0$. Since $s \in S$ and $s \neq 0$ it follows that $ord(s) = \infty$, so there exists some $n \in \mathsf{N}$ such that $ns = 1$. By Corollary 4.1.5, $n(s \wedge a) = ns \wedge na$ and we infer that $na = 0$. It follows that $a = 0$ and $Rad(A) = \{0\}$, so A is a simple MV-algebra. □

EXAMPLE 4.3.31. Let G be an ℓ-group, then $[\langle 0, 0 \rangle, \langle 2, 0 \rangle]_{Z \times_{lex} G}$ is a local MV-algebra which is singular and not totally ordered.

We further characterize the perfect MV-algebras.

PROPOSITION 4.3.32. *For an* MV*-algebra A, the following are equivalent:*

(a) A *is perfect,*

(b) $A = \langle Rad(A)\rangle = Rad(A) \cup Rad(A)^*$,

(c) $A/Rad(A) \simeq \mathbf{Ł}_2$.

Proof. (a) ⇒ (b) If A is perfect, then $Fin(A) = \emptyset$. Hence $A = \langle Rad(A)\rangle$ by Proposition 4.3.16.

(b) ⇒ (c) For any $x \in A$ we have $x \in Rad(A)$ or $x \in Rad(A)^*$. Hence $[x]_{Rad(A)} = [0]_{Rad(A)}$ or $[x]_{Rad(A)} = [1]_{Rad(A)}$ and the conclusion follows.

(c) ⇒ (a) If $x \in A$, then $[x]_{Rad(A)} = [0]_{Rad(A)}$ or $[x]_{Rad(A)} = [1]_{Rad(A)}$. Thus we get $x \in Rad(A)$ or $x \in Rad(A)^*$. If $x \in Rad(A)$, then $ord(x) = \infty$ and so $ord(x^*) = 2 < \infty$ by Lemma 3.5.2 (a). If $x \in Rad(A)^*$, then $ord(x^*) = \infty$ and $ord(x) = 2 < \infty$. We proved that A is perfect. □

EXAMPLE 4.3.33. Let Z be the Abelian ℓ-group of integers and G an Abelian ℓ-group. Note that $\langle 1, 0\rangle$ is a strong unit of the lexicographic product $Z \times_{lex} G$ and

$$[\langle 0,0\rangle, \langle 1,0\rangle] = \{\langle 0,g\rangle \mid g \in G, g \geq 0\} \cup \{\langle 1,g\rangle \mid g \in G, g \leq 0\}.$$

If we denote $\Delta(G) = [\langle 0,0\rangle, \langle 1,0\rangle]_{Z\times_{lex}G}$, the interval MV-algebra defined as in Lemma 2.5.1, then

$$\langle 0,g\rangle^* = \langle 1,-g\rangle \text{ for any } g \geq 0 \quad \text{and} \quad \langle 1,g\rangle^* = \langle 0,-g\rangle \text{ for any } g \leq 0.$$

One can easily prove that $ord(\langle 0,g\rangle) = \infty$ for every $g \geq 0$ and $ord(\langle 1,g\rangle) = 2$ for every $g \leq 0$. Thus $\Delta(G)$ is a perfect MV-algebra and $Rad(\Delta(G)) = \{\langle 0,g\rangle \mid g \in G, g \geq 0\}$.

In Section 5.5 we will prove that for any perfect MV-algebra A there exists an Abelian ℓ-group G such that A and $\Delta(G)$ are isomorphic MV-algebras. Moreover, such an ℓ-group is unique up to isomorphism.

FACT 4.3.34. *One can easily prove that the* MV*-algebra $\Delta(Z)$ is isomorphic with Chang's* MV*-algebra $\boldsymbol{C}$ and the isomorphism is*

$$\langle 0,n\rangle \mapsto nc \quad \textit{and} \quad \langle 1,-n\rangle \mapsto 1 - nc$$

for any $n \in \mathsf{N}$.

FACT 4.3.35. *The class of perfect* MV*-algebras is obviously closed to subalgebras and is closed under homomorphic images. Still the class of perfect* MV*-algebras is not equational, since it is not closed under direct products. To see this, let A be any perfect* MV*-algebra. By Corollary 3.5.9, $Rad(A \times A) = Rad(A) \times Rad(A)$ and it is obvious that $\langle 1,0\rangle \notin Rad(A \times A) \cup Rad(A \times A)^*$. Hence $A \times A$ is not a perfect* MV*-algebra. In Section 5.5 we will prove that the equational class generated by perfect* MV*-algebras is in fact generated by Chang's* MV*-algebra $\boldsymbol{C}$.*

5 MV-algebras and Abelian ℓu-groups

This section contains some of the most important results from the theory of MV-algebras. The first one is Mundici's categorical equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit [20, 59]. This result led to a considerable development of the domain.

The correspondence between MV-chains and totally ordered Abelian ℓ-groups with strong unit was established by Chang in [13]. His goal was an algebraic proof for the completeness of Łukasiewicz's ∞-valued logic. Chang's proof of the completeness theorem is presented in this section.

The last subsection contains Di Nola's representation theorem [21]. The representation of the MV-algebras as subdirect product of chains is very useful in practice, but it offers few information about their structure. Di Nola's representation theorem asserts that any MV-algebra is isomorphic to an algebra of nonstandard real valued functions.

5.1 The functor Γ

Recall that $\mathbb{MV}$ is the category of MV-algebras. The objects of $\mathbb{MV}$ are MV-algebras and the morphisms are the MV-algebra homomorphisms. We will denote by $\mathbb{ALG}\mathrm{u}$ the category of ℓu-groups. The elements of this category are pairs $\langle G, u\rangle$ where G is an Abelian ℓ-group and u is a strong unit of G. The morphisms will be ℓ-group homomorphisms which preserve the strong unit. This means that $h\colon \langle G, u\rangle \to \langle H, v\rangle$ is an ℓu-group homomorphism if $h\colon G \to H$ is an ℓ-groups homomorphism and $h(u) = v$.

In order to prove the categorical equivalence we define two functors:

$$\Gamma\colon \mathbb{ALG}\mathrm{u} \to \mathbb{MV} \quad \text{and} \quad \Xi\colon \mathbb{MV} \to \mathbb{ALG}\mathrm{u}.$$

The definition of the functor Γ is straightforward:

$$\begin{aligned} \Gamma(G,u) &:= [0,u]_G \text{ if } \langle G,u\rangle \text{ is an } \ell u\text{-group},\\ \Gamma(h) &:= h|_{[0,u]} \text{ if } h\colon \langle G,u\rangle \to \langle H,v\rangle \text{is an } \ell u\text{-group homomorphism.} \end{aligned}$$

It is more difficult to define the functor Ξ, i.e. given an MV-algebra A to construct an ℓu-group G_A with a strong unit u such that A is isomorphic with $[0,u]_{G_A}$. Then we have to prove that an ℓu-group G with strong unit u is isomorphic with $G_{[0,u]_G}$. For any MV-algebra A, the group G_A is usually called *the Chang's group of* A because, in the particular case when A is an MV-chain, the ℓ-group G_A was firstly defined by Chang [13].

The construction in the general case and the categorical equivalence between MV-algebras and ℓu-groups are due to Mundici [59]. An alternative construction for G_A is given in [22] using the notion of *clan*.

If A is an MV-algebra and 1 is its greatest element, then the MV-algebras isomorphism between A and $[0,1]_{G_A}$ follows by an easy computation. If G is an ℓu-group with a strong unit u, then the ℓu-groups isomorphism between G and $G_{[0,u]_G}$ follows by the fact that $[0,u]$ generates G.

DEFINITION 5.1.1. *For an* MV*-chain A we define G_A as the set of all the ordered pairs $\langle m, a\rangle$ with $m \in \mathsf{Z}$ and $a \in A$. If on G_A we define*

$$\langle m+1, 0\rangle = \langle m, 1\rangle,$$

$$\langle m, a\rangle + \langle n, b\rangle = \begin{cases} \langle m+n, a \oplus b\rangle & \text{if } a \oplus b < 1, \\ \langle m+n+1, a \odot b\rangle & \text{if } a \oplus b = 1, \end{cases}$$

$$-\langle m, a\rangle = \langle -m-1, a^*\rangle,$$

then $\langle G_A, +, \langle 0, 0\rangle\rangle$ is a group. Moreover if we set

$$\langle m, a\rangle \leq \langle n, b\rangle \ \text{ iff either } m < n \quad \text{or} \quad m = n \text{ and } a \leq b,$$

then G_A becomes an ℓ-group, $\langle 0, 1\rangle$ is a strong unit and A is isomorphic with the MV-*algebra $[\langle 0, 0\rangle, \langle 0, 1\rangle]_{G_A}$.*

In the sequel, A is an MV-algebra.

DEFINITION 5.1.2. *A sequence $\mathbf{a} = \{a_k \mid k \in \mathsf{Z}\}$ in A is* good *if the following properties are satisfied:*

(1) $a_k \oplus a_{k+1} = a_k$ *for all $k \in \mathsf{Z}$,*

(2) *there is some $n \in \mathsf{N}$ such that $a_k = 0$ for all $k \geq n$ and $a_k = 1$ for all $k < -n$.*

Let G_A be the set of all the good sequences in A. On G_A we define the group operations as follows.

DEFINITION 5.1.3. *Let $\mathbf{a} = \{a_k \mid k \in \mathsf{Z}\}$ and $\mathbf{b} = \{b_k \mid k \in \mathsf{Z}\}$ be two good sequences in A. The group operations, $\mathbf{a} + \mathbf{b}$ and $-\mathbf{a}$, are given by*

$$(\mathbf{a}+\mathbf{b})_k := \bigoplus_{i+j=k-1} (a_i \odot b_j) \quad \text{and} \quad (-\mathbf{a})_k := (a_{-k-1})^* \text{ for all } k \in \mathsf{Z}.$$

Some particular sequences will play a special role, so we introduce the following notations:

NOTATION 5.1.4. If $m \in \mathsf{Z}$ and $a \in A$ we will denote by $\langle m, a\rangle$ the sequence defined as follows:

$$\langle m, a\rangle_k := \begin{cases} 1 & \text{if } k < m, \\ a & \text{if } k = m, \\ 0 & \text{if } k > m. \end{cases}$$

Thus $\langle m, a\rangle = \ldots, 1, 1, \overset{m}{a}, 0, 0, \ldots$ We also introduce the particular notations:

$$\mathbf{o} := \langle 0, 0\rangle \quad \text{and} \quad \mathbf{u} := \langle 0, 1\rangle.$$

LEMMA 5.1.5. *If* $\mathbf{a} = \langle m, a\rangle$ *and* $\mathbf{b} = \langle n, b\rangle$ *are good sequences in* A*, then the group operations from Definition 5.1.3 are given by*

(1) $$(\langle m, a\rangle + \langle n, b\rangle)_k = \begin{cases} 1 & \text{if } k < m+n, \\ a \oplus b & \text{if } k = m+n, \\ a \odot b & \text{if } k = m+n+1, \\ 0 & \text{if } k > m+n+1, \end{cases}$$

(2) $-\langle m, a\rangle = \langle -m-1,\ a^*\rangle.$

Proof. (1) If we denote $\mathbf{a} = \langle m, a\rangle$ and $\mathbf{b} = \langle m, b\rangle$ and $\mathbf{c} = \mathbf{a} + \mathbf{b}$, then

$$c_k = \bigoplus_{i+j=k-1} (a_i \odot b_j)\,,$$

for every $k \in \mathsf{Z}$. Suppose $k, i, j \in \mathsf{Z}$ such that $i + j = k - 1$.

Case 1: $k > m+n+1$ It follows that $i > m$ or $j > n$ so $a_i \odot b_j = 0$. Thus $c_k = 0$.

Case 2: $k < m+n$ If $i = m-1$, then $j = k-1-i < n$ so $a_i \odot b_j = 1$. It follows that $c_k = 1$.

Case 3: $k = m+n$ If $i > m$, then $a_i \odot b_j = 0$. If $i < m-1$, then $j = k-1-i > n$ so $a_i \odot b_j = 0$. If $i = m-1$, then $j = n$ and $a_i \odot b_j = b$. If $i = m$, then $j = n-1$ and $a_i \odot b_j = a$. It follows that $c_{m+n} = a \oplus b$.

Case 4: $k = m+n+1$ If $i > m$, then $a_i \odot b_j = 0$. If $i < m$, then $j > n$ so $a_i \odot b_j = 0$. Thus $c_k = a_m \odot b_n = a \odot b$.

The above considerations led us to the desired conclusion.

(2) Straightforward. □

PROPOSITION 5.1.6. *If* A *is an* MV*-chain, then:*

(a) *the good sequences in* A *have the form* $\langle m, a\rangle$ *with* $m \in \mathsf{Z}$ *and* $a \in A$*,*

(b) $$\langle m, a\rangle + \langle n, b\rangle = \begin{cases} \langle m+n, a \oplus b\rangle & \text{if } a \oplus b < 1, \\ \langle m+n+1, a \odot b\rangle & \text{if } a \oplus b = 1, \end{cases}$$

where $\langle m, a\rangle$ *and* $\langle n, b\rangle$ *are good sequences in* A*,*

(c) $\langle G_A, +, \mathbf{o}\rangle$ *is the group defined by Chang.*

Proof. (a) Let $\{a_k \mid k \in \mathsf{Z}\}$ be a good sequence in A and $k \in \mathsf{Z}$. Because $a_k \oplus a_{k+1} = a_k$ we get ${a_k}^* \wedge a_{k+1} = {a_k}^* \odot (a_k \oplus a_{k+1}) = 0$. Since A is linearly ordered, it follows that ${a_k}^* = 0$ or $a_{k+1} = 0$. Thus $a_k = 1$ or $a_{k+1} = 0$ for every $k \in \mathsf{Z}$. Let $m \in \mathsf{Z}$ be the greatest element $k \in \mathsf{Z}$ such that $a_k = 1$, which exists by Definition 5.1.2, condition (2). We get $a_k = 1$ if $k \le m$ and $a_k = 0$ if $k > m+1$. We proved that $\{a_k \mid k \in \mathsf{Z}\} = \langle m+1, a_{m+1}\rangle$.

(b) We have $b^* \le a$ or $a \le b^*$, thus $a \oplus b = 1$ or $a \odot b = 0$. The desired result follows by Lemma 5.1.5 (1).

(c) By (a), G_A is the set of all the ordered pairs $\langle m, a\rangle$ with $m \in \mathsf{Z}$ and $a \in A$. By (b) and Lemma 5.1.5 (2), the group operations on G_A coincide with Chang's operations. The condition $\langle m+1, 0\rangle = \langle m, 1\rangle$ is obvious by Notation 5.1.4. □

Now we are able to prove that, for any MV-algebra A, the set of all the good sequences in A has a group structure.

PROPOSITION 5.1.7. *(a) The group operations from Definition 5.1.3 are well defined, i.e. the sum of good sequences is a good sequence and the opposite of a good sequence is a good sequence.*

(b) $\langle G_A, +, \mathbf{o}\rangle$ *is a group.*

Proof. (a) Let $\mathbf{a}$ and $\mathbf{b}$ be two good sequences in A and $m, n \in \mathsf{N}$ such that condition (2) from Definition 5.1.2 is satisfied, respectively, for $\mathbf{a}$ and $\mathbf{b}$. Then the same condition is trivially satisfied for the $-\mathbf{a}$ and n. The condition is also satisfied for $\mathbf{a} + \mathbf{b}$ and $m+n+1$. This can be proved similarly with Lemma 5.1.5 (1), Cases 1 and 2. In order to prove that $\mathbf{a} + \mathbf{b}$ and $-\mathbf{a}$ verify condition (1) from Definition 5.1.2 we will use Chang's representation theorem which states that every MV-algebra is a subdirect product of MV-chains. Thus, it suffices to show that the desired equality holds in MV-chains. This immediately follows for $-\mathbf{a}$ by Lemma 5.1.5 (2), and for $\mathbf{a}+\mathbf{b}$ by Proposition 5.1.6 (b).
(b) Straightforward by Chang's representation theorem and Proposition 5.1.6. □

In the sequel we will prove that G_A is an ℓu-group. Let $\mathbf{a}$ and $\mathbf{b}$ be two good sequences in A. We define

$$\mathbf{a} \leq \mathbf{b} \text{ iff } a_k \leq b_k \text{ for all } k \in \mathsf{Z}.$$

It is easy to see that $\leq$ is an order relation on G_A. Moreover, since the MV-algebra operations are isotone, it follows that the group translation is isotone, so G_A is a partially ordered group. We only have to define the lattice operations on G_A.

LEMMA 5.1.8. *The lattice operations on G_A are defined on components:*

$$(\mathbf{a} \vee \mathbf{b})_k := a_k \vee b_k \quad \textit{and} \quad (\mathbf{a} \wedge \mathbf{b})_k := a_k \wedge b_k \qquad \textit{for all } k \in \mathsf{Z}.$$

Proof. Firstly we will prove that the sequence $\mathbf{a} \wedge \mathbf{b}$ is good. The condition (2) from Definition 5.1.2 is obviously satisfied. For every $k \in \mathsf{Z}$ we have

$$\begin{aligned}(a_k \wedge b_k) \oplus (a_{k+1} \wedge b_{k+1}) &= (a_k \oplus a_{k+1}) \wedge (b_k \oplus b_{k+1}) \wedge (a_k \oplus b_{k+1}) \wedge (b_k \oplus a_{k+1}) \\ &= a_k \wedge b_k \wedge (a_k \oplus b_{k+1}) \wedge (b_k \oplus a_{k+1}) \\ &= a_k \wedge b_k,\end{aligned}$$

so the condition (1) from Definition 5.1.2 holds. Moreover, $\mathbf{a} \wedge \mathbf{b}$ is the infimum of $\mathbf{a}$ and $\mathbf{b}$, since the order relation is defined on components. Since G_A is a partially ordered group, the supremum of $\mathbf{a}$ and $\mathbf{b}$ is the good sequence given by $\mathbf{a} \vee \mathbf{b} = -(-\mathbf{a} \wedge -\mathbf{b})$. For any $k \in \mathsf{Z}$ we get

$$\begin{aligned}(\mathbf{a} \vee \mathbf{b})_k = (-(-\mathbf{a} \wedge -\mathbf{b}))_k = ((-\mathbf{a} \wedge -\mathbf{b})_{-k-1})^* &= ((-\mathbf{a})_{-k-1} \wedge (-\mathbf{b})_{-k-1})^* \\ &= ({a_k}^* \wedge {b_k}^*)^* = a_k \vee b_k. \quad \square\end{aligned}$$

LEMMA 5.1.9. *The good sequence $\mathbf{u} = \langle 0, 1\rangle$ is a strong unit of G_A.*

Proof. By Lemma 5.1.5 (1), we get $n\mathbf{u} = \langle n-1, 1\rangle$ for every natural number $n > 0$. Thus, if $\mathbf{a}$ is a good sequence and $n \in \mathsf{N}$ such that $a_k = 0$ for all $k \geq n$ and $a_k = 1$ for all $k < -n$, it follows that $\mathbf{a} \leq n\mathbf{u}$. □

We proved that G_A is an ℓu-group for any MV-algebra A. Now we define the functor $\Xi\colon \mathbb{MV} \to \mathbb{ALG}\mathrm{u}$ as follows:

$$\begin{aligned} \Xi(A) &:= G_A, \\ \Xi(h) &:= \widetilde{h} \text{ where } h\colon A \to B \text{ is an MV-algebra homomorphism} \\ &\quad\ \text{and } \widetilde{h}(\{a_k \mid k \in \mathsf{Z}\}) = \{h(a_k) \mid k \in \mathsf{Z}\}. \end{aligned}$$

PROPOSITION 5.1.10. *The* MV*-algebras A and $[\mathbf{o}, \mathbf{u}]_{G_A}$ are isomorphic and the isomorphism is $\theta_A\colon A \to [\mathbf{o}, \mathbf{u}]_{G_A}$, defined by $\theta_A(a) = \langle 0, a\rangle$ for any $a \in A$.*

Proof. Let $\mathbf{a}$ be a good sequence in A such that $\mathbf{o} = \langle 0, 0\rangle \leq \mathbf{a} \leq \langle 0, 1\rangle = \mathbf{u}$. Since the order relation is defined on components it follows that $\mathbf{a} = \langle 0, a\rangle$ with $a \in A$. Thus θ_A is well defined and bijective. Since $\theta_A(0) = \langle 0, 0\rangle = \mathbf{o}$ we only have to prove that $\theta_A(a \oplus b) = \theta_A(a) \oplus \theta_A(b)$ and $\theta_A(a^*) = \theta_A(a)^*$ for any $a, b \in A$. By Lemma 5.1.5 we obtain $\theta_A(a)^* = \mathbf{u} - \theta_A(a) = \langle 0, 1\rangle - \langle 0, a\rangle = \langle 0, 1\rangle + \langle -1, a^*\rangle = \langle 0, a^*\rangle = \theta_A(a^*)$ and also

$$\begin{aligned} (\theta_A(a) \oplus \theta_A(b))_k &= ((\langle 0, a\rangle + \langle 0, b\rangle) \wedge \langle 0, 1\rangle)_k \\ &= \begin{cases} 1 & \text{if } k < 0, \\ (a \oplus b) \wedge 1 & \text{if } k = 0, \\ (a \odot b) \wedge 0 & \text{if } k = 1, \\ 0 & \text{if } k > 1, \end{cases} \\ &= \langle 0, a \oplus b\rangle = \theta_A(a \oplus b), \end{aligned}$$

Thus, θ_A is an MV-algebra isomorphism. $\square$

In the sequel G will designate an Abelian ℓu-group with strong unit u, and A will designate $[0, u]_G$. In order to complete the proof of the categorical equivalence between MV-algebras and ℓu-groups we have to show that G and G_A are isomorphic ℓu-groups.

DEFINITION 5.1.11. *For any integer k let $\pi_k\colon G \to A$ be defined by the rule*

$$\pi_k(g) \equiv (g \wedge (k+1)u) \vee ku - ku = ((g - ku) \wedge u) \vee 0.$$

PROPOSITION 5.1.12. *For every $f, g \in G$ and $k \in \mathsf{Z}$ the following properties hold:*

- *(1) π_0 is the identity map on A,*
- *(2) if $g \leq ku$, then $\pi_k(g) = 0$,*
- *(3) if $g \geq (k+1)u$, then $\pi_k(g) = u$,*
- *(4) if $g \leq 0$, then $\pi_k(g) = 0$ for all $k \geq 0$,*
- *(5) if $g \geq 0$, then $\pi_k(g) = u$ for all $k < 0$,*
- *(6) there must be some $n \in \mathsf{N}$ such that $\pi_k(g) = 0$ for all $k \geq n$ and $\pi_k(g) = u$ for all $k < -n$,*
- *(7) $\pi_k(g) \geq \pi_{k+1}(g)$ for all $k \in \mathsf{Z}$,*
- *(8) $\pi_k(f \vee g) = \pi_k(f) \vee \pi_k(g)$ and $\pi_k(f \wedge g) = \pi_k(f) \wedge \pi_k(g)$ for all $k \in \mathsf{Z}$,*
- *(9) $\pi_k(g) \oplus \pi_{k+1}(g) = \pi_k(g)$ for all $k \in \mathsf{Z}$,*

(10) $\pi_k(-g) = \pi_{-k-1}(g)^*$ *for all* $k \in \mathsf{Z}$*,*

(11) $\pi_k(f+g) = \bigoplus_{i+j=k-1}(\pi_i(f) \odot \pi_j(g))$ *for all* $k \in \mathsf{Z}$*,*

(12) *if* $g \geq 0$*, then* $g = \sum_{0 \leq k} \pi_k(g)$*,*

(13) $\pi_k(g_+) = \pi_k(g)$ *for* $k \geq 0$ *and* $\pi_k(g_+) = u$ *for* $k < 0$*,*

(14) $\pi_k(g_-) = \pi_{-k-1}(g)^*$ *for* $k \geq 0$ *and* $\pi_k(g_-) = u$ *for* $k < 0$*,*

(15) $g = \sum_{0 \leq k} \pi_k(g) - \sum_{k<0} \pi_k(g)^*$*,*

(16) $g = \sum_{-n \leq k < n} \pi_k(g) - nu$ *for any* $n \in \mathsf{N}$ *as in (6),*

(17) *if all the necessary suprema and infima exist, then*

$$\pi_k\Big(\bigvee\{g_i \mid i \in I\}\Big) = \bigvee\{\pi_k(g_i) \mid i \in I\}$$
$$\pi_k\Big(\bigwedge\{g_i \mid i \in I\}\Big) = \bigwedge\{\pi_k(g_i) \mid i \in I\}$$

Proof. (1), (2), (3) Follow directly from the definition of π_k.
(4) Follows from (2)
(5) Follows from (3).
(6) Since u is a strong unit for G, there is $n \in \mathsf{N}$ such that $|g| \leq nu$, so

$$-nu \leq -|g| \leq g \leq |g| \leq nu.$$

The desired properties for π_k follow from (2) and (3).
(7) and (8) Follows directly from the definition of π_k.
(9) Denoting $t := g - ku$ we get

$$\begin{aligned}
\pi_k(g) \oplus \pi_{k+1}(g) &= [((t \wedge u) \vee 0) + (((t-u) \wedge u) \vee 0)] \wedge u \\
&= [((t\wedge) + ((t-u) \wedge u)) \vee (t \wedge u) \vee ((t-u) \wedge u) \vee 0] \wedge u \\
&= [((2t-u) \wedge (t+u) \wedge t \wedge 2u) \vee (t \wedge u) \vee 0] \wedge u \\
&\leq [(t \wedge 2u) \vee (t \wedge u) \vee 0] \wedge u \\
&= (t \wedge 2u \wedge u) \vee (t \wedge u) \vee 0 \\
&= (t \wedge u) \vee 0 = \pi_k(g)
\end{aligned}$$

Since $\pi_k(g) \oplus \pi_{k+1}(g) \geq \pi_k(g)$, the desired equality is proved.
(10) Straightforward:

$$\begin{aligned}
\pi_{-k-1}(g)^* &= u - [((g + (k+1)u) \wedge u) \vee 0] \\
&= [(u - (g + (k+1)u)) \vee 0] \wedge u \\
&= ((-g - ku) \vee 0) \wedge u \\
&= ((-g - ku) \wedge u) \vee 0 = \pi_k(-g).
\end{aligned}$$

(11) Will be (together with (12)) proved in R, since an equation holds in all the Abelian ℓ-groups iff it holds in R. Given real numbers f and g, let i' and j' be the unique integers such that $i' < f \leq i' + 1$ and $j' < g \leq j' + 1$. Then $i' + j' < f + g \leq i' + j' + 2$,

and $\pi_i(f) \odot \pi_j(g)$ is 0 for $i > i'$ because $\pi_i(f) = 0$ and for $j > j'$ because $\pi_j(g) = 0$. Therefore if $k > i' + j' + 1$, then all terms in the sum $\bigoplus_{i+j=k-1}(\pi_i(f) \odot \pi_j(g))$ are 0 and so the sum itself is 0, which is in agreement with $\pi_k(f + g)$ because $f + g \leq i' + f' + 2 \leq k$. If $k \leq i' + j' - 1$, then the sum contains the term $\pi_{i'-1}(f) \odot \pi_{k-i'}(g)$, and this term is 1 because each factor is 1. The result is that the entire sum is 1, which is in agreement with $\pi_k(f + g)$ because $f + g > i' + j' \geq k + 1$. Only two cases remain. If $k = i' + j'$, then the sum reduces to

$$\begin{aligned}(\pi_{i'}(f) \odot \pi_{j'-1}(g)) \oplus (\pi_{i'-1}(f) \odot \pi_{j'}(g)) &= \pi_{i'}(f) \oplus \pi_{j'}(g)\\ &= ((f - i') + (g - j'))_+ \wedge 1\\ &= ((f + g) - k)_+ \wedge 1 = \pi_k(f + g).\end{aligned}$$

If $k = i' + j' + 1$, then the sum reduces to

$$\pi_{i'}(f) \odot \pi_{j'}(g) = ((f - i') + (g - j') - 1)_+ = ((f + g) - k)_+ \wedge 1 = \pi_k(f + g).$$

This completes the proof.
(12) If $g = 0$, then the equality is obvious. Let $g > 0$ be a real number and n the first natural number such that $n < g \leq n + 1$. By (2), $\pi_k(g) = 0$ for $k \geq n + 1$ so we have to prove that

$$g = \sum_{0 \leq k \leq n} \pi_k(g).$$

We will prove this identity by induction on n. If $n = 0$, then $g \in A$ and $\pi_0(g) = g$. Suppose $n \geq 1$ and $n + 1 < g \leq n + 2$. It follows that $n < g - 1 < n + 1$ so, by induction

$$g - 1 = \sum_{0 \leq k \leq n} \pi_k(g - 1).$$

An easy computation shows that $\pi_k(g - 1) = \pi_{k+1}(g)$ and $\pi_0(g) = (g \wedge 1) \vee 0 = 1 \vee 0 = 1$ so

$$g = 1 + \sum_{0 \leq k \leq n} \pi_{k+1}(g) = \sum_{0 \leq k \leq n+1} \pi_k(g).$$

(13) Since $g_+ \geq 0$ we get $\pi_k(g_+) = u$ for $k < 0$. If $k \geq 0$ then

$$\begin{aligned}\pi_k(g_+) &= [((g \vee 0) - ku) \wedge u] \vee 0\\ &= ((g - ku) \wedge u) \vee ((-ku) \wedge u) \vee 0\\ &= ((g - ku) \wedge u) \vee (-ku) \vee 0\\ &= ((g - ku) \wedge u) \vee 0 = \pi_k(g).\end{aligned}$$

(14) If $k \geq 0$, then

$$\begin{aligned}\pi_k(g_-)^* &= u - [(((-g) \vee 0) - ku) \wedge u] \vee 0\\ &= u - [(-g - ku) \wedge u] \vee [(-ku) \wedge u] \vee 0\\ &= u - [(-g - ku) \wedge u] \vee (-ku) \vee 0\\ &= u - [(-g - ku) \wedge u] \vee 0\\ &= ((g + (k + 1)u) \vee 0) \wedge u\\ &= ((g + (k + 1)u) \wedge u) \vee 0 = \pi_{-k-1}(g).\end{aligned}$$

(15) For any $g \in G$, $g = g_+ - g_-$. In the sequel we will use (12) for g_+ and g_- and the desired equality will follow by (14) and (15):

$$\begin{array}{rcl} g &=& g_+ - g_- \\ &=& \sum_{0\leq k} \pi_k(g_+) - \sum_{0\leq k} \pi_k(g_-) \\ &=& \sum_{0\leq k} \pi_k(g) - \sum_{0\leq k} \pi_{-k-1}(g)^* \\ &=& \sum_{0\leq k} \pi_k(g) - \sum_{k<0} \pi_k(g)^*. \end{array}$$

(16) Let $n \in \mathsf{N}$ such that $\pi_k(g) = 0$ for all $k \geq n$ and $\pi_k(g) = u$ for all $k < -n$. By (15) we get:

$$\begin{array}{rcl} g &=& \sum_{0\leq k} \pi_k(g) - \sum_{k<0} \pi_k(g)^* \\ &=& \sum_{0\leq k<n} \pi_k(g) - \sum_{-n\leq k<0} \pi_k(g)^* \\ &=& \sum_{0\leq k<n} \pi_k(g) - \sum_{-n\leq k<0} (u - \pi_k(g)) \\ &=& \sum_{-n\leq k<n} \pi_k(g) - nu. \end{array}$$

(17) We will prove the identity for suprema.

$$\begin{array}{rcl} \pi_k(\bigvee\{g_i \mid i \in I\}) &=& (((\bigvee g_i) - ku) \wedge u) \vee 0 = ((\bigvee(g_i - ku)) \wedge u) \vee 0 \\ &=& \bigvee((g_i - ku) \wedge u \vee 0) = \bigvee\{\pi_k(g_i) \mid i \in I\}. \end{array}$$

The identity for infima follows similarly. □

The crucial point is that each $g \in G$ can be uniquely coded as the sequence $\{\pi_k(g) \mid k \in \mathsf{Z}\}$ in A.

LEMMA 5.1.13. *For any element $g \in G$ the sequence $\{\pi_k(g) \mid k \in \mathsf{Z}\}$ is the unique good sequence $\{a_k \mid k \in \mathsf{Z}\}$ in A such that*

$$g = \sum_{0\leq k} a_k - \sum_{k<0} {a_k}^*.$$

Proof. The fact that $\{\pi_k(g) \mid k \in \mathsf{Z}\}$ is a good sequence for g which satisfies the intended equality follows from Proposition 5.1.12 (9), (6) and (15). Suppose that $\{a_k\}$ and $\{b_k\}$ are good sequences for g such that

$$g = \sum_{0\leq k} a_k - \sum_{k<0} {a_k}^* = \sum_{0\leq k} b_k - \sum_{k<0} {b_k}^*.$$

We can assume that there is some $n \in \mathsf{N}$ such that $a_k = b_k = 0$ for all $k \geq n$ and $a_k = b_k = u$ for all $k < -n$, so

$$g = \sum_{-n\leq k<n} a_k - nu = \sum_{-n\leq k<n} b_k - nu.$$

We get

$$\begin{array}{rcl} a_{-n} &=& \bigoplus_{-n\leq k<n} a_k = \sum_{-n\leq k<n} a_k \wedge u \\ &=& \sum_{-n\leq k<n} b_k \wedge u = \bigoplus_{-n\leq k<n} b_k = b_{-n}. \end{array}$$

By induction it follows that $a_k = b_k$ for any $-n \leq k < n$. □

PROPOSITION 5.1.14. *The ℓu-groups $\langle G, u\rangle$ and $\langle G_A, \mathbf{u}\rangle$ are isomorphic and the isomorphism is $\eta_G\colon G \to G_A$, defined by $\eta_G(g) = \{\pi_k(g) \mid k \in \mathsf{Z}\}$.*

Proof. It is obvious that $\eta_G(0) = \mathbf{o}$ and $\eta_G(u) = \mathbf{u}$. Moreover, from Definition 5.1.3 and Proposition 5.1.12 (10) and (11), it follows that $\eta_G(-g) = -\eta_G(g)$ and $\eta_G(g+h) = \eta_G(g)+\eta_G(h)$, so η_G is an ℓu-group homomorphism. If g and $h \in G$ such that $\eta_G(g) = \eta_G(h)$, then $\pi_k(g) = \pi_k(h)$ for every $k \in \mathsf{Z}$ so, by Proposition 5.1.12 (15), $g = h$ and η_G is injective. Let $\mathbf{a} = \{a_k \mid k \in \mathsf{Z}\}$ be a good sequence in G_A and $g = \sum_{0\leq k} a_k - \sum_{k<0} a_k{}^*$. By Lemma 5.1.13, $a_k = \pi_k(g)$ for every $k \in \mathsf{Z}$, so $\eta_G(g) = \{a_k \mid k \in \mathsf{Z}\}$. Thus η_G is also surjective. We prove that η_G is an ℓu-group isomorphism between $\langle G, u\rangle$ and $\langle G_A, \mathbf{u}\rangle$. □

COROLLARY 5.1.15. $\Gamma(\eta_G) = \theta_{\Gamma(G,u)}$.

Proof. The proof is straightforward using the definitions. □

The proof of the categorical equivalence between Abelian ℓ-groups with strong unit and MV-algebras is finished by the following theorem.

THEOREM 5.1.16. *The functors $\Gamma\colon \mathbb{ALG}\mathrm{u} \to \mathbb{MV}$ and $\Xi\colon \mathbb{MV} \to \mathbb{ALG}\mathrm{u}$ establish a categorical equivalence.*

Proof. In Proposition 5.1.10 we proved that, for any MV-algebra A, there is an MV-algebras isomorphism, θ_A, between A and $\Gamma(\Xi(A))$. By Proposition 5.1.14, for any ℓu-group $\langle G, u\rangle$ there is an ℓu-group isomorphism, η_G, between G and $\Xi(\Gamma(G, u))$. Let A and B be two MV-algebras and $h\colon A \to B$ an MV-algebra homomorphism. We have to prove that $\Gamma(\Xi(h)) \circ \theta_A = \theta_B \circ h$. For $a \in A$ we get

$$\Gamma(\Xi(h)) \circ \theta_A(a) = \Xi(h)(\langle 0, a\rangle) = \widetilde{h}(\langle 0, a\rangle) = \langle 0, h(a)\rangle = \theta_B(h(a)).$$

Similarly, if $\langle G, u\rangle$ and $\langle H, v\rangle$ are two ℓu-groups and $h\colon \langle G, u\rangle \to \langle H, v\rangle$ is an ℓu-group homomorphism, then we will show that $\Xi(\Gamma(h)) \circ \eta_G = \eta_H \circ h$. Since h is a homomorphism, we have $h(\pi_k(g)) = \pi_k(h(g))$ for every $g \in G$ and $k \in \mathsf{Z}$. Thus, for any $g \in G$, we get

$$\begin{aligned}\Xi(\Gamma(h)) \circ \eta_G(g) &= \Xi(\Gamma(h))(\{\pi_k(g) \mid k \in \mathsf{Z}\}) = \widetilde{\Gamma(h)}(\{\pi_k(g) \mid k \in \mathsf{Z}\})\\ &= \{\Gamma(h)(\pi_k(g)) \mid k \in \mathsf{Z}\} = \{h(\pi_k(g)) \mid k \in \mathsf{Z}\}\\ &= \{\pi_k(h(g)) \mid k \in \mathsf{Z}\} = \eta_h(h(g)).\end{aligned}$$

The proof of the categorical equivalence is now complete. □

5.2 Properties of $\Gamma(G, u)$

In the sequel $\langle G, u\rangle$ is an ℓu-group and $A = \Gamma(G, u)$. We recall that an ℓu-subgroup of G is a subset $H \subseteq G$ which contains u and it is closed under the ℓ-group operations.

PROPOSITION 5.2.1.

(a) If $H \subseteq G$ is an ℓu-subgroup of G, then $H \cap A$ is an MV*-subalgebra of A. Moreover, H is the ℓu-subgroup generated by $H \cap A$ in G.*

(b) *If $B \subseteq A$ is an* MV*-subalgebra of A , then*

$$H = \{g \in G \mid \pi_k(g) \in B \text{ for all } k \in \mathsf{Z}\}$$

is the ℓu-subgroup generated by B in G. Moreover, $H \cap A = B$.

(c) *There is a bijective correspondence between the set of all the* MV*-subalgebras of A and the set of all the ℓu-subgroups of G.*

Proof. (a) Let H be an ℓu-subgroup of G. It is easy to see that $H \cap A$ is an MV-subalgebra of A. In order to prove that H is the ℓu-subgroup generated by $H \cap A$ in G, we consider V another ℓu-subgroup of G that contains $H \cap A$ and $h \in H$. Since H is an ℓ-subgroup which contains u, it follows that $\pi_k(h) \in H$ for any $k \in \mathsf{Z}$, so $\pi_k(h) \in H \cap A \subseteq V$ for any $k \in \mathsf{Z}$. By Proposition 5.1.12 (16), we get $h \in V$. Thus $H \subseteq V$ and H is the smallest ℓu-subgroup that contains $H \cap A$.
(b) Since B is closed under the MV-algebras operations, by Proposition 5.1.12 (10), (11) and (17), we infer that H is an ℓ-subgroup of G. By Proposition 5.1.12 (1), we get $B \in H$, so $u \in H$. Thus, H is an ℓu-subgroup of G that includes B. Let V be another ℓu-subgroup of G containing B and $h \in H$. We have $\pi_k(h) \in B \subseteq V$ for any $k \in \mathsf{Z}$, so $h \in V$ by Proposition 5.1.12 (16). Hence $H \in V$ and we proved that H is the ℓu-subgroup generated by B in G. Let h be an element in $H \cap A$. Because $h \in A$ we get $\pi_k(h) = h$ for any $k \in \mathsf{Z}$. Since $h \in H$, we get $\pi_k(h) \in B$ for any $k \in \mathsf{Z}$. Thus $h \in B$, so $H \cap A = B$.
(c) Straightforward by (a) and (b). □

PROPOSITION 5.2.2.

(a) *If $H \subseteq G$ is an ℓ-ideal of G, then $H \cap A$ is an ideal of A. Moreover, $H = (H \cap A]_G$, where $(H \cap A]_G$ is the ℓ-ideal generated by $H \cap A$ in G.*

(b) *If $I \subseteq A$ is an ideal of A, then*

$$(I\,]_G = \{g \in G \mid \pi_k(|g|) \in I \text{ for all } k \geq 0\} = \{g \in G \mid |g| \wedge u \in I\}$$

is the ℓ-ideal generated by I in G. Moreover, $(I\,]_G \cap A = I$.

(c) *If $I, J \subseteq A$ are two ideals, then*

$$I \subseteq J \quad \text{iff} \quad (I\,]_G \subseteq (J\,]_G.$$

(d) *There is a bijective correspondence between the set of all the ideals of A and the set of all the ℓ-ideals of G, which maps the maximal ideals of A into maximal ideals of G.*

Proof. Let I be an ideal of A. Note that $\pi_0(|g|) = |g| \wedge u$ for any $g \in G$. By Proposition 5.1.12 (7), it follows that $|g| \wedge u \in I$ iff $\pi_k(|g|) \in I$ for all $k \geq 0$. In the sequel we will only prove that

$$H = \{g \in G \mid \pi_k(|g|) \in I \text{ for all } k \geq 0\}$$

is an ℓ-ideal of G. We recall that, in an ℓ-group G, $|g| = g \vee (-g)$, $|g+h| \leq |g| + |h|$ and $|g \vee h| \leq |g| \vee |h| \leq |g| + |h|$ for any $g, h \in G$. Since $|g| = |-g|$ for any $g \in G$ we have $g \in H$ implies $-g \in H$. For $g, h \in H$ and $k \geq 0$ we get $\pi_k(|g| + |h|) \in I$ by Proposition 5.1.12 (11) and the fact that I is an ideal. Since $\pi_k(|g+h|) \leq \pi_k(|g|+|h|)$, it follows that $\pi_k(|g+h|) \in I$ for any $k \geq 0$, so $g + h \in H$. Similarly, we can prove that $g \vee h \in H$, since $\pi_k(|g \vee h|) \leq \pi_k(|g| + |h|) \in I$ for any $k \geq 0$. Thus, H is an ℓ-subgroup of G. In order to prove that H is convex, we consider $0 \leq g \leq h \in H$. It follows that $|g| = g$, $|h| = h$ and $\pi_k(|g|) \leq \pi_k(|h|) \in I$ for any $k \geq 0$. Since I is an ideal, we get $\pi_k(|g|) \in I$ for any $k \geq 0$, so $g \in H$. We proved that H is an ℓ-ideal of G. The rest of the proof is straightforward. □

COROLLARY 5.2.3. *The topological spaces $Max(A)$ and $Max(G)$ (with the spectral topologies) are homeomorphic.*

Proof. It follows directly by Proposition 5.2.2 (d). □

PROPOSITION 5.2.4. *Let $\langle G, u\rangle$ be an ℓu-group. The* MV*-algebra A is an* MV*-chain iff G is linearly ordered.*

Proof. If G is linearly ordered, then A is linearly ordered, since the order relation on A is a restriction of the order relation on G. Now, suppose that A is linearly ordered. By Proposition 5.1.6 and Proposition 5.1.12 (16) we infer that for every $g \in G$ there is $a \in A$ and $m \in \mathsf{Z}$ such that $g = a + mu$. Moreover, if g and v are in G, $g = a + mu$ and $v = b + nu$ where $a, b \in A$ and $m, n \in \mathsf{Z}$, then

$$g \leq v \quad \text{iff} \quad m \leq n \ \text{ or } \ m = n \text{ and } a \leq b.$$

This is obviously a linear order relation. □

If a is an element of A we will denote

$$(na)_A = \underbrace{a \oplus \cdots \oplus a}_{n} \quad \text{and} \quad (na)_G = \underbrace{a + \cdots + a}_{n}.$$

The two values can be different: for example $(nu)_A = u < (nu)_G$ for $n \geq 2$. Still, if there is no possible confusion or the two values coincide, we will not use any determination.

In the sequel, we define the divisible MV-algebras, which will be essentially used in the proof of Chang's Completeness Theorem for MV-algebras (Theorem 5.3.7).

PROPOSITION 5.2.5. *For an* MV*-algebra A the following are equivalent:*

(a) for any $a \in A$ and for any natural number $n \geq 1$ there is $x \in A$ such that $nx = a$ and $a^* \oplus (n-1)x = x^*$,

(b) for any $a \in A$ and for any natural number $n \geq 1$ there is $x \in A$ such that $(nx)^+$ is defined and $(nx)^+ = nx = a$, where $+$ is the partial addition defined in Section 2.9.

Proof. We have $x^* = a^* \oplus (n-1)x = (nx)^* \oplus (n-1)x = (x \oplus (n-1)x)^* \oplus (n-1)x = x^* \odot ((n-1)x)^* \oplus (n-1)x = x^* \vee (n-1)x$. Thus, the condition $a^* \oplus (n-1)x = x^*$ is equivalent with $(n-1)x \leq x^*$.

(a) $\Rightarrow$ (b) According to the definition of the partial addition $(n-1)x+x$ is defined and $(n-1)x+x=(n-1)x\oplus x=nx$. Moreover, for any $1\leq k\leq n-1$ we have $kx\leq(n-1)x\leq x^*$, so $kx+x$ is defined and $kx+x=kx\oplus x=(k+1)x$. Thus,

$$a=nx=(n-1)x+x=(n-2)x+x+x=\cdots=(nx)^+.$$

(b) $\Rightarrow$ (a) Since $(nx)^+$ is defined, we get $((n-1)x)^+$ is defined and $((n-1)x)^+=(n-1)x\leq x^*$. The desired conclusion is straightforward. $\square$

DEFINITION 5.2.6. *An* MV*-algebra A is called* divisible *if it satisfies one of the equivalent condition from Proposition 5.2.5.*

EXAMPLE 5.2.7. The MV-algebras $[0,1]$, $\mathsf{Z}\cap[0,1]$ and $\mathbf{Ł}_{n+1}$ are divisible.

PROPOSITION 5.2.8. *The* MV*-algebra A is divisible iff G is a divisible ℓ-group.*

Proof. Note that the partial addition on A defined in Section 2.9 coincide with the group addition. Let G be a divisible ℓ-group, a an arbitrary element from A and $n\geq 1$. Then there is $x\in G$ such that $(nx)_G=a$. We will prove that $x\geq 0$. In G we have

$$\begin{aligned}n(x\wedge 0)_G&=(nx)_G\wedge((n-1)x)_G\wedge\cdots\wedge 0=a\wedge((n-1)x)_G\wedge\cdots\wedge 0\\&=((n-1)x)_G\wedge\cdots\wedge 0=(n-1)(x\wedge 0)_G.\end{aligned}$$

It follows that $x\wedge 0=0$, so $x\geq 0$. Thus $x\leq nx\leq a$, so $x\in A$ and $(nx)_G=(nx)_A=a$. Conversely, let A be a divisible MV-algebra and $g\in G$. By Proposition 5.1.12 (16) there is $n\in\mathsf{N}$ such that $g=\sum_{-n\leq l<n}\pi_l(g)-nu$. Let $k\neq 0$ be a natural number. Then, for any $-n\leq l<n$ there is $a_l\in A$ such that $(ka_l)_G=(ka_l)_A=\pi_l(g)$. Moreover, there is $a\in A$ such that $ka=u$. Thus, if we denote $x=\sum_{-n\leq l<n}a_l-na$ we get $(kx)_G=g$. If $k<0$ is an integer, then there is $x\in G$ such that $((-k)x)_G=g$. It follows that $(k(-x))_G=g$, so G is a divisible ℓ-group. $\square$

PROPOSITION 5.2.9. *An* MV*-algebra A is semisimple iff G is an Archimedean ℓ-group.*

Proof. Suppose G is Archimedean and let $a\in A$ such that $(na)_A\leq a^*$ for any $n\in N$. If $a^*=1$, then $a=0$. If $a^*<1$, then $(na)_A=(na)_G\leq a^*$. By hypothesis it follows that $a\leq 0$. Since a is also positive we get $a=0$, so A is Archimedean. Conversely, suppose that A is semisimple, hence A is an Archimedean MV-algebra. Let $g,v\in G$ such that $g,v\geq 0$ and $ng\leq v$ for any $n\in N$. We know that there are $k\in N$ and $a_1,\ldots,a_k$ in A such that $v=a_1+\cdots+a_k$. We will prove that $g=0$ by induction on k. If $k=1$, then $v,g\in A$ so $(ng)_G=(ng)_A\leq v\leq u$. For any $n\geq 1$ we get $(n-1)g\leq u-g=g^*$ so, by hypothesis, we get $g=0$. In order to prove the induction step we suppose that our assertion holds for any $g,v\geq 0$ such that v is a sum of less than k elements from A and we assume that $ng\leq v=a_1+\cdots+a_{k-1}+a$ for any $n\in N$, where $a_1,\ldots,a_{k-1},a\in A$. For $n\in N$ we get $ng-a\leq a_1+\cdots+a_{k-1}$. If we denote $t=ng-a$, then, for any $m\in N$, we have $mt=mng-ma\leq mng\leq a_1+\cdots+a_{k-1}$. Thus $m(t\vee 0)=mt\vee(m-1)t\vee\cdots\vee t\vee 0\leq a_1+\cdots+a_{k-1}$ for any $m\in N$. By induction hypothesis, $t\vee 0=0$ so $t\leq 0$, i.e. $ng\leq a$. Since n was arbitrary in N, it follows that $ng\leq a$ for any $n\in N$. Using the induction hypothesis for $k=1$ we conclude that $g=0$. $\square$

5.3 Chang's completeness theorem

Following Theorem 5.1.16, we can identify an MV-algebra A with the interval $[0, u]$ of some ℓu-group.

COROLLARY 5.3.1. *Any* MV*-algebra can be embedded into a divisible* MV*-algebra. In particular, any* MV*-chain can be embedded into a divisible* MV*-chain.*

Proof. Let A be an MV-algebra and $\langle G, u\rangle$ an ℓu-group such that A is isomorphic to $[0, u]_G$. By [3, Chapter 3], there is an Abelian divisible ℓ-group H and an ℓ-group embedding $h\colon G \to H$. Moreover, if G is totally ordered, then H is totally ordered by [34]. Since h is injective, we get $h(u) > 0$, so we can consider the MV-algebra $D = [0, h(u)]_H$. It is obvious that the restriction of h to $[0, u] \subseteq G$ is an MV-algebra embedding of $[0, u]_G$ into D. Thus, the MV-algebra A can be embedded into D. By Proposition 5.2.8, D is a divisible MV-algebra. If A is an MV-chain, then D will be also an MV-chain by Proposition 5.2.4. □

Recall now the first-order theories of MV-algebras and ℓ-groups.

FACT 5.3.2 (The theory of MV-algebras). *The language of* MV*-algebras is* $\Sigma_{\mathrm{MV}} = \{0, \neg, \oplus\}$ *where* 0 *is a constant symbol,* $\neg$ *is a unary function symbol and* $\oplus$ *is a binary function symbol. The theory of* MV*-algebras has the following axioms:*

(1) $(\forall xyz)x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(2) $(\forall x)x \oplus 0 = x$,

(3) $(\forall xy)x \oplus y = y \oplus x$,

(4) $(\forall x)\neg(\neg x) = x$,

(5) $(\forall x)x \oplus (\neg 0) = \neg 0$,

(6) $(\forall xy)x \oplus \neg(x \oplus \neg y) = y \oplus \neg(y \oplus \neg x)$.

If we add the axiom

(7) $(\forall xy)(x \oplus \neg(x \oplus \neg y) = x \vee x \oplus \neg(x \oplus \neg y) = y)$

we get the theory of MV*-chains. Moreover, the theory of divisible* MV*-chains is obtained adding an infinite number of axioms: for any* $n \geq 1$ *we add the axiom*

(8_n) $(\forall x)(\exists y)(ny = x \wedge (\neg x) \oplus (n-1)y = \neg y)$.

Note that the interpretation of $\neg$ *in an* MV*-algebra is the unary operation* *.

FACT 5.3.3 (The theory of Abelian ℓ-groups). *The language of* ℓ*-groups is* $\Sigma_{lg} = \{0, -, +, \vee, \wedge, \leq\}$ *where* 0 *is a constant symbol,* $-$ *is a unary operation,* $+$*,* $\vee$*,* $\wedge$ *are binary function symbols and* $\leq$ *is a binary relation symbol. The theory of Abelian* ℓ*-groups has the following axioms:*

(1) $(\forall xyz)x + (y + z) = (x + y) + z$,

(2) $(\forall x)x + 0 = x$,

(3) $(\forall x)x + (-x) = 0$,

(4) $(\forall xy)x + y = y + x$,

(5) $(\forall x)x \leq x$,

(6) $(\forall xy)(x \leq y \wedge y \leq x \rightarrow x = y)$,

(7) $(\forall xyz)(x \leq y \wedge y \leq z \rightarrow x \leq z)$,

(8) $(\forall xy)(x \vee y = y \leftrightarrow x \leq y)$,

(9) $(\forall xy)(x \wedge y = x \leftrightarrow x \leq y)$,

(10) $(\forall xyz)(x \leq y \rightarrow (x + z) \leq (y + z))$.

It is possible to give other descriptions using either the relation symbol $\leq$ or the function symbols $\vee$ and $\wedge$. Moreover, we can remove $-$ from our language and replace the axiom (3) with axiom

(3′) $(\forall x)(\exists y)x + y = 0$.

The axioms we proposed are more convenient for our purpose. The theory of totally ordered Abelian ℓ-groups is obtained adding the axiom

(11) $(\forall xy)(x \leq y \vee y \leq x)$.

The theory of totally ordered divisible Abelian ℓ-groups can be described adding for any $n \geq 1$ the axiom

(12_n) $(\forall x)(\exists y)(ny = x)$.

The following result is a generalization of [13, Lemma 7].

PROPOSITION 5.3.4. *For any sentence σ of Σ_{MV} there is a formula with only one free variable $\widetilde{\sigma}(v)$ of Σ_{lg} such that for any MV-algebra A we have*

$$A \models \sigma \quad \textit{iff} \quad G \models \widetilde{\sigma}[u]$$

for any Abelian ℓ-group G and $u > 0$ in G such that A is isomorphic with $[0, u]_G$.

Proof. Since any two isomorphic MV-algebras are elementarily equivalent, it suffices to prove the desired result for $A = [0, u]_G$, where G is an Abelian ℓ-group and $u > 0$ in G. Let $t(v_1, \ldots, v_k)$ be a term of Σ_{MV} and v a propositional variable different from v_1, …, v_k. We define $\widetilde{t}$ as follows:

- if $t = 0$, then $\widetilde{0}$ is 0,
- if $t = \neg t_1$, then $\widetilde{t}$ is $v - \widetilde{t_1}$,
- if $t = t_1 \oplus t_2$, then $\widetilde{t}$ is $(t_1 + t_2) \wedge v$.

Let $\varphi(v_1, \ldots, v_k)$ be a formula of Σ_{MV} such that all the free and bound variables of φ are in $\{v_1, \ldots, v_k\}$ and v a propositional variable different from v_1, …, v_k. We define $\widetilde{\varphi}$ as follows:

- if φ is $t_1 = t_2$, then $\widetilde{\varphi}$ is $\widetilde{t1} = \widetilde{t2}$,
- if φ is $\neg\psi$, then $\widetilde{\varphi}$ is $\neg\widetilde{\psi}$,
- if φ is $\psi \vee \chi$, then $\widetilde{\varphi}$ is $\widetilde{\psi} \vee \widetilde{\chi}$ and similarly for $\wedge$, $\rightarrow$, $\leftrightarrow$,
- if φ is $(\forall v_i)\psi$, then $\widetilde{\varphi}$ is $(\forall v_i)(0 \leq v_i \wedge v_i \leq v \rightarrow \widetilde{\psi})$,
- if φ is $(\exists v_i)\psi$, then $\widetilde{\varphi}$ is $(\exists v_i)(0 \leq v_i \wedge v_i \leq v \rightarrow \widetilde{\psi})$.

Thus to any formula $\varphi(v_1, \ldots, v_k)$ of Σ_{MV} corresponds a formula $\widetilde{\varphi}(v_1, \ldots, v_k, v)$ of Σ_{lg}. As a consequence, to any sentence σ of Σ_{MV} corresponds a formula with only one free variable $\widetilde{\sigma}(v)$ of Σ_{lg}.

Since $A = [0, u]_G$, we recall that, for any $a, b \in A$ we have $a \oplus b = (a + b) \wedge u$ and $a^* = u - a$. We prove that for any term $t(v_1, \ldots, v_n)$ of Σ_{MV} and any $a_1, \ldots, a_n \in A$, $t[a_1, \ldots, a_n] = \widetilde{t}[a_1, \ldots, a_n, u]$. Thus,

- if $t = 0$ it is obvious,
- if $t = \neg t_1$, then

$$\begin{aligned}\widetilde{t}[a_1, \ldots, a_n, u] &= u - \widetilde{t_1}[a_1, \ldots, a_n, u] = u - t_1[a_1, \ldots, a_n] \\ &= t_1[a_1, \ldots, a_n]^* = (\neg t_1)[a_1, \ldots, a_n] \\ &= t[a_1, \ldots, a_n],\end{aligned}$$

- if $t = t_1 \oplus t_2$, then

$$\begin{aligned}\widetilde{t}[a_1, \ldots, a_n, u] &= (\widetilde{t_1}[a_1, \ldots, a_n, u] + \widetilde{t_2}[a_1, \ldots, a_n, u]) \\ &\ u(t_1[a_1, \ldots, a_n] + t_2[a_1, \ldots, a_n]) \wedge u \\ &= (t_1[a_1, \ldots, a_n] \oplus t_2[a_1, \ldots, a_n]) \\ &= t[a_1, \ldots, a_n].\end{aligned}$$

Now, we will prove that for any formula $\varphi(v_1, \ldots, v_n)$ of Σ_{MV} and any $a_1, \ldots, a_n \in A$, $A \models \varphi[a_1, \ldots, a_n]$ iff $G \models \widetilde{\varphi}[a_1, \ldots, a_n, u]$. If φ is of the form $t_1 = t_2$ or $\neg\psi$ or $\psi \vee \chi$ the proof is straightforward. We give the details when φ is $(\forall v_i)\psi$. The case when φ is $(\exists v_i)\psi$ is similar, since $a \in A$ is equivalent to $0 \leq a \leq u$ in G. Thus, if φ is $(\forall v_i)\psi$, then

$$\begin{aligned}G \models \widetilde{\varphi}[a_1, \ldots, a_n, u] \quad &\text{iff} \quad \text{for any } a \in G,\ 0 \leq a \leq u \text{ implies} \\ &\phantom{\text{iff}} \quad G \models \psi[a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n, u] \\ &\text{iff} \quad \text{for any } a \in G,\ 0 \leq a \leq u \text{ implies} \\ &\phantom{\text{iff}} \quad A \models \psi[a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n] \\ &\text{iff} \quad \text{for any } a \in A,\ A \models \psi[a_1, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n] \\ &\text{iff} \quad A \models (\forall v_i)\psi[a_1, \ldots, a_n] \\ &\text{iff} \quad A \models \varphi[a_1, \ldots, a_n].\end{aligned}$$

The desired conclusion is obtained in the particular case when φ is a sentence. □

The following result is known for Abelian ℓ-groups.

THEOREM 5.3.5 ([79, Theorem 3.1.2]). *Any non-trivial divisible totally ordered ℓ-group is elementarily equivalent with the group Q of rationals.*

We prove a similar result for MV-algebras.

THEOREM 5.3.6. *Any non-trivial divisible* MV*-chain is elementarily equivalent with* $[0, 1]_Q$.

Proof. Let A be a divisible MV-chain and $\langle G, u\rangle$ an ℓu-group such that A is isomorphic with $\Gamma(G, u) = [0, u]_G$. By Proposition 5.2.4 and 5.2.8, G is a totally ordered divisible ℓu-group. We have to prove that for any sentence σ of Σ_{MV}, $A \models \sigma$ iff $[0, 1]_Q \models \sigma$. If $A \models \sigma$ then, by Proposition 5.3.4, $G \models \widetilde{\sigma}[u]$ where $\widetilde{\sigma}(v)$ is a formula of Σ_{lg} with only one free variable. If τ is the formula $(\exists v)(0 \leq v \wedge (\neg(v = 0)) \wedge \widetilde{\sigma})$ then τ is a sentence and $G \models \tau$. By Theorem 5.3.5, $Q \models \tau$, so there is a rational number $r \in Q$ such that $r > 0$ and $Q \models \widetilde{\sigma}[r]$. Using again Proposition 5.3.4, the MV-algebra

$[0,r]_Q \models \sigma$. If we define $f\colon [0,1] \to [0,r]$ by $f(x) = x \cdot r$, where $\cdot$ is the real numbers product operation, then it is obvious that f is a bijective function. Moreover, f is an MV-algebra isomorphism between $[0,1]_Q$ and $[0,r]_Q$. It follows that $[0,1]_Q$ and $[0,r]_Q$ are elementary equivalent, so $[0,1]_Q \models \sigma$. Conversely, let us suppose that $[0,1]_Q \models \sigma$. Using our previous observation, $[0,1]_Q$ is isomorphic to $[0,r]_Q$ for any rational number $r > 0$. Thus, $[0,r]_Q \models \sigma$ for any $r > 0$. By Proposition 5.3.4, $Q \models \widetilde{\sigma}[r]$ for any $r > 0$. If θ is the sentence $(\forall v)(0 \leq v \wedge (\neg(v = 0)) \to \widetilde{\sigma})$, then $Q \models \theta$. By Theorem 5.3.5 we infer that $G \models \theta$. Since $u > 0$ in G, it follows that $G \models \widetilde{\sigma}[u]$ which is equivalent to $A \models \sigma$. □

In fact, the previous theorem asserts that the theory of nontrivial divisible MV-chains is complete. We are ready to prove Chang's completeness theorem. Note that the present proof uses the full equivalence between MV-algebras and ℓu-groups, as well as model-theoretical properties. In [18] one can see a self-contained proof, which involves only a minimal fragment of Γ-theory.

THEOREM 5.3.7 (Chang's completeness theorem). *For any sentence σ of Σ_{MV},*

$$A \models \sigma \text{ for any } \mathrm{MV}\text{-algebra } A \text{ iff } [0,1]_Q \models \sigma \text{ iff } [0,1] \models \sigma.$$

Proof. One implication is obvious. To prove the other one, let A be an MV-algebra such that $A \not\models \sigma$. By Theorem 4.1.4 and Corollary 5.3.1, we can safely assume that A is a divisible MV-chain. Hence, using Theorem 5.3.6, we get that $[0,1]_Q \not\models \sigma$. The equivalence with $[0,1] \models \sigma$ follows by Theorem 5.3.6, since $[0,1]$ is a non-trivial divisible MV-chain. □

5.4 Functional representation of semisimple MV-algebras

In this section we give concrete representations for finite, simple and semisimple MV-algebras.

PROPOSITION 5.4.1. *Any simple* MV*-algebra is isomorphic to a subalgebra of* $[0,1]$.

Proof. By Propositions 4.2.9 and 4.2.5 it follows that any simple MV-algebra A is linearly ordered and Archimedean. Thus, following Propositions 5.2.4 and 5.2.9, we infer that the corresponding ℓu-group $\langle G, u\rangle$ is linearly ordered and Archimedean. By Holder's Theorem, there is a subgroup H of real numbers such that G and H are isomorphic. If $h \in H$ is the corresponding element for u, then h is a strong unit of H. Consider $H' = \{x/h \mid x \in H\}$ where $/$ stands for the real numbers division. One can easily see that H' is isomorphic to H and 1 is a strong unit for H'. We get $A \simeq \Gamma(G,u) \simeq \Gamma(H,h) \simeq \Gamma(H',1)$. Thus, A is isomorphic to a subalgebra of $[0,1]$. □

COROLLARY 5.4.2. *Any finite and simple* MV*-algebra is isomorphic to* $\mathbf{Ł}_n$ *for some* $n \geq 2$.

Proof. By Proposition 5.4.1 and Corollary 4.1.11. □

COROLLARY 5.4.3. *Any finite* MV*-chain is isomorphic to* $\mathbf{Ł}_n$ *for some* $n \geq 2$.

Proof. If A is a finite MV-chain, then, by Proposition 4.2.11 (b), A is a simple MV-algebra. the desired result follows by Corollary 5.4.2. □

PROPOSITION 5.4.4. *For any finite* MV*-algebra A the following properties hold:*

(a) $Spec(A) = Max(A)$*,*

(b) *A is semisimple,*

(c) $A \simeq \mathbf{Ł}_{k_1} \times \cdots \times \mathbf{Ł}_{k_n}$ *for some natural numbers $n \geq 1$ and $k_1, \ldots, k_n \geq 2$.*

Proof. Let A be a finite MV-algebra.
(a) For any prime ideal $P \subseteq Spec(A)$, the quotient A/P is a finite MV-chain. By Corollary 5.4.3, A/P is a simple MV-algebra, so P is a maximal ideal.
(b) Straightforward by (a).
(c) We have $M_1 \cap \cdots \cap M_n = \{0\}$, where $Max(A) = \{M_1, \ldots, M_n\}$. If $i \neq j$, then $M_i \subset M_i \vee M_j$, so $M_i \vee M_j = A$. Now we can apply Corollary 3.2.12 and we get that A is isomorphic to the direct product $(A/M_1) \times \cdots \times (A/M_n)$. For any $i \in \{1, \ldots, n\}$, A/M_i is a finite simple MV-algebra. the desired result follows by Corollary 5.4.2. □

We focus now on the representation of semisimple MV-algebras.

DEFINITION 5.4.5. *For a set X, the* MV*-algebra* $[0, 1]^X$ *will be called the* bold algebra of fuzzy subsets of X.

Assume A is a semisimple MV-algebra. Then for any $M \in Max(A)$, by Propositions 4.2.10 and 5.4.1, A/M is isomorphic with a subalgebra of $[0, 1]$. Hence for any $a \in A$ the congruence class $[a]_M$ is mapped into an element $\hat{a}_M \in [0, 1]$. In this, for any $a \in A$, we define a map $\hat{a}\colon Max(A) \to [0, 1]$, $\hat{a}(M) = \hat{a}_M$ for any $M \in Max(A)$.

THEOREM 5.4.6. *For a semisimple* MV*-algebra A, the function*

$$\iota_A\colon A \to [0, 1]^{Max(A)} \text{ defined as } \iota_A(a) = \hat{a} \text{ for any } a \in A$$

is an MV*-algebra embedding. As consequence, any semisimple* MV*-algebra is isomorphic with a subalgebra of a bold algebra of fuzzy sets.*

Proof. It is straightforward by Propositions 3.2.11, Proposition 5.4.1 and the above considerations. □

By Theorem 3.6.10, the maximal ideal space $Max(A)$ endowed with the spectral topology is a compact Hausdorff space. The MV-algebra of continuous functions $C(Max(A), [0, 1])$ with pointwise operations is an MV-subalgebra of $[0, 1]^{Max(A)}$. We prove that, for any $a \in A$, the function $\hat{a}\colon Max(A) \to [0, 1]$ is continuous. As consequence, we represent the semisimple MV-algebra A an MV-algebra of $[0, 1]$-valued continuous functions defined on its maximal ideal space.

THEOREM 5.4.7. *Any semisimple* MV*-algebra A is, up to isomorphism, an* MV*-algebra of $[0, 1]$-valued continuous functions defined on some nonempty compact Hausdorff space X (with pointwise operations). Moreover, A is separating, i.e. for any $x \neq y$ in X there is $f \in A$ such that $f(x) \neq f(y)$.*

Proof. Using the notations from Theorem 5.4.6, we firstly prove that $\iota_A(a) = \hat{a}$ is continuous for any $a \in A$. Without loss of generality one can assume that $A = \Gamma(G, u)$ for some ℓu-group $\langle G, u\rangle$. For $a \in A \subseteq G$ one can easily prove that $[a]_M = [a]_{(M]_G}$ for any maximal ideal $M \in Max(A)$. By Corollary 5.2.3 there exists a homeomorphism $h\colon Max(G) \to Max(A)$ and we define the function $\tilde{a}\colon Max(G) \to [0,1]$, $\tilde{a} = \hat{a} \circ h$. Since $\tilde{a}$ is a continuous function by [7, Theorem 13.2.4], it follows that $\hat{a}$ is also a continuous function. Hence $\iota_A(A)$ is an MV-subalgebra of $C(Max(A), [0,1])$. In order to prove that $\iota_A(A)$ is separating, assume that $M \neq N$ in $Max(A)$. There is $a \in A$ such that $a \in M$ and $a \notin N$, so $\hat{a}(M) = 0 \neq \hat{a}(N)$. □

5.5 Perfect MV-algebras and Abelian ℓ-groups

We will prove that the category of perfect MV-algebras is equivalent with the category of Abelian ℓ-groups. Let $\mathbb{P}\mathrm{erf}$ denote the category whose objects are perfect MV-algebras and whose morphisms are MV-algebra homomorphisms. Note that $\mathbb{P}\mathrm{erf}$ is a full subcategory of $\mathbb{MV}$. Denote $\mathbb{ALG}$ the category whose objects are Abelian ℓ-groups and whose morphisms are ℓ-group homomorphism. The above mentioned categorical equivalence will be established by the functors

$$\Delta\colon \mathbb{ALG} \to \mathbb{P}\mathrm{erf} \quad \text{and} \quad \mathcal{D}\colon \mathbb{P}\mathrm{erf} \to \mathbb{ALG}$$

defined as in the following. For an Abelian ℓ-group G, let $\Delta(G)$ be the perfect MV-algebra from Example 4.3.33. If $h\colon G \to H$ is an ℓ-group homomorphism, we define

$$\Delta(h)\colon \Delta(G) \to \Delta(H) \quad \text{and} \quad \Delta(h)(k, g) := \langle k, h(g)\rangle \text{ for any } \langle k, g\rangle \in \Delta(G).$$

One can easily prove that $\Delta(h)$ is an MV-algebra homomorphism and Δ is a functor. In order to define the functor $\mathcal{D}$ we need to remind some classical result of ℓ-group theory.

THEOREM 5.5.1. *Let M be a partially ordered Abelian monoid. Then M is the positive cone of a partially ordered Abelian group iff the following properties hold:*

(a) *M is cancellative,*

(b) *the order on M is natural, i.e. $a \leq b$ iff there exists $x \in M$ such that $b = a + x$.*

Proof. We only remind the construction of G. For a detailed proof see [34]. Define the following equivalence on $M \times M$:

$$\langle a, b\rangle \sim \langle c, d\rangle \quad \text{iff} \quad a + d = c + b$$

and we consider $G = M \times M/{\sim}$, the set of all the equivalence classes. If we denote by $[a, b]$ the equivalence class of the element $\langle a, b\rangle$, then the group operations on G will be defined by:

$$[a, b] + [c, d] = [a + c, d + b] \quad \text{and} \quad - [a, b] = [b, a].$$

It is obvious that M can be identified with the set $\{[a, 0] \mid a \in M\}$. The order on G is defined by

$$[a, b] \leq [c, d] \quad \text{iff} \quad [c, d] - [a, b] \in M \text{ and } - [a, b] + [c, d] \in M.$$

One can prove that $\langle G, +, [0, 0], \leq\rangle$ is an Abelian partially ordered group and M is isomorphic with the positive cone of G. □

FACT 5.5.2. *Under the hypothesis of the previous theorem, if M is a lattice ordered monoid then G is an Abelian ℓ-group. By [8], a partially ordered group G is an ℓ-group iff for every $g \in G$, $g_+ = g \vee 0$ exists in G. Then $g \vee h = (g-h)_+ + h$ and $g \wedge h = -((-h) \vee (-g))$ for every $g, h \in G$. In our particular case one can prove that $[a,b]_+ = [a \vee b, b]$. It follows that the lattice operations on G are defined by*

$$[a,b] \vee [c,d] = [(a+d) \vee (c+b), d+b] \quad \textit{and} \quad [a,b] \wedge [c,d] = [a+c, (d+a) \vee (b+c)]$$

If A is a perfect MV-algebra then, by Corollary 3.5.3, $Rad(A)$ is a latticial monoid that satisfies the hypothesis of Theorem 5.5.1. Let $\mathcal{D}(A) := Rad(A) \times Rad(A)/\sim$ be defined as in the proof of Theorem 5.5.1. Hence $\mathcal{D}(A)$ is an Abelian ℓ-group such that $\mathcal{D}(A)_+$ and $Rad(A)$ are isomorphic latticial monoids.

If A and B are perfect MV-algebras and $f\colon A \to B$ is an MV-algebra homomorphism we define a mapping $\mathcal{D}(f)\colon \mathcal{D}(A) \to \mathcal{D}(B)$ as

$$\mathcal{D}(f)([a,b]) := [f(a), f(b)] \text{ for any } [a,b] \in \mathcal{D}(A).$$

Hence $\mathcal{D}(f)$ is a well-defined ℓ-group homomorphism. One can easily prove that $\mathcal{D}$ is a functor.

PROPOSITION 5.5.3. *If A is a perfect* MV*-algebra, then A and $\Delta(\mathcal{D}(A))$ are isomorphic* MV*-algebras.*

Proof. Let us denote $G = \mathcal{D}(A)$ and $B = \Delta(G)$. It follows that $Rad(A)$ and G_+ are isomorphic lattice monoids. By Example 4.3.33, B is a perfect MV-algebra and G_+ is also isomorphic with $Rad(B)$. So there exists $f\colon Rad(A) \to Rad(B)$ an isomorphism of lattice monoids. We define $F\colon A \to B$ by

$$F(a) = \begin{cases} f(a) & \text{if } a \in Rad(A), \\ f(a^*)^* & \text{if } a \in Rad(A)^*. \end{cases}$$

It is obvious that F is bijective. We only have to prove that F is an MV-algebra homomorphism. Note that $F(0) = f(0) = 0$ and $F(1) = f(0)^* = 1$. Since $b = b^{**}$ for every $b \in B$, we get

$$\begin{aligned} F(a)^* &= \begin{cases} f(a)^* & \text{if } a \in Rad(A), \\ f(a^*)^{**} & \text{if } a \in Rad(A)^* \end{cases} \\ &= \begin{cases} f(a)^* & \text{if } a \in Rad(A), \\ f(a^*) & \text{if } a \in Rad(A)^* \end{cases} \\ &= \begin{cases} f(a)^* & \text{if } a^* \in Rad(A)^*, \\ f(a^*) & \text{if } a^* \in Rad(A) \end{cases} \\ &= F(a^*). \end{aligned}$$

We will prove that $F(a \vee b) = F(a) \vee F(b)$. If $a, b \in Rad(A)$, then the relation is obvious, since f is a lattice morphism. If $a, b \in Rad(A)^*$, then $F(a \vee b) = f(a^* \wedge b^*)^* = (f(a^*) \wedge f(b^*))^* = f(a^*)^* \vee f(b^*)^* = F(a) \vee F(b)$. If $a \in Rad(A)$ and $b \in Rad(A)^*$, then $F(a) \in Rad(B)$ and $F(b) \in Rad(B)^*$. By Lemma 3.5.2, $a \vee b = b$ and $F(a) \vee F(b) = F(b)$, so $F(a \vee b) = F(a) \vee F(b)$.

Now we are able to prove that $F(a \oplus b) = F(a) \oplus F(b)$. If $a, b \in Rad(A)$ the equality is obvious since f is a homomorphism of monoids. If $a, b \in Rad(A)^*$, then $F(a), F(b) \in Rad(B)^*$. By Lemma 3.5.2 (a), $a \oplus b = 1$ and $F(a) \oplus F(b) = 1$ so the desired relation is obvious. If $a \in Rad(A)$ and $b \in Rad(A)^*$, then we get:

$$F(a \vee b^*) = F(a) \vee F(b^*),$$
$$F((b^* \odot a^*) \oplus a) = (F(b^*) \odot F(a)^*) \oplus F(a),$$
$$F(b^* \odot a^*) \oplus F(a) = (F(b^*) \odot F(a^*)) \oplus F(a).$$

To prove the last equality we use the fact that $b^* \odot a^*, a \in Rad(A)$. The last equality is in $Rad(B)$, which is a cancellative monoid, so $F(b^* \odot a^*) = F(b^*) \odot F(a^*)$. Thus $F(a \oplus b) = F(b^* \odot a^*)^* = (F(b^*) \odot F(a^*))^* = F(a^=) \oplus F(b^=) = F(a) \oplus F(b)$. □

PROPOSITION 5.5.4. *If G is an Abelian ℓ-group, then G and $\mathcal{D}(\Delta(G))$ are isomorphic ℓ-groups.*

Proof. We denote $A = \Delta(G)$ and $H = \mathcal{D}(A)$. Then G_+ and H_+ are isomorphic as latticial monoids, since they are both isomorphic with $D(A)$. If $f\colon G_+ \to H_+$ is an isomorphism of latticial monoids, then the function $F\colon G \to H$ defined by $F(g) = f(g_+) - f(g_-)$ is the unique ℓ-group isomorphism such that $F(g) = f(g)$ for any $g \in G_+$. □

THEOREM 5.5.5. *The functors Δ and $\mathcal{D}$ establish a categorical equivalence between the category $\mathbb{P}\mathrm{erf}$ of perfect MV-algebras and the category $\mathbb{ALG}$ of Abelian ℓ-groups.*

Proof. By Propositions 5.5.3 and 5.5.4. □

In the sequel we give a complete characterization for MV-algebras of finite rank.

THEOREM 5.5.6. *The following are equivalent for any MV-algebra A and $n \geq 1$:*

(a) *A is an MV-algebra of rank n,*

(b) *$A \simeq \Gamma(Z \times_{lex} G, \langle n, g\rangle)$ with G Abelian ℓ-group and $g \in G$.*

Proof. It is easy to check that $\Gamma(Z \times_{lex} G, \langle n, g\rangle)$ has rank n.

Let A be an MV-algebra of rank n and let ψ be the isomorphism from $A/Rad(A)$ onto $\mathbf{Ł}_{n+1}$ and let $\{0, b, 2b, \ldots, nb\}$ be defined as in Proposition 4.3.24.

Further, let $G = \mathcal{D}(\langle Rad(A)\rangle)$. We set $A' = \Gamma(Z \times_{lex} G, \langle n, g\rangle)$ with $g = [(nb)^*, 0]$ and we want to show that A is isomorphic to A'. Define a map $\varphi\colon A \to A'$ in the following way: if $\psi([x]_{Rad(A)}) = r/n$ with $0 \leq r \leq n$ by Lemma 3.2.6 we have $x = (rb \oplus \epsilon) \odot \mu^*$ with $\epsilon, \mu \in Rad(A)$. We set $\varphi(x) = \langle r, [\epsilon, \mu]\rangle$. Note that in particular, if $x \in Rad(A)$, $\varphi(x) = \langle 0, [x, 0]\rangle$.

In order to prove that φ is an isomorphism, note that Lemmas 3.2.6 and 3.5.12 insure that φ is well-defined and it is a bijection by Proposition 4.3.24.

The element $1 \in A$ is such that $1 = nb \oplus (nb)^*$ with $(nb)^* \in Rad(A)$ hence $\varphi(1) = \langle n, [(nb)^*, 0]\rangle$. Let us prove that $\varphi((x)^*) = \varphi(x)^*$. Let $x = (rb \oplus \epsilon) \odot \mu^*$, hence $x^* = ((rb \oplus \epsilon) \odot \mu^*)^* = ((rb \odot \mu^*) \oplus \epsilon)^* = ((rb)^* \oplus \mu) \odot \epsilon^*$. By Proposition 4.3.24, $(rb)^* = (n-r)b \oplus (nb)^*$ hence $x^* = ((n-r)b \oplus ((nb)^* \oplus \mu)) \odot \epsilon^*)$. So $\varphi(x^*) =$

$\langle n-r, [(nb)^* \oplus \mu, \epsilon]\rangle$. On the other side, $\varphi(x)^* = \langle r, [\epsilon, \mu]\rangle^* = \langle n-r, [(nb)^* \oplus \mu, \epsilon]\rangle$ hence the claim follows.

Let us prove that $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y)$. Let $x = (rb \oplus \epsilon) \odot \mu^*$ and $y = (sb \oplus \epsilon') \odot (\mu')^*$ and first consider the case $r + s \leq n$. Then by Lemma 3.2.6 and Proposition 4.3.24 $x \oplus y = (rb \oplus sb \oplus \epsilon \oplus \epsilon') \odot (\mu \oplus \mu') = ((r+s)b \oplus \epsilon \oplus \epsilon') \odot (\mu \oplus \mu')$ hence $\varphi(x \oplus y) = \langle r+s, [\epsilon \oplus \epsilon', \mu \oplus \mu']\rangle = \varphi(x) \oplus \varphi(y)$. If $r + s > n$, by definition $\varphi(x) \oplus \varphi(y) = \langle n, [(nb)^*, 0]\rangle$. Further, since $r > n-s$, we have $x > y^*$ hence $x \oplus y = 1$ and $\varphi(x \oplus y) = \langle n, [(nb)^*, 0]\rangle$. □

By Remark 4.3.35, the class $\mathbb{P}\mathrm{erf}$ is not equational, since it is not closed under direct products. Using Theorem 5.5.5 we will characterize the equational class generated by $\mathbb{P}\mathrm{erf}$ as the variety generated by Chang's MV-algebra $\mathbf{C}$.

Denote $\mathbf{V}(\mathbb{P}\mathrm{erf})$ the variety generated by the class of perfect MV-algebras and by $\mathbf{V}(\mathbf{C})$ the variety generated by Chang's MV-algebra.

PROPOSITION 5.5.7. $\mathbf{V}(\mathbb{P}\mathrm{erf}) = \mathbf{V}(\mathbf{C})$.

Proof. Since $\mathbf{C}$ is a perfect MV-algebra, it is obvious that $\mathbf{V}(\mathbb{P}\mathrm{erf})$ contains $\mathbf{V}(\mathbf{C})$. In order to prove the other inclusion, it suffices to show that any perfect MV-algebra is in $\mathbf{V}(\mathbf{C})$. Let A be a perfect MV-algebra. Then, by Theorem 5.5.5, A is isomorphic with $\Delta(G)$ for some Abelian ℓ-group G. Any Abelian ℓ-group is a homomorphic image of a subdirect product of groups isomorphic with Z. Then there exists an Abelian ℓ-group K, an ℓ-group homomorphism $h\colon K \to G$ and a set I such that $G = h(K)$ and K is an ℓ-subgroup of Z^I. It is easy to see that $\Delta(G) = \Delta(h)(\Delta(K))$, so A is a homomorphic image of $\Delta(K)$. But $\Delta(K)$ is an MV-subalgebra of $\Delta(Z^I)$, which is an MV-subalgebra of $\Delta(Z)^I$. Since $\Delta(Z)$ is isomorphic with $\mathbf{C}$ (see Remark 4.3.34) it follows that $\Delta(K)$ is isomorphic with a subalgebra of $\mathbf{C}^I$, so $\Delta(K)$ is in $\mathbf{V}(\mathbf{C})$. Hence A is in $\mathbf{V}(\mathbf{C})$ since it is a homomorphic image of $\Delta(K)$. We proved that any perfect MV-algebra is in $\mathbf{V}(\mathbf{C})$, so the desired conclusion follows. □

DEFINITION 5.5.8. *An* MV*-algebra* A *has an* Archimedean radical *if for every* a, $b \in Rad(A)$, $na \leq b$ *for any* $n \in \mathsf{N}$ *implies* $a = 0$.

PROPOSITION 5.5.9. *If* G *is an Abelian* ℓ*-group and* $A = \Delta(G)$*, then the following are equivalent:*

(a) G *is an Archimedean* ℓ*-group,*

(b) A *has an Archimedean radical.*

Proof. Since $Rad(A) = \{\langle 0, g\rangle \mid g \geq 0\}$, the desired result is straightforward. □

In the following we prove that any MV-algebra A is, up to isomorphism, an interval algebra in the radical of a perfect MV-algebra.

LEMMA 5.5.10. *Let* P *be a perfect* MV*-algebra,* $G = \mathcal{D}(P)$ *and* $u \in G_+$*. Then for any latticial monoids isomorphism* $f\colon G_+ \to Rad(P)$ *the following are equivalent:*

(a) $f(u)$ *generates* $Rad(P)$*,*

(b) u *is a strong unit of* G*.*

Proof. (a) $\Rightarrow$ (b) If $g \in G_+$ then, by hypothesis, there is $n \in \mathsf{N}$ such that $f(g) \leq nf(u) = f(nu)$. Since f is an isomorphism we get $g \leq nu$. We have proved that for any $g \in G_+$ there exists $n \in \mathsf{N}$ such that $g \leq nu$. Hence, u is a strong unit of G.
(b) $\Rightarrow$ (a) Follows similarly. □

Recall that for an MV-algebra A and for $0 < a$ in A the interval MV-algebra $A(0, a)$ is defined in Example 2.4.6.

PROPOSITION 5.5.11. *For every* MV*-algebra A there is a perfect* MV*-algebra P and $a \in P$ such that $Rad(P)$ is generated by $\{a\}$ and A is isomorphic with the interval algebra $P(0, a)$.*

Proof. If A is an MV-algebra, then A is isomorphic with $\Gamma(G, u)$ for some Abelian ℓu-group $\langle G, u\rangle$. If we consider $P = \Delta(G)$, then, by Theorem 5.5.5, G and $\mathcal{D}(P)$ are isomorphic ℓ-groups and there exists a latticial monoids isomorphism $h\colon G_+ \to Rad(P)$. Since u is a strong unit of G, by Lemma 5.5.10, $a = h(u)$ is a generator for $Rad(P)$. We will prove that $f\colon \Gamma(G, u) \to P(0, a)$ defined by $f(x) := h(x)$ for any $x \in [0, u]_G$ is an MV-algebra isomorphism. If $x, y \in [0, u]_G$, then

$$f(x \oplus y) = f((x+y) \wedge u) = f(x+y) \wedge f(u) = (f(x) \oplus f(y)) \wedge a = f(x) \oplus_{[0,a]} f(y),$$

$$f(u-x) \oplus f(x) = f(u-x+x) = f(u) = a = a \vee f(x) = a \odot f(x)^* \oplus f(x).$$

Since $Rad(P)$ is a cancellative monoid, we get $f(x^*) = f(u-x) = a \odot f(x)^* = f(x)^{*[0,a]}$. Hence f is an MV-algebra isomorphism and A is isomorphic with the MV-algebra $P(0, a)$. □

5.6 Representations by ultrapowers

The representation of the MV-algebras as subdirect product of chains is very useful in practice, but it offers few information about their structure. representation theorem asserts that any MV-algebra is isomorphic to an algebra of nonstandard real valued functions.

THEOREM 5.6.1. *For any* MV*-algebra A there is an ultrapower ${}^*[0,1]$ of the* MV*-algebra $[0,1]$ such that A can be embedded into the product $({}^*[0,1])^{Spec(A)}$.*

Proof. Let A be an MV-algebra. By Theorem 4.1.4, A can be embedded into the direct product $\prod\{A/P \mid P \in Spec(A)\}$. If $P \in Spec(A)$, then A/P is an MV-chain, so A/P can be embedded into a divisible MV-chain D_P by Corollary 5.3.1. Now we consider the set $\mathcal{F} = \{D_P \mid P \in Spec(A)\}$. By Theorem 5.3.6 any two MV-algebras from $\mathcal{F}$ are elementarily equivalent. Using the joint embedding property [14, Proposition 3.1.4], there is an MV-algebra D such that D_P can be elementarily embedded in D for any $P \in Spec(A)$. It follows that D is also elementarily equivalent with the MV-algebras of $\mathcal{F}$. But $[0, 1]$ also is elementarily equivalent with the MV-algebras of $\mathcal{F}$, since $[0, 1]$ is a divisible MV-chain. Thus, by Frayne's Theorem [14, Corollary 4.3.13], there is an ultrapower ${}^*[0, 1]$ of $[0, 1]$ in which D is elementarily embedded. For any $P \in Spec(A)$ we get

$$A/P \hookrightarrow D_P \hookrightarrow D \hookrightarrow {}^*[0,1]$$

and we denote by $\iota_P : A/P \hookrightarrow {}^*[0,1]$ the resulting embedding. Hence, if we define $\iota : A \to ({}^*[0,1])^{Spec(A)}$ as

$$\iota(a) = \{\iota_P(a) \mid P \in Spec(A)\},$$

we get the desired embedding for the MV-algebra A. □

We will focus on the dependency of *non-standard* representation of an MV-algebra A on the cardinality of A, showing how to get, for certain classes of MV-algebras, a *non-standard* representation having as target algebra a unique ultrapower of the MV-algebra $[0,1]$. Actually, we explore classes of MV-algebras which are representable via algebras of functions from a set X to a fixed *regular* ultrapower of $[0,1]$.

From now on, $\mathbf{k}$ is an infinite cardinal. Recall that a model $\mathcal{A}$ is α-universal if and only if for every model $\mathcal{B}$ of power less than α which is elementarily equivalent to $\mathcal{A}$ is elementarily embedded in $\mathcal{A}$.

THEOREM 5.6.2 ([14, Theorem 4.3.12]). *Let $|\mathcal{L}| \leq \alpha$ and D be a ultrafilter which is α-regular. Then, for every model $\mathcal{A}$, the ultrapower $\prod_D \mathcal{A}$ is α^+-universal.*

LEMMA 5.6.3. *Let A be an MV-algebra and $\langle G, u\rangle$ an Abelian ℓu-group such that $A \simeq \Gamma(G,u)$. Let $\mathbf{k}$ be an infinite cardinal and $|A| = \mathbf{k}$, then $|G| = \mathbf{k}$.*

Proof. Suppose $|A| = \mathbf{k}$, where $\mathbf{k}$ an infinite cardinal. Let $\mathbf{a} = \{a_i \mid i \in Z\}$ be a good sequence. Then there is an integer n_a such that $a_i = 1, i < -n_a$ and $a_i = 0$, $i > n_a$. Thus we can identify the good sequence $\mathbf{a}$ with a $2n_a + 1$ tuple, $< a_{-n_a}, a_{-n_a+1}, \ldots, a_{n_a-1}, a_{n_a} >$. Call n_a the index of a. Then for the set S_n of all good sequences of a given index n, we have $|S_n|$ less than or equal to $\mathbf{k}^{2n+1}$. But $|\mathbf{k}^{2n+1}| = \mathbf{k}$. Thus S_n will have cardinality at most $\mathbf{k}$. The set S of all good sequences can be identified with the union of all S_n. Hence the cardinality of S, will be $|S| \leq \mathbf{k}|Z|$ which is just $\mathbf{k}$ since $|Z|$ is denumerable. So the set of all good sequences S we have $|S| < \mathbf{k}$ or $|S| = \mathbf{k}$.

To see that $|S| = \mathbf{k}$, just consider the good sequences of index $1, S_1$. Let a be an element of A and define $a_0 = a$, $a_1 = a_2 = \cdots = 0$, $a_{-1} = a_{-2} = \cdots = 1$. Thus the sequence looks like $< \ldots, 1, 1, a, 0, 0, \ldots >$. This is a good sequence as $a_{-2} + a_1 = 1 = a_{-2}a_{-1} + a_0 = 1 + a = 1 = a_{-1}a_0 + a_1 = a + 0 = a = a_0a_1 + a_2 = 0 = a_1$, etc. Clearly then $|S_1| = |A| = \mathbf{k}$ and so $|S| = \mathbf{k}$. □

LEMMA 5.6.4. *Let G be an Abelian ℓ-group and $\mathbf{k}$ be an infinite cardinal such that $|G| = \mathbf{k}$. Then G can be embedded into a divisible Abelian ℓ-group D_G such that $|D_G| = \mathbf{k}$.*

Proof. First we show that any Abelian group can be embedded in an Abelian divisible group of cardinality α, where $\alpha = \max\{\aleph_0, |G|\}$. Indeed, let X be a generating set of G and let F be the free Abelian group with basis X. Obviously, there exists some $H \leq F$ such that G is isomorphic to F/H. Let D be the direct product of $|X|$ copies of Q. Since F is isomorphic to the direct product of $|X|$ copies of Z we can embed it in D, in natural way. Hence G embeds in the Abelian divisible group D/H. Moreover $|D/H| \leq |X||\mathsf{Q}| \leq \alpha$ and obviously $|G| \leq |D/H|$. On the other hand, since D/H

is divisible it is infinite and $\aleph_0 \leq |D/H|$. D/H can be converted into an ℓ-group as follows: call an element h of D/H positive if $nh \in G^+$ for some positive integer n; this makes D/H an ℓ-group. □

THEOREM 5.6.5. *For every cardinal* $\mathbf{k}$ *there is an ultrapower* $\mathcal{U}_{\mathbf{k}}$ *of the* MV*-algebra* $[0,1]$*, such that every* MV*-chain* A *of cardinality* $\mathbf{k}$ *embeds in* $\mathcal{U}_{\mathbf{k}}$ *via an ultrafilter* $\mathbf{k}$*-regular over* $\mathbf{k}$*.*

Proof. Let A be an infinite MV-chain such that $|A| = \mathbf{k}$ and $A \simeq \Gamma(G,u)$, then, by Lemma 5.6.3, G is an ordered Abelian group with strong unit u and $|G| = \mathbf{k}$. Hence $\langle G,u \rangle$ can be embedded into a divisible ordered group D_G with strong unit u_D, in addition, $|D_G| = \mathbf{k}$. Since D_G is elementarily equivalent to $\mathbf{R}$, we get that the MV-chain $A_d = \Gamma(D_G,u_D)$ is elementarily equivalent to $[0,1]$, A can be embedded into A_d, i.e. $A_d = \Gamma(D_G,u_D) \equiv [0,1]$ and $A \hookrightarrow A_d$.

Now, by Frayne's Theorem, A_d can be elementarily embedded into an ultrapower $\prod_F [0,1]$ of $[0,1]$ via an ultrafilter F. Since the cardinality of A is $\mathbf{k}$, then, in the light of the proof of Frayne's Theorem, F can be $\mathbf{k}$-regular over $\mathbf{k}$. Hence A can be embedded into $\mathcal{U}_{\mathbf{k}} = \prod_F [0,1]$, which is the desired ultrapower. □

THEOREM 5.6.6. *For every cardinal* $\mathbf{k}$ *there is an ultrapower,* $\mathcal{U}_{\mathbf{k}}$*, of the* MV*-algebra* $[0,1]$*, obtained via a ultrafilter* $\mathbf{k}$*-regular over* $\mathbf{k}$*, such that every* MV*-algebra* A *of cardinality* $\mathbf{k}$ *embeds into an* MV*-algebra of functions from a set to* $\mathcal{U}_{\mathbf{k}}$*.*

Proof. Let A be an infinite MV-algebra of cardinality $\mathbf{k}$. Then by Chang' representation theorem we have $A \hookrightarrow \prod_{P \in Spec(A)} A/P$. Moreover, $|A/P| \leq \mathbf{k}$ for every prime ideal P of A.

Let F be a ultrafilter $\mathbf{k}$-regular over $\mathbf{k}$. By Theorem 5.6.2, the ultrapower $\prod_F [0,1]$ is $\mathbf{k}^+$-universal. Then for every $P \in$ *Spec*(A) the MV-chain A/P can be embedded into the ultrapower $\prod_F [0,1]$ of the MV-algebra $[0,1]$. Indeed the ultrafilter F is independent of A/P. Hence A can be embedded into $(\prod_F [0,1])^{Spec(A)}$, then we get the representation claimed by the theorem. □

6 Łukasiewicz ∞-valued logic

6.1 The syntax of Ł

The *language* of the propositional calculus Ł consists of: denumerable many propositional variables: $v_1, \ldots, v_n, \ldots$ (the set of all the propositional variables will be denoted by V); logical connectives: $\to$ and $\neg$; and parenthesis: (and).

The *formulas* are defined inductively as follows:

(f1) every propositional variable is a formula,

(f2) if φ is a formula, then $\neg\varphi$ is a formula,

(f3) if φ and ψ are formulas, then $(\varphi \to \psi)$ is a formula,

(f4) a string of symbols is a formula of Ł iff it can be shown to be a formula by a finite number of applications of (f1), (f2), and (f3).

We will denote by $Fm_{Ł}$ the set of all formulas of Ł. The particular four *axiom schemes* of this propositional calculus are:

(A1) $\varphi \to (\psi \to \varphi)$,

(A2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(A3) $((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$,

(A4) $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$.

The *deduction rule* is *modus ponens* (MP): from φ and $\varphi \to \psi$ infer ψ.

DEFINITION 6.1.1 (Syntactic consequences, theorems). *Let Θ be a set of formulas and φ a formula. A* Θ-proof *for φ is a finite sequence of formulas $\varphi_1, \ldots, \varphi_n = \varphi$ such that, for any $i \in \{1, \ldots, n\}$, at least one of the following conditions holds:*

(c1) φ_i *is an axiom,*

(c2) $\varphi_i \in \Theta$,

(c3) *there are $j, k < i$ such that φ_k is $\varphi_j \to \varphi_i$ (the formula φ_i follows from φ_j and φ_k using* modus ponens*).*

We will say that φ is a syntactic consequence *of Θ (or φ is* provable *from Θ) if there exists a Θ-proof for φ. We write $\Theta \vdash \varphi$. The set of all the syntactic consequences of Θ will be denoted by $Theor(\Theta)$.*

A formula φ will be called a theorem *(or* provable formula*) if it is provable from the empty set. This will be denoted by $\vdash \varphi$. In this case, a proof for φ will be a sequence of formulas $\varphi_1, \ldots, \varphi_n = \varphi$ such that for any $i \in \{1, \ldots, n\}$, one of the above conditions (c1) or (c3) is satisfied. The set of all the theorems will be denoted by $Theor$.*

LEMMA 6.1.2 (Syntactic compactness). *If Θ is a set of formulas and φ is a formula such that $\Theta \vdash \varphi$, then $\Gamma \vdash \varphi$ for some finite subset $\Gamma \subseteq \Theta$.*

Proof. If $\Theta \vdash \varphi$, then there exists a Θ-proof $\varphi_1, \ldots, \varphi_n = \varphi$ for φ. We consider $\Gamma = \Theta \cap \{\varphi_1, \ldots, \varphi_n\}$. It is straightforward by Definition 6.1.1 that $\varphi_1, \ldots, \varphi_n = \varphi$ is a Γ-proof for φ. $\square$

In the sequel we will prove some theorems of Ł. We present a detailed proof for the theorem

(1) $\varphi \to ((\varphi \to \psi) \to \psi)$.

The proof is:

$$\begin{array}{lr}
\vdash \varphi \to ((\psi \to \varphi) \to \varphi) & (\mathrm{A1}) \\
\vdash (\varphi \to ((\psi \to \varphi) \to \varphi)) \to (((\psi \to \varphi) \to \varphi) \to ((\varphi \to \psi) \to \psi)) \to & \\
\qquad\qquad (\varphi \to ((\varphi \to \psi) \to \psi))) & (\mathrm{A2}) \\
\vdash (((\psi \to \varphi) \to \varphi) \to ((\varphi \to \psi) \to \psi)) \to (\varphi \to ((\varphi \to \psi) \to \psi)) & (\mathrm{MP}) \\
\vdash ((\psi \to \varphi) \to \varphi) \to ((\varphi \to \psi) \to \psi) & (\mathrm{A3}) \\
\vdash \varphi \to ((\varphi \to \psi) \to \psi) & (\mathrm{MP})
\end{array}$$

In the sequel, when we will use an axiom (or a previous proved theorem), like (A2) in the previous proof, we will not write the axiom (or the theorem) obtained by making some particular substitution. We will only indicate the axiom (theorem) used and how many time we apply *modus ponens*. For example, the previous proof will be:

$\vdash \varphi \to ((\psi \to \varphi) \to \varphi)$	(A1)
$\vdash ((\psi \to \varphi) \to \varphi) \to ((\varphi \to \psi) \to \psi)$	(A3)
$\vdash \varphi \to ((\varphi \to \psi) \to \psi)$	(A2), 2 (MP)

PROPOSITION 6.1.3. *In Ł the following formulas are theorems:*

(2) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$,

(3) $\varphi \to \varphi$,

(4) $(\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$,

(5) $\neg\neg\varphi \to (\psi \to \varphi)$,

(6) $\neg\neg\varphi \to \varphi$,

(7) $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$,

(8) $\varphi \to \neg\neg\varphi$,

(9) $(\neg\varphi \to \psi) \to (\neg\psi \to \varphi)$,

(10) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$.

Proof. Let α be an axiom and θ the formula $((\psi \to \chi) \to \chi) \to (\varphi \to \chi)$ then

(2)	$\vdash (\psi \to ((\psi \to \chi) \to \chi)) \to (\theta \to (\psi \to (\varphi \to \chi)))$	(A2)
	$\vdash \theta \to (\psi \to (\varphi \to \chi))$	(1), (MP)
	$\vdash (\varphi \to (\psi \to \chi)) \to \theta$	(A2)
	$\vdash (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$	(A2), 2 (MP)
(3)	$\vdash \varphi \to (\alpha \to \varphi)$	(A1)
	$\vdash (\varphi \to (\alpha \to \varphi)) \to (\alpha \to (\varphi \to \varphi))$	(2)
	$\vdash \alpha \to (\varphi \to \varphi)$	(MP)
	$\vdash \alpha$	axiom
	$\vdash \varphi \to \varphi$	(MP)
(4)	$\vdash (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$	(A1), (2), (MP)
(5)	$\vdash \neg\neg\varphi \to (\neg\neg\psi \to \neg\neg\varphi)$	(A1)
	$\vdash (\neg\neg\psi \to \neg\neg\varphi) \to (\neg\varphi \to \neg\psi)$	(A4)
	$\vdash \neg\neg\varphi \to (\neg\varphi \to \neg\psi)$	(A2), 2 (MP)
	$\vdash (\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$	(A4)
	$\vdash \neg\neg\varphi \to (\psi \to \varphi)$	(A2), 2 (MP)
(6)	$\vdash \neg\neg\varphi \to (\alpha \to \varphi)$	(5)
	$\vdash \alpha \to (\neg\neg\varphi \to \varphi)$	(2), (MP)
	$\vdash \alpha$	axiom
	$\vdash \neg\neg\varphi \to \varphi$	(MP)

$$\begin{array}{lll}
(7) & \vdash \neg\neg\varphi \to \varphi & (6) \\
 & \vdash (\varphi \to \neg\psi) \to (\neg\neg\varphi \to \neg\psi) & (\mathrm{A2}), (\mathrm{MP}) \\
 & \vdash (\neg\neg\varphi \to \neg\psi) \to (\psi \to \neg\varphi) & (\mathrm{A4}) \\
 & \vdash (\varphi \to \neg\psi) \to (\psi \to \neg\varphi) & (\mathrm{A2}), 2\ (\mathrm{MP}) \\
(8) & \vdash \neg\varphi \to \neg\varphi & (3) \\
 & \vdash (\neg\varphi \to \neg\varphi) \to (\varphi \to \neg\neg\varphi) & (7) \\
 & \vdash \varphi \to \neg\neg\varphi & (\mathrm{MP}) \\
(9) & \vdash \psi \to \neg\neg\psi & (8) \\
 & \vdash (\neg\varphi \to \psi) \to (\neg\varphi \to \neg\neg\psi) & (4), (\mathrm{MP}) \\
 & \vdash (\neg\varphi \to \neg\neg\psi) \to (\neg\psi \to \varphi) & (\mathrm{A4}) \\
 & \vdash (\neg\varphi \to \psi) \to (\neg\psi \to \varphi) & (\mathrm{A2}), 2\ (\mathrm{MP}) \\
(10) & \vdash \neg\neg\varphi \to \varphi & (6) \\
 & \vdash (\varphi \to \psi) \to (\neg\neg\varphi \to \psi) & (\mathrm{A2}), (\mathrm{MP}) \\
 & \vdash (\neg\neg\varphi \to \psi) \to (\neg\psi \to \neg\varphi) & (9) \\
 & \vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi) & (\mathrm{A2}), 2\ (\mathrm{MP})
\end{array}$$

□

We define other logical connectives as follows (for any $n \geq 1$):

$$\begin{array}{rcl@{\qquad}rcl}
\varphi \oplus \psi & := & \neg\varphi \to \psi & \varphi \odot \psi & := & \neg(\varphi \to \neg\psi), \\
\varphi \vee \psi & := & (\varphi \to \psi) \to \psi & \varphi \wedge \psi & := & \varphi \odot (\varphi \to \psi), \\
\varphi^n & := & \underbrace{\varphi \odot \cdots \odot \varphi}_{n} & n\varphi & := & \underbrace{\varphi \oplus \cdots \oplus \varphi}_{n}
\end{array}$$

One can easily see that $(\mathrm{A3})$ is equivalent to $\varphi \vee \psi \to \psi \vee \varphi$ and (1) is equivalent to $\varphi \to \varphi \vee \psi$.

PROPOSITION 6.1.4. *The following formulas are theorems of Ł:*

(11) $(\varphi \to (\psi \to \chi)) \to ((\varphi \odot \psi) \to \chi)$,

(12) $((\varphi \odot \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$,

(13) $(\varphi \odot \psi) \to (\psi \odot \varphi)$,

(14) $\varphi \to (\psi \to (\psi \odot \varphi))$,

(15) $\varphi \to (\psi \to (\varphi \odot \psi))$,

(16) $(\varphi \odot \neg\varphi) \to \psi$,

(17) $(\varphi \odot \psi) \to \varphi$,

(18) $(\varphi \to \psi) \to ((\chi \odot \varphi) \to (\chi \odot \psi))$,

(19) $((\chi \odot \varphi) \to (\chi \odot \psi)) \to ((\varphi \odot \chi) \to (\psi \odot \chi))$,

(20) $(\varphi \odot (\varphi \to \psi)) \to \psi$,

(21) $(\varphi \to \psi) \to ((\varphi \vee \chi) \to (\psi \vee \chi))$,

(22) $((\varphi \vee \chi) \to (\psi \vee \chi)) \to ((\chi \vee \varphi) \to (\chi \vee \psi))$,

(23) $(\varphi \vee \varphi) \to \varphi$.

Proof. The following sequences are formal proofs in Ł.

(11)	$\vdash (\psi \to \chi) \to (\neg\chi \to \neg\psi)$	(10)
	$\vdash (\varphi \to (\psi \to \chi)) \to (\varphi \to (\neg\chi \to \neg\psi))$	(4), (MP)
	$\vdash (\varphi \to (\neg\chi \to \neg\psi)) \to (\neg\chi \to (\varphi \to \neg\psi))$	(2)
	$\vdash (\varphi \to (\psi \to \chi)) \to (\neg\chi \to (\varphi \to \neg\psi))$	(A2), 2 (MP)
	$\vdash (\neg\chi \to (\varphi \to \neg\psi)) \to (\neg(\varphi \to \neg\psi) \to \chi)$	(9)
	$\vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \odot \psi) \to \chi)$	(A2), 2 (MP)
(12)	$\vdash ((\varphi \odot \psi) \to \chi) \to (\neg\chi \to (\varphi \to \neg\psi))$	(9)
	$\vdash (\neg\chi \to (\varphi \to \neg\psi)) \to (\varphi \to (\neg\chi \to \neg\psi))$	(2)
	$\vdash ((\varphi \odot \psi) \to \chi) \to (\varphi \to (\neg\chi \to \neg\psi))$	(A2), 2 (MP)
	$\vdash (\varphi \to (\neg\chi \to \neg\psi)) \to (\varphi \to (\psi \to \chi))$	(A4), (4)
	$\vdash ((\varphi \odot \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$	(A2), 2 (MP)
(13)	$\vdash (\psi \to \neg\varphi) \to (\varphi \to \neg\psi)$	(7)
	$\vdash \neg(\varphi \to \neg\psi) \to \neg(\psi \to \neg\varphi)$	(10)
	$\vdash (\varphi \odot \psi) \to (\psi \odot \varphi)$	
(14)	follows from (13) and (12) using (MP)	
(15)	follows from (14) and (2) using (MP)	
(16)	$\vdash \varphi \to (\neg\psi \to \varphi)$	(A1)
	$\vdash (\neg\psi \to \varphi) \to (\neg\varphi \to \psi)$	(9)
	$\vdash \varphi \to (\neg\varphi \to \psi)$	(A2), 2 (MP)
	$\vdash (\varphi \odot \neg\varphi) \to \psi$	(11), (MP)
(17)	$\vdash (\varphi \odot \neg\varphi) \to \neg\psi$	(16)
	$\vdash \varphi \to (\neg\varphi \to \neg\psi)$	(12), (MP)
	$\vdash \neg\varphi \to (\varphi \to \neg\psi)$	(2), (MP)
	$\vdash \neg(\varphi \to \neg\psi) \to \varphi$	(9), (MP)
	$\vdash (\varphi \odot \psi) \to \varphi$	
(18)	$\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$	(10)
	$\vdash (\neg\psi \to \neg\varphi) \to ((\chi \to \neg\psi) \to (\chi \to \neg\varphi))$	(4)
	$\vdash (\varphi \to \psi) \to ((\chi \to \neg\psi) \to (\chi \to \neg\varphi))$	(A2), 2 (MP)
	$\vdash ((\chi \to \neg\psi) \to (\chi \to \neg\varphi)) \to ((\chi \odot \varphi) \to (\chi \odot \psi))$	(10)
	$\vdash (\varphi \to \psi) \to ((\chi \odot \varphi) \to (\chi \odot \psi))$	(A2), 2 (MP)
(19)	$\vdash (\varphi \odot \chi) \to (\chi \odot \varphi)$	(13)
	$\vdash ((\chi \odot \varphi) \to (\chi \odot \psi)) \to ((\varphi \odot \chi) \to (\chi \odot \psi))$	(A2), (MP)
	$\vdash (\chi \odot \psi) \to (\psi \odot \chi)$	(13)
	$\vdash ((\varphi \odot \chi) \to (\chi \odot \psi)) \to ((\varphi \odot \chi) \to (\psi \odot \chi))$	(A4), (MP)
	$\vdash ((\chi \odot \varphi) \to (\chi \odot \psi)) \to ((\varphi \odot \chi) \to (\psi \odot \chi))$	(A2), 2 (MP)
(20)	$\vdash (\varphi \to \psi) \to (\varphi \to \psi)$	(3)
	$\vdash (\varphi \odot (\varphi \to \psi)) \to \psi$	(11), (MP)
(21)	$\vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	(A2)
	$\vdash ((\psi \to \chi) \to (\varphi \to \chi)) \to$ $(((\varphi \to \chi) \to \chi) \to ((\psi \to \chi) \to \chi))$	(A2)
	$\vdash (\varphi \to \psi) \to (((\varphi \to \chi) \to \chi) \to ((\psi \to \chi) \to \chi))$	(A2), 2 (MP)
	$\vdash (\varphi \to \psi) \to ((\varphi \vee \chi) \to (\psi \vee \chi))$	

(22) is similar with (19), but we use $(\mathrm{A3})$ instead of (13).

(23)
$$\begin{array}{ll}
\vdash (\varphi \to \varphi) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)) & (\mathrm{A1}) \\
\vdash (\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi) & (3), (\mathrm{MP}) \\
\vdash ((\varphi \to \varphi) \to \varphi) \to \varphi & (\mathrm{A3}), (\mathrm{MP}) \\
\vdash (\varphi \vee \varphi) \to \varphi &
\end{array}$$
$\square$

LEMMA 6.1.5. *Let* Θ *be a set of formulas such that* $\Theta \vdash \varphi \to \psi$ *and* $\Theta \vdash \alpha \to \beta$. *Then:*

(a) $\Theta \vdash (\varphi \odot \alpha) \to (\psi \odot \beta)$,

(b) $\Theta \vdash (\varphi \vee \alpha) \to (\psi \vee \beta)$.

Proof. (a) The following sequence is formal proof in Ł.

$$\begin{array}{ll}
\Theta \vdash \varphi \to \psi & \\
\Theta \vdash (\alpha \odot \varphi) \to (\alpha \odot \psi) & (18), (\mathrm{MP}) \\
\Theta \vdash (\varphi \odot \alpha) \to (\psi \odot \alpha) & (19), (\mathrm{MP}) \\
\Theta \vdash \alpha \to \beta & \\
\Theta \vdash (\psi \odot \alpha) \to (\psi \odot \beta) & (18), (\mathrm{MP}) \\
\Theta \vdash (\varphi \odot \alpha) \to (\psi \odot \beta) & (\mathrm{A2}), 2\ (\mathrm{MP})
\end{array}$$

(b) Similar to (a), but we use (21) instead of (18) and (22) instead of (19). $\square$

LEMMA 6.1.6. *If* φ *is a formula and* Θ *is a set of formulas, then:*

(a) $\vdash \varphi^n \to \varphi$ *for any* $n \geq 1$,

(b) $\Theta \vdash \varphi$ *implies* $\Theta \vdash \varphi^n$ *for any* $n \geq 1$,

(c) *if* $\Theta \vdash \varphi^n$ *for some* $n \geq 1$, *then* $\Theta \vdash \varphi$.

Proof. (a) If $n = 1$, then $\vdash \varphi \to \varphi$ by (3). If $n > 1$, we get $\vdash \varphi \odot \varphi^{n-1} \to \varphi$ by (17).
(b) We prove that $\Theta \vdash \varphi^n$ for any $n \geq 1$ by induction. If $n = 1$, then the intended result follows by (3). Now we suppose that $\Theta \vdash \varphi^n$. It follows that

$$\begin{array}{ll}
\Theta \vdash \varphi^{n+1} \to \varphi^{n+1} & (3) \\
\Theta \vdash (\varphi^n \odot \varphi) \to \varphi^{n+1} & \\
\Theta \vdash \varphi^n \to (\varphi \to \varphi^{n+1}) & (12), (\mathrm{MP}) \\
\Theta \vdash \varphi^n & \text{induction hypothesis} \\
\Theta \vdash \varphi & \\
\Theta \vdash \varphi^{n+1} & 2\ (\mathrm{MP})
\end{array}$$

(c) Follows by (a). $\square$

THEOREM 6.1.7 (Implicational deduction theorem). *If* $\Theta \subseteq Fm_{\text{Ł}}$ *and* $\varphi, \psi \in Fm_{\text{Ł}}$, *then*

$$\Theta \cup \{\varphi\} \vdash \psi \textit{ iff there is } n \geq 1 \textit{ such that } \Theta \vdash \varphi^n \to \psi.$$

Proof. If $\Theta \vdash \varphi^n \to \psi$ for some $n \geq 1$, then

$$\begin{array}{ll}
\Theta \cup \{\varphi\} \vdash \varphi & \\
\Theta \cup \{\varphi\} \vdash \varphi^n & \text{Lemma 6.1.6 (b)} \\
\Theta \cup \{\varphi\} \vdash \varphi^n \to \psi & (\Theta \subseteq \Theta \cup \{\varphi\}) \\
\Theta \cup \{\varphi\} \vdash \psi & (\mathrm{MP})
\end{array}$$

Conversely, suppose that $\Theta\cup\{\varphi\} \vdash \psi$ and let $\alpha_1, \ldots, \alpha_k = \psi$ be a $\Theta\cup\{\varphi\}$-proof for ψ. We prove by induction that, for any $1 \le i \le k$, there is $n_i \ge 1$ such that $\Theta \vdash \varphi^{n_i} \to \alpha_i$. For $i = 1$ we have three possible cases:

– if α_1 is an axiom, then

$\Theta \vdash \alpha_1 \to (\varphi \to \alpha_1)$ (A1)
$\Theta \vdash \alpha_1$ axiom
$\Theta \vdash \varphi \to \alpha_1$ (MP)

– if $\alpha_1 \in \Theta$ the proof is similar,

– if $\alpha_1 = \varphi$, then $\Theta \vdash \varphi \to \varphi$ by (3). Thus, if $i = 1$, then $n_1 = 1$. Let $1 < i \le k$ and suppose the desired result holds for $j < i$. If α_i is an axiom or $\alpha_i \in \Theta \cup \{\varphi\}$, then we get $n_i = 1$ like above. Suppose that α_i is follows from α_j and α_t using (MP), where $j, t < i$. By induction, there are $n_j, n_t \ge 1$ such that $\Theta \vdash \varphi^{n_j} \to \alpha_j$ and $\Theta \vdash \varphi^{n_t} \to \alpha_t$. We assume that $\alpha_t = \alpha_j \to \alpha_i$. It follows that $\Theta \vdash \varphi^{n_j} \to \alpha j$ and $\Theta \vdash \varphi^{n_t} \to (\alpha_j \to \alpha_i)$. By Lemma 6.1.5 (a), we get $\Theta \vdash (\varphi^{n_j} \odot \varphi^{n_t}) \to (\alpha_j \odot (\alpha_j \to \alpha_i))$. We consider $n_i = n_j + n_t$. Thus,

$\Theta \vdash \varphi^{n_i} \to (\alpha_j \odot (\alpha_j \to \alpha_i))$
$\Theta \vdash (\alpha_j \odot (\alpha_j \to \alpha_i)) \to \alpha_i$ (20)
$\Theta \vdash \varphi^{n_i} \to \alpha_i$ (A2), 2 (MP) □

6.2 The Lindenbaum–Tarski algebra (Ł(Θ), Ł)

In the sequel Θ is a fixed set of formulas. For any two formulas φ and ψ we define

$$\varphi \sim_\Theta \psi \quad \text{iff} \quad \Theta \vdash \varphi \to \psi \text{ and } \Theta \vdash \psi \to \varphi.$$

LEMMA 6.2.1. *The relation $\sim_\Theta$ is an equivalence relation on $Fm_{\text{Ł}}$.*

Proof. The relation $\sim_\Theta$ is reflexive by (3) and it is symmetric by definition. In order to prove the transitivity, we suppose that $\varphi \sim_\Theta \psi$ and $\psi \sim_\Theta \chi$. It follows that $\Theta \vdash \varphi \to \psi$ and $\Theta \vdash \psi \to \chi$. Using (A2) and applying twice (MP), we infer that $\Theta \vdash \varphi \to \chi$. Similarly, it follows that $\Theta \vdash \chi \to \varphi$, so $\varphi \sim_\Theta \chi$. □

For any formula $\varphi \in Fm_{\text{Ł}}$ we will denote by $[\varphi]_\Theta$ the equivalence class of φ with respect to $\sim_\Theta$, i.e. $[\varphi]_\Theta = \{\psi \mid \Theta \vdash \varphi \to \psi \text{ and } \Theta \vdash \psi \to \varphi\}$. Consequently, $Fm_{\text{Ł}}/{\sim_\Theta} = \{[\varphi]_\Theta \mid \varphi \in Fm_{\text{Ł}}\}$ is the quotient of $Fm_{\text{Ł}}$ with respect to $\sim_\Theta$.

LEMMA 6.2.2. *For any formulas φ and ψ the following properties hold:*

(a) $\Theta \vdash \varphi$ *iff* $[\varphi]_\Theta = \mathit{Theor}(\Theta)$,

(b) *if* $\Theta \vdash \varphi$, *then* $[\varphi \to \psi]_\Theta = [\psi]_\Theta$.

Proof. (a) We suppose that $\Theta \vdash \varphi$ and we consider $\psi \in \mathit{Theor}(\Theta)$. It follows that $\Theta \vdash \psi$. By (A1), we get $\Theta \vdash \psi \to (\varphi \to \psi)$ and $\Theta \vdash \varphi \to (\psi \to \varphi)$. Using (MP), we infer that $\Theta \vdash \psi \to \varphi$ and $\Theta \vdash \varphi \to \psi$, so $\mathit{Theor}(\Theta) \subseteq [\varphi]_\Theta$. Conversely, if $\psi \in [\varphi]_\Theta$, then $\Theta \vdash \varphi \to \psi$. Using again (MP), we get $\Theta \vdash \psi$, so $\psi \in \mathit{Theor}(\Theta)$. We proved that $[\varphi]_\Theta \subseteq \mathit{Theor}(\Theta)$. Now we suppose that $[\varphi]_\Theta = \mathit{Theor}(\Theta)$. Let α be an axiom. Then $\alpha \in \mathit{Theor}(\Theta)$ and $\alpha \sim_\Theta \varphi$. We get $\Theta \vdash \alpha$ and $\Theta \vdash \alpha \to \varphi$, so $\Theta \vdash \varphi$ by (MP).

(b) By (A1), it follows that $\Theta \vdash \psi \to (\varphi \to \psi)$ and $\Theta \vdash \varphi \to ((\psi \to \varphi) \to \varphi)$. By hypothesis and (MP), we infer that $\Theta \vdash (\psi \to \varphi) \to \varphi$. Using (A3) and (MP), it follows that $\Theta \vdash (\varphi \to \psi) \to \psi$. Thus $\Theta \vdash \psi \to (\varphi \to \psi)$ and $\Theta \vdash (\varphi \to \psi) \to \psi$, so $[\varphi \to \psi]_\Theta = [\psi]_\Theta$. □

On $Fm_Ł/\sim_\Theta$ we define the following operations:

$$[\varphi]^*_\Theta := [\neg\varphi]_\Theta \qquad [\varphi]_\Theta \to [\psi]_\Theta := [\varphi \to \psi]_\Theta \qquad [\varphi]_\Theta \oplus [\psi]_\Theta := [(\neg\varphi) \to \psi]_\Theta.$$

PROPOSITION 6.2.3.
The structure $\langle Fm_Ł/\sim_\Theta, \to, ^*, Theor(\Theta)\rangle$ is a Wajsberg algebra.

Proof. If α is an axiom, then $\Theta \vdash \alpha$ so, by Lemma 6.2.2 (a), $[\alpha]_\Theta = Theor(\Theta)$. Thus the Wajsberg algebras axioms (W2)–(W4) from Definition 2.3.6 follows by Lemma 6.2.2 (a) applied to axioms (A2)–(A4). The axiom (W1) follows by Lemma 6.2.2 (b). □

COROLLARY 6.2.4. *The structure $\langle Fm_Ł/\sim_\Theta, \oplus, ^*, Theor(\Theta)^*\rangle$ is an* MV*-algebra.*

Proof. It is straightforward by Proposition 2.3.10. □

In the sequel, we will denote $\mathbf{Ł}(\Theta) = \langle Fm_Ł/\sim_\Theta, \oplus, ^*, Theor(\Theta)^*\rangle$ and this MV-algebra will be called *the Lindenbaum–Tarski algebra of* Θ. Note that, for any set of formulas Θ, the Lindenbaum–Tarski algebras of Θ and $Theor(\Theta)$ coincide, i.e. $\mathbf{Ł}(\Theta) = \mathbf{Ł}(Theor(\Theta))$. In the particular case when $\Theta = \emptyset$, the relation $\sim_\emptyset$ will be simply denoted by $\sim$. Thus, we have

$$\varphi \sim \psi \quad \text{iff} \quad \vdash \varphi \to \psi \text{ and } \vdash \psi \to \varphi.$$

The equivalence class of a formula φ with respect to $\sim$ will be denoted by $[\varphi]$. Consequently, the MV-algebra $\mathbf{Ł}(\emptyset) = \mathbf{Ł}(Theor)$ will be denoted by $\mathbf{Ł}$ and this is the Lindenbaum–Tarski algebra of the ∞-valued Łukasiewicz propositional calculus. In order to simplify the notation, $\mathbf{Ł}(\Theta)$ will denote both the Lindenbaum–Tarski algebra and its support.

6.3 Consistent sets, linear sets, and deductive systems

DEFINITION 6.3.1. *Let Θ be a set of formulas. We say that Θ is* consistent *if there is a formula φ such that $\Theta \nvdash \varphi$. Otherwise, Θ is called* inconsistent*. One can easily see that Θ is inconsistent iff $Theor(\Theta) = Fm_Ł$. The set Θ is* maximally consistent *if it cannot be strictly included in any consistent set.*

LEMMA 6.3.2. *For a set Θ of formulas the following are equivalent:*

(a) *Θ is inconsistent,*

(b) *$\Theta \vdash \varphi$ and $\Theta \vdash \neg\varphi$ for some $\varphi \in Fm_Ł$,*

(c) *$\Theta \vdash \varphi \odot \neg\varphi$ for some $\varphi \in Fm_Ł$.*

Proof. (a) ⇒ (b) Straightforward, since $Theor(\Theta) = Fm_Ł$.
(b) ⇒ (c) By hypothesis, (15) and twice (MP).
(c) ⇒ (a) By hypothesis and (16), using (MP), we get $\Theta \vdash \psi$ for any formula ψ. □

COROLLARY 6.3.3. *Any consistent set of formulas is contained in a maximally consistent set of formulas.*

Proof. Let Θ be a consistent set of formulas and

$$\mathcal{T} = \{\Gamma \subseteq Fm_Ł \mid \Gamma \text{ is consistent and } \Theta \subseteq \Gamma\}.$$

We will prove that $\mathcal{T}$ is inductively ordered. Thus, the desired conclusion follows by Zorn's Lemma. Let $\{\Gamma_i \mid i \in I\}$ be a totally ordered (with respect to the set inclusion) family from $\mathcal{T}$. We have to prove that this family has an upper bound in $\mathcal{T}$. This upper bound will be $\Gamma = \bigcup\{\Gamma_i \mid i \in I\}$. We only have to prove that Γ is consistent. Suppose that Γ is inconsistent. By Lemma 6.3.2, we get $\Gamma \vdash \varphi \odot \neg\varphi$. Than there exists a Γ-proof $\varphi_1, \ldots, \varphi_n$ for $\varphi \odot \neg\varphi$. Let $i_1, \ldots, i_n \in I$ such that $\varphi_j \in \Gamma_{i_j}$ for any $j \in \{1, \ldots, n\}$. Since the family $\{\Gamma_i \mid i \in I\}$ is totally ordered, there is $k \in \{1, \ldots, n\}$ such that $\Gamma_{i_j} \subseteq \Gamma_{i_k}$, for any $j \in \{1, \ldots, n\}$. Thus $\varphi_1, \ldots, \varphi_n$ is a Γ_{i_k}-proof for $\varphi \odot \neg\varphi$ so, by Lemma 6.3.2, Γ_{i_k} is inconsistent, which is a contradiction. Hence, Γ is consistent and $\mathcal{T}$ is inductively ordered. □

LEMMA 6.3.4. *If Θ is a set of formulas and φ is a formula, then the following are equivalent:*

- *(a)* $\Theta \cup \{\varphi\}$ *is inconsistent,*
- *(b)* $\Theta \vdash \neg(\varphi^n)$ *for some* $n \geq 1$.

Proof. (a) ⇒ (b) Let α be any axiom. By hypothesis, $\Theta \cup \{\varphi\} \vdash \neg\alpha$ so, by Theorem 6.1.7, there is $n \geq 1$ such that $\Theta \vdash \varphi^n \to \neg\alpha$. Using (7) and (MP), we get $\Theta \vdash \alpha \to \neg(\varphi^n)$. Since α is a theorem, $\Theta \vdash \neg(\varphi^n)$.
(b) ⇒ (a) Let $n \geq 1$ such that $\Theta \vdash \neg(\varphi^n)$, then $\Theta \cup \{\varphi\} \vdash \neg(\varphi^n)$. By Lemma 6.1.6 (b), it follows that $\Theta \cup \{\varphi\} \vdash \varphi^n$ for any $n \geq 1$. Hence, using (15) and twice (MP), we get $\Theta \cup \{\varphi\} \vdash \varphi^n \odot \neg(\varphi^n)$. The desired conclusion follows by Lemma 6.3.2. □

LEMMA 6.3.5. *If Θ is a maximally consistent set of formulas, then $\Theta = Theor(\Theta)$.*

Proof. Let φ be a formula such that $\varphi \in Theor(\Theta)$ and $\varphi \notin \Theta$. Since Θ is maximally consistent, we infer that $\Theta \cup \{\varphi\}$ is inconsistent. By Lemma 6.3.4, there is $n \geq 1$ such that $\Theta \vdash \neg(\varphi^n)$. Because $\Theta \vdash \varphi$, by Lemma 6.1.6 (b), it follows that $\Theta \vdash \varphi^n$. By Lemma 6.3.2, Θ is inconsistent, which is a contradiction. Thus, $Theor(\Theta) \subseteq \Theta$. Since $\Theta \subseteq Theor(\Theta)$ for any set of formulas, we get $\Theta = Theor(\Theta)$. □

LEMMA 6.3.6. *For a consistent set of formulas Θ, the following are equivalent:*

- *(a)* Θ *is maximally consistent,*
- *(b)* *if* $\varphi \in Fm_Ł$ *and* $\varphi \notin \Theta$, *then* $\neg(\varphi^n) \in \Theta$ *for some* $n \geq 1$.

Proof. (a) ⇒ (b) If $\varphi \notin \Theta$, then $\Theta \cup \{\varphi\}$ is inconsistent. By Lemma 6.3.4, there is $n \geq 1$ such that $\Theta \vdash \neg(\varphi^n)$. Using Lemma 6.3.5, this means that $\neg(\varphi^n) \in \Theta$.
(b) ⇒ (a) By hypothesis and Lemma 6.3.4, we infer that $\Theta \cup \{\varphi\}$ is inconsistent for any $\varphi \notin \Theta$. It is obvious that Θ is maximally consistent. □

DEFINITION 6.3.7. *A set of formulas* Θ *is called* linear *if* $\Theta \vdash \varphi \to \psi$ *or* $\Theta \vdash \psi \to \varphi$ *for any two formulas* φ *and* ψ.

LEMMA 6.3.8. *For any* $n \geq 1$*, the following formula is a theorem of* $\mathcal{L}$*:*

(24) $(\varphi \to \psi)^n \vee (\psi \to \varphi)^n$.

Proof. By Lemma 2.3.1 (e), the identity

$$(a \to b)^n \vee (b \to a)^n = 1$$

holds in any MV-algebra, so it holds in the Lindenbaum–Tarski algebra **Ł**. Hence, for any two formulas φ and ψ, we get

$$([\varphi] \to [\psi])^n \vee ([\psi] \to [\varphi])^n = Theor \quad \text{and} \quad [(\varphi \to \psi)^n \vee (\psi \to \varphi)^n] = Theor.$$

The desired conclusion follows by Lemma 6.2.2 (a). □

PROPOSITION 6.3.9. *For any consistent set of formulas* Θ *and formula* φ *such that* $\Theta \not\vdash \varphi$*, there is a consistent and linear set of formulas* Γ *such that* $\Theta \subseteq \Gamma$ *and* $\Gamma \not\vdash \varphi$.

Proof. We inductively define a sequence $\{\Gamma_n \mid n \in \mathsf{N}\}$ with the following properties:

- $\Theta \subseteq \Gamma_n$ for any $n \in \mathsf{N}$,
- $\Gamma_n \not\vdash \varphi$ for any $n \in \mathsf{N}$,
- for any two formulas ψ and χ there is $n \in \mathsf{N}$ such that $\Gamma_n \vdash \psi{\to}\chi$ or $\Gamma_n \vdash \chi{\to}\psi$,
- $\Gamma_n \subseteq \Gamma_{n+1}$ for any $n \in \mathsf{N}$.

Let $\{(\psi_n, \chi_n) \mid n \in \mathsf{N}\}$ be an enumeration for $Fm_{Ł} \times Fm_{Ł}$. We define the sequence $\{\Gamma_n \mid n \in \mathsf{N}\}$ as follows. For $n = 0$ we set $\Gamma_0 = \Theta$. Now we suppose that Γ_n is defined having the above properties. We prove that

$$(*) \qquad \Gamma_n \cup \{\psi_n \to \chi_n\} \not\vdash \varphi \quad \text{or} \quad \Gamma_n \cup \{\chi_n \to \psi_n\} \not\vdash \varphi.$$

If $\Gamma_n \cup \{\psi_n \to \chi_n\} \vdash \varphi$ and $\Gamma_n \cup \{\chi_n \to \psi_n\} \vdash \varphi$ then, by Theorem 6.1.7, there exist two natural numbers r and s such that

$$\Gamma_n \vdash (\psi_n \to \chi_n)^r \to \varphi \quad \text{and} \quad \Gamma_n \vdash (\chi_n \to \psi_n)^s \to \varphi.$$

If $k = \max\{r, s\}$, then, using (17) and (A2), one can easily prove that

$$\Gamma_n \vdash (\psi_n \to \chi_n)^k \to \varphi \quad \text{and} \quad \Gamma_n \vdash (\chi_n \to \psi_n)^k \to \varphi.$$

Using Lemma 6.1.5 (b), we infer that

$$\Gamma_n \vdash ((\psi_n \to \chi_n)^k \vee (\chi_n \to \psi_n)^k) \to \varphi \vee \varphi.$$

By (23) and (A2), using twice (MP), we get

$$\Gamma_n \vdash ((\psi_n \to \chi_n)^k \vee (\chi_n \to \psi_n)^k) \to \varphi.$$

From Lemma 6.3.8 it follows that $\Gamma_n \vdash \varphi$, which is a contradiction. Hence, the condition (*) is proved. We define

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\psi_n \to \chi_n\} & \text{if } \Gamma_n \cup \{\psi_n \to \chi_n\} \not\vdash \varphi, \\ \Gamma_n \cup \{\chi_n \to \psi_n\} & \text{otherwise.} \end{cases}$$

It is obvious that Γ_{n+1} satisfies the required conditions. Let $\Gamma = \bigcup\{\Gamma_n \mid n \in \mathsf{N}\}$. It is obvious that $\Theta \subseteq \Gamma$. We also have that $\Gamma \not\vdash \varphi$, since otherwise, by Lemma 6.1.2, there would be a finite subset $\Delta \subseteq \Gamma$ such that $\Delta \vdash \varphi$. Since Δ is finite, then there is some $n \in \mathsf{N}$ such that $\Delta \subseteq \Gamma_n$, so $\Gamma_n \vdash \varphi$ which is a contradiction. Thus, Γ is also consistent. We only have to prove that Γ is linear. If ψ and χ are two arbitrary formulas, then there is some $n \in \mathsf{N}$ such that $(\psi, \chi) = (\psi_n, \chi_n)$ and $\psi \to \chi \in \Gamma_{n+1} \subseteq \Gamma$ or $\chi \to \psi \in \Gamma_{n+1} \subseteq \Gamma$. □

COROLLARY 6.3.10. *Any maximally consistent set is linear.*

Proof. Let Θ be a maximally consistent set. By Proposition 6.3.9, there is a consistent and linear set of formulas, Γ, such that $\Theta \subseteq \Gamma$. Since Θ is maximally consistent, we get $\Theta = \Gamma$. Hence, Θ is linear. □

DEFINITION 6.3.11. *We call* deductive system *a set Θ of formulas which contains the axioms and it is closed with respect to* (MP).

PROPOSITION 6.3.12. *If Θ is a set of formulas, then the following are equivalent:*

(a) Θ *is a deductive system,*

(b) $\Theta = \mathit{Theor}(\Theta)$.

Proof. (a) ⇒ (b) We have to prove that $\mathit{Theor}(\Theta) \subseteq \Theta$. If $\varphi \in \mathit{Theor}(\Theta)$, then $\Theta \vdash \varphi$. Let $\varphi_1, \ldots, \varphi_n = \varphi$ be a Θ-proof for φ. We prove by induction that $\varphi_i \in \Theta$ for any $i \in \{1, \ldots, n\}$. For $i = 0$, we have $\varphi_0 \in \Theta$ or φ_0 is an axiom. In this case we also get $\varphi_0 \in \Theta$, since Θ is a deductive system. We suppose that $\varphi_j \in \Theta$ for any $j < i$ and we prove that $\varphi_i \in \Theta$. If φ_i is an axiom or $\varphi_i \in \Theta$, then the desired conclusion is straightforward. Otherwise, there are $k, j < i$ such that $\varphi_k = \varphi_j \to \varphi_i$. By induction, φ_k and φ_j are in Θ. Since Θ is closed with respect to (MP), we get $\varphi_i \in \Theta$. Hence, φ is in Θ, so $\Theta = \mathit{Theor}(\Theta)$.
(b) ⇒ (a) If α is an axiom, then $\Theta \vdash \alpha$, so $\alpha \in \mathit{Theor}(\Theta) = \Theta$. If φ and $\varphi \to \psi$ are in $\Theta = \mathit{Theor}(\Theta)$, then $\Theta \vdash \varphi$ and $\Theta \vdash \varphi \to \psi$. By (MP) we get $\Theta \vdash \psi$, so $\psi \in \mathit{Theor}(\Theta) = \Theta$. □

COROLLARY 6.3.13. *Any maximally consistent set of formulas is a deductive system.*

Proof. By Lemma 6.3.5 and Proposition 6.3.12. □

In the sequel, we will provide an algebraic interpretation for the particular sets of formulas presented above. If $\Theta \subseteq Fm_{\text{Ł}}$ and $F \subseteq \mathbf{Ł}$ then we define

$$[\Theta] := \{[\varphi] \mid \varphi \in \Theta\} \quad \text{and} \quad F^* := \{\varphi \mid [\varphi] \in F\}.$$

The following result will be very useful.

LEMMA 6.3.14. *If Θ is a deductive system and φ is a formula, then the following are equivalent:*

(a) $\varphi \in \Theta$,

(b) $[\varphi] \in [\Theta]$.

Proof. (a) ⇒ (b) Straightforward by the definition of $[\Theta]$.
(b) ⇒ (a) If $[\varphi] \in [\Theta]$, then there is $\psi \in \Theta$ such that $[\varphi] = [\psi]$. It follows that $\psi \in \Theta$ and $\vdash \psi \to \varphi$. Since Θ is a deductive system, we get $\varphi \in \Theta$. □

We recall that, in an MV-algebra, the notions of filter, prime filter and ultrafilter (maximal filter) are the dual notions of those of ideal, prime ideal and maximal ideal, respectively.

PROPOSITION 6.3.15. *If Θ is a deductive system and F is a filter of* $\mathbf{Ł}$*, then the following properties hold:*

(a) $[\Theta]$ *is a filter in* $\mathbf{Ł}$*,*

(b) F^* *is a deductive system,*

(c) $[\Theta]^* = \Theta$*,*

(d) $[F^*] = F$*,*

(e) Θ *is maximally consistent iff* $[\Theta]$ *is an ultrafilter.*

Proof. One can see Definition 2.7.4 and Proposition 2.7.3 (a) for the notion of filter.
(a) If α is an axiom, then $\alpha \in \Theta$, so $[\alpha] = Theor \in [\Theta]$. Thus, $[\Theta]$ is a nonempty set of $\mathbf{Ł}$. If $[\varphi]$ and $[\varphi] \to [\psi] = [\varphi \to \psi]$ are in $[\Theta]$, then φ and $\varphi \to \psi$ are in Θ by Lemma 6.3.14. Since Θ is a deductive system, we get $\psi \in \Theta$. Thus, $[\psi] \in [\Theta]$ and $[\Theta]$ is a filter in $\mathbf{Ł}$.
(b) Since F is a filter and $Theor$ is the last element of $\mathbf{Ł}$, it follows that $Theor \in F$. If α is an axiom, then $[\alpha] = Theor$ by Lemma 6.2.2 (a), so $\alpha \in F^*$. Hence, F^* contains the axioms. We have to prove that it is closed with respect to *modus ponens*. If φ and $\varphi \to \psi$ are in F^*, then $[\varphi]$ and $[\varphi \to \psi] = [\varphi] \to [\psi]$ are in F. By Proposition 2.7.3 (a), we get $[\psi] \in F$, so $\psi \in F^*$. We proved that F^* is a deductive system.
(c) If φ is a formula then, using Lemma 6.3.14, it follows that:

$$\varphi \in [\Theta]^* \quad \text{iff} \quad [\varphi] \in [\Theta] \quad \text{iff} \quad \varphi \in \Theta.$$

(d) Let φ be a formula. Using (b) and Lemma 6.3.14 we get:

$$[\varphi] \in [F^*] \quad \text{iff} \quad \varphi \in F^* \quad \text{iff} \quad [\varphi] \in F.$$

(e) Firstly, we have $\varphi \notin \Theta$ iff $[\varphi] \notin [\Theta]$ by Lemma 6.3.14. Thus, Θ is consistent iff $[\Theta]$ is a proper filter. Let Θ be a maximal consistent set of formulas and suppose that $[\Theta] \subseteq F$, where F is a proper filter in $\mathbf{Ł}$. By (c), we infer $\Theta = [\Theta]^* \subseteq F^*$ and that F^* is consistent. Thus, $\Theta = F^*$ and $[\Theta] = [F^*] = F$ by (d). We proved that $[\Theta]$ is an ultrafilter. The converse implication follows similarly. □

COROLLARY 6.3.16. *There is an order isomorphism between the set of all the deductive systems of* Ł *and the set of all the filters of* $\mathbf{Ł}$ *(ordered by set theoretical inclusion). Moreover, the maximally consistent sets correspond to ultrafilters.*

Proof. By Proposition 6.3.15. □

PROPOSITION 6.3.17. *Let Θ be a set of formulas. It follows that:*

(a) Θ *is consistent iff* $[\mathit{Theor}(\Theta)]$ *is a proper filter,*

(b) Θ *is consistent and linear iff* $[\mathit{Theor}(\Theta)]$ *is a prime filter.*

Proof. By Proposition 6.3.12, $\mathit{Theor}(\Theta)$ is a deductive system, therefore $[\mathit{Theor}(\Theta)]$ is a filter of **Ł**.
(a) If Θ is consistent, then there is a formula φ such that $\Theta \nvdash \varphi$. It follows that $\varphi \notin \mathit{Theor}(\Theta)$. By Lemma 6.3.14, we get $[\varphi] \notin [\mathit{Theor}(\Theta)]$, so $[\mathit{Theor}(\Theta)]$ is proper. The converse implication is straightforward.
(b) Let Θ be a consistent and linear set of formulas. By (a), $[\mathit{Theor}(\Theta)]$ is a proper filter. Since Θ is linear, we get $\Theta \vdash \varphi \to \psi$ or $\Theta \vdash \psi \to \varphi$ for any two formulas φ and ψ. It follows that $\varphi \to \psi \in \mathit{Theor}(\Theta)$ or $\psi \to \varphi \in \mathit{Theor}(\Theta)$. Hence, in **Ł** we get

$$[\varphi] \to [\psi] \in [\mathit{Theor}(\Theta)] \quad \text{or} \quad [\psi] \to [\varphi] \in [\mathit{Theor}(\Theta)],$$

which is the dual of condition (a) from Proposition 3.3.1. It follows that $[\mathit{Theor}(\Theta)]$ is a prime filter. The converse implication can be proved similarly. □

LEMMA 6.3.18. *If Θ is a set of formulas and $F = [\mathit{Theor}(\Theta)]$, then the following are equivalent for any two formulas φ and ψ:*

(a) $\varphi \sim_\Theta \psi$,

(b) $[\varphi] \sim_F [\psi]$,

where $\sim_F$ is the congruence relation associated to the filter F in **Ł**.

Proof. (a) ⇒ (b) If $\varphi \sim_\Theta \psi$, then $\Theta \vdash \varphi \to \psi$ and $\Theta \vdash \psi \to \varphi$. It follows that $[\varphi] \to [\psi] \in F$ and $[\psi] \to [\varphi] \in F$. By Remark 3.2.13, this means that $[\varphi] \sim_F [\psi]$.
(b) ⇒ (a) Straightforward using Lemma 6.3.14. □

COROLLARY 6.3.19. *Let Θ be a set of formulas. Then* $\mathbf{Ł}(\Theta)$ *and the quotient algebra* $\mathbf{Ł}/[\mathit{Theor}(\Theta)]$ *are isomorphic* MV*-algebras.*

Proof. It follows directly from Lemma 6.3.18. □

COROLLARY 6.3.20. *If Θ is a set of formulas, then the following are equivalent:*

(a) Θ *is linear,*

(b) $\mathbf{Ł}(\Theta)$ *is an* MV*-chain.*

Proof. If Θ is inconsistent, then $\mathbf{Ł}(\Theta)$ is a trivial MV-algebra, so our conclusion is obvious. If Θ is a consistent set, then the desired equivalence follows by Proposition 6.3.17 (b), Corollary 6.3.19 and Proposition 4.1.3. □

COROLLARY 6.3.21. *If Θ is a set of formulas, then the following are equivalent:*

(a) Θ *is maximally consistent,*

(b) $\mathbf{Ł}(\Theta)$ *is a simple* MV*-algebra.*

Proof. It follows by Proposition 6.3.15 (e), Corollary 6.3.19, Proposition 3.4.2 (the filter version) and Proposition 4.2.9. □

6.4 The semantics of Ł

DEFINITION 6.4.1 (Evaluation). *Let A be an* MV*-algebra. An* A-evaluation *is a function* $e\colon Fm_{Ł} \to A$ *which satisfies the following conditions:*

(e1) $e(\varphi \to \psi) = e(\varphi) \to_A e(\psi)$,

(e2) $e(\neg\varphi) = e(\varphi)^*$,

where $\to_A$ *and* * *are the implication and the negation operations in* A*. In the sequel, the operations of an* MV*-algebra* A *will be simply denoted* $\oplus$, $\odot$, *, $\to$, $\vee$, $\wedge$, *without any qualification. The reader can deduce from the context if a symbol operations is a logical one, or it denotes an* MV*-algebra operation. A function* $e\colon Fm_{Ł} \to [0,1]$ *will be called an* evaluation *if it is a* $[0,1]$*-evaluation.*

FACT 6.4.2. *If* A *is an* MV*-algebra, then it suffices to define an* A*-evaluation on propositional variables. Given* $e\colon V \to A$*, we can inductively define an evaluation* $e^{\#}\colon Fm_{Ł} \to A$ *as follows:*

$$e^{\#}(\varphi) = \begin{cases} e(\varphi) & \text{if } \varphi \in V, \\ e^{\#}(\psi) \to e^{\#}(\chi) & \text{if } \varphi \text{ is } \psi \to \chi, \\ e^{\#}(\psi)^* & \text{if } \varphi \text{ is } \neg\psi. \end{cases}$$

Thus, an A*-evaluation is uniquely determined by the values of the propositional variables. In the following we will use the same notation for a function* $e\colon V \to A$ *and the corresponding* A*-evaluation* $e\colon Fm_{Ł} \to A$.

DEFINITION 6.4.3 (Tautologies). *If* A *is an* MV*-algebra, we say that a formula* φ *is an* A-tautology *if* $e(\varphi) = 1$ *for any* A*-evaluation* $e\colon Fm_{Ł} \to A$*. We say that a formula* φ *is a* tautology *if it is a* $[0,1]$*-tautology. We will denote by* $\models_A \varphi$ *the fact that* φ *is an* A*-tautology and by* $\models \varphi$ *the fact that* φ *is a tautology. In the sequel,* $Taut_A$ *will be the set of all the* A*-tautologies and* $Taut$ *will be the set of all the tautologies of* Ł.

DEFINITION 6.4.4 (Semantic consequence). *Let* Θ *be a set of formulas and* φ *a formula. For an* MV*-algebra* A*, we say that* φ *is an* A-semantic consequence *of* Θ *if*

$$e(\Theta) = \{e(\psi) \mid \psi \in \Theta\} = \{1\} \text{ implies } e(\varphi) = 1,$$

for any A*-evaluation* $e\colon Fm_{Ł} \to A$*. In this case we write* $\Theta \models_A \varphi$*. If* φ *is a* $[0,1]$*-semantic consequence of* Θ*, we will simply say that* φ *is a semantic consequence of* Θ*, and we write* $\Theta \models \varphi$*. The set of all the* A*-semantic consequences of* Θ *will be denoted by* $Taut_A(\Theta)$ *and, consequently, the set of all the semantic consequences of* Θ *will be* $Taut(\Theta)$*. One can see that* A*-tautologies (tautologies) coincide with* A*-semantic consequences (semantic consequences) of the empty set.*

PROPOSITION 6.4.5 (Soundness. Completeness w.r.t. MV-chains). *If* Θ *is a set of formulas and* φ *is a formula, then the following are equivalent:*

(a) $\Theta \vdash \varphi$,

(b) $\Theta \models_A \varphi$ *for any* MV*-algebra* A,

(c) $\Theta \models_A \varphi$ *for any* MV*-chain* A.

Proof. (a) ⇒ (b) Let $\Theta \vdash \varphi$ and let $\varphi_1, \ldots, \varphi_n = \varphi$ be a Θ-proof for φ. For any MV-algebra A and for any A-evaluation e such that $e(\Theta) = \{1\}$, we will prove that $e(\varphi_i) = 1$ for any $i \in \{1, \ldots, n\}$ using mathematical induction. If $i = 1$, then it follows that $\varphi_1 \in \Theta$ or φ_1 is an axiom. If $\varphi_1 \in \Theta$ it is obvious that $e(\varphi_1) = 1$. If φ_1 is one of the axioms (A2), (A3) or (A4), then $e(\varphi_1) = 1$ from Proposition 2.3.5. If φ_1 is (A1), then $e(\varphi_1) = 1$ by Corollary 2.3.7 and Proposition 2.3.8 (e). Now, we suppose that $e(\varphi_j) = 1$ for any $j < i$ and we prove that $e(\varphi_i) = 1$. If $\varphi_i \in \Theta$ or φ_i is an axiom, then $e(\varphi_i) = 1$ as above. Otherwise, there are $j, k < i$ such that φ_k is $\varphi_j \to \varphi_i$. By induction hypothesis, we infer that $1 = e(\varphi_k) = e(\varphi_j) \to e(\varphi_i) = 1 \to e(\varphi_i)$. It follows that $e(\varphi_i) = 1$.
(b) ⇒ (c) Obvious.
(c) ⇒ (a) We suppose that $\Theta \not\vdash \varphi$. By Proposition 6.3.9, it follows that there is a linear set of formulas Γ such that $\Theta \subseteq \Gamma$ and $\Gamma \not\vdash \varphi$. We denote $C = \mathbf{Ł}(\Gamma)$, the Lindenbaum–Tarski algebra of Γ. By Corollary 6.3.20, C is an MV-chain. Let $e\colon Fm_{Ł} \to C$ defined by $e(\psi) = [\psi]_\Gamma$, which obviously is a C-evaluation. Note that $e(\Gamma) = \{1\}$, so $e(\Theta) = \{1\}$ since $\Theta \subseteq \Gamma$. By hypothesis, it follows that $e(\varphi) = 1$, which means that $[\varphi]_\Gamma = \mathit{Theor}(\Gamma)$. Thus, $\Gamma \vdash \varphi$, which is a contradiction. We proved that $\Theta \vdash \varphi$. □

COROLLARY 6.4.6. *If Θ is a set of formulas, then*

$$\mathit{Theor}(\Theta) = \bigcap\{\mathit{Taut}_A(\Theta) \mid A \textit{ is an } \mathrm{MV}\textit{-algebra}\}.$$

Proof. By Proposition 6.4.5. □

COROLLARY 6.4.7. *$\mathit{Theor} \subseteq \mathit{Taut}$.*

Proof. It follows by Corollary 6.4.6, when $\Theta = \emptyset$ and $A = [0, 1]$. □

COROLLARY 6.4.8. *The classical deduction theorem does not hold in* Ł.

Proof. We recall that in classical (Boolean) logic the deduction theorem asserts that for any set of formulas Θ and for any two formulas φ and ψ, $\Theta \cup \{\varphi\} \vdash \psi$ iff $\Theta \vdash \varphi \to \psi$.

If $\Theta \vdash \varphi \to \psi$, then we also have $\Theta \cup \{\varphi\} \vdash \psi$ by Theorem 6.1.7 (the deduction theorem in Ł). We give an counter-example for the converse implication. By (14), we have $\vdash \varphi \to (\varphi \to (\varphi \odot \varphi))$, so $\{\varphi\} \vdash \varphi \odot \varphi$. If we suppose that $\vdash \varphi \to (\varphi \odot \varphi)$ then, by Corollary 6.4.7, $e(\varphi \to (\varphi \odot \varphi)) = 1$ for any evaluation $e\colon Fm_{Ł} \to [0, 1]$. It follows that $1 = e(\varphi)^* \oplus (e(\varphi) \odot e(\varphi)) = e(\varphi)^* \vee e(\varphi)$. If φ is a propositional variable and we choose $e(\varphi) = 1/2$, then we get a contradiction. Thus, $\{\varphi\} \vdash \varphi \odot \varphi$ does not imply that $\vdash \varphi \to (\varphi \odot \varphi)$. □

COROLLARY 6.4.9. *The empty set $\emptyset$ is consistent.*

Proof. We suppose that the $\emptyset$ is inconsistent. By Lemma 6.3.2, there is a formula φ such that $\vdash \varphi$ and $\vdash \neg\varphi$. If $e\colon Fm_{Ł} \to [0, 1]$ is an evaluation then, by Corollary 6.4.7, $e(\varphi) = 1$ and $e(\neg\varphi) = 1$. Thus, we get $1 = e(\varphi) = e(\neg\varphi) = e(\varphi)^* = 0$, which is a contradiction. We proved that $\emptyset$ is consistent. □

COROLLARY 6.4.10. *For a nonempty set of formulas Θ the following properties are equivalent:*

(a) Θ *is consistent,*

(b) *there is an evaluation* $e\colon Fm_{\text{Ł}} \to [0,1]$ *such that* $e(\Theta) = \{1\}$.

Proof. (a) ⇒ (b) By Corollary 6.3.3, there is a maximally consistent set of formulas Γ such that $\Theta \subseteq \Gamma$. From Corollary 6.3.21, we infer that $\mathbf{Ł}(\Gamma)$ is a simple MV-algebra, so it is isomorphic with a subalgebra of $[0,1]$ by Proposition 5.4.1. Thus, there is an injective homomorphism $f\colon \mathbf{Ł}(\Gamma) \to [0,1]$. We define $e\colon Fm_{\text{Ł}} \to [0,1]$ by $e(\varphi) = f([\varphi]_\Gamma)$. If $\varphi \in \Theta \subseteq \Gamma$, then $[\varphi]_\Gamma = 1$, so $e(\varphi) = 1$. Hence, e is the desired evaluation.
(b) ⇒ (a) Let e be an evaluation such that $e(\Theta) = \{1\}$. If Θ is inconsistent then, by Lemma 6.3.2, there is a formula φ such that $\Theta \vdash \varphi$ and $\Theta \vdash \neg\varphi$. By Proposition 6.4.5, $e(\varphi) = e(\neg\varphi) = 1$, so $0 = 1$ in $[0,1]$, which is a contradiction. It follows that Θ is a consistent set of formulas. ☐

COROLLARY 6.4.11. *If Θ is a set of formulas such that for any finite subset $\Gamma \subseteq \Theta$ there is an evaluation e_Γ such that $e_\Gamma(\Gamma) = \{1\}$, then there is an evaluation e such that $e(\Theta) = \{1\}$.*

Proof. By hypothesis and Corollary 6.4.10, every finite subset of Θ is consistent. We suppose that there is no evaluation e such that $e(\Theta) = \{1\}$. By Corollary 6.4.10, Θ is inconsistent. From Lemma 6.3.2, there is a formula φ such that $\Theta \vdash \varphi \odot \neg\varphi$. Using Lemma 6.1.2, there is a finite set of formulas $\Gamma \subseteq \Theta$ such that $\Gamma \vdash \varphi \odot \neg\varphi$. Thus, Γ is also inconsistent, which is a contradiction. ☐

Recall that $[0,1]_Q$ is the MV-algebra of the rational numbers from the unit interval $[0,1]$.

THEOREM 6.4.12 (Completeness). *If Θ is a finite set of formulas and φ is a formula, then the following assertions are equivalent:*

(a) $\Theta \vdash \varphi$,

(b) $\Theta \models \varphi$,

(c) $\Theta \models_{\mathbf{Ł}_n} \varphi$ *for any* $n \geq 2$,

(d) $\Theta \models_{[0,1]_Q} \varphi$.

Proof. (a) ⇒ (b) Follows by Proposition 6.4.5.
(b) ⇒ (c) Let $n \geq 2$ and let $e\colon Fm_{\text{Ł}} \to Ł_n$ be an $\mathbf{Ł}_n$-evaluation such that $e(\Theta) = \{1\}$. Since $\mathbf{Ł}_n$ is a subalgebra of $[0,1]$, using (b), it follows that $e(\varphi) = 1$.
(c) ⇒ (d) We suppose that $\Theta \not\models_{[0,1]_Q} \varphi$, so there is a $[0,1]_Q$-evaluation e such that $e(\Theta) = \{1\}$ and $e(\varphi) \neq 1$. As we already noticed, e is uniquely determined by the values $e(v)$, where v is a propositional variable. Let $v_1, \ldots, v_n$ be all the propositional variables that appear in φ or in the formulas from Θ and let $e(v_i) = m_i/d_i \in [0,1]_Q$ for any $i \in \{1,\ldots,n\}$. Hence, m_i and d_i are natural numbers such that $d_i \neq 0$ for any $i \in \{1,\ldots,n\}$. If we consider d the least common multiple of $d_1,\ldots,d_n$, then $m_i/d_i = p_i/d$ and $0 \leq p_i \leq d$ for any $i \in \{1,\ldots,n\}$. It follows that $e(v_i) \in \{0, 1/d, \ldots, (d-1)/d, 1\} = L_{d+1}$ for any $i \in I$. We consider an $\mathbf{Ł}_{d+1}$-evaluation

$e'\colon Fm_Ł \to L_{d+1}$ with the property that $e'(v_i) = e(v_i) = p_i/d$ for any $i \in \{1, \dots, n\}$. Since, for any formula ψ, the value $e'(\psi)$ is determined by the values of the propositional variables that appear in ψ, we infer that $e'(\varphi) = e(\varphi) \neq 1$ and $e'(\Theta) = e(\Theta) = \{1\}$. Thus, $\Theta \not\models_{\mathbf{Ł}_{d+1}} \varphi$, so we get a contradiction of (c). It follows that $\Theta \models_{[0,1]_Q} \varphi$.
(d) $\Rightarrow$ (a) Let us suppose that $\Theta \not\vdash \varphi$. By Proposition 6.4.5, there is an MV-chain A and there is an A-evaluation e such that $e(\Theta) = \{1\}$ and $e(\varphi) \neq 1$. By Corollary 5.3.1, we can safely assume that A is divisible. Assume $\Theta = \{\theta_1, \dots, \theta_k\}$ and let $v_1, \dots, v_n$ be all the propositional variables that appear in φ or in a formula from Θ. We define the following sentence in the first order language of MV-algebras:

$$\sigma \text{ is } (\forall v_1) \dots (\forall v_n)((\theta_1 = 1 \wedge \dots \wedge \theta_k = 1) \to \varphi = 1).$$

If we consider the formulas $\varphi = 1(v_1, \dots, v_n)$ and $\theta_i = 1(v_1, \dots, v_n)$ for any $i \in \{1, \dots, k\}$, then the previous assumption asserts that $A \models \theta_i = 1[a_1, \dots, a_n]$ for any $i \in \{1, \dots, k\}$ and $A \not\models \varphi = 1[a_1, \dots, a_n]$, where $a_j = e(v_j)$ for any $j \in \{1, \dots, n\}$. It follows that $A \not\models \sigma$. By Theorem 5.3.6, we infer that $[0,1]_Q \not\models \sigma$, so there are rational numbers $r_1, \dots, r_n$ in $[0,1]$ such that for any $i \in \{1, \dots, k\}$ we have:

$$[0,1]_Q \models \theta_i = 1[r_1, \dots, r_n] \quad \text{and} \quad [0,1]_Q \not\models \varphi = 1[r_1, \dots, r_n].$$

Let $e'\colon Fm_Ł \to [0,1]_Q$ be a $[0,1]_Q$-evaluation such that $e'(v_j) = r_j$ for any $j \in \{1, \dots, n\}$. We get $e'(\theta_i) = 1$ for any $i \in \{1, \dots, k\}$ and $e'(\varphi) \neq 1$, which is a contradiction of (d). Hence, $\Theta \vdash \varphi$. □

COROLLARY 6.4.13. *If Θ is a finite set of formulas, then $Theor(\Theta) = Taut(\Theta)$. In particular, $Theor = Taut$.*

Proof. By Theorem 6.4.12. □

In the sequel we will provide a characterization of those sets of formulas Θ with the property that the syntactic consequences and the semantic consequences of Θ coincide. The following example shows that this property does not hold in general.

EXAMPLE 6.4.14 ([80]). Let v and u be two propositional variables and

$$\Theta = \{nv \to u \mid n \geq 1\} \cup \{(\neg v) \to u\}.$$

We will prove that $\Theta \models u$, but $\Theta \not\vdash u$.

Let $e\colon Fm_Ł \to [0,1]$ be an evaluation such that $e(\Theta) = \{1\}$. Thus $e(nv \to u) = e(\neg v \to u) = 1$ for any $n \geq 1$, so $e(v)^* \leq e(u)$ and $ne(v) \leq e(u)$ for any $n \geq 1$. If $e(v) = 0$, then $e(v)^* = 1$ and we get $e(u) = 1$. If $e(v) > 0$, since $[0,1]$ is a simple MV-algebra, there is $n \geq 1$ such that $ne(v) = 1$, so $e(u) = 1$. We proved that $e(\Theta) = \{1\}$ implies $e(u) = 1$, which means that $\Theta \models u$.

We suppose that $\Theta \vdash u$. By Lemma 6.1.2, there is a finite set $\Gamma \subseteq \Theta$ such that $\Gamma \vdash u$. From Theorem 6.4.12 we infer that $\Gamma \models u$. We will define an evaluation e such that $e(\Gamma) = \{1\}$, but $e(u) \neq 1$, so we will get a contradiction. Note that Γ is not empty, since u is a propositional variable. We define $M = \{n \mid nv \to u \in \Gamma\}$. Let $m = \max\{n \mid n \in M\}$ if $M \neq \emptyset$ and $m = 1$, otherwise. We consider e an evaluation such that $e(v) = 1/(m+1)$ and $e(u) = m/(m+1)$. If $nv \to u \in \Gamma$ then, since $n \leq m$, it follows that

$$e(nv \to u) = ne(v) \to e(u) = (n/(m+1)) \to (m/(m+1)) = 1.$$

If $\neg v \to u \in \Gamma$, then

$$\begin{aligned} e(\neg v \to u) &= e(v)^* \to e(u) = (1 - (1/(m+1))) \to (m/(m+1)) \\ &= (m/(m+1)) \to (m/(m+1)) = 1. \end{aligned}$$

Thus, $e(\Gamma) = \{1\}$, but $e(u) = m/(m+1) \neq 1$.

Example 6.4.14 also shows that the semantical compactness does not hold in Ł. This means that $\Theta \models \varphi$ does not imply that there is a finite subset $\Gamma \subseteq \Theta$ such that $\Gamma \models \varphi$. For a geometric representation of the same phenomenon we refer to [61]. We also note that in classical (Boolean) logic, the semantical compactness is equivalent to the previous Corollary 6.4.11.

LEMMA 6.4.15. *If Θ is an arbitrary set of formulas, then for any* MV*-algebra A and for any A-evaluation $e\colon Fm_{\text{Ł}} \to A$ such that $e(\Theta) = \{1\}$, there is a unique* MV*-algebra homomorphism $f_e\colon \mathbf{Ł}(\Theta) \to A$ with $f_e([\varphi]_\Theta) = e(\varphi)$ for any $\varphi \in Fm_{\text{Ł}}$.*

Proof. Let us define $f_e\colon \mathbf{Ł}(\Theta) \to A$ by $f_e([\varphi]_\Theta) = e(\varphi)$ for any $\varphi \in Fm_{\text{Ł}}$. We have to prove that f_e is well defined. If φ and ψ are formulas such that $[\varphi]_\Theta = [\psi]_\Theta$, then $\Theta \vdash \varphi \to \psi$ and $\Theta \vdash \psi \to \varphi$. By Proposition 6.4.5 and our hypothesis, it follows that $e(\varphi \to \psi) = e(\psi \to \varphi) = 1$, so $e(\varphi) = e(\psi)$. Thus, f_e is well defined. The fact that f_e is a homomorphism and its uniqueness are straightforward. $\square$

COROLLARY 6.4.16. *Let A be an* MV*-algebra, Θ a set of formulas and φ a formula. The following are equivalent:*

(a) $\Theta \models_A \varphi$,

(b) $f([\varphi]_\Theta) = 1$ *for any* MV*-algebra homomorphism* $f\colon \mathbf{Ł}(\Theta) \to A$.

Proof. (a) ⇒ (b) If $f\colon \mathbf{Ł}(\Theta) \to A$ is an MV-algebra homomorphism, then we define an A-evaluation by $e(\psi) = f([\psi]_\Theta)$. For any $\psi \in \Theta$ we get $e(\psi) = 1$, so $e(\varphi) = 1$ by hypothesis. Thus, $f([\varphi]_\Theta) = 1$.
(b) ⇒ (a) Let $e\colon Fm_{\text{Ł}} \to A$ be an A-evaluation such that $e(\Theta) = \{1\}$ and let f_e be the homomorphism uniquely determined by e from Lemma 6.4.15. By hypothesis, it follows that $f_e([\varphi]_\Theta) = 1$, so $e(\varphi) = 1$. $\square$

LEMMA 6.4.17. *If $v_1 \neq v_2$ are propositional variables, then $[v_1] \neq [v_2]$.*

Proof. Let A be an MV-algebra and $e\colon Fm_{\text{Ł}} \to A$ an A-evaluation such that $e(v_1) \neq e(v_2)$. By Lemma 6.4.15 there is a unique MV-algebra homomorphism $f_e\colon \mathbf{Ł} \to A$ with $f_e([v_1]) = e(v_1) \neq e(v_2) = f_e([v_2])$. If we assume that $[v_1] = [v_2]$ then, by completeness, $\models_A v_1 \to v_2$ and $\models_A v_2 \to v_1$. Hence, by Corollary 6.4.16, we get $f_e([v_1]) = f_e([v_2])$, which contradicts our hypothesis. We proved that $[v_1] = [v_2]$. $\square$

LEMMA 6.4.18. *The following are equivalent for any* MV*-algebra A and $a \in A$:*

(a) $a \in U$ *for any ultrafilter U of A,*

(b) $f(a) = 1$ *for any* MV*-algebra homomorphism* $f\colon A \to [0,1]$.

Proof. (a) ⇒ (b) If $f\colon A \to [0,1]$ is an MV-algebra homomorphism, then $U = \{x \in A \mid f(x) = 1\}$ is a filter. By Theorem 3.2.8, A/U is isomorphic with $f(A)$. Since $f(A)$ is a subalgebra of $[0,1]$, it is a simple MV-algebra, so U is an ultrafilter. By hypothesis, $a \in U$, so $f(a) = 1$.
(b) ⇒ (a) If U is an ultrafilter of A, then A/U is a simple MV-algebra, so there is an injective homomorphism $f'\colon A/U \to [0,1]$. We define $f\colon A \to [0,1]$ by $f(x) = f'([x]_U)$. It is obvious that f is an MV-algebra homomorphism, so $f(a) = 1$ by hypothesis. Thus, $f'([a]_U) = 1$, so $[a]_U = 1$ since f' is injective. We proved that $a \in U$. □

THEOREM 6.4.19. *If Θ is a set of formulas, then the following are equivalent:*

(a) $Theor(\Theta) = Taut(\Theta)$*,*

(b) $\mathbf{Ł}(\Theta)$*, the Lindenbaum–Tarski algebra of* Θ*, is semisimple,*

(c) $[Theor(\Theta)] = \bigcap\{U \subseteq \mathbf{Ł}(\Theta) \mid [Theor(\Theta)] \subseteq U,\ U$ *ultrafilter*$\}$.

Proof. (a) ⇔ (b) By Corollary 6.4.16 and Lemma 6.4.18, we get $[\varphi]_\Theta \in U$ if and only if $\varphi \in Taut(\Theta)$ for any ultrafilter U of $\mathbf{Ł}(\Theta)$. By Lemma 6.2.2 (a) it follows that, if $Taut(\Theta) = Theor(\Theta)$, the intersection of all the ultrafilters of $\mathbf{Ł}(\Theta)$ is $\{Theor(\Theta)\}$. Thus, in the MV-algebra $\mathbf{Ł}(\Theta)$, the intersection of all the ultrafilters contains only the last element of the algebra, so $\mathbf{Ł}(\Theta)$ is semisimple. Conversely, if $\mathbf{Ł}(\Theta)$ is semisimple, then the intersection of all the ultrafilters is $Theor(\Theta)$. Thus, if $\varphi \in Taut(\Theta)$, then $[\varphi]_\Theta = Theor(\Theta)$, which means that $\varphi \in Theor(\Theta)$. We proved that $Taut(\Theta) \subseteq Theor(\Theta)$. Since $Theor(\Theta) \subseteq Taut(\Theta)$ by Corollary 6.4.6, (a) is proved.
(b) ⇔ (c) Follows by Corollary 6.3.19 and the dual of Proposition 3.2.7. □

7 Varieties of MV-algebras

In this section we describe equational classes, i.e. varieties of MV-algebras, that is classes V of MV-algebras for which exists a set E of MV-equations such that an MV-algebra A is in V if and only if every equation from E holds in A. We recall that a variety of algebras is a class of algebras which is closed under subalgebras, homomorphic images and direct products. The class of all MV-algebras, denoted by $\mathbb{MV}$, is obviously an equational class. Let $\{A_i\}_{i\in I}$ be a family of MV-algebras. Then by $\mathbf{V}(\{A_i\}_{i\in I})$ we denote the variety generated by $\{A_i\}_{i\in I}$, i.e., the smallest variety of MV-algebras containing $\{A_i\}_{i\in I}$. The algebras A_i will be called generators of $\mathbf{V}(\{A_i\}_{i\in I})$. Also we write $\mathbf{V}(A)$ to denote the variety generated by $\{A\}$ or $\mathbf{V}(A_1,\ldots,A_n)$ to denote the variety generated by $\{A_1,\ldots,A_n\}$, for any positive integer n. When $\mathbf{V}(A_1,\ldots,A_n)$ cannot be generated by a proper subset of $\{A_1,\ldots,A_n\}$, we say that $\{A_1,\ldots,A_n\}$ is an irreducible set of generators. In the sequel we write just $\mathbf{V}(A_1,\ldots,A_n)$ to denote a variety generated by an irreducible set of generators $\{A_1,\ldots,A_n\}$. In the present section we mainly show that every proper subvariety of the variety $\mathbb{MV}$ of all MV-algebras is generated by a finite set of MV-chains and that it is axiomatizable by a finite set of equations in a single variable.

Throughout this section we write $rank(A) = n$ whenever A is an MV-algebra of rank n.

7.1 Komori's Theorem

Let A be an MV-algebra and τ, σ MV-terms in the variables $x_1, \ldots, x_n$ such that for all $a_1, \ldots, a_n \in A$:

$$\tau_A(a_1, \ldots, a_n) = \sigma_A(a_1, \ldots, a_n).$$

In this case we say that the MV-algebra A *satisfies the* MV*-equation* $\tau = \sigma$, or that the MV-equation $\tau = \sigma$ holds in A, and write $A \models \tau = \sigma$. We simply write $\tau = \sigma$ to refer to the pair $\langle \tau, \sigma \rangle$. The following proposition is a standard result of universal algebra.

PROPOSITION 7.1.1. *Let A be an* MV*-algebra which is a subdirect product of the* MV*-algebras $\{A_i\}_i \in I$ and σ, τ* MV*-terms in the variables $x_1, \ldots, x_n$. Then the following are equivalent:*

(a) $A \models \tau = \sigma$,

(b) $A_i \models \tau = \sigma$ *for every* $i \in I$.

THEOREM 7.1.2. *Every variety of* MV*-algebras is generated by the collection of its* MV*-chains.*

Proof. By Theorem 4.1.4 every MV-algebra A is subalgebra of the product of MV-chains A_i, $i \in I$, where each A_i is a quotient of A by a prime ideal of A. Furthermore, by Proposition 7.1.1 any MV-equation $\tau = \sigma$ is valid in A iff it is valid in A_i, for every $i \in I$. The proof is indeed complete. $\square$

PROPOSITION 7.1.3. *For any infinite subalgebra A of* $[0, 1]$, $\mathbf{V}(A) = \mathbb{MV}$.

Proof. By Corollary 4.1.12, A is dense in $[0, 1]$. Since for any MV-term τ, the function $\tau_{[0,1]}$ is continuous, an equation $\tau = \sigma$ is valid in A iff it is valid in $[0, 1]$. $\square$

PROPOSITION 7.1.4. *If* $2 < n_0 < n_1 < \cdots$ *is an infinite sequence of natural numbers, then* $\mathbf{V}(\{\text{Ł}_{n_i} \mid i = 0, 1, \ldots\}) = \mathbb{MV}$.

Proof. Let $S = (\bigcup L_{n_i})_{i=0,1,\ldots}$ and assume that the MV-equation $\sigma = \tau$, in k variables, is satisfied in each Ł_{n_i}. Let $u_1, \ldots, u_k \in [0, 1]$. Since S is infinite, for every $u_i \in \{u_1, \ldots, u_k\}$ there is an increasing infinite sequence of elements of S, $y_{i_1} < \cdots < y_{i_m} \cdots$ such that $y_{i_m} < u_i$ for every positive integer m. From the continuity of MV-terms on $[0, 1]^k$ we get that $\sigma = \tau$ holds in $[0, 1]$ too. Vice versa, trivially, every MV-equation valid in $[0, 1]$ is valid in each Ł_{n_i}, then the proposition is proved. $\square$

PROPOSITION 7.1.5. *Let $\mathbb{W}$ be a proper subvariety of* $\mathbb{MV}$. *Then there is an integer $m \geq 1$ such that for each* MV*-chain A of $\mathbb{W}$ $rank(A) \leq m$.*

Proof. Let A be an MV-chain and $A \in \mathbb{W}$. Then $A/Rad(A) \in \mathbb{W}$. By Proposition 7.1.3 $A/Rad(A)$ has to be finite, i.e., A is of finite rank. By Proposition 7.1.4 the set of finite chains from $\mathbb{W}$ has to be finite, so the set of ranks of chains from W is finite too. This proves the proposition. $\square$

For any integer $n \geq 1$ we set $H_n = \Gamma(\mathsf{Z} \times_{lex} \mathsf{Z}, \langle n, 1 \rangle)$.

LEMMA 7.1.6. *For each integer $n \geq 1$ and each nonsimple* MV*-chain A of rank n, $H_n \in \mathbf{V}(A)$.*

Proof. By Proposition 7.1.5 we assume $A \simeq \Gamma(\mathsf{Z} \times_{lex} G, \langle n, g\rangle)$ for some totally ordered Abelian group G and $0 < g \in G$. The mapping $h\colon H_n \to \Gamma(\mathsf{Z} \times_{lex} G, \langle n, g\rangle)$ defined by $h(\langle j, m\rangle) = \langle j, mg\rangle$ for each $\langle j, m\rangle$ is an injective MV-homomorphism and $h(\langle n, 1\rangle) = \langle n, g\rangle$. Hence it follows that $H_n \in \mathbf{V}(A)$. □

Note that $\mathbf{Ł}_{n+1} \simeq \Gamma(\mathsf{Z}, n)$. In the rest of this section, $\mathbf{Ł}_{n+1}$ is to be intended as $\Gamma(\mathsf{Z}, n)$, instead of the subalgebra of $[0,1]$ with universe $\{0, 1/n, \ldots, (n-1)/n, 1\}$. For any integer $n \geq 1$ we set $\boldsymbol{K}_{n+1} = \Gamma(\mathsf{Z} \times_{lex} \mathsf{Z}, \langle n, 0\rangle)$.

LEMMA 7.1.7. *Let A be an* MV*-algebra and m, n positive integers. If A contains an isomorphic image of $\mathbf{Ł}_{n+1}$ and an isomorphic image of $\boldsymbol{K}_{m+1}$, then A contains an isomorphic image of $\boldsymbol{K}_{n+1}$.*

Proof. Let $f\colon \mathbf{Ł}_{n+1} \to A$ and Let $g\colon \boldsymbol{K}_{m+1} \to A$ be embeddings. Set

$$\begin{aligned}
T &= f(L_{n+1}) \cup g(\langle Rad(\boldsymbol{K}_{m+1})\rangle,\\
W &= \{f(k/n) \odot (g(0,r))^*\}_{0<k<n, r>0},\\
V &= \{f(k/n) \oplus g(0,r)\}_{0<k<n, r>0},\\
S &= T \cup W \cup V.
\end{aligned}$$

Let us prove that S is a totally ordered subset of A. Indeed, it is easy to see that T is totally ordered. So we show now that W is totally ordered. Let $a, b \in W$ such that

$$a = f(k/n) \odot (g(0,r))^* \quad \text{and} \quad b = f(h/n) \odot (g(0,s))^*.$$

If $h = k$, trivially a and b are comparable. If $k < h$, by Lemma 3.5.13,

$$b = (f(k/n) \odot ((g(0,s))^*) \oplus f((h-k)/n).$$

For $r = s$,

$$b = a \oplus f((h-k)/n) > a.$$

For $r < s$, by Lemma 3.5.11 (a) and Lemma 3.5.12,

$$b = (f(k/n) \odot (g(0,r))^*) \oplus g(0,s) > a.$$

For $r > s$,

$$b > f(k/n) \odot (g(0,r))^*) = a.$$

Hence, W is a totally ordered set. Now, let $a, b \in V$ and

$$a = f(k/n) \oplus g(0,r) \qquad b = f(h/n) \oplus g(0,s).$$

Then, as above, if $k < h$ we have:

$$b > f(k/n) \oplus f((h-k)/n) > a.$$

If $h = k$, then $a < b$ if and only if $r < s$, and $a > b$ if and only if $r > s$. Hence V is totally ordered. To show that S is totally ordered we distinguish the following cases:

(1) $a \in T$ and $b \in W$,

(2) $a \in T$ and $b \in V$,

(3) $a \in W$ and $b \in V$.

Assume (1) *and* $b = f(k/n) \odot (g(0,r))^*)$.

For $a = f(k/n)$, if $k = h$, trivially we have $b < a$. If $k > h$, then $a = f(k/n) \oplus f((h-k)/n) > b$. If $k < h$, then by Lemma 3.5.13:

$$\begin{aligned} b &= (f(k/n) \oplus f((h-k)/n)) \odot (g(0,r))^*) = \\ &= f(k/n) \oplus (f((h-k)/n)) \odot g(0,r))^*) > a. \end{aligned}$$

For $a = g(0,s)$ and $b = f(k/n) \odot (g(0,r))^*)$, by Lemma 3.5.11 (c), $a < b$.
For $a = (g(0,s))^*$, $a^* = g(0,s) < (f(k/n))^* \oplus g(0,r) = b^*$. Hence $b < a$.

Assume (2) *and* $b = f(k/n) \odot g(0,r)$.

For $a = f(k/n)$, if $k \leq h$, then $a \leq b$. If $k > h$, then by Lemma 3.5.11 (a), $a > b$.
For $a = g(0,s)$, by Lemma 3.5.11 (a), $a < b$.
For $a = (g(0,s))^*$, $a^* = g(0,s) < (f(k/n))^* \odot (g(0,r))^* = b^*$. Hence $b < a$.

Assume (3) $a = f(k/n) \odot (g(0,s))^*$ *and* $b = f(k/n) \oplus g(0,r)$.

If $k \leq h$, then $a < b$. If $k > h$, by Lemma 3.5.11 (a) and (g)

$$\begin{aligned} a &= (f(k/n) \odot (g(0,s))^*) \oplus f((h-k)/n) > \\ &> ((f(k/n) \odot (g(0,s))^*) \oplus g(0,s)) \oplus g(0,r) = \\ &= (f(k/n) \vee g(0,s)) \oplus g(0,r) = b. \end{aligned}$$

Hence, we proved that S is a totally ordered subset of A. Let us prove, now, that S is a subalgebra of A. It is clear that $0 = f(0) \in S$, and $1 = f(1) \in S$. To prove that S is closed under the operation * it is just a matter of checking. To show that S is closed under $\oplus$ operation we limit ourselves to analyze the cases listed below, the remaining cases are trivial. We assume that $x, y \in S$.

Case 1. $x = f(k/n) \in T \quad \textit{and} \quad y = (g(0,r))^* \in T$

Then, by Lemma 3.5.11 (a), $x \oplus y = 1$.

Case 2. $x = f(k/n) \odot (g(0,r))^* \in W \quad \textit{and} \quad y = f(h/n) \odot (g(0,s))^* \in W$

If $h + k = n$ then, by Lemma 3.5.11 (e), $x \oplus y = (g(0, r+s))^* \in T$. If $h + k \geq n + 1$, then

$$\begin{aligned} x \oplus y &\geq (f(k/n) \odot (g(0,r))^*) \oplus ((f(k/n)^* \oplus (f(1/n)) \odot (g(0,s))^*) \geq \\ &\geq (f(k/n) \odot (g(0,r))^*) \oplus ((f(k/n)^* \oplus g(0,r) \oplus g(0,s)) \odot g(0,r))^*) = \\ &= f(k/n) \oplus f(k/n)^* = 1. \end{aligned}$$

Case 3. $x = f(k/n) \in T \quad \textit{and} \quad y = f(h/n) \odot (g(0,r))^* \in W$

If $h + k = n$, then, by Lemma 3.5.11 (e) $x \oplus y = (g(0,r))^* \in T$. If $h + k < n$, then by Lemma 3.5.13 $x \oplus y \in W$. If $h + k \geq n + 1$, then by Lemma 3.5.11 (a) $x \oplus y = 1$.

Case 4. $x = f(k/n) \odot (g(0,s))^* \in W \quad \textit{and} \quad y = f(h/n) \oplus g(0,r) \in V$

If $h + k \geq n$, as above, by Lemma 3.5.11 (a) $x \oplus y = 1$. If $h + k < n$, then by Lemma 3.5.13, if $r > s$, $x \oplus y = f((h+k)/n) \odot (g(0, r-s))^* \in V$, if $r = s$, $x \oplus y = f((h+k)/n) \in T$, if $r < s$, $x \oplus y = f((h+k)/n) \odot (g(0, s-r))^* \in W$.

Hence, we proved that S is a subalgebra of A. Finally, we provide the desired isomorphism between S and $\boldsymbol{K}_{n+1}$ checking, by a direct inspection, that the mapping ψ defined below is an isomorphism:

$$\begin{array}{lll} \psi(x) = \langle f^{-1}(x), 0\rangle & \text{if} & x \in f(L_{n+1}), \\ \psi(x) = g^{-1}(x) & \text{if} & x \in g(\langle Rad(\boldsymbol{K}_{m+1})\rangle\,, \\ \psi(x) = \langle k/n, -r\rangle & \text{if} & x \in W \text{ and } x = f(k/n) \odot (g(0,r))^*, \\ \psi(x) = \langle k/n, r\rangle & \text{if} & x \in V \text{ and } x = f(k/n) \oplus (g(0,r))^*. \end{array} \qquad \square$$

LEMMA 7.1.8. *Let A be an* MV*-algebra and m, n positive integers. If A contains an isomorphic image of $\boldsymbol{K}_{n+1}$ and an isomorphic image of $\boldsymbol{K}_{m+1}$, then A contains an isomorphic image of $\boldsymbol{K}_{q+1}$, where $q = l.c.m.(m, n)$.*

Proof. Let $f\colon \boldsymbol{K}_{n+1} \to A$ and $g\colon \boldsymbol{K}_{m+1} \to A$ be isomorphisms. Then the restrictions of f and g to L_{n+1} and to L_{m+1}, respectively, are isomorphisms. It follows that A contains a copy of $\mathbf{Ł}_{q+1}$. The thesis follows by Lemma 7.1.7. $\square$

PROPOSITION 7.1.9. *Let G be an Abelian ℓ-group, $0 < b \in G$. Then for each $n \geq 1$,*

$$\Gamma(\mathsf{Z} \times_{lex} G, \langle n, b\rangle) \in \mathbf{V}(\boldsymbol{K}_{n+1}).$$

Proof. Let G be an Abelian ℓ-group, $0 < b \in G$ and $n > 0$. For each integer $n > 0$, and $0 < b \in G$, $\Gamma(\mathsf{Z} \times_{lex} G, \langle n, b\rangle)$ can be embedded into $\Gamma(\mathsf{Z} \times_{lex} G, \langle n, 0\rangle)$. Indeed, such an embedding can be obtained via the map φ defined as: $\varphi(m, y) = (m, ny - mb)$. Hence we only need to show that $\Gamma(\mathsf{Z} \times_{lex} G, \langle n, 0\rangle) \in \mathbf{V}(\boldsymbol{K}_{n+1})$. By assumption, G is an Abelian ℓ-group. There exists a subgroup K of the Abelian ℓ-group Z^I, for some index set I and a group ℓ-homomorphism f such that $f(K) = G$. Via the functor Δ, we have:

$$(j) \qquad \Delta(f)(\Delta(K)) = \Delta(G),$$

i.e., $\Delta(G)$ is a perfect MV-algebra which is a homomorphic image of $\Delta(K)$, and

$$(jj) \qquad \Delta(K) \subseteq \Delta(\mathsf{Z}^I).$$

We define a homomorphism h as follows:

$h(0, a) = \langle 0, f(a)\rangle$ for every $a \in K_+$,

$h(m, a) = \langle m, f(a)\rangle$ for every $0 < m < n$ and $a \in K$,

$h(n, a) = \langle n, f(a)\rangle$ for every $a \in K_-$.

Note that

$$(jjj) \qquad h(\Gamma(\mathsf{Z} \times_{lex} K, \langle n, 0\rangle)) = \Gamma(\mathsf{Z} \times_{lex} G, \langle n, 0\rangle).$$

Hence we proved that $\Gamma(\mathsf{Z} \times_{lex} G, \langle n, g\rangle) \in \mathbf{V}(\boldsymbol{K}_{n+1})$. $\square$

LEMMA 7.1.10. *For any integer $n \geq 1$, $\boldsymbol{K}_{n+1} \in \mathbf{V}(H_n)$.*

Proof. The Lemma is proved once we will show that if an MV-equation holds in H_n it has to be valid in $\boldsymbol{K}_{n+1}$ too. By Proposition 2.6.1 (d2), we can assume that the equation has the form $\pi(x_1, \ldots, x_k) = 0$, for some MV-term π in the variables $x_1, \ldots, x_k$. Assume that the equation $\pi(x_1, \ldots, x_k) = 0$ does not hold in $\boldsymbol{K}_{n+1}$, for some MV-term π. Then there are elements $c_1, \ldots, c_k \in K_{n+1}$ such that

$$\pi_{K_{n+1}}(c_1, \ldots, c_k) \geq \langle 0, 1\rangle. \tag{1.1}$$

For each integer $m \geq 1$, let $f_m \colon \mathsf{Z} \times_{lex} \mathsf{Z} \to \mathsf{Z} \times_{lex} \mathsf{Z}$ be the ℓ-group homomorphism defined by

$$f_m(v, w) = \langle v, mw\rangle \quad for \quad all \quad \langle v, w\rangle \in \mathsf{Z}.$$

For an MV-term σ in the variables $x_1, \ldots, x_k$ let $g(\sigma)$ be the number of symbols * and $\oplus$ occurring in σ. By induction on $g(\sigma)$, for each k-tuple $\langle b_1, \ldots, b_k\rangle \in K_{n+1} \subseteq H_n$, we will prove the following system of inequalities:

$$\begin{aligned} f_m(\sigma_{K_{n+1}}(b_1, \ldots, b_k)) - (0, g(\sigma)) \ &\leq\ \sigma_{H_n}(f_m(b_1), \ldots, f_m(b_k)), \\ \sigma_{H_n}(f_m(b_1), \ldots, f_m(b_k)) \ &\leq\ f_m(\sigma_{K_{n+1}}(b_1, \ldots, b_k)) + (0, g(\sigma)). \end{aligned} \tag{1.2}$$

Indeed, if $g(\sigma) = 0$, then $\sigma = x_i$, i.e., σ is a single variable or $\sigma = 0$, the constant 0. The latter case is trivial, so we have to show that

$$f_m(\sigma_{K_{n+1}}(b_i)) \leq \sigma_{H_n}(f_m(b_i))).$$

Let $b_i = \langle v, w\rangle \in K_{n+1} \subseteq H_n$, then $\sigma_{K_{n+1}}(b_i) = b_i$ and $\sigma_{H_n}(f_m(b_i))) = f_m(b_i)$. Hence

$$f_m(\sigma_{K_{n+1}}(b_1, \ldots, b_k)) = f_m(b_i) = (v, mw) = \sigma_{H_n}(f_m(b_1), \ldots, f_m(b_k)).$$

Thus the desired inequalities are proved in the case $g(\sigma) = 0$. We proceed by induction on the number of symbols * and $\oplus$ occurring in MV-terms. Suppose that for some integer $d > 0$ (1.2) holds for all MV-terms ξ in the variables $x_1, \ldots, x_k$ such that $g(\xi) < d$. Let $\sigma(x_1, \ldots, x_k)$ be an MV-term such that $g(\sigma) = d$. Only the following cases occur.

Case 1. There is an MV-term $\tau(x_1, \ldots, x_k)$ such that $\sigma = \tau^*$.

Case 2. There are MV-terms $\mu(x_1, \ldots, x_k)$, $\nu(x_1, \ldots, x_k)$ such that $\sigma = \mu \oplus \nu$.

We can safely restrict ourselves to considering MV-terms in one variable. Assume the Case 1 holds. Let us prove that:

$$\sigma_{H_n}(f_m(b)) \leq f_m(\sigma_{K_{n+1}}(b)) + \langle 0, g(\sigma)\rangle. \tag{1}$$

Indeed, we have $g(\sigma) = 1 + g(\tau)$,

$$(i) \qquad \sigma_{H_n}(f_m(b)) = \langle n-1, 1\rangle - \tau_{H_n}(f_m(b))$$

and, by induction hypothesis

$$(ii) \qquad \tau_{H_n}(f_m(b)) \leq f_m(\tau_{K_{n+1}}(b)) + \langle 0, g(\tau)\rangle$$

$$(iii) \qquad f_m(\tau_{K_{n+1}}(b)) - \langle 0, g(\tau)\rangle \leq \tau_{H_n}(f_m(b)).$$

By (i) and (ii) we get:

$$\begin{aligned}\sigma_{H_n}(f_m(b)) &\geq \langle n-1,1\rangle - (f_m(\tau_{K_{n+1}}(b)) - \langle 0,g(\tau)\rangle = \\ &= \langle n-1,1\rangle - f_m(\tau_{K_{n+1}}(b)) + \langle 0,1\rangle - \langle 0,g(\sigma)\rangle \geq \\ &\geq f_m(\sigma_{K_{n+1}}(b)) - \langle 0,g(\sigma)\rangle.\end{aligned}$$

Similarly, by (i) and (iii):

$$\begin{aligned}\sigma_{H_n}(f_m(b)) &\leq \langle n-1,1\rangle - f_m(\tau_{K_{n+1}}(b)) + \langle 0,g(\tau\rangle) = \\ &= \langle n-1,0\rangle - f_m(\tau_{K_{n+1}}(b)) + \langle 0,1\rangle + \langle 0,g(\tau)\rangle = \\ &= f_m(\sigma_{K_{n+1}}(b)) + \langle 0,g(\sigma)\rangle.\end{aligned}$$

Hence σ satisfies (1). In Case 2, we have $g(\sigma) = g(\mu) + g(\nu) + 1$,

$$(iv) \quad \sigma_{H_n}(f_m(b)) \quad = \quad \langle n-1,1\rangle \wedge (\mu_{H_n}(f_m(b)) + \nu_{H_n}(f_m(b)))$$

and by induction hypothesis:

$$\begin{aligned}(v) \quad \sigma_{H_n}(f_m(b)) \quad &\leq \quad \langle n-1,1\rangle \wedge (f_m(\mu_{K_{n+1}}(b)) + \\ &\qquad +f_m(\nu_{K_{n+1}}(b)) + \langle 0,g(\mu)\rangle + \langle 0,g(\nu)\rangle),\end{aligned}$$

$$\begin{aligned}(vi) \quad \sigma_{H_n}(f_m(b)) \quad &\geq \quad \langle n-1,1\rangle \wedge (f_m(\mu_{K_{n+1}}(b)) + \\ &\qquad +f_m(\nu_{K_{n+1}}(b)) - \langle 0,g(\nu)\rangle - \langle 0,g(\mu)\rangle).\end{aligned}$$

Thus, by (iv) and (v) we get

$$\begin{aligned}\sigma_{H_n}(f_m(b)) &\leq \langle n-1,1\rangle \wedge (f_m(\mu_{K_{n+1}}(b)) + f_m(\nu_{K_{n+1}}(b))) + \langle 0,g(\mu)+g(\nu)\rangle \leq \\ &\leq \langle n-1,0\rangle \wedge (f_m(\mu_{K_{n+1}}(b)) + f_m(\nu_{K_{n+1}}(b))) + \langle 0,g(\mu)+g(\nu)+1\rangle \\ &= f_m(\sigma_{K_{n+1}}(b)) + \langle 0,g(\sigma)\rangle.\end{aligned}$$

Similarly, by (iv) and (vi) we get

$$\sigma_{H_n}(f_m(b)) \geq f_m(\sigma_{K_{n+1}}(b)) - \langle 0,g(\sigma)\rangle.$$

Thus, we showed that (1) holds. Hence, by (1.1) and (1) we obtain

$$(2) \quad \langle 0,m-g(\pi)\rangle \leq f_m(\pi_{K_{n+1}}(b_1,\dots,b_k)) - \langle 0,g(\pi)\rangle \leq \pi_{H_n}(f_m(b_1),\dots,f_m(b_k)).$$

Hence, if $m > g(\pi)$, then $\pi_{H_n}(f_m(b_1),\dots,f_m(b_k)) > \langle 0,1\rangle$ and H_n does not satisfy the equation $\pi(x_1,\dots,x_k) = 0$. We conclude that $\boldsymbol{K}_{n+1}$ must satisfy all equations that are satisfied by H_n. □

THEOREM 7.1.11. *For each integer* $n \geq 1$ *and each non simple* MV*-chain* A *of rank* n, $\mathbf{V}(A) = \mathbf{V}(\boldsymbol{K}_{n+1})$.

Proof. From Theorem 5.5.6 and Lemmas 3.5.11 and 3.5.12. □

For each $n = 1,2,\dots$ let $\mathbb{T}(n)$ denote the subvariety of $\mathbb{MV}$ defined by the identity:

$Eq(T) \qquad nx = (n+1)x.$

LEMMA 7.1.12. *Let m be an integer such that $1 \leq m \leq n$. Then $\mathbf{Ł}_{m+1} \in \mathbb{T}(n)$.*

Proof. Trivial checking. □

LEMMA 7.1.13. *Let m be an integer such that $1 \leq n < m$. Then $\mathbf{Ł}_{m+1} \notin \mathbb{T}(n)$.*

Proof. It is easy to check that for $m > 1$, the atom $1/m \in \mathbb{T}(n)$, does not satisfy $Eq(T)$. □

LEMMA 7.1.14. *Let A be an infinite simple MV-chain. Then $A \notin \mathbb{T}(n)$.*

Proof. Suppose $A \in \mathbb{T}(n)$. Then, by Proposition 7.1.3, A generates all the variety $\mathbb{MV}$. By Lemma 7.1.13, $\mathbb{T}(n)$ is proper, absurd. Hence $A \notin \mathbb{T}(n)$. □

LEMMA 7.1.15. *Let A be an infinite non simple MV-chain. Then $A \notin \mathbb{T}(n)$.*

Proof. Since A is not simple, then $Rad(A) \neq \{0\}$. Let $x \in Rad(A) \setminus \{0\}$, then it is easy to check that x does not satisfy $Eq(T)$. □

PROPOSITION 7.1.16. $\mathbb{T}(n) = \mathbf{V}(\mathbf{Ł}_2, \ldots, \mathbf{Ł}_{n+1})$.

Proof. By Lemmas 7.1.10, 7.1.12, and 7.1.13 and Theorems 7.1.2, 7.1.11. □

THEOREM 7.1.17. *Let A be an MV-algebra and $n \geq 1$ an integer. Then $A \in \mathbb{T}(n)$ if and only if A is subdirect product of $\mathbf{Ł}_{k+1}$ algebras with $1 \leq k \leq n$.*

Proof. By Proposition 4.1.3 and Lemma 7.1.14. □

For each $n = 1, 2, \ldots$ let $\mathbb{W}(n)$ denote the subvariety of $\mathbb{MV}$ defined by the identity:

$Eq(E)$ $\quad ((n+1)x^n)^2 = 2x^{n+1}$.

LEMMA 7.1.18. *If A is an infinite atomless subalgebra of $[0, 1]$, then $A \notin \mathbb{W}(n)$.*

Proof. We will show that there exists an element $a \in A$ which does not satisfy $Eq(E)$. Let $q_1 = (1 + 2(n^2 - 1))/(2n(n + 1))$ and $q_2 = n/(n + 1)$, then we get $q_1 < q_2$ and $q_1, q_2 \in \mathsf{Z} \cap [0, 1]$. Since, by Proposition 4.1.9, A is dense in $[0, 1]$, then there is $a \in A$ such that $q_1 < a < q_2$. It is easy to check that the element $a \in A$ does not satisfy $Eq(E)$. Hence $A \notin \mathbb{W}(n)$. □

LEMMA 7.1.19. *For every integer $1 \leq m \leq n$, $\boldsymbol{K}_{m+1} \in \mathbb{W}(n)$.*

Proof. For $n = 1$ the result follows from a direct verification. In the case $n > 1$, if $y \in (Rad(\boldsymbol{K}_{m+1}))^*$, it is easy to check that $((n + 1)y^n)^2 = 2y^{n+1} = 1$. Let $y = \langle m - 1, h\rangle \in K_{m+1}$, with $h \in \mathsf{Z}^+$. Then $((n + 1)y^n)^2 = 0$. On the other hand $y^{n+1} = 0$, and $Eq(E)$ holds for every element of the set $B = (Rad(\boldsymbol{K}_{m+1}))^* \cup \{\langle m - 1, h\rangle \mid h \in \mathsf{Z}^+\}$. Finally, if $y \in K_{m+1} \setminus B$, then we can write $y < z$, for some $z \in \{\langle m - 1, h\rangle \mid h \in \mathsf{Z}^+\} \subseteq K_{m+1}$. By monotonicity we get $((n + 1)y^n)^2 = 2y^{n+1} = 0$. □

LEMMA 7.1.20. *Let $1 \leq n < m$. Then $\mathbf{Ł}_{m+1} \notin \mathbb{W}(n)$.*

Proof. Let $1 \le n < m$ be integers with $m < 2(n+1)$. Then the MV-algebra $\mathbf{Ł}_m$ does not satisfy the identity $((n+1)y^n)^2 = 2y^{n+1}$. Indeed, let $y = m-1 \in L_{m+1}$. Then y is the co-atom of $\mathbf{Ł}_{m+1}$ and $((n+1)y^n)^2 = 1$. On the other hand, $y^{n+1} = m-n-1$, whence, from $m < 2(n+1)$ and $2 < ord(y^{n+1})$ we get $2y^{n+1} < 1$. Assume, now, $2(n+1) \le m$. Let the integers q and r be given by $m = 2q(n+1)+r$, $0 < q$, $0 \le r < 2(n+1)$, and $y = m-(q+1) \in L_{m+1}$. Then it is easy to see that $m - n(q+1) \ge m/2 + q - n \ge q+1 > 0$, $y^n = m - n(q+1) > 0$, and $(n+1)y^n = m(n+1)(q+1)$. By direct inspection we have:

$$m(n+1) - n((n+1)q + (n+1)) \ge m + n(m/2 - (n+1)).$$

Thus, by hypothesis $2(n+1) \le m$ we obtain $m(n+1) - n((n+1)q + (n+1)) \ge m$. Hence $(n+1)y^n = ((n+1)y^n)^2 = 1$. Now we are going to prove that $2y^{n+1} \neq 1$. If $y^{n+1} = 0$, we are done. Suppose $y^{n+1} > 0$. Then, $y^{n+1} = m - (n+1)(q+1)$. But,

$$m - q(n+1) - (n+1) = m + r/2 - m/2 - (n+1) < m/2.$$

In conclusion, $2y^{n+1} < 1$, whence $((n+1)y^n)^2 \neq 2y^{n+1}$, as required. $\square$

LEMMA 7.1.21. *Let $1 \le n$ be an integer and A an* MV*-chain. Then A satisfies $Eq(E)$ if and only if $rank(A) \le n$.*

Proof. By Lemma 7.1.18, $Eq(E)$ holds in the algebra $\boldsymbol{K}_{n+1}$, for $1 \le m \le n$. Also, by Lemma 7.1.10, $Eq(E)$ holds in all MV-chains of rank $m \le n$. By Lemma 7.1.18, for $n < m$, the MV-chain $\mathbf{Ł}_{m+1}$ does not satisfy $Eq(E)$. Hence, every MV-chain of rank $m > n$ cannot satisfy $Eq(E)$. $\square$

THEOREM 7.1.22. $\mathbb{W}(n) = \mathbf{V}(\boldsymbol{K}_2, \ldots, \boldsymbol{K}_{n+1})$.

Proof. By Lemma 7.1.19 and Propositions 7.1.9, 4.1.3, and 7.1.1. $\square$

THEOREM 7.1.23 (Komori's theorem). *Let $\mathbb{W}$ be a proper variety of* MV*-algebras. Then there are two finite sets I and J of integers greater than* 2 *such that $I \cup J \neq \emptyset$ and $\mathbb{W} = \mathbf{V}(\{\mathbf{Ł}_{i+1}\}_{i\in I}, \{\boldsymbol{K}_{i+1}\}_{i\in J})$.*

Proof. By Proposition 7.1.5 and Lemmas 7.1.15 and 7.1.20 we know that a variety of MV-algebras is proper if and only if it is generated by a set of MV-chains of bounded rank. An application of Proposition 7.1.9 then yields the theorem. $\square$

7.2 Equational characterization of varieties of MV-algebras

For any $i \in \mathsf{Z}^+ = \{1, 2, \ldots\}$ we set

$$\delta(i) = \{n \in \mathsf{Z} \mid 1 \le n \text{ is a divisor of } i\}.$$

For J nonempty finite subset of Z^+ and $i = \{1, 2, \ldots\}$ we let

$$\Delta(i, J) = \{d \in \delta(i) \setminus \bigcup_{j \in J} \delta(j)\}.$$

If $J = \emptyset$ we define $\Delta(i, J) = \delta(i)$. For each integer $n \ge 3$, let $\mathbb{W}(n,p)$ denote the variety defined by the identity:

$Eq(2p) \qquad (px^{p-1})^{n+1} = (n+1)x^p \qquad \text{with } 1 < p < n.$

THEOREM 7.2.1. *For all $n \geq 3$, and $1 < p < n$, $\boldsymbol{K}_{n+1} \in \mathbb{W}(n,p)$.*

Proof. Assume $y \in K_{n+1}$ and $y^{p-1} = 0$. Then $y^p = 0$ and y satisfies $Eq(2p)$. Assume now $y = \langle s, z\rangle \in K_{n+1}$ and $0 < y^{p-1}$. If $y^p = 0$, then $s < (p-1)n/p$. Hence, $ps(p-1) - pn(p-2) < n$. It follows that $py^{p-1} = (ps(p-1) - pn(p-2), pz(p-1)) \notin (Rad(\boldsymbol{K}_{n+1}))^*$, and hence $(py^{p-1})^{n+1} = 0$. If, on the other hand, $y^p > 0$, then $s > (p-1)n/p$, whence $py^{p-1} = 1$. We conclude that, for all $y \in K_{n+1}$, $Eq(2p)$ holds. □

THEOREM 7.2.2. *Let $n \geq 3$ and $1 < p < n$. Then the following are equivalent for all $1 \leq q < n$:*

(i) $\text{Ł}_{q+1} \in \mathbb{W}(n,p)$;

(ii) *p does not divide q.*

Proof. Assume that $\text{Ł}_{q+1} \in \mathbb{W}(n,p)$ and $p = kq$ for some $k \in \mathsf{Z}^+$. Consider $y = k(p-1) \in L_{q+1} \setminus \{1\}$. Then we have $py^{p-1} = pk(p-1)^2 - pq(p-2) = q$, and $y^p = pk(p-1) - q(p-1) = 0$, a contradiction.

Conversely, assume p does not divide q. and pick an arbitrary element $y \in L_{q+1}$. If $y^{p-1} = 0$, then $y^q = 0$ and $Eq(2p)$ is satisfied. If, on the other hand, $0 < y^{p-1}$, then we proceed by cases. If $y^p = 0$, then $y < q(p-1)/p$ and $py(p-1) - pq(p-2) < q$. Thus $(py^{p-1})^{n+1} = 0$. If $y^p > 0$, then $y > q(p-1)/p$ which implies $py^{p-1} = 1$. Therefore, $Eq(2p)$ holds for Ł_{q+1}. □

COROLLARY 7.2.3. *For $n \geq 3$, and $1 < p < n$, $\text{Ł}_{p+1} \notin \mathbb{W}(n,p)$.*

For $n \geq 3$ we define the equational class $H(n)$ by the stipulation:

$$\mathbb{H}(n) = \mathbb{W}(n) \cap (\bigcap\{\mathbb{W}(n,p) \mid 1 < p < n \text{ and } p \text{ is not a divisor of } n\}).$$

THEOREM 7.2.4. *For $n \geq 3$, $\mathbf{V}(\boldsymbol{K}_{n+1}) = \mathbb{H}(n)$.*

Proof. From Theorem 7.1.22 and Theorem 7.2.1, it follows that $\mathbf{V}(\boldsymbol{K}_{n+1}) \subseteq \mathbb{H}(n)$. Now, we show the reverse inclusion. From Lemma 7.1.18, $\mathbb{H}(n)$ is a proper subvariety of $\mathbb{MV}$. By Lemma 7.1.21, there exist two subsets $P = \{p_1, \ldots, p_r\}$, and $N = \{n_1, \ldots, n_t\}$ of Z^+, with $P \cup N \neq \emptyset$ and $r, t \in \mathsf{Z}^+$, such that

$$\mathbb{H}(n) = \mathbf{V}(\text{Ł}_{p_1+1}, \ldots, \text{Ł}_{p_r+1}, \boldsymbol{K}_{p_1+1}, \ldots, \boldsymbol{K}_{p_t+1}).$$

Since, by Lemma 7.1.17 and Theorem 7.1.22, $\boldsymbol{K}_{n+1} \in \mathbb{H}(n)$, $n \in \{n_1, \ldots, n_t\}$; it is no loss of generality to assume $n = n_1$. From Lemma 7.1.18, we have $n_j \leq n$ for every $j \in \{1, \ldots, t\}$. Then, by Corollary 7.2.3, each n_j divides n, whence $\mathbb{H}(n) = \mathbf{V}(\text{Ł}_{p_1+1}, \ldots, \text{Ł}_{p_r+1}, \boldsymbol{K}_{n+1})$. Again, by Corollary 7.2.3, $\mathbb{H}(n) = \mathbf{V}(\boldsymbol{K}_{n+1})$. □

COROLLARY 7.2.5. *For $n \geq 3$, $\mathbf{V}(\boldsymbol{K}_{n+1})$ is characterized by the equations:*

$$((n+1)x^n)^2 = 2x^{n+1},$$
$$(px^{p-1})^{n+1} = (n+1)x^p$$

for every positive integer $1 < p < n$ such that p is not a divisor of n.

For $n_1 < \cdots < n_t$, positive integers, assume that $t \geq 2$, $n_t \geq 5$ and that n_i is not a divisor on n_j whenever $i < j \leq t$. Let Σ denote the system of equations

$$((n_t+1)x^{n_t})^2 = 2x^{n_t+1},$$
$$(px^{p-1})^{n_t+1} = (n_t+1)x^p,$$

for every integer $1 < p < n_t$ such that p is not a divisor of any n_i, $i = 1, 2, \ldots, t$, and let $\mathbb{L}(n_1, \ldots, n_t)$ denote the variety defined by Σ. Moreover we define

$$\mathbb{L}(n_t) = \mathbb{H}(n_t) \quad \text{if} \quad t = 1,$$
$$\mathbb{L}(2,3) = \mathbb{W}(3),$$
$$\mathbb{L}(3,4) = \mathbb{W}(4).$$

THEOREM 7.2.6. *For $n_1 < \cdots < n_t$, positive integers, such that $t \geq 2$, $n_t \geq 5$,*

$$\mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1}) = \mathbb{L}(n_1, \ldots, n_t).$$

Proof. Firstly, we show that $\mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1}) \subseteq \mathbb{L}(n_1, \ldots, n_t)$. We note that $\boldsymbol{K}_{n_i+1} \in \mathbb{L}(n_1, \ldots, n_t)$ for every $i \in \{1, \ldots, t\}$. If $i = t$, then the result follows from Theorems 7.1.17 and 7.1.22. If $i < t$, the result follows from Theorems 7.1.17 and 7.1.23, because p is not a divisor of any n_t, $i \in \{1, \ldots, t\}$.

Now, by Lemma 7.1.21, we obtain $\mathbb{L}(n_1, \ldots, n_t) = \mathbf{V}(\mathbf{Ł}_{s_1+1}, \ldots, \mathbf{Ł}_{s_j+1}, \boldsymbol{K}_{r_1+1}, \ldots, \boldsymbol{K}_{r_g+1})$ for some positive integers $s_1, \ldots, s_j, r_1, \ldots, r_g$, and from Lemma 7.1.18, $r_g \leq n_t$. Let $q < n_t$ be a positive integer not dividing any n_i, $i = 1, \ldots, t$. Choose $y = q - 1 \in L_{q+1}$. Then $qy^{q-1} = 1$ and $y^q = 0$. This implies that each element of the set $s_1, \ldots, s_j, r_1, \ldots, r_g$ is a divisor of some index n_i. Thus, we have $\mathbb{L}(n_1, \ldots, n_t) \subseteq \mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1})$. This completes the proof. □

COROLLARY 7.2.7. *For $n_1 < \cdots < n_t$, positive integers, assume that n_i is not a divisor of n_j, whenever $i < j \leq t$. Then the subvariety $\mathbf{V}(\mathbf{Ł}_{n_1+1}, \ldots, \mathbf{Ł}_{n_t+1})$ is characterized by the identities:*

$$((n_t+1)x^{n_t})^2 = 2x^{n_t+1},$$
$$(px^{p-1})^{n_t+1} = (n_t+1)x^p,$$

where $1 < p < n_t$ and p does not divide any n_i, $i = 1, \ldots, t$.

Let $I = \{\alpha_1, \ldots, \alpha_s\} \neq \emptyset$, with $\alpha_1 < \cdots < \alpha_s$, α_i not dividing α_j, whenever $i < j \leq s$. Let $J = \{\beta_1, \ldots, \beta_t\}$ with $\beta_1 < \cdots < \beta_t$ and $I, J \subset \mathsf{Z}^+$. In case $J \neq \emptyset$, assume, for every $i = 1, \ldots, s$, α_i does not divide β_j, for each $j = 1, \ldots, t$. Furthermore, assume that α_i is not a divisor of α_j whenever $i < j \leq s$, and β_i is not a divisor of β_j whenever $i < j \leq t$. Let $n = \max\{I \cup J\}$. Let Σ' denote the system of equations:

$$((n+1)x^n)^2 = 2x^{n+1},$$
$$(px^{p-1})^{n+1} = (n+1)x^p,$$

for every positive integer p such that $1 < p < n$ and p does not divide i whenever $i \in I \cup J$. Let Σ'' denote the system of equations:

$$(n+1)x^q = (n+2)x^q,$$

for every q such that $q \in \bigcup_{\alpha_i \in I} \Delta(\alpha_i, J)$. Let $\mathbb{H}^*$ be the variety defined by $\Sigma' \cup \Sigma''$ and

$$\bar{\mathbb{H}} = \mathbf{V}(\mathbf{Ł}_{\alpha_1+1}, \ldots, \mathbf{Ł}_{\alpha_n+1}, \boldsymbol{K}_{\beta_1+1}, \ldots, \boldsymbol{K}_{\beta_m+1}).$$

THEOREM 7.2.8. $\bar{\mathbb{H}} = \mathbb{H}^*$.

Proof. The MV-algebras $\mathbf{Ł}_{\alpha_1+1}, \ldots, \mathbf{Ł}_{\alpha_n+1}, \boldsymbol{K}_{\beta_1+1}, \ldots, \boldsymbol{K}_{\beta_m+1}$ are all members of $\mathbb{H}^*$. Thus by Corollary 7.2.3, they verify all the equations of Σ'. Furthermore, for each $\alpha_i \in I$, $y \in L_{\alpha_i+1}$, and $q \in \bigcup_{\alpha_i \in I} \Delta(\alpha_i, J)$, we have either $0 = y^q$ or $ord(y^q) \leq n$.

Thus, $(n+1)y^q = (n+2)y^q$ for every $q \in \bigcup_{\alpha_i \in I} \Delta(\alpha_i, J)$. This proves that $\mathbf{Ł}_{\alpha_1+1}, \ldots, \mathbf{Ł}_{\alpha_n+1}$ verify Σ''. Now let $\beta_h \in J$ and $y \in K_{\beta_h+1}$. Since q does not divide β_h, then either $y^q = 0$ or $ord(y^q) \leq n+1$, for every $q \in \bigcup_{\alpha_i \in I} \Delta(\alpha_i, J)$. Thus, it follows that $(n+1)y^q = (n+2)y^q$. So, also $\boldsymbol{K}_{\beta_j+1}$ satisfies Σ'', for every $\beta_j \in J$. Hence $\bar{\mathbb{H}} \subseteq \mathbb{H}^*$.

By Theorem 7.1.22, $\mathbb{H}^*$ is a proper subvariety, hence by Lemma 7.1.21, $\mathbb{H}^* = \mathbf{V}(\mathbf{Ł}_{q_1+1}, \ldots, \mathbf{Ł}_{q_h+1}, \boldsymbol{K}_{p_1+1}, \ldots, \boldsymbol{K}_{p_t+1})$ for $q_1, \ldots, q_h, p_1, \ldots, p_t, h, t \in \mathsf{Z}^+$.

Each index $q_1, \ldots, q_h, p_1, \ldots, p_t$, by Lemma 7.1.18, is smaller than or equal to n; while, by Corollary 7.2.3, it divides some element of $I \cup J$. Hence, we infer that each MV-algebra $\mathbf{Ł}_{q_1+1}, \ldots, \mathbf{Ł}_{q_h+1}, \boldsymbol{K}_{p_1+1}, \ldots, \boldsymbol{K}_{p_t+1}$ is a subalgebra of some MV-algebra, $\mathbf{Ł}_{\alpha_1+1}, \ldots, \mathbf{Ł}_{\alpha_n+1}, \boldsymbol{K}_{\beta_1+1}, \ldots, \boldsymbol{K}_{\beta_m+1}$. Now we show that $\mathbb{H}^* \subseteq \bar{\mathbb{H}}$. We will show that the greatest subalgebra of $\boldsymbol{K}_{\alpha_i+1}$, which is member of $\mathbb{H}^*$, is $\boldsymbol{K}_{\alpha_i+1}$, for each $\in I$. Indeed, choose $y = \langle \alpha_i - 1, z \rangle \in K_{\alpha_i+1}$, for $z \in \mathsf{Z}^+$; we have $y^{\alpha_i} \in Rad(\boldsymbol{K}_{\alpha_i+1}) \setminus \{0\}$; then $(n+1)y^{\alpha_i} = (n+1)(0, \alpha_i z) \neq (n+2)y^{\alpha_i}$. Thus, the element $y \in K_{\alpha_i+1}$ is not a solution of the equation $(n+1)x^{\alpha_i} = (n+2)x^{\alpha_i}$ of Σ''. The thesis is now proved. $\square$

We prove the main theorem of this section.

THEOREM 7.2.9. *Let* $\mathbb{W}$ *be a proper subvariety of* $\mathbb{MV}$. *Then there exist finite sets* I *and* J *of integers* ≥ 1 *with* $I \cup J \neq \emptyset$ *such that for any* MV*-algebra* A *the following are equivalent:*

(a) $A \in \mathbb{W}$,

(b) *A satisfies the equations:*

$$\begin{array}{lll} Eq(1) & ((n+1)x^n)^2 = 2x^{n+1} & \textit{with } n = \max\{I \cup J\}, \\ Eq(2) & (px^{p-1})^{n+1} = (n+1)x^p, & \\ Eq(3) & (n+1)x^q = (n+2)x^q, & \end{array}$$

for every positive integer $1 < p < n$ *such that* p *is not a divisor of any* $i \in I \cup J$ *and for every* $q \in \bigcup_{i \in I} \Delta(i, J)$.

Proof. By Lemma 7.1.21 there exist integers $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ such that

$$\mathbb{W} = \mathbf{V}(\text{\L}_{\alpha_1}, \ldots, \text{\L}_{\alpha_n}, \boldsymbol{K}_{\beta_1}, \ldots, \boldsymbol{K}_{\beta_m}).$$

By Theorem 7.2.8 $Eq(1)$, $Eq(2)$ and $Eq(3)$ characterize $\mathbb{W}$. □

THEOREM 7.2.10. *Let A be an* MV*-algebra. Then the following are equivalent:*

(a) $A \in \mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1})$,

(b) $A/Rad(A) \in \mathbf{V}(\text{\L}_{n_1+1}, \ldots, \text{\L}_{n_t+1})$.

Proof. (a) ⇒ (b) By Chang's representation theorem, A can be subdirectly embedded into a direct product $\prod_{i \in I} A_i$ where $A_i = A/P$ for every $P \in Spec(A)$. Let ϕ be the MV-embedding mapping defined s follows:

$$\phi(x) = (x_i/Rad(A_i))_{i \in I},$$

where $x = (x_i)_{i \in I}$. By (j), for every $i \in I$, $A_i/Rad(A_i) \in \mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1})$. Hence, by Proposition 7.1.5 $rank(A_i/Rad(A_i)) = k_i$ for some positive integer k_i. But $A_i/Rad(A_i)$ is simple, and thus $A_i/Rad(A_i) \simeq \text{\L}_{k_i+1}$. Therefore we obtain $\text{\L}_{k_i+1} \in \mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1})$. We can safely assume that $n_1 < \cdots < n_t$, hence

$$\mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1}) \subseteq \mathbf{V}(\boldsymbol{K}_2, \ldots, \boldsymbol{K}_{n_t+1}) = \mathbb{W}(n_t),$$

by Lemma 7.1.18 $k_i = n_t$ and $\text{\L}_{k_i+1} \in \mathbf{V}(\text{\L}_2, \ldots, \text{\L}_{n_t+1}) = \mathbb{T}(n_t)$ by Lemmas 7.1.10 and 7.1.14. Assume k_i not be a divisor of n_t. Then, by Corollary 7.2.5 the following equation

$$(k_i x^{k_i-1})^{n_t+1} = (n_t+1)x^{k_i}$$

has to be true for $\text{\L}_{k_i+1}$. But this is not true, indeed. In fact, the element $(k_i-1)/k_i$ of L_{k_i+1} does not satisfy the above equation. Hence k_i is a divisor of n_t. Then $A_i/Rad(A_i)$ can be embedded into $\text{\L}_{n_t+1}$. Thus $A/Rad(A) \in \mathbf{V}(\text{\L}_{n_1+1}, \ldots, \text{\L}_{n_t+1})$.
(b) ⇒ (a) Let $P \in Spec(A)$ and $Rad(A) \subseteq P$, then by (jj),

$$(*) \qquad A/P \simeq (A/Rad(A))/(P/Rad(A)) \in \mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1}).$$

If $Rad(A) \nsubseteq P$, then $(A/P)/(Rad(A/P)) \simeq A/M$, where M is the unique maximal ideal of A such that $P \subseteq M$. By (jj) and Proposition 7.1.5, $A/M \simeq \text{\L}_{p+1}$ for some positive integer p and $\text{\L}_{p+1} \in \mathbf{V}(\text{\L}_{n_1+1}, \ldots, \text{\L}_{n_t+1})$. Assume again that $n_1 < \cdots < n_t$, hence by Lemma 7.1.14,

$$\text{\L}_{p+1} \in \mathbf{V}(\text{\L}_{n_1+1}, \ldots, \text{\L}_{n_t+1}) \subseteq \mathbf{V}(\text{\L}_2, \ldots, \text{\L}_{n_t+1}) = \mathbb{T}(n_t),$$

by Lemma 7.1.10, $p \leq n_t$. Again, by Corollary 7.2.5, as above, we get p is a divisor of n_t. Hence A/P can be embedded into $\text{\L}_{n_t+1}$, hence

$$(**) \qquad A/P \in \mathbf{V}(\text{\L}_{n_1+1}, \ldots, \text{\L}_{n_t+1}) \subseteq \mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1}).$$

By $(*)$ and $(**)$ we get $A \in \mathbf{V}(\boldsymbol{K}_{n_1+1}, \ldots, \boldsymbol{K}_{n_t+1})$. □

8 Historical remarks and further reading

MV-algebra theory has its origin in the study of infinite-valued Łukasiewicz propositional logic. The completeness theorem of this logic was firstly published by Rose and Rosser in 1958 [77]. An earlier proof given by Wajsberg was never published.

MV-*algebras* were defined by Chang [12] in 1958, developing an algebraic version of Łukasiewicz propositional logic. He also gave an algebraic proof of the completeness theorem [13].

MV-algebras, therefore, stand in relation to the Łukasiewicz infinite valued logic as Boolean algebras stand in relation to classical 2-valued logic. Boolean algebras are not glued to their origin in logic. Their well-recognized use in other areas of mathematics led to an extensive investigation of their structure.

The MV-algebra theory followed a similar development. Establishing connections with other areas of mathematics was a mainstream of investigation [4, 26, 59, 60]. In parallel, the comprehensive study of their intrinsic structure has shown the existence of a rich variety of classes of MV-algebras and this research continues in present.

The study of MV-algebras was quiescent for about thirty years after their appearance. They have been considered with increasing attention after successful applications of Mundici in the study of AF C^*-algebras [59], lattice-ordered Abelian groups [60, 63] and of Belluce in Bold Fuzzy Set Theory [4].

In [60], Mundici proved the categorical equivalence between MV-algebras and lattice-ordered Abelian groups with strong unit. This theorem is the origin of a considerable number of remarkable results in the theory of MV-algebras. It turns out that the functor Γ plays a pivotal role in the theory of MV-algebras. In the same vein, a different categorical equivalence was established by Di Nola and Lettieri [26], showing that the category of lattice-ordered Abelian groups is equivalent to the category of *perfect* MV-*algebras*.

The composition of the functor Γ with Grothendieck's functor K_0 yields a one-one correspondence between MV-algebras and a class of AF C^*-algebras. Such a correspondence was explored by Mundici in several papers, see, for example, [59] and [62]. In particular, the author introduced in [62], the AF C^*-algebra $\mathfrak{M}_1$ corresponding to the free MV-algebra of one variable, rediscovered twenty years later by Boca in [9].

A first important class of MV-algebras was characterized by Chang [12]. He proved that the linearly ordered structures, MV-chains, are exactly quotients of MV-algebras with respect to the prime ideals. Successively, to cite some authors, Lacava [46], Belluce [4–6], Cignoli [17], Hoo [40] gave further contributions to the classification of MV-algebras. Further results concerning characterizations of classes of MV-algebras can be found in more recent papers, see for example [19, 23, 26].

MV-algebras form a variety, so the description of its subvarieties is an important issue. In [43] Komori gave a complete description of all subvarieties of MV-algebras. Concerning equational bases for such varieties, a first step was made by Grigolia [38, 39], who axiomatized equationally the class of $\mathbf{Ł}_n$-algebras. With different approaches, Di Nola and Lettieri [27] and Panti [73] provided finite equational bases for each subvariety of MV-algebras.

The representation theory of MV-algebras has a rich history. In [12], Chang proved the first representation theorem: any MV-algebra is isomorphic with a subdirect product of MV-chains. In the context of many-valued logic, the algebras of $[0,1]$-valued functions have a special role. These are the semisimple MV-algebras, i.e. algebras without infinitesimals. Their functional representation was pointed out by Chang [12] and Belluce [4]. The representation of semisimple MV-algebras as algebras of continuous functions is proved in [18, 29]. A functional representation for all MV-algebras is due to Di Nola [21]: any MV-algebra can be seen as an MV-algebra of functions from a set to an ultrapower of $[0,1]$.

Another key result in the MV-algebra theory is *McNaughton theorem* [56], which led to a functional representation of free MV-algebras. Subsequent deep studies are due to Aguzzoli, Cignoli, Marra, Mundici and Panti. One can see [66] for a constructive proof of McNaughton Theorem and [72] for a geometrical proof.

In the paper [67] the author opened the studies on *states of* MV*-algebras*. Quite recently, studies on states of MV-algebras received a renewed impulse, showing that such states play an analogous fruitful role for MV-algebras like probabilities do for Boolean algebras, connecting phenomena coming from groups, measure theory, topology and *betting*, being states a generalization of probabilities. In the last few years, the theory of states was studied by many experts in MV-algebras, see e.g. [44, 45, 74, 76]. Kühr and Mundici studied states using the notion of a coherent state by De Finetti with the motivation in Dutch book making, see [45]. The Kroupa-Panti theorem yields a one-one correspondence between states of an MV-algebra A and the regular Borel probability measures on the spectrum of maximal ideals of A, see [44, 74]. MV-algebraic probability theory is now a large area of studies which involves many topics, among them we find continuous states, conditional probabilities, Caratheodory algebraic probability, metric completion, axiomatics, entropy, complexity theory, invariant states, abstract Lebesgue integration, MV-algebraic ergodic theory, see [49, 53, 55, 68–70].

Of great interest are the recent studies on the category of *finitely presented* MV*-algebras*. A combination of usage of the Γ functor and of methods coming from the theory of finitely presented Abelian ℓ-groups and polyhedral geometry has been applied to reach extremely interesting results. Indeed, using the Γ functor, in [51] it is proved that finitely generated projective ℓ-groups coincide with ℓ-groups presented by a word only using the lattice operations. This strengthens a well known corollary of the celebrated *Baker–Beynon duality:* an Abelian ℓ-group G is finitely presented (by a word only using the additive subtractive structure of G) if and only if it is finitely generated projective. On the other hand, projective unital ℓ-groups and projective MV-algebras are rather the exception among finitely presented algebras (see [10, 11]). In his two papers [41, 42], Jeřábek relates projective MV-algebras with *unification problems* in Łukasiewicz logic.

The non-commutative generalizations of MV-algebras were introduced, independently, by Georgescu and Iorgulescu [35], under the name *pseudo* MV*-algebras*, and by Rachůnek [75], under the name *generalized* MV*-algebras*. The MV-algebraic operations $\oplus$ and $\odot$ are not necessarily commutative, so the new structures are categorically equivalent with lattice-ordered groups with strong unit, as proved by Dvurečenskij in [30]. Consequently, the corresponding propositional calculus developed in [48] has two implications and it provides a non-commutative approach to Łukasiewicz logic.

Another research direction was to characterize the class of structures generated by $[0,1]$ in the language of MV-algebras enriched with the real product. These investigations led to the definition of PMV-*algebras* (*product* MV-*algebras*) [22, 57]. The analogue of Mundici's theorem for these structures was obtained by Di Nola and Dvurečenskij [22]: there exists a categorical equivalence between PMV-algebras and lattice-ordered rings with strong unit. In [58], Montagna axiomatized the quasivariety generated by $[0,1]$ in the language of PMV-algebras. It seems quite natural to introduce "modules" over PMV-algebras. The structure of MV-*module* is defined and studied in [24, 47]. The MV-modules over $[0,1]$ are particularly interesting, since they are categorically equivalent with a class of Riesz spaces. A common generalization of MV-modules and product MV-algebras is obtained in [32], where MV-algebras with operators are defined.

In [36] the author defines DMV-*algebras* and their corresponding propositional calculus, *Rational Łukasiewicz logic*. DMV-algebras are obtained by adding a countable family of operators to the structure of MV-algebra. Consequently, the Rational Łukasiewicz logic is an extension of the infinite valued Łukasiewicz calculus.

MV-algebras are also intimately connected with other apparently remote domains of mathematics like toric varieties [1, 54], multisets [19], semirings [25, 37] or quantum structures [31]. One can also find in literature applications of Łukasiewicz logic to error correcting codes and fault-tolerant search [15, 16], based on the game-theoretic semantics for Łukasiewicz logic introduced by Mundici [64, 65] or connections with neural networks [2], image processing [28] and automata [37, 78].

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Chapter VII: Gödel–Dummett Logics

MATTHIAS BAAZ AND NORBERT PREINING

1 Introduction and History

The logics we present in this chapter, Gödel logics, can be characterized in a rough-and-ready way as follows: The language is a standard (propositional, quantified propositional, first-order) language. The logics are many-valued, and the sets of truth values considered are (closed) subsets of $[0, 1]$ which contain both 0 and 1. 1 is the 'designated value,' i.e., a formula is valid if it receives the value 1 in every interpretation. The truth functions of conjunction and disjunction are minimum and maximum, respectively, and in the first-order case quantifiers are defined by infimum and supremum over subsets of the set of truth values. The characteristic operator of Gödel logics, the Gödel conditional, is defined by $a \to b = 1$ if $a \leq b$ and $= b$ if $a > b$. Because the truth values are ordered (indeed, in many cases, densely ordered), the semantics of Gödel logics is suitable for formalizing *comparisons*. It is related in this respect to a more widely known many-valued logic, Łukasiewicz (or 'fuzzy') logic (see Chapter VI)—although the truth function of the Łukasiewicz conditional is defined not just using comparison, but also addition. In contrast to Łukasiewicz logic, which might be considered a logic of *absolute* or *metric comparison*, Gödel logics are logics of *relative comparison*.

There are other reasons why the study of Gödel logics is important. As noted, Gödel logics are related to other many-valued logics of recognized importance. Indeed, Gödel logic is one of the three basic t-norm based logics which have received increasing attention in the last 15 or so years (the others are Łukasiewicz and product logic; see [26]). Yet Gödel logic is also closely related to intuitionistic logic:[1] it is the logic of linearly-ordered Heyting algebras. In the propositional case, infinite-valued Gödel logic can be axiomatized by the intuitionistic propositional calculus extended by the axiom schema $(A \to B) \vee (B \to A)$. This connection extends also to Kripke semantics for intuitionistic logic: Gödel logics can also be characterized as logics of (classes of) linearly ordered and countable intuitionistic Kripke structures with constant domains [19]. Furthermore, the infinitely valued propositional Gödel logic can be embedded into the box fragment of LTL in the same way as intuitionistic propositional logic can be embedded into $\mathrm{S4}$.

We want to start here with an observation concerning implications for many-valued logics, that spotlights why Gödel logics behave well in some cases in contrast to other many-valued logics, namely that they are based on the only implication that admits both modus ponens and the deduction theorem, as can be seen from the following observation of Gaisi Takeuti.

[1] Editorial footnote: The notation of this chapter conforms the one used in the study of Gödel logics and therefore differs from the rest of the handbook.

1.1 The Gödel conditional

LEMMA 1.1.1. *Suppose we have a standard language containing a 'conditional'* $\twoheadrightarrow$ *interpreted by a truth-function into* $[0,1]$*, and some entailment relation* $\models$*. Suppose further that*

1. *a conditional evaluates to* 1 *if the truth value of the antecedent is less or equal to the truth value of the consequent, i.e., if* $\mathcal{I}(A) \leq \mathcal{I}(B)$*, then* $\mathcal{I}(A \twoheadrightarrow B) = 1$*;*
2. *if* $\Gamma \models B$*, then* $\mathcal{I}(\Gamma) \leq \mathcal{I}(B)$*;*
3. *the deduction theorem holds, i.e.,* $\Gamma \cup \{A\} \models B \Leftrightarrow \Gamma \models A \twoheadrightarrow B$*.*

Then $\twoheadrightarrow$ *is the Gödel conditional.*

Proof. From (1), we have that $\mathcal{I}(A \twoheadrightarrow B) = 1$ if $\mathcal{I}(A) \leq \mathcal{I}(B)$. Since $\models$ is reflexive, $B \models B$. Since it is monotonic, $B, A \models B$. By the deduction theorem, $B \models A \twoheadrightarrow B$. By (2),

$$\mathcal{I}(B) \leq \mathcal{I}(A \twoheadrightarrow B).$$

From $A \twoheadrightarrow B \models A \twoheadrightarrow B$ and the deduction theorem, we get $A \twoheadrightarrow B, A \models B$. By (2),

$$\min\{\mathcal{I}(A \twoheadrightarrow B), \mathcal{I}(A)\} \leq \mathcal{I}(B).$$

Thus, if $\mathcal{I}(A) > \mathcal{I}(B), \mathcal{I}(A \twoheadrightarrow B) \leq \mathcal{I}(B)$. □

A large class of many-valued logics can be developed from the theory of t-norms [26]. The class of t-norm based logics includes not only (standard) Gödel logic, but also Łukasiewicz and product logic. In these logics, the conditional is defined as the residuum of the respective t-norm, and the logics differ only in the definition of their t-norm and the respective residuum, i.e., the conditional (see Chapter V). The truth function for the Gödel conditional is of particular interest as it can be 'deduced' from simple properties of the evaluation and the entailment relation, as shown above.

Note that all usual conditionals (Gödel, Łukasiewicz, product conditionals) satisfy condition (1). So, in some sense, the Gödel conditional is the only many-valued conditional which validates both directions of the deduction theorem for $\models$. For instance, for the Łukasiewicz conditional $\rightarrow_{Ł}$ the right-to-left direction fails: $A \rightarrow_{Ł} B \models A \rightarrow_{Ł} B$, but $A \rightarrow_{Ł} B, A \not\models B$. (With respect to $\Vdash$, the left-to-right direction of the deduction theorem fails for $\rightarrow_{Ł}$.)

One of the surprising facts about Gödel logics is that whereas there is only one infinite-valued propositional Gödel logic, and already uncountably many different logics when considering propositional entailments [16] or quantification over propositions [15], there are only countably many different infinite-valued first-order Gödel logics depending on the choice of the set of truth values (Theorem 3.5.1). For both quantified propositional and first-order Gödel logics, different sets of truth values with different order-theoretic properties in general result in different sets of valid formulas.

Besides the logical and computational interest in Gödel logics, they also provide an interesting playground for various areas of more traditional mathematics, like topology, esp. Polish spaces and Order theory.

1.2 History of Gödel logics

Gödel logics are one of the oldest families of many-valued logics. Propositional finite-valued Gödel logics were introduced by Gödel [25] to show that there are infinitely many logics between intuitionistic and classical logic. They provide the first examples of intermediate logics (intermediate, that is, in strength between classical and intuitionistic logics). Dummett [22] was the first to study infinite-valued propositional Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom $(A \to B) \vee (B \to A)$. Hence, infinite-valued propositional Gödel logic is also sometimes called Gödel–Dummett logic or Dummett's $\mathbf{LC}$. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders. The entailment relation in propositional Gödel logics was investigated in [16] and Gödel logics with quantifiers over propositions in [7].

Standard first-order Gödel logic $\mathbf{G}_\mathbb{R}$—the one based on the full interval $[0, 1]$—has been discovered and studied by several people independently. Alfred Horn [27] was probably the first: He discussed this logic under the name *logic with truth values in a linearly ordered Heyting algebra*, and gave an axiomatization and the first completeness proof. [38] called $\mathbf{G}_\mathbb{R}$ *intuitionistic fuzzy logic* and gave a sequent calculus axiomatization for which they proved completeness. This system incorporates the density rule

$$\frac{\Gamma \vdash A \vee (C \to p) \vee (p \to B)}{\Gamma \vdash A \vee (C \to B)}$$

(where p is any propositional variable not occurring in the lower sequent.) The rule is redundant for an axiomatization of $\mathbf{G}_\mathbb{R}$, as was shown by Takano [36], who gave a streamlined completeness proof of Takeuti–Titani's system without the rule. A syntactical proof of the elimination of the density rule was later given in [17]. Other proof-theoretic investigations of Gödel logics can be found in [2] and [3]. The density rule is nevertheless interesting: It forces the truth value set to be dense in itself (in the sense that, if the truth value set is not dense in itself, the rule does not preserve validity). This contrasts with the expressive power of formulas: no formula is valid only for truth value sets which are dense in themselves.

Recent developments have clarified many long standing questions like the classification of axiomatizability, the relation to Kripke frames, and the status of satisfiability of the monadic class.

1.3 Syntax and semantics for propositional Gödel logics

When considering propositional Gödel logics we fix a standard propositional language $\mathscr{L}^0$ with countably many propositional variables p_i, and the connectives $\wedge$, $\vee$, $\to$ and the constant $\bot$ (for 'false'); negation is introduced as an abbreviation: we let $\neg p \equiv (p \to \bot)$. For convenience, we also define $\top \equiv \bot \to \bot$. We will sometimes use the unary connective $\triangle$, introduced in [1]. Furthermore we will use $p \prec q$ as an abbreviation for $(q \to p) \to q$.

DEFINITION 1.3.1. *Let* $V \subseteq [0,1]$ *be some set of truth values which contains* 0 *and* 1. *A propositional Gödel valuation* $\mathcal{I}^0$ *(short valuation) based on* V *is a function from the set of propositional variables into* V *with* $\mathcal{I}^0(\bot) = 0$. *This valuation can be extended to a function mapping formulas from* $\mathrm{Frm}(\mathscr{L}^0)$ *into* V *as follows:*

$$\mathcal{I}^0(A \wedge B) = \min\{\mathcal{I}^0(A), \mathcal{I}^0(B)\},$$
$$\mathcal{I}^0(A \vee B) = \max\{\mathcal{I}^0(A), \mathcal{I}^0(B)\},$$
$$\mathcal{I}^0(\triangle A) = \begin{cases} 1 & \mathcal{I}^0(A) = 1, \\ 0 & \mathcal{I}^0(A) < 1, \end{cases}$$
$$\mathcal{I}^0(A \to B) = \begin{cases} \mathcal{I}^0(B) & \textit{if } \mathcal{I}^0(A) > \mathcal{I}^0(B), \\ 1 & \textit{if } \mathcal{I}^0(A) \le \mathcal{I}^0(B). \end{cases}$$

A formula is called valid with respect to V *if it is mapped to* 1 *for all valuations based on* V. *The set of all formulas which are valid with respect to* V *will be called the propositional Gödel logic based on* V *and will be denoted by* G^0_V.

The validity of a formula A *with respect to* V *will be denoted by*

$$\models^0_V A \quad \textit{or} \quad \models_{\mathrm{G}^0_V} A.$$

REMARK 1.3.2. *The extension of the valuation* $\mathcal{I}^0$ *to formulas provides the following truth functions:*

$$\mathcal{I}^0(\neg A) = \begin{cases} 0 & \textit{if } \mathcal{I}^0(A) > 0, \\ 1 & \textit{otherwise,} \end{cases}$$
$$\mathcal{I}^0(A \prec B) = \begin{cases} 1 & \textit{if } \mathcal{I}^0(A) < \mathcal{I}^0(B) \textit{ or } \mathcal{I}^0(A) = \mathcal{I}^0(B) = 1, \\ \mathcal{I}(B) & \textit{otherwise.} \end{cases}$$

Thus, the intuition behind $A \prec B$ *is that* A *is strictly less than* B, *or both are equal to* 1.

1.4 Syntax and semantics for first-order Gödel logics

When considering first-order Gödel logics we fix a standard first-order language $\mathscr{L}$ with finitely or countably many predicate symbols P and finitely or countably many function symbols f for every finite arity k. In addition to the connectives of propositional Gödel logics the two quantifiers $\forall$ and $\exists$ are used.

In the first-order case, where quantifiers will be interpreted as infima and suprema, we require the truth value set to be a closed subset of $[0,1]$ (and as before $0, 1 \in V$).

DEFINITION 1.4.1 (Gödel set). *A* Gödel set *is a closed set* $V \subseteq [0,1]$ *which contains* 0 *and* 1.

The semantics of Gödel logics, with respect to a fixed Gödel set as set of truth values and a fixed language $\mathscr{L}$ of predicate logic, is defined using the extended language $\mathscr{L}^U$, where U is the universe of the interpretation $\mathcal{I}$. $\mathscr{L}^U$ is $\mathscr{L}$ extended with constant symbols for each element of U.

DEFINITION 1.4.2 (Semantics of Gödel logic). *Let V be a Gödel set. An* interpretation $\mathcal{I}$ into V*, or a V-*interpretation*, consists of*

1. *a nonempty set $U = U^{\mathcal{I}}$, the 'universe' of $\mathcal{I}$,*
2. *for each k-ary predicate symbol P, a function $P^{\mathcal{I}}\colon U^k \to V$,*
3. *for each k-ary function symbol f, a function $f^{\mathcal{I}}\colon U^k \to U$,*
4. *for each variable v, a value $v^{\mathcal{I}} \in U$.*

Given an interpretation $\mathcal{I}$, we can naturally define a value $t^{\mathcal{I}}$ for any term t and a truth value $\mathcal{I}(A)$ for any formula A of $\mathscr{L}^U$. For a term $t = f(u_1, \ldots, u_k)$ we define $\mathcal{I}(t) = f^{\mathcal{I}}(u_1^{\mathcal{I}}, \ldots, u_k^{\mathcal{I}})$. For atomic formulas $A \equiv P(t_1, \ldots, t_n)$, we define $\mathcal{I}(A) = P^{\mathcal{I}}(t_1^{\mathcal{I}}, \ldots, t_n^{\mathcal{I}})$. For composite formulas A we extend the truth definitions from the propositional case for the new syntactic elements by:

$$\mathcal{I}(\forall x\, A(x)) = \inf\{\mathcal{I}(A(u)) \mid u \in U\}$$
$$\mathcal{I}(\exists x\, A(x)) = \sup\{\mathcal{I}(A(u)) \mid u \in U\}.$$

If $\mathcal{I}(A) = 1$, we say that $\mathcal{I}$ satisfies *A, and write $\mathcal{I} \models A$. If $\mathcal{I}(A) = 1$ for every V-interpretation $\mathcal{I}$, we say A is* valid *in $\mathbf{G}_V$ and write $\mathbf{G}_V \models A$.*

If Γ is a set of sentences, we define $\mathcal{I}(\Gamma) = \inf\{\mathcal{I}(A) \mid A \in \Gamma\}$.

Abusing notation slightly, we will often define interpretations simply by defining the truth values of atomic formulas in $\mathscr{L}^U$.

DEFINITION 1.4.3. *If Γ is a set of formulas (possibly infinite), we say that* Γ entails A in $\mathbf{G}_V$, $\Gamma \models_V A$ *iff for all $\mathcal{I}$ into V, $\mathcal{I}(\Gamma) \leq \mathcal{I}(A)$.*

Γ 1-entails A *in $\mathbf{G}_V$, $\Gamma \Vdash_V A$, iff, for all $\mathcal{I}$ into V, whenever $\mathcal{I}(B) = 1$ for all $B \in \Gamma$, then $\mathcal{I}(A) = 1$.*

We will write $\Gamma \models A$ instead of $\Gamma \models_V A$ in case it is obvious which truth value set V is meant.

DEFINITION 1.4.4. *For a Gödel set V we define the* first-order Gödel logic $\mathbf{G}_V$ *as the set of all pairs (Γ, A) such that $\Gamma \models_V A$.*

One might wonder whether a different definition of the entailment relation in Gödel logic might give different results. But as the following proposition shows, the above two definitions yield the same result, allowing us to use the characterization of $\models$ or $\Vdash$ as convenient.

PROPOSITION 1.4.5. $\Pi \models_V A$ *iff* $\Pi \Vdash_V A$.

Proof. See [16, Proposition 2.2]. □

Note that in the presence of $\triangle$, Proposition 1.4.5 does not hold and we will use the 1-entailment. Furthermore, it is important to mention that the (1-)satisfiability in the case without $\triangle$ does *not* define the entailment, which changes when adding $\triangle$.

1.5 Axioms and deduction systems for Gödel logics

In this section we introduce deduction systems for Gödel logics, and we show soundness and completeness.

Most of the time we concentrate on Hilbert-style deduction systems, for proof theory and Gentzen style systems see Chapter III. The only time a Gentzen style proof system will be used in this chapter is when proving the strong completeness. In this proof system the notion of sequent, written as

$$A_1, \dots, A_n \Rightarrow B$$

is introduced which we will consider as an abbreviation for

$$A_1 \to A_2 \to \dots \to A_n \to B,$$

and $A_1, \dots, A_n \Rightarrow$ as an abbreviation for $A_1, \dots, A_n \Rightarrow \bot$.

We will denote by **IL** the following complete axiom system for intuitionistic logic, where $B^{(x)}$ means that x is not free in B:

I1	$\bot \to A$	I8	$(A \to B) \to [(C \to A) \to (C \to B)]$
I2	$A \to (B \to A)$	I9	$[A \to (C \to B)] \to [C \to (A \to B)]$
I3	$(A \wedge B) \to A$	I10	$(A \to C) \wedge (B \to C) \to ((A \vee B) \to C)$
I4	$(A \wedge B) \to B$	I11	$(C \to A) \wedge (C \to B) \to (C \to (A \wedge B))$
I5	$A \to (B \to (A \wedge B))$	I12	$(A \to (B \to C)) \to (A \wedge B \to C)$
I6	$A \to (A \vee B)$	I13	$[A \to (A \to B)] \to (A \to B)$
I7	$B \to (A \vee B)$		
IQ1	$\dfrac{B^{(x)} \to A(x)}{B^{(x)} \to \forall x A(x)}$	IQ2	$\forall x A(x) \to A(t)$
IQ3	$A(t) \to \exists x A(x)$	IQ4	$\dfrac{A(x) \to B^{(x)}}{\exists x A(x) \to B^{(x)}}$
MP	$\dfrac{A \quad A \to B}{B}$		

The following formulas will play an important rôle when axiomatizing Gödel logics. Their names can be explained as follows: QS stands for 'quantifier shift', LIN for 'linearity', ISO_0 for 'isolation axiom of 0', ISO_1 for 'isolation axiom of 1', and $\text{FIN}(n)$ for 'finite with n elements'.

QS	$\forall x(C^{(x)} \vee A(x)) \to (C^{(x)} \vee \forall x A(x))$
LIN	$(A \to B) \vee (B \to A)$
ISO_0	$\forall x \neg\neg A(x) \to \neg\neg \forall x A(x)$
ISO_1	$\forall x \triangle A(x) \to \triangle \exists x A(x)$
$\text{FIN}(n)$	$(\top \to p_1) \vee (p_1 \to p_2) \vee \dots \vee (p_{n-2} \to p_{n-1}) \vee (p_{n-1} \to \bot)$

We will use additionally, referred to as $\mathrm{AX}\Delta$

$$
\begin{array}{ll}
\Delta 1 & \triangle A \vee \triangle A \\
\Delta 2 & \triangle(A \vee B) \to (\triangle A \vee \triangle B) \\
\Delta 3 & \triangle A \to A \\
\Delta 4 & \triangle A \to \triangle\triangle A \\
\Delta 5 & \triangle(A \to B) \to (\triangle A \to \triangle B) \\
\Delta 6 & \dfrac{A}{\triangle A}
\end{array}
$$

DEFINITION 1.5.1. *If $\mathcal{A}$ is an axiom system, we denote by $\mathcal{A}^0$ the propositional part of $\mathcal{A}$, i.e. all the axioms which do not contain quantifiers.*

With $\mathcal{A}\Delta$ we denote the axiom system obtained from $\mathcal{A}$ by adding the axioms $\mathrm{AX}\Delta$.

With $\mathcal{A}_n$ we denote the axiom system obtained from $\mathcal{A}$ by adding the axiom $\mathrm{FIN}(n)$.

We denote by $\mathbf{H}$ the axiom system $\mathbf{IL} + \mathrm{QS} + \mathrm{LIN}$.

EXAMPLE 1.5.2. $\mathbf{IL}^0$ is $\mathbf{IPL}$. $\mathbf{H}^0$ is Dummett's $\mathbf{LC}$.

For all these axiom systems the general notion of deducibility can be defined:

DEFINITION 1.5.3. *If a formula/sequent Γ can be deduced from an axiom system $\mathcal{A}$ we denote this by*

$$\vdash_{\mathcal{A}} \Gamma.$$

PROPOSITION 1.5.4 (Soundness). *Suppose Γ contains only closed formulas, and all axioms of $\mathcal{A}$ are valid in $\mathbf{G}_V$. Then, if $\Gamma \vdash_{\mathcal{A}} A$ then $\Gamma \models_V A$. In particular, $\mathbf{H}$ is sound for $\models_V$ for any Gödel set V; $\mathbf{H}_n$ is sound for $\models_V$ if $|V| = n$; $\mathbf{H} + \mathrm{ISO}_0$ is sound for $\models_V$ if 0 is isolated in V; and $\mathbf{H}\triangle + \mathrm{ISO}_1$ is sound for $\models_V$ with $\triangle$.*

1.6 Topology and order

In the following we will recall some definitions and facts from topology and order theory which will be used later on in many places.

1.6.1 Perfect sets

All the following notations, lemmas, theorems are carried out within the framework of Polish spaces, which are separable, completely metrizable topological spaces. For our discussion it is only necessary to know that $\mathbb{R}$ and all its closed subsets are Polish spaces (hence, every Gödel set is a Polish space). For a detailed exposition see [28, 30].

DEFINITION 1.6.1 (Limit point, perfect space, perfect set). *A* limit point *of a topological space is a point that is not isolated, i.e. for every open neighborhood U of x there is a point $y \in U$ with $y \neq x$. A space is* perfect *if all its points are limit points. A set $P \subseteq \mathbb{R}$ is* perfect *if it is closed and together with the topology induced from $\mathbb{R}$ is a perfect space.*

It is obvious that all (non-trivial) closed intervals are perfect sets, as well as all countable unions of (non-trivial) intervals. But all these sets generated from closed intervals have the property that they are 'everywhere dense,' i.e., contained in the closure of their inner component. There is a well-known example of a perfect set that is nowhere dense, the Cantor set:

EXAMPLE 1.6.2 (Cantor Set). The set of all numbers in the unit interval which can be expressed in triadic notation only by digits 0 and 2 is called the *Cantor set* $\mathbb{D}$.

A more intuitive way to obtain this set is to start with the unit interval, take out the open middle third and restart this process with the lower and the upper third. Repeating this you get exactly the Cantor set because the middle third always contains the numbers which contain the digit 1 in their triadic notation.

This set has a lot of interesting properties, the most important one for our purposes is that it is a perfect set:

PROPOSITION 1.6.3. *The Cantor set is perfect.*

It is possible to embed the Cauchy space into any perfect space, yielding the following proposition:

PROPOSITION 1.6.4 ([28, Corollary 6.3]). *If X is a nonempty perfect Polish space, then $|X| = 2^{\aleph_0}$. All nonempty perfect subsets of $[0, 1]$ have cardinality $2^{\aleph_0}$.*

It is possible to obtain the following characterization of perfect sets (see [40]):

PROPOSITION 1.6.5 (Characterization of perfect sets in $\mathbb{R}$). *For any perfect subset of $\mathbb{R}$ there is a unique partition of the real line into countably many intervals such that the intersections of the perfect set with these intervals are either empty, the full interval or isomorphic to the Cantor set.*

So we see that intervals and Cantor sets are prototypical for perfect sets and the basic building blocks of more complex perfect sets.

Every Polish space can be partitioned into a perfect kernel and a countable rest. This is the well-known Cantor–Bendixon Theorem:

THEOREM 1.6.6 (Cantor–Bendixon). *Let X be a Polish space. Then X can be uniquely written as $X = P \cup C$, with P a perfect subset of X and C countable and open. The subset P is called the* perfect kernel *of X (denoted by X^∞).*

As a corollary we obtain that any uncountable Polish space contains a perfect set, and therefore, has cardinality $2^{\aleph_0}$.

2 Propositional Gödel logics

As already mentioned Gödel introduced this family of logics on the propositional level to analyze intuitionistic logic. This allows the approach to Gödel logics via restricting the possible accessibility relations of Kripke models of intuitionistic logic. Two somehow reasonable restrictions of the Kripke structures are the restriction to constant domains and the restriction that the Kripke worlds are linearly ordered and of order type ω. One can now ask what sentences are valid in this restricted class of Kripke models. This question has been settled by Dummett [22] for the propositional case by adding to a complete axiomatization of intuitionistic logic the axiom of linearity

$$\text{LIN} \qquad (p \to q) \vee (q \to p)$$

It is interesting to note that p and q in the linearity scheme are propositional formulas. It is *not* enough to add this axiom for atomic p and q. For an axiom scheme only necessary for atomic formulas we have to use

$$((p \to q) \to p) \vee (p \to (p \to q))$$

to obtain completeness [15].

Another interesting distinction between $\mathbf{LC}$, which is $\mathrm{G}^0_\downarrow$, and other propositional Gödel logics, is the fact that while $\mathrm{G}^0_\downarrow$ and $\mathrm{G}^0_\mathbb{R}$ have the same set of tautologies, the entailment relation of the former is not compact, while the one of the latter is. The logic Dummett discussed, the logic of linearly ordered Kripke frames of order type ω, corresponds to $\mathrm{G}^0_\downarrow$. Therefore, Dummett proved only weak completeness (see Section 4.4).

One of the important properties of propositional Gödel logics is that the set of tautologies for any infinitely valued propositional Gödel logic coincides with the intersection of the sets of tautologies of all finitely valued propositional Gödel logics. Therefore, there is, with respect to the set of tautologies, only one infinite valued propositional Gödel logic, in contrast to entailment, quantified propositional logics, and first-order.

2.1 Completeness of H^0 for LC

[22] proved that a formula of propositional Gödel logic is valid in any infinite truth value set if it is valid in one infinite truth value set. Moreover, all the formulas valid in these sets are axiomatized by any axiomatization of intuitionistic propositional logic extended with the linearity axiom scheme $(p \to q) \vee (q \to p)$. The proof given here is a simplified proof of the completeness of $\mathbf{H}^0$ taken from [27].

DEFINITION 2.1.1. *An algebra* $\mathbf{P} = \langle P, \cdot, +, \to, \mathbf{0}, \mathbf{1} \rangle$ *is a* Heyting algebra *if the reduct* $\langle P, \cdot, +, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice with least element* $\mathbf{0}$*, largest element* $\mathbf{1}$ *and* $x \cdot y \leq z$ *iff* $x \leq (y \to z)$.

DEFINITION 2.1.2. *An* L-algebra *is a Heyting algebra in which*

$$(x \to y) + (y \to x) = \mathbf{1}$$

is valid for all x, y.

It is obvious that if we take L-algebras as our reference models for completeness, the proof of completeness is trivial. Generally, it is not very interesting to define algebras fitting to logics like a second skin, and then proving completeness with respect to this class (Ł-algebras, ...), without giving any connection to well-known algebraic structures or already accepted reference models. In our case we want to show completeness with respect to the real interval $[0, 1]$ or one of its sub-orderings. More generally we aim at completeness with respect to chains, which are special Heyting algebras:

DEFINITION 2.1.3. *A* chain *is a linearly ordered Heyting algebra.*

Chains are exactly what we are looking for as every chain (with cardinality less or equal to the continuum) is isomorphic to a sub-ordering of the $[0,1]$ interval, and vice versa. Our aim is now to show completeness of the above axiomatization with respect to chains. Furthermore we will exhibit that the length of the chains for a specific formula can be bounded by the number of propositional variables in the formula. More precisely:

THEOREM 2.1.4. *A formula α is provable in $\mathbf{H}^0 = \mathbf{LC}$ if and only if it is valid in all chains with at most $n+2$ elements, where n is the number of propositional variables in α.*

Proof. As usual we define the relation $\alpha \preceq \beta$ equivalent to $\vdash \alpha \to \beta$ and $\alpha \equiv \beta$ as $\alpha \preceq \beta$ and $\beta \preceq \alpha$. It is easy to verify that $\equiv$ is an equivalence relation. We denote $\alpha/\!\equiv$ with $|\alpha|$. It is also easy to show that with $|\alpha| + |\beta| = |\alpha \vee \beta|$, $|\alpha| \cdot |\beta| = |\alpha \wedge \beta|$, $|\alpha| \to |\beta| = |\alpha \to \beta|$ the set $\mathcal{F}/\!\equiv$ becomes a Heyting algebra, and due to the linearity axiom it is also an L-algebra. Furthermore note that $|\alpha| = \mathbf{1}$ if and only if α is provable in $\mathbf{H}^0$ ($\mathbf{1} = |p \to p|$, $|\alpha| = |p \to p|$ gives $\vdash (p \to p) \to \alpha$ which in turn gives $\vdash \alpha$).

If our aim would be completeness with respect to L-algebras the proof would be finished here, but we aim at completeness with respect to chains, therefore, we will take a close look at the structure of $\mathcal{F}/\!\equiv$ as L-algebra. Assume that a formula α is given, which is not provable, we want to give a chain where α is not valid. We already have an L-algebra where α is not valid, but how to obtain a chain?

We could use the general result from [27], Theorem 1.2, that a Heyting algebra is an L-algebra if and only if it is a subalgebra of a direct product of chains, but we will exhibit how to find explicitly a suitable chain. The idea is that the L-algebra $\mathcal{F}/\!\equiv$ describes all possible truth values for all possible orderings of the propositional variables in α. We want to make this more explicit:

DEFINITION 2.1.5. *We denote with*

$$\mathcal{C}(\bot, p_{i_1}, \ldots, p_{i_n}, \top)$$

the chain with these elements and the ordering

$$\bot \leq p_{i_1} < \ldots < p_{i_n} \leq \top.$$

If $\mathcal{C}$ is a chain we denote with $|\alpha|_{\mathcal{C}}$ the evaluation of the formula in the chain $\mathcal{C}$.

LEMMA 2.1.6. *The L-algebra $\mathcal{F}/\!\equiv$ is a subalgebra of the following direct product of chains*

$$X = \prod_{i=1}^{n!} \mathcal{C}(\bot, \pi_i(p_1, \ldots, p_n), \top)$$

where π_i ranges over the set of permutations of n elements. We will use $\mathcal{C}_i$ to denote $\mathcal{C}(\bot, \pi_i(p_1, \ldots, p_n), \top)$.

Proof. Define $\phi \colon \mathcal{F}/\!\equiv \to X$ as follows:

$$\phi(|\alpha|) = (|\alpha|_{\mathcal{C}_1}, \ldots, |\alpha|_{\mathcal{C}_{n!}}).$$

We have to show that ϕ is well defined, is a homomorphism and is injective. First assume that $\beta \in |\alpha|$ but $\phi(|\alpha|) \neq \phi(|\beta|)$, i.e.

$$(|\alpha|_{\mathcal{C}_1}, \ldots, |\alpha|_{\mathcal{C}_{n!}}) \neq (|\beta|_{\mathcal{C}_1}, \ldots, |\beta|_{\mathcal{C}_{n!}})$$

but then there must be an i such that

$$|\alpha|_{\mathcal{C}_i} \neq |\beta|_{\mathcal{C}_i}.$$

Without loss of generality, assume that $|\alpha|_{\mathcal{C}_i} < |\beta|_{\mathcal{C}_i}$. From the fact that $|\alpha| = |\beta|$ we get $\vdash \beta \to \alpha$. From this we get that $|\beta \to \alpha|_{\mathcal{C}_i} < \mathbf{1}$ and from $\vdash \beta \to \alpha$ we get that $|\beta \to \alpha|_{\mathcal{C}_i} = \mathbf{1}$, which is a contradiction. This proves the well-definedness.

To show that ϕ is a homomorphism we have to prove that

$$\begin{aligned}
\phi(|\alpha| \cdot |\beta|) &= \phi(|\alpha|) \cdot \phi(|\beta|)\\
\phi(|\alpha| + |\beta|) &= \phi(|\alpha|) + \phi(|\beta|)\\
\phi(|\alpha| \to |\beta|) &= \phi(|\alpha|) \to \phi(|\beta|).
\end{aligned}$$

This is a straightforward computation using $|\alpha \wedge \beta|_{\mathcal{C}} = \phi(|\alpha|_{\mathcal{C}}) \cdot \phi(|\beta|_{\mathcal{C}})$.

Finally we have to prove that ϕ is injective. Assume that $\phi(|\alpha|) = \phi(|\beta|)$ and that $|\alpha| \neq |\beta|$. From the former we obtain that $|\alpha|_{\mathcal{C}_i} = |\beta|_{\mathcal{C}_i}$ for all $1 \leq i \leq n!$, which means that

$$\mathcal{I}_{\mathcal{C}_i}(\alpha) = \mathcal{I}_{\mathcal{C}_i}(\beta) \quad \text{for all } 1 \leq i \leq n!.$$

On the other hand we know from the latter that there is an interpretation $\mathcal{I}$ such that $\mathcal{I}(\alpha) \neq \mathcal{I}(\beta)$. Without loss of generality assume that

$$\bot \leq \mathcal{I}(p_{i_1}) < \cdots < \mathcal{I}(p_{i_n}) \leq \top.$$

There is an index k such that the $\mathcal{C}_k$ is exactly the above ordering with

$$\mathcal{I}_{\mathcal{C}_k}(\alpha) \neq \mathcal{I}_{\mathcal{C}_k}(\beta),$$

this is a contradiction.

This completes the proof that $\mathcal{F}/\equiv$ is a subalgebra of the given direct product of chains. □

EXAMPLE 2.1.7. For $n = 2$ the chains are $\mathcal{C}(\bot, p, q, \top)$ and $\mathcal{C}(\bot, q, p, \top)$. The product of these two chains looks as given in Figure 1, p. 596. The labels below the nodes are the products, the formulas above the nodes are representatives for the class $\alpha/\equiv$.

Now the proof of Theorem 2.1.4 is trivial since, if $|\alpha| \neq \mathbf{1}$, there is a chain $\mathcal{C}_i$ where $|\alpha|_{\mathcal{C}_i} \neq \mathbf{1}$. □

This yields the following theorem:

THEOREM 2.1.8. *A propositional formula is valid in any infinite chain iff it is derivable in* $\mathbf{LC} = \mathbf{H}^0$.

Going on to finite truth value sets we can give the following theorem:

THEOREM 2.1.9. *A formula is valid in any chain with at most n elements iff it is provable in* $\mathbf{LC}_n$.

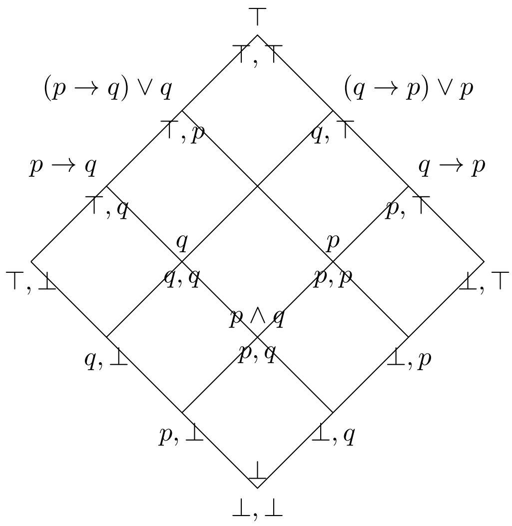

Figure 1. L-algebra of $\mathcal{C}(\bot, p, q, \top) \times \mathcal{C}(\bot, q, p, \top)$. Labels below the nodes are the elements of the direct product, formulas above the node are representatives for the class $\alpha/{\equiv}$.

Proof. Assuming that $\mathbf{H}_n^0 \nvdash \alpha$ and using the deduction theorem we can proceed as follows:

$$\mathbf{H}_n^0 \nvdash \alpha$$
$$\mathbf{H}^0 + \text{FIN}(n) \nvdash \alpha$$
$$\mathbf{H}^0 \nvdash \text{FIN}(n) \to \alpha$$

From this we know that there is an interpretation $\mathcal{I}$ such that

$$\mathcal{I}(\text{FIN}(n) \to \alpha) < 1$$

which is equivalent to

$$\mathcal{I}(\text{FIN}(n)) = 1 \text{ and } \mathcal{I}(\alpha) < 1.$$

The first formula ensures that the domain has at most n elements. Therefore, $\mathcal{I}$ is an interpretation with a domain with at most n elements and which evaluates α to a value less than 1. □

As a simple consequence of these results the following corollaries settle the number of propositional Gödel logics and their relation:

COROLLARY 2.1.10. *The propositional Gödel logics* G_n^0 *and* $\mathrm{G}_{\mathbb{R}}^0$ *are all different, thus there are countable many different propositional Gödel logics, and*

$$\bigcap_{n \in \mathbb{N}} \mathrm{G}_n^0 = \mathrm{G}_{\mathbb{R}}^0.$$

2.2 The Delta operator

For Gödel logics there is an asymmetry between 0 and 1 because 0 can be distinguished from other values (by using the negation), while 1 cannot be distinguished. The reason is that all connectives and quantifiers are continuous at 1. To overcome this asymmetry the operator $\triangle$ has been introduced in [1] with the following truth function:

$$\phi(\triangle A) = \begin{cases} 1 & \text{if } \phi(A) = 1, \\ 0 & \text{o.w.} \end{cases}$$

THEOREM 2.2.1. *There is only one infinitely valued Gödel logic with $\triangle$, and it is axiomatized by $\mathbf{H}^0\triangle$ (see page 591), and this logic is the intersection of the finitely valued logics.*

It is important to note that adding $\triangle$ to the language is an actual extension, i.e., the $\triangle$ operator cannot be defined.

3 First-order Gödel logics

After some preliminaries we discuss the relationships between different Gödel logics in Section 3.2, characterize the axiomatizable first-order Gödel logics in Sections 3.3.1, 3.3.2, and 3.3.3, followed by the characterization of those logics that are not recursively enumerable in Sections 3.3.4 and 3.3.5.

Following this complete characterization of axiomatizability we explicate the relation between (linear) Kripke frames based logics and Gödel logics in Section 3.4, and briefly discuss the very surprising result on the number of different first-order Gödel logics in Section 3.5.

All the results in the following sections are from [3, 14, 16–19].

3.1 Preliminaries

We will be concerned below with the relationships between Gödel logics, here considered as entailment relations. Note that $\mathbf{G}_V \models A$ iff $(\emptyset, A) \in \mathbf{G}_V$, so in particular, showing that $\mathbf{G}_V \subseteq \mathbf{G}_W$ also shows that every valid formula of $\mathbf{G}_V$ is also valid in $\mathbf{G}_W$. On the other hand, to show that $\mathbf{G}_V \nsubseteq \mathbf{G}_W$ it suffices to show that for some A, $\mathbf{G}_V \models A$ but $\mathbf{G}_W \not\models A$.

REMARK 3.1.1. *The case that a formula A evaluates to 1 under a certain interpretation $\mathcal{I}$ depends only on the* relative ordering *of the truth values of the atomic formulas (in $\mathscr{L}^{\mathcal{I}}$), and not directly on the set V or on the specific values of the atomic formulas. If $V \subseteq W$ are both Gödel sets, and $\mathcal{I}$ is a V-interpretation, then $\mathcal{I}$ can be seen also as a W-interpretation, and the values generated during the computation of $\mathcal{I}(A)$ do not depend on whether we view $\mathcal{I}$ as a V-interpretation or a W-interpretation. Consequently, if $V \subseteq W$, there are more interpretations into W than into V. Hence, if $\Gamma \models_W A$, then also $\Gamma \models_V A$ and $\mathbf{G}_W \subseteq \mathbf{G}_V$.*

This can be generalized to embeddings between Gödel sets other than inclusion. First, we make precise which formulas are involved in the computation of the truth-value of a formula A in an interpretation $\mathcal{I}$:

DEFINITION 3.1.2. *The only subformula of an atomic formula A in $\mathscr{L}^U$ is A itself. The subformulas of $A \star B$ for $\star \in \{\to, \land, \lor\}$ are the subformulas of A and of B, together with $A \star B$ itself. The subformulas of $\forall x\, A(x)$ and $\exists x\, A(x)$ with respect to a universe U are all subformulas of all $A(u)$ for $u \in U$, together with $\forall x\, A(x)$ (or, $\exists x\, A(x)$, respectively) itself.*

The set of truth-values of subformulas of A under a given interpretation $\mathcal{I}$ is denoted by

$$\mathrm{Val}(\mathcal{I}, A) = \{\mathcal{I}(B) \mid B \textit{ subformula of } A \textit{ w.r.t. } U^{\mathcal{I}}\} \cup \{0, 1\}$$

if Γ is a set of formulas, then $\mathrm{Val}(\mathcal{I}, \Gamma) = \bigcup\{\mathrm{Val}(\mathcal{I}, A) \mid A \in \Gamma\}$.

LEMMA 3.1.3. *Let $\mathcal{I}$ be a V-interpretation, and let $h\colon \mathrm{Val}(\mathcal{I}, \Gamma) \to W$ be a mapping satisfying the following properties:*

1. *$h(0) = 0$, $h(1) = 1$;*
2. *h is strictly monotonic, i.e., if $a < b$, then $h(a) < h(b)$;*
3. *for every $X \subseteq \mathrm{Val}(\mathcal{I}, \Gamma)$, $h(\inf X) = \inf h(X)$ and $h(\sup X) = \sup h(X)$ (provided $\inf X, \sup X \in \mathrm{Val}(\mathcal{I}, \Gamma)$).*

Then the W-interpretation $\mathcal{I}_h$ with universe $U^{\mathcal{I}}$, $f^{\mathcal{I}_h} = f^{\mathcal{I}}$, and for atomic $B \in \mathscr{L}^{\mathcal{I}}$,

$$\mathcal{I}_h(B) = \begin{cases} h(\mathcal{I}(B)) & \textit{if } \mathcal{I}(B) \in \mathrm{dom}\, h, \\ 1 & \textit{otherwise} \end{cases}$$

satisfies $\mathcal{I}_h(A) = h(\mathcal{I}(A))$ for all $A \in \Gamma$.

Proof. By induction on the complexity of A. If $A \equiv \bot$, the claim follows from (1). If A is atomic, it follows from the definition of $\mathcal{I}_h$. For the propositional connectives the claim follows from the strict monotonicity of h (2). For the quantifiers, it follows from property (3). □

PROPOSITION 3.1.4 (Downward Löwenheim–Skolem). *For any interpretation $\mathcal{I}$ with $U^{\mathcal{I}}$ infinite, there is an interpretation $\mathcal{I}' \prec \mathcal{I}$ with a countable universe $U^{\mathcal{I}'}$.*

LEMMA 3.1.5. *Let $\mathcal{I}$ be an interpretation into V, $w \in [0, 1]$, and let $\mathcal{I}_w$ be defined by*

$$\mathcal{I}_w(B) = \begin{cases} \mathcal{I}(B) & \textit{if } \mathcal{I}(B) < w, \\ 1 & \textit{otherwise} \end{cases}$$

for atomic formulas B in $\mathscr{L}^{\mathcal{I}}$. Then $\mathcal{I}_w$ is an interpretation into V. If $w \notin \mathrm{Val}(\mathcal{I}, A)$, then $\mathcal{I}_w(A) = \mathcal{I}(A)$ if $\mathcal{I}(A) < w$, and $\mathcal{I}_w(A) = 1$ otherwise.

Proof. By induction on the complexity of formulas A in $\mathscr{L}^{\mathcal{I}}$. The condition that $w \notin \mathrm{Val}(\mathcal{I}, A)$ is needed to prove the case of $A \equiv \exists x\, B(x)$, since if $\mathcal{I}(\exists x\, B(x)) = w$ and $\mathcal{I}(B(d)) < w$ for all d, we would have $\mathcal{I}_w(\exists x\, B(x)) = w$ and not $= 1$. □

Using this lemma we can obtain the following observation.

LEMMA 3.1.6. *If for all interpretations $\mathcal{I}(A) = 1$ iff $\mathcal{I}(B) = 1$, then already $A \leftrightarrow B$ is valid.*[2]

Proof. If $A \leftrightarrow B$ is not valid, then there is a real number w strictly between the valuations of A and B, such that either w is not a truth value, or w does not occur in the set of valuations of all sub-formulas of A and B. Assuming w.l.o.g. that $\mathcal{I}(A) > \mathcal{I}(B)$ we obtain that $I_w(A) = 1$ while $I_w(B) = I(B) < 1$. □

The following lemma was originally proved in [33], where it was used to extend the proof of recursive axiomatizability of the 'standard' Gödel logic $\mathbf{G}_{\mathbb{R}}$ to Gödel logics with a truth value set containing a perfect set in the general case. The following simpler proof is inspired by [18]:

LEMMA 3.1.7. *Suppose that $M \subseteq [0,1]$ is countable and $P \subseteq [0,1]$ is perfect. Then there is a strictly monotone continuous map $h\colon M \to P$ (i.e., infima and suprema already existing in M are preserved). Furthermore, if $\inf M \in M$, then one can choose h such that $h(\inf M) = \inf P$.*

Proof. Let σ be the mapping which scales and shifts M into $[0,1]$, i.e. the mapping $x \mapsto (x - \inf M)/(\sup M - \inf M)$ (assuming that M contains more than one point). Let w be an injective monotone map from $\sigma(M)$ into 2^ω, i.e. $w(m)$ is a fixed binary representation of m. For dyadic rational numbers (i.e. those with different binary representations) we fix one possible.

Let i be the natural bijection from 2^ω (the set of infinite $\{0,1\}$-sequences, ordered lexicographically) onto $\mathbb{D}$, the Cantor set. i is an order preserving homeomorphism. Since P is perfect, we can find a continuous strictly monotone map c from the Cantor set $\mathbb{D} \subseteq [0,1]$ into P, and c can be chosen so that $c(0) = \inf P$. Now $h = c \circ i \circ w \circ \sigma$ is also a strictly monotone map from M into P, and $h(\inf M) = \inf P$, if $\inf M \in M$. Since c is continuous, existing infima and suprema are preserved. □

COROLLARY 3.1.8. *A Gödel set V is uncountable iff it contains a non-trivial dense linear subordering.*

Proof. If: Every countable non-trivial dense linear order has order type η, $\mathbf{1} + \eta$, $\eta + \mathbf{1}$, or $\mathbf{1} + \eta + \mathbf{1}$ [35, Corollary 2.9], where η is the order type of $\mathbb{Q}$. The completion of any ordering of order type η has order type λ, the order type of $\mathbb{R}$ [35, Theorem 2.30], thus the truth value set must be uncountable. Only if: By Theorem 1.6.6, V^∞ is non-empty. Take $M = \mathbb{Q} \cap [0,1]$ and $P = V^\infty$ in Lemma 3.1.7. The image of M under h is a non-trivial dense linear subordering in V. □

THEOREM 3.1.9. *Suppose V is a truth value set with non-empty perfect kernel P, and let $W = V \cup [\inf P, 1]$. Then $\Gamma \models_V A$ iff $\Gamma \models_W A$, i.e., $\mathbf{G}_V = \mathbf{G}_W$.*

Proof. As $V \subseteq W$ we have $\mathbf{G}_W \subseteq \mathbf{G}_V$ (cf. Remark 3.1.1). Now assume that $\mathcal{I}$ is a W-interpretation which shows that $\Gamma \models_W A$ does *not* hold, i.e., $\mathcal{I}(\Gamma) > \mathcal{I}(A)$. By Proposition 3.1.4, we may assume that $U^{\mathcal{I}}$ is countable. The set $\mathrm{Val}(\mathcal{I}, \Gamma \cup A)$ has

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cardinality at most $\aleph_0$, thus there is a $w \in [0,1]$ such that $w \notin \mathrm{Val}(\mathcal{I}, \Gamma \cup A)$ and $\mathcal{I}(A) < w < 1$. By Lemma 3.1.5, $\mathcal{I}_w(A) < w < 1$. Now consider $M = \mathrm{Val}(\mathcal{I}_w, \Gamma \cup A)$: these are all the truth values from $W = V \cup [\inf P, 1]$ required to compute $\mathcal{I}_w(A)$ and $\mathcal{I}_w(B)$ for all $B \in \Gamma$. We have to find some way to map them to V so that the induced interpretation is a counterexample to $\Gamma \models_V A$.

Let $M_0 = M \cap [0, \inf P)$ and $M_1 = (M \cap [\inf P, w]) \cup \{\inf P\}$. By Lemma 3.1.7 there is a strictly monotone continuous (i.e. preserving all existing infima and suprema) map h from M_1 into P. Furthermore, we can choose h such that $h(\inf M_1) = \inf P$.

We define a function g from $\mathrm{Val}(\mathcal{I}_w, \Gamma \cup A)$ to V as follows:

$$g(x) = \begin{cases} x & 0 \leq x \leq \inf P, \\ h(x) & \inf P \leq x \leq w, \\ 1 & x = 1. \end{cases}$$

Note that there is no $x \in \mathrm{Val}(\mathcal{I}_w, \Gamma \cup A)$ with $w < x < 1$. This function has the following properties: $g(0) = 0$, $g(1) = 1$, g is strictly monotonic and preserves existing infima and suprema. Using Lemma 3.1.3 we obtain that $\mathcal{I}_g$ is a V-interpretation with $\mathcal{I}_g(C) = g(\mathcal{I}_w(C))$ for all $C \in \Gamma \cup A$, thus also $\mathcal{I}_g(\Gamma) > \mathcal{I}_g(A)$. □

3.2 Relationships between Gödel logics

We now establish some results regarding the relationships between various first-order Gödel logics. For this, it is useful to consider several 'prototypical' Gödel sets.

$$\begin{aligned} V_{\mathbb{R}} &= [0,1] \qquad V_0 = \{0\} \cup [1/2, 1] \\ V_{\downarrow} &= \{1/k \mid k \geq 1\} \cup \{0\} \\ V_{\uparrow} &= \{1 - 1/k \mid k \geq 1\} \cup \{1\} \\ V_n &= \{1 - 1/k \mid 1 \leq k \leq m-1\} \cup \{1\} \end{aligned}$$

The corresponding Gödel logics are $\mathbf{G}_{\mathbb{R}}$, $\mathbf{G}_0$, $\mathbf{G}_{\downarrow}$, $\mathbf{G}_{\uparrow}$, and $\mathbf{G}_n$. $\mathbf{G}_{\mathbb{R}}$ is the *standard* Gödel logic.

The logic $\mathbf{G}_{\downarrow}$ also turns out to be closely related to some temporal logics [9, 10]. $\mathbf{G}_{\uparrow}$ is the intersection of all finite-valued first-order Gödel logics as shown in Theorem 3.2.4.

PROPOSITION 3.2.1. *Intuitionistic predicate logic* **IL** *is contained in all first-order Gödel logics.*

Proof. The axioms and rules of **IL** are sound for the Gödel truth functions. □

As a consequence of this proposition, we will be able to use any intuitionistically sound rule and intuitionistically valid formula when working in any of the Gödel logics.

PROPOSITION 3.2.2. $\mathbf{G}_{\mathbb{R}} = \bigcap_V \mathbf{G}_V$*, where V ranges over all Gödel sets.*

Proof. If $\Gamma \models_V A$ for every Gödel set V, then it does so in particular for $V = [0,1]$. Conversely, if $\Gamma \not\models_V A$ for a Gödel set V, there is a V-interpretation $\mathcal{I}$ with $\mathcal{I}(\Gamma) > \mathcal{I}(A)$. Since $\mathcal{I}$ is also a $[0,1]$-interpretation, $\Gamma \not\models_{\mathbb{R}} A$. □

PROPOSITION 3.2.3. *The following strict containment relationships hold:*

1. $\mathbf{G}_n \supsetneq \mathbf{G}_{n+1}$,
2. $\mathbf{G}_n \supsetneq \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\mathbb{R}$,
3. $\mathbf{G}_n \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R}$,
4. $\mathbf{G}_0 \supsetneq \mathbf{G}_\mathbb{R}$.

Proof. The only non-trivial part is proving that the containments are strict. For this note that

$$\textsc{fin}(n) \equiv (\top \to A_1) \vee \ldots \vee (A_{n-1} \to \bot)$$

is valid in $\mathbf{G}_n$ but not in $\mathbf{G}_{n+1}$. Furthermore, let

$$C_\uparrow = \exists x(A(x) \to \forall y\, A(y)) \text{ and}$$
$$C_\downarrow = \exists x(\exists y\, A(y) \to A(x)).$$

$C_\downarrow$ is valid in all $\mathbf{G}_n$ and in $\mathbf{G}_\uparrow$ and $\mathbf{G}_\downarrow$; $C_\uparrow$ is valid in all $\mathbf{G}_n$ and in $\mathbf{G}_\uparrow$, but not in $\mathbf{G}_\downarrow$; neither is valid in $\mathbf{G}_0$ or $\mathbf{G}_\mathbb{R}$ [10, Corollary 2.9].

$\mathbf{G}_0 \models \textsc{iso}_0$ but $\mathbf{G}_\mathbb{R} \not\models \textsc{iso}_0$. □

The formulas $C_\uparrow$ and $C_\downarrow$ are of some importance in the study of first-order infinite-valued Gödel logics. $C_\uparrow$ expresses the fact that the infimum of any subset of the set of truth values is contained in the subset (every infimum is a minimum), and $C_\downarrow$ states that every supremum (except possibly 1) is a maximum. The intuitionistically admissible quantifier shifting rules are given by the following implications and equivalences:

$$(\forall x\, A(x) \wedge B) \leftrightarrow \forall x(A(x) \wedge B) \tag{1}$$
$$(\exists x\, A(x) \wedge B) \leftrightarrow \exists x(A(x) \wedge B) \tag{2}$$
$$(\forall x\, A(x) \vee B) \to \forall x(A(x) \vee B) \tag{3}$$
$$(\exists x\, A(x) \vee B) \leftrightarrow \exists x(A(x) \vee B) \tag{4}$$
$$(B \to \forall x\, A(x)) \leftrightarrow \forall x(B \to A(x)) \tag{5}$$
$$(B \to \exists x\, A(x)) \leftarrow \exists x(B \to A(x)) \tag{6}$$
$$(\forall x\, A(x) \to B) \leftarrow \exists x(A(x) \to B) \tag{7}$$
$$(\exists x\, A(x) \to B) \leftrightarrow \forall x(A(x) \to B) \tag{8}$$

The remaining three are:

$$(\forall x\, A(x) \vee B) \leftarrow \forall x(A(x) \vee B) \tag{S_1}$$
$$(B \to \exists x\, A(x)) \to \exists x(B \to A(x)) \tag{S_2}$$
$$(\forall x\, A(x) \to B) \to \exists x(A(x) \to B) \tag{S_3}$$

Of these, S_1 is valid in any Gödel logic. S_2 and S_3 imply and are implied by $C_\downarrow$ and $C_\uparrow$, respectively (take $\exists y\, A(y)$ and $\forall y\, A(y)$, respectively, for B). S_2 and S_3 are, respectively, both valid in $\mathbf{G}_\uparrow$, invalid and valid in $\mathbf{G}_\downarrow$, and both invalid in $\mathbf{G}_\mathbb{R}$.

Note that since we defined $\neg A \equiv A \to \bot$, the quantifier shifts for $\to$ (7, 8, S_3) include the various directions of De Morgan's laws as special cases. Specifically, the only direction of De Morgan's laws which is not valid in all Gödel logics is the one corresponding to (S_3), i.e., $\neg\forall x\, A(x) \to \exists x \neg A(x)$. This formula is equivalent to ISO_0. For, $\mathbf{G}_V \models \forall x \neg\neg A(x) \leftrightarrow \neg\exists\neg A(x)$ by (8). We get ISO_0 using $\neg\exists x \neg A(x) \to \neg\neg\forall x\, A(x)$, which is an instance of (S_3). The other direction is given in Lemma 3.3.6.

We now also know that $\mathbf{G}_\uparrow \neq \mathbf{G}_\downarrow$. In fact, we have $\mathbf{G}_\downarrow \subsetneq \mathbf{G}_\uparrow$; this follows from the following theorem.

THEOREM 3.2.4 ([14, Theorem 23]).

$$\mathbf{G}_\uparrow = \bigcap_{n \geq 2} \mathbf{G}_n.$$

Proof. By Proposition 3.2.3, $\mathbf{G}_\uparrow \subseteq \bigcap_{n\geq 2} \mathbf{G}_n$. We now prove the reverse inclusion. Suppose $\Gamma \nvDash_{V_\uparrow} A$, i.e., there is a $V_\uparrow$-interpretation $\mathcal{I}$ such that $\mathcal{I}(\Gamma) > \mathcal{I}(A)$. Let $\mathcal{I}(A) = 1 - 1/k$, and pick w somewhere between $1 - 1/k$ and $1 - 1/(k+1)$. Then the interpretation $\mathcal{I}_w$ given by Lemma 3.1.5 is so that $\mathcal{I}(\Gamma) = 1$ and $\mathcal{I}(A) = 1 - 1/k$. Since there are only finitely many truth values below w in $V_\uparrow$, $\mathcal{I}_w$ is also a $\mathbf{G}_{k+1}$ interpretation which shows that $\Gamma \nvDash_{V_{k+1}} A$. Hence, $(\Gamma, A) \notin \bigcap_{n\geq 2} \mathbf{G}_n$. $\square$

COROLLARY 3.2.5. $\mathbf{G}_n \supsetneq \bigcap_n \mathbf{G}_n = \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R} = \bigcap_V \mathbf{G}_V$.

Note that also $\mathbf{G}_\uparrow \supsetneq \mathbf{G}_0 \supsetneq \mathbf{G}_\mathbb{R}$ by the above, and that neither $\mathbf{G}_0 \subseteq \mathbf{G}_\downarrow$ nor $\mathbf{G}_\downarrow \subseteq \mathbf{G}_0$ (counterexamples are ISO_0 or $\neg\forall x\, A(x) \to \exists\neg A(x)$, and $C_\downarrow$, respectively).

LEMMA 3.2.6. *If all infima in the truth value set are minima or A contains no quantifiers, and A evaluates to some $v < 1$ in $\mathcal{I}$, then A also evaluates to v in $\mathcal{I}_v$ where*

$$\mathcal{I}_v(P) = \begin{cases} 1 & \text{if } \mathcal{I}(P) > v, \\ \mathcal{I}(P) & \text{otherwise,} \end{cases}$$

for P atomic sub-formula of A.

Proof. We prove by induction on the complexity of formulas that any sub-formula F of A with $\mathcal{I}(F) \leq v$ has $\mathcal{I}'(F) = \mathcal{I}(F)$. This is clear for atomic sub-formulas. We distinguish cases according to the logical form of F:

$F \equiv D \wedge E$. If $\mathcal{I}(F) \leq v$, then, without loss of generality, assume $\mathcal{I}(F) = \mathcal{I}(D) \leq \mathcal{I}(E)$. By induction hypothesis, $\mathcal{I}'(D) = \mathcal{I}(D)$ and $\mathcal{I}'(E) \geq \mathcal{I}(E)$, so $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then $\mathcal{I}(D) > v$ and $\mathcal{I}(E) > v$, by induction hypothesis $\mathcal{I}'(D) = \mathcal{I}'(E) = 1$, thus, $\mathcal{I}'(F) = 1$.

$F \equiv D \vee E$. If $\mathcal{I}(F) \leq v$, then, without loss of generality, assume $\mathcal{I}(F) = \mathcal{I}(D) \geq \mathcal{I}(E)$. By induction hypothesis, $\mathcal{I}'(D) = \mathcal{I}(D)$ and $\mathcal{I}'(E) = \mathcal{I}(E)$, so $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then, again without loss of generality, $\mathcal{I}(F) = \mathcal{I}(D) > v$, by induction hypothesis $\mathcal{I}'(D) = 1$, thus, $\mathcal{I}'(F) = 1$.

$F \equiv D \to E$. Since $v < 1$, we must have $\mathcal{I}(D) > \mathcal{I}(E) = \mathcal{I}(F)$. By induction hypothesis, $\mathcal{I}'(D) \geq \mathcal{I}(D)$ and $\mathcal{I}'(E) = \mathcal{I}(E)$, so $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then $\mathcal{I}(D) \geq \mathcal{I}(E) = \mathcal{I}(F) > v$, by induction hypothesis $\mathcal{I}'(D) = \mathcal{I}'(E) = \mathcal{I}'(F) = 1$.

$F \equiv \exists x D(x)$. First assume that $\mathcal{I}(F) \le v$. Since $D(c)$ evaluates to a value less or equal to v in $\mathcal{I}$ and, by induction hypothesis, in $\mathcal{I}'$ also the supremum of these values is less or equal to v in $\mathcal{I}'$, thus $\mathcal{I}'(F) = \mathcal{I}(F)$. If $\mathcal{I}(F) > v$, then there is a c such that $\mathcal{I}(D(c)) > v$, by induction hypothesis $\mathcal{I}'(D(c)) = 1$, thus, $\mathcal{I}'(F) = 1$.

$F \equiv \forall x D(x)$. This is the crucial part. First assume that $\mathcal{I}(F) < v$. Then there is a witness c such that $\mathcal{I}(F) \le \mathcal{I}(D(c)) < v$ and, by induction hypothesis, also $\mathcal{I}'(D(c)) < v$ and therefore, $\mathcal{I}'(F) = \mathcal{I}(F)$. For $\mathcal{I}(F) > v$ it is obvious that $\mathcal{I}'(F) = \mathcal{I}(F) = 1$. Finally assume that $\mathcal{I}(F) = v$. If this infimum would be proper, i.e. no minimum, then the value of all witnesses under $\mathcal{I}'$ would be 1, but the value of F under $\mathcal{I}'$ would be v, which would contradict the definition of the semantic of the $\forall$ quantifier. Since all infima are minima, there is a witness c such that $\mathcal{I}(D(c)) = v$ and therefore, also $\mathcal{I}'(D(c)) = v$ and thus $\mathcal{I}'(F) = \mathcal{I}(F)$. □

As we will see later, the axioms FIN(n) axiomatize exactly the finite-valued Gödel logics. In these logics the quantifier shift axiom QS is not necessary. Furthermore, all quantifier shift rules are valid in the finite valued logics. Since $\mathbf{G}_\uparrow$ is the intersection of all the finite ones, all quantifier shift rules are valid in $\mathbf{G}_\uparrow$. Moreover, any infinite-valued Gödel logic other than $\mathbf{G}_\uparrow$ is defined by some V which either contains an infimum which is not a minimum, or a supremum (other than 1) which is not a maximum. Hence, in V either $C_\uparrow$ or $C_\downarrow$ will be invalid, and therewith either S_3 or S_2. We have:

COROLLARY 3.2.7. *In $\mathbf{G}_V$ all quantifier shift rules are valid iff there is a strictly monotone and continuous embedding from V to $V_\uparrow$, i.e., V is either finite or order isomorphic to $V_\uparrow$.*

This means that it is in general not possible to transform formulas to equivalent prenex formulas in the usual way. Moreover, in general there is not even a recursive procedure for mapping formulas to equivalent, or even just validity-equivalent formulas in prenex form, since for some V, $\mathbf{G}_V$ is not r.e. whereas the corresponding prenex fragment is r.e. $V = \{0\} \cup \{1/n \mid n \in \mathbb{N}\} \cup [0.5, 1]$ is such an example.

3.3 Axiomatizability results

3.3.1 Axiomatizable case 1: 0 is contained in the perfect kernel

If V is uncountable, and 0 is contained in V^∞, then $\mathbf{G}_V$ is axiomatizable. Indeed, Theorem 3.1.9 showed that all such logics $\mathbf{G}_V$ coincide. Thus, it is only necessary to establish completeness of the axioms system $\mathbf{H}$ with respect to $\mathbf{G}_\mathbb{R}$. This result has been shown by several researchers over the years. We give here a generalization of the proof of [36]. Alternative proofs can be found in [26, 27, 38]. The proof of [27], however, does not give strong completeness, while the proof of [38] is specific to the Gödel set $[0, 1]$. Our proof is self-contained and applies to Gödel logics directly, making an extension of the result easier.

THEOREM 3.3.1 ([36], [14, Theorem 37], Strong completeness of Gödel logic).
If $\Gamma \models_\mathbb{R} A$, then $\Gamma \vdash_\mathbf{H} A$.

Proof. Assume that $\Gamma \nvdash A$, we construct an interpretation $\mathcal{I}$ in which $\mathcal{I}(A) = 1$ for all $B \in \Gamma$ and $\mathcal{I}(A) < 1$. Let $y_1, y_2, \ldots$ be a sequence of free variables which do not occur in $\Gamma \cup \triangle$, let $\mathcal{T}$ be the set of all terms in the language of $\Gamma \cup \Delta$ together with the new variables $y_1, y_2, \ldots$, and let $\mathcal{F} = \{F_1, F_2, \ldots\}$ be an enumeration of the formulas in this language in which y_i does not appear in $F_1, \ldots, F_i$ and in which each formula appears infinitely often.

If Δ is a set of formulas, we write $\Gamma \Rightarrow \Delta$ if for some $A_1, \ldots, A_n \in \Gamma$, and some $B_1, \ldots, B_m \in \Delta$, $\vdash_{\mathbf{H}} (A_1 \wedge \ldots \wedge A_n) \to (B_1 \vee \ldots \vee B_m)$ (and $\not\Rightarrow$ if this is not the case). We define a sequence of sets of formulas Γ_n, Δ_n such that $\Gamma_n \not\Rightarrow \Delta_n$ by induction. First, $\Gamma_0 = \Gamma$ and $\Delta_0 = \{A\}$. By the assumption of the theorem, $\Gamma_0 \not\Rightarrow \Delta_0$.

If $\Gamma_n \Rightarrow \Delta_n \cup \{F_n\}$, then $\Gamma_{n+1} = \Gamma_n \cup \{F_n\}$ and $\Delta_{n+1} = \Delta_n$. In this case, $\Gamma_{n+1} \not\Rightarrow \Delta_{n+1}$, since otherwise we would have $\Gamma_n \Rightarrow \Delta_n \cup \{F_n\}$ and $\Gamma_n \cup \{F_n\} \Rightarrow \Delta_n$. But then, we'd have that $\Gamma_n \Rightarrow \Delta_n$, which contradicts the induction hypothesis (note that $\vdash_{\mathbf{H}} (A \to B \vee F) \to ((A \wedge F \to B) \to (A \to B)))$.

If $\Gamma_n \not\Rightarrow \Delta_n \cup \{F_n\}$, then $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n \cup \{F_n, B(y_n)\}$ if $F_n \equiv \forall x\, B(x)$, and $\Delta_{n+1} = \Delta_n \cup \{F_n\}$ otherwise. In the latter case, it is obvious that $\Gamma_{n+1} \not\Rightarrow \Delta_{n+1}$. In the former, observe that by I10 and QS, if $\Gamma_n \Rightarrow \Delta_n \cup \{\forall x\, B(x), B(y_n)\}$ then also $\Gamma_n \Rightarrow \Delta_n \cup \{\forall x\, B(x)\}$ (note that y_n does not occur in Γ_n or Δ_n).

Let $\Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta^* = \bigcup_{i=0}^{\infty} \Delta_i$. We have:

1. $\Gamma^* \not\Rightarrow \Delta^*$, for otherwise there would be a k such that $\Gamma_k \Rightarrow \Delta_k$.
2. $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$ (by construction).
3. $\Gamma^* = \mathcal{F} \setminus \Delta^*$, since each F_n is either in Γ_{n+1} or Δ_{n+1}, and if for some n, $F_n \in \Gamma^* \cap \Delta^*$, there would be a k so that $F_n \in \Gamma_k \cap \Delta_k$, which is impossible since $\Gamma_k \not\Rightarrow \Delta_k$.
4. If $\Gamma^* \Rightarrow B_1 \vee \ldots \vee B_n$, then $B_i \in \Gamma^*$ for some i. For suppose not, then for $i = 1, \ldots, n$, $B_i \notin \Gamma^*$, and hence, by (3), $B_i \in \Delta^*$. But then $\Gamma^* \Rightarrow \Delta^*$, contradicting (1).
5. If $B(t) \in \Gamma^*$ for every $t \in \mathcal{T}$, then $\forall x\, B(x) \in \Gamma^*$. Otherwise, by (3), $\forall x\, B(x) \in \Delta^*$ and so there is some n so that $\forall x\, B(x) = F_n$ and Δ_{n+1} contains $\forall x\, B(x)$ and $B(y_n)$. But, again by (3), then $B(y_n) \notin \Gamma^*$.
6. Γ^* is closed under provable implication, since if $\Gamma^* \Rightarrow A$, then $A \notin \Delta^*$ and so, again by (3), $A \in \Gamma^*$. In particular, if $\vdash_{\mathbf{H}} A$, then $A \in \Gamma^*$.

Define relations $\preceq$ and $\equiv$ on $\mathcal{F}$ by

$$B \preceq C \Leftrightarrow B \to C \in \Gamma^* \quad \text{and} \quad B \equiv C \Leftrightarrow B \preceq C \wedge C \preceq B.$$

Then $\preceq$ is reflexive and transitive, since for every B, $\vdash_{\mathbf{H}} B \to B$ and so $B \to B \in \Gamma^*$, and if $B \to C \in \Gamma^*$ and $C \to D \in \Gamma^*$ then $B \to D \in \Gamma^*$, since $B \to C, C \to D \Rightarrow B \to D$ (recall (6) above). Hence, $\equiv$ is an equivalence relation on $\mathcal{F}$. For every B in $\mathcal{F}$ we let $|B|$ be the equivalence class under $\equiv$ to which B belongs, and $\mathcal{F}/{\equiv}$ the set of all equivalence classes. Next we define the relation $\leq$ on $\mathcal{F}/{\equiv}$ by

$$|B| \leq |C| \Leftrightarrow B \preceq C \Leftrightarrow B \to C \in \Gamma^*.$$

Obviously, $\leq$ is independent of the choice of representatives A, B.

LEMMA 3.3.2. *$\langle \mathcal{F}/\equiv, \leq \rangle$ is a countably linearly ordered structure with distinct maximal element $|\top|$ and minimal element $|\bot|$.*

Proof. Since $\mathcal{F}$ is countably infinite, $\mathcal{F}/\equiv$ is countable. For every B and C, $\vdash_{\mathbf{H}} (B \to C) \vee (C \to B)$ by LIN, and so either $B \to C \in \Gamma^*$ or $C \to B \in \Gamma^*$ (by (4)), hence $\leq$ is linear. For every B, $\vdash_{\mathbf{H}} B \to \top$ and $\vdash_{\mathbf{H}} \bot \to B$, and so $B \to \top \in \Gamma^*$ and $\bot \to B \in \Gamma^*$, hence $|\top|$ and $|\bot|$ are the maximal and minimal elements, respectively. Pick any A in Δ^*. Since $\top \to \bot \Rightarrow A$, and $A \notin \Gamma^*$, $\top \to \bot \notin \Gamma^*$, so $|\top| \neq |\bot|$. □

We abbreviate $|\top|$ by $\mathbf{1}$ and $|\bot|$ by $\mathbf{0}$.

LEMMA 3.3.3. *The following properties hold in $\langle \mathcal{F}/\equiv, \leq \rangle$:*

1. $|B| = \mathbf{1} \Leftrightarrow B \in \Gamma^*$.
2. $|B \wedge C| = \min\{|B|, |C|\}$.
3. $|B \vee C| = \max\{|B|, |C|\}$.
4. $|B \to C| = \mathbf{1}$ *if* $|B| \leq |C|$, $|B \to C| = |C|$ *otherwise.*
5. $|\neg B| = \mathbf{1}$ *if* $|B| = \mathbf{0}$*;* $|\neg B| = \mathbf{0}$ *otherwise.*
6. $|\exists x B(x)| = \sup\{|B(t)| \mid t \in \mathcal{T}\}$.
7. $|\forall x B(x)| = \inf\{|B(t)| \mid t \in \mathcal{T}\}$.

Proof. (1) If $|B| = \mathbf{1}$, then $\top \to B \in \Gamma^*$, and hence $B \in \Gamma^*$. And if $B \in \Gamma^*$, then $\top \to B \in \Gamma^*$ since $B \Rightarrow \top \to B$. So $|\top| \leq |B|$. It follows that $|\top| = |B|$ as also $|B| \leq |\top|$.

(2) From $\Rightarrow B \wedge C \to B$, $\Rightarrow B \wedge C \to C$ and $D \to B, D \to C \Rightarrow D \to B \wedge C$ for every D, it follows that $|B \wedge C| = \inf\{|B|, |C|\}$, from which (2) follows since $\leq$ is linear. (3) is proved analogously.

(4) If $|B| \leq |C|$, then $B \to C \in \Gamma^*$, and since $\top \in \Gamma^*$ as well, $|B \to C| = \mathbf{1}$. Now suppose that $|B| \not\leq |C|$. From $B \wedge (B \to C) \Rightarrow C$ it follows that $\min\{|B|, |B \to C|\} \leq |C|$. Because $|B| \not\leq |C|$, $\min\{|B|, |B \to C|\} \neq |B|$, hence $|B \to C| \leq |C|$. On the other hand, $\vdash C \to (B \to C)$, so $|C| \leq |B \to C|$.

(5) If $|B| = \mathbf{0}$, $\neg B = B \to \bot \in \Gamma^*$, and hence $|\neg B| = \mathbf{1}$ by (1). Otherwise, $|B| \not\leq |\bot|$, and so by (4), $|\neg B| = |B \to \bot| = \mathbf{0}$.

(6) Since $\vdash_{\mathbf{H}} B(t) \to \exists x\, B(x)$, $|B(t)| \leq |\exists x\, B(x)|$ for every $t \in \mathcal{T}$. On the other hand, for every D without x free,

	$\lvert B(t)\rvert \leq \lvert D\rvert$	for every $t \in \mathcal{T}$
$\Leftrightarrow$	$B(t) \to D \in \Gamma^*$	for every $t \in \mathcal{T}$
$\Rightarrow$	$\forall x(B(x) \to D) \in \Gamma^*$	by property (5) of Γ^*
$\Rightarrow$	$\exists x\, B(x) \to D \in \Gamma^*$	since $\forall x(B(x) \to D) \Rightarrow \exists x\, B(x) \to D$
$\Leftrightarrow$	$\lvert \exists x\, B(x)\rvert \leq \lvert D\rvert$.	

(7) is proved analogously. □

The rest of the proof of Theorem 3.3.1. $\langle \mathcal{F}/\!\equiv, \leq \rangle$ is countable, let $\mathbf{0} = a_0, \mathbf{1} = a_1, a_2, \ldots$ be an enumeration. Define $h(\mathbf{0}) = 0$, $h(\mathbf{1}) = 1$, and define $h(a_n)$ inductively for $n > 1$: Let $a_n^- = \max\{a_i \mid i < n \text{ and } a_i < a_n\}$ and $a_n^+ = \min\{a_i \mid i < n \text{ and } a_i > a_n\}$, and define $h(a_n) = (h(a_n^-) + h(a_n^+))/2$ (thus, $a_2^- = \mathbf{0}$ and $a_2^+ = \mathbf{1}$ as $\mathbf{0} = a_0 < a_2 < a_1 = \mathbf{1}$, hence $h(a_2) = \frac{1}{2}$). Then $h\colon \langle \mathcal{F}/\!\equiv, \leq \rangle \to \mathbb{Q} \cap [0,1]$ is a strictly monotone map which preserves infs and sups. By Lemma 3.1.7 there exists a $\mathbf{G}$-embedding h' from $\mathbb{Q} \cap [0,1]$ into $\langle [0,1], \leq \rangle$ which is also strictly monotone and preserves infs and sups. Put $\mathcal{I}(B) = h'(h(|B|))$ for every atomic $B \in \mathcal{F}$ and we obtain a $V_{\mathbb{R}}$-interpretation.

Note that for every B, $\mathcal{I}(B) = 1$ iff $|B| = \mathbf{1}$ iff $B \in \Gamma^*$. Hence, we have $\mathcal{I}(B) = 1$ for all $B \in \Gamma$ while if $A \notin \Gamma^*$, then $\mathcal{I}(A) < 1$, so $\Gamma \not\models A$. Thus we have proven that on the assumption that if $\Gamma \nvdash A$, then $\Gamma \not\models A$ ☐

This completeness proof can be adapted to hypersequent calculi for Gödel logics (Chapter III, [3, 20]), even including the $\triangle$ projection operator [13].

As already mentioned we obtain from this completeness proof together with the soundness theorem (Theorem 1.5.4) and Theorem 3.1.9 the characterization of recursive axiomatizability:

THEOREM 3.3.4 ([14, Theorem 40]). *Let V be a Gödel set with 0 contained in the perfect kernel of V. Suppose that Γ is a set of closed formulas. Then $\Gamma \models_V A$ iff $\Gamma \vdash_{\mathbf{H}} A$.*

COROLLARY 3.3.5 (Deduction theorem for Gödel logics). *Suppose that Γ is a set of formulas, and A is a closed formula. Then*

$$\Gamma, A \vdash_{\mathbf{H}} B \quad \textit{iff} \quad \Gamma \vdash_{\mathbf{H}} A \to B.$$

Proof. Use the soundness and completeness theorems (Theorem 1.5.4 and 3.3.4, resp.) and a straight-forward semantic deduction. Another proof would be by induction on the length of the proof. See [26, Theorem 2.2.18]. ☐

3.3.2 Axiomatizable case 2: 0 is isolated

In the case where 0 is isolated in V, and thus also not contained in the perfect kernel, we will transform a counter example in $\mathbf{G}_{\mathbb{R}}$ for $\Gamma, \Pi \models A$, where Π is a set of sentences stating that every infimum is a minimum, into a counterexample in $\mathbf{G}_V$ to $\Gamma \models A$.

LEMMA 3.3.6. *Let $x, \bar{y}$ be the free variables in A.*

$$\vdash_{\mathbf{H}_0} \forall \bar{y}(\neg \forall x\, A(x, \bar{y}) \to \exists x\, \neg A(x, \bar{y})).$$

Proof. It is easy to see that in all Gödel logics the following weak form of the law of excluded middle is valid: $\neg\neg A(x) \vee \neg A(x)$. By quantification we obtain $\forall x \neg\neg A(x) \vee \exists x \neg A(x)$ and, by $\textsc{iso}_0$, $\neg\neg\forall x\, A(x) \vee \exists \neg A(x)$. Using the intuitionistically valid schema $(\neg A \vee B) \to (A \to B)$ we can prove $\neg \forall x\, A(x) \to \exists x \neg A(x)$. A final quantification of the free variables concludes the proof. ☐

THEOREM 3.3.7 ([14, Theorem 43]). *Let V be an uncountable Gödel set where 0 is isolated. Suppose Γ is a set of closed formulas. Then $\Gamma \models_V A$ iff $\Gamma \vdash_{\mathbf{H}_0} A$.*

Proof. If: Follows from soundness (Theorem 1.5.4) and the observation that ISO_0 is valid for any V where 0 is isolated.

Only if: We already know from Theorem 3.1.9 that the entailment relations of V and $V \cup [\inf P, 1]$ coincide, where P is the perfect kernel of V. So we may assume without loss of generality that V already is of this form, i.e., that $w = \inf P$ and $V \cap [w, 1] = [w, 1]$. Let $V' = [0, 1]$. Define

$$\Pi = \{\forall \bar{y}(\neg\forall x\, A(x, \bar{y}) \to \exists x \neg A(x, \bar{y})) \mid A(x, \bar{y}) \text{ has } x, \bar{y} \text{ free}\}$$

where $A(x, \bar{y})$ ranges over all formulas with free variables x and $\bar{y}$. We consider the entailment relation in V'. Either $\Pi, \Gamma \models_{V'} A$ or $\Pi, \Gamma \not\models_{V'} A$. In the former case we know from the strong completeness of $\mathbf{H}$ for $\mathbf{G}_{\mathbb{R}}$ that there are finite subsets Π' and Γ' of Π and Γ, respectively, such that $\Pi', \Gamma' \vdash_{\mathbf{H}} A$. Since all the sentences in Π are provable in $\mathbf{H}_0$ (see Lemma 3.3.6) we obtain that $\Gamma' \vdash_{\mathbf{H}_0} A$. In the latter case there is an interpretation $\mathcal{I}'$ such that $\mathcal{I}'(\Pi \cup \Gamma) > \mathcal{I}'(A)$.

It is obvious from the structure of the formulas in Π that their truth value will always be either 0 or 1. Combined with the above we know that for all $B \in \Pi$, $\mathcal{I}'(B) = 1$. Next we define a function $h(x)$ which maps values from $\mathrm{Val}(\mathcal{I}', \Gamma \cup \Pi \cup \{A\})$ into V:

$$h(x) = \begin{cases} 0 & x = 0, \\ w + x/(1 - w) & x > 0. \end{cases}$$

We see that h satisfies conditions (1) and (2) of Lemma 3.1.3, but we cannot use this Lemma directly, as not all existing infima and suprema are necessarily preserved.

Consider as in Lemma 3.1.3 the interpretation $\mathcal{I}_h(B) = h(\mathcal{I}'(B))$ for atomic subformulas of $\Gamma \cup \Pi \cup \{A\}$. We want to show that the identity $\mathcal{I}_h(B) = h(\mathcal{I}'(B))$ extends to all subformulas of $\Gamma \cup \Pi \cup \{A\}$. For propositional connectives and the existentially quantified formulas this is obvious. The important case is $\forall x\, A(x)$. First assume that $\mathcal{I}'(\forall x\, A(x)) > 0$. Then it is obvious that $\mathcal{I}_h(\forall x\, A(x)) = h(\mathcal{I}'(\forall x\, A(x)))$. In the case where $\mathcal{I}'(\forall x\, A(x)) = 0$ we observe that $A(x)$ contains a free variable and therefore $\neg\forall x\, A(x) \to \exists x \neg A(x) \in \Pi$, thus $\mathcal{I}'(\neg\forall x\, A(x) \to \exists x \neg A(x)) = 1$. This implies that there is a witness u such that $\mathcal{I}'(A(u)) = 0$. Using the induction hypothesis we know that $\mathcal{I}_h(A(u)) = 0$, too. We obtain that $\mathcal{I}_h(\forall x\, A(x)) = 0$, concluding the proof.

Thus we have shown that $\mathcal{I}_h$ is a counterexample to $\Gamma \models_V A$. $\square$

3.3.3 Axiomatizable case 3: Finite Gödel sets

In this section we show that entailment over finite truth value sets are axiomatized by $\mathbf{H}_n$.

THEOREM 3.3.8 ([14, Theorem 45]). *Suppose Γ contains only closed formulas. Then $\Gamma \models_{V_n} A$ iff $\Gamma \vdash_{\mathbf{H}_n} A$.*

Proof. If: By Theorem 1.5.4, since every instance of $\text{FIN}(n)$ is valid in $\mathbf{G}_n$.

Only if: Suppose $\Gamma \nvdash_{\mathbf{H}_n} A$, and consider the set Π of closed formulas of the form

$$\forall \bar{x}_1 \dots \bar{x}_{n-1}((A_0(\bar{x}_0) \to A_1(\bar{x}_1)) \vee \dots \vee (A_{n-1}(\bar{x}_{n-1}) \to A_n(\bar{x}_n)))),$$

where $A_0, \dots, A_n$ ranges over all sequences (with repetitions) of length $n+1$ where each A_i is $P(\bar{x})$ for some predicate symbol P occurring in Γ or A. Each formula in Π follows from an instance of $\textsc{fin}(n)$ by generalization. Hence, $\Gamma, \Pi \nvdash_{\mathbf{H}} A$. From the (strong) completeness (Theorem 3.3.4) of $\mathbf{H}$ for $\mathbf{G}_{\mathbb{R}}$ we know there is an interpretation $\mathcal{I}_{\mathbb{R}}$ (into $[0,1]$) such that $\mathcal{I}_{\mathbb{R}}(B) = 1$ for all $B \in \Gamma \cup \Pi$ and $\mathcal{I}_{\mathbb{R}}(A) < 1$.

For sake of brevity let $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Delta)$ for a set of formulas Δ be the set of all truth values of atomic subformulas of formulas in Δ, i.e., $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Delta) = \{\mathcal{I}_{\mathbb{R}}(P(\bar{u})) \mid \bar{u}$ constants from $\mathscr{L}^{\mathcal{I}}\}$. We claim that $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\})$ contains at most n elements. To see this, assume that it contains more than n elements. Then there exist atomic subformulas (w.r.t. $\mathcal{I}$) $B_0, \dots, B_n$ of A or of formulas in Γ such that $\mathcal{I}_{\mathbb{R}}(B_i) > \mathcal{I}_{\mathbb{R}}(B_{i+1})$ for $i = 0, \dots, n-1$. Thus, $\mathcal{I}_{\mathbb{R}}((B_0 \to B_1) \vee \dots \vee (B_{n-1} \to B_n)) < 1$. But this formula is an instance of a formula in Π, and so we have a contradiction with $\mathcal{I}_{\mathbb{R}}(B) = 1$.

Now let $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\}) = \{0, v_1, \dots, v_k, 1\}$ be sorted in increasing order, and let $h(0) = 0$, $h(1) = 1$, and $h(v_i) = 1 - 1/(i+1)$. Note that any truth value occurring in $\mathrm{Val}(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\})$ must be one of the elements of $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\})$. This is easily seen by induction on the complexity of subformulas of $\Gamma \cup \{A\}$ w.r.t. $\mathcal{I}_{\mathbb{R}}$, as the inf and sup of any subset of the finite set $\mathrm{Val}^a(\mathcal{I}_{\mathbb{R}}, \Gamma \cup \{A\})$ is a member of the finite set. By Lemma 3.1.3, $\mathcal{I}_h$ is a V_n-interpretation with $\mathcal{I}_h(B) = h(\mathcal{I}_{\mathbb{R}}) = 1$ for all $B \in \Gamma$ and $\mathcal{I}_h(A) = h(\mathcal{I}_{\mathbb{R}}) < 1$. $\square$

3.3.4 Not recursively enumerable case 1: Countable Gödel sets

In this section we show that the first-order Gödel logics where the set of truth values does not contain a dense subset are not r.e. We establish this result by reducing the classical validity of a formula in all finite models to the validity of a formula in Gödel logic (the set of these formulas is not r.e. by Trakhtenbrot's Theorem).

DEFINITION 3.3.9. *A formula is called* crisp *if all its atomic subformulas occur either negated or double-negated in it.*

LEMMA 3.3.10. *If A and B are crisp and classically equivalent, then also $\mathbf{G}_V \models A \leftrightarrow B$, for any Gödel set V. Specifically, if $A(x)$ and $B^{(x)}$ are crisp, then*

$$\mathbf{G}_V \models (\forall x\, A(x) \to B^{(x)}) \leftrightarrow \exists x(A(x) \to B^{(x)}) \quad \textit{and}$$
$$\mathbf{G}_V \models (B^{(x)} \to \exists x\, A(x)) \leftrightarrow \exists x(B^{(x)} \to A(x)).$$

Proof. Given an interpretation $\mathcal{I}$, define $\mathcal{I}'(C) = 1$ if $\mathcal{I}(C) > 0$ and $= 0$ if $\mathcal{I}(C) = 0$ for atomic C. It is easily seen that if A, B are crisp, then $\mathcal{I}(A) = \mathcal{I}'(A)$ and $\mathcal{I}(B) = \mathcal{I}'(B)$. But $\mathcal{I}'$ is a classical interpretation, so by assumption $\mathcal{I}'(A) = \mathcal{I}'(B)$. $\square$

THEOREM 3.3.11 ([14, Theorem 36]). *If V is countably infinite, then the set of validities of $\mathbf{G}_V$ is not r.e.*

Proof. By Theorem 3.1.8, V is countably infinite iff it is infinite and does not contain a non-trivial densely ordered subset. We show that for every sentence A there is a sentence A^g s.t. A^g is valid in $\mathbf{G}_V$ iff A is true in every finite (classical) first-order structure.

We define A^g as follows: Let P be a unary and L be a binary predicate symbol not occurring in A and let $Q_1, \dots, Q_n$ be all the predicate symbols in A. We use the abbreviations $x \in y \equiv \neg\neg L(x, y)$ and $x \prec y \equiv (P(y) \to P(x)) \to P(y)$. Note that for any interpretation $\mathcal{I}$, $\mathcal{I}(x \in y)$ is either 0 or 1, and as long as $\mathcal{I}(P(x)) < 1$ for all x (in particular, if $\mathcal{I}(\exists z\, P(z)) < 1$), we have $\mathcal{I}(x \prec y) = 1$ iff $\mathcal{I}(P(x)) < \mathcal{I}(P(y))$. Let $A^g \equiv$

$$\left\{ \begin{array}{l} S \wedge c_1 \in 0 \wedge c_2 \in 0 \wedge c_2 \prec c_1 \wedge \\ \quad \forall i \big[\forall x, y \forall j \forall k \exists z\, D \vee \forall x \neg (x \in s(i)) \big] \end{array} \right\} \to (A' \vee \exists u\, P(u))$$

where S is the conjunction of the standard axioms for 0, successor and $\leq$, with double negations in front of atomic formulas,

$$D \equiv \begin{array}{l} (j \leq i \wedge x \in j \wedge k \leq i \wedge y \in k \wedge x \prec y) \to \\ \quad \to (z \in s(i) \wedge x \prec z \wedge z \prec y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(i) \equiv \exists x (x \in i)$.

Intuitively, L is a predicate that divides a subset of the domain into levels, and $x \in i$ means that x is an element of level i. If the antecedent is true, then the true standard axioms S force the domain to be a model of the reduct of PA to the language without $+$ and $\times$, which could be either a standard model (isomorphic to $\mathbb{N}$) or a non-standard model ($\mathbb{N}$ followed by copies of $\mathbb{Z}$). P orders the elements of the domain which fall into one of the levels in a subordering of the truth values.

The idea is that for any two elements in a level $\leq i$ there is an element in a not-empty level $j \geq i$ which lies strictly between those two elements in the ordering given by $\prec$. If this condition cannot be satisfied, the levels above i are empty. Clearly, this condition can be satisfied in an interpretation $\mathcal{I}$ only for finitely many levels if V does not contain a dense subset, since if more than finitely many levels are non-empty, then $\bigcup_i \{\mathcal{I}(P(d)) \mid \mathcal{I} \models d \in i\}$ gives a dense subset. By relativizing the quantifiers in A to the indices of non-empty levels, we in effect relativize to a finite subset of the domain. $\square$

This shows that no infinite-valued Gödel logic whose set of truth values does not contain a dense subset, i.e., no countably infinite Gödel logic is r.e.

3.3.5 Not recursively enumerable case 2: 0 not isolated but not in the perfect kernel

In the preceding sections, we gave axiomatizations for the logics based on those uncountably infinite Gödel sets V where 0 is either isolated or in the perfect kernel of V. It remains to determine whether logics based on uncountable Gödel sets where 0 is neither isolated nor in the perfect kernel are axiomatizable. The answer in this case is negative. If 0 is not isolated in V, 0 has a countably infinite neighborhood. Furthermore, any sequence $(a_n)_{n\in\mathbb{N}} \to 0$ is so that, for sufficiently large n, $V \cap [0, a_n]$ is countable and hence, by (the proof of) Theorem 3.1.8, contains no densely ordered subset. This fact is the basis for the following non-axiomatizability proof, which is a variation on the proof of Theorem 3.3.11.

THEOREM 3.3.12 ([14, Theorem 48]). *If V is uncountable, 0 is not isolated in V, but not in the perfect kernel of V, then the set of validities of $\mathbf{G}_V$ is not r.e.*

Proof. We show that for every sentence A there is a sentence A^h s.t. A^h is valid in $\mathbf{G}_V$ iff A is true in every finite (classical) first-order structure.

The definition of A^h mirrors the definition of A^g in the proof of Theorem 3.3.11, except that the construction there is carried out infinitely many times for $V \cap [0, a_n]$, where $(a_n)_{n\in\mathbb{N}}$ is a strictly descending sequence, $0 < a_n < 1$ for all n, which converges to 0. Let P be a binary and L be a ternary predicate symbol not occurring in A and let $R_1, \ldots, R_n$ be all the predicate symbols in A. We use the abbreviations $x \in_n y \equiv \neg\neg L(x, y, n)$ and $x \prec_n y \equiv (P(y, n) \to P(x, n)) \to P(y, n)$. As before, for a fixed n, provided $\mathcal{I}(\exists x\, P(x, n)) < 1$, $\mathcal{I}(x \prec_n y) = 1$ iff $\mathcal{I}(P(x, n)) < \mathcal{I}(P(y, n))$, and $\mathcal{I}(x \in_n y)$ is always either 0 or 1. We also need a unary predicate symbol $Q(n)$ to give us the descending sequence $(a_n)_{n\in\mathbb{N}}$: Note that $\mathcal{I}(\neg\forall n\, Q(n)) = 1$ iff $\inf\{\mathcal{I}(Q(d)) \mid d \in U_{\mathcal{I}}\} = 0$ and $\mathcal{I}(\forall n\, \neg\neg Q(n)) = 1$ iff $0 \notin \{\mathcal{I}(Q(d)) \mid d \in U_{\mathcal{I}}\}$.

Let $A^h \equiv$

$$\left\{\begin{array}{l} S \wedge \forall n((Q(n) \to Q(s(n))) \to Q(n)) \wedge \\ \quad \neg\forall n\, Q(n) \wedge \forall n\, \neg\neg Q(n) \wedge \\ \quad \forall n\forall x((Q(n) \to P(x, n)) \to Q(n)) \wedge \\ \quad \forall n\exists x\exists y(x \in_n 0 \wedge y \in_n 0 \wedge x \prec_n y) \wedge \\ \qquad \forall n\forall i\big[\forall x, y\forall j\forall k\exists z\, E \vee \forall x\neg(x \in_n s(i))\big] \end{array}\right\} \to (A' \vee \exists n\exists u\, P(u, n) \vee \exists n\, Q(n))$$

where S is the conjunction of the standard axioms for 0, successor and $\leq$, with double negations in front of atomic formulas,

$$E \equiv \begin{array}{c} (j \leq i \wedge x \in_n j \wedge k \leq i \wedge y \in_n k \wedge x \prec_n y) \to \\ \to (z \in_n s(i) \wedge x \prec_n z \wedge z \prec_n y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(n) \equiv \forall i\exists x(x \in_n i)$.

The idea here is that an interpretation $\mathcal{I}$ will define a sequence $(a_n)_{n\in\mathbb{N}} \to 0$ by $a_n = \mathcal{I}(Q(n))$ where $a_n > a_{n+1}$, and $0 < a_n < 1$ for all n. Let $L_n^i = \{x \mid \mathcal{I}(x \in_n i)\}$ be the i-th n-level. $P(x, n)$ orders the set $\bigcup_i L_n^i = \{x \mid \mathcal{I}(\exists i\, x \in_n i) = 1\}$ in a subordering of $V \cap [0, a_n]$: $x \prec_n y$ iff $\mathcal{I}(x \prec_n y) = 1$. Again we force that whenever $x, y \in L_n^i$ with $x \prec_n y$, there is a $z \in L_n^{i+1}$ with $x \prec_n z \prec_n y$, or, if no possible such z exists, $L_n^{i+1} = \emptyset$. Let $r(n)$ be the least i so that L_n^i is empty, or ∞ otherwise. If $r(n) = \infty$ then there is a densely ordered subset of $V \cap [0, a_n]$. So if 0 is not in the perfect kernel, for some sufficiently large L, $r(n) < \infty$ for all $n > L$. $\mathcal{I}(R(n)) = 1$ iff $r(n) = \infty$ hence $\{n \mid \mathcal{I}(R(n)) = 1\}$ is finite whenever the interpretations of P, L, and Q are as intended.

Now if A is classically false in some finite structure $\mathcal{I}$, we can again choose a $\mathbf{G}_V$-interpretation $\mathcal{I}^h$ so that there are as many n with $\mathcal{I}^h(R(n)) = 1$ as there are elements in the domain of $\mathcal{I}$, and the predicates of A behave on $\{n \mid \mathcal{I}(R(n)) = 1\}$ just as they do on $\mathcal{I}$.

For instance, we can define $\mathcal{I}^h$ as follows. We may assume that the domain of $\mathcal{I} = \{0, \dots, m\}$. Let $U^{\mathcal{I}^h} = \mathbb{N}$, and $\mathcal{I}^h(B) = \mathcal{I}(B)$ for B an atomic subformula of A in the language $\mathscr{L}^{\mathcal{I}}$. Pick a strictly monotone descending sequence $(a_n)_{n\in\mathbb{N}}$ in V with $\lim a_n = 0$ so that $a_0, \dots, a_{m+1} \in V^\infty$, $a_0 < 1$, $a_{m+1} = \inf V^\infty$, and let $\mathcal{I}^h(Q(n)) = a_n$. This guarantees that $\mathcal{I}^h(\forall n((Q(n) \to Q(s(n))) \to Q(n))) = 1$ (because $a_n > a_{n+1}$), $\mathcal{I}^h(\neg\forall n\, Q(n)) = 1$ (because $\inf a_n = 0$), $\mathcal{I}^h(\forall n \neg\neg Q(n)) = 1$ (because $a_n > 0$), and $\mathcal{I}^h(\exists n\, Q(n)) < 1$ (because $a_0 < 1$). Then $V \cap [0, a_n]$ is uncountable if $n \leq m$, and countable if $n > m$. For $n \leq m$, let $D_n \subseteq V \cap [0, a_n)$ be countable and densely ordered, and let $j_n \colon \mathbb{N} \to D_n$ be bijective.

For $n > m$, let $j_n(0) = a_{n+1}$, and $j_n(i) = 0$ for $i > 0$. Define $\mathcal{I}^h(P(i,n)) = j_n(i)$. Then, since $j_n(i) < a_n$ for all i, $\mathcal{I}^h(\forall n \forall x((Q(n) \to P(x,n)) \to Q(n))) = 1$, and, since $j_n(i) < a_0 < 1$, $\mathcal{I}^h(\exists n \exists u\, P(u,n)) < 1$. Finally, let $\mathcal{I}^h(L(x,y,n)) = 1$ for all x, $y \in \mathbb{N}$ if $n \leq m$ (i.e., $L^i_n = \mathbb{N}$), and if $n > m$ let $\mathcal{I}^h(L(0,0,n)) = \mathcal{I}^h(L(1,0,n)) = 1$ and $\mathcal{I}^h(L(x,y,n)) = 0$ if $x > 1$ and $y \in \mathbb{N}$, and if $x \in \mathbb{N}$ and $y > 0$ (i.e., $L^0_n = \{0,1\}$, $L^i_n = \emptyset$ for $i > m$). This makes the rest of the antecedent of A^h true and ensures that $\mathcal{I}^h(R(n)) = \mathcal{I}^h(\forall i \exists x(x \in_n i)) = 1$ if $n \leq m$ and $= 0$ otherwise. Hence $\mathcal{I}^h(A') = 0$ and $\mathcal{I}^h \nvDash A^h$.

On the other hand, if $\mathcal{I} \nvDash A^h$, then the value of the consequent is < 1. Then as required, for all x, n, $\mathcal{I}(P(x,n)) < 1$ and $\mathcal{I}(Q(n)) < 1$. Since the antecedent, as before, must be $= 1$, this means that $x \prec_n y$ expresses a strict ordering of the elements of L^i_n and $\mathcal{I}((Q(n) \to Q(s(n))) \to Q(n)) = 1$ for all n guarantees that $\mathcal{I}(Q(s(n))) = a_{n+1} < a_n = \mathcal{I}(Q(n))$. The other conditions are likewise seen to hold as intended, so that we can extract a finite countermodel for A based on the interpretation of the predicate symbols of A on $\{n \mid \mathcal{I}(R(n)) = 1\}$, which must be finite. $\square$

3.4 Relation to Kripke frames

For propositional logic the truth value sets on which Gödel logics are based can be considered as linear Heyting algebras (or pseudo-Boolean algebras). By taking the prime filters of a Heyting algebra as the Kripke frame it is easy to see that the induced logics coincide (see [23, 31]). This direct method does not work for first order logics as the structure of the prime filters does not coincide with the possible evaluations in the first-order case.

[19] showed that the class of logics defined by countable linear Kripke frames on constant domains and the class of all Gödel logics coincide. More precisely, for every countable Kripke frame we will *construct* a truth value set such that the logic induced by the Kripke frame and the one induced by the truth value set coincide, and vice versa (Theorems 3.4.1 and 3.4.6). As corollaries a complete characterisation of axiomatisability of logics based on countable linear Kripke frames with constant domains (Corollaries 3.4.7 and 3.4.8) have been obtained. Furthermore, we obtain that there are only countable many different logics based on countable linear Kripke frames with constant domains (Corollary 3.4.9). This is especially surprising for at least two reasons: Due to

a result obtained in [16] there are uncountably many different propositional quantified Gödel logics, and thus also uncountably many propositional quantified logics based on countable linear Kripke frames. Furthermore, the number of all intermediate (predicate) logics extending the basic linear logic with constant domains is uncountable.

In the following we will state the central results and give proof ideas and sketches for these results.

THEOREM 3.4.1 ([19, Theorem 18]). *For every countable linear Kripke frame K there is a Gödel set V_K such that* $\mathbf{L}(K) = \mathbf{G}_{V_K}$.

Proof. Let $K = (W, \preceq)$ be a countable linear Kripke frame. The construction of V_K will be in three steps: First, we will enlarge K by doubling all limit worlds; then we will apply the Horn monomorphism (Lemma 3.1.7) to embed the enlarged Kripke frame to $\mathbb{Q} \cap [0,1]$; finally, V_K will be the completion of the range of this embedding.

Let W^* be a disjoint copy of W whose elements can be accessed by the bijection ${}^*\colon W \to W^*$. Elements of W^* serve as names for points which we may have to add. We extend $\preceq$ to a total order $\preceq^*$ on $W \cup W^*$ by putting w^* as the direct successor of w for each $w \in W$, see Figure 2.

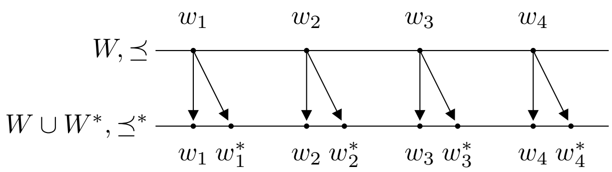

Figure 2. Extending $(W, \preceq)$ to $(W \cup W^*, \preceq^*)$.

Formally we define $\preceq^*$ as follows:

$$\preceq^* \quad := \quad \preceq \cup \{(v^*, w^*) \mid v \preceq w\} \cup \{(v, w^*) \mid v \preceq w\} \cup \{(v^*, w) \mid v \prec w\}\,.$$

Let $\mathrm{Lim}(W)$ denote the set of limit worlds in W:

$$w \in \mathrm{Lim}(W) \quad \text{iff} \quad (\forall w' \succ w)(\exists w'' \succ w)(w'' \prec w').$$

Observe that a maximal element of K, if it exists, would be in $\mathrm{Lim}(W)$. We define

$$W' \quad := \quad W \cup \{w^* \mid w \in \mathrm{Lim}(W)\}$$

and we define $\preceq'$ as the restriction of $\preceq^*$ to W':

$$\preceq' \quad := \quad \preceq^* \cap (W' \times W').$$

Let $K' := (W', \preceq')$. Next, we apply the Horn monomorphism from the proof of Lemma 3.1.7 to the converse of K', i.e. to $(W', \succeq')$. We obtain an embedding σ from $(W', \succeq')$ to $(\mathbb{Q} \cap [0,1], \leq)$ which satisfies the following form of continuity: for any subsets X and Y of W', if

$$\{w \in W' \mid \exists x \in X w \preceq' x\} \cap \{w \in W' \mid \exists y \in Y y \preceq' w\} = \emptyset$$

and

$$\{w \in W' \mid \exists x \in X w \preceq' x\} \cup \{w \in W' \mid \exists y \in Y y \preceq' w\} = W'$$

then $\sup \sigma(Y) = \inf \sigma(X)$.

To finish our construction, let V_K be the closure of $\sigma(W')$:

$$V_K \quad := \quad \overline{\sigma(W')}.$$

It remains to show that the logics $\mathbf{L}(K)$ and $\mathbf{G}_{V_K}$ coincide, which can be shown using notions and lemmas from [19]. $\square$

The following example considers the logic of the Kripke frame with set of worlds $\mathbb{Q}$. Takano [37] has shown that this logic is axiomatised by any complete axiom system for first-order intuitionistic logic (see e.g. [39]) plus the axiom scheme of linearity $(A \to B) \vee (B \to A)$ and the axiom scheme of constant domain (or quantifier shift) $\forall x(A \vee B(x)) \to (A \vee \forall x B(x))$, where x must not occur free in A. This axiomatisation is the same as the one for the standard first-order Gödel logic, i.e. the one based on the full interval $[0,1]$ (cf. [27]). Hence, we can expect that our construction derives a related Gödel set from the Kripke frame $\mathbb{Q}$.

EXAMPLE 3.4.2 (The logic $\mathbf{L}(\mathbb{Q})$). Let $K_{\mathbb{Q}} = (\mathbb{Q}, \leq)$ be the Kripke frame of $\mathbb{Q}$. We want to describe the Gödel set $V_{\mathbb{Q}}$ corresponding to $K_{\mathbb{Q}}$ which is obtained by the construction given in the proof of the previous Theorem. $V_{\mathbb{Q}}$ will be isomorphic to the set of upsets of $\mathbb{Q}$.

Note that for every element $q \in \mathbb{Q}$ there are two designated upsets in $\mathrm{Up}(K_{\mathbb{Q}})$, $q^{\uparrow}$, and $q^{\uparrow} \setminus \{q\}$. Between these two upsets there is no other upset in $\mathrm{Up}(K_{\mathbb{Q}})$. Thus, $q^{\uparrow}$ and $q^{\uparrow} \setminus \{q\}$ under the isomorphism between $\mathrm{Up}(K_{\mathbb{Q}})$ and $V_{\mathbb{Q}}$ determine an open interval of $[0,1]$ which will never contain a point during our construction. Hence, doing this for all elements of $\mathbb{Q}$, countably many disjoint open intervals are generated which are densely ordered, which is achieved by a set isomorphic to the Cantor set.

To be more precise: For every $q \in \mathbb{Q}$ the upset $q^{\uparrow} \setminus \{q\}$ is of type β. Thus, our construction from the last proof duplicates all the rational number, i.e. $\mathbb{Q}' = \mathbb{Q} \cup \{q^* \mid q \in \mathbb{Q}\}$ and $\leq' = \leq^*$. Now fix a particular enumeration of $\mathbb{Q} = \{q_1, q_2, \dots\}$ and consider the following enumeration induced on $\mathbb{Q}' = \{q_1, q_1^*, q_2, q_2^*, \dots\}$. The images of the pairs q_1, q_1^*, q_2, q_2^*, etc., under the Horn function h determine a sequence of disjoint open intervals of $[0,1]$ which are removed from $[0,1]$. This obviously mimics Cantor's middle third construction of repeatedly removing the middle thirds of line segments of $[0,1]$. Hence the image of $\mathbb{Q}'$ under the Horn function h for this enumeration is a set isomorphic to the set of boundary points of the Cantor set, and the completion of $h(\mathbb{Q}')$ is a set isomorphic to the Cantor set.

Now, the Gödel logic $\mathbf{G}_{\mathbb{C}_{[0,1]}}$ generated by the Cantor set $\mathbb{C}_{[0,1]}$ is equal to the Gödel logic of the full interval, $\mathbf{G}_{[0,1]}$ (Theorem 3.3.4). To obtain an idea for this, first observe that obviously $\mathbf{G}_{[0,1]} \subseteq \mathbf{G}_{\mathbb{C}_{[0,1]}}$. Furthermore, for each $\varphi \notin \mathbf{G}_{[0,1]}$ we can find a valuation based on a countable model which makes φ false; hence the occurring truth values form a countable set (not necessarily closed!) which can be embedded into $\mathbb{C}_{[0,1]}$ such that existing infima and suprema are preserved. This gives rise to an interpretation based on $\mathbb{C}_{[0,1]}$ which also makes φ false. Hence, also $\varphi \notin \mathbf{G}_{\mathbb{C}_{[0,1]}}$.

Going from Gödel sets to Kripke frames is not as complicated as the other direction. First we consider countable Gödel sets. For the general case of uncountable Gödel sets we will use Example 3.4.2 and a splitting lemma (Lemma 3.4.5) which divides uncountable Gödel sets into a countable part and a part containing a perfect set.

LEMMA 3.4.3. *For every countable Gödel set V there is a countable linear Kripke frame K_V such that $\mathbf{G}_V = \mathbf{L}(K_V)$.*

Proof. Since V is countable and closed, it can be viewed as a complete and completely distributive linear lattice. Every element of V is either an isolated point, or it is the limit of some isolated points. Thus every element of V is the join of a set of completely join-irreducible elements and V is isomorphic to a complete linear ring of sets (see [34] for definitions of join-irreducibility and this result). Furthermore, a lattice is isomorphic to a complete ring of sets if and only if it is isomorphic to the lattice of order ideals of some partial order P (see e.g. [21] for the definition of order ideals and this result). It is an easy exercise to show now that the logics $\mathbf{G}_V$ and $\mathbf{L}(P)$ coincide. □

REMARK 3.4.4. *It is worth explicitly describing the construction of the Kripke frame underlying the previous proof. This is useful for finding Kripke frames for concretely given Gödel sets such that the logics defined by the Kripke frames are the same as the logics defined by the Gödel sets.*

Let V be a countable Gödel set. By removing proper suprema from V we obtain a corresponding Kripke frame K_V: Let $\mathrm{Sups}(V)$ be the set of all suprema of V,

$$\mathrm{Sups}(V) := \{p \in V \mid \exists (p_n) \subset V \textit{ strictly increasing to } p\} \cup \{0\}.$$

We define the set of worlds as $W_V := V \setminus \mathrm{Sups}(V)$. Then the Kripke frame $K_V := (W_V, \geq)$ defines the same logic as the Gödel set V. This construction works because a supremum of V will reoccur in $\mathrm{Up}(K_V)$ as the upset of all elements smaller than that supremum.

For the treatment of general, i.e. uncountable, Gödel sets we need the following splitting Lemma which allows to split Kripke frames into parts and consider the logics of these parts only.

LEMMA 3.4.5. *Let V_1 and V_2 be Gödel sets and $K_1 = (W_1, \preceq_1)$ and $K_2 = (W_2, \preceq_2)$ be Kripke frames such that $(V_i, \leq)$ and $(\mathrm{Up}(K_i), \subseteq)$ are isomorphic. Assume $W_1 \cap W_2 = \emptyset$. Let $\alpha \in (0,1)$, define*

$$V := \alpha V_1 \cup ((1-\alpha)V_2 + \alpha)$$

and $K := (W_2 \cup W_1, \preceq)$ with

$$\preceq := \preceq_2 \cup \preceq_1 \cup \{(w_2, w_1) \mid w_2 \in W_2, w_1 \in W_1\},$$

see Figure 3. Then $(V, \leq)$ and $(\mathrm{Up}(K), \subseteq)$ are isomorphic, too.

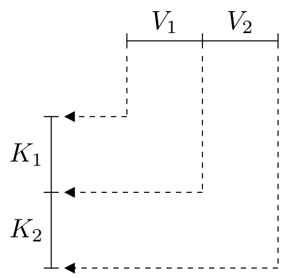

Figure 3. The relation of V_1, V_2 to K_1, K_2.

Proof. Let f_i be the isomorphism from V_i to $\mathrm{Up}(K_i)$. We define $f\colon V \to \mathrm{Up}(K)$ as follows: If $v \in [0,\alpha] \cap V$ then $f(v) = f_1(v/\alpha)$. If $v \in [\alpha, 1] \cap V$ then $f(v) = W_1 \cup f_2((v-\alpha)/(1-\alpha))$.

First observe that f is well defined: the only critical point is at α where we have two ways to compute $f(\alpha)$:

$$f(\alpha) = f_1(\alpha/\alpha) = f_1(1) = W_1$$

and

$$f(\alpha) = W_1 \cup f_2((\alpha-\alpha)/(1-\alpha)) = W_1 \cup f_2(0) = W_1 \cup \emptyset = W_1.$$

It is easy to verify that f is a $(V, \leq)$–$(\mathrm{Up}(K), \subseteq)$ isomorphism: f being bijective is reduced to f_1 and f_2 being bijective, and it is also immediate from the construction that f is a $\leq$–$\subseteq$ homomorphism. □

THEOREM 3.4.6 ([19, Theorem 25]). *For every Gödel set V there is a countable linear Kripke frame K_V such that $\mathbf{G}_V = \mathbf{L}(K_V)$.*

Proof. In Corollary 1.6.6 the Cantor-Bendixon representation of V gives a countable set C and a perfect set P such that $V = C \cup P$ and $C \cap P = \emptyset$. If V is countable, then P is empty and the Gödel logic induced by V can already be represented using Lemma 3.4.3. So assume that V is not countable, which means P is not empty. Let $\alpha := \inf P$, $V'' := V \cup [\alpha, 1]$ and $V' := (V \cap [0,\alpha]) \cup \mathbb{C}_{[\alpha,1]}$, where $\mathbb{C}_I$ is the Cantor middle-third set on the interval I, which is a perfect set. Using Theorem 3.1.9 we obtain that $\mathbf{G}_V = \mathbf{G}_{V''} = \mathbf{G}_{V'}$. Hence, it is enough to consider V'.

In the case that $\alpha = 0$ we have $V' = \mathbb{C}_{[0,1]}$, in which case the Gödel logic based on V' is the same as the $\mathbf{L}(\mathbb{Q})$, see Example 3.4.2.

Otherwise let $V_1 := (1/\alpha)(V \cap [0,\alpha])$ and $V_2 := \mathbb{C}_{[0,1]}$. Then we can write V' as

$$V' = \alpha V_1 \cup ((1-\alpha)V_2 + \alpha).$$

By construction of α, V_1 is countable and due to V being closed V_1 is also closed. Hence, by the proof of Lemma 3.4.3 we can find a countable linear Kripke frame K_1 such that $(V_1, \leq)$ and $(\mathrm{Up}(K_1), \subseteq)$ are isomorphic. Due to Example 3.4.2 we know that $(V_2, \leq)$ and $(\mathrm{Up}(K_{\mathbb{Q}}), \subseteq)$ are isomorphic. Applying Lemma 3.4.5 we obtain a countable Kripke frame K such that $(V', \leq)$ and $(\mathrm{Up}(K), \subseteq)$ are isomorphic. Finally, it is easy to see that the induced logics agree:

$$\mathbf{G}_V = \mathbf{G}_{V'} = \mathbf{L}(\mathrm{Up}(K)) = \mathbf{L}(K). \qquad \square$$

It is worth pointing out some structural consequences which can be inferred from our constructions. Let K be a countable linear Kripke frame and let V_K be the corresponding Gödel set. K having a top element is equivalent to 0 being isolated in V_K, and K having a bottom element is equivalent to 1 being isolated in V_K. Let K *ends with* $\mathbb{Q}$ denote that there is an embedding σ of $\mathbb{Q}$ into K such that $\forall k \in K\ \exists q \in \mathbb{Q}\ k \preceq \sigma(q)$. In this case we have that $\mathbf{L}(K) = \mathbf{L}(\mathbb{Q})$. To see this observe that, as in Example 3.4.2, the condition 'K ends with $\mathbb{Q}$' implies that V_K contains a Cantor set which contains 0. But then Theorem 3.1.9 shows that the induced Gödel logic $\mathbf{G}_{V_K}$ is the same as the Gödel logic of the full unit interval, hence

$$\mathbf{L}(K) = \mathbf{G}_{V_K} = \mathbf{G}_{[0,1]} = \mathbf{L}(\mathbb{Q}).$$

It is interesting to note that Theorem 3.4.6 cannot be deduced from the Löwenheim–Skolem Theorems in [32] and Lemma 3.4.3. Rather, the results presented in the present paper indicate that the Löwenheim–Skolem Theorem in [32, Theorem 4.8], which deals with reducing the cardinality of the pseudo-Boolean algebra, cannot be strengthened in the form that it is reduced to the cardinality of the universe (assuming it is infinite), i.e. in terms of [32, Theorem 4.8], $\lambda' = 2^\lambda$ cannot be replaced by λ in general. To see this observe that the pseudo-Boolean algebra $[0, 1]$ cannot be replaced by any countable pseudo-Boolean algebra: the Gödel logic of the former is axiomatisable (see above), where the Gödel logic of any countable truth value set is not axiomatisable (see Theorem 3.3.11).

As consequences of previous results on axiomatizability we obtain:

COROLLARY 3.4.7. *Let K be a countable linear Kripke frame. The intermediate predicate logic defined by K on constant domains is axiomatisable if and only if K is finite, or if $\mathbb{Q}$ can be embedded into K, and either K has a top element or ends with a copy of $\mathbb{Q}$.*

COROLLARY 3.4.8. *Let K be a countable linear Kripke frame. If K is either not finite and $\mathbb{Q}$ cannot be embedded into K (i.e., K is scattered), or $\mathbb{Q}$ can be embedded into K, but K does not end with $\mathbb{Q}$ and K does not have a top element, then the intermediate predicate logic defined by K on constant domains is not recursively enumerable.*

COROLLARY 3.4.9. *The set of intermediate predicate logics defined by countable linear Kripke frames on constant domains is countable.*

Another surprising aspect from the point of view of the last corollary is that while there are uncountably many different countable linear orderings (which can be taken as Kripke frames), the class of logics defined by them on constant domains only contains countably many elements. Furthermore, the last result is contrasted by the fact that the number of all intermediate logics extending the basic linear logic with constant domains is uncountable.

In a similar way one can show that the logics of scattered Kripke frames with constant domains are not recursively enumerable.

3.5 Number of different Gödel logics

Following the last remark we will now consider the number of different logics. A reasonable argumentation for a lower bound on it would be as follows: If we have a basic logic with extensions in which each of countable many principles can be either true or false, then we would expect uncountably many different logics. As an example let us consider the class of all intermediate predicate logics, i.e. all those logics which are between intuitionistic logic and classical logic (cf. [32]). Here, we have a common basic logic, intuitionistic logic, and extensions of it by different principles. And in fact there are uncountably many intermediate predicate logics. Another example is the class of modal logics which has K as its common basic logic.

Considering Gödel logics, there is a common basic logic, the logic of the full interval, which is included in all other Gödel logics. On the side of logics defined by linear Kripke frames on constant domains this corresponds to the logic determined by a set of worlds of order-type $\mathbb{Q}$. There are still countably many extension principles but, surprisingly, in total only countably many different logics. This has been proven recently by formulating and solving a variant of a Fraïssé Conjecture [24] on the structure of countable linear orderings w.r.t. continuous embeddability.

The ordering relation in this article is smc-embeddability, which is a strictly monotone and continuous embedding of one countable closed subset of the reals into another.

THEOREM 3.5.1 ([18, Corollary 40]). *The set of Gödel logics*

(a) is countable,

(b) is a (lightface) Σ^1_2 set,

(c) is a subset of Gödel's constructible universe L.

Proof. (a) First note that the set of countable Gödel logics (i.e. those with countable truth value set), ordered by $\supseteq$, is a wqo. To see this, assume that $\langle \mathbf{G}_n \mid n \in \omega \rangle$ is a sequence of countable Gödel logics. Take the sequence of countable Gödel sets $\langle V_n \mid n \in \omega \rangle$ generating these logics and define the respective Q-labeled countable closed linear ordering (cclo) (also denoted with V_n) with $Q = \{0, 1\}$, $0 <_Q 1$ and $V_n(0) = V_n(1) = 1$, and $V_n(x) = 0$ otherwise. Using results from [18] one shows that original sequence of Gödel logics $\langle \mathbf{G}_n \mid n \in \omega \rangle$ is good, i.e., it is neither an infinite anti-chain or infinitely decreasing chain with respect to embeddability.

As each countable Gödel logic is a subset of a fixed countable set (the set of all formulas), the family of countable Gödel logics cannot contain a copy of ω_1. So by [18, Lemma 39], the family of countable Gödel logics must be countable.

According to Theorem 3.1.9 any uncountable Gödel logic, i.e. Gödel logic determined by an uncountable Gödel set, such that 0 is not included in the prefect kernel P of the Gödel set is completely determined by the countable part $V \cap [0, \inf P]$. So the total number of Gödel logics is at most two times the number of countable Gödel logics plus 1 for the logic based on the full interval, i.e. countable.

(b) First, note that the set

$$\{(v, \varphi, v(\varphi)) \mid M^v = \mathbb{N}\}$$

is a Borel set, since we can show by induction on the quantifier complexity of φ that the sets $\{(v, q) \mid M^v = \mathbb{N}, v(\varphi) \geq q\}$ are Borel sets (even of finite rank).

Next, a set G of formulas is a Gödel logic iff there *exists* a closed set $V \subseteq [0, 1]$ (say, coded as the complement of a sequence of finite intervals) such that:

- For every $\varphi \in G$, *for every* v with $M^v = \mathbb{N}$, $v(\varphi) = 1$, and
- For every $\varphi \notin G$, there exists v with $M^v = \mathbb{N}$, $v(\varphi) < 1$.

(We can restrict our attention to valuations v with $v^M = \mathbb{N}$ because of Proposition 3.1.4.) Counting quantifiers we see that this is a Σ^1_2 property.

(c) Follows from (a) and (b) by the Mansfield–Solovay Theorem (see [29], [30, 8G.1, and 8G.2]). $\Box$

4 Further topics

In the following we shortly mention some further topics and observations and refer the interested reader to the cited references.

4.1 The Delta operator

With respect to the set of valid formulas the $\triangle$ operator extends the recursively enumerable infinitely valued Gödel logics.

THEOREM 4.1.1 ([14]). *A Gödel logic with $\triangle$ is axiomatizable iff the truth value set is one of:*

• $\{0, 1\} \subset V^\infty$	*axiomatization:* $\mathbf{H}\Delta$
• $0 \in V^\infty$, 1 *isolated*	*axiomatization:* $\mathbf{H}\Delta + \text{ISO}_0$
• 0 *isolated,* $1 \in V^\infty$	*axiomatization:* $\mathbf{H}\Delta + \text{ISO}_1$
• 0 *and* 1 *isolated and* $V^\infty \neq \emptyset$ *(thus V is uncountable)*	*axiomatization:* $\mathbf{H}\Delta + \text{ISO}_0 + \text{ISO}_1$
• *V finite*	*axiomatization:* $\mathbf{H}\Delta + \text{FIN}(n)$

Another observation is that $G^{\Delta}_{\uparrow}$ is not anymore the intersection of the finitely valued Gödel logics with Delta.

4.2 The Takeuti–Titani rule and quantified propositional logics

As already mentioned and discussed in the introduction, Takeuti and Titani in [38] introduced the following rule

$$\frac{C \vee (A \to x) \vee (x \to B)}{C \vee A \to B}$$

where the variable x does not occur in the conclusion. Recall that this rule can be semantically and syntactically eliminated from proofs, see [17, 36]. This shows that using dependent rules certain semantic properties can forced.

Using this rule we can introduce quantified propositional Gödel logics on $[0, 1]$.

DEFINITION 4.2.1 (Propositional quantified Gödel logic over V). *The language is the propositional language with quantifiers binding propositional variables. The propositional quantifiers range over all truth values.*

THEOREM 4.2.2 ([15]). *The quantified propositional Gödel logic on* $[0,1]$ *admits quantifier elimination and is axiomatized by the quantified propositional variant of* $\mathbf{H}$ *together with the Takeuti–Titani rule.*

Note that in this case the Takeuti–Titani rule cannot be eliminated, but can be replaced by the equivalent formulation as axiom.

By coding open and closed intervals in the language one obtains the following result:

PROPOSITION 4.2.3 ([15]). *There are uncountable ($\aleph_1$) many quantified propositional Gödel logics.*

Other quantified propositional logics that admit quantifier elimination are $G_\uparrow$ and $G_\downarrow$, but in these cases a syntactical extension of the language by an unary operator is necessary [7, 11].

Considering the intersection of quantified propositional Gödel logics a varied image is shown:

- the intersection of all finitely valued q.p. without $\triangle$ is the q.p. logic over $V_\uparrow$
- the intersection of all finitely valued q.p. with $\triangle$ is not a Gödel logic (with $\triangle$ and propositional quantifiers finiteness can be expressed)
- the intersection of all q.p. without $\triangle$ is not a Gödel logic.

We conclude this further topics with an observation of rarely considered logics where first-order and quantified propositional quantifiers are mixed.

First note that in all first-order Gödel logics the following two implications are valid:

$$\begin{array}{lcl} \forall x A(x) \vee B & \to & \exists x (A(x) \to C \vee C \to B) \\ A \to \exists x B(x) & \to & \exists x (A \to C \vee C \to B(x)) \end{array}$$

Thus, in the quantified propositional Gödel logic over $[0,1]$ the following equivalences are valid (where the first quantifier can be a propositional or first-order quantifier)

$$\begin{array}{lcl} (\forall x A(x) \to B) & \leftrightarrow & \forall p \exists x (A(x) \to p \vee p \to B) \\ (A \to \exists x B(x)) & \leftrightarrow & \forall p \exists x (A \to p \vee p \to B(x) \end{array}$$

Combining that with the fact that all other quantifier shifts are valid we obtain:

THEOREM 4.2.4. *In the Gödel logic over* $[0,1]$ *with first-order and quantified propositional quantifiers all formulas can be transformed into equivalent prenex formulas.*

4.3 Fragments of Gödel logics

THEOREM 4.3.1 ([12]). *The bottom-less fragment, the prenex fragment, and the existential fragment for infinitely valued Gödel logics are recursively enumerable if and only if the truth value set is uncountable. The resulting sets of valid formulas coincide.*

4.4 Entailment

The compactness of the underlying entailment relation is of central importance for the deductive properties of the logic under consideration. If the entailment is not compact, no effective representation of the entailment can be constructed [16].

DEFINITION 4.4.1 (Compactness). G_V^0 *is* compact *if, whenever* $\Pi \models_V A$ *there is a finite* $\Pi' \subset \Pi$ *such that* $\Pi' \models_V A$.

It is important to mention that if we consider entailment relations or compactness, the underlying truth value set has to be closed under infima.

In the case of propositional tautologies, all logics of infinite truth value sets are the same (Theorem 2.1.8). The case for the entailment relation is similar with dense linear subset taking the position of the infinite subset.

It is an easy but fundamental result that $\mathrm{Taut}(V) = \mathrm{G}_V^0$ and $\mathrm{Ent}(V)$, the set of valid entailment relations, depend only on the order type of V. This central property of Gödel logics is dependent on the specific definition of the Gödel implication, other definitions of implication might not allow this kind of equivalence (see Lemma 3.1.3).

THEOREM 4.4.2 ([16, Proposition 3.2]). *If* V *is finite then* G_V *is compact.*

Proof. We are discussing the entailment $\Pi \Vdash A$. Let $\Pi = \{B_1, B_2, \ldots\}$, and let $X = \{p_0, p_1, \ldots\}$, be an enumeration of variables occurring in Π, A such that all variables in B_i occur before the variables in B_{i+1}. We show that either $\{B_1, \ldots, B_k\} \Vdash A$ for some $k \in \mathbb{N}$ or $\Pi \nVdash A$.

Let T be the complete semantic tree on X, i.e. $T = V^{<\omega}$. An element of T of length k is a valuation of $p_0, \ldots, p_{k-1}$. Since V is finite, T is finitary. Let T' be the subtree of T defined by: $v \in T'$ if for every initial segment v' of v and every k such that all the variables in $A, B_1, \ldots, B_k$ are among $p_0, \ldots, p_{\ell(v')}$,

$$v'(\{B_1, \ldots, B_k\}) = \min\{v(B_1), \ldots, v(B_k)\} > v'(A).$$

In other words, branches in T' terminate at nodes v', where

$$v'(\{B_1, \ldots, B_k\}) \leq v(A).$$

Now if T' is finite, there is a k such that $B_1, \ldots, B_k \Vdash_V A$. Otherwise, since T' is finitary, it contains an infinite branch. Let v be the limit of the partial valuations in that branch. Obviously, since V is finite, $v(\Pi) > v(A)$ and so $\Pi \nVdash_V A$. □

THEOREM 4.4.3 ([16, Theorem 3.4]). *If* V *is uncountable, then* G_V *is compact.*

Proof. Let W be a densely ordered, countable subset of V. Such a subset exists according to Proposition 1.6.4. Let X be a set of variables. A *chain on* X is an arrangement of X in a linear order. Formally, a chain C on X is a sequence of pairs $\langle p_i, o_i \rangle$ where $o_i \in \{<, =, >\}$ where p_i appears exactly once. A valuation $\mathcal{I}$ respects C if $\mathcal{I}(p_i) = \mathcal{I}(p_{i+1})$ if o_i is $=$, $\mathcal{I}(p_i) > \mathcal{I}(p_{i+1})$ if o_i is $>$, and $\mathcal{I}(p_i) < \mathcal{I}(p_{i+1})$ if o_i is $<$. If X is finite, there are only finitely many chains on X.

We consider the entailment relation $\Pi \Vdash A$ and construct a tree in stages as follows: The initial node is labeled by $0 < 1$ and an empty valuation. Stage $n+1$: A node N constructed in stage n is labeled by a chain on the variables $p_1, \ldots, p_n$ and a valuation $\mathcal{I}_N$ of $p_1, \ldots, p_n$ respecting the chain. N receives successor nodes, one for each possibility of extending the chain by inserting p_{n+1}. The labels of each successor node N' are the corresponding extended chain and an extension of $\mathcal{I}_N$ which respects the extended chain. The value $\mathcal{I}_{N'}(p_{n+1})$ is chosen inside W, i.e. the endpoints of W may not be chosen as values. Since W is densely ordered, this ensures that such a choice can be made at every stage.

We call a branch of T *closed at node* N (constructed at stage n) if for some finite $\Pi' \subseteq \Pi$ such that $\mathrm{var}(\Pi') \cup \mathrm{var}(A) \subseteq \{p_1, \ldots, p_n\}$ it holds that $\mathcal{I}_N(\Pi') \leq \mathcal{I}_N(A)$. T is *closed* if it is closed on every branch. In that case, for some finite $\Pi' \subseteq \Pi$, we have $\Pi' \Vdash A$.

If T is not closed, it contains an infinite branch. Let $\mathcal{I}$ be the limit of the $\mathcal{I}_N$ of nodes N on the infinite branch. It holds that $\mathcal{I}(B) > \mathcal{I}(A)$ for all $B \in \Pi$, for otherwise the branch would be closed at the first stage where all the variables in A were assigned values. Let $w = \mathcal{I}(A)$. By Lemma 3.2.6, $\mathcal{I}_w(A) = \mathcal{I}(A)$ and $\mathcal{I}_w(\Pi) = \inf\{\mathcal{I}_w(B) \mid B \in \Pi\} = 1$, and so $\Pi \nVdash A$, a contradiction. □

THEOREM 4.4.4 ([16, Theorem 3.5]). *If V is countably infinite, then* G_V *is not compact.*

Proof. Note that if V is countable, it cannot contain a densely ordered subset, since truth-value sets for entailment have to be closed (under infima). We define a sequence of formulas Γ_k as follows:

$$\begin{aligned}
\Gamma'_k &= \{p_{0/2^k} \prec p_{1/2^k} \prec \cdots \prec p_{(2^k-1)/2^k} \prec p_{2^k/2^k}\} \\
\Gamma''_k &= \{p_{0/2^k} \to q, \ldots, p_{2^k/2^k} \to q\} \\
\Gamma_k &= \Gamma'_k \cup \Gamma''_k \\
\Gamma &= \bigcup_{k \in \omega} \Gamma_k
\end{aligned}$$

Intuitively, Γ'_k expresses that the $p_r = p_{i/2^k}$ are linearly ordered and $\bigcup_{k\in\omega} \Gamma'_k$ expresses that the variables p_r are densely ordered. Since V does not contain a densely ordered subset, we have

$$\Gamma \Vdash_V q.$$

In fact the only $\mathcal{I}$ such that $\mathcal{I}(\Gamma) = 1$ is $\mathcal{I}(p_r) = 1$ for all r, and $\mathcal{I}(q) = 1$. Now assume a finite $\Gamma' \subset \Gamma$ such that

$$\Gamma' \Vdash_V q.$$

There is a $\Gamma_k \supseteq \Gamma'$. Since V is infinite we can choose at least $2^k + 2$ different truth values $v_0 < \cdots < v_{2^k+1} < 1$. Define the valuation $\mathcal{I}$ as

$$\begin{aligned}
\mathcal{I}(p_{i/2^k}) &= v_i, \\
\mathcal{I}(q) &= v_{2^k+1}.
\end{aligned}$$

Then we have $\mathcal{I}(\Gamma_k) = \mathcal{I}(\Gamma') = 1$, but $\mathcal{I}(q) < 1$ and therefore, $\Gamma' \nVdash_V q$. □

Thus, we have succeeded in characterizing the compact propositional Gödel logics. They are all those where the set of truth values V is either finite or contains a nontrivial densely ordered subset.

Although the number of *first-order Gödel logics* has been settled to countable (see Theorem 3.5.1), it is possible to prove that there are uncountably ($2^{\aleph_0}$) many entailment relations:

PROPOSITION 4.4.5. *The number of different entailment relations of first-order Gödel logics is* $2^{\aleph_0}$.

Sketch. It is possible to express ordinals and their orderings with entailment relations. Observing that there are uncountably many such orderings concludes the proof. □

4.5 Satisfiability

In the case of Gödel logics, the connection between satisfiability and entailment one is used to from classical logic breaks down. It is not the case that $\models A$ iff $\{\neg A\}$ is unsatisfiable. For instance, $B \vee \neg B$ is not a tautology, but can also never take the value 0, hence $\mathcal{I}(\neg(B \vee \neg B)) = 0$ for all $\mathcal{I}$, i.e., $\neg(B \vee \neg B)$ is unsatisfiable. So entailment cannot be defined in terms of satisfiability in the same way as in classical logic. Yet, satisfiability can be defined in terms of entailment: Γ is satisfiable iff $\Gamma \not\models \bot$. Hence also for Gödel logics, establishing soundness and strong completeness for entailment yields the familiar versions of soundness and completeness in terms of satisfiability: a set of formulas Γ is satisfiable iff it is consistent.

In the following we concentrate on satisfiability as many practical problems are connected to it: large ontologies are always checked for consistency, i.e., satisfiability. For the applicability of Gödel logics it is essential that satisfiability in many cases means *classical* satisfiability, and therefore usual automated theorem provers may be used for consistency checks.

THEOREM 4.5.1 ([5]). *In the following cases satisfiability in Gödel logics is classical satisfiability:*

- *in the propositional case,*
- 0 *is isolated in the truth value set,*
- *prenex fragment for any truth value set,*
- *existential fragment for any truth value set.*

Proof. For the first two cases let Q be any formula of G_V. If Q is satisfiable in classical logic then Q is satisfiable in G_V. For the converse direction, consider any interpretation $\mathcal{I}_{\mathbf{G}}$ of G_V such that $\mathcal{I}_{\mathbf{G}}(Q) = 1$. An interpretation $\mathcal{I}_{\mathbf{CL}}$ of classical logic such that $\mathcal{I}_{\mathbf{CL}}(Q) = 1$ is defined as follows: for any atomic formula A

$$\mathcal{I}_{\mathbf{CL}}(A) = \begin{cases} 1 & \text{if } \mathcal{I}_{\mathbf{G}}(A) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that for each formula P, $(*)$ $\mathcal{I}_{\mathbf{G}}(P) = 0$ if and only if $\mathcal{I}_{\mathbf{CL}}(P) = 0$ and $\mathcal{I}_{\mathbf{G}}(P) > 0$ if and only if $\mathcal{I}_{\mathbf{CL}}(P) = 1$. The proof proceeds by induction on the complexity of P and all cases go though for *all* Gödel logics except when P has the form $\forall x P_1(x)$; in this case, being 0 an isolated point in V, $\mathcal{I}_{\mathbf{G}}(P) = 0$ if and only if there is an element u in the domain of $\mathcal{I}_{\mathbf{G}}$ such that $\mathcal{I}_{\mathbf{G}}(P_1(u)) = 0$; by induction hypothesis $\mathcal{I}_{\mathbf{CL}}(P_1(u)) = 0$ and hence $\mathcal{I}_{\mathbf{CL}}(\forall x P_1(x)) = 0$.

Considering the prenex case, let $Q = \mathcal{Q}\bar{x}P$ be any prenex formula, were $\mathcal{Q}\bar{x}$ is the formula prefix and P does not contain quantifiers. Assume that $\mathcal{I}_{\mathbf{G}}(Q) = 1$. As above we can prove $(*)$ for P. $\mathcal{I}_{\mathbf{CL}}(Q) = 1$ easily follows by induction on the number n of quantifiers in $\mathcal{Q}\bar{x}$. $\square$

4.5.1 Recursively enumerability of (un)satisfiability

This is an area with many open questions. The only known results are

THEOREM 4.5.2.

- *For finitely valued logics, unsatisfiability is r.e.*
- *For the prenex fragment with $\triangle$ over the truth value set $[0, 1]$, unsatisfiability is r.e. [8].*
- *For the class FO^1_{mon} consisting of all formulas in the first-order language with $\triangle$ of the form*

$$\bigvee_{i=1}^{n} (\exists x A_1^i(x) \wedge \ldots \wedge \exists x A_{n_i}^i(x) \wedge \forall x B_1^i(x) \wedge \ldots \wedge \forall x B_{m_i}^i(x)),$$

  satisfiability is decidable [6].

It can be easily shown that with respect to satisfiability there are at least countably many logics, but the upper limit is not known.

We conclude this part with two observations, the first concerning the Löwenheim–Skolem Theorem. Consider

$$\neg(x = y) \rightarrow \neg\triangle(P(x) \leftrightarrow P(y)).$$

which implies that the upward Löwenheim–Skolem Theorem does not hold for infinitely valued Gödel logics with $\triangle$.

To show the contrast between validity and satisfiability, we emphasize the following theorem:

THEOREM 4.5.3. *The prenex fragment of the monadic class (with at least 2 predicate symbols) for infinitely valued Gödel logics is*

- *undecidable (with the possible exceptions of $\mathrm{G}_\uparrow$) with respect to validity (the construction in [4] is easily adaptable to the prenex case);*
- *decidable with respect to satisfiability (see Theorem 4.5.1).*

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Chapter VIII: Fuzzy Logics with Enriched Language

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1 Introduction

The basic language of t-norm based fuzzy logics is composed of the strong conjunction $\&$, the lattice (or additive) conjunction $\wedge$, the implication $\to$ and the truth-constant $\overline{0}$ denoting *falsum*. From these primitive connectives,[1] other usual connectives are definable: the truth-constant $\overline{1}$ is defined as $\overline{0} \to \overline{0}$; the residual negation $\neg$, where $\neg\varphi$ is defined as $\varphi \to \overline{0}$; the equivalence $\leftrightarrow$, where $\varphi \leftrightarrow \psi$ is defined as $(\varphi \to \psi) \,\&\, (\psi \to \varphi)$; or the lattice disjunction $\vee$, where $\varphi \vee \psi$ is defined as $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$. The properties of these connectives may heavily vary depending on the particular semantics of the different t-norm based logics we consider. As a matter of example, consider the negation $\neg$. It turns out that e.g. in Łukasiewicz logic Ł, this negation is involutive, so $\neg\neg\varphi$ is equivalent to φ, while in Gödel logic G, the negation behaves very differently: it is, in fact, a pseudo-complementation and satisfies the axiom $\neg(\varphi \wedge \neg\varphi)$. Therefore, in some cases, we might need to use an involutive negation in the framework of Gödel logic, or, vice versa, we might need to use a Gödel negation in the framework of Łukasiewicz logic. Thus, in order to increase the expressive power of a given logic, it might be interesting to study expansions of a logic with different additional connectives. Indeed, developments in the field of fuzzy logic in a broad sense (like the study of De Morgan triples, the use of linguistic hedges and evaluated formulas in fuzzy logic applications, etc.) have led to the study of a number of expansions of fuzzy logics with additional connectives with varying arity. In this chapter we have selected some of the most relevant systems among such expansions.

In Section 2, we consider expansions of a logic L_* of a continuous t-norm $*$ with a set of truth-constants $\overline{r}$ for each r belonging to a countable subalgebra $\boldsymbol{C}$ of the standard L_*-algebra $[0,1]_*$. In Section 3, we deal with expansions of core fuzzy logics with truth-stressing and truth-depressing hedges, modelled as unary connectives. Their intended interpretations on standard algebras are non-decreasing mappings $h\colon [0,1] \to [0,1]$ such that $h(0) = 0$ and $h(1) = 1$: so, they respect the Boolean truth-values; moreover h is required to be subdiagonal ($h(x) \leq x$ for all $x \in [0,1]$) in case of a truth-stresser, and superdiagonal ($h(x) \geq x$ for all $x \in [0,1]$) in case of a truth-depresser. In Section, 4 we consider expansions of core fuzzy logics with an involutive negation $\sim$, that is, a negation such that $\sim\sim\varphi$ is equivalent to φ, which is not usually the case with the negation $\neg$

[1] In logics above BL, the lattice conjunction $\wedge$ is also definable, namely $\varphi \wedge \psi$ is $\varphi \,\&\, (\varphi \to \psi)$.

definable from the implication and the truth-constant $\overline{0}$, where $\neg\varphi$ is $\varphi \to \overline{0}$. Finally, in Section 5 we consider specially relevant expansions for Łukasiewicz logic, namely, the so-called Rational Łukasiewicz logic and different expansions including the product conjunction, eventually leading to the logics ŁΠ and ŁΠ$\frac{1}{2}$.

2 Expansions with truth-constants

T-norm based fuzzy logics are basically logics of *comparative truth*. In fact, the residuum $\Rightarrow$ of a (left-continuous) t-norm $*$ satisfies the condition $x \Rightarrow y = 1$ if, and only if , $x \leq y$ for all $x, y \in [0, 1]$. This means that a formula $\varphi \to \psi$ is a logical consequence of a theory if the truth degree of φ is at most as high as the truth degree of ψ in any interpretation which is a model of the theory. In fact, the logic of continuous t-norms presented in Hájek's seminal book [46] only deals with valid formulas and deductions taking 1 as the only truth value to be preserved by inference (in the sense of yielding true consequences from true premises for each interpretation). This line has been followed by the majority of papers written since then in the setting of many-valued systems of mathematical fuzzy logic, including this handbook. However, in general, these truth-preserving logics do not exploit in depth neither the idea of comparative truth nor the potentiality of dealing with explicit partial truth that a many-valued logic setting offers.

The idea of comparative truth is pushed forward in the so-called *logics preserving truth-degrees*, studied in [5, 37], where a deduction is valid if, and only if, the degree of truth of the premises is less than or equal to the degree of truth of the conclusion: in fact they preserve lower bounds of truth values. Actually, since Gödel logic is the only t-norm based logic enjoying the classical deduction-detachment theorem, it is the only case where both notions of logic coincide.

On the other hand, in some situations one might be also interested in explicitly representing and reasoning with intermediate degrees of truth. A way to do so, while keeping the truth preserving framework, is to introduce truth-constants into the language. This approach actually goes back to Pavelka [75], who built a propositional many-valued logical system which turned out to be equivalent to the expansion of Łukasiewicz logic obtained by adding to the language a truth-constant $\overline{r}$ for each real $r \in [0, 1]$, together with some additional axioms. Pavelka proved that his logic is strongly complete in a non-finitary sense (known as Pavelka-style completeness), heavily relying on the continuity of Łukasiewicz truth-functions.

Similar expansions with truth-constants for other propositional t-norm based fuzzy logics can be analogously defined. However, Pavelka-style completeness cannot be obtained in those cases, since Łukasiewicz logic is the only t-norm based logic whose truth-functions are continuous. A more general approach was developed in a series of papers [12, 26, 30–33, 78] where, rather than Pavelka-style completeness, the authors focused on the usual notion of completeness of a logic. It is interesting to note that in this approach: (1) the logic to be expanded with truth-constants has to be the logic of a given left-continuous t-norm; (2) the expanded logic is still a truth-preserving logic, but its richer language admits formulas of type $\overline{r} \to \varphi$, implying that, when their evaluation equals 1, the truth degree of φ is greater or equal than r; and (3) the expanded logic is still algebraizable in the sense of Blok and Pigozzi.

In this section we describe the expansions with truth-constants of logics of continuous t-norms in a general setting.[2] Actually, we provide a full description of completeness results for the expansions of logics of continuous t-norms with a set of truth-constants $\{\overline{r} \mid r \in C\}$, for a suitable countable $C \subseteq [0, 1]$, when (i) the t-norm is a finite ordinal sum of Łukasiewicz, Gödel and Product components and (ii) the set of truth-constants *covers* the whole unit interval in the sense that each component contains at least one value of C in its interior.

This section is structured as follows. After this introduction, for historical reasons we first introduce Rational Pavelka logic, a simplified version of Pavelka logic defined by Hájek [46]. In Section 2.2 we introduce a general notion of expanded logics with truth-constants and their algebraic semantics. In Sections 2.3 and 2.4, we describe the structure and relevant algebraic properties of the expanded linearly ordered algebras, which are needed to obtain the completeness results reported in Sections 2.5 and 2.6. Section 2.7 deals with completeness results when restricting the language to evaluated formulas. In Section 2.8, we also consider expansions with the Monteiro–Baaz $\triangle$ connective as well as an alternative approach to the use of the $\triangle$ connective. Finally, after mentioning some open questions in Section 2.9, we overview in Section 2.10 the expansions of first-order logics and their main results.

2.1 Rational Pavelka logic

Hájek [46] showed that Pavelka's logic could be significantly simplified while keeping the completeness results. Indeed, Rational Pavelka logic (RPL), see [45, 46], is the expansion of Łukasiewicz logic Ł by adding a truth-constant $\overline{r}$ for each rational $r \in [0, 1]$ together with the following book-keeping axioms for truth-constants:

$$
\begin{array}{ll}
(\mathrm{RPL1}) & \overline{r} \,\&\, \overline{s} \;\leftrightarrow\; \overline{r *_{Ł} s} \\
(\mathrm{RPL2}) & \overline{r} \to \overline{s} \;\leftrightarrow\; \overline{r \Rightarrow_{Ł} s}
\end{array}
$$

where $*_{Ł}$ and $\Rightarrow_{Ł}$ are Łukasiewicz t-norm and implication respectively. An evaluation e of propositional variables into the real unit interval $[0, 1]$ is extended to an RPL-evaluation of arbitrary formulas as in Łukasiewicz logic with the additional requirement that $e(\overline{r}) = r$ for each rational r.

Notice that a formula of the form $\overline{r} \to \varphi$ gets value 1 under an evaluation e whenever φ gets a value by e greater or equal than r. Therefore, the RPL-formula $\overline{r} \to \varphi$ expresses that the truth-value of φ is at least r. Similarly, $\varphi \to \overline{r}$ expresses that the truth-value of φ is at most r.

As usual, a *theory* T over RPL is just a set of formulas. The notion of proof, denoted $\vdash_{\mathrm{RPL}}$, is defined as usual from the axioms of RPL and *modus ponens*. A theory T is *consistent* if $T \not\vdash \overline{0}$. Furthermore, a theory T is *linear* if $T \vdash (\varphi \to \psi)$ or $T \vdash (\psi \to \varphi)$ for each pair of RPL-formulas φ, ψ.

Given a theory T, the *truth degree* of a formula φ in T is defined as

$$||\varphi||_T = \inf\{e(\varphi) \mid e \text{ is a model of } T\},$$

[2] For the case of expansions of some left-continuous t-norms the reader is referred to [30, 33].

and the *provability degree* of φ over T as

$$|\varphi|_T = \sup\{r \mid T \vdash_{\mathrm{RPL}} \overline{r} \to \varphi\}.$$

REMARK 2.1.1. The provability degree is a supremum, which is not necessarily a maximum; for an infinite T, $|\varphi|_T = 1$ does not always imply $T \vdash \varphi$. (Still, this works for a finite T, see [53] and [46, Theorem 3.3.14].)

The (Pavelka-style) form of strong completeness for RPL says that the provability degree of φ in T equals the truth degree of φ over T, that is, $||\varphi||_T = |\varphi|_T$. To prove this we need some preliminary lemmas (that the reader can consult in [46]).

LEMMA 2.1.2.

(1) $T \vdash \overline{0}$ *iff* $T \vdash \overline{r}$ *for some* $r < 1$.

(2) *Each consistent theory* T *can be extended to a consistent and complete theory* T'.

(3) *If* T *does not prove* $(\overline{r} \to \varphi)$ *then* $T \cup \{\varphi \to \overline{r}\}$ *is consistent.*

(4) *If* T *is consistent and complete, then* $\sup\{r \mid T \vdash \overline{r} \to \varphi\} = \inf\{r \mid T \vdash \varphi \to \overline{r}\}$.

Proof. (1) It easily follows from the fact that if $r < 1$, then there exists n such that $r *_{Ł} \overset{n}{\ldots} *_{Ł} r = 0$.

(2) This is a well-known fact for Łukasiewicz logic, which is not invalidated by the presence of the truth-constants.

(3) Assume $T \cup \{\varphi \to \overline{r}\}$ is inconsistent, hence $T \cup \{\varphi \to \overline{r}\} \vdash \overline{0}$. By the local deduction theorem of Łukasiewicz logic, there is n such that $T \vdash (\varphi \to \overline{r})^n \to \overline{0}$. But $(\varphi \to \psi)^n \vee (\psi \to \varphi)^n$ is a theorem of Łukasiewicz logic, therefore RPL proves $(\varphi \to \overline{r})^n \vee (\overline{r} \to \varphi)^n$ as well. Thus obviously $T \vdash (\overline{0})^n \vee (\varphi \to \overline{r})^n$. Hence $T \vdash (\varphi \to \overline{r})^n$. Therefore, we obtain a contradiction.

(4) Since for each r, either $T \vdash \varphi \to \overline{r}$ or $T \vdash \overline{r} \to \varphi$, it suffices to show that $T \vdash \overline{r} \to \varphi$ and $T \vdash \varphi \to \overline{s}$ implies $r \leq s$. Assume $r > s$. Then we would get $T \vdash \overline{r} \to \overline{s}$, i.e. $T \vdash \overline{r \Rightarrow_{Ł} s}$, but $r \Rightarrow_{Ł} s < 1$, and thus T would be inconsistent. □

LEMMA 2.1.3. *If* T *is consistent and complete, the provability degree commutes with the connectives, i.e.*

$$|\overline{r}|_T = r, \qquad |\neg\varphi|_T = 1 - |\varphi|_T, \qquad |\varphi \to \psi|_T = |\varphi|_T \Rightarrow_{Ł} |\psi|_T.$$

Proof. The case of truth-constants is easy. For the case of the negation, we have $|\neg\varphi|_T = \sup\{r \mid T \vdash \overline{r} \to \neg\varphi\} = \sup\{r \mid T \vdash \varphi \to \overline{1-r}\} = \sup\{1 - r \mid T \vdash \varphi \to \overline{r}\} = 1 - \inf\{r \mid T \vdash \varphi \to \overline{r}\} = 1 - |\varphi|_T$.

For the case of the implication, we show the two inequalities:

(a) $|\varphi|_T \Rightarrow_{Ł} |\psi|_T = \inf\{s \mid T \vdash \varphi \to \overline{s}\} \Rightarrow_{Ł} \sup\{r \mid T \vdash \overline{r} \to \psi\} = \sup\{s \Rightarrow_{Ł} r \mid T \vdash \varphi \to \overline{s}, T \vdash \overline{r} \to \psi\} \leq \sup\{t \mid T \vdash \overline{t} \to (\varphi \to \psi)\} = |\varphi \to \psi|_T$. Notice that here the continuity of $\Rightarrow_{Ł}$ plays a crucial role.

(b) Assume there exist rationals $t, t' \in [0,1]$ such that $|\varphi|_T \Rightarrow_{*_Ł} |\psi|_T < t < t' < |\varphi \to \psi|_T$. Express t as $r \Rightarrow_Ł s$ for some $r < |\varphi|_T$ and some $s > |\psi|_T$. Then $T \vdash \overline{r} \to \varphi$ and $T \vdash \psi \to \overline{s}$, and hence $T \vdash (\varphi \to \psi) \to (\overline{r} \to \overline{s})$, $T \vdash (\varphi \to \psi) \to \overline{t}$, $T \vdash \overline{t'} \to (\varphi \to \psi)$, and thus $T \vdash \overline{t'} \to \overline{t}$, i.e. $T \vdash \overline{t' \Rightarrow_Ł t}$. But since $t' > t$, we have $t' \Rightarrow_Ł t < 1$ and thus T is inconsistent. Therefore $|\varphi|_T \Rightarrow_Ł |\psi|_T \geq |\varphi \to \psi|_T$. □

From these lemmas, one can finally prove the following Pavelka's style completeness for RPL.

THEOREM 2.1.4. *In* RPL *we have* $||\varphi||_T = |\varphi|_T$*, for any theory* T *and any formula* φ*.*

Proof. The inequality $|\varphi|_T \leq ||\varphi||_T$ is derivable from the soundness of RPL. To prove the other inequality it is enough to show that for each rational $r < \|\varphi\|_T$, $T \vdash \overline{r} \to \varphi$, or equivalently, if $T \not\vdash \overline{r} \to \varphi$ then $r \geq \|\varphi\|_T$. But if $T \not\vdash \overline{r} \to \varphi$, then $T \cup \{\varphi \to \overline{r}\}$ is consistent. In that case, $T \cup \{\varphi \to \overline{r}\}$ has a consistent complete extension T', and by Lemma 2.1.3, the evaluation defined as $e(p_i) = |p_i|_{T'}$ is a model of T' and $e(\varphi \to \overline{r}) = 1$, and thus $e(\varphi) \leq r$ and hence $\|\varphi\|_T \leq \|\varphi\|_{T'} \leq r$. □

Actually in his papers [75], Pavelka proved a more general completeness result. In fact what he proves is that one can expand the logic with an arbitrary set of additional connectives whose real semantics are defined by "fitting" (finitary) operations on the real unit interval $[0, 1]$.[3] In the framework of RPL, a very similar result is obtained in [49] for the expansion of RPL with product conjunction. Namely, the logic RPL^+ is defined as the expansion of RPL with a new connective $\odot$ and having as axioms those of RPL plus:

(RPL^+1) $\quad (\varphi \to \psi) \to ((\varphi \odot \chi) \to (\psi \odot \chi))$

(RPL^+2) $\quad (\varphi \to \psi) \to ((\chi \odot \varphi) \to (\chi \odot \psi))$

(RPL^+3) $\quad \overline{r} \odot \overline{s} \leftrightarrow \overline{r \cdot s}$

where $\cdot$ denotes product of reals. The first two axioms clearly stand for the monotonicity conditions of $\odot$ and the third is the book-keeping axiom on truth-constants with the product operation. Evaluations e of RPL^+ formulas are defined as in RPL together with the additional requirement that $e(\varphi \odot \psi) = e(\varphi) \cdot e(\psi)$. Moreover, the notions of truth and provability degrees of RPL^+-formulas in a theory are defined in the same way as in RPL. Then the following completeness theorem holds.

THEOREM 2.1.5. *In* RPL^+ *we have* $||\varphi||_T = |\varphi|_T$*, for any theory* T *and any formula* φ*.*

Proof. The proof mimics the one for RPL and basically one has to extend Lemma 2.1.3 to $\odot$, that is, one has to prove that $|\varphi \odot \psi|_T = |\varphi|_T \cdot |\psi|_T$. Remark that due to axioms (RPL^+1) and (RPL^+2), in RPL^+ we have that if $T \vdash \varphi_1 \to \psi_1$ and $T \vdash \varphi_2 \to \psi_2$ then we also have $T \vdash \varphi_1 \odot \varphi_2 \to \psi_1 \odot \psi_2$.

[3] An operation $O\colon [0,1]^n \to [0,1]$ *fits* the standard MV-chain $[0,1]_Ł$ if there is exists natural numbers k_1, $\ldots$, k_n such that for any $x_1, \ldots, x_n, y_1, \ldots, y_n \in [0,1]$, holds: $(x_1 \Leftrightarrow_Ł y_1)^{k_1} \otimes \cdots \otimes (x_n \Leftrightarrow_Ł y_n)^{k_n} \leq O(x_1, \ldots, x_n) \Leftrightarrow_Ł O(y_1, \ldots, y_n)$, where $\Leftrightarrow_Ł$ is defined as $x \Leftrightarrow_Ł y = \min\{x \Rightarrow_Ł y, y \Rightarrow_Ł x\}$.

(a) $|\varphi|_T \cdot |\psi|_T = \sup\{s \mid T \vdash \overline{s} \to \varphi\} \cdot \sup\{r \mid T \vdash \overline{r} \to \psi\} = \sup\{s \cdot r \mid T \vdash \overline{s} \to \varphi, T \vdash \overline{r} \to \psi\} \leq \sup\{t \mid T \vdash \overline{t} \to (\varphi \odot \psi)\} = |\varphi \odot \psi|_T$.

(b) Assume there exist rationals $t, t' \in [0,1]$ such that $|\varphi|_T \cdot |\psi|_T < t < t' < |\varphi \odot \psi|_T$. Clearly we can express t as $r \cdot s$ for some $r > |\varphi|_T$ and some $s > |\psi|_T$. Then $T \vdash \varphi \to \overline{r}$ and $T \vdash \psi \to \overline{s}$, and hence $T \vdash (\varphi \odot \psi) \to (\overline{r} \odot \overline{s})$, $T \vdash (\varphi \odot \psi) \to \overline{t}$, $T \vdash \overline{t'} \to (\varphi \odot \psi)$, and thus $T \vdash \overline{t'} \to \overline{t}$, i.e. $T \vdash \overline{t' \Rightarrow_{Ł} t}$. But since $t' > t$, we have $t' \Rightarrow_{Ł} t < 1$ and thus T is inconsistent.

Therefore $|\varphi|_T \cdot |\psi|_T = |\varphi \odot \psi|_T$. □

Looking at the above proof, one realizes that the same proof would apply if the conjunction $\odot$ is semantically interpreted by another continuous t-norm $*$ closed over the rationals, that is, if we replace axiom (RPL^+3) by

(RPL^+_*3) $\quad \overline{r} \odot \overline{s} \leftrightarrow \overline{r * s}$

then the resulting logic would enjoy the same Pavelka's style completeness. Therefore, this shows that the monotonicity axioms (RPL^+1) and (RPL^+2) plus the book-keeping axiom (RPL^+_*3) suffice to axiomatize (à la Pavelka) any continuous t-norm closed over the rationals.

2.2 Expansions of the logic of a continuous t-norm with truth-constants

A complete analogy between RPL and other logics L_* of continuous t-norms $*$ is not possible since Łukasiewicz logic is the only logic L_* with continuous real truth-functions. Still, one can consider similar expansions with analogous book-keeping axioms and investigate classical completeness properties. This is the main goal of the rest of this section.

Let L_* be the logic of a continuous t-norm $*$, i.e., the extension of BL such that for any finite set of formulas $\Gamma \cup \{\varphi\}$,

$$\Gamma \vdash_{\mathrm{L}_*} \varphi \qquad \text{iff} \qquad \Gamma \models_{[0,1]_*} \varphi.$$

As proved in Chapter V, whenever $*$ is a continuous t-norm, the logic L_* is finitely axiomatizable. Moreover, Chapter V gives a finite axiomatization of L_* as an axiomatic extension of BL.

The goal of this section is to define and study the expansion of any L_* by adding a countable set of truth-constants.

DEFINITION 2.2.1. *Let $[0,1]_*$ be the real L_*-chain and $\boldsymbol{C}$ its countable subalgebra. We define the expanded language $\mathcal{L}_C = \mathcal{L} \cup \{\overline{r} \mid r \in C \setminus \{0,1\}\}$, where $\mathcal{L}$ is the language of L_*. The logic $\mathrm{L}_*(\boldsymbol{C})$ is the expansion of L_* in the language of $\mathcal{L}_C$ obtained by adding the so-called book-keeping axioms:*

$$\overline{r} \mathbin{\&} \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow_* s}$$

where $\Rightarrow_*$ is the residuum of $*$.

Since these logics are core fuzzy logics, sharing *modus ponens* as the only inference rule, they have the same local deduction-detachment theorem as BL. In fact, the proof for BL also applies here.

THEOREM 2.2.2. *For every* $\Gamma \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}_C}$, $\Gamma, \varphi \vdash_{\mathrm{L}_*(\boldsymbol{C})} \psi$ *if, and only if, there is a natural* $k \geq 1$ *such that* $\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi^k \to \psi$.

The algebraic counterpart of the $\mathrm{L}_*(\boldsymbol{C})$ logics is defined in the natural way.

DEFINITION 2.2.3. *Let* $*$ *be a continuous t-norm and* $\boldsymbol{C}$ *a countable subalgebra of* $[0,1]_*$. *A structure* $\boldsymbol{A} = \langle A, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \{\overline{r}^{\boldsymbol{A}} \mid r \in C\}\rangle$ *is an* $\mathrm{L}_*(\boldsymbol{C})$*-algebra if:*

(1) $\langle A, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ *is an* L_**-algebra, and*

(2) *for every* $r, s \in C$ *the following identities hold:*

$$\overline{r}^{\boldsymbol{A}} \,\&^{\boldsymbol{A}}\, \overline{s}^{\boldsymbol{A}} = \overline{r * s}^{\boldsymbol{A}}$$
$$\overline{r}^{\boldsymbol{A}} \to^{\boldsymbol{A}} \overline{s}^{\boldsymbol{A}} = \overline{r \Rightarrow_* s}^{\boldsymbol{A}}.$$

Given $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_{\boldsymbol{C}}}$, *we define* $\Gamma \models_{\boldsymbol{A}} \varphi$ *iff for all evaluations* e *on* $\boldsymbol{A}$ *(i.e. such that* $e(\overline{r}) = \overline{r}^{\boldsymbol{A}}$*), we have* $e(\varphi) = \overline{1}^{\boldsymbol{A}}$ *whenever* $e(\psi) = \overline{1}^{\boldsymbol{A}}$ *for all* $\psi \in \Gamma$.

Real chains *are* $\mathrm{L}_*(\boldsymbol{C})$*-chains whose underlying domain is the real unit interval* $[0,1]$. *In particular the* canonical $\mathrm{L}_*(\boldsymbol{C})$*-chain is* $[0,1]_{\mathrm{L}_*(\boldsymbol{C})} = \langle [0,1], *, \Rightarrow_*, \min, \max, \{r \mid r \in C\}\rangle$*, i.e. the* $\mathcal{L}_{\boldsymbol{C}}$*-expansion of* $[0,1]_*$ *where the truth-constants are interpreted as themselves.*

We denote the set of interpretations of truth-constants over $\boldsymbol{A}$ *by* $\boldsymbol{C}^{\boldsymbol{A}}$.

Notice that $\boldsymbol{C}^{\boldsymbol{A}}$ is closed under the operations of the algebra $\boldsymbol{A}$, i.e. $\langle \boldsymbol{C}^{\boldsymbol{A}}, \&^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\rangle$ is a subalgebra of $\boldsymbol{A}$.

It is worth noticing that it is not always possible to equip any L_*-algebra with an arbitrary set of constants from a subalgebra of $[0,1]_*$. For instance, it is not possible to equip a finite MV-chain with truth-constants from the whole subalgebra of rationals of $[0,1]_{\text{Ł}}$.

Since $\mathrm{L}_*(\boldsymbol{C})$ is an expansion of L_* without new rules of inference, by [21], $\mathrm{L}_*(\boldsymbol{C})$ is a semilinear logic. As a consequence, each $\mathrm{L}_*(\boldsymbol{C})$-algebra is a subdirect product of chains and thus the logic $\mathrm{L}_*(\boldsymbol{C})$ is complete not only with respect to the full variety but also with respect to the chains of the variety.

To describe real completeness results requires a deeper insight into $\mathrm{L}_*(\boldsymbol{C})$-chains. This is done in the next subsection. Actually, for technical reasons (see the remarks at the end of this section), we will restrict ourselves to logics $\mathrm{L}_*(\boldsymbol{C})$ satisfying the following two conditions:

($\mathbf{C1}$) $*$ is a continuous t-norm that is a *finite* ordinal sum of the basic components (we will denote by **CONT-fin** the set of such continuous t-norms).

($\mathbf{C2}$) each component of the t-norm contains at least one value of C different from the bounds of the component.

2.3 About the structure of real $\mathrm{L}_*(C)$-chains

Suppose that $*$ is a continuous t-norm in $\mathbf{CONT\text{-}fin}$ whose decomposition as ordinal sum of isomorphic copies of the three basic components is $\bigoplus_{i\in I}[a_i,b_i]_{*_i}$.

DEFINITION 2.3.1. *Let $\boldsymbol{A}$ be an $\mathrm{L}_*(\boldsymbol{C})$-chain. $\boldsymbol{C}^{\boldsymbol{A}}$ will denote the subalgebra of $\boldsymbol{A}$ defined over $\{\overline{r}^{\boldsymbol{A}} \mid r \in C\}$ and $F_{\boldsymbol{C}}(\boldsymbol{A})$ will denote the set of the truth-constants interpreted as 1 in $\boldsymbol{A}$, i.e. $F_{\boldsymbol{C}}(\boldsymbol{A}) = \{r \in C \mid \overline{r}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}\}$.*

LEMMA 2.3.2. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be non-trivial $\mathrm{L}_*(\boldsymbol{C})$-chains with the same $\mathcal{L}$-reduct (i.e. possibly differing only on the interpretation of truth-constants). Then:*

(i) *$F_{\boldsymbol{C}}(\boldsymbol{A})$ is a proper filter of $\boldsymbol{C}$.*

(ii) *$\boldsymbol{C}/F_{\boldsymbol{C}}(\boldsymbol{A}) \cong \boldsymbol{C}^{\boldsymbol{A}}$.*

(iii) *If $\boldsymbol{A} \cong \boldsymbol{B}$, then $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B})$.*

(iv) *If $r, s \in C \setminus F_{\boldsymbol{C}}(\boldsymbol{A})$ and $r < s$, then $\overline{r}^{\boldsymbol{A}} < \overline{s}^{\boldsymbol{A}}$.*

Proof. (i) Clearly $1 \in F_{\boldsymbol{C}}(\boldsymbol{A})$. If $r \in F_{\boldsymbol{C}}(\boldsymbol{A})$ and $s \in C$, $s > r$, then $s \in F_{\boldsymbol{C}}(\boldsymbol{A})$ because by the book-keeping axioms and the definability of $\min$ and $\max$ we have $\overline{s}^{\boldsymbol{A}} = \max\{\overline{r}^{\boldsymbol{A}}, \overline{s}^{\boldsymbol{A}}\} = \overline{1}^{\boldsymbol{A}}$. Moreover if $r, s \in F_{\boldsymbol{C}}(\boldsymbol{A})$ then $r * s \in F_{\boldsymbol{C}}(\boldsymbol{A})$, since $\overline{r * s}^{\boldsymbol{A}} = \overline{r}^{\boldsymbol{A}} \,\&\, \overline{s}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}$.

(ii) Consider the function $f\colon \boldsymbol{C} \to \boldsymbol{C}^{\boldsymbol{A}}$ defined by $f(r) = \overline{r}^{\boldsymbol{A}}$. It is clear that f is a surjective homomorphism and $Ker f = F_{\boldsymbol{C}}(\boldsymbol{A})$, so $\boldsymbol{C}/F_{\boldsymbol{C}}(\boldsymbol{A}) \cong \boldsymbol{C}^{\boldsymbol{A}}$.

(iii) If $\boldsymbol{A} \cong \boldsymbol{B}$, then it is clear that $\boldsymbol{C}^{\boldsymbol{A}} \cong \boldsymbol{C}^{\boldsymbol{B}}$, so $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B})$.

(iv) If $r < s \notin F_{\boldsymbol{C}}(\boldsymbol{A})$, then $\overline{r}^{\boldsymbol{A}} \leq \overline{s}^{\boldsymbol{A}}$ since the book-keeping axioms imply that the order must be preserved. On the other hand, if $\overline{r}^{\boldsymbol{A}} = \overline{s}^{\boldsymbol{A}}$, then $[r]_{F_{\boldsymbol{C}}(\boldsymbol{A})} = [s]_{F_{\boldsymbol{C}}(\boldsymbol{A})}$, which implies $s \to r \in F_{\boldsymbol{C}}(\boldsymbol{A})$. So, we obtain a contradiction. In fact:

- (a) If $r, s \in (a_i, b_i)$ and $[a_i, b_i]$ is a Łukasiewicz component, then $s \to r$ belongs to $F_{\boldsymbol{C}}(\boldsymbol{A})$, which implies that the minimum of the component also belongs to $F_{\boldsymbol{C}}(\boldsymbol{A})$. Therefore $[a_i, b_i] \subseteq F_{\boldsymbol{C}}(\boldsymbol{A})$, i.e. a contradiction.
- (b) If $r, s \in (a_i, b_i)$ and $[a_i, b_i]$ is a Product component, then $s \to r \in F_{\boldsymbol{C}}(\boldsymbol{A})$, which implies: if $r = 0$ then $0 \in F_{\boldsymbol{C}}(\boldsymbol{A})$, which is a contradiction; and if $r \neq 0$ then there exists n such that $r > (s \to r)^n$, and, thus, $r, s \in F_{\boldsymbol{C}}(\boldsymbol{A})$, i.e., again, a contradiction.
- (c) Finally, if $r * s = \min\{r, s\}$ then $s \to r = r \in F_{\boldsymbol{C}}(\boldsymbol{A})$, a contradiction. □

Notice that the first three properties in the previous lemma also hold for the general case of $*$ being a left-continuous t-norm. However, we do make use of the continuity of $*$ in the proof of the last one.[4] Actually this lemma describes all possible interpretations of the truth-constants over $\mathrm{L}_*(\boldsymbol{C})$-chains. For instance, for every filter F we can define

[4] In [30] it is proved that (iv) is also valid when $*$ is a Weak Nilpotent Minimum t-norm but it could fail for a general left-continuous t-norm.

an $\mathrm{L}_*(\boldsymbol{C})$-algebra over $[0,1]_*$ interpreting $\overline{r}$ as 1 if $r \in F$ and as r otherwise. We will denote this algebra by $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$. An easy computation shows that it is indeed an $\mathrm{L}_*(\boldsymbol{C})$-chain. Notice that the canonical real algebra corresponds to the case $F = \{1\}$. Moreover, if the t-norm has only Łukasiewicz or Product components, there are as many $\mathrm{L}_*(\boldsymbol{C})$-algebras over $[0,1]_*$ (up to isomorphism) as proper filters of $\boldsymbol{C}$.

PROPOSITION 2.3.3. *Let $*$ be a continuous t-norm that is a finite ordinal sum of Łukasiewicz and Product components. Let $X = \{[\boldsymbol{A}] \mid \boldsymbol{A}$ is a real $\mathrm{L}_*(\boldsymbol{C})$-algebra over $[0,1]_*\}$ be the set of isomorphism classes of $\mathrm{L}_*(\boldsymbol{C})$-algebras over $[0,1]_*$ and let $Fi(\boldsymbol{C})$ be the set of proper filters of $\boldsymbol{C}$. Then, the function $\Phi\colon X \to Fi(\boldsymbol{C})$ such that for every $\boldsymbol{A} \in X$, $\Phi([\boldsymbol{A}]) = F_{\boldsymbol{C}}(\boldsymbol{A})$, is a bijection.*

Proof. Φ is well-defined because of (iii) of Lemma 2.3.2. For an easier notation we will simply write $\Phi(\boldsymbol{A})$ instead of $\Phi([\boldsymbol{A}])$. Φ is clearly onto because $\Phi([0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}) = F$. Thus, we have to prove that Φ is also injective. Suppose that $\Phi(\boldsymbol{A}) = \Phi(\boldsymbol{B})$, i.e. $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B})$. Then, we have $\boldsymbol{C}^{\boldsymbol{A}} \cong \boldsymbol{C}/F_{\boldsymbol{C}}(\boldsymbol{A}) = \boldsymbol{C}/F_{\boldsymbol{C}}(\boldsymbol{B}) \cong \boldsymbol{C}^{\boldsymbol{B}}$. In the following, denoting by h the isomorphism between $\boldsymbol{C}^{\boldsymbol{A}}$ and $\boldsymbol{C}^{\boldsymbol{B}}$, we show how to extend it as a function $h\colon [0,1] \to [0,1]$ making $\boldsymbol{A}$ and $\boldsymbol{B}$ isomorphic as well.

(1) If $* = *_{\text{Ł}}$ (the Łukasiewicz t-norm), the only proper filter of $\boldsymbol{C}$ is $\{1\}$, and thus $\boldsymbol{C}^{\boldsymbol{A}} \cong \boldsymbol{C}^{\boldsymbol{B}} \cong \boldsymbol{C}$. But taking into account that any two isomorphic subalgebras of $[0,1]_{*_{\text{Ł}}}$ coincide (see e.g. [10, Corollary 7.2.6]), $\boldsymbol{C}^{\boldsymbol{A}} = \boldsymbol{C}^{\boldsymbol{B}} = \boldsymbol{C}$, and thus necessarily $\boldsymbol{A} = \boldsymbol{B}$.

(2) If $* = *_{\Pi}$ (the product t-norm), there are only two proper filters, $\{1\}$ and $C \setminus \{0\}$ and thus we have two types of $\Pi(\boldsymbol{C})$-chains over $[0,1]_{\Pi}$ corresponding to the cases that $F = \{1\}$ ($\Pi(\boldsymbol{C})$-chains such that for each pair $r < s$ in C, $\overline{r}^{\mathcal{A}} < \overline{s}^{\boldsymbol{A}}$) and the case $C \setminus \{0\}$ ($\Pi(\boldsymbol{C})$-chains such that $\overline{r}^{\mathcal{A}} = \overline{1}^{\mathcal{A}}$ for all $r \neq 0$). If $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B}) = \{1\}$, then $\boldsymbol{C}^{\boldsymbol{A}} \cong \boldsymbol{C}^{\boldsymbol{B}} \cong \boldsymbol{C}$, and by [78, Theorem 2] we obtain $\boldsymbol{A} \cong \boldsymbol{B}$. If $F_{\boldsymbol{C}}(\boldsymbol{A}) = F_{\boldsymbol{C}}(\boldsymbol{B}) = C \setminus \{0\}$, the result is trivial.

(3) If $*$ is any continuous t-norm that is a finite ordinal sum of Łukasiewicz or Product components, then all possible proper filters are either of the form $[a, 1]$ where a is the minimum of a Łukasiewicz or product component, or of the form $(a, 1]$ where a is the minimum of a product component. The result is proved by applying the previous cases to each component of its decomposition not included in the filter. □

Notice that the last result is not valid when $*$ is the minimum t-norm, as the following counterexample shows. Take $C = \mathsf{Q} \cap [0,1]$, $F = \{1\}$ and the following chains over $[0,1]_{\mathrm{G}}$:

(1) The canonical $\mathrm{G}(\boldsymbol{C})$-chain, i.e. the chain $\boldsymbol{A}$ obtained by interpreting each $\overline{r}$ as its intended value r.

(2) The real $\mathrm{G}(\boldsymbol{C})$-chain $\boldsymbol{B}$ obtained by interpreting the truth-constants as:

$$\overline{r}^{\boldsymbol{B}} = r \quad \text{if} \quad r > \frac{1}{2} \qquad \text{and} \qquad \overline{r}^{\boldsymbol{B}} = \frac{r}{2} \quad \text{if} \quad r \leq \frac{1}{2}.$$

It is clear that $\boldsymbol{C} = \boldsymbol{C}^{\boldsymbol{A}} \equiv \boldsymbol{C}^{\boldsymbol{B}}$, but it is impossible to extend the isomorphism between $\boldsymbol{C}^{\boldsymbol{A}}$ and $\boldsymbol{C}^{\boldsymbol{B}}$ to an isomorphism of the full interval $[0,1]_*$.

From now on, for every filter F of $\boldsymbol{C}$ we will say that an $\mathrm{L}_*(\boldsymbol{C})$-chain $\boldsymbol{A}$ is of *type* F if $F = F_{\boldsymbol{C}}(\boldsymbol{A})$.

To finish this section, we point out that, as we mentioned in the proof of Proposition 2.3.3, any subalgebra $\boldsymbol{C}$ of $[0,1]_\Pi$ has only two filters: $F = \{1\}$ and $F = C \setminus \{0\}$, and hence we have only two types of $\Pi(\boldsymbol{C})$-algebras, which will be referred to (as in [78]) as type I and type II, respectively. If we restrict ourselves to the real chains, there is only one $\Pi(\boldsymbol{C})$-chain of type I, which is the canonical $\Pi(\boldsymbol{C})$-chain $[0,1]_{\Pi(\boldsymbol{C})}$, and also only one chain of type II, denoted $[0,1]^*_{\Pi(\boldsymbol{C})}$. The following result, that will be used in Section 2.6, relates these two real $\Pi(\boldsymbol{C})$-chains. The interested reader may find the proof in [78].

PROPOSITION 2.3.4. *The real* $\Pi(\boldsymbol{C})$*-algebra of type II,* $[0,1]^*_{\Pi(\boldsymbol{C})}$*, belongs to the variety generated by the (canonical)* $\Pi(\boldsymbol{C})$*-algebra of type I,* $[0,1]_{\Pi(\boldsymbol{C})}$*, and hence the variety generated by the class of real* $\Pi(\boldsymbol{C})$*-chains is* $\mathbb{V}([0,1]_{\Pi(\boldsymbol{C})})$.

2.4 Partial embeddability property

In order to study completeness of $\mathrm{L}_*(\boldsymbol{C})$ logics, we need results about the partial embeddability $\mathrm{L}_*(\boldsymbol{C})$-chains into real ones. In this section we will show that most of these logics enjoy this partial embeddability property with respect to their classes $\mathrm{L}_*(\boldsymbol{C})$-chains over $[0,1]$.

DEFINITION 2.4.1. *We say that a logic* $\mathrm{L}_*(\boldsymbol{C})$ *has the partial embeddability property if, and only if, for every filter* F *of* $\boldsymbol{C}$ *and every subdirectly irreducible* $\mathrm{L}_*(\boldsymbol{C})$*-chain* $\boldsymbol{A}$ *of type* F*,* $\boldsymbol{A}$ *is partially embeddable into* $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$.

PROPOSITION 2.4.2. $\mathrm{G}(\boldsymbol{C})$ *has the partial embeddability property.*

Proof. Let $\boldsymbol{A}$ be a linearly ordered $\mathrm{G}(\boldsymbol{C})$-algebra of type F, and let X be a finite subset of A. Let g be an order-preserving injection of X into $[0,1]$ satisfying

$$g(\overline{r}^{\boldsymbol{A}}) = \begin{cases} 1 & \text{if } r \in F, \\ r & \text{otherwise.} \end{cases}$$

So defined, g clearly gives a partial embedding of $\boldsymbol{A}$ into the real $\mathrm{G}(\boldsymbol{C})$-chain of type F, $[0,1]^F_{\mathrm{G}(\boldsymbol{C})}$. □

PROPOSITION 2.4.3. $\Pi(\boldsymbol{C})$ *has the partial embeddability property.*

Proof. For linearly ordered $\Pi(\boldsymbol{C})$-algebras of type II, the problem reduces to the well-known partial embeddability property of product chains into the standard product chain $[0,1]_{*_\Pi}$.

Therefore, let $\boldsymbol{A}$ be a linearly ordered $\Pi(\boldsymbol{C})$-algebra of type I, and let E be a finite subset of A. Denote by C_E the set $\{r \in C \mid \overline{r}^{\boldsymbol{A}} \in E\}$. We have to show that there exists a one-to-one mapping $h\colon E \to [0,1]$ satisfying the following conditions:

(i) h preserves the order,

(ii) $h(\overline{r}^{\boldsymbol{A}}) = r$ for all $r \in C_E$,

(iii) if $x, y, z \in E$ and $z = x * y$ then $h(x) \cdot h(y) = h(z)$,

(iv) If $x, y, z \in E$ and $z = x \Rightarrow y$ then $h(x) \Rightarrow_\Pi h(y) = h(z)$.

Let $\widetilde{\boldsymbol{C}_E}$ be the Π-algebra generated by C_E. Note that the Π-algebra generated by E is naturally a $\Pi(\widetilde{\boldsymbol{C}_E})$-algebra, which will be denoted by $\boldsymbol{A}_E$.

CLAIM 2.4.4. *$\boldsymbol{A}_E$ is isomorphic to a $\Pi(\widetilde{\boldsymbol{C}_E})$-algebra $\boldsymbol{D}$ such that the following conditions are satisfied:*

(1) $\boldsymbol{D} = \mathbf{P}(\mathcal{G})$ with $\mathcal{G}$ being a subgroup of $(\mathsf{R}^+)^k_{lex}$, where k is a natural number,

(2) there is an integer l and a real number $\alpha > 0$, such that, for every positive $r \in \widetilde{C_E}$, we have $\overline{r}^{\boldsymbol{D}} = \omega_{k,l}(r^\alpha)$,

where, for any $x \in (0,1]$ and natural $1 \leq l \leq k$, $\omega_{k,l}(x) = \langle 1, \ldots, 1, x, 1, \ldots, 1\rangle \in (\mathsf{R}^+)^k$, with x being at coordinate with index l.

In this claim, $\mathbf{P}(\mathcal{G})$ denotes the Π-algebra defined from the negative cone of the linearly ordered Abelian group $\mathcal{G}$,[5] and $(\mathsf{R}^+)^k_{lex}$ denotes the ordered Abelian group obtained as the lexicographic product of k copies of the multiplicative group of positive reals. The proof of this claim is rather technical and can be found in [78, Proposition 12].

CLAIM 2.4.5. *For every* finite *subset E' of D, there is a mapping $\delta\colon E' \to [0,1]$ satisfying the following conditions:*

(i) *δ preserves the order,*

(ii) *$\delta(\overline{r}^{\boldsymbol{D}}) = r$ for all $r \in C_E$,*

(iii) *if $x, y, x * y \in E$ then $\delta(x) \cdot \delta(y) = \delta(x * y)$,*

(iv) *if $x, y, x \Rightarrow y \in E$ then $\delta(x) \Rightarrow_\Pi \delta(y) = \delta(x \Rightarrow y)$.*

Proof. The candidates for δ are restrictions to E of functions $g\colon \mathcal{G} \to \mathsf{R}^+$ of the form

$$g((x_1, x_2, \ldots, x_k)) = (x_1^{\varepsilon_1} \cdot x_2^{\varepsilon_2} \cdot \cdots \cdot x_k^{\varepsilon_k})^\beta,$$

where $\varepsilon_i, \beta > 0$. Each of these functions is a homomorphism w.r.t. the product of $\mathcal{G}$. Hence, for every choice of ε_i and β, the restriction of g to E satisfies (iii). By the assumption, for every $r \in C^*$, $\overline{r}^{\boldsymbol{D}} = \omega_{k,l}(r^\alpha)$. Therefore, for every choice of ε_i and β, we have $g(\overline{r}^{\boldsymbol{D}}) = r^{\alpha \cdot \varepsilon_l \cdot \beta}$, where $\alpha \cdot \varepsilon_l \cdot \beta > 0$. By choosing $\beta = 1/(\alpha \cdot \varepsilon_l)$, we obtain that the restriction of g to E' satisfies (ii).

[5] Product algebras are closely related to ordered Abelian groups [46], in fact a linearly ordered Π-algebra without the bottom element can be identified with the negative cone of a linearly ordered Abelian group.

Let us prove that it is possible to choose the ε_i in such a way that the restriction of g to E' satisfies (i). We classify the pairs of distinct values in E' according to the first index i_0, where the values are different. Pairs which satisfy $i_0 = k$ are ordered correctly for any positive value of ε_k. Pairs satisfying $i_0 = k-1$ may be put into the right order by choosing $\varepsilon_{k-1} = 1$ and ε_k small enough to guarantee that the difference (measured as a ratio) in the $(k-1)$-th coordinate is always larger than the difference in the k-th coordinate. In fact, if the exponents $\varepsilon_{k-1} = 1, \varepsilon_k$ guarantee the right order of the pairs with $i_0 = k-1$, then the exponents $\varepsilon_{k-1} = t, t\cdot\varepsilon_k$, for any positive t, guarantee the order as well. Hence, when it is necessary to put the pairs with $i_0 = k-2$ into the right order, we choose $\varepsilon_{k-2} = 1$ and t small enough so that the difference in the $(k-2)$-th coordinate is always larger than the differences contributed by $(k-1)$-th and k-th coordinates. Since we preserve the ratio between ε_{k-1} and ε_k, we do not destroy the already correct order of pairs with $i_0 = k-1$. We proceed in a similar way for pairs with smaller and smaller i_0.

The condition (iv), the preservation of existing implications in E', is a consequence of h being order preserving (i), and the preservation of existing products (iii). This ends the proof of the claim. □

Now, take E' to be the image of E under the isomorphism between $\boldsymbol{A}_E$ and $\boldsymbol{D}$. Applying Claim 2.4.5 to $\boldsymbol{D}$ and E' with $C = \widetilde{C_E}$, we obtain an embedding δ, whose composition with the above isomorphism has the required properties of h. □

PROPOSITION 2.4.6. *Ł$(\boldsymbol{C})$ has the partial embeddability property into the canonical Ł$(\boldsymbol{C})$-chain.*

Proof. Let X be a finite subset of an Ł$(\boldsymbol{C})$-chain $\boldsymbol{A} = \langle A, \wedge, \vee, \otimes, \to, \{\overline{r}^{\boldsymbol{A}} \mid r \in C\}\rangle$. We have to prove that there is a mapping $f\colon X \to [0,1]$ such that:

- if $x, y, x \circ y \in X$, then $f(x \circ y) = f(x) \circ' f(y)$
  for $\circ = \otimes$ and $\circ' = *_{\text{Ł}}$, or for $\circ' = \to$ and $\circ' = \Rightarrow_{\text{Ł}}$,
- for any $r \in C$ such that $\overline{r}^{\boldsymbol{A}} \in X$, $f(\overline{r}^{\boldsymbol{A}}) = r$.

It is well known that if an MV-algebra $\boldsymbol{A}$ is isomorphic to $\Gamma(\mathcal{G}, u)$ for some ℓ-group $\mathcal{G}$ with strong unit u, and if $\boldsymbol{S}$ is a subalgebra of $\boldsymbol{A}$, then there is a (unique) sub-ℓ-group $\mathcal{E}$ of $\mathcal{G}$ such that $u \in \mathcal{E}$ and $\mathcal{S} \cong \Gamma(\mathcal{E}, u)$ (see [10]).

Since $\boldsymbol{C}$ is a countable subalgebra of the standard MV-algebra $\Gamma(\mathsf{R}, 1) = [0,1]_{\text{Ł}}$, it is isomorphic to $\Gamma(\mathcal{H}, 1)$ for a *unique* sub-ℓ-group $\mathcal{H}$ of R such that $1 \in H$. Moreover, the product chain $\mathbf{P}(\mathcal{H})$ is a product subalgebra of $\mathbf{P}(\mathsf{R})$. Notice that, since R is an Archimedean group, each element of the negative cone H^- can be written as $-n + r$, with $r \in C$ and $n \in \mathsf{N}$. The mapping

$$f\colon \mathbf{P}(\mathsf{R}) \to [0,1]_{\Pi}$$

defined by $f(x) = e^x$ for $x < 0$ and $f(\perp) = 0$ is indeed an isomorphism of product algebras, and therefore, $C^\star = \{e^{-n+r} \mid r \in C, n \in \mathsf{N}\} \cup \{0\}$ is the domain of a subalgebra of $[0,1]_{\Pi}$ isomorphic to $\mathbf{P}(\mathcal{H})$. Hence, we can consider the expanded logic $\Pi(\boldsymbol{C}^\star)$ and its canonical $\Pi(\boldsymbol{C}^\star)$-algebra $[0,1]_{\Pi(\boldsymbol{C}^\star)}$.

Therefore, we have seen that for each countable subalgebra $\boldsymbol{C}$ of the standard MV-algebra $[0,1]_{\text{Ł}}$, we can define a corresponding countable subalgebra $\boldsymbol{C}^{\star}$ of the real Π-algebra $[0,1]_{\Pi}$. Hence, we can associate to the canonical Ł$(\boldsymbol{C})$-chain the canonical $\Pi(\boldsymbol{C}^{\star})$-chain.

If $\boldsymbol{A}$ is a Ł$(\boldsymbol{C})$-algebra, then there is an ℓ-group $\mathcal{G}$, a sub-ℓ-group $\mathcal{L}$ and an order unit u of $\mathcal{G}$ such that $\boldsymbol{A} \cong \Gamma(\mathcal{G}, u)$ and $C^{\boldsymbol{A}} \cong \Gamma(\mathcal{L}, u)$. But $\Gamma(\mathcal{G}, u)$ is also isomorphic to the MV-algebra $\Gamma^{-}(\mathcal{G}, u)$ defined on the interval $[-u, 0]$ with the mirror operations. $\Gamma^{-}(\mathcal{L}, u)$ is analogously defined and it is also isomorphic to $\Gamma(\mathcal{L}, u)$. Since $C^{\boldsymbol{A}}$ is isomorphic to a subalgebra of the real MV-algebra, it follows that $\mathcal{L}$ is isomorphic to a sub-ℓ-group $\mathcal{H}$ of R, and since u is an order unit, all the elements of the negative cone L^{-} can be written as $-nu + \overline{r}^{\mathcal{A}}$, for $n \in \mathsf{N}$ and $r \in C$. Thus we can consider the product algebra $\mathcal{P}(\mathcal{G})$ as a $\Pi(\mathcal{C}^{\star})$-algebra, with $\overline{e^{-n+r}}^{\mathcal{P}(\mathcal{G})} = -nu + \overline{r}^{\mathcal{A}}$.

Let X be a finite subset of A. From now on, we identify $\boldsymbol{A}$ and $\Gamma(\mathcal{G}, u)$ (hence taking $\overline{0}^{\boldsymbol{A}} = 0_{\boldsymbol{G}}$ and $\overline{1}^{\boldsymbol{A}} = u$), and, without loss of generality, we can assume $u \in X$. Let $i \colon \Gamma(\mathcal{G}, u) \to \Gamma^{-}(\mathcal{G}, u)$ be defined by $i(x) = x - u$. By the partial embeddability property of Product logic with constants, the $\Pi(\boldsymbol{C}^{\star})$-chain $\mathbf{P}(G)$ is partially embeddable into the canonical $[0,1]_{\Pi(\boldsymbol{C}^{\star})}$. Therefore, considering $i(X)$, as a subset of the $\Pi(\boldsymbol{C}^{\star})$-chain $\mathbf{P}(G)$, there is a partial embedding from $i(X)$ into $[0,1]_{\Pi(\boldsymbol{C}^{\star})}$ such that $\overline{r}^{\boldsymbol{A}} - u = i(\overline{r}^{\boldsymbol{A}}) \mapsto e^{r-1}$, for each $\overline{r}^{\boldsymbol{A}} \in X$. In particular, $-u = i(\overline{0}^{\boldsymbol{A}}) \mapsto e^{-1}$ and $0_G = i(\overline{1}^{\boldsymbol{A}}) \mapsto e^{0} = 1$, thus all the elements of $i(X)$ go to the segment $[e^{-1}, 1]$. Applying natural logarithms, we obtain a partial embedding of $i(X)$ into $\Gamma^{-}(\mathsf{R}, 1)$ such that $i(\overline{r}^{\boldsymbol{A}}) \mapsto r - 1$ for each $\overline{r}^{\boldsymbol{A}} \in X$. Thus, composing i with this embedding and finally with the isomorphism from $\Gamma^{-}(\mathsf{R}, 1)$ to $\Gamma(\mathsf{R}, 1)$ mapping $r - 1 \mapsto r$, we obtain a partial embedding of $X \subset \boldsymbol{A}$ into the canonical Ł$(\boldsymbol{C})$-chain $[0,1]_{\text{Ł}(\boldsymbol{C})}$. This ends the proof. □

THEOREM 2.4.7. *Let $*$ be a continuous t-norm which is a finite ordinal sum and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. Then* $\mathrm{L}_*(\boldsymbol{C})$ *enjoys the partial embeddability property.*

Proof. Suppose that $[0,1]_* = \bigoplus_{i=1}^{n} \boldsymbol{A}_i$. We know that the subdirectly irreducible chains of $\mathbf{V}([0,1]_*)$ are members of

$$\mathbf{HSP}_U(\boldsymbol{A}_1) \cup (\mathbf{ISP}_U(\boldsymbol{A}_1) \oplus \mathbf{HSP}_U(\boldsymbol{A}_2)) \cup \cdots \cup (\textstyle\bigoplus_{i=1}^{n-1} \mathbf{ISP}_U(\boldsymbol{A}_i) \oplus \mathbf{HSP}_U(\boldsymbol{A}_n))$$

(see Chapter V). From this fact, we can use the previous results concerning expansions of G, Ł, and Π to prove the theorem. □

In the next three subsections, we describe different kinds of real completeness properties for the family of logics $\mathrm{L}_*(\boldsymbol{C})$, where $*$ and C satisfy the conditions $(\mathbf{C1})$ and $(\mathbf{C2})$. In the next subsection, we focus on (finite) strong completeness results, while in Section 2.6, we refine the results by determining which logics are canonical real complete. Finally, in Section 2.7, we study the completeness properties when we restrict to evaluated formulas.

2.5 About (finite) strong real completeness

The partial embeddability property allows us to prove both real completeness and conservativeness results.

THEOREM 2.5.1. *Let $*$ be a continuous t-norm and let $\boldsymbol{C}$ be a subalgebra of $[0,1]_*$. If $\mathrm{L}_*(\boldsymbol{C})$ satisfies the partial embeddability property, then $\mathrm{L}_*(\boldsymbol{C})$ has the $\mathrm{FS}\mathcal{R}\mathrm{C}$. In fact, for every finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_\mathcal{C}}$,*

$$\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathbb{K}} \varphi,$$

where $\mathbb{K} = \{[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})} \mid F$ proper filter of $\boldsymbol{C}\}$.

Proof. It is a consequence of Theorem 2.4.7. □

PROPOSITION 2.5.2. *$\mathrm{L}_*(\boldsymbol{C})$ is a conservative expansion of L_*.*

Proof. Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$ be arbitrary formulas and suppose that $\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$. Then, there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$, and this implies that $\Gamma_0 \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \varphi$. Since the new truth-constants do not occur in $\Gamma_0 \cup \{\varphi\}$, we have $\Gamma_0 \models_{[0,1]_*} \varphi$, and by $\mathrm{FS}\mathcal{R}\mathrm{C}$ of L_*, $\Gamma_0 \vdash_{\mathrm{L}_*} \varphi$, and hence $\Gamma \vdash_{\mathrm{L}_*} \varphi$. □

As a consequence of Theorem 2.5.1, we obtain that $\mathrm{Ł}(\boldsymbol{C})$ enjoys the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}$, because the algebra $\boldsymbol{C}$ is simple and so there is only one real algebra up to isomorphisms: the canonical one. In the case of expansions of Product logic $\Pi(\boldsymbol{C})$, $\boldsymbol{C}$ (if it has more than two elements) has only two proper filters $F_1 = \{1\}$ and $F_2 = \{r \in C \mid r > 0\}$. Let us denote by $[0,1]^\star_{\Pi(\boldsymbol{C})}$ the $\Pi(\boldsymbol{C})$-algebra $[0,1]^{F_2}_{\Pi((\boldsymbol{C})}$. Then an immediate consequence is the following proposition.

PROPOSITION 2.5.3. *For any $\Pi(\boldsymbol{C})$-formula φ and any finite set of $\Pi(\boldsymbol{C})$-formulas Γ, we have $\Gamma \vdash_{\Pi(\boldsymbol{C})} \varphi$ iff $\Gamma \models_{[0,1]_{\Pi(\boldsymbol{C})}} \varphi$ and $\Gamma \models_{[0,1]^\star_{\Pi(\boldsymbol{C})}} \varphi$.*

As for the $\mathrm{S}\mathcal{R}\mathrm{C}$, we obtain the following results.

THEOREM 2.5.4. *L_* has the $\mathrm{S}\mathcal{R}\mathrm{C}$ if, and only if, $\mathrm{L}_*(\boldsymbol{C})$ has the $\mathrm{S}\mathcal{R}\mathrm{C}$.*

Proof. The right-to-left direction is a consequence of $\mathrm{L}_*(\boldsymbol{C})$ being a conservative expansion of L_* (Proposition 2.5.2). To prove the left-to-right direction, we follow an idea from [16, Lemma 3.4.4]. Assume L_* has the $\mathrm{S}\mathcal{R}\mathrm{C}$ property and that $\Gamma \nvdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$. Let BK the set of instances of the book-keeping axioms over $\boldsymbol{C}$. Then, it is easy to check that over L_* (i.e. considering the truth-constants as fresh propositional variables) φ remains not provable from $\Gamma \cup \mathrm{BK}$, i.e. $\Gamma \cup \mathrm{BK} \nvdash_{\mathrm{L}_*} \varphi$. Since L_* is $\mathrm{S}\mathcal{R}\mathrm{C}$, there is a real L_*-chain $\boldsymbol{A}$ and an evaluation e into $\boldsymbol{A}$ such that e is a model of $\Gamma \cup \mathrm{BK}$ and $e(\varphi) < 1^{\boldsymbol{A}}$. Now expand the signature of the algebra $\boldsymbol{A}$ with 0-ary operators $\overline{r}$, one for each $r \in C$, and set $\overline{r}^{\boldsymbol{A}} = e(\overline{r})$. The resulting algebra, call it $\boldsymbol{A}'$, is an $\mathrm{L}_*(\boldsymbol{C})$-chain, and e becomes an evaluation into $\boldsymbol{A}'$ such that it is a model of Γ but $e(\varphi) < 1^{\boldsymbol{A}} = 1^{\boldsymbol{A}'}$, and, consequently, $\Gamma \not\models_{\mathrm{L}_*(\boldsymbol{C})} \varphi$. □

As a consequence, $\mathrm{G}(\boldsymbol{C})$ has the $\mathrm{S}\mathcal{R}\mathrm{C}$ since Gödel logic has the $\mathrm{S}\mathcal{R}\mathrm{C}$. However, $\Pi(\boldsymbol{C})$ and $\mathrm{Ł}(\boldsymbol{C})$ enjoy the $\mathrm{FS}\mathcal{R}\mathrm{C}$, but they fail to satisfy the $\mathrm{S}\mathcal{R}\mathrm{C}$. In general we obtain the following result.

	$\mathrm{G}(\boldsymbol{C})$	$\Pi(\boldsymbol{C})$	Ł$(\boldsymbol{C})$	$\mathrm{L}_*(\boldsymbol{C})$
$\mathcal{R}$C	Yes	Yes	Yes	Yes
FS$\mathcal{R}$C	Yes	Yes	Yes	Yes
S$\mathcal{R}$C	Yes	No	No	No
Canonical FS$\mathcal{R}$C	No	No	Yes	No
Canonical S$\mathcal{R}$C	No	No	No	No

Table 1. Real completeness results for logics with truth-constants enjoying the partial embeddability property (where $*$ denotes a continuous t-norm which is a finite ordinal sum of at least two basic components).

THEOREM 2.5.5. *If* $*$ *is a continuous t-norm, then* $\mathrm{L}_*(\boldsymbol{C})$ *enjoys:*

(i) *the* S$\mathcal{R}$C *if, and only if,* $* = \min$*,*

(ii) *the* FS$\mathcal{R}$C *if* $* \in \mathbf{CONT\text{-}fin}$*,*

(iii) *the canonical* FS$\mathcal{R}$C *if, and only if,* $*$ *is the Łukasiewicz t-norm.*

Proof. First of all, since L_* does not have the S$\mathcal{R}$C when $[0,1]_*$ contains as a component a copy of Łukasiewicz or product t-norms, it is clear that in such a case also the logic $\mathrm{L}_*(\boldsymbol{C})$ does not have the S$\mathcal{R}$C. Thus, (i) is proved.

Result (ii) is an obvious consequence of Theorem 2.4.7.

To prove (iii), we show that if $\boldsymbol{C}$ has a non-trivial proper filter, then the logic $\mathrm{L}_*(\boldsymbol{C})$ does not enjoy the canonical FS$\mathcal{R}$C. Namely, since $F \neq \{1\}$, there exists $r \in F$, $r \notin \{0,1\}$. Then, the following semantical deduction[6] is valid over the canonical real $\mathrm{L}_*(\boldsymbol{C})$-chain but not over $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$:

$$(p \to q) \to \overline{r} \models q \to p.$$

To prove it, take into account that for every evaluation e over the canonical real chain, $e((p \to q) \to \overline{r}) = 1$ iff $e(p \to q) \leq r < 1$, and this implies $e(q) < e(p)$: so, the deduction is valid. However, over the chain $\boldsymbol{A} = [0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$, the formula $(p \to q) \to \overline{r}$ is always satisfied (remember that $\overline{r}^{\boldsymbol{A}} = 1$), and thus the deduction is not valid. Therefore, $q \to p$ is not provable from $(p \to q) \to \overline{r}$ in the logic $\mathrm{L}_*(\boldsymbol{C})$, and, consequently, this logic does not have the canonical FS$\mathcal{R}$C. Taking into account that $\boldsymbol{C}$ is simple if, and only if, $*$ is the Łukasiewicz t-norm, Theorem 2.4.6 proves (iii). $\square$

Notice that for a continuous t-norm $*$, $\mathrm{L}_*(\boldsymbol{C})$ does not have the canonical S$\mathcal{R}$C. Indeed, if $\mathrm{L}_*(\boldsymbol{C})$ had the canonical S$\mathcal{R}$C, then it would also enjoy the S$\mathcal{R}$C and the canonical FS$\mathcal{R}$C, which is impossible, as shown by the previous theorem.

All these completeness results are collected in Table 1.

[6]Actually there are simpler examples that could have been used in this proof, like $\overline{r} \models \overline{0}$, but the one chosen here will be useful later in Section 2.7.

2.6 About canonical real completeness

The logics $\mathrm{L}_*(\boldsymbol{C})$ considered in the last section do not have the canonical FS$\mathcal{R}$C, except for $\text{Ł}(\boldsymbol{C})$. However, some of them enjoy the weaker property of canonical $\mathcal{R}$C, i.e. their theorems are exactly the tautologies of their corresponding canonical algebra over $[0,1]$. In this section, we show which logics do have the canonical $\mathcal{R}$C.

THEOREM 2.6.1 ([30]). $\mathrm{G}(\boldsymbol{C})$ *has the canonical* $\mathcal{R}$C.

Proof. As usual, soundness is trivial. To prove completeness, suppose $\nvdash_{\mathrm{G}(\boldsymbol{C})} \varphi$. Then, by completeness of $\mathrm{G}(\boldsymbol{C})$ w.r.t. the $\mathrm{G}(\boldsymbol{C})$-chains, there exist a countable $\mathrm{G}(\boldsymbol{C})$-chain $\boldsymbol{A}$ and an evaluation e over $\boldsymbol{A}$ such that $e(\varphi) < \overline{1}^{\boldsymbol{A}}$. We have to show there is an evaluation e' on the real algebra $[0,1]_{\mathrm{G}(\boldsymbol{C})}$ such that $e'(\varphi) < 1$.

Let $s = \min\{r \in C \mid r = 1 \text{ or } \overline{r} \text{ subformula of } \varphi \text{ with } \overline{r}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}\}$. Clearly $s > 0$. Let $g\colon A \to [0,s]$ be an order-preserving injection such that $g(\overline{0}^{\boldsymbol{A}}) = 0$, $g(\overline{1}^{\boldsymbol{A}}) = s$ and $g(\overline{r}^{\boldsymbol{A}}) = r$ for $\overline{r}$ a subformula of φ with $r < s$. Then, we define a $\mathrm{G}(\boldsymbol{C})$-evaluation e' on the real $\mathrm{G}(\boldsymbol{C})$-algebra $[0,1]$ as follows: for all propositional variables p, $e'(p) = g(e(p))$. Then e' is extended to $\mathrm{G}(\boldsymbol{C})$-formulas as usual (of course with $e'(\overline{r}) = r$, for each $r \in \boldsymbol{C}$).

CLAIM 2.6.2. *For each* ψ *subformula of* φ*:*

(1) *if* $e(\psi) = \overline{1}^{\boldsymbol{A}}$ *then* $e'(\psi) \geq s$*,*

(2) *if* $e(\psi) < \overline{1}^{\boldsymbol{A}}$ *then* $e'(\psi) = g(e(\psi)) < s$*.*

Proof. The claim is clear for variables and for truth-constants $\overline{r}$ subformulas of φ. The induction step for $\wedge$ is trivial. Let us consider the case of $\to$. If $e(\gamma \to \delta) = e(\delta) < \overline{1}^{\boldsymbol{A}}$ then $e'(\delta) = g(e(\delta)) < s$. Now, if $e(\gamma) = \overline{1}^{\boldsymbol{A}}$ then $e'(\gamma) \geq s$ and $e'(\gamma \to \delta) = e'(\delta) < s$; and if $e(\gamma) < \overline{1}^{\boldsymbol{A}}$ then $e'(\gamma) = g(e(\gamma)) > g(e(\delta)) = e'(\delta)$, thus again $e'(\gamma \to \delta) = e'(\delta) < s$. On the other hand, assume $e(\gamma \to \delta) = \overline{1}^{\boldsymbol{A}}$, thus $e(\gamma) \leq e(\delta)$. If $e(\delta) = \overline{1}^{\boldsymbol{A}}$ then $e'(\gamma \to \delta) \geq e'(\delta) \geq s$. And if $e(\delta) < \overline{1}^{\boldsymbol{A}}$ then $e'(\gamma) = g(e(\gamma)) \leq g(e(\delta)) = e'(\delta)$ and $e'(\gamma \to \delta) = 1 \geq s$. This proves the claim. □

This also finishes the proof of the theorem; indeed, since $e(\varphi) < \overline{1}^{\boldsymbol{A}}$, then $e'(\varphi) < 1$ as required. □

THEOREM 2.6.3 ([78]). $\Pi(\boldsymbol{C})$ *has the canonical* $\mathcal{R}$C.

Proof. Let φ be a $\Pi(\boldsymbol{C})$ formula such that $\nvdash_{\Pi(\boldsymbol{C})} \varphi$. We can further assume that φ contains some truth constant $\overline{r}$ with $0 < r < 1$ as a subformula, otherwise the real completeness of product logic does the job. By general completeness, there is a linearly ordered $\Pi(\boldsymbol{C})$-algebra $\boldsymbol{A}$ and an evaluation e on $\boldsymbol{A}$ such that $e(\varphi) < \overline{1}^{\boldsymbol{A}}$. The task is to find an evaluation e' on the canonical real $\Pi(\boldsymbol{C})$-algebra $[0,1]_{\Pi(\boldsymbol{C})}$ such that $e'(\varphi) < 1$. Let $E = \{e(\psi) \mid \psi \text{ is a subformula of } \varphi\} \cup \{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$. We consider the following cases:

Case 1: $\boldsymbol{A}$ is of type I.

By applying Proposition 2.4.3 we obtain a partial embedding h of E into $[0,1]$. Now define a $[0,1]_{\Pi(\boldsymbol{C})}$-evaluation e' by putting

$$e'(p) = \begin{cases} h(e(p)) & \text{if } p \text{ is a propositional variable in } \varphi, \\ arbitrary & \text{otherwise.} \end{cases}$$

It is easy to check, by the properties of h, that $e'(\varphi) = h(e(\varphi)) < 1$.

Case 2: $\boldsymbol{A}$ is of type II.

By well-known results on Π-algebras (see [13]), there is a partial embedding f of E into the real Π-algebra $[0,1]_\Pi$ and the evaluation v on $[0,1]_\Pi$ defined as follows

$$v(p) = \begin{cases} f(e(p)) & \text{if } p \text{ is a propositional variable in } \varphi, \\ arbitrary & \text{otherwise,} \end{cases}$$

is such that $v(\varphi^*) < 1$, where φ^* is the Π-formula obtained from φ by replacing all truth-constants $\overline{r}$ with $0 < r$ by $\overline{1}$. Now, the evaluation v' on the real $\Pi(\boldsymbol{C})$-algebra of type II such that $v'(p) = v(p)$ for all propositional variables p satisfies $v(\varphi^*) = v'(\varphi) < 1$. Then, by Proposition 2.3.4, there is also an evaluation e' on the canonical real $\Pi(\boldsymbol{C})$-algebra $[0,1]_{\Pi(\boldsymbol{C})}$ such that $e'(\varphi) < 1$. This ends the proof of Case 2 and of the theorem as well. □

Notice that the canonical $\mathcal{R}$C is not valid in general for expansions of other logics of a continuous t-norm. First, we will show that the canonical $\mathcal{R}$C fails for a large family of logics giving a counterexample, i.e. showing a formula φ that is a tautology of the canonical real algebra but not of the algebra $[0,1]^F_{\text{Ł}_*(\boldsymbol{C})}$ for some proper filter F of $\boldsymbol{C}$. Suppose that the first component of $[0,1]_*$ is defined on the interval $[0,a]$.

(1) If the first component of the t-norm $*$ is a copy of Łukasiewicz t-norm (and $a \in C$), then, an easy computation shows that the formula

$$\overline{a} \to (\neg\neg p \to p)$$

is valid in the canonical real algebra but is not valid in the real chain defined by the filter $F = [a,1] \cap C$ (where $\overline{a}$ is interpreted as 1).

(2) If the first component of the t-norm $*$ is a copy of product t-norm, take b as any element of $C \cap (0,a)$. Then, an easy computation shows that the formula

$$\overline{b} \to \neg p \vee ((p \to p \,\&\, p) \to p)$$

is valid in the canonical real algebra but is not valid in the real chain defined by the filter $F = (0,1] \cap C$ (where $\overline{b}$ is interpreted as 1).

(3) If the first component is the minimum t-norm, take b as any element of $C \cap (0, a)$. Then, the formula

$$\overline{b} \to (p \to p \,\&\, p)$$

is valid in the canonical real algebra but is not valid in the real chain where $\overline{b}$ is interpreted as 1.

Observe that for a t-norm whose decomposition begins with two copies of the Łukasiewicz t-norm, the idempotent element a separating the two components must belong to the truth-constants subalgebra $\boldsymbol{C}$. Indeed, take into account that, by assumption, C must contain a-non idempotent element c of the second component, and for this element there exists a natural number n such that $c^n = a$ and thus $a \in C$. Hence, this case is subsumed in the above first item.

The remaining cases (when the first component is Łukasiewicz but its upper bound a does not belong to C) will be divided in two different groups:

(1) If $[0, 1]_* = [0, a]_{Ł} \oplus [a, 1]_{\mathrm{G}}$ or $[0, 1]_* = [0, a]_{Ł} \oplus [a, 1]_{\Pi}$, then the logic $\mathrm{L}_*(\boldsymbol{C})$ has the canonical $\mathcal{R}$C. Actually, in that case, the filters of $\boldsymbol{C}$ are the same as the filters of $C \cap [a, 1]_{\mathrm{G}}$ or $C \cap [a, 1]_{\Pi}$ respectively. Therefore, a modified version (given in the next two theorems) of the proof of the canonical $\mathcal{R}$C for $\mathrm{G}(\boldsymbol{C})$ and $\Pi(\boldsymbol{C})$ applies.

(2) If $[0, 1]_*$ is an ordinal sum of three or more components, then $\mathrm{L}_*(\boldsymbol{C})$ does not have the canonical $\mathcal{R}$C, as the following examples show:

(2.1) If $[0, 1]_* = [0, a]_{Ł} \oplus [a, b]_{\mathrm{G}} \oplus \boldsymbol{A}$, take $d \in F = (a, 1] \cap C$ in the second component. Then the formula

$$\overline{d} \to (\neg\neg p \to p) \vee (p \to p \,\&\, p)$$

is a tautology of the canonical real algebra but not of $[0, 1]^F_{\mathrm{L}_*(\boldsymbol{C})}$.

(2.2) If $[0, 1]_* = [0, a]_{Ł} \oplus [a, b]_{\Pi} \oplus \boldsymbol{A}$, take $d \in F = (a, b] \cap C$ in the second component. Then the formula

$$\begin{aligned} \overline{d} \to [(\neg\neg p \to p) \vee (\neg\neg q \to q) \vee (p \to p \,\&\, p) \\ \vee (q \to q \,\&\, q) \vee ((p \to p \,\&\, q) \to q)] \end{aligned}$$

is a tautology of the canonical real algebra and not of $[0, 1]^F_{\mathrm{L}_*(\boldsymbol{C})}$.[7]

The remaining cases enjoy the $\mathcal{R}$C.

THEOREM 2.6.4. *Let either* $[0, 1]_* = [0, a]_{Ł} \oplus [a, 1]_{\Pi}$ *or* $[0, 1]_* = [0, a]_{Ł} \oplus [a, 1]_{\mathrm{G}}$. *Then the logic* $\mathrm{L}_*(\boldsymbol{C})$ *has the canonical* $\mathcal{R}$C *if, and only if, the minimum element of the second component does not belong to* C.

[7] We thank Franco Montagna, who pointed out that the formula in [26], claimed to be a tautology of the canonical real algebra and not of $[0, 1]^F_{\mathrm{L}_*(\boldsymbol{C})}$, does not satisfy the required conditions. Here, we provide a new formula satisfying the conditions and prove that the claimed result is true.

$[0,1]_*$	Canonical $\mathcal{R}$C for $\mathrm{L}_*(\boldsymbol{C})$
$[0,1]_{Ł}$	Yes
$[0,1]_{\mathrm{G}}$	Yes
$[0,1]_{\Pi}$	Yes
$[0,a]_{Ł} \oplus [a,1]_{\mathrm{G}}$, $\quad a \notin C$	Yes
$[0,a]_{Ł} \oplus [a,1]_{\Pi}$, $\quad a \notin C$	Yes
$[0,a]_*$, for other $* \in \mathbf{CONT\text{-}fin}$	No

Table 2. Canonical real completeness results for logics $\mathrm{L}_*(\boldsymbol{C})$ when $*$ is a finite ordinal sum of the three basic components.

Proof. The proof can easily be obtained by combining the proofs for $\mathrm{G}(\boldsymbol{C})$, $\Pi(\boldsymbol{C})$ and $Ł(\boldsymbol{C})$ (see [26, Theorems 17 and 18] for the details). □

Summarizing (see Table 2), the canonical $\mathcal{R}$C holds for the expansion of the logic of a continuous t-norm $*$, which is a finite ordinal sum of the three basic ones, by a set of truth-constants if, and only if, $[0,1]_*$ is either one of the three basic algebras ($[0,1]_{Ł}$, $[0,1]_{\mathrm{G}}$ or $[0,1]_{\Pi}$) or $[0,1]_* = [0,a]_{Ł} \oplus [a,1]_{\Pi}$ or $[0,1]_* = [0,a]_{Ł} \oplus [a,1]_{\mathrm{G}}$ (with $a \notin C$).

2.7 Completeness results for evaluated formulas

This section deals with completeness results when we restrict to what we call *evaluated formulas*, i.e., formulas of type $\overline{r} \to \varphi$, where φ is a formula without new truth-constants. We denote by $\mathcal{R}\mathrm{C}_{ev}$, $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$ and $\mathrm{S}\mathcal{R}\mathrm{C}_{ev}$ the restriction of the properties we have been studying in the previous section to evaluated formulas. From the previous sections, we know that FS$\mathcal{R}$C is true for the expansion of L_* with a subalgebra of truth-constants (not only for evaluated formulas), but the canonical FS$\mathcal{R}$C is only true for expansions of Łukasiewicz logic. The next theorem states the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$ for the expansions of Gödel and Product logics with truth-constants.

THEOREM 2.7.1. $\mathrm{G}(\boldsymbol{C})$ *and* $\Pi(\boldsymbol{C})$ *have the canonical* $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$*, i.e., for any formulas* $\varphi_1, \ldots, \varphi_n, \psi$ *and values* $r_1, \ldots, r_n, s \in C$*, and* $\Gamma = \{\overline{r}_i \to \varphi_i \mid 1 \le i \le n\}$*, we have:*

(i) $\Gamma \vdash_{\mathrm{G}(\boldsymbol{C})} \overline{s} \to \psi$ *if, and only if,* $\Gamma \models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \overline{s} \to \psi$.

(ii) $\Gamma \vdash_{\Pi(\boldsymbol{C})} \overline{s} \to \psi$ *if, and only if,* $\Gamma \models_{[0,1]_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$.

Proof. (i) We start by stating the following previous result whose proof is not difficult (see [30] for the details).

CLAIM 2.7.2. *Let* $a \in (0,1]$ *and define a mapping* $f_a \colon [0,1] \to [0,1]$ *as follows:*

$$f_a(x) = \begin{cases} 1 & \text{if } x \ge a, \\ x & \text{otherwise.} \end{cases}$$

Then f_a *is a homomorphism of real Gödel chains. Therefore, if* e *is a* G*-evaluation of formulas, then* $e_a = f_a \circ e$ *is another* G*-evaluation.*

To prove the statement it is enough to show the following:

$$\Gamma \models_{[0,1]_{G(\boldsymbol{C})}} \overline{s} \to \psi \text{ iff } \models_{[0,1]_{G(\boldsymbol{C})}} (\bigwedge_{i=1}^{n} (\overline{r_i} \to \varphi_i)) \to (\overline{s} \to \psi).$$

One direction is easy. As for the non trivial one, it is enough to prove that if there is an evaluation e which is not a model of $(\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$, then we can find another evaluation e' that is a model of $\{\overline{r_1} \to \varphi_1, \ldots, \overline{r_n} \to \varphi_n\}$ and not of $\overline{s} \to \psi$.

So, let e be such that $e((\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)) < 1$. If e is a model of every $\overline{r_i} \to \varphi_i$ for $i = 1, \ldots, n$, then we can take $e' = e$ and the problem is solved. Otherwise, there exists some $1 \leq j \leq n$ for which $r_j > e(\varphi_j)$ and thus $e(\overline{r_j} \to \varphi_j) = e(\varphi_j) < 1$. Let $J = \{j \mid r_j > e(\varphi_j)\}$ and $a = e(\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) = \min\{e(\varphi_j) \mid j \in J\}$. Then, the $G(\boldsymbol{C})$-evaluation e' such that $e' = e_a$ over the propositional variables does the job. Namely, by Claim 2.7.2, over Gödel formulas we have $e' = e_a \geq e$, so e' is still model of those $\overline{r_i} \to \varphi_i$'s for $i \in \{1, \ldots, n\} \setminus J$. But now, $e'(\varphi_j) = 1$ for every $j \in J$, so e' is also a model of $(\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$. On the other hand, since $e((\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)) < 1$, it must be $s > e(\psi)$ and $a = e(\bigwedge_{i=1}^{n} \overline{r_i} \to \varphi_i) > e(\psi)$. Now, by the above claim, $e'(\psi) = e_a(\psi) = e(\psi)$, hence $e'(\overline{s} \to \psi)) = e(\overline{s} \to \psi) < 1$.

(ii) Due to Corollary 2.5.3, we only need to prove that if $\Gamma \models_{[0,1]_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$ then $\Gamma \models_{[0,1]^{\star}_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$. Without loss of generality we may assume $r_i > 0$ for all i and $s > 0$. Suppose $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]^{\star}_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$. Then, there exists a $[0,1]^{\star}_{\Pi(\boldsymbol{C})}$-evaluation e such that $e(\overline{r_1} \to \varphi_1) = \cdots = e(\overline{r_n} \to \varphi_n) = 1$ and $e(\overline{s} \to \psi) < 1$. Since $e(r_i) = e(s) = 1$ for all i, we also have $e(\varphi_1) = \cdots = e(\varphi_n) = 1$ and $e(\psi) < 1$.

Assume $e(\psi) = 0$. Then, letting e' be the $[0,1]_{\Pi(\boldsymbol{C})}$-evaluation defined by $e'(p) = e(p)$ for any propositional variable p, we have $1 = e'(\overline{r_1} \to \varphi_1) = \cdots = e'(\overline{r_n} \to \varphi_n)$ and $e'(\psi) = 0$, hence $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$.

Assume $e(\psi) > 0$. Let $\alpha \in \mathsf{R}^+$ so that $(e(\psi))^\alpha < s$.[8] Then, the $[0,1]_{\Pi(\boldsymbol{C})}$-evaluation e', where $e'(p) = (e(p))^\alpha$ for any propositional variable p, is such that $e'(\overline{r_i} \to \varphi_i) = 1$ for all i but $e'(\overline{s} \to \psi) < 1$, and thus $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]_{\Pi(\boldsymbol{C})}} \overline{s} \to \psi$. □

One could wonder whether these restricted completeness results hold for formulas of type $\varphi \to \overline{r}$ such that φ does not contain a truth-constant different from $\overline{0}$ and $\overline{1}$. Actually, the situation is different for $G(\boldsymbol{C})$ and $\Pi(\boldsymbol{C})$:

- As for $G(\boldsymbol{C})$, the result does not hold. It is easy to check that

$$\neg\neg p \to \overline{0.7} \models_{[0,1]_{G(\boldsymbol{C})}} p \to \overline{0.2},$$

since the premise is only true if $e(p) = 0$, while

$$\neg\neg p \to \overline{0.7} \not\vdash_{G(\boldsymbol{C})} p \to \overline{0.2}.$$

[8] Observe that $e(\psi) \in [0,1]$ and thus $(e(\psi))^\alpha$ is the usual α power of $e(\psi)$. Notice that, for any real α, the mappings $f(x) = x^\alpha$ are automorphisms of $[0,1]_\Pi$.

In fact, by the deduction-detachment theorem and the canonical $\mathcal{R}$C of the logic $\mathrm{G}(\boldsymbol{C})$ this is equivalent to show that

$$\not\models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} (\neg\neg p \to \overline{0.7}) \to (p \to \overline{0.2}),$$

which is true, since, if $e(p) = c$ for $c > 0.2$, an easy computation shows that $e((\neg\neg p \to \overline{0.7}) \to (p \to \overline{0.2})) = 0.2$.

- As for $\Pi(\boldsymbol{C})$, the result holds true when the formulas $\varphi \to \overline{r}$ are such that $r > 0$ (see [78]), since in such a case these formulas are trivially satisfied in the non-canonical real $\Pi(\boldsymbol{C})$-algebra $[0,1]^{F}_{\Pi(\boldsymbol{C})}$ for $F = (0,1]$.

In any case, the result is not true if we allow formulas of both types together. Indeed, given $r \neq 1$, it is obvious that the semantical deduction (already used in the proof of Theorem 2.5.5)

$$(p \to q) \to \overline{r} \models \overline{1} \to (q \to p)$$

is valid over the canonical real chain but not over a real chain where $\overline{r}$ is interpreted as 1.

Now we will study the canonical $\mathcal{R}\mathrm{C}_{ev}$ and the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$ for other logics. Suppose that $*$ is a t-norm that is a non-trivial finite ordinal sum of the basic components, and suppose that the first component is defined on the interval $[0,a]$. For the following cases we can refute the canonical $\mathcal{R}\mathrm{C}_{ev}$ (and hence the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$ as well):

(1) The first component of the t-norm $*$ is a copy of the Łukasiewicz t-norm and $a \in C$.

(2) The first component of the t-norm $*$ is a copy of the product t-norm.

(3) The first component of the t-norm $*$ is a copy of the minimum t-norm.

(4) There are more than two components and the second component is a copy of the minimum t-norm.

(5) There are more than two components and the second component is a copy of the product t-norm.

Indeed, for all these cases we can use the same counterexample that was given in the previous section to show that the corresponding logics do not enjoy canonical $\mathcal{R}\mathrm{C}_{ev}$, because the counterexamples were actually evaluated formulas.

The following theorem deals with the remaining case of ordinal sums of two basic components. The case $[0,1]_* = [0,a]_{\text{Ł}} \oplus [a,1]_{\text{Ł}}$ is not considered here since in such a situation, under the working hypothesis that there exists $b \in (a,1]$ such that $b \in C$, $a \in C$ necessarily as well.

THEOREM 2.7.3. *The restriction to evaluated formulas of the logic* $\mathrm{L}_*(\boldsymbol{C})$*, when either* $[0,1]_* = [0,1]_{\text{Ł}} \oplus [0,1]_{\mathrm{G}}$ *or* $[0,1]_* = [0,1]_{\text{Ł}} \oplus [0,1]_{\Pi}$ *and the minimum element of the second component does not belong to C, has the canonical* $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$.

Proof. The proof is an easy modification of the proofs given in [30] for $\mathrm{G}(\boldsymbol{C})$ and in [78] for $\Pi(\boldsymbol{C})$. Here, we only sketch the proof for $[0,1]_* = [0,1]_{Ł} \oplus [0,1]_{\Pi}$. Let $\Gamma = \{\overline{r_1} \to \varphi_1, \ldots, \overline{r_n} \to \varphi_n\}$. What we want to prove is:

$$\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \overline{s} \to \psi \quad \text{if, and only if,} \quad \Gamma \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$$

where φ_i and ψ are L_*-formulas, i.e., formulas not containing truth-constants different from $\overline{0}$ and $\overline{1}$. Actually, as always, one direction (soundness) is obvious. To prove the converse direction

$$\text{if } \Gamma \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi, \text{ then } \Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \overline{s} \to \psi$$

it is enough to combine the $\mathrm{FS}\mathcal{R}\mathrm{C}$ of $\mathrm{L}_*(\boldsymbol{C})$ with the following result:

CLAIM 2.7.4. *If* $\Gamma \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$ *then* $\Gamma \models_{[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$*, where* $F = (a,1] \cap C$ *and* a *is the idempotent separating the first and second component of* $*$.

To prove it, without loss of generality, we may assume $r_i > 0$ for all i and $s > 0$. Suppose $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$. Then, there exists a $[0,1]^F_{\mathrm{L}_*(\boldsymbol{C})}$-evaluation e such that $e(\overline{r_1} \to \varphi_1) = \cdots = e(\overline{r_n} \to \varphi_n) = 1$ and $e(\overline{s} \to \psi) < 1$. Then we consider the following two cases:

(i) If $s \in (0,a]$, and hence $e(\overline{s}) = s$ and $e(\psi) < s$, take the evaluation e' over the canonical real chain defined by $e'(p) = e(p)$ for any propositional variable p. Notice that, since $e(\overline{r}) \geq e'(\overline{r})$ and $e(\varphi) = e'(\varphi)$, it is easy to compute that $e'(\overline{r_1} \to \varphi_1) = \cdots = e'(\overline{r_n} \to \varphi_n) = 1$ and $e'(\overline{s} \to \psi) = e(\overline{s} \to \psi) < 1$.

(ii) If $s \in (a,1]$, and hence $e(\overline{s}) = 1$ and $e(\psi) < 1$, we can assume $e(\psi) \geq s$, otherwise the above evaluation e' does the job. Then, take the family of evaluations e'_t (being t any natural number) over the canonical real chain defined by $e'_t(p) = k_t(e(p))$ for any propositional variable p, where $k_t\colon [0,1] \to [0,1]$ is the mapping

$$k_t(z) = \begin{cases} z & \text{if } z \in [0,a], \\ h^{-1}((h(z))^t) & \text{otherwise,} \end{cases}$$

where h is a bijection from $[a,1]$ to $[0,1]$ (e.g. the one defined by $h(x) = \frac{x-a}{1-a}$). By definition of k_t, it is easy to find a large enough t such that $a < e'_t(\psi) < s$, and hence $e'_t(\overline{s} \to \psi) < 1$. Moreover, it is easy to check that we still have $e'_t(\overline{r_1} \to \varphi_1) = \cdots = e'_t(\overline{r_n} \to \varphi_n) = 1$. Indeed, if $r_i \in (a,1]$, then $e(\overline{r}_i) = 1$ and $e(\varphi) = 1$, hence $e'_t(\varphi) = 1$ as well. If $r_i \in (0,a]$, then $e'_t(\overline{r_i}) = e(\overline{r_i}) = r_i$ and $e(\varphi_i) \geq r_i$. Now, if $e(\varphi_i) \leq a$ then $e'_t(\varphi_i) = e(\varphi_i)$, otherwise, if $e(\varphi_i) > a$ then $e'_t(\varphi_i) > a$ as well. In any case, $e'_t(\varphi_i) \geq r_i$, hence $e'_t(\overline{r_i} \to \varphi_i) = 1$.

Therefore in both cases $\{\overline{r}_1 \to \varphi_1, \ldots, \overline{r}_n \to \varphi_n\} \not\models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \overline{s} \to \psi$ and hence the claim and the theorem are proved. □

A final (partially negative) result that deserves some comments concerns the $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$ property. This property obviously fails for those logics $\mathrm{L}_*(\boldsymbol{C})$ such that L_* does not enjoy the $\mathrm{S}\mathcal{R}\mathrm{C}$: therefore it clearly makes sense to investigate what happens with the logics $\mathrm{G}(\boldsymbol{C})$. When $C = [0,1] \cap \mathsf{Q}$, it is easy to notice that all the logics $\mathrm{L}_*(\boldsymbol{C})$ under our scope fail to satisfy the $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$, as it can be seen with the following counterexample. Let $\Gamma = \{\overline{(\frac{n}{n+1})} \to \varphi \mid n \in \mathsf{N}\}$. For every logic $\mathrm{L}_*(\boldsymbol{C})$ we have $\Gamma \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \varphi$. If $\Gamma \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$ then, since the logic is finitary, there would exist $n_0 \in \mathsf{N}$ such that $\overline{(\frac{n_0}{n_0+1})} \to \varphi \vdash_{\mathrm{L}_*(\boldsymbol{C})} \varphi$, hence, we would have $\overline{(\frac{n_0}{n_0+1})} \to \varphi \models_{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}} \varphi$, i.e. a contradiction. An analogous counter-example works as well for the case when the algebra $\boldsymbol{C}$ has an accumulation point r that is the supremum of a strictly increasing sequence $(r_i)_{i \in \mathsf{N}}$ of points of $\boldsymbol{C}$. We call sup-*accessible* such an accumulation point r. Notice that for expansions $\mathrm{G}(\boldsymbol{C})$ where $\boldsymbol{C}$ does not have sup-accessible points, the $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$ holds [32, Theorem 6]: there the theorem is proved for rational semantics, but the same proof also works for the real semantics.

THEOREM 2.7.5. *The logic* $\mathrm{G}_*(\boldsymbol{C})$ *where* C *does not have* sup-*accessible points has the* $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$.

Proof. Soundness is obvious as usual. For completeness we have to prove that if a (possibly infinite) family of evaluated formulas $\{\overline{r}_i \to \varphi_i \mid i \in I\}$ does not prove an evaluated formula $\overline{s} \to \psi$ then there is an evaluation v over the canonical chain such that for every $i \in I$, $v(\overline{r}_i \to \varphi_i) = 1$ and $v(\overline{s} \to \psi) < 1$.

By the algebraizability of the logic with truth-constants, if the syntactical deduction is not valid there is a countable $\mathrm{G}(\boldsymbol{C})$-chain $\boldsymbol{A}$ and an evaluation e over it such that, for every $i \in I$, $e(\overline{r}_i \to \varphi_i) = \overline{1}^{\boldsymbol{A}}$ and $e(\overline{s} \to \psi) < \overline{1}^{\boldsymbol{A}}$. Suppose this is a chain of type F, that is, F is a filter of $\boldsymbol{C}$ such that for every $r \in F$, $\overline{r}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}$. Observe that, since the elements of C are not sup-accessible, for each point $r \in C$ there is an interval $I_r^- = (r - \delta, r)$ (with countably many elements) such that $I_r^- \cap C = \emptyset$. To build the desired evaluation v we need to study two cases:

(1) Suppose $s \in F$. In such a case, define the mapping $f\colon A \to [0,1]$ as follows: $f(\overline{1}^{\boldsymbol{A}}) = 1, f(\overline{0}^{\boldsymbol{A}}) = 0$ and f restricted to $A \setminus \{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$ is an embedding into I_s^-. An easy computation shows that f is a morphism of G-chains (without truth-constants). Define the $[0,1]_{\mathrm{G}(\boldsymbol{C})}$-evaluation v as $v(p) = f(e(p))$ for every propositional variable p. Such a v satisfies the required conditions since: if $r_i \in F$ then $v(\varphi_i) = e(\varphi_i) = 1 \geq r_i$, and if $r_i \notin F$ then $v(\varphi_i) \in \{1\} \cup I_s^-$, and thus $v(\varphi_i) \geq r_i$ as well. Moreover, since $e(\psi) < 1$, we have $v(\psi) \in I_s^- \cup \{0\}$ and thus $v(\psi) < s$.

(2) Suppose $s \notin F$. In such a case, define the mapping $f\colon A \to [0,1]$ as follows: $f(\overline{1}^{\boldsymbol{A}}) = 1, f(\overline{0}^{\boldsymbol{A}}) = 0$ and $f(\overline{s}^{\boldsymbol{A}}) = s$ and f restricted to $(\overline{s}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}})$ is an embedding into I_1^- and f restricted to $(\overline{0}^{\boldsymbol{A}}, \overline{s}^{\boldsymbol{A}})$ is an embedding into I_s^-. An easy computation shows that f is a morphism of G-chains (without truth-constants). Define the $[0,1]_{\mathrm{G}(\boldsymbol{C})}$-evaluation v as $v(p) = f(e(p))$ for every propositional variable p. Such a v satisfies the required conditions since: if $r_i \in F$, then $v(\varphi_i) = e(\varphi_i) = 1 \geq r_i$; if $r_i \notin F$, $r_i > s$, then $e(\varphi_i) > \overline{s}^{\boldsymbol{A}}$ and thus $v(\varphi_i) \in I_1^- \cup \{1\}$, which implies $v(\varphi_i) \geq r_i$; if there is some $r_i = s$, obviously $v(\varphi_i) \geq s$; if $r_i < s$ then $v(\varphi_i) \in \{1\} \cup I_1^- \cup I_s^-$, which implies $v(\varphi_i) \geq r_i$. Finally, since $e(\psi) < \overline{s}^{\boldsymbol{A}}$, we have $v(\psi) \in I_s^- \cup \{0\}$ and thus $v(\psi) < s$. $\square$

$[0,1]_*$	$\mathrm{Can}\mathcal{R}\mathrm{C}_{ev}$	$\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$	$\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$
$[0,1]_{Ł}$	Yes	Yes	No
$[0,1]_{\mathrm{G}}, C \notin \mathit{SupAcc}$	Yes	Yes	Yes
$[0,1]_{\mathrm{G}}, C \in \mathit{SupAcc}$	Yes	Yes	No
$[0,1]_{\Pi}$	Yes	Yes	No
$[0,a]_{Ł} \oplus [a,1]_{\mathrm{G}}, a \notin C$	Yes	Yes	No
$[0,a]_{Ł} \oplus [a,1]_{\Pi}, a \notin C$	Yes	Yes	No
other cases	No	No	No

Table 3. Canonical $\mathrm{S}\mathcal{R}\mathrm{C}_{ev}$ and $\mathrm{FS}\mathcal{R}\mathrm{C}_{ev}$ results for logics $\mathrm{L}_*(\boldsymbol{C})$ when $*$ is a finite ordinal sum of the three basic components.

All the completeness results for evaluated formulas are summarized in Table 3, where we denote by SupAcc the set of countable subalgebras of $[0,1]_*$ with sup-accessible accumulation points. Interestingly, it turns out that both the $\mathrm{Can}\mathcal{R}\mathrm{C}_{ev}$ and $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$ properties restricted to evaluated formulas become equivalent. Furthermore, comparing this table with Table 2, we realise that for a logic $\mathrm{L}_*(\mathcal{C})$ where $*$ is a finite ordinal sum of basic components, $\mathrm{Can}\mathcal{R}\mathrm{C}$ turns out to be equivalent to $\mathrm{Can}\mathcal{R}\mathrm{C}_{ev}$ (and to $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$).

2.8 Forcing the canonical interpretation of truth-constants: two approaches

In the previous subsections we have studied the logics $\mathrm{L}_*(\boldsymbol{C})$ obtained by adding truth-constants to logics of a continuous t-norms following Hájek's approach with the book-keeping axioms. One of the main drawbacks of these systems (with the exception of the expansions of Łukasiewicz logic) is the fact that different truth-constants can be interpreted to the same value. In this section, we introduce two approaches that overcome this problem and force the canonical interpretation of truth-constants (modulo an isomorphism).

2.8.1 Using the $\triangle$ operator

One possible solution is to further expand the logics with the Monteiro–Baaz $\triangle$ operator. Indeed, for every continuous t-norm $*$, we can consider the expansion of the logic L_* with $\triangle$, denoted $\mathrm{L}_{*\triangle}$. The reader may consult Chapter I for more details: there, $\mathrm{L}_{*\triangle}$ is shown to be a conservative expansion of L_*. For these expansions the following results hold (see for instance [16, 46]):

(i) The logics $\mathrm{L}_{*\triangle}$ enjoy the $\mathrm{FS}\mathcal{R}\mathrm{C}$;

(ii) The logics $\mathrm{L}_{*\triangle}$ enjoy the $\mathrm{S}\mathcal{R}\mathrm{C}$ if, and only if, $* = \min$.

Now, we will consider expansions with truth-constants for these logics with $\triangle$. Given a continuous t-norm $*$ and a countable subalgebra $\boldsymbol{C} \subseteq [0,1]_*$, we define the logic $\mathrm{L}_{*\triangle}(\boldsymbol{C})$ as the expansion of $\mathrm{L}_{*\triangle}$ in the language $\mathcal{L}_C$ obtained by adding the following book-keeping axioms:

$$\overline{r} \,\&\, \overline{s} \leftrightarrow \overline{r * s} \qquad (\overline{r} \to \overline{s}) \leftrightarrow \overline{r \Rightarrow_* s} \qquad \triangle\overline{r} \leftrightarrow \overline{\triangle(r)}$$

for every $r, s \in C$. Here, we use the symbol $\triangle$ to denote the truth function on $\boldsymbol{C}$, i.e. $\triangle(1) = 1$ and $\triangle(r) = 0$ for each $r \in C \setminus \{1\}$.

Since $\text{L}_{*\triangle}(\boldsymbol{C})$ is an expansion of $\text{L}_{*\triangle}$ with no new rules of inference then, by [21], $\text{L}_{*\triangle}(\boldsymbol{C})$ is a $\triangle$-core fuzzy logic. As a consequence, any $\text{L}_{*\triangle}(\boldsymbol{C})$-algebra is a subdirect product of chains, and so the logic $\text{L}_{*\triangle}(\boldsymbol{C})$ is complete not only with respect to the full variety of $\text{L}_{*\triangle}(\boldsymbol{C})$-algebras, but also with respect to the class of chains of the variety.

PROPOSITION 2.8.1. *For every continuous t-norm $*$ and every countable subalgebra $\boldsymbol{C} \subseteq [0,1]_*$, the logic $\text{L}_{*\triangle}(\boldsymbol{C})$ is a conservative expansion of $\text{L}_{*\triangle}$.*

Proof. It is analogous to the proof of Proposition 2.5.2. □

LEMMA 2.8.2. *Let $\boldsymbol{A}$ be a non-trivial $\text{L}_{*\triangle}(\boldsymbol{C})$-chain. Then, for every $r, s \in C$ such that $r < s$, we have $\overline{r}^{\boldsymbol{A}} < \overline{s}^{\boldsymbol{A}}$.*

Proof. If $r < s$ and $\overline{r}^{\boldsymbol{A}} = \overline{s}^{\boldsymbol{A}}$, then $\overline{1}^{\boldsymbol{A}} = \triangle\overline{1}^{\boldsymbol{A}} = \triangle(\overline{s \to r}^{\boldsymbol{A}}) = \overline{\triangle(s \to r)}^{\boldsymbol{A}} = \overline{0}^{\boldsymbol{A}}$, a contradiction. □

Therefore, if $*$ is a finite ordinal sum of Łukasiewicz and Product components, there is only one (up to isomorphism) real chain, the canonical one, that we denote by $[0,1]_{\text{L}_{*\triangle}(\boldsymbol{C})}$. The result is not true for a continuous t-norm containing a Gödel component, as the counterexample at the end of Section 2.3 shows (with the obvious changes). Nevertheless, similar to the case of $\text{L}_*(\boldsymbol{C})$ (without $\triangle$), the following results hold.

THEOREM 2.8.3. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. If $\text{L}_*(\boldsymbol{C})$ has the partial embeddability property,[9] then $\text{L}_{*\triangle}(\boldsymbol{C})$ has the canonical* $\text{FS}\mathcal{R}\text{C}$.

Proof. Take an arbitrary $\text{L}_{*\triangle}(\boldsymbol{C})$-chain $\boldsymbol{A}$. Then, its $\mathcal{L}_C$-reduct is partially embeddable into $[0,1]_{\text{L}_*(\boldsymbol{C})}$, thus obviously $\boldsymbol{A}$ is partially embeddable into $[0,1]_{\text{L}_{*\triangle}(\boldsymbol{C})}$ as well. □

PROPOSITION 2.8.4. *Let $*$ be a continuous t-norm and $\boldsymbol{C}$ a countable subalgebra of $[0,1]_*$ such that $\text{L}_*(\boldsymbol{C})$ satisfies the partial embeddability property. Then, $\text{L}_{*\triangle}(\boldsymbol{C})$ is a conservative expansion of $\text{L}_*(\boldsymbol{C})$ if, and only if, $\text{L}_*(\boldsymbol{C})$ enjoys the canonical* $\text{FS}\mathcal{R}\text{C}$.

Proof. One direction is again analogous to the proof of Proposition 2.5.2. For the converse, suppose that $\text{L}_*(\boldsymbol{C})$ does not enjoy the canonical $\text{FS}\mathcal{R}\text{C}$. Then, there is a finite set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm_{\mathcal{L}_C}$ such that $\Gamma \models_{[0,1]_{\text{L}_*(\boldsymbol{C})}} \varphi$ and $\Gamma \not\vdash_{\text{L}_*(\boldsymbol{C})} \varphi$. But then, $\Gamma \models_{[0,1]_{\text{L}_{*\triangle}(\boldsymbol{C})}} \varphi$ and hence $\Gamma \vdash_{\text{L}_{*\triangle}(\boldsymbol{C})} \varphi$, by the canonical $\text{FS}\mathcal{R}\text{C}$ of $\text{L}_{*\triangle}(\boldsymbol{C})$. Therefore, $\text{L}_{*\triangle}(\boldsymbol{C})$ is not a conservative expansion of $\text{L}_*(\boldsymbol{C})$. □

THEOREM 2.8.5. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. $\text{L}_{*\triangle}$ has the $\text{S}\mathcal{R}\text{C}$ if, and only if, $\text{L}_{*\triangle}(\boldsymbol{C})$ has the* $\text{S}\mathcal{R}\text{C}$.

Proof. The proof is analogous to the one of Theorem 2.5.4, taking into account that $\text{L}_{*\triangle}(\boldsymbol{C})$ is a conservative expansion of $\text{L}_{*\triangle}$ (Proposition 2.8.1). □

COROLLARY 2.8.6. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. $\text{L}_{*\triangle}(\boldsymbol{C})$ enjoys the $\text{S}\mathcal{R}\text{C}$ if, and only if, $* = \min$.*

[9] In particular if $\text{L}_*(\boldsymbol{C})$ satisfies conditions $(\mathbf{C1})$ and $(\mathbf{C2})$.

2.8.2 Introducing additional inference rules

An alternative approach to the use of the $\triangle$ operator, in order to force truth-constants to be interpreted in their intended values, is proposed in [6]. Given a logic of a continuous t-norm L_* and a countable subalgebra $\boldsymbol{C}$ of $[0,1]_*$, one defines the logic $\overline{L}_*(\boldsymbol{C})$ as the extension of $L_*(\boldsymbol{C})$ with the following inference rule for each $r \in C$ such that $r < 1$:

$$\text{from} \quad \varphi \vee \overline{r} \quad \text{infer} \quad \varphi.$$

The algebraic counterpart of these logical systems is the class of $\overline{L}_*(\boldsymbol{C})$-algebras, which are defined in the natural way, i.e. as $L_*(\boldsymbol{C})$-algebras satisfying the following quasiequations for $r \in C \setminus \{1\}$: if $x \vee \overline{r} = 1$ then $x = 1$. It is clear then that the class of $\overline{L}_*(\boldsymbol{C})$-algebras forms a quasivariety. Since the new inference rule is closed under $\vee$-forms, the logic $\overline{L}_*(\boldsymbol{C})$ turns out to be a semilinear logic (see Chapter II), and is therefore complete with respect to the class of $\overline{L}_*(\boldsymbol{C})$-chains of the quasivariety. Moreover, every algebra of the quasivariety is a subdirect product of chains of the quasivariety.

The presence of the new inference rule has as a consequence that (like in the case of expansions of $L_{*\triangle}$ with truth-constants) the interpretation of truth-constants in a $\overline{L}_*(\boldsymbol{C})$-chain is one-to-one.

LEMMA 2.8.7. *Let $\boldsymbol{A}$ be a non-trivial $\overline{L}_*(\boldsymbol{C})$–chain. Then, for every $r, s \in C$ such that $r < s$, we have $\overline{r}^{\boldsymbol{A}} < \overline{s}^{\boldsymbol{A}}$.*

Proof. Suppose $r < s$ and $\overline{r}^{\boldsymbol{A}} = \overline{s}^{\boldsymbol{A}}$. Let $t = s \Rightarrow r$. It is clear that $t < 1$ but $\overline{t}^{\boldsymbol{A}} = \overline{s}^{\boldsymbol{A}} \rightarrow_{\boldsymbol{A}} \overline{r}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}$, which contradicts the fulfillment of the rule. □

Therefore, analogously to what happens in the variety of $L_{*\triangle}(\boldsymbol{C})$-algebras, if $*$ is a continuous t-norm that is a finite ordinal sum of Łukasiewicz and Product components, in the quasivariety of $\overline{L}_*(\boldsymbol{C})$-algebras there is only one (up to isomorphism) real chain over $[0,1]_*$, the canonical one, denoted as $[0,1]_{L_*(\boldsymbol{C})}$. Again, this is not true if $*$ contains a Gödel component, as the counterexample at the end of Section 2.3 also shows.

Moreover the logical system $\overline{L}_*(\boldsymbol{C})$ is also a conservative expansion of L_*.

PROPOSITION 2.8.8. *For every continuous t-norm $*$ and every countable subalgebra $\boldsymbol{C} \subseteq [0,1]_*$, the logic $\overline{L}_*(\boldsymbol{C})$ is a conservative expansion of L_*.*

Proof. It is analogous to the proof of Proposition 2.5.2. □

The partial embeddability property also applies, in this setting, for continuous t-norms satisfying conditions $(\mathbf{C1})$ and $(\mathbf{C2})$. The proof is completely analogous to the proof for the case of $L_*(\boldsymbol{C})$, and, so, it is left to the reader. This implies the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}$.

THEOREM 2.8.9. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra. If $\overline{L}_*(\boldsymbol{C})$ has the partial embeddability property,[10] then $\overline{L}_*(\boldsymbol{C})$ has the canonical $\mathrm{FS}\mathcal{R}\mathrm{C}$.*

Finally the following result fully characterizes the logics satisfying the $\mathrm{S}\mathcal{R}\mathrm{C}$.

[10] In particular for any $*$ satisfying conditions $(\mathbf{C1})$ and $(\mathbf{C2})$.

THEOREM 2.8.10. *Let $*$ be a continuous t-norm and let $\boldsymbol{C} \subseteq [0,1]_*$ be a countable subalgebra.* $\bar{\mathrm{L}}_*(\boldsymbol{C})$ *enjoys the* $\mathrm{S}\mathcal{R}\mathrm{C}$ *if, and only if,* $* = \min$.

Proof. The proof is the same as the one given for $\mathrm{G}(\boldsymbol{C})$. □

Notice that the canonical $\mathrm{S}\mathcal{R}\mathrm{C}$ is not true even for $\overline{\mathrm{G}}(\boldsymbol{C})$. The above mentioned counterexample at the end of Section 2.3, with the obvious changes, proves that no countable $\overline{\mathrm{G}}(\boldsymbol{C})$-chain is embeddable into the canonical $\overline{\mathrm{G}}(\boldsymbol{C})$-chain. In fact, the completeness results for these logics coincide with those for the expansions of the logics with $\triangle$ described in the previous section.

2.9 Some open questions

In the previous subsections, we have provided a complete description of completeness results for the expansions of logics of continuous t-norms with a set of truth-constants $\{\overline{r} \mid r \in C\}$, for a suitable countable $C \subseteq [0,1]$, when (i) the t-norm is a finite ordinal sum of basic components and (ii) the set of truth-constants *covers* all the unit interval, in the sense that each component of the t-norm contains at least one value of C different from the bounds of the component. From a practical point of view, it seems that these cases are the most interesting ones for fuzzy logic-based systems, since they usually consider a set of truth values spread all over the real unit interval, and it is natural to assume that there are elements of C in each component of the t-norm. All those cases where at least one of the above two conditions (i) and (ii) is not satisfied remain to be studied. It seems that for these remaining cases (i.e., when either the t-norm has infinitely many components, or the set C does not cover $[0,1]$), a methodology similar to the one used in this section could be applied. In fact, there is a multitude of cases to be considered and the need of new definitions and tools seems unavoidable. Let us show a couple of illustrative examples: the first when the set C does not cover $[0,1]$ and the second when the t-norm has infinitely many components.

EXAMPLE 2.9.1. Let $[0,1]_* = [0,a]_\Pi \oplus [a,1]_\Pi$ and let $C = \{0,1\} \cup \{b^n \mid n \in \mathsf{N}\}$ for some $b < a$. Obviously, there are only two proper filters of C, $F_1 = \{1\}$ and $F_2 = C \setminus \{0\}$, but there are (up to isomorphism) three real $\mathrm{L}_*(\boldsymbol{C})$-chains. One, of type F_2, in the sense used in this paper, is the $\mathrm{L}_*(\boldsymbol{C})$-chain over $[0,1]_*$ where the constants different from $\overline{0}$ are interpreted as 1, and $\overline{0}$ is interpreted as 0. The other two are of type F_1. They are both $\mathrm{L}_*(\boldsymbol{C})$-chains over $[0,1]_*$, where all the constants are interpreted as different elements, either as powers of an element of the first product component or as powers of an element of the second product component. Of course, these two algebras are not isomorphic. This example shows that, in general, there is not a bijection between proper filters and real algebras and, even though it seems possible to have the partial embedding property, the notion and treatment of real chains should be modified in the case that C does not cover all components.

EXAMPLE 2.9.2. Let $[0,1]_* = \bigoplus_{n\in\mathsf{N}}[a_n, a_{n+1}]_{Ł}$, where $a_n = n/(n+1)$, be an infinite ordinal sum of Łukasiewicz components where the idempotent elements form an increasing sequence with limit 1. For a given $k > 2$, let C_i be the carrier of the k-element subalgebra of $[a_i, a_{i+1}]_{Ł}$, and denote its elements as $r_{1i} = a_i, r_{2i}, \ldots, r_{ki} = a_{i+1}$. Take $C = \cup_{i\in\mathsf{N}} C_i \cup \{1\}$. It is clear that C covers all the components but there are real algebras where the interpretations of the truth-constants do not cover all the components. Indeed, let f be any strictly increasing mapping $f\colon \mathsf{N} \to \mathsf{N}$ such that $f(1) = 1$. One real $\mathrm{L}_*(\boldsymbol{C})$-algebra is the chain over $[0,1]_*$, where $\overline{r_{ij}}$ is interpreted as $r_{f(i)j}$. An easy computation shows that this interpretation defines a real $\mathrm{L}_*(\boldsymbol{C})$-chain where the interpretations of truth-constants do not cover the real unit interval. In fact, if $f(i+1)$ is not the successor of $f(i)$ (there are some natural numbers in between), the corresponding components contain no interpretations of truth-constants.

The general case of adding truth-constants to the logic of a left-continuous t-norm $*$ is only studied in the particular case of $*$ corresponding a weak nilpotent minimum t-norm in [30, 33]. Moreover, in [32], the completeness problem of these logics with truth-constants is studied for some distinguished semantics, especially for rational and finite semantics.

2.10 First-order fuzzy logics expanded with truth-constants

The expansion of a first-order t-norm based fuzzy logic with truth-constants, in principle, could be introduced in two different ways:

- Given a left-continuous t-norm $*$ and a countable subalgebra $\boldsymbol{C} \subseteq [0,1]_*$, consider the logic $\mathrm{L}_*(\boldsymbol{C})$ and take its first-order extension $\mathrm{L}_*(\boldsymbol{C})\forall$.
- Given a left-continuous t-norm $*$, consider its associated propositional logic L_*. Take its first-order extension $\mathrm{L}_*\forall$ and now (by enhancing the language with the constants and adding the book-keeping axioms) define its expansion $\mathrm{L}_*\forall(\boldsymbol{C})$ with truth-constants from a countable algebra $\boldsymbol{C} \subseteq [0,1]_*$.

However, these two methods turn out to define the same logic. To set the notation, we will use the second one: $\mathrm{L}_*\forall(\boldsymbol{C})$.

As in the propositional case, we are interested in completeness properties of these logics and even though there are some results for $*$ being a left-continuous t-norm, we restrict ourselves to the case of continuous t-norms. We will find again some positive and some negative results. For the negative ones, we can note that the failure of a completeness property in a weaker logic implies the failure in the stronger one. To make use of this observation, an interesting result is to show that adding truth-constants to a first-order logic $\mathrm{L}_*\forall$ results into a conservative expansion. This is done in the next section.

2.10.1 Conservativeness results

In the case of Łukasiewicz t-norm, Hájek *et al.* already proved in [52] that $\mathrm{RPL}\forall$ (Rational Pavelka predicate logic[11]) is a conservative expansion of Ł$\forall$. Actually, from the proofs in [52], we can extract the following result:

LEMMA 2.10.1 ([52]). *Let $\boldsymbol{C}$ be a subalgebra of $[0,1]^{\mathsf{Q}}_{\text{Ł}}$, $\boldsymbol{A}$ be a countable MV-chain and $\mathbf{M}$ be an $\boldsymbol{A}$-safe structure in a predicate language for Ł$\forall$. Then, there is a divisible[12] MV-chain $\boldsymbol{A}'$, such that $\boldsymbol{A}$ is σ-embeddable into $\boldsymbol{A}'$, and the truth-constants from $\boldsymbol{C}$ are interpretable in $\boldsymbol{A}'$ in such a way that $\mathbf{M}$ is also an $\boldsymbol{A}'$-safe structure for Ł$\forall(\boldsymbol{C})$.*

[11] In our notation $\mathrm{RPL}\forall$ corresponds to Ł$\forall(\boldsymbol{C})$ when $C = [0,1] \cap \mathsf{Q}$.

[12] An MV-chain $\boldsymbol{A}$ is called *divisible* if for every natural m and every $x \in A$ there exists $y \in A$ such that $y \oplus \overset{m}{\ldots} \oplus y = x$ and $y \,\&\, (y \oplus \overset{m-1}{\ldots} \oplus y) = 0$.

This embeddability result is also valid for the predicate logics of the remaining basic continuous t-norms, i.e. Gödel and product logics. In fact it is also valid for the predicate logic of any SBL t-norm.[13]

LEMMA 2.10.2. *Let* $*$ *be an* SBL *t-norm,* $\boldsymbol{C}$ *be a countable subalgebra of* $[0,1]_*$ *and* $\mathbf{M}$ *be an* $\boldsymbol{A}$*-safe structure in a predicate language for* $\mathrm{L}_*\forall$*. Then, the truth-constants from* $\boldsymbol{C}$ *are interpretable in* $\boldsymbol{A}$ *in such a way that* $\mathbf{M}$ *is also an* $\boldsymbol{A}$*-safe structure for* $\mathrm{L}_*\forall(\boldsymbol{C})$.

Proof. For every $r \in C \setminus \{0\}$, interpret $\overline{r}$ as $\overline{1}^{\boldsymbol{A}}$, and $\overline{0}$ as $\overline{0}^{\boldsymbol{A}}$. This turns $\boldsymbol{A}$ into a chain for the expanded language. It is clear that $\mathbf{M}$ is also $\boldsymbol{A}$-safe in this language, since the interpretation of the constants does not give any new value. $\square$

From these lemmas, we obtain conservativeness results for logics based on continuous t-norms. In the proof of next theorem, we use the fact that an SBL t-norm is an ordinal sum either with a first component that is not a Łukasiewicz component or without a first component (see [24]).

THEOREM 2.10.3. *Let* $*$ *be a continuous t-norm and* $\boldsymbol{C}$ *a countable subalgebra of* $[0,1]_*$ *such that, if* $*$ *is not an* SBL*-t-norm, the truth-constants in the Łukasiewicz first component of the decomposition correspond to rational numbers. Then,* $\mathrm{L}_*\forall(\boldsymbol{C})$ *is a conservative expansion of* $\mathrm{L}_*\forall$.

Proof. Let $\Gamma \cup \{\varphi\}$ be a set of $\mathrm{L}_*\forall$-formulas such that $\Gamma \not\vdash_{\mathrm{L}_*\forall} \varphi$. We must show that $\Gamma \not\vdash_{\mathrm{L}_*\forall(\boldsymbol{C})} \varphi$. By hypothesis, there is some safe $\mathrm{L}_*\forall$-structure $\langle \mathbf{M}, \boldsymbol{A}\rangle$ such that $\langle \mathbf{M}, \boldsymbol{A}\rangle \models \Gamma$ and $\langle \mathbf{M}, \boldsymbol{A}\rangle \not\models \varphi$, where $\boldsymbol{A}$ is a countable L_*-chain. If $*$ is an SBL-t-norm, then $\boldsymbol{A}$ is an SBL-chain and applying Lemma 2.10.2 the problem is solved. If $*$ is not an SBL-t-norm, then $*$ is the ordinal sum of a Łukasiewicz component and a hoop $\boldsymbol{B}$. Then, by [24, Proposition 3], $\boldsymbol{A}$ must be a chain of $\mathbf{HSP}_U([0,1]_{\text{Ł}}) \cup (\mathbf{ISP}_U([0,1]_{\text{Ł}}) \oplus \mathbf{HSP}_U(\boldsymbol{B}))$. Then, $\boldsymbol{A}$ is either an MV-chain or the ordinal sum (in the sense of hoops) of an MV-chain $\boldsymbol{A}_1$ and a hoop $\boldsymbol{A}_2$ of $\mathbf{HSP}_U(\boldsymbol{B})$. Take $\boldsymbol{A}'$ as the ordinal sum of the divisible hull $\boldsymbol{A}_1'$ of $\boldsymbol{A}$ (as done in Lemma 2.10.1) and $\boldsymbol{A}_2$. Thus, we obtain a BL-chain $\boldsymbol{A}'$ belonging to $\mathbf{V}([0,1]_*)$ (again by [24, Proposition 3]). Then, we define an $\mathrm{L}_*(\boldsymbol{C})$-chain over $\boldsymbol{A}'$ interpreting the truth-constants from the Łukasiewicz first component of $\boldsymbol{C}$ as the corresponding truth-values of $\boldsymbol{A}_1'$ and the remaining truth-constants as $\overline{1}^{\boldsymbol{A}'}$. By the previous lemmas, $\boldsymbol{A}$ is σ-embeddable into $\boldsymbol{A}'$ as $\mathrm{L}_*(\boldsymbol{C})$-chains. Therefore, we have obtained a chain $\boldsymbol{A}'$ in the expanded language, such that $\mathbf{M}$ is an $\boldsymbol{A}'$-safe structure, and, consequently, $\langle \mathbf{M}, \boldsymbol{A}'\rangle \models \Gamma$, while $\langle \mathbf{M}, \boldsymbol{A}'\rangle \not\models \varphi$. So, the theorem is proved. $\square$

2.10.2 Completeness results

Using the results of previous subsection along with the fact that, for every continuous t-norm $*$ different from Gödel t-norm, the $\mathcal{R}\mathrm{C}$ fails for $\mathrm{L}_*\forall$, we have that $\mathcal{R}\mathrm{C}$ also fails for their expansions with truth-constants. However, for the minimum-t-norm based logic, we can give positive answers to some completeness problems.

THEOREM 2.10.4. *If* $* = \min$*, the logic* $\mathrm{L}_*\forall(\boldsymbol{C})$ *enjoys the* $\mathrm{S}\mathcal{R}\mathrm{C}$.

[13] An SBL t-norm is a continuous t-norm $*$ such that, for all $x \in [0,1]$, it holds that $\min\{x, x \Rightarrow_* 0\} = 0$.

Proof. In [46] it was proved that every countable G-chain $\boldsymbol{A}$ is σ-embeddable into the real G-chain $\boldsymbol{B}$. Denote by $f\colon A \to [0,1]$ one of these σ-embeddings. Assume, in addition, that $\boldsymbol{A}$ is a $\mathrm{G}(\boldsymbol{C})$-chain. For every $r \in C$, interpret $\overline{r}$ in B as $f(\overline{r}^A)$: this gives a real $\mathrm{G}(\boldsymbol{C})$-chain. Thus, we obtain the $\mathrm{S}\mathcal{R}\mathrm{C}$ for $\mathrm{G}\forall(\boldsymbol{C})$. □

Moreover, $\mathrm{G}\forall(\boldsymbol{C})$ enjoys canonical completeness.

THEOREM 2.10.5. *The logic* $\mathrm{G}\forall(\boldsymbol{C})$ *enjoys the* $\mathrm{Can}\mathcal{R}\mathrm{C}$.

Proof. Soundness is obvious, as usual. For the other direction, we will argue by contraposition, i.e. we will prove that if $\nvdash_{\mathrm{G}\forall(\boldsymbol{C})} \varphi$ for some formula φ, then there is a $\mathrm{G}\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}, [0,1]_{\mathrm{G}(\boldsymbol{C})}\rangle$ such that $(\mathbf{M}, [0,1]_{\mathrm{G}(\boldsymbol{C})}) \not\models \varphi$.

If $\nvdash_{\mathrm{L}_*\forall(\boldsymbol{C})} \varphi$, then there exists an $\mathrm{L}_*\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}, \boldsymbol{A}\rangle$ over a countable L_*-chain $\boldsymbol{A}$ and an evaluation v such that $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M},v} < \overline{1}^{\boldsymbol{A}}$. As in Theorem 2.6.1, take $s = \min(\{r \in C \mid \overline{r}^{\boldsymbol{A}} = \overline{1}^{\boldsymbol{A}}, r \text{ appears in } \varphi\} \cup \{1\})$ and define an order-preserving injection $g\colon A \to [0,1]$, also preserving existing suprema and infima, and such that $g(\overline{0}^{\boldsymbol{A}}) = 0$, $g(\overline{1}^{\boldsymbol{A}}) = s$ and $g(\overline{r}^{\boldsymbol{A}}) = r$, for every truth-constant appearing in φ such that $\overline{r}^{\boldsymbol{A}} \neq \overline{1}^{\boldsymbol{A}}$. If $\mathbf{M} = \langle M, \langle P_{\mathbf{M}}\rangle_{P\in Pred}, \langle f_{\mathbf{M}}\rangle_{f\in Funct}\rangle$, using the mapping g, we produce a structure

$$\langle \mathbf{M}', [0,1]_{\mathrm{L}_*(\boldsymbol{C})}\rangle,$$

where $\mathbf{M}' = \langle M, \langle P_{\mathbf{M}'}\rangle_{P\in Pred}, \langle f_{\mathbf{M}}\rangle_{f\in Funct}\rangle$, with $P^{\mathbf{M}'}\colon M^{ar(P)} \to [0,1]$ defined as $P_{\mathbf{M}'} = g \circ P_{\mathbf{M}}$. Therefore, for every evaluation of variables e on M one has

$$\|P(t_1, t_2, \ldots, t_n)\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',e} = g(\|P(t_1, t_2, \ldots, t_n)\|^{\boldsymbol{A}}_{\mathbf{M},e})$$

for each predicate symbol P and terms $t_1, t_2, \ldots, t_n$.

Now, we will prove by induction that given any $\mathbf{M}$ and e and their associated $\mathbf{M}'$ and e', the following statements are true for every subformula ψ of φ:

(a) if $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},e} = \overline{1}^{\boldsymbol{A}}$, then $\|\psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e} \geq s$,

(b) if $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},e} \neq \overline{1}^{\boldsymbol{A}}$, then $\|\psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e} = g(\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},e}) < s$.

The inductive steps for $\psi = \overline{r}$, $\psi = P(t_1, t_2, \ldots, t_n)$, $\psi = \alpha \,\&\, \beta$ and $\psi = \alpha \to \beta$ are proved as in the propositional case in Theorem 2.6.1. Therefore, we are left only with the steps involving quantifiers. We start with $\psi = (\forall x)\alpha$. Let $V(e)$ denote the set of evaluations v of variables such that $e(y) = v(y)$ for all variables y, except x. Recall that $\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e} = \inf\{\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid v \in V(e)\}$.

If $\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e} = \overline{1}^{\boldsymbol{A}}$, then for every such $v \in V(e)$ we have $\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} = \overline{1}^{\boldsymbol{A}}$, and hence $\|\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',v} \geq s$, which implies that $\|(\forall x)\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',e} \geq s$.

If $\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e} \neq \overline{1}^{\boldsymbol{A}}$, it suffices to consider the infimum over the set $V^+(e)$ of evaluations v such that $\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} \neq \overline{1}^{\boldsymbol{A}}$, i.e. $\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e} = \inf\{\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid v \in V^+(e)\} \neq \overline{1}^{\boldsymbol{A}}$. Then, since g preserves all the existing infima, we have: $s > g(\|(\forall x)\alpha\|^{\boldsymbol{A}}_{\mathbf{M},e}) =$

$g(\inf\{\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v} \mid v \in V^+(e)\}) = \inf\{g(\|\alpha\|^{\boldsymbol{A}}_{\mathbf{M},v}) \mid v \in V^+(e)\} = \inf\{\|\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',v} \mid v \in V^+(e)\} = \inf\{\|\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',v} \mid v \in V(e)\} = \|(\forall x)\alpha\|^{[0,1]_{\mathrm{L}_*(\boldsymbol{C})}}_{\mathbf{M}',e}$.

The reasoning in the case $\psi = (\exists x)\alpha$ is similar to the previous one (now it uses that g preserves existing suprema). □

Therefore, we have solved all the real completeness problems for first-order logics under our scope, since in the remaining cases the properties obviously do not hold as they already fail for the corresponding propositional logics with truth-constants. Table 4 collects these results.

Logic	$\mathcal{R}$C, FS$\mathcal{R}$C, S$\mathcal{R}$C	Can$\mathcal{R}$C	CanFS$\mathcal{R}$C
$\mathrm{L}_*\forall(\boldsymbol{C})$, $* \in$ **CONT-fin** $\setminus \{*_{\mathrm{G}}\}$	No	No	No
$\mathrm{G}\forall(\boldsymbol{C})$	Yes	Yes	No

Table 4. Real completeness properties for first-order t-norm based logics with truth-constants.

2.10.3 The case of evaluated formulas

In this section we restrict the completeness properties of our first-order logics to evaluated formulas in the hope of improving the completeness results we have obtained in general. These completeness properties are straightforwardly refuted in many cases. Namely, for each $* \in$ **CONT-fin** $\setminus \{*_{\mathrm{G}}\}$, there is a constant-free formula φ such that $\nvdash_{\mathrm{L}_*\forall} \varphi$ and $\models_{[0,1]_*} \varphi$, and hence, since φ is equivalent to the evaluated formula $\overline{1} \to \varphi$ and $\mathrm{L}_*\forall(C)$ is a conservative expansion of $\mathrm{L}_*\forall$, we also have a counterexample to the $\mathcal{R}\mathrm{C}_{ev}$ of $\mathrm{L}_*\forall(C)$.

In addition, the completeness properties for evaluated formulas are also refuted in those cases where they already fail at the propositional level (and hence also including the failure of $\mathrm{CanS}\mathcal{R}\mathrm{C}$ for the cases 1–5 listed before Theorem 2.7.3).

There are, nonetheless, several positive results. Regarding canonical completeness properties, the only cases that remain to be checked are those corresponding to the logics $\mathrm{G}\forall(\boldsymbol{C})$. In the rest of this section, we show that $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$ always holds for these logics, while we provide only some partial (positive) results in the case of $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$.

THEOREM 2.10.6. *The logics* $\mathrm{G}\forall(\boldsymbol{C})$ *enjoy the* $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$.

Proof. We have to show that for every formulas $\varphi_1, \dots, \varphi_k, \psi$ in the language of G∀ and positive constants $\overline{r_1}, \dots, \overline{r_k}, \overline{s}$:

$\{\overline{r_i} \to \varphi_i \mid i = 1, \dots, k\} \vdash_{\mathrm{G}\forall(\boldsymbol{C})} \overline{s} \to \psi$ if, and only if,
$\{\overline{r_i} \to \varphi_i \mid i = 1, \dots, k\} \models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \overline{s} \to \psi$.

The proof is analogous to the one of (i) of Theorem 2.7.1 with the obvious changes. Still, we include it for the sake of readability. By the deduction theorem and the canonical standard completeness for $\mathrm{G}\forall(\boldsymbol{C})$, a finite deduction of type $\{\overline{r_i} \to \varphi_i \mid i = 1, \dots, k\} \vdash_{\mathrm{G}\forall(\boldsymbol{C})} \overline{s} \to \psi$ is equivalent to $\models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \&_{i=1,\dots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$. Thus, what we need to prove is the semantical version of the deduction theorem for

$\text{Ł}_*\forall(\mathcal{C})$, i.e. the equivalence between $\{\overline{r_i} \to \varphi_i \mid i = 1, \ldots, k\} \models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \overline{s} \to \psi$ and $\models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$.

From right to left the implication is obvious. We prove the other direction by contraposition. If $\not\models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)$, there must exist (by the previous theorem) a $\mathrm{G}\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}, [0,1]_{\mathrm{G}(\boldsymbol{C})}\rangle$ and an evaluation e such that

$$\|\&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i) \to (\overline{s} \to \psi)\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} < 1.$$

We have to build a $\mathrm{G}\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}', [0,1]_{\mathrm{G}(\boldsymbol{C})}\rangle$ and an evaluation of variables e' such that $\|\&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} = 1$ and $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} < 1$. Observe first that the previous inequality implies that $\|\&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} > \|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e}$ and thus $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} = \|\psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} < 1$. We follow the proof by cases:

(i) If $\|\overline{r_i} \to \varphi_i\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} = 1$ for every $i \in \{1, \ldots, k\}$, then we just take $\mathbf{M}' = \mathbf{M}$ and $e' = e$.

(ii) Suppose there exists a non-empty set of indexes $J \subseteq \{1, \ldots, k\}$ such that for all $j \in J$, $\|\overline{r_j} \to \varphi_j\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} = \|\varphi_j\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} < 1$. Let $a = \min\{\|\varphi_j\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e} \mid j \in J\}$. Define (like in Theorem 2.7.1) f_a as the endomorphism of $[0,1]_{\mathrm{G}(\boldsymbol{C})}$ given by $f_a(x) = 1$ for every $x \geq a$ and by an order preserving bijection between $[0,a)$ and $[0,1)$ preserving existing suprema and infima. Now, we consider a structure $\mathbf{M}'$ over the same domain as $\mathbf{M}$ with the same interpretation of functional symbols, with the same evaluation of variables $e' = e$, and we will just change the interpretation of the predicate symbols. Indeed, for every n-ary predicate P and arbitrary elements of the domain $m_1, \ldots, m_n$, we define $P_{\mathbf{M}'}(m_1, \ldots, m_n) = f_a(P_{\mathbf{M}}(m_1, \ldots, m_n))$. Then, since f is a homomorphism that preserves existing suprema and infima, it is obvious that for every $\mathrm{G}\forall$-formula φ we have $\|\varphi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e} = f_a(\|\varphi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M},e})$. An easy computation shows that $\|\&_{i=1,\ldots,k}(\overline{r_i} \to \varphi_i)\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} = 1$, while $\|\overline{s} \to \psi\|^{[0,1]_{\mathrm{G}(\boldsymbol{C})}}_{\mathbf{M}',e'} < 1$. □

Regarding the properties of $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$, as already mentioned above, it remains to check the cases of logics $\mathrm{G}\forall(\boldsymbol{C})$ when the algebra of truth-constants C has no positive sup-accessible points, i.e. for each $r \in C$ there exists an open interval $(r - \epsilon, r)$ containing no element of C (otherwise $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$ already fails in the propositional case). Two paradigmatic particular examples of algebras of truth-constants satisfying this condition are the case when C is finite (which is obvious) and the case when $C \setminus \{0\}$ is a strictly decreasing sequence with limit 0 (addressed in next theorem).

THEOREM 2.10.7. *Let $\boldsymbol{C}$ be such that $C \setminus \{0\} = \{t_n \mid n \in \mathsf{N}\}$, where $\langle t_n\rangle_{n\in\mathsf{N}}$ is a strictly decreasing sequence with limit* 0. *Then, the logic* $\mathrm{G}\forall(\boldsymbol{C})$ *enjoys* $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$.

Proof. We have to show that for every set of formulas $\{\varphi_i \mid i \in I\} \cup \{\psi\}$, in the language of $\mathrm{G}\forall$ and positive constants $\{\overline{r_i} \mid i \in I\} \cup \{\overline{s}\}$:

$$\{\overline{r_i} \to \varphi_i \mid i \in I\} \vdash_{\mathrm{G}\forall(\boldsymbol{C})} \overline{s} \to \psi \text{ if, and only if, } \{\overline{r_i} \to \varphi_i \mid i \in I\} \models_{[0,1]_{\mathrm{G}(\boldsymbol{C})}} \overline{s} \to \psi.$$

In one direction the implication is obvious. We prove the other one by contraposition. If $\{\overline{r_i} \to \varphi_i \mid i \in I\} \nvdash_{G\forall(\boldsymbol{C})} \overline{s} \to \psi$, there must exist a countable G$\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}, \boldsymbol{A}\rangle$, and an evaluation e over $\boldsymbol{A}$ such that $\|\overline{r_i} \to \varphi_i\|^{A}_{\mathbf{M},e} = \overline{1}^{A}$ for all $i \in I$ and $\|\overline{s} \to \psi\|^{A}_{\mathbf{M},e} < \overline{1}^{A}$. We have to build a G$\forall(\boldsymbol{C})$-structure $\langle \mathbf{M}', [0,1]_{G(\boldsymbol{C})}\rangle$ and an evaluation of variables e' such that $\|\overline{r_i} \to \varphi_i\|^{[0,1]_{G(\boldsymbol{C})}}_{\mathbf{M}',e'} = 1$ for all $i \in I$ and $\|\overline{s} \to \psi\|^{[0,1]_{G(\boldsymbol{C})}}_{\mathbf{M}',e'} < 1$.

The proof will consist in taking the same domain of individuals $\mathbf{M}' = \mathbf{M}$, the same evaluation $e' = e$, and defining for every n-ary predicate P and arbitrary elements of the domain $m_1, \dots, m_n$, $P_{\mathbf{M}'}(m_1, \dots, m_n) = f(P_{\mathbf{M}}(m_1, \dots, m_n))$, where f is a σ-embedding of $\mathcal{A}$ as G-algebra into $[0,1]_{G}$ satisfying:

(i) $f(\overline{r_i}^{\mathcal{A}}) \geq r_i$ for all $i \in I$,

(ii) $f(\|\psi\|^{A}_{\mathbf{M},e}) < s$.

Notice that such a mapping f solves our problem, since being a σ-embedding it holds that, for any G$\forall$-formula φ, $\|\varphi\|^{[0,1]_{G(\boldsymbol{C})}}_{\mathbf{M}',e} = f(\|\varphi\|^{A}_{\mathbf{M},e})$, and so by (i) we obtain that $\|\varphi_i\|^{[0,1]_{G(\boldsymbol{C})}}_{\mathbf{M}',e} \geq r_i$ for all $i \in I$, and (ii) gives us $\|\psi\|^{[0,1]_{G(\boldsymbol{C})}}_{\mathbf{M}',e} < s$. Therefore, the rest of the proof is devoted to building the σ-embedding f.

Since $\boldsymbol{A}$ is a G$(\boldsymbol{C})$-chain, it defines a filter $F_A = \{r \in C \mid \overline{r}^{A} = \overline{1}^{A}\}$ of $\mathcal{C}$ such that $\overline{p}^{A} < \overline{q}^{A}$ for any $p, q \notin F_A$ and $p < q$. We consider the following cases:

(1) $s \in F_A$ and $\inf_n \overline{t_n}^{A} = \overline{0}^{A}$.

Let t_m be the greatest element of $C \setminus F_A$. We split the construction of f in two parts. The restriction of f to the interval $[\overline{0}^{A}, \overline{t_m}^{A}]$ is taken as any σ-embedding into $[0, t_m]$ such that $f(\overline{t_k}^{A}) = t_k$ for each $k \geq m$. On the other hand, if $\|\psi\|^{A}_{\mathbf{M},e} \leq \overline{t_m}^{A}$, the restriction of f to $[\overline{t_m}^{A}, \overline{1}^{A}]$ is taken as any σ-embedding into $[t_m, 1]$. Otherwise, let $\delta \in [0,1]$ be such that $\delta < s$ and $[\delta, s) \cap C = \emptyset$. Then the restriction of f to $[\overline{t_m}^{A}, \overline{1}^{A}]$ is taken as any σ-embedding into $[t_m, 1]$ such that $f(\|\psi\|^{A}_{\mathbf{M},e}) = \delta$.

(2) $s \in F_A$ and there exists $\overline{0}^{A} < \alpha \in A$ such that $\overline{t_n}^{A} > \alpha$ for each n.

The construction of the restriction of f to $[\overline{t_m}^{A}, \overline{1}^{A}]$ is exactly the same as in (1). Now, the restriction of f to $[\overline{0}^{A}, \overline{t_m}^{A}]$ is defined as any σ-embedding into $[0, t_m]$ such that $f(\alpha) = t_{m-1}$. In this case, it holds that $f(\overline{t_k}^{A}) \geq t_k$ for $k \geq m$.

(3) $s \notin F_A$.

In this case, the restriction of f to $[\overline{s}^{A}, \overline{1}^{A}]$ can be taken as any σ-embedding into $[s, 1]$ such that $f(\overline{t_i}^{A}) = t_i$ for all $t_i \notin F_A$ and $t_i \geq s$ (there are finitely many). The restriction of f to $[\overline{0}^{A}, \overline{s}^{A}]$ depends on whether $\inf_n \overline{t_n}^{A} = \overline{0}^{A}$ or there exists $\overline{0}^{A} < \alpha \in A$ such that $\overline{t_n}^{A} > \alpha$ for each n. Taking t_m as s, the restriction of f to $[\overline{0}^{A}, \overline{s}^{A}]$ in the former case is defined as in (1) and in the latter case as in (2). $\square$

The proof of the above theorem can be easily adapted to the cases considered in the next corollary, and, thus, we omit the proofs.

COROLLARY 2.10.8. *The logic* $\mathrm{G}\forall(\boldsymbol{C})$ *also enjoys the* $\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$ *in the following cases:*

- C *is finite,*
- $C \setminus \{0\} = \{t_n \mid n \in \mathsf{N}\}$*, where* $\langle t_n \rangle_{n\in\mathsf{N}}$ *is a strictly decreasing sequence without limit in* C*,*
- $C \setminus \{0\} = \{t_n \mid n \in \mathsf{N}\} \cup \{\alpha\}$*, where* $\langle t_n \rangle_{n\in\mathsf{N}}$ *is a strictly decreasing sequence with limit* $\alpha \in C$*.*

However, it is still unknown whether these positive results also hold for the general case of $\boldsymbol{C}$ having no positive $\sup$-accessible points. In Table 5, we summarize the available results about canonical completeness properties. We do not include there the results about non-canonical completeness for evaluated formulas since, as already discussed in the beginning of this section, they turn out to be the same as for arbitrary formulas.

Logic	$\mathrm{Can}\mathcal{R}\mathrm{C}_{ev}$, $\mathrm{CanFS}\mathcal{R}\mathrm{C}_{ev}$	$\mathrm{CanS}\mathcal{R}\mathrm{C}_{ev}$
$\mathrm{L}_*\forall(\boldsymbol{C})$, $* \in \mathbf{CONT\text{-}fin} \setminus \{*_{\mathrm{G}}\}$	No	No
$\mathrm{G}\forall(\boldsymbol{C})$ C^+ has sup-accessible points	Yes	No
$\mathrm{G}\forall(\boldsymbol{C})$ C^+ has no sup-accessible points	Yes	?

Table 5. Canonical real and rational completeness properties for first-order t-norm based logics with truth-constants restricted to evaluated formulas.

3 Expansions with truth-stressing and truth-depressing hedges

Typical examples of fuzzy truth-values in the sense of Zadeh (see [87]) are "very true", "quite true", "more or less true", "slightly true", etc. They are represented in fuzzy logic in narrow sense as fuzzy subsets on the set of truth values, typically the real unit interval. In order to cope with these fuzzy truth values in the setting of mathematical fuzzy logic, Hájek proposed in [47] to understand them as truth functions of new unary connectives called either *truth-stressing* or *truth-depressing hedges* (depending on whether they reinforce or weaken the truth value). The intuitive interpretation of a truth-stressing (resp. depressing) hedge like *very true* (resp. *slightly true*) on a chain of truth-values is a subdiagonal (resp. superdiagonal) non-decreasing function preserving 0 and 1. From now on, such functions will be called *hedge functions*. Notice that the well-known globalization operator $\triangle$ (introduced independently first by Monteiro in the context of intuitionistic logic [72] and later by Baaz in the context of Gödel–Dummett logics [2]) is a limit case of a truth-stresser, since, over a chain, it maps 1 to 1 and all the other elements to 0, and its intuitive interpretation would correspond to *definitely true*.

Hájek [47] and Vychodil [86] proposed an axiomatization of truth-stressing and depressing hedges respectively as expansions of BL (and of some of its prominent extensions, like Łukasiewicz, Product or Gödel logics) by new unary connectives vt, for *very true*, and st, for *slightly true*, respectively. The logics they define are shown to be algebraizable and to enjoy completeness with respect to the classes of chains of their corresponding varieties. However the axiomatization proposed by Hájek (also used by Vychodil) is quite restrictive, since not any BL-chain expanded with a hedge function is a model of the proposed logic, as one would expect from the traditional use of hedges in fuzzy logic in a wide sense. Moreover, the defined logics are not proved to enjoy general standard completeness, except for the case of logics expanding Gödel logic. One of the main reasons behind both problems is the presence in the axiomatizations of the well-known modal axiom K for the vt connective, which puts quite a lot of constraints on the hedges to be models of these logics without a natural algebraic interpretation.

Next (based on the preliminary paper [34]) we show simple and general axiomatizations with very intuitive properties and nice completeness results based on the abstract logical approach to fuzzy logic (in the sense of semilinear residuated logics) fully described in Chapter II of this handbook.

3.1 The logic L_S of truth-stressing hedges

Let L be a core fuzzy logic, and let L_S be the expansion of L with a new unary connective s (for *stresser*) defined by the following additional axioms:

(VTL1) $s\varphi \to \varphi$
(VTL2) $s\overline{1}$

and the following additional inference rule:

(MON) from $(\varphi \to \psi) \vee \chi$ infer $(s\varphi \to s\psi) \vee \chi$.

If we denote by $\vdash_{\mathrm{L}_S}$ the notion of deduction defined as usual from the above axioms and rules, one can easily show the following:

LEMMA 3.1.1. *The following deductions are valid in* L_S*:*

(i) $\vdash_{\mathrm{L}_S} \neg s\overline{0}$

(ii) $\varphi \to \psi \vdash_{\mathrm{L}_S} s\varphi \to s\psi$

(iii) $\psi \vdash_{\mathrm{L}_S} s\psi$

(iv) $s\varphi, \varphi \to \psi \vdash_{\mathrm{L}_S} s\psi$.

Proof. (i) follows directly from (VTL1) taking $\varphi = \overline{0}$.
(ii) follows directly from (MON) taking $\chi = \overline{0}$.
(iii) follows directly from (ii) taking $\varphi = \overline{1}$ and using (VTL2).
(iv) is easily derivable using (ii) and *modus ponens*. □

Notice that (iv) is a kind of stronger version of *modus ponens:* if φ implies ψ and φ is s-true (for instance "very true"), then one can derive that ψ is s-true (very true) as well. On the other hand, (ii) shows that s satisfies the congruence property (see Section 3.3 in Chapter I). Therefore, the logic L_S is Rasiowa-implicative and its equivalent algebraic semantics is the class of L_S-algebras. An algebra $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, s, \overline{0}, \overline{1}\rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ is an L_S-algebra if it is an L-algebra expanded by a unary operator $s\colon A \to A$ (truth-stressing hedge) that satisfies, for all $x, y, z \in A$,

(1) $s(\overline{1}) = \overline{1}$,

(2) $s(x) \leq x$,

(3) if $(x \to y) \vee z = \overline{1}$ then $(s(x) \to s(y)) \vee z = \overline{1}$.

It is clear that the class of L_S-algebras forms a quasivariety (call it $\mathbb{L}_S$). Notice that if $\langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1}\rangle$ is a totally ordered L-algebra and $s\colon A \to A$ is any non-decreasing mapping such that $s(\overline{1}) = \overline{1}$ and $s(a) \leq a$ for any $a \in A$, then the expanded structure $\langle A, \&, \to, \wedge, \vee, s, \overline{0}, \overline{1}\rangle$ is an L_S-chain. In other words, in L_S-chains the quasiequation (3) turns out to be equivalently expressed by this simplified form: if $x \to y = \overline{1}$ then $s(x) \to s(y) = \overline{1}$, and this condition simply expresses that s is non-decreasing.

Moreover, since the rule (MON) is closed under $\vee$-forms we know that $\vee$ keeps being a disjunction in the expanded logic. On the other hand, since $(\varphi \to \psi) \vee (\psi \to \varphi)$ was already valid in L, we obtain that L_S is also semilinear and hence it is complete with respect to the semantics given by all L_S-chains (see Chapter II, Section 3.2), for the role of disjunction in semilinear logics).

THEOREM 3.1.2. L_S *is strongly complete with respect to the class of all* L_S*-chains, that is,* L_S *is* $\mathrm{S}\mathbb{K}\mathrm{C}$*, with* $\mathbb{K}$ *being the class of* L_S*-chains.*

COROLLARY 3.1.3. *The following deductions are valid in* L_S*:*

(v) $\vdash_{\mathrm{L}_S} s(\varphi \vee \psi) \leftrightarrow s\varphi \vee s\psi$

(vi) $\vdash_{\mathrm{L}_S} s(\varphi \wedge \psi) \leftrightarrow s\varphi \wedge s\psi$.

Proof. Both properties can be easily seen to hold on L_S-chains. □

One might wonder whether one or both corresponding equations for the monotonicity of s (i.e. $s(x \wedge y) = s(x) \wedge s(y)$ and/or $s(x \vee y) = s(x) \vee s(y)$) may substitute the quasiequation (3) in the definition of L_S-algebras. Notice first that over algebras satisfying (1) and (2), the two equations for monotonicity are not equivalent,[14] as the following examples show.

EXAMPLE 3.1.4. Let $\boldsymbol{A}$ be the 5-element Gödel algebra $\{0, a, b, c, 1\}$, where 0 is the bottom, a is an atom, $b \wedge c = a$, $b \vee c = 1$ and 1 is the top element.

- Take s as $s(b) = s(c) = s(a) = 0$. Monotonicity is satisfied for the infimum but not for the supremum since $s(c \vee b) = s(1) = 1$ and $s(c) \vee s(b) = 0 \vee 0 = 0$.

[14] We thank Franco Montagna for pointing out this fact to us.

- Take s as the identity operator except for $s(a) = 0$. This mapping satisfies the monotonicity for the supremum but not for the infimum since $s(c \wedge b) = s(0) = 0$ and $s(c) \wedge s(b) = c \wedge b = a$.

By Corollary 3.1.3, the monotonicity equations are both satisfied in L_S. Hence, the right question is whether the two monotonicity equations may substitute the quasiequation (3). In other words, does the quasivariety $\mathbb{L}_s$ coincide with the variety $\mathbb{V}$ of expansions of L-algebras satisfying the equations (1), (2) and the two monotonicity equations of s? The answer is negative as shown by the following example.

EXAMPLE 3.1.5. Let $\boldsymbol{A}$ be the same Gödel algebra as in Example 3.1.4 and define s as the truth-stresser given by $s(1) = s(b) = 1$ and $s(a) = s(c) = s(0) = 0$. Take the filter $F = \{c, a, b, 1\}$. An easy computation shows that (A, F) is a model of the logic defined by (1), (2) and the monotonicity equations, but the rule (3), even in its simplified form (from $\varphi \to \psi$ deduce $s(\varphi) \to s(\psi)$), is not sound for this model since $a \to b, b \to a \in F$ and $s(b) \to s(a) = 1 \to 0 = 0 \notin F$.

Thus, $\mathbb{V}$ and $\mathbb{L}_s$ coincide over chains but they are different. While L_s is semilinear due to the rule (MON), the logic associated to $\mathbb{V}$ is not. This also shows that in the presentation of L_S, (MON) cannot be substituted by the simpler rule: from $\varphi \to \psi$ infer $s\varphi \to s\psi$ (which, as we have just seen, is sound in L_S-chains but not for all L_S-algebras).

Similarly, inspired by the well-known presentation of logics with $\triangle$, one might also ask whether (MON) could be substituted by the globalization rule: from φ infer $s\varphi$. The answer is again negative.

EXAMPLE 3.1.6. Let $\boldsymbol{C}$ be the finite MTL-chain defined over $C = \{0, 1, 2, 3, 4, 5\}$ with the natural order and the following monoidal operation:

$\&$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	1	1	2	2
3	0	1	1	1	2	3
4	0	1	2	2	4	4
5	0	1	2	3	4	5

Take the MTL-filter $F = \{4, 5\}$ and the following unary operation s:

x	0	1	2	3	4	5
$s(x)$	0	1	1	3	4	5

It is clear that s is subdiagonal, maps the top element to itself and is non-decreasing. Moreover, for every $x \in F$, $s(x) \in F$, i.e. it is sound w.r.t. the globalization rule. However, it is not sound w.r.t. (MON): indeed, $3 \to 2 = 4 \in F$, while $s(3) \to s(2) = 3 \to 1 = 3 \notin F$.

We consider now the issue of completeness of L_S with respect to distinguished semantics of L_S-chains. One can prove that if L has the finite strong real completeness property (FS$\mathcal{R}$C), then L_S has it as well. As usual, this can be done by showing that any L_S-chain is partially embeddable into a standard L_S-chain.

THEOREM 3.1.7 (Finite strong real completeness). *If* L *is a finite strong real complete (*FS$\mathcal{R}$C*) core fuzzy logic, then the logic* L_S *is finite strong real complete as well.*

Proof. Assume that L has the FS$\mathcal{R}$C. Take any L_S-chain $\boldsymbol{A} = \langle A, \&, \rightarrow, \wedge, \vee, s, \overline{0}, \overline{1}\rangle$, and let B be a finite partial subalgebra of $\boldsymbol{A}$. We have to show that there exist a real L_S-chain $\langle [0,1], \wedge, \vee, *, \Rightarrow, s', 0, 1\rangle$ and a mapping $f\colon B \rightarrow [0,1]$ preserving the existing operations. By assumption, the s-free reduct of $\boldsymbol{A}$ is partially embeddable into a real L-chain $\langle [0,1], \wedge, \vee, *, \Rightarrow, 0, 1\rangle$. Denote this embedding by f, and consider any non-decreasing and subdiagonal function $s'\colon [0,1] \rightarrow [0,1]$ satisfying $s'(f(x)) = f(s(x))$ for every $x \in B$ such that $s(x) \in B$. There are obviously many such functions s' interpolating the set of points $P = \{\langle f(x), f(s(x))\rangle \mid x, s(x) \in B\}$ (a linear interpolant, for instance). Another interpolant can be defined as follows: let $0 = z_1 < \cdots < z_n < 1$ be the set of elements of $[0,1]$ such that $\langle z_i, \cdot\rangle \in P$ and define $s'(1) = 1$ and, for all $z \in [0,1)$,

$$s'(z) = f(s(x_i)), \text{ if } z_i \leq z < z_{i+1}$$

where $x_i \in B$ is such that $z_i = f(x_i)$. In any case, s' makes $\langle [0,1], \wedge, \vee, *, \Rightarrow, s', 0, 1\rangle$ an L_S-chain and f a partial embedding of L_S-chains. □

Actually, this theorem can be generalized to arbitrary classes of L-chains and their s-expansions, proved in a completely analogous way, and yielding a more general result.

COROLLARY 3.1.8. *Let* L *be a core fuzzy logic,* $\mathbb{K}$ *a class of* L*-chains, and* $\mathbb{K}_S$ *the class of the* L_S*-chains whose* s*-reducts are in* $\mathbb{K}$*. If* L *has the* FS$\mathbb{K}$C*, then* L_S *has the* FS$\mathbb{K}_S$C *as well.*

THEOREM 3.1.9 (Strong real completeness). *If* L *is a strong real complete (*S$\mathcal{R}$C*) core fuzzy logic, then the logic* L_S *is strong real complete as well.*

Proof. Let L have the S$\mathcal{R}$C. We have to show that any countable L_S-chain can be embedded into a standard L_S-chain. Let $\boldsymbol{A}$ be a countable L_S-chain. By the assumption, the s-free reduct of $\boldsymbol{A}$ is embeddable into a standard L-chain $\boldsymbol{B} = \langle [0,1], \wedge, \vee, *, \Rightarrow, 0, 1\rangle$. Denote this embedding by f and define $s'\colon B \rightarrow B$ in the following way: for each $z \in [0,1]$, $s'(z) = \sup\{f(s(x)) \mid x \in A, f(x) \leq z\}$. So defined, s' is a non-decreasing and subdiagonal function such that $s'(f(x)) = f(s(x))$ for any $x \in A$. Therefore, $\boldsymbol{B}$ expanded with s' is a standard L_S-chain where $\boldsymbol{A}$ can be embedded. □

Observe that the proof of the previous theorem can be repeated whenever the linear order of the chains is complete. Therefore we obtain the following corollary.

COROLLARY 3.1.10. *Let* L *be a core fuzzy logic,* $\mathbb{K}$ *a class of completely ordered* L*-chains, and* $\mathbb{K}_S$ *the class of the* L_S*-chains whose* s*-reducts are in* $\mathbb{K}$*. If* L *has the* S$\mathbb{K}$C*, then* L_S *has the* S$\mathbb{K}_S$C.

3.2 On the logics L_S and their associated quasivarieties

In the previous subsection, we have seen that if L is a core fuzzy logic, then L_S is a semilinear logic (complete with respect to chains of the associated quasivariety). However, this does not imply that it has either a global or local deduction-detachment theorem (denoted, from now on, GDDT and LDDT respectively). In this subsection, we present two families of logics L_S that enjoy the GDDT and one family enjoying the LDDT. Moreover, we prove that the quasivarieties associated to these families of logics are, in fact, varieties.

3.2.1 The case of L being the logic of a finite BL-chain

The first family we consider is that of the logics L_S where L is the logic of a finite BL-chain $\boldsymbol{A}$ having n elements, i.e. $\boldsymbol{A}$ is an ordinal sum of copies of finite MV-chains ($\boldsymbol{\L}_k$) and finite Gödel chains ($\boldsymbol{G}_r$).

PROPOSITION 3.2.1. *If* L *is the logic of a finite* BL*-chain* $\boldsymbol{A}$*, then:*

(1) *The chains of the variety generated by* $\boldsymbol{A}$ *are the subalgebras of* $\boldsymbol{A}$*.*

(2) *Given a* BL*-filter* F *of* $\boldsymbol{A}$ *, the congruence defined by it,* $\equiv_F$*, is defined by:* $x \equiv_F y$ *iff either* $x = y$ *or* $x, y \in F$*, i.e. the congruence classes are* F *and the singletons* $\{x\}$ *for any* $x \notin F$*.*

(3) *The set of* L_S*-filters of* $\boldsymbol{A}$ *coincides with the set of* L*-filters that are closed under* s*.*

Proof. The first claim is a consequence of [24, Theorem 1], taking into account that every finite BL-chain is subdirectly irreducible and the fact that any chain belonging to the variety generated by a finite Gödel or MV-chain is a subalgebra of it.

The proof of the second claim is easy since if $x \geq y$, then $x \equiv_F y$ iff $x \to y \in F$. The filters of $\boldsymbol{A}$ are the principal filters defined by an element a that either belongs to a Gödel component or is the bottom of an MV component. Thus, an easy computation shows that $x \to y \in F$ iff either $x = y$ or $x, y \in F$.

In order to prove the third claim observe first that, if F is an L_S-filter of $\boldsymbol{A}$, then, it is closed under s, since if $\bar{a} \in F$, then $1 \to \bar{a} \in F$, and thus $1 \to s(\bar{a}) = s(\bar{a}) \in F$. On the other hand, suppose that F is a BL-filter closed under s. Then F is a L_S-filter. Remember that if F is a BL-filter over a finite BL-chain then $a \equiv_F b$ iff $a = b$ or $a, b \in F$. Therefore, if F is closed under s, then $s(a) \equiv_F s(b)$. □

LEMMA 3.2.2. *Let* L *be the logic of a finite* BL*-chain* $\boldsymbol{L}$*, and let* L_S *be the expansion of* L *with a truth-stressing hedge as defined in Section 3.1. Then, in any* L_S*-algebra* $\boldsymbol{A}$*, the* L_S*-filter* $F(\bar{a})$ *generated by an element* $\bar{a} \in \boldsymbol{A}$ *is principal, i.e. there is an element* $t(\bar{a})$ *such that* $F(\bar{a}) = [t(\bar{a}), \bar{1}] \cap A$*.*

Proof. If $\boldsymbol{A}$ is an L_S-algebra, then $\boldsymbol{A}$ can be embedded into a direct product $\prod_{i \in I} \boldsymbol{L}$ (remember that any L_S-chain is a subalgebra of $\boldsymbol{L}$, and suppose that $\boldsymbol{L}$ has n elements, k components and m is the maximum length of an MV component). Given an element $\bar{a} \in A$, take the element $t(\bar{a}) = (s^n(\overset{k}{\ldots} s^n(\bar{a}^m))^m \ldots)^m$. An easy computation shows that $t(\bar{a})$ is idempotent and it is a fixed point by s. Then, we will prove that $F(\bar{a})$ is the principal filter defined by $t(\bar{a})$. The proof follows from the following facts:

(i) $t(\bar{a}) \in F(\bar{a})$,

(ii) if $t(\bar{a})_i$ is the i-projection of $t(\bar{a})$, then $F(t(\bar{a})_i) = \{x \in L \mid x \geq t(\bar{a})_i\}$ is the filter of L generated by $t(\bar{a})_i$, and

(iii) $F(t(\bar{a})) = A \cap \prod_{i\in I} F(t(\bar{a})_i)$, by definition. □

THEOREM 3.2.3. *Let* L *be the logic of a finite* BL*-chain* $\boldsymbol{L}$ *(with* n *elements,* k *components and with* m *being the maximum length of an* MV *component), and let the logic* L_S *be its expansion with a truth-stressing hedge as defined in Section 3.1. Then, the logic* L_S *enjoys the GDDT, i.e. given a set* $\Gamma \cup \{\varphi, \psi\}$ *of formulas, there is a formula* $t(\varphi) = (s^n(\overset{k}{\ldots} s^n(\varphi^m))^m \ldots)^m$ *such that,*

$$\Gamma, \varphi \vdash_{\mathrm{L}_S} \psi \quad \textit{iff} \quad \Gamma \vdash_{\mathrm{L}_S} t(\varphi) \to \psi.$$

Proof. The right-to-left direction follows easily from the observation that $\varphi \vdash_{\mathrm{L}_S} t(\varphi)$. Let us prove the other direction by reasoning semantically, using completeness: i.e., we assume $\Gamma, \varphi \models_{\mathbb{L}_s} \psi$, and we show $\Gamma \models_{\mathbb{L}_s} t(\varphi) \to \psi$. Take any L_S-algebra $\boldsymbol{A}$ and any $\boldsymbol{A}$-evaluation e such that $e[\Gamma] \subseteq \{\bar{1}^A\}$. Consider the matrix L_S-model $\langle \boldsymbol{A}, Fe[\Gamma], e(\varphi)\rangle$. By soundness $e(\psi) \in F(e[\Gamma], e(\varphi))$, i.e. $e(\psi) \in F(e(\varphi)) = [t(e(\varphi)), \bar{1}^A]$. Then, $t(e(\varphi)) \leq e(\psi)$, and so $e(t(\varphi) \to \psi) = \bar{1}^{\boldsymbol{A}}$. □

From GDDT the following result is obvious.

COROLLARY 3.2.4. *The quasivariety associated to the logic of a finite* BL*-chain is a variety.*

Some remarks are in order here:

- The results in this section are valid for any logic of a finite MTL-chain with the condition that L_S-filters on L_S-chains coincide with MTL-filters closed under s.
- A sufficient condition for an MTL-filter on an L_S-chain closed under s to be an L_S-filter is the fact that $a \equiv_F b$ iff either $a = b$ or $a, b \in F$. For example, any finite WNM-chain $\boldsymbol{L}$ (with n elements) satisfies this condition, and so the logic L_S enjoys the GDDT (with the formula $t(\varphi) = (s^n(\varphi))^2$), and hence the quasivariety corresponding to the logic of a finite WNM-chain with a truth-stresser is a variety.
- The following example proves that there are finite MTL-chains with MTL-filters closed under s that are not L_S filters.

EXAMPLE 3.2.5. Take a 6-element chain $\boldsymbol{A}$ such that $(1 > a > b > c > d > 0)$, and define the operation $*$ by (assuming that $*$ is determined when one value is 0 or 1) $a * a = a$, and $x * y = d$ otherwise. Then the MTL-filters are $\{1\}, \{1, a\}, \{1, a, b, c, d\}$ and A itself. Define the operator s by (the values of 0 and 1 are determined) $s(a) = a, s(b) = b, s(c) = s(d) = 0$. It is obvious that the MTL-filters closed under s are $\{1\}, \{1, a\}$ and A. But $\{1, a\}$ is not an L_S-filter since $b \to c = a$ and $s(b) \to s(c) = b \to 0 = 0 \notin \{1, a\}$.

3.2.2 The case of logics L_S where the operator $\triangle$ is definable

The second family we consider is the family of logics L_S where the Monteiro–Baaz $\triangle$ operator is definable. In such a case it is obvious that the $\triangle$ detachment-deduction theorem (that is global) is valid. Then, having L_S a GDDT, by a general result of algebraic logic, the quasivariety of L_S-algebras enjoys the congruence extension property, and, consequently, the class of L_S-algebras forms a variety.

Indeed, if $\triangle$ is definable in L_S, then the (MON) inference rule in L_s can equivalently be replaced by the axiom

$$(\mathrm{MON}_\triangle) \quad \triangle(\varphi \to \psi) \to (s\varphi \to s\psi)$$

and so the quasivariety $\mathbb{L}_S$ is in fact defined by a family of equations and thus it is a variety.

Core fuzzy logics L where $\triangle$ is definable include e.g. the n-valued Łukasiewicz logic $\mathbf{Ł}_n$ or the axiomatic extensions of MTL by the axiom $\neg(\varphi)^n \vee \varphi$, called $\mathrm{S}_n\mathrm{MTL}$. In these cases, $\triangle\varphi$ is defined as φ^n. In both cases, we have a sequence of nested logics, $\mathrm{Boolean} = Ł_2 \subset Ł_3 \subset \cdots \subset Ł_n \subset \cdots$ and $\mathrm{Boolean} = \mathrm{S}_2\mathrm{MTL} \subset \mathrm{S}_3\mathrm{MTL} \subset \cdots \subset \mathrm{S}_n\mathrm{MTL} \subset \cdots$ respectively. On the other hand, given a core fuzzy logic L, one can also consider the family of axiomatic extensions of L_S with the axiom $\neg(s^n(\overset{n}{\ldots}(s^n(\varphi^n))^n \ldots))^n \vee \varphi$, where $\triangle$ is also definable. Of course, these logics, denoted $\mathrm{S_nL_S}$, are parameterized by n, and, hence, we obtain again a sequence of nested logics $\mathrm{S_2L_S} \subset \mathrm{S_3L_S} \subset \cdots \subset \mathrm{S_nL_S} \subset \cdots$. In all these logics, $\triangle$ is definable by $\triangle\varphi := (s^n(\overset{n}{\ldots}(s^n(\varphi^n))^n \ldots))^n$.

3.2.3 The case of L_S satisfying the modal axiom K

The third family we consider consists of the logics L_{SK} defined over any core fuzzy logic L (as Hájek did in [47] over any axiomatic extension of BL) by adding a unary (truth-stressing) connective s satisfying the axioms,

$$(\mathrm{VE1}) \quad s\varphi \to \varphi$$
$$(\mathrm{VE2}) \quad s(\varphi \to \psi) \to (s\varphi \to s\psi)$$
$$(\mathrm{VE3}) \quad s(\varphi \vee \psi) \to (s\varphi \vee s\psi)$$

with *modus ponens* and necessitation for s (from φ derive $s(\varphi)$) as inference rules.

Axiom (VE3) is a formula that is derivable in the logic L_S. Axiom (VE2) is the well-known axiom K of modal logics for the truth-stresser s. In our setting it means that if both φ and $\varphi \to \psi$ are "very true" then so is ψ. Moreover, it also implies that the interpretation of s over an MTL-chain is a non-decreasing mapping, as it is in our general system studied in this chapter. However, axiom (VE2) is not always sound in our general framework, i.e. the L_S logic. Take for example the L_S-chain defined over the standard MV-chain $[0,1]_Ł$ by an operator s such that it is non-decreasing, $s(0) = 0$, $s(1) = 1$, $s(x) \le x$ (it is a truth-stressing hedge), and suppose there are two elements $a, b \in [0,1]_Ł$ such that $a > b$ and $s(a) < a$ and $s(b) = b$. Then, $s(a) \to s(b) = 1 - s(a) + s(b) > 1 - a + b = a \to b \ge s(a \to b)$ in contradiction with (VE2).

Next, we prove that L_{SK} is an axiomatic extension of L_S.

LEMMA 3.2.6.

(1) *The following formulas are provable in* L_{SK}*:*

(i) $\neg s\bar{0}$

(ii) $(s\varphi \,\&\, s\psi) \to s(\varphi \,\&\, \psi)$

(iii) $s(\varphi \vee \psi) \leftrightarrow (s\varphi \vee s\psi)$.

(2) *The rule of inference* (MON) *is derivable in* L_{SK}*:*

$$\textit{from } (\varphi \to \psi) \vee \chi \textit{ infer } (s(\varphi) \to s(\psi)) \vee \chi.$$

Proof. By (VE1) $\vdash_{\mathrm{L}_{SK}} s\bar{0} \to \bar{0}$ and so (i) is proved. From $\vdash_{\mathrm{L}_{SK}} \varphi \to (\psi \to (\varphi \& \psi))$, applying necessitation and (VE2), we obtain $\vdash_{\mathrm{L}_{SK}} s\varphi \to (s\psi \to s(\varphi \& \psi))$. Therefore, (ii) is proved as well. Clearly $\vdash_{\mathrm{L}_{SK}} s\varphi \to (s\varphi \vee s\psi)$ and $\vdash_{\mathrm{L}_{SK}} s\psi \to (s\varphi \vee s\psi)$, then $\vdash_{\mathrm{L}_{SK}} (s\varphi \vee s\psi) \to (s\varphi \vee s\psi)$, and, taking into account (VE3), (iii) is proved.

Finally from $(\varphi \to \psi) \vee \chi$, using necessitation and (ii) of this lemma, we infer $s(\varphi \to \psi) \vee s\chi$, and, by (VE1), $s(\varphi \to \psi) \vee \chi$ and, by (VE3), we infer $(s\varphi \to s\psi) \vee \chi$. Consequently, (2) is also proved. □

COROLLARY 3.2.7. L_{SK} *is the axiomatic extension of* L_S *by adding the axiom* (VE2).

Now, following Hájek in [47], we prove a deduction-like theorem (similar to the one proved for $\triangle$). We will need an auxiliary notation: $\tau\varphi$ stands for $s(\varphi \,\&\, \varphi)$ and $\tau^n\varphi$ stands for $\tau(\ldots\tau(\tau\varphi)\ldots)$.

LEMMA 3.2.8. *In* L_{SK} *the following formulas are provable:*

(i) $\tau^{n+1}\varphi \to \tau^n\varphi$,

(ii) $\tau\varphi \to s\varphi$, $\tau\varphi \to \varphi \,\&\, \varphi$,

(iii) $\tau(\varphi \vee \psi) \leftrightarrow (\tau\varphi \vee \tau\psi)$.

THEOREM 3.2.9 (LDDT). *Let T be a theory and let* φ, ψ *be formulas. Then:* $T \cup \{\varphi\} \vdash_{\mathrm{L}_{SK}} \psi$*, iff, for some* n*,* $T \vdash_{\mathrm{L}_{SK}} \tau^n\varphi \to \psi$.

Proof. As usual, let us check the deduction rules. If $T \vdash_{\mathrm{L}_{SK}} \tau^n\varphi \to \alpha$ and also $T \vdash_{\mathrm{L}_{SK}} \tau^n\varphi \to (\alpha \to \beta)$, then $T \vdash_{\mathrm{L}_{SK}} (\tau^n\varphi \& \tau^n\varphi) \to \beta$, thus $T \vdash_{\mathrm{L}_{SK}} \tau^{n+1}\varphi \to \beta$. Similarly, if $T \vdash_{\mathrm{L}_{SK}} \tau^n\varphi \to \beta$, then $T \vdash_{\mathrm{L}_{SK}} s(\tau^n\varphi) \to s\beta$, thus $T \vdash_{\mathrm{L}_{SK}} \tau^{n+1}\varphi \to s\beta$. □

The corresponding algebraic structures are the L_{SK}-algebras. An algebra $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, s, 0, 1\rangle$ is an L_{SK}-algebra if it is an L-algebra expanded with a unary operator s (truth-stressing hedge) that satisfies, for all $x, y \in A$,

(ve1) $s(x) \leq x$

(ve2) $s(x \to y) \leq (s(x) \to s(y))$

(ve3) $s(x \vee y) \leq (s(x) \vee s(y))$

(ve4) $s(1) = 1$.

From the above remarks, an L_{SK}-algebra is just an L_S-algebra also satisfying the property (ve2). In this case, it is obvious that L_{SK}-algebras form a variety (recall that, like in the expansion with $\triangle$, the inference rules of the logic are MP and necessitation). On the other hand, as usual, for each left-continuous t-norm $*$, the chain obtained by adding to $[0,1]_*$ a truth-stressing hedge s satisfying the above properties is an L_{SK}-chain called a real chain.

Next, we give some examples of truth-stressers on real chains $[0,1]_*$ satisfying axiom (VE2). We will call them K-truth-stressers.

EXAMPLE 3.2.10. (1) *The function $s(x) = x * \overset{n}{\ldots} * x$ (x^n for short) is a K-truth-stressing function over $[0,1]_*$ for any left-continuous t-norm $*$. Obviously, this truth-stressing function is continuous if so is the t-norm, and it is the identity if the t-norm corresponds to the minimum.*

(2) *The function $s(x) = x \cdot x$ (product of reals) is also a K-truth-stressing function for the three basic continuous t-norms. Observe that this function coincides with the one of the previous example whenever $*$ is the product t-norm and $n = 2$.*

(3) *The function defined by the Łukasiewicz t-norm as $s(x) = x*x = \max\{0, 2x-1\}$ is a K-truth-stressing function for Łukasiewicz and minimum t-norms but not for the product. This function coincides with the first example for the Łukasiewicz t-norm and $n = 2$.*

(4) *For any $k \in [0,1]$, the function $s(x) = k \cdot x$ for $x < 1$ and $s(1) = 1$ is a K-truth-stressing function for the three basic continuous t-norms. Observe that when $k = 0$, this is the $\triangle$ operator.*

Since it is an axiomatic extension of L_S, the logic L_{SK} is semilinear and so it is complete with respect to the quasivariety of L_{SK}-algebras and with respect to the class of L_{SK}-chains. The problem of standard completeness for the logics L_{SK} is far from being solved. When L is the logic of a Gödel chain (for continuous t-norms) or a WNM-chain (for the general MTL-chains) the problem is easy, since we have the following result.

PROPOSITION 3.2.11. *Let* L *be the logic of a given* WNM-*chain.*[15] *Then the* L_{SK} *logic coincides with the logic* L_S.

Proof. It is only necessary to prove that axiom (VE2) is valid over each L_S-chain. This is easy, because, if $a \leq b$, then $s(a \to b) = 1 = s(a) \to s(b)$, and, if $a > b$, then either $s(a) = s(b)$ and then $s(a \to b) \leq s(a) \to s(b) = 1$, or $s(a \to b) = s(\neg a \vee b) = s(\neg a) \vee s(b) \leq \neg s(a) \vee s(b) = s(a) \to s(b)$ (take into account that $s(\neg a) \leq \neg a \leq \neg s(a)$). □

Applying Corollary 3.1.8 we obtain the following result.

COROLLARY 3.2.12. *Let* L *be the logic of a given* WNM-*chain. Then* L_{SK} *is (finite) strong real complete whenever* L *is (finite) strong real complete.*

The only logic of a continuous t-norm that satisfies (VE2) is Gödel logic, and thus G_{SK} is strong real complete. For the rest of logics L of continuous t-norms, the problem of real completeness, both for the general case L_{SK} and for $Ł_{SK}$ or Π_{SK}, is still open.

[15] Notice that a Gödel chain is a particular case of a WNM-chain.

3.3 The case of truth-depressers

Similar to the case of truth-stressers, we can proceed to define an axiomatization for truth-depressers just by replacing axioms (VTL1) and (VTL2) with their dual versions (STL1) and (STL2) (for *slightly true*). Namely, given a core fuzzy logic L, we define L_D as the expansion of L with a new unary connective d (for *depresser*), the following additional axioms

(STL1) $\varphi \to d\varphi$
(STL2) $\neg d\overline{0}$

and the following additional inference rule

(MON) from $(\varphi \to \psi) \vee \chi$ infer $(d\varphi \to d\psi) \vee \chi$.

Since L_D is a kind of dual version of L_S, many properties are proved in a completely analogous way.

LEMMA 3.3.1. *The following deductions are valid in* L_D*:*

(i) $\vdash_{\mathrm{L}_D} d\overline{1}$

(ii) $\varphi \to \psi \vdash_{\mathrm{L}_D} d\varphi \to d\psi$

(iii) $\neg\varphi \vdash_{\mathrm{L}_D} \neg d\varphi$

(iv) $\vdash_{\mathrm{L}_D} \neg d\varphi \to \neg\varphi$

(v) $d\varphi, \varphi \to \psi \vdash_{\mathrm{L}_D} d\psi$.

Proof. (i) follows directly from (STL1) taking $\varphi = \overline{0}$.
(ii) follows directly from (MON) taking $\chi = \overline{0}$.
(iii) follows from (ii) for $\psi = \overline{0}$ and (STL2).
(iv) follows directly from (STL1) using the fact that $\vdash_{\mathrm{MTL}} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$.
(v) is very easy using (ii) and *modus ponens*. □

Notice that (v) is a kind of weaker or modified version of *modus ponens:* if φ implies ψ and φ is slightly true, then one can derive that ψ is slightly true as well.

Again, (ii) shows that the congruence condition is satisfied for the new unary connective too. Therefore, the logic L_D is Rasiowa-implicative (see Chapter II, Section 2), and its equivalent algebraic semantics is the class of L_D-algebras. An algebra $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, d, \overline{0}, \overline{1}\rangle$ of type $\langle 2, 2, 2, 2, 1, 0, 0\rangle$ is an L_D-algebra if it is an L-algebra expanded with a unary operator $d\colon A \to A$ (truth-depressing hedge) that satisfies, for all $x, y, z \in A$,

(1′) $d(0) = 0$,

(2′) $x \leq d(x)$,

(3′) if $(x \to y) \vee z = \overline{1}$ then $(d(x) \to d(y)) \vee z = \overline{1}$.

Also, since the lattice disjunction still satisfies the (PCP) in the expanded logic, L_D is semilinear and hence complete with respect to the semantics of all L_D-chains. As a straightforward consequence, we have:

LEMMA 3.3.2. *The following deductions are valid in* L_D*:*

(vi) $\vdash_{\mathrm{L}_D} d(\varphi \vee \psi) \leftrightarrow d\varphi \vee d\psi$

(vii) $\vdash_{\mathrm{L}_D} d(\varphi \wedge \psi) \leftrightarrow d\varphi \wedge d\psi.$

Taking $s(a,b) = \langle a \vee b, a \vee b\rangle$, Example 3.1.5 shows that in the context of truth-depressers the rule (MON) cannot be substituted by simple monotonicity. Similarly, Example 3.1.6 can be modified by taking the function d:

x	0	1	2	3	4	5
$d(x)$	0	1	2	4	4	5

showing that in the presentation of L_D the rule (MON) cannot be substituted by the following rule: from $\neg\varphi$ infer $\neg d\varphi$. Indeed, d is superdiagonal, maps the bottom element to itself, is non-decreasing, and satisfies $\neg d(x) \in F$ whenever $\neg x \in F$. However, $3 \to 2 = 4 \in F$, while $d(3) \to d(2) = 4 \to 2 = 3 \notin F$.

Finally, analogous proofs makes it possible to prove this theorem about preservation of completeness properties:

THEOREM 3.3.3 ((Real) completeness properties). *Let* L *be a core fuzzy logic,* $\mathbb{K}$ *a class of* L*-chains and* $\mathbb{K}_D$ *the class of* L_D*-chains whose* d*-free reducts are in* $\mathbb{K}$*. Then:*

(i) *If* L *has the* $\mathrm{FS}\mathbb{K}\mathrm{C}$*, then* L_D *has the* $\mathrm{FS}\mathbb{K}_D\mathrm{C}$.

(ii) *If* L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$ *and all the chains in* $\mathbb{K}$ *are completely ordered, then* L_D *has the* $\mathrm{S}\mathbb{K}_D\mathrm{C}$.

4 Expansions with an involutive negation

In all the t-norm based fuzzy logics studied in the previous chapters, the negation connective $\neg$ is defined from the implication $\to$ and the truth constant $\bar{0}$, namely $\neg\varphi$ is $\varphi \to \bar{0}$. However, this negation may behave quite differently in different varieties of algebras. Indeed, for instance, the associated negation function is involutive in any IMTL chain (in particular in algebras associated to Łukasiewicz logic), but it may not be involutive outside the variety of IMTL-algebras. The most paradigmatic cases are the chains of the variety of SMTL-algebras, where $\neg$ is interpreted as the so-called Gödel's negation n_G, defined by:

$$n_G(x) = x \Rightarrow 0 = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this section, we will define and study the expansion of any axiomatic extension of $\mathrm{MTL}_\triangle$ with an independent involutive negation. Some particularly interesting cases are those of $\mathrm{SMTL}_\sim$ and its axiomatic extensions $\mathrm{G}_\sim$ and $\Pi_\sim$, where $\triangle$ is definable as a composition of the two negations (residuated and involutive) defined there.

Notice that having an involutive negation in the logic enriches, in a non-trivial way, the expressive power of the logical language. For instance, in the enriched language:

- a strong disjunction $\varphi \veebar \psi$ is definable as $\sim(\sim\varphi \,\&\, \sim\psi)$, thus having a truth function in real algebras $[0,1]_*$ defined by the *dual t-conorm* $\oplus$ given by $x \oplus y = n(n(x) * n(y))$;
- a contrapositive implication $\varphi \hookrightarrow \psi$ is definable as $\sim\varphi \veebar \psi$, thus having a truth function corresponding to the *strong implication* function $\overset{c}{\Rightarrow}$ defined as $x \overset{c}{\Rightarrow} y = \sim x \oplus y$.

Although these new connectives are interesting for future developments and are already present in early fuzzy logic papers (see for example, [3, 83, 88]), we shall make no further use of them in the rest of the chapter.

4.1 Expanding a $\triangle$-core fuzzy logic with an involutive negation

Following [35], we define the expansion of a $\triangle$-core fuzzy logic with an involutive negation as follows.

DEFINITION 4.1.1. *Let* L *be a* $\triangle$*-core fuzzy logic. Then the logic* $\mathrm{L}_\sim$ *is the axiomatic expansion of* L *obtained adding a new unary connective* $\sim$ *satisfying the following two additional axioms:*

(~ 1)	$(\sim\sim\varphi) \leftrightarrow \varphi$	*(Involution)*
(~ 2)	$\triangle(\varphi \to \psi) \to (\sim\psi \to \sim\varphi)$	*(Order Reversing)*

LEMMA 4.1.2. *In* $\mathrm{L}_\sim$ *the following inference rule is derivable and the following formulas are provable:*

(AMON)	*from* $\varphi \to \psi$ *infer* $\sim\psi \to \sim\varphi$	*(Antimonotonicity)*
$(\mathrm{DM1})$	$\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$	*(De Morgan law for* $\wedge$*)*
$(\mathrm{DM2})$	$\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$	*(De Morgan law for* $\vee$*)*

Proof. As for the inference rule (AMON), from $\varphi \to \psi$, by the necessitation rule for $\triangle$, $\mathrm{L}_\sim$ proves $\triangle(\varphi \to \psi)$, and, by axiom ($\sim 2$), it also proves $\sim\psi \to \sim\varphi$. Let us prove $(\mathrm{DM1})$. Clearly, L proves $\varphi \wedge \psi \to \varphi$ and $\varphi \wedge \psi \to \psi$. By (AMON), $\mathrm{L}_\sim$ proves $\sim\varphi \to \sim(\varphi \wedge \psi)$ and $\sim\psi \to \sim(\varphi \wedge \psi)$, and thus it proves $(\sim\varphi \vee \sim\psi) \to \sim(\varphi \wedge \psi)$ as well. Analogously, we can prove $\sim(\varphi \vee \psi) \to (\sim\varphi \wedge \sim\psi)$. Substituting φ and ψ by $\sim\varphi$ and $\sim\psi$ in the last formula we have $\sim(\sim\varphi \vee \sim\psi) \to \varphi \wedge \psi$, and by (AMON) we infer $\sim(\varphi \wedge \psi) \to (\sim\varphi \vee \sim\psi)$. This ends the proof of $(\mathrm{DM1})$. The proof for $(\mathrm{DM2})$ is analogous. $\square$

It is very easy to show that $\mathrm{L}_\sim$ is itself a $\triangle$-core fuzzy logic.

THEOREM 4.1.3. *Let* L *be a* $\triangle$*-core fuzzy logic. Then* $\mathrm{L}_\sim$ *is itself a* $\triangle$*-core fuzzy logic, that is, the following conditions are satisfied:*

(1) $\mathrm{L}_\sim$ *has the congruence property for* $\sim$*, i.e. for any formulas* φ, ψ *of* $\mathrm{L}_\sim$ *it holds:*

$$\varphi \leftrightarrow \psi \vdash_{\mathrm{L}_\sim} {\sim}\varphi \leftrightarrow {\sim}\psi.$$

(2) $\mathrm{L}_\sim$ *satisfies the* $\triangle$*-deduction theorem, i.e. for each theory* T *over* $\mathrm{L}_\sim$ *and formula* φ *of* $\mathrm{L}_\sim$*, it holds:*

$$T \cup \{\varphi\} \vdash_{\mathrm{L}_\sim} \psi \textit{ iff } T \vdash_{\mathrm{L}_\sim} \triangle\varphi \to \psi.$$

Proof. The congruence property is an immediate consequence of the (AMON) inference rule. Also the $\triangle$-deduction theorem for $\mathrm{L}_\sim$ is a direct consequence of the fact that $\mathrm{L}_\sim$ is in fact an axiomatic expansion of L, i.e. no new inference rules are added. □

The corresponding algebraic semantics for $\mathrm{L}_\sim$ is given by the class of $\mathrm{L}_\sim$-algebras, defined in the natural way.

DEFINITION 4.1.4. *Let* L *be a* $\triangle$*-core fuzzy logic. An* $\mathrm{L}_\sim$*-algebra is an* L*-algebra expanded with a unary operation* $\sim$ *satisfying the following conditions:*

$(A_\sim 1)$ $\quad {\sim}{\sim}x = x,$

$(A_\sim 2)$ $\quad$ *if* $x \leq y$*, then* ${\sim}y \leq {\sim}x$*.*

Since $\mathrm{L}_\sim$ is a $\triangle$-core fuzzy logic, then we know that the class of $\mathrm{L}_\sim$-algebras is in fact a variety (see e.g. Section 3.2 in Chapter I), since $(A_\sim\ 2)$ can be equivalently expressed as an equation using $\triangle$. Moreover, we get also for free that $\mathrm{L}_\sim$-algebras are representable as subdirect products of $\mathrm{L}_\sim$-chains and that $\mathrm{L}_\sim$ is strongly complete with respect to the class of $\mathrm{L}_\sim$-chains.

Since, by definition, the $\sim$-free reducts of $\mathrm{L}_\sim$-chains are L-chains, chain completeness readily yields that for each $\triangle$-core fuzzy logic L, $\mathrm{L}_\sim$ is a conservative expansion of L.

PROPOSITION 4.1.5. *Let* L *be any* $\triangle$*-core fuzzy logic. Then the logic* $\mathrm{L}_\sim$ *is a conservative expansion of* L*.*

As usual, the $\mathrm{L}_\sim$-chains over the unit real interval, that will be called *real* $\mathrm{L}_\sim$-chains, are especially interesting. If $\boldsymbol{A}$ is a real $\mathrm{L}_\sim$-chain, then it is the expansion of its L-reduct with a strong negation function $n\colon [0,1] \to [0,1]$, that is a strictly decreasing function n such that $n(0) = 1$ and such that $n(n(x)) = x$ for all $x \in [0,1]$.

REMARK 4.1.6. *It is well known that all strong negation functions on* $[0,1]$ *are isomorphic to each other (see [81]), that is, if* n *and* n' *are strong negation functions, there is a strictly increasing mapping* $h\colon [0,1] \to [0,1]$*, with* $h(0) = 0$ *and* $h(1) = 1$*, such that* $n'(x) = h^{-1}(n(h(x)))$ *for all* $x \in [0,1]$*. In particular, all strong negation functions are isomorphic to the so-called* standard negation*, defined as* $n_s(x) = 1 - x$*. Accordingly, real* $\mathrm{L}_\sim$*-chains having* n_s *as involutive negation will be called* standard $\mathrm{L}_\sim$*-chains. Notice that if* $\boldsymbol{A}$ *is a real* $\mathrm{L}_\sim$*-chain, there always exists a standard* $\mathrm{L}_\sim$*-chain* $\boldsymbol{A}'$ *which is isomorphic to* $\boldsymbol{A}$*. Indeed, if* h *is the mapping such that* $n_s = h^{-1} \circ n \circ h$*, where* n *is the involutive negation in* $\boldsymbol{A}$*, then the operations of* $\boldsymbol{A}'$ *are obtained applying the same* *transformation, i.e., for instance if $\star$ is a binary operation in $\boldsymbol{A}$, the corresponding operation in $\boldsymbol{A}'$ is defined as $\star' = h^1 \circ \star \circ (h \times h)$. Therefore, when talking later about different kinds of completeness properties of logics $\mathrm{L}_\sim$ with respect to the whole class of real $\mathrm{L}_\sim$-chains, like $\mathrm{S}\mathcal{R}\mathrm{C}$ or $\mathrm{FS}\mathcal{R}\mathrm{C}$, we can always restrict ourselves to the subclass of real chains with the standard negation.*

Some comments are in order here. In [28] the authors give an axiomatization of $\mathrm{SBL}_\sim$, and of their main axiomatic extensions $\mathrm{G}_\sim$ and $\Pi_\sim$, as expansions of SBL, G and Π, respectively, with an involutive negation. In these logics the initial negation $\neg$ is Gödel negation and the operator $\triangle$ is definable as the composition of the two negations, i.e. $\triangle\varphi$ is defined as $\neg{\sim}\varphi$, but the axiomatization needs the addition of the necessitation rule for $\triangle$. Hence, even though $\triangle$ is definable, in some sense, the logic is an expansion of a logic with $\triangle$.

The axiomatization of $\mathrm{L}_\sim$ for a $\triangle$-core fuzzy logic L presented in this section makes heavily use of the $\triangle$ operator. An interesting question is the possibility of obtaining an axiomatization without $\triangle$. An approach, suggested in [35], would be to take the axiom (~ 1) together with the previously mentioned rule (AMON). However, this axiomatization produces a logic which is not semilinear as the following example shows.

EXAMPLE 4.1.7. Let $\boldsymbol{B}_4^\sim$ be the algebra obtained by expanding the four element Boolean algebra $\boldsymbol{B}_4$ with the involutive negation $\sim$ defined by ${\sim}0 = 1$, ${\sim}1 = 0$, ${\sim}a = a$, ${\sim}b = b$. Let $\mathrm{L}_{\boldsymbol{B}_4^\sim}$ be the finitary logic given by the matrix $\langle \boldsymbol{B}_4^\sim, \{1\}\rangle$. This logic is an expansion of Classical logic that is not semilinear. Indeed, the $\vee$-form of the (AMON) inference rule, i.e.

$$\text{from } \ (\varphi \to \psi) \vee \chi \quad \text{infer} \quad ({\sim}\psi \to {\sim}\varphi) \vee \chi$$

is not sound. Namely, take for instance φ, ψ, χ to be three different propositional variables and an evaluation e such that $e(\varphi) = 1, e(\psi) = b$ and $e(\chi) = a$. Then we have $(1 \to b) \vee a = b \vee a = 1$, while $({\sim}b \to {\sim}1) \vee a = a \vee a = a$. Thus, the logic is not semilinear (see Chapter II).

On the other hand, another possibility considered in the same paper amounts to axiomatizing $\sim$ with axiom (~ 1) and the axiom

$$(\sim 3) \quad (\varphi \to \psi) \to ({\sim}\psi \to {\sim}\varphi).$$

However, the authors explicitly mention that this solution might not be completely satisfactory. Indeed, for any element a of an algebra of the variety corresponding to this logic, the new axiom (~ 3) implies $\neg a = a \to 0 = 1 \to {\sim}a = {\sim}a$. So, axiom (~ 3) forces the two negations to coincide, and thus $\neg$ becomes involutive as well. Therefore, the expansion of a core fuzzy logic L with an additional negation $\sim$ together with the axioms (~ 1) and (~ 3) turns out to be equivalent to the axiomatic extension of L with the involutiveness axiom

$$(\neg\neg) \quad \neg\neg\varphi \to \varphi$$

for the residual negation $\neg$ of L. Thus L must be an axiomatic extension of an IMTL logic.

4.2 Real completeness

As shown in [35], the Jenei-Montagna method for embedding a MTL-chain into a real MTL-chain [59]) can be extended to the case of MTL-chains with an involutive negation. Based on this result we can show the following general completeness result.

THEOREM 4.2.1. *Let* L *be the expansion with* $\triangle$ *of an axiomatic extension* L' *of* MTL. *If the Jenei-Montagna completion method provides a way to embed any countable* L'*-chain into a real* L'*-chain, then the logic* $\mathrm{L}_\sim$ *has the* $\mathrm{S}\mathcal{R}\mathrm{C}$.

Proof. The proof is an easy extension of the Jenei-Montagna method (a particular case of the embedding given in Lemma 4.1.4 of Chapter IV; see the original construction in [59] and its adaptation the involutive case in [25]). Let $\boldsymbol{C}$ be a countable $\mathrm{L}_\sim$, let $\boldsymbol{C}'$ be the completion of its L-reduct given by the Jenei-Montagna completion method, and let $\boldsymbol{D}$ be the real L-chain where $\boldsymbol{C}'$ embeds. Call this last embedding h. Then we can define an involutive negation n on $\boldsymbol{C}'$ as follows: for each $\langle s, q\rangle \in C'$ (with $s \in C$ and $q \in (0,1] \cap \mathrm{Q}$),

$$n(s,q) = \begin{cases} \langle \sim s, 1\rangle & \text{if } q = 1, \\ \langle succ(\sim s), 1-q\rangle & \text{otherwise,} \end{cases}$$

where $\sim$ denotes the involutive negation in the original $\mathrm{L}_\sim$-chain $\boldsymbol{C}$, and $succ(s)$ denotes the successor of s, if it exists, otherwise $succ(s) = s$. The expansion of $\boldsymbol{C}'$ with n makes it an $\mathrm{L}_\sim$-chain. Then, it is easy to check that the embedding of $\boldsymbol{C}$ into $\boldsymbol{C}'$, defined by $s \mapsto \langle s, 1\rangle$, is indeed also a morphism with respect to the involutive negations, and hence $\boldsymbol{C}$ embeds into the dense chain $\boldsymbol{C}'$ also as $\mathrm{L}_\sim$-chains. We can extend n over an involution on the real unit interval $\bar{n}$ by defining $\bar{n}(x) = \inf\{h(n(z)) \mid z \in \boldsymbol{C}', h(z) \leq x\}$. Again, expanding $\boldsymbol{D}$ with $\bar{n}$ makes it a real $\mathrm{L}_\sim$-chain where $\boldsymbol{C}'$ embeds (as $\mathrm{L}_\sim$-chains). Finally, from the compound embedding of the original chain $\boldsymbol{C}$ into $\boldsymbol{D}$ as $\mathrm{L}_\sim$-chains, the $\mathrm{S}\mathcal{R}\mathrm{C}$ of $\mathrm{L}_\sim$ immediately follows (see Chapter II, Section 3.4). □

As a direct consequence of this theorem we get the $\mathrm{S}\mathcal{R}\mathrm{C}$ for all the logics $\mathrm{L}_\sim$ with $\mathrm{L} \in \{\mathrm{MTL}_\triangle, \mathrm{SMTL}_\triangle, \mathrm{IMTL}_\triangle, \mathrm{WNM}_\triangle, \mathrm{NM}_\triangle, \mathrm{G}_\triangle\}$.

For the case of L being $\mathrm{G}_\triangle$, there is a stronger result. Indeed, in this case the logic $\mathrm{L}_\sim$, that we will denote by $\mathrm{G}_\sim$, is not only strongly complete with respect to the class of real $\mathrm{G}_\sim$-chains but also with respect the *standard* $\mathrm{G}_\sim$-chain, i.e. with respect to the real $\mathrm{G}_\sim$-chain where the involutive negation is the standard one, $n_s(x) = 1-x$. This is due to the fact that, as already mentioned in Remark 4.1.6, all involutive negations on $[0,1]$ are isomorphic to each other and there is only one real G-chains that is in fact the standard G-chain $[0,1]_G$.

PROPOSITION 4.2.2. *The logic* $\mathrm{G}_\sim$ *is strongly standard complete, i.e. strongly complete w.r.t. the standard* $\mathrm{G}_\sim$*-chain.*

Theorem 4.2.1 says nothing about expansions of other logics like $\Pi\mathrm{MTL}_\sim$, $\mathrm{BL}_\sim$, $\Pi_\sim$, or $\mathrm{SBL}_\sim$ since on these logics the Jenei-Montagna method does not apply. Nevertheless, all the expansions $\mathrm{L}_\sim$ where L is the $\triangle$-expansion of the logic of a continuous t-norm enjoy the $\mathrm{FS}\mathcal{R}\mathrm{C}$. In fact we have the following more general result.

THEOREM 4.2.3. *Let* L *be the* $\triangle$*-core fuzzy logic with a finite language enjoying the* $\mathrm{FS}\mathcal{R}\mathrm{C}$. *Then the logic* $\mathrm{L}_\sim$ *has the* $\mathrm{FS}\mathcal{R}\mathrm{C}$ *as well.*

Proof. Taking into account that, under the assumption of finite language, $\mathrm{FS}\mathcal{R}\mathrm{C}$ is equivalent to the partial embeddability property into real chains [18], it is enough to prove that this property extends from L to $\mathrm{L}_\sim$. Let X be any finite subset of an $\mathrm{L}_\sim$-chain $\boldsymbol{A}$ and let $Y = X \cup \{0,1\} \cup \{\sim x \mid x \in X\}$. Being Y finite, there exists a partial embedding of Y into a real L-chain $\boldsymbol{A}'$, call it h. We can always define an involution n on $\boldsymbol{A}'$ coinciding with $\sim$ over $h[Y]$, i.e. such that $n(h(x)) = h(\sim x)$ for every $x \in Y$. □

A direct consequence of this result is that e.g. $\mathrm{SBL}_\sim$ and $\Pi_\sim$ enjoy the $\mathrm{FS}\mathcal{R}\mathrm{C}$.

As we have already pointed out, all real $\mathrm{G}_\sim$-chains are isomorphic to each other. However, this does not hold in general, e.g. this is not true for $\Pi_\sim$. Indeed, in [28], in order to show that $\Pi_\sim$ is not standard complete, i.e. it is not complete with respect to the standard $\Pi_\sim$-chain $[0,1]_{\Pi_\sim} = \langle [0,1], \max, \min, *_\Pi, \Rightarrow_\Pi, n_s, 0, 1\rangle$, the authors check that the formula $(\sim\varphi \& \varphi) \to (\sim(\sim\varphi\&\varphi))^3$ (where ψ^3 means $\psi\&\psi\&\psi$) is a 1-tautology over $[0,1]_{\Pi_\sim}$ but it is not a 1-tautology in some real $\Pi_\sim$-chain with a strong negation different from n_s. In fact, $\mathrm{SBL}_\sim$ is not complete with respect to only one real chain.

4.3 The lattice of subvarieties of $\Pi_\sim$-algebras and of $\mathrm{SBL}_\sim$-algebras

The fact that the logics $\mathrm{SBL}_\sim$ and $\Pi_\sim$ are only $\mathrm{FS}\mathcal{R}\mathrm{C}$ complete, i.e. complete with respect to all the real $\mathrm{SBL}_\sim$- and $\Pi_\sim$-chains respectively, makes the study of the logics of families of these chains (and equivalently, the subvarieties generated by families of these chains) interesting.

We start by studying the lattice of subvarieties generated by real $\Pi_\sim$-chains. To do so, the next proposition highlights the important role played by the set $\mathcal{I}$ of the $\Pi_\sim$-chains defined over the standard Π-chain by adding an involutive negation with $\frac{1}{2}$ as its fixed point. Since a $\Pi_\sim$-chain $\langle [0,1], *_\Pi, \Rightarrow_\Pi, \min, \max, n, 0, 1\rangle$ of $\mathcal{I}$ is determined by the involutive negation n, in what follows we will denote it by $[0,1]_{\Pi,n}$.

PROPOSITION 4.3.1. *Any real* $\Pi_\sim$*-chain is isomorphic to a* $\Pi_\sim$*-chain from* $\mathcal{I}$.

Proof. Let $\boldsymbol{D}$ be a real $\Pi_\sim$-chain with n being its involutive negation. As recalled in Remark 4.1.6, any real Π-chain is isomorphic to the standard one. Let f be the isomorphism between the Π-chain reduct of $\boldsymbol{D}$ and the standard Π-chain. Then $\boldsymbol{D}$ is isomorphic to the $\Pi_\sim$-chain defined over the standard Π-chain adding the involutive negation $\bar{n} = f^{-1} \circ n \circ f$. Denote by s the fixed point of $\bar{n}$. On the other hand, any automorphism g of the standard Π-chain is of the form $x \to x^a$ for a fixed $a \in \mathsf{R}^+$. Let g be the automorphism defined by taking an a such that $s^a = \frac{1}{2}$. Then, $g \circ f$ gives the desired isomorphism, since g transforms the involutive negation $\bar{n}$ into a new involutive negation with fixed point $\frac{1}{2}$. □

Thus, in order to study the subvarieties generated by real $\Pi_\sim$-chains, one only needs to consider as generators the chains belonging to $\mathcal{I}$. The main result in this section is stated in the following theorem.

THEOREM 4.3.2. *The lattice of subvarieties generated by real* $\Pi_\sim$*-chains has infinite height and infinite (uncountable) width.*

The result is really surprising if we take into account that the lattice of subvarieties of Π-algebras contains Boolean algebras as the only proper subvariety and, thus, the addition of an involutive negation gives rise to a continuum of subvarieties.

The proof of the next theorem is based on results from [38].[16] Actually, in that paper only the infinite (uncountable) width result is proved, while the infinite height result is proved in [19]. Next, we follow [38] for the proof of the uncountable width result, and from there we provide a new proof of the infinite height result.

THEOREM 4.3.3. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be two real $\Pi_\sim$-chains. The variety generated by $\boldsymbol{C}$ is comparable with the variety generated by $\boldsymbol{D}$ if, and only if, $\boldsymbol{C}$ and $\boldsymbol{D}$ are isomorphic.*

By the previous result we can restrict ourselves to chains in $\mathcal{I}$. Obviously, any two chains from $\mathcal{I}$ are isomorphic to each other if, and only if, they are the same chain. Thus, the theorem says that two different chains in $\mathcal{I}$ generate incomparable subvarieties. To prove this statement we need several lemmas.

LEMMA 4.3.4. *If $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, then $n(\frac{1}{2^k}) = \eta(\frac{1}{2^k})$ for all $k \in \mathsf{N}$.*

Proof. Consider for any $k, l, m \in \mathsf{N}$, the equations,[17]

$$(\sim((x \vee \sim(x))^k))^l \leq (y \vee \sim(y))^m \tag{1}$$

$$(\sim((x \wedge \sim(x))^k))^l \geq (y \wedge \sim(y))^m. \tag{2}$$

Equation (1) is valid over $[0,1]_{\Pi,n}$ if the inequality holds for any $a, b \in [0,1]$ which is equivalent to

$$\max_{a\in[0,1]}(n(a \vee n(a))^k)^l \leq \min_{b\in[0,1]}(b \vee n(b))^m.$$

It is obvious that these extreme values are obtained at $\frac{1}{2}$, the fixed point of the negations. Thus (1) holds in $[0,1]_{\Pi,n}$ if, and only if,

$$n\left(\frac{1}{2^k}\right) \leq \left(\frac{1}{2}\right)^{\frac{m}{l}}.$$

Similarly, we can prove that (2) holds in $[0,1]_{\Pi,n}$ if, and only if,

$$n\left(\frac{1}{2^k}\right) \geq \left(\frac{1}{2}\right)^{\frac{m}{l}}.$$

If $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, the inequalities that hold for $[0,1]_{\Pi,n}$ also must hold for $[0,1]_{\Pi,\eta}$, and being the set $\{(\frac{1}{2})^{\frac{m}{l}} \mid l, m \in \mathsf{N}\}$ dense in the real unit interval, we conclude that for each $k \in \mathsf{N}$, $n(\frac{1}{2^k}) = \eta(\frac{1}{2^k})$. □

Now for any involutive negation n with fixed point $\frac{1}{2}$, define the set

$$M(n) = \left\{ \left(n\left(\frac{1}{2^k}\right)\right)^l \mid k, l \in \mathsf{N} \right\}.$$

[16] The paper studies logics of strict De Morgan triples, $\Pi_\sim$-chains without residuated implication, but the result is also valid when the implication is included.

[17] Remember that $a \leq b$ is equivalent to $a \wedge b = a$.

LEMMA 4.3.5. *For any involutive negation n with fixed point $\frac{1}{2}$, $M(n)$ is dense in the real unit interval.*

Proof. Obviously $\left\{n(\frac{1}{2^k}) \mid k \in \mathsf{N}\right\}$ is an increasing sequence with limit 1, and thus for any $\epsilon > 0$ there is k_0 such that $1 - n(\frac{1}{2^{k_0}}) < \epsilon$. But $1 - b < \epsilon$ implies $b^m - b^{m+1} = b^m(1-b) < (1-b) < \epsilon$. Therefore, it follows that for each element of the real unit interval there is an element of the sequence $\left\{n(\frac{1}{2^k}))^l \mid k, l \in \mathsf{N}\right\}$ whose difference from it is at most ϵ. $\square$

LEMMA 4.3.6. *If $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, then n and η coincide on $M(n)$.*

Proof. If $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, from Lemma 4.3.4, we know that for all $k, l \in \mathsf{N}$, $n((\frac{1}{2^k}))^l = \eta((\frac{1}{2^k}))^l$. Thus, the sets $M(n)$ and $M(\eta)$ coincide. Now, consider the inequalities:

$$(\sim((\sim((x \wedge \sim(x))^k))^l))^r \leq (y \vee \sim(y))^m \tag{3}$$

and

$$(\sim((\sim((x \vee \sim(x))^k))^l))^r \geq (y \wedge \sim(y))^m. \tag{4}$$

By an argument similar to the one used in Lemma 4.3.4, we obtain that (3) holds in $[0,1]_{\Pi,n}$ if, and only if,

$$n\left(\left(n\left(\frac{1}{2^k}\right)\right)^l\right) \leq \left(\frac{1}{2}\right)^{\frac{m}{r}},$$

and that (4) holds in $[0,1]_{\Pi,n}$ if, and only if,

$$n\left(\left(n\left(\frac{1}{2^k}\right)\right)^l\right) \geq \left(\frac{1}{2}\right)^{\frac{m}{r}}.$$

The same conditions are valid for η and thus, reasoning as in Lemma 4.3.4, we obtain that for all $a \in M(n) = M(\eta)$, $n(a) = \eta(a)$. $\square$

We have shown that if $[0,1]_{\Pi,n}$ belongs to the variety generated by $[0,1]_{\Pi,\eta}$, then n and η agree on a dense set, and since involutive negations are continuous functions, they coincide over the whole real unit interval. This ends the proof of Theorem 4.3.3.

The two families of equations used in the proofs above have been considered separately. However the first family is a special case of the second. Namely, taking $k = 1$, equation (3) becomes (1), and (4) becomes (2) as an easy computation shows. Thus, in fact, we only have one family of equations used to separate the subvarieties. From Theorem 4.3.3 it follows that there are as many incomparable subvarieties as there are involutive negations with $\frac{1}{2}$ as a fixed point. Of course there are uncountable many of the latter. Summarizing, we have the following result.

COROLLARY 4.3.7. *The set of subvarieties of the variety generated by a single real $\Pi_\sim$-chain contains an uncountable set of pairwise incomparable subvarieties. Furthermore, these subvarieties are separated by the following family of equations:*

$$(\sim((\sim((x \wedge \sim(x))^k))^l))^r \leq (y \vee \sim(y))^m.$$

Now, we will prove the infinite height part of Theorem 4.3.2. Set $k_0, m_0 \in \mathsf{N}$ and a strictly increasing sequence of naturals $\{l_i\}_{i\in\mathsf{N}}$. Then, define the sequence $\{T_i\}_{i\in\mathsf{N}}$ of subsets of $\mathcal{I}$ as

$$T_i = \left\{ [0,1]_{\Pi,n} \in \mathcal{I} \mid n\left(\frac{1}{2^{k_0}}\right) \leq \left(\frac{1}{2}\right)^{\frac{m_0}{l_i}} \right\}.$$

Since $\{\frac{m_0}{l_i}\}_{i\in\mathsf{N}}$ is a decreasing sequence with limit 0, for all $i \in \mathsf{N}$, $T_i \subset T_{i+1}$, and the same inclusions hold true for the varieties generated by these families. Finally, an easy observation shows that these inclusions are proper, since the equation (1) for k_0, m_0, and l_i is valid for T_i but not for T_{i+1}. Thus, we have an infinite sequence of strict inclusions of subvarieties, and so the height of the lattice of subvarieties is, at least, countable.

In [19] and [55], the authors offer further insight into the subvarieties of $\mathrm{SBL}_\sim$- and $\Pi_\sim$-algebras. In order to separate the subvarieties, Cintula et al. [19] use a different family of equations. They define for each natural n, the equation[18]

$$\sim((\sim(x^n))^n) = x \tag{D_n}$$

and prove, using these equations, that the lattice of subvarieties of $\mathrm{SBL}_\sim$ and $\Pi_\sim$-algebras contain a sublattice isomorphic to the lattice of natural numbers $\langle \mathsf{N}, \preceq\rangle$, with the order $\preceq$ defined by: $1 \preceq n$ for all $n \in \mathsf{N}$ and $n \preceq m$ if there is a natural k such that $n^k = m$. It is clear that, under this definition, $\langle \mathsf{N}, \preceq\rangle$ has infinite width and infinite height.

Haniková and Savický [55] generalize Theorem 4.3.3 from real $\Pi_\sim$-chains to SBL-chains defined by ordinal sums with a finite number of components in the following way. Let $[0,1]_{*,n}$ denote the SBL-chain defined by a strict Archimedean t-norm $*$ and an involutive negation n.

THEOREM 4.3.8 ([55]). *Let $*$ be a t-norm with a finite number of idempotents. Then, if $*$ is of either of type Π, or $\Pi \oplus j.$Ł, or $\Pi \oplus i.$Ł $\oplus\, \Pi \oplus j.$Ł (where $\oplus$ is interpreted as the ordinal sum, and $i.$Ł means the ordinal sum of i copies of a Łukasiewicz component) then the following two conditions hold for arbitrary involutive negations n_1 and n_2:*

(1) *the varieties generated by $[0,1]_{*,n_1}$ and $[0,1]_{*,n_2}$ coincide iff $[0,1]_{*,n_1}$ is isomorphic to $[0,1]_{*,n_2}$;*

(2) *if the varieties generated by $[0,1]_{*,n_1}$ and $[0,1]_{*,n_2}$ do not coincide then they are incomparable.*

Otherwise, if $*$ is of type $\Pi \oplus i.$Ł $\oplus\, \Pi$ or it contains at last three product components, then (1) does not hold for $*$.

Finally one interesting question is to know whether or not there is an axiomatization with finitely-many axiom schemes for the logic that is complete with respect to a chain $[0,1]_{\Pi,n}$, or equivalently, whether there is a finite equational basis for the subvariety generated by $[0,1]_{\Pi,n}$. The question makes sense because in each of the cited papers [20, 38, 55] the authors give different sets of separating equations, i.e. defining different

[18] We have applied our notation to the equation.

subvarieties, but in each case they need an infinite number of equations to axiomatize the subvariety generated by each one of the real chains.[19] As far as we know, only the case of the standard $\Pi_\sim$ chain, the one defined by the standard negation $n_s(x) = 1-x$, has been proved to be finitely axiomatizable. The proof of this result is not trivial and is based on the study of the logic ŁΠ [29] (see Section 5.2 for details). The original definition of ŁΠ was given in a language with four basic connectives, i.e. the Łukasiewicz and product conjunctions and implications, and it was shown to be complete with respect to the standard ŁΠ-chain $[0,1]_{\text{Ł}\Pi} = \langle [0,1], *_{\text{Ł}}, \Rightarrow_{\text{Ł}}, *_\Pi, \Rightarrow_\Pi, \max, \min, 0, 1\rangle$. However, a very nice result due to Cintula [14] proves that ŁΠ is also complete with respect to the standard $\Pi_\sim$-chain (modulo term equivalence). Indeed, he observes that, over the standard $\Pi_\sim$-chain, the standard Łukasiewicz conjunction and implication operations are definable as follows:

(I) $x \Rightarrow_{\text{Ł}} y = n_s(x *_\Pi n_s(x \Rightarrow_\Pi y))$

(C) $x *_{\text{Ł}} y = x *_\Pi n_s(x \Rightarrow_\Pi n_s(y))$

and, following this idea, he proves that ŁΠ can be defined as an axiomatic extension of the logic $\Pi_\sim$ by adding the axiom:

$$(\varphi \to_{\text{Ł}} \psi) \to_{\text{Ł}} ((\psi \to_{\text{Ł}} \chi) \to_{\text{Ł}} (\varphi \to_{\text{Ł}} \psi)).$$

This result was later complemented by Vetterlein [84], who showed that one could alternatively add the axiom:

$$\varphi \,\&_{\text{Ł}}\, \psi \to_{\text{Ł}} \psi \,\&_{\text{Ł}}\, \varphi.$$

Therefore, the logic that is (standard) complete with respect to $[0,1]_{\Pi_\sim}$ is in fact ŁΠ, and hence it is finitely axiomatizable (since ŁΠ is). A final remark is that if in the above definitions (C) and (I) one takes an involutive negation different from n_s, the resulting operation in (C) is no longer commutative, and, analogously, the resulting function in (I) is not transitive anymore.

5 Expansions of Łukasiewicz logic

Some of the most remarkable properties of Łukasiewicz logic and MV-algebras come from their connection with ordered Abelian groups (see Chapter VI). Chang's Completeness Theorem and McNaughton's Theorem are clear examples of results deriving from this connection. It is then natural to investigate whether the addition of new connectives to Łukasiewicz logic, and new operators to MV-algebras, can lead to finding similar relations to richer and well-known structures like rings and fields.

Apart from purely technical motivations, the interest in exploring expansions of Łukasiewicz logic also comes from the fact that the addition of new connectives significantly increases the expressive power of the logic. This allows to notably enhance the spectrum of definable functions in the algebras over the reals associated to the expansions, yielding richer and more complex logical systems.

[19] In [55], the authors prove that this infinite set of equations does not axiomatize the logic due to the fact that the logic is finitary, and, consequently, there are tautologies that cannot be derived from a finite number of axioms.

A first step is obtained by expanding Łukasiewicz logic with divisibility connectives δ_n, defining the logic RL. The linearly ordered algebras related to RL, called DMV-chains, are related to ordered divisible Abelian groups the same way MV-chains are related to ordered Abelian groups.

Even richer systems come from adding to Łukasiewicz logic the product connective and the product implication. The logics $\mathrm{P}Ł, \mathrm{P}Ł', Ł\Pi$ and $Ł\Pi\frac{1}{2}$ are the results of this expansion. As imagined, the algebras related to these logics, i.e. $\mathrm{PMV}, \mathrm{PMV}^+, Ł\Pi$ and $Ł\Pi\frac{1}{2}$ algebras bear a strong relation with certain rings, integral domains, and fields.

Among the logics mentioned so far, $Ł\Pi\frac{1}{2}$ is the system with greater expressive power. In fact, the first-order theory of real numbers can be interpreted within the equational theory of $Ł\Pi\frac{1}{2}$, making $Ł\Pi\frac{1}{2}$ a powerful framework for the interpretation of other logical systems.

The purpose of this part of the chapter is to explore the above mentioned expansions, providing the basic notions and results, and making their relation with groups, rings and fields explicit.

5.1 Rational Łukasiewicz logic

Rational Łukasiewicz logic RL is an expansion of Łukasiewicz logic obtained by adding the unary connectives δ_n, for each $n \geq 1$, plus the following axioms:

(D1) $n(\delta_n\varphi) \leftrightarrow \varphi$

(D2) $\neg\delta_n\varphi \oplus \neg(n-1)(\delta_n\varphi)$,

with $n\psi := \underbrace{\psi \oplus \cdots \oplus \psi}_{n}$. As in the case of Łukasiewicz logic, other connectives are definable as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
\varphi \,\&\, \psi & := & \neg(\varphi \to \neg\psi) & \varphi \oplus \psi & := & \neg(\neg\varphi \,\&\, \neg\psi) \\
|\varphi - \psi| & := & (\varphi \ominus \psi) \oplus (\psi \ominus \varphi) & \varphi \vee \psi & := & \varphi \oplus (\psi \ominus \varphi) \\
\varphi \wedge \psi & := & \varphi \ominus (\varphi \ominus \psi) & \varphi \ominus \psi & := & \neg(\neg\varphi \oplus \psi) \\
\varphi \leftrightarrow \psi & := & (\varphi \to \psi) \,\&\, (\psi \to \varphi) & \overline{1} & := & \varphi \to \varphi.
\end{array}
$$

The algebraic semantics of RL is given by DMV-*algebras* (Divisible MV-algebras), i.e. structures $\boldsymbol{A} = \langle A, \oplus, \neg, \{\delta_n\}_{n\in\mathsf{N}}, 0\rangle$ of type $\langle 2, 1, 1, 0\rangle$ such that $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra and the following equations hold for all $x \in A$ and $n \geq 1$:

$$(\delta_n 1)\;\; n(\delta_n x) = x, \qquad (\delta_n 2)\;\; \delta_n x \odot (n-1)(\delta_n x) = 0,$$

with $nx := \underbrace{x \oplus \cdots \oplus x}_{n}$. As in the case of MV-algebras, other operations can be defined as follows:

$$
\begin{array}{rcl@{\qquad}rcl}
x \to y & := & \neg x \oplus y & x \odot y & := & \neg(x \to \neg y) \\
x \ominus y & := & \neg(\neg x \oplus y) & |x - y| & := & (x \ominus y) \oplus (y \ominus x) \\
x \wedge y & := & x \ominus (x \ominus y) & x \vee y & := & x \oplus (y \ominus x) \\
x \leftrightarrow y & := & (x \to y) \odot (y \to x). & & &
\end{array}
$$

The class $\mathbb{DMV}$ of DMV-algebras is a variety.

$[0,1]_{\mathrm{DMV}}$ denotes the standard DMV-chain over the real unit interval $[0,1]$, and its operations are defined as follows:

$$
\begin{array}{rclrcl}
x \rightarrow y & = & \min\{1-x+y,1\} & \neg x & = & 1-x \\
x \oplus y & = & \min\{x+y,1\} & x \odot y & = & \max\{x+y-1,0\} \\
x \ominus y & = & \max\{x-y,0\} & |x-y| & = & \max\{x-y,y-x\} \\
x \vee y & = & \max\{x,y\} & x \wedge y & = & \min\{x,y\} \\
x \leftrightarrow y & = & 1-|x-y| & \delta_n x & = & \frac{n}{x}.
\end{array}
$$

An evaluation e of RL-formulas into a DMV-algebra is simply an extension of an evaluation for Łukasiewicz logic (see Chapter VI) for the connectives δ_n, so that $e(\delta_n\varphi) = \delta_n(e(\varphi))$. Notice that in RL, all rationals in $[0,1]$ are definable as truth-constants in the following way:

- $\frac{1}{n}$ is definable as $\delta_n\overline{1}$, and
- $\frac{m}{n}$ is definable as $m(\delta_n\overline{1})$,

since for every evaluation e into the real unit interval $[0,1]$,

$$e(\delta_n\overline{1}) = \frac{1}{n} \qquad \text{and} \qquad e(m(\delta_n\overline{1})) = m\left(\frac{1}{n}\right) = \frac{m}{n}.$$

The definition of 'ideal' for DMV-algebras coincides with the one for MV-algebras.

DEFINITION 5.1.1. *Given an* MV*-algebra* $\boldsymbol{A}$*, a non-empty set* $I \subseteq A$ *is an* ideal *whenever the following properties are satisfied:*

(1) $a \leq b$ *and* $b \in I$ *imply* $a \in I$*,*

(2) $a, b \in I$ *implies* $a \oplus b \in I$.

As a consequence, the proof of the next theorem is almost identical to the proof of Chang's Representation Theorem (see Chapter VI).

THEOREM 5.1.2. *Every* DMV*-algebra is isomorphic to a subdirect product of linearly ordered* DMV*-algebras.*

Every MV-chain is well-known to be isomorphic to the MV-chain defined over the unit interval of an ordered Abelian group with strong unit. A similar result holds from DMV-chains w.r.t. ordered divisible Abelian groups with strong unit.

LEMMA 5.1.3. *Let* $\boldsymbol{A}$ *be a* DMV*-chain. Then there exists an ordered divisible Abelian group* $\boldsymbol{G}$ *with a strong unit* u *such that* $\boldsymbol{A} \cong \Gamma(\boldsymbol{G}, u)$.

Proof. Let $\boldsymbol{A}$ be a DMV-chain and $\boldsymbol{A}'$ its MV-reduct. Clearly, $\boldsymbol{A}'$ is an MV-chain isomorphic to $\Gamma(\boldsymbol{G}_{A'}, (1,0))$ (see Chapter VI). Let $u = (1,0)$. For any $a \in G_{A'}$, there exists an $n \in \mathsf{N}$ such that $nu \leq_{G_{A'}} a \leq_{G_{A'}} (n+1)u$. Let $b = a - nu$. Since $b, u \in [0,u]$, for any $m \in \mathsf{N}$, there exist $c, d \in [0,u]$ such that $b = mc$ and $u = md$, respectively. Then $a = nu + b = n(md) + mc = m(nd + c)$, which means that $\mathbf{G}_{A'}$ is indeed divisible. $\square$

The fact that each DMV-chain is definable over the unit interval of an ordered divisible Abelian group with strong unit and that ordered divisible Abelian groups are *elementarily equivalent* to each other, i.e. they satisfy the same first-order sentences in the language of ordered groups $\langle +, -, 0, < \rangle$, make it possible to prove the next theorems.

THEOREM 5.1.4. $\mathbb{DMV}$ *is generated by the standard* DMV*-chain* $[0,1]_{\mathrm{DMV}}$.

Proof. Let $\boldsymbol{A}$ be a DMV-algebra and $\varphi(\overline{x})$ an equation not valid in $\boldsymbol{A}$. $\boldsymbol{A}$ is a subdirect product of DMV-chains $\boldsymbol{B}_i$, so there is at least one $\boldsymbol{B}_i$ in which $\varphi(\overline{x})$ fails. $\boldsymbol{B}_i$ is isomorphic to the DMV-algebra of an ordered divisible Abelian group $\boldsymbol{G}_{\boldsymbol{B}_i}$. Ordered divisible Abelian groups are elementarily equivalent to each other, and to the group of reals R [8], in particular. Since the operations of a DMV-algebra are definable in the related ordered group (see Chapter VI), this implies that $\varphi(\overline{x})$ does not hold in the reals. $\square$

Moreover, we have:

THEOREM 5.1.5. RL *has the* $\mathrm{FS}\mathcal{R}\mathrm{C}$.

Proof. We just need to show that every DMV-chain $\boldsymbol{A}$ is embeddable into an ultrapower of $[0,1]_{\mathrm{DMV}}$ (see Chapter I). All ordered divisible Abelian groups are elementarily equivalent to each other, and so are all DMV-chains, being structures defined over the interval of ordered divisible Abelian groups. This means that every DMV-chain $\boldsymbol{A}$ is elementarily equivalent to $[0,1]_{\mathrm{DMV}}$, and so, by Frayne's Theorem , it can be embedded into an ultrapower of $[0,1]_{\mathrm{DMV}}$ (see [8]). $\square$

5.2 Expansions with the product connective

We are now going to study expansions of Łukasiewicz logic that include the product conjunction. We will study some of their basic algebraic properties and their relationship to certain classes of ordered commutative rings to provide completeness results. We will devote special attention to the logic $\text{Ł}\Pi\frac{1}{2}$, that combines both Product and Łukasiewicz logics in a unified framework, whose algebraic semantics bears a strong relation to ordered fields.

5.2.1 The axiomatic systems and their algebraic semantics

The language of the logic $\mathrm{P\text{Ł}}$ (Product Łukasiewicz) is the language of Łukasiewicz logic, i.e. $\{\to, \neg\}$, plus the Product connective $\&_\Pi$. All the connectives definable in Łukasiewicz logic are obviously definable in $\mathrm{P\text{Ł}}$ (see Chapter VI and Section 5.1 above). The axioms of the logic $\mathrm{P\text{Ł}}$ are those of Łukasiewicz logic, plus the following axioms:

(P1) $(\varphi \,\&_\Pi\, \psi) \ominus (\varphi \,\&_\Pi\, \chi) \leftrightarrow \varphi \,\&_\Pi\, (\psi \ominus \chi)$
(P2) $\varphi \,\&_\Pi\, (\psi \,\&_\Pi\, \chi) \leftrightarrow (\varphi \,\&_\Pi\, \psi) \,\&_\Pi\, \chi$
(P3) $\varphi \to (\varphi \,\&_\Pi\, \overline{1})$
(P4) $(\varphi \,\&_\Pi\, \psi) \to \varphi$
(P5) $(\varphi \,\&_\Pi\, \psi) \to (\psi \,\&_\Pi\, \varphi)$.

The only deduction rule is *modus ponens* for $\&$ and $\to$.

The logic $\mathrm{P}Ł'$ is obtained from $\mathrm{P}Ł$ by adding the deduction rule:

(ZD) $\neg(\varphi \,\&_\Pi\, \varphi) \vdash_{\mathrm{P}Ł'} \neg\varphi$.

The logic $Ł\Pi$ is defined from $\mathrm{P}Ł$ by adding the product implication connective $\to_\Pi$ to the language, along with the following axioms:

(ŁΠ1) $((\varphi \,\&_\Pi\, \psi) \to_\Pi \chi) \leftrightarrow (\varphi \to_\Pi (\psi \to_\Pi \chi))$

(ŁΠ2) $(\varphi \,\&_\Pi\, (\varphi \to_\Pi \psi)) \leftrightarrow (\varphi \wedge \psi)$

(ŁΠ3) $\varphi \to_\Pi \varphi$

(ŁΠ4) $(\varphi \to_\Pi (\varphi \wedge \psi)) \leftrightarrow (\varphi \to_\Pi \psi)$.

Other definable connectives are the following:

$$\neg_\Pi\varphi := \varphi \to_\Pi 0 \qquad \triangle\varphi := \neg_\Pi\neg\varphi \qquad \nabla\varphi := \neg\neg_\Pi\varphi.$$

The logic $Ł\Pi\frac{1}{2}$ is an expansion of $Ł\Pi$ obtained by adding the constant $\overline{\frac{1}{2}}$ and the following axiom:

(ŁΠ5) $\overline{\frac{1}{2}} \leftrightarrow \neg\overline{\frac{1}{2}}$.

The following Deduction Theorem is easily seen to hold (see Chapter I).

THEOREM 5.2.1.

(1) *Let Γ be a theory over $\mathrm{P}Ł$ and φ, ψ be formulas. Then $\Gamma \cup \{\varphi\} \vdash_{\mathrm{P}Ł} \psi$ iff there is an n such that $\Gamma \vdash_{\mathrm{P}Ł} \varphi^n \to \psi$.*

(2) *Let L denote either $Ł\Pi$ or $Ł\Pi\frac{1}{2}$, Γ be a theory over L, and φ, ψ be L-formulas. Then $\Gamma \cup \varphi \vdash_{\mathrm{L}} \psi$ iff $\Gamma \vdash_{\mathrm{L}} \triangle\varphi \to \psi$.*

We will see later (Corollary 5.3.21) that $\mathrm{P}Ł'$ does not satisfy the same Deduction Theorem as $\mathrm{P}Ł$.

We now introduce the classes of $\mathrm{PMV}, \mathrm{PMV}^+, Ł\Pi$, and $Ł\Pi\frac{1}{2}$ algebras that constitute the algebraic semantics for $\mathrm{P}Ł, \mathrm{P}Ł', Ł\Pi$, and $Ł\Pi\frac{1}{2}$, respectively.

DEFINITION 5.2.2. *A PMV-algebra $\boldsymbol{A} = \langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a structure of type $\langle 2, 1, 2, 0, 0\rangle$, where $\langle A, \oplus, \neg, 0\rangle$ is an MV-algebra, $\langle A, \cdot, 1\rangle$ is a commutative monoid, and the following equation holds:*

$$(x \cdot y) \ominus (x \cdot z) = x \cdot (y \ominus z). \tag{1}$$

A PMV^+-algebra $\boldsymbol{A} = \langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a PMV-algebra in which the following quasiequation holds:

$$\textit{if } x \cdot x = 0, \textit{ then } x = 0. \tag{2}$$

An $Ł\Pi$-algebra $\boldsymbol{A} = \langle A, \oplus, \neg, \cdot, \to_\Pi, 0, 1\rangle$ is a structure of type $\langle 2, 1, 2, 2, 0, 0\rangle$, where $\langle A, \oplus, \neg, \cdot, 0, 1\rangle$ is a PMV-algebra, and the following equations hold

$$(x \cdot y) \to_\Pi z = x \to_\Pi (y \to_\Pi z), \tag{3}$$

$$x \cdot (x \to_\Pi y) = x \wedge y, \tag{4}$$

$$x \to_\Pi x = 1, \tag{5}$$

$$x \to_\Pi (x \wedge y) = x \to_\Pi y. \tag{6}$$

An $\text{Ł}\Pi\frac{1}{2}$*-algebra* $\boldsymbol{A}=\langle A,\oplus,\neg,\cdot,\to_\Pi,0,1,\frac{1}{2}\rangle$ *is a structure of type* $\langle 2,1,2,2,0,0,0\rangle$*, where* $\langle A,\oplus,\neg,\cdot,\to_\Pi,0,1\rangle$ *is an* $\text{Ł}\Pi$*-algebra, and the following equation holds*

$$\neg\tfrac{1}{2}=\tfrac{1}{2}. \tag{7}$$

In the classes of structures introduced above, the operations

$$x\to y \qquad x\odot y \qquad x\ominus y \qquad |x-y| \qquad x\wedge y \quad x\vee y \qquad x\leftrightarrow y$$

are defined as in the case of MV-algebras (see Chapter VI and Section 5.1). Other operations can be defined as follows:

$$\neg_\Pi x:=x\to_\Pi 0 \qquad \triangle x:=\neg_\Pi\neg x \qquad \nabla x:=\neg\neg_\Pi x.$$

The following proposition makes the connection between $\text{Ł}\Pi$ and PMV^+ algebras explicit, and will be (implicitly) used throughout the rest of the chapter.

PROPOSITION 5.2.3. *The quasiequation 'if* $x\cdot x=0$*, then* $x=0$*' holds in every* $\text{Ł}\Pi$*-algebra.*

Proof. Suppose $x\cdot x=0$. Then, $(x\cdot x)\cdot\neg_\Pi(x\cdot x)\le 0$, and so $\neg_\Pi(x\cdot x)\le x\to_\Pi\neg_\Pi x$. We show that $x\to_\Pi\neg_\Pi x\le\neg_\Pi x$.

From $x\to_\Pi\neg_\Pi x\le\neg_\Pi\neg_\Pi x\to_\Pi\neg_\Pi x$ we derive $\neg_\Pi\neg_\Pi x\le(x\to_\Pi\neg_\Pi x)\to_\Pi\neg_\Pi x$, and so $\neg_\Pi x\le(x\to_\Pi\neg_\Pi x)\to_\Pi\neg_\Pi x$. From the fact that $\neg_\Pi x\vee\neg_\Pi\neg_\Pi x=1$ holds in every $\text{Ł}\Pi$-algebra (easy to check), we obtain that $x\to_\Pi\neg_\Pi x\le\neg_\Pi x$.

So we have $\neg_\Pi(x\cdot x)\le\neg_\Pi x$, and since $\neg\neg x=x$, $\triangle\neg(x\cdot x)\le\neg_\Pi x$. From the assumption $x\cdot x=0$, we derive $\triangle\neg(x\cdot x)=1$, which means that $\neg_\Pi x=1$, and so $x=0$. □

The notions of evaluation, model, and proof for the above logics are defined as usual. In particular, the operations $x\cdot y,\ x\to_\Pi y,\ \neg_\Pi x,\ \triangle x,\ \nabla x$ have the following interpretation over the real unit interval $[0,1]$:

$$x\cdot y = xy \qquad\qquad x\to_\Pi y=\begin{cases}1 & \text{if } x\le y\\ \frac{y}{x} & \text{if } x>y\end{cases}$$

$$\neg_\Pi x=\begin{cases}1 & \text{if } x=0\\ 0 & \text{if } x>0\end{cases} \qquad\qquad \triangle x=\begin{cases}1 & \text{if } x=1\\ 0 & \text{if } x<1\end{cases}$$

$$\nabla x=\begin{cases}1 & \text{if } x>0\\ 0 & \text{if } x=0\end{cases}$$

$[0,1]_{\mathrm{PMV}}$ denotes the standard PMV-algebra over the real unit interval $[0,1]$, where the operations of its MV-reduct are obviously interpreted as in the standard MV-chain, and $\cdot$ is interpreted as the product of reals. Clearly, $[0,1]_{\mathrm{PMV}}$ is also a PMV^+-algebra. The standard $\text{Ł}\Pi$ and $\text{Ł}\Pi\frac{1}{2}$ algebras over $[0,1]$ are denoted by $[0,1]_{\text{Ł}\Pi}$ and $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$, respectively, and their operations correspond to those given for $[0,1]_{\mathrm{PMV}}$ along with the interpretation of the product implication $\to_\Pi$ given above as the residuum of the product t-norm.

The classes of PMV, $\text{Ł}\Pi$, and $\text{Ł}\Pi\frac{1}{2}$ algebras are varieties, denoted by $\mathbb{PMV}$, $\mathbb{LP}$, and $\mathbb{LP}\frac{1}{2}$, respectively, while the class of PMV^+-algebras $\mathbb{PMV}^+$ is a quasivariety. At the end of the next section, we will see that $\mathbb{PMV}^+$ cannot constitute a variety.

5.3 Completeness results

The goal of this section is to prove several completeness results for $\mathrm{P}Ł$, $\mathrm{P}Ł'$, $Ł\Pi$, and $Ł\Pi\frac{1}{2}$, including both completeness w.r.t. evaluations into the related class of linearly ordered algebras and evaluations into the related algebra over the real unit interval. Linearly ordered PMV, PMV^+, and $Ł\Pi$ algebras will be shown to be structures definable on certain classes of ordered rings, and $Ł\Pi\frac{1}{2}$-chains will be shown to be definable on ordered fields. These connections will be exploited in order to prove completeness w.r.t. real evaluations for the related logics.

It is fairly easy to see that $\mathrm{P}Ł$, $\mathrm{P}Ł'$, $Ł\Pi$, and $Ł\Pi\frac{1}{2}$ are algebraizable logics in the sense of Blok and Pigozzi [4]. As an immediate consequence, we obtain:

THEOREM 5.3.1. *Let* L *denote any among the logics* $\mathrm{P}Ł$*,* $\mathrm{P}Ł'$*,* $Ł\Pi$*, and* $Ł\Pi\frac{1}{2}$*. Let* $\mathbb{K}$ *denote any among the classes* $\mathbb{PMV}$*,* $\mathbb{PMV}^+$*,* $\mathbb{LP}$*, and* $\mathbb{LP}\frac{1}{2}$*. Then* L *has the* $\mathrm{S}\mathbb{K}\mathrm{C}$.

COROLLARY 5.3.2. $\mathrm{P}Ł$ *and* $\mathrm{P}Ł'$ *are conservative extensions of Łukasiewicz logic.*

Proof. Let φ be a formula of Łukasiewicz logic which is a theorem of $\mathrm{P}Ł$. Then, $e(\varphi) = 1$ for every evaluation e into any PMV-algebra $\boldsymbol{A}$, and, in particular, for each evaluation into $[0,1]_{\mathrm{PMV}}$. Since φ is a formula of Łukasiewicz logic, it is also a tautology w.r.t. evaluations into the MV-algebra over $[0,1]$. Using the fact that Łukasiewicz logic has the $\mathcal{R}\mathrm{C}$ we conclude that φ is a theorem of Łukasiewicz logic.

The same holds for $\mathrm{P}Ł'$. □

We are now going to prove that each member of the classes of PMV-, $Ł\Pi$-, and $Ł\Pi\frac{1}{2}$-algebras is a subdirect product of chains of the related class.[20]

DEFINITION 5.3.3 ([44]). *Let* $\boldsymbol{A}$ *be any algebra having a constant* 0*. A* 0*-ideal of* $\boldsymbol{A}$ *is a subset* J *of* A *for which there is a congruence* θ *of* $\boldsymbol{A}$ *such that* $J = \{a \in A \mid a\,\theta\,0\}$.

In the rest of this chapter, we will refer to 0-ideals simply as "ideals".

LEMMA 5.3.4. *Let* $\boldsymbol{A}$ *be a* PMV*-algebra and* $\boldsymbol{B}$ *its underlying* MV*-algebra. Then:*

(1) $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *have the same congruences.*[21]

(2) $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *have the same ideals, and for every ideal* J *of* $\boldsymbol{A}$*, there is exactly one congruence* θ *of* $\boldsymbol{A}$ *such that* J *is the congruence class of* 0 *with respect to* θ*. This congruence is defined by* $x\ \theta\ y$ *iff* $|x-y| \in J$.

Proof. To prove the first claim it is sufficient to prove that for every congruence θ of $\boldsymbol{B}$, and for all $x,y,u,v \in A$, if $x\ \theta\ y, u\ \theta\ v$, then $(x\cdot u)\ \theta\ (y\cdot v)$. Given the assumptions above, and since $|x-y| = (x\vee y)\ominus(x\wedge y)$ and $|u-v| = (u\vee v)\ominus(u\wedge v)$, we have $|x-y|\ \theta\ 0$ and $|u-v|\ \theta\ 0$. It follows that

$$\begin{aligned}|x\cdot u - y\cdot v| \leq\ & ((x\vee y)\cdot(u\vee v))\ominus((x\wedge y)\cdot(u\wedge v)) \leq \\ \leq\ & (((x\vee y)\cdot(u\vee v))\ominus((x\vee y)\cdot(u\wedge v))) \\ & \oplus(((x\vee y)\cdot(u\wedge v))\ominus((x\wedge y)\cdot(u\wedge v))) =\end{aligned}$$

[20]Notice that for PMV the result follows from the fact that $\mathrm{P}Ł$ is a core fuzzy logic, while $Ł\Pi$ and $Ł\Pi\frac{1}{2}$ are $\triangle$-core fuzzy logics (see Chapter I). Still, we are going to provide a direct proof of these facts.

[21]See Chapter VI for the definition of "ideal" and "congruence" for MV-algebras.

$$= ((x \vee y) \cdot ((u \vee v) \ominus (u \wedge v))) \oplus ((u \wedge v) \cdot ((x \vee y) \ominus (x \wedge y))) \leq$$
$$\leq ((u \vee v) \ominus (u \wedge v)) \oplus ((x \vee y) \ominus (x \wedge y)) =$$
$$= |u - v| \oplus |x - y|.$$

Since $(|u - v| \oplus |x - y|)\ \theta\ (0 \oplus 0)$, we have $|x \cdot u - y \cdot v|\ \theta\ 0$, and, consequently, $(x \cdot u)\ \theta\ (y \cdot v)$, as desired.

The second claim: From (1) and Definition 5.3.3, we have that $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same ideals. Let θ be any congruence of $\boldsymbol{A}$, and let J be the congruence class of 0 modulo θ. If $x\ \theta\ y$, then $(x \ominus y)\ \theta\ (y \ominus y)$, which means that $(x \ominus y)\ \theta\ 0$. By a similar argument, we have that $y \ominus x\ \theta\ 0$. So, if $x\ \theta\ y$, then $|x - y|\ \theta\ 0$. Conversely, if $|x-y|\ \theta\ 0$, and since $((x \ominus y) \wedge |x-y|)\ \theta\ ((x \ominus y) \wedge 0)$ and $(x \ominus y) = (x \ominus y) \wedge |x-y|$, we have that $x \ominus y\ \theta\ 0$. Similarly, we obtain $(y \ominus x)\ \theta\ 0$, and so $(x \vee y)\ \theta\ x$, given that $x \vee y = (x \oplus (y \ominus x))$ and $(x \oplus (y \ominus x))\ \theta\ (x \oplus 0)$. From $(x \vee y)\ \theta\ y$ we get $x\ \theta\ y$. In conclusion, for all $x, y \in A$ we have $x\ \theta\ y$ iff $|x - y| \in J$. □

LEMMA 5.3.5. *Every* PMV*-algebra is isomorphic to a subdirect product of a family of* PMV*-chains.*

Proof. Let $\boldsymbol{A}$ be any subdirectly irreducible PMV-algebra. Since, every PMV-algebra is isomorphic to a subdirect product of subdirectly irreducible PMV-algebras, it is sufficient to prove that every subdirectly irreducible PMV-algebra is linearly ordered. The congruence lattice of $\boldsymbol{A}$ has a minimum non-zero element, i.e. the monolith, so there is a minimum non-zero ideal J. Clearly, J is generated by a single element $c > 0$. Suppose by contradiction that there are $a, b \in A$ such that neither $a \leq b$ nor $b \leq a$ holds. Thus, $a \ominus b > 0$ and $b \ominus a > 0$. It follows that c belongs both to the ideal $I_{a \ominus b}$ generated by $a \ominus b$ and to the ideal $I_{b \ominus a}$ generated by $b \ominus a$. By Lemma 5.3.4, there is $n \in \mathsf{N}$ such that $c \leq n(a \ominus b)$ and $c \leq n(b \ominus a)$. So, $c \leq n(a \ominus b) \wedge n(b \ominus a)$. Since $(a \ominus b) \wedge (b \ominus a) = 0$, we conclude that $c \leq n(a \ominus b) \wedge n(b \ominus a) = 0$, which clearly is a contradiction. □

The same result can be proven for $\mathbb{LP}$ and $\mathbb{LP}\frac{1}{2}$ by using a similar argument.

LEMMA 5.3.6. *Let J be a subset of an* ŁΠ*-algebra $\boldsymbol{A}$. J is an ideal of $\boldsymbol{A}$ iff it is an ideal of the underlying* MV*-algebra and is closed under* $\bigtriangledown$.

Proof. Let θ be any congruence of $\boldsymbol{A}$, and let $J = \{x \in A \mid x\ \theta\ 0\}$. If $x \in J$, then $x\ \theta\ 0$, therefore $\neg_\Pi x\ \theta\ \neg_\Pi 0$, and $\bigtriangledown x\ \theta\ 0$. So, $\bigtriangledown x \in J$, and J is closed under $\bigtriangledown$. Conversely, let J be an ideal of the underlying MV-algebra which is closed under $\bigtriangledown$, and define: $x\ \theta\ y$ iff $|x - y| \in J$. By Lemma 5.3.4, θ is a congruence of the underlying PMV-algebra. Now suppose that $x\ \theta\ y$ and $u\ \theta\ v$. Then, $|x - y| \in J$, $|u - v| \in J$. So, $\bigtriangledown(|x - y|) \in J$, and $\bigtriangledown(|u - v|) \in J$. It follows that $\bigtriangledown(|x - y|) \vee \bigtriangledown(|u - v|) \in J$, and $|(u \to x) - (v \to y)| \in J$. In fact, note that $|(u \to x) - (v \to y)| \leq \bigtriangledown(|x - y|) \vee \bigtriangledown(|u - v|)$ holds in all PMV-algebras (see [66]). Thus, $|(u \to x) - (v \to y)|\ \theta\ 0$, and $(u \to x)\ \theta\ (v \to y)$. So, θ is a congruence of $\boldsymbol{A}$, and clearly $J = \{x \in A \mid x\ \theta\ 0\}$. □

LEMMA 5.3.7. *Let $\boldsymbol{A}$ be an* ŁΠ*-algebra and $a \in A$ an arbitrary element. Then the ideal J_a generated by a is given by $J_a = \{x \in A \mid x \leq \bigtriangledown(a)\}$.*

Proof. Let $I = \{x \in A \mid x \leq \nabla a\}$. Clearly, I is a lattice ideal. It is easily seen that $\nabla a \oplus \nabla a = \nabla a \vee \nabla a = \nabla a$, therefore I is closed under $\oplus$. Finally, it is easy to see that if $x \leq \nabla a$, then $\nabla x \leq \nabla a$. So, I is closed under ∇. Therefore, I is an ideal of $\boldsymbol{A}$. Conversely, if J is an ideal and $a \in J$, then $\nabla a \in J$, therefore $I \subseteq J$. □

LEMMA 5.3.8. *Every* ŁΠ*-algebra* (*every* $\text{Ł}\Pi\frac{1}{2}$*-algebra*) *is isomorphic to a subdirect product of a family of* ŁΠ*-chains* ($\text{Ł}\Pi\frac{1}{2}$*-chains*).

Proof. We show that every subdirectly irreducible ŁΠ algebra is linearly ordered. Suppose that an ŁΠ-algebra $\boldsymbol{A}$ is subdirectly irreducible but not linearly ordered. Let J be a minimal non-zero ideal. By Lemma 5.3.7 and by the minimality of J, there is $c \neq 0$ such that $J = \{x \in A \mid x \leq \nabla c\}$. Now, suppose that for some $a, b \in A$, neither $a \leq b$ nor $b \leq a$. Then, both $a \ominus b$ and $b \ominus a$ generate non-trivial ideals $I_{a \ominus b}$ and $I_{b \ominus a}$ respectively. Both ideals contain J, and so, by Lemma 5.3.7, $\nabla c \leq \nabla(a \ominus b)$, and $\nabla c \leq \nabla(b \ominus a)$. Consequently, $c \leq \nabla c \leq \nabla(a \ominus b) \wedge \nabla(b \ominus a)$. Notice that, in all ŁΠ-algebras, $x \wedge y = 0$ implies $\nabla x \wedge \nabla y = 0$. Since $(a \ominus b) \wedge (b \ominus a) = 0$, we have that $c \leq \nabla(a \ominus b) \wedge \nabla(b \ominus a) = 0$, which is a contradiction. □

As an immediate consequence of the above results (see Chapter I), we have:

THEOREM 5.3.9. *Let* L *denote any among the logics* $\text{PŁ}, \text{Ł}\Pi, \text{Ł}\Pi\frac{1}{2}$. *Then* L *has the* $\text{S}\mathbb{K}\text{C}$*, where* $\mathbb{K}$ *denotes the class of chains of the corresponding class of algebras.*

We are now going to prove that some of the varieties introduced above are generated by their algebra over the reals (up to isomorphism). As a consequence, the related logics will be shown to have $\text{FS}\mathcal{R}\text{C}$. These results will be obtained by exploiting the connection between the linearly ordered members of the above varieties and certain classes of ordered rings.

An *ordered ring* is a structure $\mathbf{R} = \langle R, +, -, \cdot, 0, 1, \leq \rangle$ where $\langle R, +, -, \cdot, 0, 1 \rangle$ is a commutative ring with unit, and $\langle R, \leq \rangle$ is a totally ordered set such that, for all $x, y, z \in R$, if $x \leq y$ then $x + z \leq y + z$; and if $0 \leq x$ and $0 \leq y$, then $0 \leq x \cdot y$. The notions of *ordered integral domain* and *ordered field* are analogously defined.

Now, given an ordered ring $\mathbf{R} = \langle R, +, -, \cdot, 0, 1, \leq \rangle$, the following structure is a PMV-chain:

$$\boldsymbol{A}_{\mathbf{R}} = \langle [0,1]_R, \oplus, \neg, \cdot, 0, 1 \rangle,$$

where $[0,1]_R = \{x \in R \mid 0 \leq x \leq 1\}$, and

$$x \oplus y = \min\{x + y, 1\},$$
$$\neg x = 1 - x,$$
$$x \cdot y = x \cdot y.$$

Given an ordered integral domain $\mathbf{D} = \langle D, +, -, \cdot, 0, 1, \leq \rangle$, the following structure (defined in the same way as $\boldsymbol{A}_{\mathbf{R}}$) is a PMV^+-chain:

$$\boldsymbol{A}_{\mathbf{D}} = \langle [0,1]_D, \oplus, \neg, \cdot, 0, 1 \rangle.$$

Given an ordered field $\mathbf{F} = \langle F, +, -, \cdot, 0, 1, \leq \rangle$, define the structure

$$\boldsymbol{A}_{\mathbf{F}} = \langle [0,1]_F, \oplus, \neg, \cdot, \rightarrow_\Pi, 0, 1 \rangle,$$

where $\langle [0,1]_F, \oplus, \neg, \cdot, 0, 1\rangle$ is defined as above and

$$x \to_\Pi y = \begin{cases} 1 & \text{if } x \le y, \\ y \cdot x^{-1} & \text{otherwise,} \end{cases}$$

where x^{-1} corresponds to the multiplicative inverse of x.

Similarly, from an ordered field $\mathbf{F}$, we can define the structure

$$\boldsymbol{A}'_{\mathbf{F}} = \langle [0,1]_F, \oplus, \neg, \cdot, \to_\Pi, 0, 1, \tfrac{1}{2}\rangle,$$

where $\frac{1}{2} = 2^{-1}$. It is easily seen that $\boldsymbol{A}_{\mathbf{F}}$ and $\boldsymbol{A}'_{\mathbf{F}}$ are an $\text{Ł}\Pi$ and an $\text{Ł}\Pi\frac{1}{2}$-chain respectively.

The structures above defined are called *interval chains*, being defined over the unit interval of certain ordered rings.

LEMMA 5.3.10.

(1) *Every* PMV*-chain is isomorphic to the interval* PMV*-chain of an ordered ring.*

(2) *Every* PMV^+*-chain is isomorphic to the interval* PMV^+*-chain of an ordered integral domain.*

(3) *Every* $\text{Ł}\Pi$*-chain with more than two elements is an* $\text{Ł}\Pi\frac{1}{2}$*-chain (modulo an extension by definition), and contains an isomorphic copy of the interval* $\text{Ł}\Pi$*-chain of the field of rationals* Q.

(4) *Every* $\text{Ł}\Pi\frac{1}{2}$*-chain is isomorphic to the interval* $\text{Ł}\Pi\frac{1}{2}$*-chain of an ordered field.*

(5) *Every* $\text{Ł}\Pi$*-chain is isomorphic either to the interval* $\text{Ł}\Pi$*-chain of an ordered field or to the interval* $\text{Ł}\Pi$*-chain of* Z.

Proof. (1) Take any PMV-chain $\boldsymbol{A} = \langle A, \oplus, \neg, \cdot, 0, 1\rangle$ and define a structure

$$\mathbf{R}_{\boldsymbol{A}} = \langle R_A, +, -, \cdot, 0_{\mathbf{R}_{\boldsymbol{A}}}, \le_{\mathbf{R}_{\boldsymbol{A}}}\rangle,$$

where $R_A = \{\langle n, x\rangle \mid n \in \mathsf{Z}, x \in A\backslash\{1\}\}$, and $0_{\mathbf{R}_{\boldsymbol{A}}} = \langle 0, 0\rangle$,

$$\langle n,x\rangle + \langle m,y\rangle = \begin{cases} \langle n+m, x\oplus y\rangle & \text{if } x\oplus y < 1, \\ \langle n+m+1, \neg(\neg x \oplus \neg y)\rangle & \text{if } x \oplus y = 1, \end{cases}$$

$$-\langle n,x\rangle = \begin{cases} \langle -n, 0\rangle & \text{if } x = 0, \\ \langle -(n+1), \neg x\rangle & \text{if } 0 < x < 1, \end{cases}$$

$$\langle n,x\rangle \cdot \langle m,y\rangle = \langle nm, x\cdot y\rangle + m\langle 0,x\rangle + n\langle 0,y\rangle,$$

$$\langle n,x\rangle \le_{R_A} \langle m,y\rangle \quad \text{if} \quad n < m, \text{ or } n = m \text{ and } x \le y.$$

$\langle R_A, +, -, 0_{\mathbf{R}_{\boldsymbol{A}}}, \le_{\mathbf{R}_{\boldsymbol{A}}}\rangle$ corresponds to the Chang group built from the MV-chain $\langle A, \oplus, \neg, 0, 1\rangle$ (see Chapter VI). It is fairly easy to see $\mathbf{R}_{\boldsymbol{A}}$ is indeed an ordered ring.

(2) We just need to prove that $\mathbf{R}_{\boldsymbol{A}}$ has no zero-divisors. From $\langle n,x\rangle\cdot\langle m,y\rangle=\langle 0,0\rangle$ we can deduce that either $n=0$ or $m=0$. If $n=m=0$, then $x\cdot y=0$, and either $x=0$ or $y=0$, since $\boldsymbol{A}$ is a linearly ordered PMV^{+}-algebra. Now suppose $n=0$ and $m\neq 0$. Then $\langle n,x\rangle\cdot\langle m,y\rangle=m\langle 0,x\rangle+\langle 0,x\cdot y\rangle$. Suppose $x\neq 0$. Let, for $\langle n,x\rangle\in R_A$, $|\langle n,x\rangle|$ denote $\langle n,x\rangle$ if $\langle n,x\rangle\geq_{\boldsymbol{R_A}}\langle 0,0\rangle$ and $-\langle n,x\rangle$ otherwise. We obtain:

$$|n\langle 0,x\rangle+\langle 0,x\cdot y\rangle|\geq_{\mathbf{R}_{\boldsymbol{A}}}|\langle 0,x\ominus(x\cdot y)\rangle|=|\langle 0,x\cdot\neg y\rangle|.$$

Now, $y\neq 1$, and therefore $\neg y\neq 0$, $x\cdot\neg y\neq 0$ (being $\boldsymbol{A}$ a linearly ordered PMV^{+}-algebra). So, we obtain a contradiction. The case where $m=0$ and $n\neq 0$ is symmetric.

(3) Let $\boldsymbol{A}$ be any linearly ordered ŁΠ algebra with more than two elements. By taking its $\rightarrow_{\Pi}$-free reduct, we can safely assume that $\boldsymbol{A}$ is the interval algebra of an ordered integral domain $\mathbf{D}_{\boldsymbol{A}}$. Let $a\in A$ be such that $0<a<1$, and let $b=\min\{a,1-a\}$. Clearly, $0<b<b+b=b\oplus b\leq 1$. Let $c=(b\oplus b)\rightarrow_{\Pi} b$. It is easy to check that: $b=b\wedge(b\oplus b)=((b\oplus b)\rightarrow_{\Pi} b)\cdot(b\oplus b)$. So, in $\mathbf{D}_{\boldsymbol{A}}$ we have: $b=c\cdot b+c\cdot b$, and $b\cdot(1-(c+c))=0$. Then, it follows that $1-(c+c)=0$, $c=1-c$, and finally $c=\neg c$. Clearly, there is a unique c such that $c=\neg c$: call it $\frac{1}{2}$. Thus, $\boldsymbol{A}$ is an $\text{Ł}\Pi\frac{1}{2}$-algebra.

We show that $\boldsymbol{A}$ contains an isomorphic copy of the ŁΠ-chain of rationals Q. We have shown that we already have the element $\frac{1}{2}$. For $n\geq 2$, we define

$$\frac{1}{n+1}=\left((n+1)\left(\frac{1}{2}\cdot\frac{1}{n}\right)\right)\rightarrow_{\Pi}\left(\frac{1}{2}\cdot\frac{1}{n}\right).$$

Define the map h from $\mathsf{Q}\cap[0,1]$ into A as: $h(0)=0$; $h(\frac{n}{m})=n\frac{1}{m}$ if $0<n\leq m$. We check that h is a one-to-one homomorphism from the interval ŁΠ-algebra of Q into $\boldsymbol{A}$. It is sufficient to show that

$$\text{for every positive } n\in\mathsf{N}, n\cdot\frac{1}{n}=1. \tag{8}$$

Indeed, from (8) and distributivity, we easily obtain that for all $x\in D_A$,

$$\underbrace{x\cdot\frac{1}{n}+\cdots+x\cdot\frac{1}{n}}_{n}=x.$$

It follows that the group $\boldsymbol{G}$ underlying $\mathbf{D}_{\boldsymbol{A}}$ is Abelian, torsion-free and divisible. Let $\boldsymbol{G}'$ be the smallest divisible subgroup of $\boldsymbol{G}$ containing 1. Let $\frac{x}{n}$ denote 0 if $x=0$, and the unique y such that $ny=x$ otherwise. Then, $\boldsymbol{G}'$ consists of 0 plus all elements of the form $\pm\frac{m}{n}$ with $n,m>0$. We have that the map h' defined by $h'(0)=0$, and $h'(\pm\frac{m}{n})=\pm\frac{m}{n}$ if $n,m>0$ is an isomorphism from the additive group of Q onto $\boldsymbol{G}'$. Using distributivity, we can also see that h' is compatible with the product $\cdot$. Finally, h' is order-preserving, and so it is an embedding of the ordered ring Q into $\mathbf{D}_{\boldsymbol{A}}$. The claim follows from the fact that h is a restriction of h' to the interval $\text{Ł}\Pi\frac{1}{2}$-algebra of Q.

(4) If $\boldsymbol{A}$ is any linearly ordered Ł$\Pi\frac{1}{2}$-algebra, then the algebra $\boldsymbol{A}'$ obtained from $\boldsymbol{A}$ by omitting the interpretations of $\rightarrow_{\Pi}$ and $\frac{1}{2}$ is the interval algebra of an ordered integral domain, $\mathbf{D}_{\boldsymbol{A}}$. Let $\boldsymbol{F}$ be the fraction field of $\mathbf{D}_{\boldsymbol{A}}$. It suffices to prove that every $c \in F \cap [0,1]$ is in $\boldsymbol{A}$. Writing, for $z \in \mathsf{Z}$ and for $\alpha \in A$, $z + \alpha$ instead of (z, α), we can represent any $c \in F \cap [0,1]$ as $c = \frac{z+\alpha}{y+\beta}$, where $\alpha, \beta \in A, \alpha < 1, \beta < 1, z, y \in \mathsf{Z}, z \geq 0, y \geq 0$, if $y = 0$, then $\beta \neq 0$, and either $z < y$, or $z = y$ and $\alpha \leq \beta$. Now, if $y = 0$, then $z = 0$, and $c = \beta \rightarrow_{\Pi} \alpha \in A$. Otherwise, let

$$d = \left(\frac{1}{2} \oplus \left(\frac{1}{2} \cdot \frac{1}{y} \cdot \beta\right)\right) \rightarrow_{\Pi} \left(\frac{1}{2} \cdot \frac{1}{y}\right).$$

It is easy to check that $c = zd \oplus (\alpha \cdot d) \in A$.

(5) Let $\boldsymbol{A}$ be an ŁΠ-chain. If $\boldsymbol{A}$ has more than two elements, it is an Ł$\Pi\frac{1}{2}$-chain by (3). If that is not the case, then $\boldsymbol{A}$ is obviously isomorphic to the interval Ł$\Pi\frac{1}{2}$-algebra of Z. □

The above results show the strong connection between Ł$\Pi\frac{1}{2}$-chains and ordered fields. We are now going to see that each Ł$\Pi\frac{1}{2}$-chain can be formally defined within the related ordered field.

From now on, we will denote Ł$\Pi\frac{1}{2}$-algebras by $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, while we will use $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$ for ordered fields. A subscript will make explicit the relation between an Ł$\Pi\frac{1}{2}$-chain and an ordered field, i.e.: given an Ł$\Pi\frac{1}{2}$-chain $\boldsymbol{A}$, the related field is denoted by $\mathbf{F}_{\boldsymbol{A}}$; conversely, given an ordered field $\mathbf{F}$, the related Ł$\Pi\frac{1}{2}$-chain is denoted by $\boldsymbol{A}_{\mathbf{F}}$.

Let Th denote a first-order classical theory in some language $\mathcal{L}$. We use $\sqcap$, $\sqcup$, $\backsim$, and $\Longrightarrow$ for classical conjunction, disjunction, negation, and implication, respectively. Denote by $\mathsf{Th}(\mathrm{OF})$ the first-order theory of ordered fields in the language

$$\langle +, \cdot, -, <, 0, 1 \rangle$$

(see [57]). Denote by $\mathsf{Th}(Ł\Pi\frac{1}{2})$ the first-order theory of linearly ordered Ł$\Pi\frac{1}{2}$-algebras in the language

$$\langle \oplus, \neg, \cdot, \rightarrow_{\Pi}, 0, 1, \tfrac{1}{2}, < \rangle.$$

$\mathsf{Th}(Ł\Pi\frac{1}{2})$ is axiomatized by the universal closure of the equations defining the variety of Ł$\Pi\frac{1}{2}$-algebras plus the sentence defining the linearity of the order relation $<$. We are going to show that $\mathsf{Th}(Ł\Pi\frac{1}{2})$ can be interpreted into $\mathsf{Th}(\mathrm{OF})$.

Let $\mathcal{L}$ be a signature of the form $\langle <, f_1, \ldots, f_n, c_1, \ldots, c_m \rangle$, where each f_i is a function symbol and each c_j is a constant symbol. $\mathcal{L}$ will be assumed to include no relation symbol but $<$ (and, of course, =). By an unnested atomic formula in $\mathcal{L}$ we mean one of the following formulas:

- $x = y, x < y$;
- $x = c$, $c = x$, $x < c$, $c < x$, for some constant symbol $c \in \mathcal{L}$;
- $f(\overline{x}) = y$, $y = f(\overline{x})$, $f(\overline{x}) < y$, $y < f(\overline{x})$, for some function symbol $f \in \mathcal{L}$.

A formula is called unnested if all its atomic subformulas are unnested. Then, it is straightforward to prove that (see [57]):

LEMMA 5.3.11. *For a first-order language* $\mathcal{L} = \langle <, f_1, \ldots, f_n, c_1, \ldots, c_m \rangle$*, every formula is equivalent to an unnested formula.*

The following definition sets what it means for a theory Th_1 in the language $\mathcal{L}_1$ to be interpretable in a theory Th_2 in the language $\mathcal{L}_2$.

DEFINITION 5.3.12. *Let* Th_1 *and* Th_2 *be two theories in the languages* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, respectively.* Th_1 *is* interpretable *in* Th_2 *if*

- (i) *there exists an* $\mathcal{L}_2$*-formula* $\Xi(z)$*,*
- (ii) *there exists a map* $\sharp$ *from the set of unnested atomic* $\mathcal{L}_1$*-formulas into the set of* $\mathcal{L}_2$ *formulas,*
- (iii) *there exists a map* $\star$ *from the set of models of* Th_1 *into the set of models of* Th_2*,*

such that, for every $\mathbf{M} \models \mathsf{Th}_1$ *(i.e., for every model* $\mathbf{M}$ *of* Th_1*),*

1. *there exists a bijection* $h_{\mathbf{M}} \colon M \to \{a \mid \mathbf{M}^\star \models \Xi(a)\}$ *from the domain of* $\mathbf{M}$ *into the set defined by* $\Xi(z)$ *over the domain of* $\mathbf{M}^\star$*;*
2. *for all* $\bar{b} \in M$ *and each unnested atomic* $\mathcal{L}_1$*-formula* Φ

$$\mathbf{M} \models \Phi(\bar{b}) \quad \textit{iff} \quad \mathbf{M}^\star \models \Phi^\sharp(h_{\mathbf{M}}(\bar{b})).$$

$\mathbf{M}^\star$ *is called the* complementary model of $\mathbf{M}$.

The above definition together with Lemma 5.3.11 yields that the interpretation of Th_1 into Th_2 can be extended to arbitrary formulas.

LEMMA 5.3.13. *Let* Th_1 *and* Th_2 *be two theories in the languages* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*, respectively. Suppose that* Th_1 *is interpretable in* Th_2*. Then, the mapping* $\sharp$ *can be applied to the whole set of* $\mathcal{L}_1$*-formulas. In other words, for each* $\mathcal{L}_1$*-formula* $\Phi(\overline{x})$ *there exists an* $\mathcal{L}_2$*-formula* $\Phi^\sharp(\overline{x})$ *so that, for every* $\mathbf{M} \models \mathsf{Th}_1$ *and all* $\bar{b} \in M$

$$\mathbf{M} \models \Phi(\bar{b}) \quad \textit{iff} \quad \mathbf{M}^\star \models \Phi^\sharp(h_{\mathbf{M}}(\bar{b})).$$

Proof. This can be simply proved by induction on the complexity of the formulas.[22] By Lemma 5.3.11, every formula in the language $\mathcal{L}_1$ is equivalent to an unnested formula, and so all its atomic subformulas are unnested. Definition 5.3.12 sets the case of unnested atomic formulas. For the case of compound formulas, define:

$$\begin{aligned} (\backsim\Phi)^\sharp &:= \backsim(\Phi^\sharp), \\ (\Phi \sqcap \Psi)^\sharp &:= \Phi^\sharp \sqcap \Psi^\sharp, \\ (\forall x\, \Phi)^\sharp &:= \forall x\, \Xi(x) \Longrightarrow \Phi^\sharp, \\ (\exists x\, \Phi)^\sharp &:= \exists x\, \Xi(x) \sqcap \Phi^\sharp, \end{aligned}$$

where $\Xi(x)$ is the formula defining the domain of each $\mathbf{M} \models \mathsf{Th}_1$ into the related complementary model $\mathbf{M}^\star$. □

[22] The proof is basically the same as the proof of Theorem 5.3.2 in [57].

THEOREM 5.3.14. $\mathsf{Th}(\text{Ł}\Pi\frac{1}{2})$ *is interpretable into* $\mathsf{Th}(\mathrm{OF})$.

Proof. Let $\Phi(\bar{x})$ be any formula in the language $\langle \oplus, \neg, \cdot, \to_\Pi, 0, 1, \frac{1}{2}, < \rangle$, and let $\boldsymbol{A}$ be an $\text{Ł}\Pi\frac{1}{2}$-chain.

Any unnested atomic formula of $\mathsf{Th}(\text{Ł}\Pi\frac{1}{2})$ can be translated into a formula in the language of ordered fields as follows (the translation of inequalities is similar):

$$\begin{aligned}
x = 0 \quad &\mapsto \quad x = 0\\
x = 1 \quad &\mapsto \quad x = 1\\
x = \tfrac{1}{2} \quad &\mapsto \quad x = \tfrac{1}{2}\\
x \oplus y = z \quad &\mapsto \quad ((x + y \leq 1) \sqcap (x + y = z)) \sqcup ((x + y \geq z) \sqcap (z = 1))\\
\neg x = y \quad &\mapsto \quad 1 - x = y\\
x \cdot y = z \quad &\mapsto \quad x \cdot y = z\\
x \to_\Pi y = z \quad &\mapsto \quad ((x \leq y) \sqcap (z = 1)) \sqcup ((x > y) \sqcap (y = z \cdot x)).
\end{aligned}$$

The formula $\Xi(x) := (0 \leq x) \sqcap (x \leq 1)$ obviously defines over $\mathbf{F}_{\boldsymbol{A}}$ an order-isomorphic copy of the domain of $\boldsymbol{A}$. By Lemma 5.3.13, the above translation, and the fact that every $\text{Ł}\Pi\frac{1}{2}$-chain is the interval algebra of an ordered field, we conclude that there exists a formula $\Phi^\sharp(\overline{x})$ in the language of ordered fields such that, for all $\bar{a} \in A$:

$$\boldsymbol{A} \models \Phi(\overline{a}) \quad \text{iff} \quad \mathbf{F}_{\boldsymbol{A}} \models \Phi^\sharp(\overline{a}). \qquad \square$$

A *real closed field* $\overline{\mathbf{F}} = \langle F, +, -, \cdot, \leq, 0, 1 \rangle$ is a field with a unique ordering whose positive cone $\{x \mid x \geq 0\}$ is the set of squares of F, and every polynomial of $F[X]$, of odd degree, has a root in F. In other words, real closed fields are ordered fields satisfying the following sentences for each $n \geq 0$:

$$\forall x_1 \ldots \forall x_n \, (x_1^2 + \cdots + x_n^2 + 1) \neq 0,$$
$$\forall x \exists y \, ((y^2 = x) \sqcup (y^2 + x = 0)),$$
$$\forall x_0 \ldots \forall x_{2n} \exists y \, \left(y^{2n+1} + \sum_{i=0}^{2n} x_i y^i \right) = 0.$$

Given an ordered field $\mathbf{F}$, the *real closure* of $\mathbf{F}$ is an algebraic extension $\overline{\mathbf{F}}$ which is a real closed field and with a unique ordering extending the ordering of $\mathbf{F}$. The field of real numbers R is a real closed field, while the field of rational numbers Q is not. The real closure of Q is the field of real algebraic numbers A, i.e. the real roots of polynomials

$$a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$$

with integer coefficients $a_0, \ldots, a_m$. The first-order theory of real closed fields, denoted as $\mathsf{Th}(\mathrm{RCF})$, admits *quantifier elimination* in the language $\langle +, -, \cdot, <, 0, 1 \rangle$, Tarski [80], i.e. any first-order formula in the language of ordered fields is equivalent to a quantifier-free formula in the same language. As a consequence, the theory of real closed fields is complete, i.e. for every formula Φ, either $\mathsf{Th}(\mathrm{RCF}) \vdash \Phi$ or $\mathsf{Th}(\mathrm{RCF}) \vdash \backsim\Phi$, and decidable (see [8]). Moreover, $\mathsf{Th}(\mathrm{RCF})$ is *model-complete*, i.e. every embedding between any of its models is elementary, i.e.: for any $\mathbf{F}, \mathbf{G} \models \mathsf{Th}(\mathrm{RCF})$, any embedding $f\colon \mathbf{F} \to \mathbf{G}$, any formula $\Phi(x_1, \ldots, x_m)$, and $a_1, \ldots, a_m \in F$,

$$\mathbf{F} \models \Phi(a_1, \ldots, a_m) \quad \text{iff} \quad \mathbf{G} \models \Phi(f(a_1), \ldots, f(a_m)).$$

A is (elementarily) embeddable into every real closed field, and therefore all real closed field are elementarily equivalent to A, and, thus, to each other.

We are now ready to prove that ŁΠ and ŁΠ$\frac{1}{2}$ are generated by their related structure over the real unit interval $[0, 1]$.

THEOREM 5.3.15.

(1) $\mathbb{LP}$ *is generated by* $[0, 1]_{\text{ŁΠ}}$.

(2) $\mathbb{LP}\frac{1}{2}$ *is generated by* $[0, 1]_{\text{ŁΠ}\frac{1}{2}}$.

Proof. We prove the case of $\mathbb{LP}$. The proof for $\mathbb{LP}\frac{1}{2}$ is almost identical. Let $\boldsymbol{A}$ be any ŁΠ-algebra, and let $\phi(\overline{x})$ be an equation not valid in $\boldsymbol{A}$. $\boldsymbol{A}$ is a subdirect product of ŁΠ-chains $\boldsymbol{B}_i$, and so there is at least one $\boldsymbol{B}_i$ in which $\phi(\overline{x})$ fails. By Lemma 5.3.10, $\boldsymbol{B}_i$ is either the interval algebra of Z or the interval algebra of an ordered field $\mathbf{F}_{\boldsymbol{B}}$. If $\boldsymbol{B}_i$ is the interval algebra of Z, then it can be trivially embedded into the interval algebra of Q. In both cases, Q and $\mathbf{F}_{\boldsymbol{B}}$ can be embedded into their respective (unique) real closure. This means that $\boldsymbol{B}_i$ is embeddable into the interval ŁΠ-algebra of a real closed field. Real closed fields are elementarily equivalent to each other, and to R, in particular. By Theorem 5.3.14, we know that every ŁΠ-chain is definable into its related ordered field. This means that $\phi(\overline{x})$ does not hold in $[0, 1]_{\text{ŁΠ}}$. □

Recall that, given a set A, a discriminator function on A is the function $d\colon A^3 \to A$ defined by

$$d(a, b, c) = \begin{cases} a & \text{if } a \neq b, \\ c & \text{if } a = b. \end{cases}$$

A ternary term $d(x, y, z)$ representing the discriminator function on a structure $\boldsymbol{A}$ is called a *discriminator term* for $\boldsymbol{A}$. Let $\mathbb{K}$ be a class of algebras with a common discriminator term $d(x, y, z)$. The variety generated by $\mathbb{K}$ is called a *discriminator variety* (see [7]). Both $\mathbb{LP}$ and $\mathbb{LP}\frac{1}{2}$ are discriminator varieties, with discriminator term

$$d(x, y, z) := (\triangle(x \leftrightarrow y) \wedge y) \vee \neg((\triangle(x \leftrightarrow y)) \wedge x).$$

We use the above fact to show that:

THEOREM 5.3.16.

(1) $\mathbb{LP}$ *is generated as a quasivariety by* $[0, 1]_{\text{ŁΠ}}$.

(2) $\mathbb{LP}\frac{1}{2}$ *is generated as a quasivariety by* $[0, 1]_{\text{ŁΠ}\frac{1}{2}}$.

Proof. We use the Theorem 5.3.15 and the fact that the variety $\mathbb{V}$ generated by any discriminator algebra $\boldsymbol{A}$ is generated by $\boldsymbol{A}$ as a quasivariety (see [7] for the details). We give an explicit proof, for the sake of completeness.

Being $\mathbb{LP}$ a discriminator variety means that it also is congruence-distributive, and, by Jónsson's Lemma [7], every subdirectly irreducible algebra in $\mathbb{LP}$ is a homomorphic image of a subalgebra $\boldsymbol{B}$ of an ultraproduct of $[0,1]_{\text{Ł}\Pi}$. $\boldsymbol{B}$ is a discriminator algebra and, therefore, it is simple. Thus, every subdirectly irreducible member of $\mathbb{LP}$ embeds into an ultraproduct of $[0,1]_{\text{Ł}\Pi}$. It follows that every member of $\mathbb{LP}$ is isomorphic to a subdirect product of subalgebras of ultraproducts of $[0,1]_{\text{Ł}\Pi}$, i.e.: $\mathbb{LP}$ is generated by $[0,1]_{\text{Ł}\Pi}$ as a quasivariety.

The same argument applies to $\mathbb{LP}\frac{1}{2}$. □

The above results now allow us to prove that $\mathbb{PMV}^+$ is a quasivariety generated by $[0,1]_{\mathrm{PMV}}$.

LEMMA 5.3.17.

(1) $\mathbb{PMV}^+$ *coincides with the quasivariety generated by* $[0,1]_{\mathrm{PMV}}$. *Therefore, every quasiequation* Φ *is true in* $[0,1]_{\mathrm{PMV}}$ *iff it is true in all* PMV^+*-algebras.*

(2) PMV^+*-chains are precisely the isomorphic images of subalgebras of ultrapowers of* $[0,1]_{\mathrm{PMV}}$. *Therefore, every universal formula* Ψ *is true in* $[0,1]_{\mathrm{PMV}}$ *iff it is true in all* PMV^+*-chains.*

Proof. First of all, every PMV^+-chain $\boldsymbol{A}$ is the interval algebra of an ordered integral domain $\mathbf{D}_{\boldsymbol{A}}$ which is embeddable in an ordered field $\boldsymbol{F}$. The interval algebra $\boldsymbol{A}_{\mathbf{F}}$ of $\mathbf{F}$ is an ŁΠ-chain that contains $\boldsymbol{A}$ as a subreduct. So, every linearly ordered PMV^+-algebra is a subreduct of a linearly ordered ŁΠ-algebra.

(1) $\mathbb{PMV}^+$ is a quasivariety containing $[0,1]_{\mathrm{PMV}}$, therefore it contains the quasivariety generated by $[0,1]_{\mathrm{PMV}}$. Conversely, by Theorem 5.3.16, the variety of ŁΠ-algebras is generated as a quasivariety by $[0,1]_{\text{Ł}\Pi}$. Hence, every PMV^+-algebra is a subreduct of a direct product of ultrapowers of $[0,1]_{\text{Ł}\Pi}$; therefore, it is a subalgebra of a direct product of ultrapowers of $[0,1]_{\mathrm{PMV}}$.

(2) Every linearly ordered PMV^+-algebra $\boldsymbol{A}$ is a subreduct of a linearly ordered ŁΠ-algebra $\boldsymbol{B}$. The ordered field $\mathbf{F}_{\boldsymbol{B}}$ related to $\boldsymbol{B}$ (see Lemma 5.3.10) embeds into its real closure $\overline{\mathbf{F}_{\boldsymbol{B}}}$, which is elementarily equivalent to R. By Frayne's Theorem (see [8]), $\overline{\mathbf{F}_{\boldsymbol{B}}}$ embeds into an ultrapower R^* of R. Consequently, $\boldsymbol{B}$ embeds into the interval algebra $\boldsymbol{C}_{\mathsf{R}^*}$ of R^*, which is isomorphic to an ultrapower of $[0,1]_{\text{Ł}\Pi}$. Clearly, $\boldsymbol{A}$ is an isomorphic image of subalgebras of ultrapowers of $[0,1]_{\mathrm{PMV}}$, and every universal formula Ψ is true in $[0,1]_{\mathrm{PMV}}$ iff it is true in all PMV^+-chains. □

As an immediate consequence of the above lemma, we have:

THEOREM 5.3.18. $\mathbb{PMV}^+$ *is generated as a quasivariety by the class of* PMV^+*-chains and by* $[0,1]_{\mathrm{PMV}}$.

THEOREM 5.3.19. $\mathbb{PMV}^+$ *is not a variety.*

Proof. Let $[0,1]^*$ be a non-trivial ultraproduct of $[0,1]_{\text{ŁΠ}}$, and let ϵ be a (strictly) positive infinitesimal, i.e., a non-zero element of $[0,1]^*$ such that for every positive natural number n, one has: $(n)\epsilon \leq 1-\epsilon$. Call $[0,1]^-$ the PMV-reduct of $[0,1]^*$, and let J be a subset of $[0,1]^-$ consisting of all z for which there is a natural number n such that $z \leq (n)\epsilon^2$. J is an MV-ideal, therefore, by Lemma 5.3.4, it determines a congruence θ of $[0,1]^-$. Now let $[0,1]^-/\theta$ denote the quotient of $[0,1]^-$ modulo θ, and for $a \in [0,1]^-$, let a_θ denote the equivalence class of a modulo θ. Then, in $[0,1]^-/\theta$ we have that $\epsilon_\theta^2 = 0$ and $\epsilon_\theta \neq 0$. On the other hand, the quasi identity $\forall x\ (x^2 = 0) \implies (x = 0)$ is true in $[0,1]_{\text{ŁΠ}}$, and in every PMV^+-algebra. Consequently, $[0,1]^-/\theta$ cannot be a PMV^+-algebra. Indeed, $\mathbb{PMV}^+$ is not closed under homomorphic images, and, so, it does not constitute a variety. □

As an immediate consequence of Theorem 5.3.16 and Theorem 5.3.18, we now have:

THEOREM 5.3.20. $\text{PŁ}'$*,* ŁΠ*, and* $\text{ŁΠ}\frac{1}{2}$ *have the* $\text{FS}\mathcal{R}\text{C}$*.*

We can now show a negative result about the Deduction Theorem for $\text{PŁ}'$.

COROLLARY 5.3.21. $\text{PŁ}'$ *does not satisfy the same Deduction Theorem as* PŁ*.*

Proof. Suppose that the Deduction Theorem is valid. Since $\neg(\varphi \,\&_\Pi\, \varphi) \vdash \neg\varphi$, we have that for some n, $(\neg(\varphi \,\&_\Pi\, \varphi))^n \to \neg\varphi$ is a theorem of $\text{PŁ}'$. This means that $(\neg(\varphi \,\&_\Pi\, \varphi))^n \to \neg\varphi$ is a tautology for evaluations into the reals, i.e., there is an n such that $(\neg(x \cdot x))^n \leq \neg x$ for each $x \in [0,1]$. Notice that the derivatives of $(\neg(x \cdot x))^n$ and $\neg x$ at the point 0 are equal to 0 and -1, respectively. This implies that for each n, there exists an x such that $(\neg(x \cdot x))^n > \neg x$, which leads to a contradiction. □

5.4 The expressive power of $\text{ŁΠ}\frac{1}{2}$

$\text{ŁΠ}\frac{1}{2}$ is certainly one of the most expressive ($\triangle$-)core fuzzy logics. We are going to make this even clearer by proving that the first-order theory $\mathsf{Th}(\text{OF})$ of ordered fields can be interpreted into the theory $\mathsf{Th}(\text{ŁΠ}\frac{1}{2})$ of $\text{ŁΠ}\frac{1}{2}$-chains. As a consequence we will obtain that the theory of the reals in interpretable into the equational theory of $\text{ŁΠ}\frac{1}{2}$. Therefore, functions definable in the theory of the reals, in the language of ordered fields, can be defined by means of $\text{ŁΠ}\frac{1}{2}$-equations (in a sense that will be made clear later on). This will allow us to show that many ($\triangle$-)core fuzzy logics can be interpreted within $\text{ŁΠ}\frac{1}{2}$.

Let **F** be any ordered field, and let $(0,1)_{\mathbf{F}} = F \cap (0,1)$ ($[0,1]_{\mathbf{F}} = F \cap [0,1]$) denote the open (closed) unit interval of **F**. Let $\sigma\colon (0,1)_{\mathbf{F}} \to F$ be the following strictly increasing surjective mapping, continuous w.r.t. the order topology:

$$\sigma(x) = \begin{cases} \frac{(2x-1)}{2x} & \text{if } 0 < x \leq \frac{1}{2}, \\ \frac{1-2x}{2(x-1)} & \text{if } \frac{1}{2} \leq x < 1, \end{cases}$$

whose inverse is

$$\sigma^{-1}(y) = \begin{cases} \frac{1}{2(1-y)} & \text{if } -\infty < y \leq 0, \\ \frac{2y+1}{2y+2} & \text{if } 0 \leq y < +\infty. \end{cases}$$

We use the function h to define an isomorphic copy of $\mathbf{F}$ over $(0,1)_{\mathbf{F}}$. This will allow, in turn, to define an interpretation of $\mathbf{F}$ over $\boldsymbol{A}_{\mathbf{F}}$. More precisely, we define:

$$\begin{aligned} x +_0 y &= \sigma^{-1}(\sigma(x) + \sigma(y)) \\ x \cdot_0 y &= \sigma^{-1}(\sigma(x) \cdot \sigma(y)) \\ -_0 x &= \sigma^{-1}(-\sigma(x)) \\ 0_0 &= \sigma^{-1}(0) = \tfrac{1}{2} \\ 1_0 &= h^{-1}(1) \\ x \leq_0 y &\text{ iff } \sigma(x) \leq \sigma(y). \end{aligned}$$

Clearly, σ is an isomorphism from

$$\mathbf{F}_0 = \langle (0,1) \cap F, +_0, \cdot_0, -_0, \leq_0, 0_0, 1_0 \rangle \text{ onto } \mathbf{F} = \langle F, +, \cdot, -, \leq, 0, 1 \rangle.$$

LEMMA 5.4.1. $\mathsf{Th}(\mathrm{OF})$*is interpretable into* $\mathsf{Th}(\text{Ł}\Pi\frac{1}{2})$.

Proof. First, let $\star$ be the map associating each ordered field $\mathbf{F}$ to its interval Ł$\Pi\frac{1}{2}$-chain $\boldsymbol{A}_{\mathbf{F}}$. Let $\Xi(x)$ be the Ł$\Pi\frac{1}{2}$-formula $(0 < x) \sqcap (x < 1)$. Then, clearly, $\sigma^{-1} \colon F \to \{a \mid \boldsymbol{A}_{\mathbf{F}} \models \chi(a)\}$. We need to show that for all $\bar{b} \in F$ and unnested atomic formulas $\Phi(\overline{x})$

$$\mathbf{F} \models \Phi(\bar{b}) \quad \text{iff} \quad \boldsymbol{A}_{\mathbf{F}} \models \Phi^{\sharp}(\sigma^{-1}(\bar{b})).$$

Now,

- if $\Phi(\overline{x})$ is $x = y$ or $x < y$, then $\Phi^{\sharp}(\overline{x})$ is $x = y$ or $x < y$, respectively;
- if $\Phi(\overline{x})$ is $x = 0$, $x = 1$, $x < 0$, or $x < 1$, then $\Phi^{\sharp}(\overline{x})$ is $x = 0$, $x = \frac{3}{4}$, $x < 0$, or $x < \frac{3}{4}$, respectively (the other cases are similar);
- if $\Phi(\overline{x})$ is $x = -y$, then $\Phi^{\sharp}(\overline{x})$ is $x = \neg y$;
- if $\Phi(\overline{x})$ is either $z = x + y$, or $z = x \cdot y$, we can find, by using the above defined functions σ and σ^{-1}, the equivalent definitions of $x +_0 y$ and $x \cdot_0 y$ over $\mathbf{F}_0$. An easy but tedious calculation shows that there exist Ł$\Pi\frac{1}{2}$-formulas $\Phi^{+_0}(x,y,z)$ and $\Phi^{\cdot_0}(x,y,z)$,[23] such that for all $a, b, c \in F$

$$\begin{aligned} \mathbf{F} \models a = b + c \quad &\text{iff} \quad \boldsymbol{A}_{\mathbf{F}} \models \Phi^{+_0}(\sigma^{-1}(a), \sigma^{-1}(b), \sigma^{-1}(c)), \\ \mathbf{F} \models a = b \cdot c \quad &\text{iff} \quad \boldsymbol{A}_{\mathbf{F}} \models \Phi^{\cdot_0}(\sigma^{-1}(a), \sigma^{-1}(b), \sigma^{-1}(c)). \end{aligned}$$ □

The next lemma shows that each quantifier-free Ł$\Pi\frac{1}{2}$-formula is equivalent to an Ł$\Pi\frac{1}{2}$-equation.

LEMMA 5.4.2. *Let* $\Phi(\overline{x})$ *be a quantifier-free formula in the language of* Ł$\Pi\frac{1}{2}$*-chains. Then there exists an* Ł$\Pi\frac{1}{2}$*-term* $t(\overline{x})$ *such that, for every* Ł$\Pi\frac{1}{2}$*-chain* $\boldsymbol{A}$*, and all* $b \in A$*:*

$$\boldsymbol{A} \models \Phi(\bar{b}) \quad \textit{iff} \quad \boldsymbol{A} \models t(\bar{b}) = 1.$$

Proof. The formula $x \leq y$ is interpreted by the term $\triangle(x \to y)$, which will be denoted as $t^{\leq}(x,y)$. The formula $x = y$ is translated by $\triangle(x \leftrightarrow y)$ (denoted as $t^{=}(x,y)$). Then, $x < y$ is interpreted by $t^{<}(x,y) = t^{\leq}(x,y) \wedge \neg(t^{=}(x,y))$.

[23] See [62] for an example of how a similar construction works.

We define for every quantifier-free formula Φ in the language of $\text{Ł}\Pi\frac{1}{2}$-chains, a term t^{Φ} in the following inductive way:

- If Φ is $x = y$, then $t^{\Phi} := t^{=}(x, y)$.
- If Φ is $x < y$, then $t^{\Phi} := t^{<}(x, y)$.
- If Φ is $\Psi \sqcup \Lambda$ ($\Psi \sqcap \Lambda$, $\backsim\Psi$ respectively), then $t^{\Phi} := t^{\Psi} \vee t^{\Lambda}$ ($t^{\Psi} \wedge t^{\Lambda}$, $\neg(t^{\Psi})$ respectively).

The claim easily follows from the above construction. □

The following corollary is an easy consequence of Lemma 5.4.1 and Lemma 5.4.2.

COROLLARY 5.4.3. *Let* $\mathbf{F}$ *be an ordered field, and let* $\Phi(x_1, \ldots, x_n)$ *be a quantifier-free formula in the language of ordered fields with coefficients in* Q. *Then, there exists an* $\text{Ł}\Pi\frac{1}{2}$*-term* $t(x_1, \ldots, x_n)$ *such that, for all* $a_1, \ldots, a_n \in F$, *the following are equivalent:*

(1) $\mathbf{F} \models \Phi(a_1, \ldots, a_n)$.

(2) $\boldsymbol{A}_{\mathbf{F}} \models t(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)) = 1$.

From the above translation, we immediately obtain the following result:

THEOREM 5.4.4. *Let* $\overline{\mathbf{F}}$ *be a real closed field, and let* $\Phi(x_1, \ldots, x_n)$ *be any formula in the language of ordered fields with coefficients in* Q. *Then, there exists an* $\text{Ł}\Pi\frac{1}{2}$*-term* $t(x_1, \ldots, x_n)$ *such that, for all* $a_1, \ldots, a_n \in F$:

$$\overline{\mathbf{F}} \models \Phi(a_1, \ldots, a_n) \quad \textit{iff} \quad \boldsymbol{A}_{\overline{\mathbf{F}}} \models t(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)) = 1.$$

Proof. The theory of real closed fields enjoys the elimination of quantifiers in the language of ordered fields, thus $\Phi(x_1, \ldots, x_n)$ is equivalent to a quantifier-free formula $\Psi(x_1, \ldots, x_n)$. The result follows from Corollary 5.4.3. □

Given a real closed field $\overline{\mathbf{F}} = \langle F, +, \cdot, -, \leq, 0, 1\rangle$, a *semialgebraic set* is a subset of F^n of the form

$$\bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in F^n \mid f_{i,j}(x) \circledcirc_{i,j} 0\} \tag{$\natural$}$$

where $f_{i,j}(x) \in F[X_1, \ldots, X_n]$ and $\circledcirc_{i,j}$ is either $<$ or $=$, for $i = 1, \ldots, s$, and $j = 1, \ldots, r_i$. It is easy to see that semialgebraic subsets of F are exactly finite unions of points and open intervals. In particular, every semialgebraic subset of F^n can be written as a finite union of semialgebraic sets of the form:

$$\{x \in F^n \mid f_1(x) = \cdots = f_l(x) = 0, g_1(x) > 0, \ldots, g_m(x) > 0\},$$

where $f_1, \ldots, f_l, g_1, \ldots, g_m \in F[X_1, \ldots, X_n]$. In other words, semialgebraic sets are subsets of a real closed field defined by a finite Boolean combination of polynomial equations and inequalities.

We call a set $S \subseteq \mathsf{R}^n$ Q-*semialgebraic* if it has the form ($\natural$), where the $f_{i,j}(x)$ are polynomials with rational coefficients (see also Chapter IX).

Recall that a set $S \subseteq \mathsf{R}^n$ is said to be *definable* in R, in the language of ordered fields, if there is a first-order formula $\Phi(x_1, \ldots, x_n)$ such that

$$S = \{\langle a_1, \ldots, a_n \rangle \mid \mathsf{R} \models \Phi(a_1, \ldots, a_n)\}.$$

A function is said to be *definable* in R iff its graph is definable in R.

DEFINITION 5.4.5.

(1) *A function $g\colon [0,1]^n \to [0,1]$ is said to be* term-definable *in* $Ł\Pi\frac{1}{2}$ *if there is a term $t(x_1, \ldots, x_n)$ of $Ł\Pi\frac{1}{2}$-algebras such that for all $a_1, \ldots, a_n \in [0,1]$:*

$$t(a_1, \ldots, a_n) = g(a_1, \ldots, a_n).$$

(2) *A set $X \subseteq [0,1]^n$ is said to be* definable *in $Ł\Pi\frac{1}{2}$ if its characteristic function is term-definable in $Ł\Pi\frac{1}{2}$.*

(3) *A function f is said to be* implicitly definable *in $Ł\Pi\frac{1}{2}$ if its graph is definable in $Ł\Pi\frac{1}{2}$.*

Recall that an $Ł\Pi\frac{1}{2}$-*hat* over $[0,1]^n$ is a function $h\colon [0,1]^n \to [0,1]$ such that there exist a Q-semialgebraic set $S \subseteq [0,1]^n$ and polynomials $f(x_1, \ldots, x_n), g(x_1, \ldots, x_n) \in \mathsf{Q}[X_1, \ldots, X_n]$ such that $g(x_1, \ldots, x_n)$ has no zeros on S, $h = \frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)}$ on S, and $h = 0$ on $[0,1]^n \backslash S$ (see Chapter IX). A function $h\colon [0,1]^n \to [0,1]$ is said to be *piecewise rational* if it is the supremum of finitely many $Ł\Pi\frac{1}{2}$-hats.

The next theorem, whose proof can be found in Chapter IX, characterizes term-definable functions.

THEOREM 5.4.6. *A function $h\colon [0,1]^n \to [0,1]$ is term-definable in $Ł\Pi\frac{1}{2}$ iff it is a piecewise rational function.*

Clearly, it follows that functions as $\sqrt{x}$ or $\sqrt{1-x^2}$ cannot be defined by terms in $Ł\Pi\frac{1}{2}$.

The next theorem gives a characterization of definable sets and, therefore, of implicitly definable functions in $Ł\Pi\frac{1}{2}$.

THEOREM 5.4.7. *A set $S \subseteq [0,1]^n$ is definable in $Ł\Pi\frac{1}{2}$ iff it is definable in R by a formula with rational coefficients iff it is Q-semialgebraic. Thus, a function $f\colon [0,1]^n \to [0,1]$ is implicitly definable in $Ł\Pi\frac{1}{2}$ iff its graph is Q-semialgebraic.*

Proof. If $S \subseteq [0,1]^n$ is definable in $Ł\Pi\frac{1}{2}$, then, by Theorem 5.3.14, there exists a formula in the language of ordered fields that defines S over R, and so S is obviously Q-semialgebraic.

Conversely, if $S \subseteq [0,1]^n$ is Q-semialgebraic, it is defined by a Boolean combination of polynomial equalities and inequalities over R, and, consequently, by Theorem 5.4.4, it is definable in $Ł\Pi\frac{1}{2}$.

It is then obvious that a function $f\colon [0,1]^n \to [0,1]$ is implicitly definable in $Ł\Pi\frac{1}{2}$ iff its graph is Q-semialgebraic. $\square$

The fact that functions definable in the theory of the real numbers can be implicitly defined in $\text{Ł}\Pi\frac{1}{2}$ can be used to show that some ($\triangle$-)core fuzzy logics have a faithful interpretation in the equational theory of $\text{Ł}\Pi\frac{1}{2}$, as shown below.

DEFINITION 5.4.8. *Let* L *be any* ($\triangle$-)*core fuzzy logic whose equivalent algebraic semantics is a variety generated by a structure whose lattice reduct is the real unit interval* $[0,1]$. L *is said to be* definable in $\text{Ł}\Pi\frac{1}{2}$ *if the interpretation of each* L*-connective over* $[0,1]$ *corresponds to an implicitly definable function.*

THEOREM 5.4.9. *Let* L *be any* ($\triangle$-) *core fuzzy logic definable in* $\text{Ł}\Pi\frac{1}{2}$. *Then, for every* L*-formula* ϕ *there exists an* $\text{Ł}\Pi\frac{1}{2}$*-formula* $\phi^\bullet$ *such that*

$$\models_{\mathrm{L}} \phi \quad \textit{iff} \quad \models_{\text{Ł}\Pi\frac{1}{2}} \phi^\bullet.$$

Proof. Let $C = \{\lambda_i\}_{1\leq i\leq n}$ be the set of basic connectives of L. By definition, the graph of each λ_i is term-definable by a formula ψ_{λ_i} in $\text{Ł}\Pi\frac{1}{2}$.

Now, let ϕ be any L-formula, and let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be the set of subformulas of ϕ. Next, to each γ_j associate a variable v_j (different variables for different subformulas). For each λ_i, let

$$\Sigma_{\lambda_i} = \{(v_\sigma, v_{\sigma_{1_i}}, \ldots, v_{\sigma_{t_i}}) \mid v_\sigma = \lambda_i(v_{\sigma_{1_i}}, \ldots, v_{\sigma_{t_i}})\},$$

where each v_{σ_j} is a variable associated to a subformula.

For each $(v_\sigma, v_{\sigma_{1_i}}, \ldots, v_{\sigma_{t_i}}) \in \Sigma_{\lambda_i}$, introduce the formulas $\psi_{\lambda_i}(v_\sigma, v_{\sigma_{1_i}}, \ldots, v_{\sigma_{t_i}})$ for each basic connective λ_i. Each ψ_{λ_i} defines the graph of λ_i.

For each $\lambda_i \in F$, denote by Θ_{λ_i} the conjunction of all the above formulas. Let $\phi^\bullet$ be the following formula:

$$\phi^\bullet := \left(\bigwedge \Theta_{\lambda_i}\right) \to v_m,$$

where v_m is the variable associated to the whole formula ϕ.

It can be to checked from the construction that

$$\models_{\mathrm{L}} \phi \quad \text{iff} \quad \models_{\text{Ł}\Pi\frac{1}{2}} \phi^\bullet. \qquad \square$$

The previous theorem shows that $\text{Ł}\Pi\frac{1}{2}$'s expressive power allows to faithfully interpret several logical systems. As an example, we are going to show that the logic associated to any continuous t-norm representable as a finite ordinal sum is definable (see Chapter I for the background notions on t-norms and ordinal sums).

THEOREM 5.4.10. *Let* $*$ *be a continuous t-norm. The following are equivalent:*

(1) *Up to isomorphism,* $*$ *is implicitly definable in* $\text{Ł}\Pi\frac{1}{2}$.

(2) *Up to isomorphism,* $*$ *is term-definable in* $\text{Ł}\Pi\frac{1}{2}$.

(3) $*$ *is representable as a finite ordinal sum of Łukasiewicz and product t-norms.*

Proof. If $*$ is term-definable it clearly also is implicitly definable. Thus, (2) implies (1).

Assume $*$ is representable as a finite ordinal sum of Łukasiewicz and Product t-norms. Without any loss of generality we can suppose that the cut points in the ordinal sum are rationals. Then, the graph of the function

$$x * y = \begin{cases} a_i + (b_i - a_i) \cdot \left(\frac{x-a_i}{b_i-a_i} *_i \frac{y-a_i}{b_i-a_i} \right) & \text{if } x, y \in (a_i, b_i]^2, \\ \min\{x, y\} & \text{otherwise} \end{cases}$$

is obviously a Q-semialgebraic set, and therefore $*$ is implicitly definable in $\text{Ł}\Pi\frac{1}{2}$, by Theorem 5.4.7. Moreover, an easy inspection shows that $*$ is a piecewise rational function, and therefore it is term-definable, by Theorem 5.4.6. Thus (3) implies both (1) and (2).

We show that (1) implies (3). Suppose that $*$ is an infinite ordinal sum of Product and Łukasiewicz components. The set Id_* of idempotent elements of $*$ is definable as $Id_* = \{x \mid x{*}x = x\}$. However, Id_* cannot be a Q-semialgebraic set, since it is not a finite union of points. This clearly implies that the graph of $*$ cannot be Q-semialgebraic, and, as a consequence, $*$ is not implicitly definable. This concludes the proof of the theorem. □

Now, we can prove:

THEOREM 5.4.11. *Let* L_* *be the logic of a continuous t-norm* $*$ *representable as a finite ordinal sum. Then* L_* *is definable in* $\text{Ł}\Pi\frac{1}{2}$.

Proof. It is easy to see that the residuum of any implicitly definable left-continuous t-norm is implicitly definable. Indeed, if $*$ is implicitly definable in $\text{Ł}\Pi\frac{1}{2}$, then its graph is definable in the theory of reals by a quantifier-free formula $\Phi(x, y, z)$, and so is the graph of its residuum $\Rightarrow_*$ by means of the first-order formula

$$\forall u \forall v (\Phi(u, x, v) \Longrightarrow (u \leq z \Longleftrightarrow v \leq y)).$$

The claim now follows by Theorem 5.4.4 and Theorem 5.4.10. □

6 Historical remarks and further reading

6.1 Expansions with truth-constants

When one is interested in explicitly representing and reasoning with intermediate degrees of truth, a convenient and elegant way is by introducing truth-constants into the language. In fact, if one introduces in the language new constant symbols $\overline{r}$ for suitable values $r \in [0, 1]$ and stipulates that $e(\overline{r}) = r$ for all truth-evaluations, then a formula of the kind $\overline{r} \to \varphi$ becomes 1-true under any evaluation e whenever $r \leq e(\varphi)$. The first formal treatment of this kind of system is due to Pavelka [75], who built a propositional many-valued logical system, which turned out to be equivalent to the expansion of Łukasiewicz logic by adding into the language a truth-constant $\overline{r}$ for each *real* $r \in [0, 1]$, together with a number of additional axioms. The resulting system was shown to be complete in a non-standard sense, later known as Pavelka-style completeness (see Section 2.1). Novák extended Pavelka's approach to Łukasiewicz first-order logic [73].

Later, Hájek [46] proved that Pavelka's logic could be significantly simplified by showing that it is enough to expand the language only with a countable set of truth-constants, one for each *rational* in $[0, 1]$, and by adding to the logic the so-called *book-keeping axioms* dealing with truth-constants. He called this new system Rational Pavelka logic (RPL), and proved it is standard complete for finite theories in the usual sense. He also defined the logic $\mathrm{RPL}\forall$, the first-order expansion of RPL, and showed that $\mathrm{RPL}\forall$ enjoys the same Pavelka-style completeness.

Several expansions à la Pavelka with truth-constants of fuzzy logics different from Łukasiewicz have also been studied, mainly related to the other two outstanding continuous t-norm based logics, namely Gödel and product logic. We may cite [46] where an expansion of $\mathrm{G}_\triangle$ with a finite number of rational truth-constants was studied, [28] where the authors define logical systems obtained by adding (rational) truth-constants to $\mathrm{G}_\sim$ (Gödel logic with an involutive negation) and to Π (product logic) and $\Pi_\sim$ (product logic with an involutive negation). In the case of the rational expansions of Π and $\Pi_\sim$ an infinitary inference rule (from $\{\varphi \to \overline{r} \mid r \in \mathsf{Q} \cap (0, 1]\}$ infer $\varphi \to \overline{0}$) is introduced in order to get Pavelka-style completeness.

Following the same line, Cintula gives in [16] a definition of what he calls *Pavelka-style extension* of a particular fuzzy logic. He considers the Pavelka-style extensions of the most popular fuzzy logics, and for each one of them he defines an axiomatic system with infinitary rules (to overcome discontinuities like in the case of Π explained above) which is proved to be Pavelka-style complete. Moreover he also considers the first-order versions of these extensions and provides necessary conditions for them to satisfy Pavelka-style completeness.

A difficulty concerning Pavelka-style completeness is that it cannot be obtained for logics different from Łukasiewicz without the introduction of infinitary rules, since Łukasiewicz logic is the only fuzzy logic whose truth-functions (conjunction and implication) are continuous functions. Due to this fact, a more general approach has been developed in a series of papers [12, 26, 30–33, 78] where, rather than Pavelka-style completeness, the authors have focused on the usual notion of completeness of a logic.

In all these works, special attention has been paid to formulas of the kind $\overline{r} \to \varphi$, where $\overline{r}$ denotes the truth-constant r and φ is a formula without any additional truth-constants. Actually, this kind of formulas has been extensively considered in other frameworks for reasoning with partial degrees of truth, like in Novák's evaluated syntax formalism based on Łukasiewicz logic (see e.g. [74]) or in fuzzy logic programming (see e.g. [85]). In particular, these formulas can be seen as a special kind of Novák's *evaluated* formulas, which are expressions a/A where a is a truth value (from a given algebra) and A is a formula that may contain truth-constants again, and whose interpretation is that the truth-value of A is at least a. Hence, our formulas $\overline{r} \to \varphi$ would be expressed as r/φ in Novák's evaluated syntax. On the other hand, formulas $\overline{r} \to \varphi$, when φ is a Horn-like rule of the form $b_1 \,\&\, \ldots \,\&\, b_n \to h$, also correspond to typical fuzzy logic programming rules $(b_1 \,\&\, \ldots \,\&\, b_n \to h, r)$, where r specifies a lower bound for the validity of the rule. Finally, truth-degrees in the syntax also appear in the Gerla's framework of abstract fuzzy logics [40], which is based on the notion of fuzzy consequence operators over fuzzy sets of formulas, where the membership degree of formulas are, again, interpreted as lower bounds of their truth-degrees.

6.2 Expansions with truth-stressing and truth-depressing hedges

There are two main references when talking about the formalization of truth-stressing hedges within the framework of mathematical fuzzy logic. The first one is Hájek's paper [47], already referred to in the previous sections, where he axiomatizes a logic for the hedge *very true* over BL. The second one is the paper by Vychodil [86], where the author extends Hájek's analysis to truth-depressing hedges.

A relevant further study of logics with truth-stressers can be found in the paper by Ciabattoni et al. [9], that makes significant contributions in various aspects. The authors basically consider expansions of MTL with a unary modality (i.e. a unary operator that satisfies axiom K and the necessitation rule), they consider three possible additional axioms to be added to Hájek axiomatics, and they develop proof systems for the new logics and study their algebraic and completeness properties. Given a logic L that is an extension of MTL, they consider the following logics particularly relevant for our purposes:

$$\begin{aligned}\text{L-KT}^r &= \mathrm{L} + (\mathrm{VE1}) + (\mathrm{VE2}) + (\mathrm{VE3}) + \mathrm{NEC},\\ \text{L-S4}^r &= \text{L-KT}^r + (\mathrm{VE4})\ s\varphi \to s(s\varphi).\end{aligned}$$

Axiom $(\mathrm{VE4})$, together with axiom $(\mathrm{VE1})$, forces the truth-stressing hedges to be closed over their image, i.e. $s\varphi$ has to be equivalent to $s(s\varphi)$ (hence s becomes a closure operator like in some previous work; see [50], for instance).

Notice that Hájek's logic BL_{SK} (called BL_{vt} in his paper) is nothing but the logic BL-KT^r. Moreover, Ciabattoni et al. prove in [9] standard completeness of the L-S4^r logics for different choices for L, namely MTL, SMTL, C_nMTL, IMTL, and $\mathrm{C}_n\mathrm{IMTL}$. Finally, observe that after adding the axiom $s\varphi \vee \neg s\varphi$ to L-KT^r, s turns to be equivalent to the well-know Monteiro–Baaz projection connective $\triangle$.

Other papers dealing with particular types of truth-stressers are:

- The paper [50], a pioneering work in the setting of truth-stressing hedges, which proves that the Yashin *strong future tense operator* can be interpreted, in our framework, as a hedge over G that is a closure operator and satisfies axiom K.

- The paper [48], which defines the logical system $\mathrm{BL}^{!}_{LU}$ obtained by adding two unary connectives, L and U, (for truth stresser and depresser) to $\mathrm{BL}_{\triangle}$ that are required to be idempotent with respect to the monoidal operation, among other technical properties. The paper contains an interesting result about the undecidability of $*$-tautologies.

- In the paper [51] the authors introduce in $\mathrm{BL}\forall$ a new unary connective At, interpreted as *almost true*, in order to analyze the *sorites* paradox in the setting of mathematical fuzzy logic. It turns out that the axioms proposed for this new connective are $(\mathrm{STL1})$ and the new axiom

  $$(\varphi \to \psi) \to (At\varphi \to At\psi)$$

  which is stronger than (MON). However, the axiom $(\mathrm{STL2})$ is not required.

- The paper [68] studies the system obtained by adding to a fuzzy logic L a unary connective called *storage operator* which has some analogies with Girard's exponentials and behaves as an idempotent truth-stresser closed over its image (it is in fact an interior operator).

Despite the undoubtable theoretical interest of these papers, truth-hedges that are either closure operators, satisfy axiom K, or are idempotent, have a quite limited behavior and can account only for some very special cases of truth-stressers.

As for truth-depressers, Vychodil [86] first introduces a logic combining both a truth-stresser and a truth-depresser. His logic, called $\mathrm{BL}_{vt,st}$, is defined as an expansion of Hájek's BL_{vt} logic with a new unary connective "slightly true" d and with the following additional axioms:

$(\mathrm{ST1})\quad \varphi \to d\varphi$

$(\mathrm{ST2})\quad d\varphi \to \neg s\neg\varphi$

$(\mathrm{ST3})\quad s(\varphi \to \psi) \to (d\varphi \to d\psi)$

This logic is proved to be complete with respect to the class of all linearly-ordered $\mathrm{BL}_{vt,st}$-algebras (defined in the obvious way). Note that axioms $(\mathrm{ST1})$ and $(\mathrm{ST2})$ put into relation both connectives s and d. Vychodil also proposes two slightly different axiomatizations (systems I and II) for the truth-depressing hedge *slightly true* alone. They are defined again as expansions of BL with the unary connective d. Namely, the system (I) has the following set of additional axioms:

$(\mathrm{DH1})\quad \varphi \to d\varphi$

$(\mathrm{DH2})\quad \neg d(\bar{0})$

$(\mathrm{DH3})\quad d(\varphi \to \psi) \to (d\varphi \to d\psi)$

while the system (II) includes the axioms $(\mathrm{DH1})$, $(\mathrm{DH2})$, and

$(\mathrm{DH4})\quad (\varphi \to \psi) \to (d\varphi \to d\psi)$

Both systems also have the following inference rule:

(RN_d) from $\neg\varphi$ infer $\neg d\varphi$

Chain-completeness for both systems is proved, but, again, the issue of real completeness is left open.

Notice that axioms $(\mathrm{DH1})$ and $(\mathrm{DH2})$ correspond exactly to $(\mathrm{STL1})$ and $(\mathrm{STL2})$ of the logic L_D, and that the inference rule (RN_d) is derivable from the rule (MON) using axiom $(\mathrm{STL2})$. So, again, the main difference between Vychodil's logics and the logics L_D is the presence of the K-like axioms $(\mathrm{DH3})$ and $(\mathrm{DH4})$, which do not appear in the logics L_D.[24]

[24] In fact for both Vychodil's systems over any axiomatic extension L of *Involutive* MTL logic IMTL the associated real chains are real L-chains taking the identity function Id as a truth-depresser. In fact, if d is a truth-depresser such that $d \neq Id$, then (DH3) and (DH4) are not satisfied. Namely, if $d \neq Id$, there exists an element $x \in (0,1)$ such that $d(x) > x$, and thus $d(x \Rightarrow 0) \geq (x \Rightarrow 0) = \neg x > \neg d(x) = d(x) \Rightarrow 0$. As a consequence, the only function d over an IMTL-chain that satisfies the axioms of either system (I) or (II) is the identity function.

Following [34], in Section 3, we have presented a more general approach to fuzzy logics with truth-stressing (depressing) hedges. The main advantage of the proposed systems with respect to the previously proposed ones is that we can show standard completeness, i.e. completeness with respect to the class of chains over the real unit interval expanded by arbitrary (stressing and depressing) hedges. The price paid in this process is that the class of corresponding algebras cannot be shown in general to be a variety any longer, but only a quasivariety. Actually it remains as an open problem to prove or disprove whether they form in fact a variety in the general case. It is only proved that L_S-algebras form a variety if either L is the logic of a finite BL-chain or it is the case that $\triangle$ is definable in L_S or when axiom $(\mathrm{VE2})$ for the s is derivable in L_S. All these cases enjoy a local or global deduction-detachment theorem.

6.3 Expansions with an involutive negation

Heyting algebras endowed with an involution were introduced by Moisil [65] already in 1942, as the algebraic models of an expansion of intuitionistic propositional calculus by means of a De Morgan negation. These algebras have been extensively investigated by Monteiro under the name of *symmetric Heyting algebras* [72]. They were also considered by Sankappanavar [77] independently from the previous work. Recently, in [11], the authors go a step further in the algebraic study of symmetric residuated lattices, in particular focusing on the properties of the combination of the two negations.

In the setting of fuzzy logic, early papers about fuzzy connectives were interested in the so-called De Morgan triples, i.e., triples formed by a t-norm, an involutive negation and the dual t-conorm (see for instance [1, 82]). In this tradition, Gehrke et al. [38] study De Morgan triples, their associated logics and the subvarieties generated by De Morgan triples over the real unit interval, with special attention to the case of De Morgan triples based on strict t-norms. In fact, the logics studied in [38] are implication-free fragments of the logics $\Pi_\sim$ and $\mathrm{SBL}_\sim$ described in this section.

In the more formal setting of mathematical fuzzy logic, the first paper on expansions with an involutive negation was [28], where Esteva et al., also independently from previous work, defined expansions of the logic SBL and their main axiomatic extensions, G and Π, with an involution. The key observation in [28] was that Monteiro–Baaz's $\triangle$ operator is definable as $\triangle\varphi := \neg{\sim}\varphi$ when $\neg$ is Gödel negation (common to SBL, G and Π) and $\sim$ is an involutive negation, and hence one can define, e.g., the expanded logic $\mathrm{SBL}_\sim$ as it was over the logic $\mathrm{SBL}_\triangle$, which makes the axiomatization much easier. The same approach works for G and Π. This line of research was continued by Flaminio and Marchioni in [35], where the more general case of adding an involution to $\mathrm{MTL}_\triangle$ and their axiomatic extensions is defined. On the other hand, Cintula et al. investigated in [19, 20] the lattice of subvarieties generated by $\mathrm{SBL}_\sim$-chains and $\Pi_\sim$-chains, while Haniková and Savický [55] went further in the study of subvarieties generated by $\mathrm{SBL}_\sim$-chains investigating isomorphisms between pairs formed by a (SBL) t-norm and different involutive negations. The main result is the characterization of families of such pairs such that they are either pairwise isomorphic or they generate incomparable subvarieties.

6.4 Expansions of Łukasiewicz logic

Section 5 covers the most important notions regarding expansions of Łukasiewicz logic and MV-algebras. We review the basic literature on the topics introduced above (not necessarily in strict chronological order), and mention where the interested reader can also find several complementary and advanced results.

Rational Łukasiewicz logic and DMV-algebras were introduced and studied by Gerla in [39], where the author also proved the basic completeness results. The category of DMV-algebras was shown to be equivalent to the category of divisible Abelian ℓ-groups with strong unit (both with homomorphisms). Moreover, the satisfiability problem for RL was proved to be $\mathbf{NP}$-complete.

Hájek, Godo, and Esteva [49] were the first to approach the problem of expanding Łukasiewicz logic with the product connective. In fact, they introduced an expansion of Rational Pavelka logic with product in order to define a logic to represent simple and conditional probability (see Section 2.1).

Later, Riečan [76] was the first to present an expansion of MV-algebras with the product operation, with the goal of defining product measures taking values in MV-algebras. A first algebraic study of MV-algebras with product was given by Di Nola and Dvurečenskij in [23]. In that work, however, the reduct $\langle A, \cdot, 1\rangle$ is not necessarily a commutative monoid. They proved a categorical equivalence result between this class of MV-algebras with product and non-commutative lattice-ordered rings.

PMV-algebras were introduced by Montagna in [66], where the author proved that the related variety is generated by the class of chains and that each PMV-chain is the interval algebra of an ordered commutative ring. Montagna explored in [69] several algebraic properties of subreducts of MV-algebras with the product conjunction and the product implication. In particular, he showed that $\mathbb{PMV}^+$ is not a variety and that it is a quasivariety generated by $[0,1]_{\mathrm{PMV}}$. The logics $\mathrm{PŁ}$ and $\mathrm{PŁ}'$ were introduced by Horčík and Cintula in [58] and shown to have finite strong completeness w.r.t. to the class of chains of the related variety and quasivariety.

ŁΠ and its related algebras were first introduced by Esteva and Godo in [27]. Montagna in [66], and Esteva, Godo, and Montagna [29] further investigated ŁΠ and introduced $\text{ŁΠ}\frac{1}{2}$ making their relation w.r.t. ordered fields explicit, and proving finite strong completeness w.r.t. evaluations into the reals. Montagna also proved in [69] that both the variety of ŁΠ and $\text{ŁΠ}\frac{1}{2}$ algebras are generated as quasivarieties by $[0,1]_{\text{ŁΠ}}$ and $[0,1]_{\text{ŁΠ}\frac{1}{2}}$, respectively. Cintula provided different equivalent axiomatizations for ŁΠ and $\text{ŁΠ}\frac{1}{2}$ in [14], and for their algebras in [17], and also studied their first-order expansion in [15].

An in-depth categorical investigation of the classes of PMV, PMV^+, ŁΠ, and $\text{ŁΠ}\frac{1}{2}$ algebras was carried out by Montagna in [66, 67, 69], where the author showed that the categories of PMV, PMV^+, ŁΠ, and $\text{ŁΠ}\frac{1}{2}$ algebras, all with homomorphisms, are equivalent to the categories of commutative lattice-ordered f-rings with strong unit, commutative lattice-ordered f-integral domains with strong unit, Q-f-semifields and f-semifields, with homomorphisms, respectively.

Montagna and Panti [70] gave a functional characterization of free ŁΠ and $\text{ŁΠ}\frac{1}{2}$ algebras in terms of piecewise rational functions (see also Chapter IX). A similar characterization for free PMV^+-algebras is still unavailable and is strictly related to the long-standing Pierce-Birkhoff conjecture in semialgebraic geometry [56].

Vetterlein gave in [84] a comprehensive study of the connections between certain classes of effect algebras, lattice-ordered rings and expansions of MV-algebras with product. Vetterlein explicitly showed the one-to-one correspondence between the class of f-product effect algebras, torsion-free f-product effect algebras, torsion-free f-product effect algebras with strict compatibility, and divisible torsion-free f-product effect algebras with strict compatibility, and the class of PMV, PMV^+, ŁΠ, and ŁΠ$\frac{1}{2}$ algebras, respectively.

Basic definitions and completeness results concerning expansions with rational truth-constants and their related bookkeeping axioms for PŁ and PŁ$'$ can be found in [58], and in [29] for ŁΠ and ŁΠ$\frac{1}{2}$. Expansions of PMV and PMV^+ algebras with $\triangle$ (and their related logics) are extensively studied in [58, 66, 67, 70], where completeness is shown along with categorical representations and functional characterizations.

The tautology problem for ŁΠ$\frac{1}{2}$ was shown to be in $\mathbf{PSPACE}$ by Hájek and Tulipani in [54] by relying on a polynomial-time translation into the universal theory of the field of reals.

The definability in ŁΠ$\frac{1}{2}$ of logics based on continuous t-norms representable as finite ordinal sums was first studied by Cintula in [14], who showed that such t-norms are term-definable in ŁΠ$\frac{1}{2}$. Marchioni and Montagna investigated functional definability issues within the equational theory of ŁΠ$\frac{1}{2}$-algebras in [62, 63], studying the definability of Q-semialgebraic sets, and triangular norms and uninorms. In particular they gave a complete characterization of term-definable and implicitly definable continuous t-norms and weak nilpotent minimum t-norms. Marchioni and Montagna also showed that the universal theory of real closed fields is definable into the equational theory of ŁΠ$\frac{1}{2}$-algebras and they both share the same computational complexity. Moreover, they proved that the logic associated to any implicitly definable uninorm is in $\mathbf{PSPACE}$, while the logic associated to any class of implicitly definable uninorms is decidable.

Marchioni investigated in [61] the lattice of subvarieties of ŁΠ$\frac{1}{2}$-algebras showing that it has the cardinality of the continuum. [61] also contains a brief study of the basic model-theoretic properties of the theory of ŁΠ$\frac{1}{2}$-chains that are interval algebras of real closed fields.

Other expansions of PMV-algebras were introduced exploiting their expressive power. The operations of the MV-algebra over the reals are continuous functions, but that is not the case for ŁΠ and ŁΠ$\frac{1}{2}$ algebras, since the product implication is obviously not continuous. The quasivariety of ŁΠ$_q$-algebras was introduced for this reason in [71], by Spada and Montagna, expanding the language of PMV-algebras with the operator $\rightarrow_q$, interpreted as a continuous approximation of the product implication.

Spada introduced in [79] μŁΠ-algebras and their logic. μŁΠ-algebras are an expansion of ŁΠ-algebras with fixed point operators $\mu x_{t(x,\overline{y})}$ for each term $t(x, \overline{y})$ not containing the product implication. Spada gave completeness results, showing that μŁΠ-chains are exactly the interval algebras of real closed fields, and that the category of μŁΠ-algebras with homomorphisms is equivalent to the category of real closed f-semifields with homomorphisms. A characterization of free μŁΠ-algebras and other model-theoretic results were given by Marchioni and Spada in [64].

Extending the approach initiated in [49] over Rational Pavelka logic, $Ł\Pi\frac{1}{2}$ has also been used for the logical representation of uncertainty measures. Esteva, Godo, and Hájek in [41, 42] defined a fuzzy modal logic over $Ł\Pi\frac{1}{2}$ to represent conditional probability and belief functions, Flaminio and Montagna defined in [36] an expansion of $Ł\Pi\frac{1}{2}$, called $SŁ\Pi$, to represent non-standard probabilities. Also, Godo and Marchioni in [43] for coherent conditional probability and Marchioni in [60] for conditional possibility, relied on $Ł\Pi\frac{1}{2}$ for building a logic to represent such classes of measures.

Finally, we mention the work [22] by Ciucci and Flaminio, where the authors use $Ł\Pi\frac{1}{2}$ to define inner and outer approximations of fuzzy sets.

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Chapter IX: Free Algebras and Functional Representation for Fuzzy Logics

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In this chapter we deal with concrete representations of free algebras in varieties that constitute the equivalent algebraic semantics of some prominent schematic extensions of BL and MTL. The chapter is organised as follows. In the first section we recall the notion of free algebra and stress its importance for logic. In the second section we shall deal with free MV-algebras, and related structures. In the third one, the subject are free product algebras, while the fourth is concerned with Gödel algebras and other subvarieties of WNM-algebras. The fifth section is concerned with free BL-algebras and related structures, while the sixth section deals with Pierce representations of free algebras in some subvarieties of MTL-algebras. We conclude the chapter with a section on open problems, and a final one on historical and bibliographical remarks.

1 Free algebras

In this section we recall some basic notions about free algebras in a variety, and their relationship with logic.

DEFINITION 1.0.1. *Let* $\mathbb{K}$ *be a class of algebras of one signature, and let* $\boldsymbol{A}$ *be an algebra of the same signature. Let* $X \subseteq A$*. Then* $\boldsymbol{A}$ *has the* universal mapping property for $\mathbb{K}$ over X *if for every* $\boldsymbol{B} \in \mathbb{K}$ *and any map* $f\colon X \to B$ *there is a homomorphism* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *extending* f*, that is* $h(x) = f(x)$ *for all* $x \in X$*.*

$\boldsymbol{A}$ *is* free for $\mathbb{K}$ over X *if* $\boldsymbol{A}$ *is generated by* X *and* $\boldsymbol{A}$ *has the universal mapping property for* $\mathbb{K}$ *over* X*.* $\boldsymbol{A}$ *is* free in $\mathbb{K}$ over X *if, in addition,* $\boldsymbol{A} \in \mathbb{K}$*. In this case* X *is called a set of* (free) generators *for* $\boldsymbol{A}$*. If* $\mathbb{K}$ *is a variety we shall call* $\boldsymbol{A}$ *the* free $\mathbb{K}$-algebra over X.

Since a homomorphism $\boldsymbol{A} \to \boldsymbol{B}$ is completely determined by how it maps the generators of $\boldsymbol{A}$, it follows that the homomorphism h extending f in Definition 1.0.1 is necessarily unique.

Note that if $\mathbb{K}$ is a variety then every free $\mathbb{K}$-algebra is in $\mathbb{K}$. In the following lemma we collect some well-known properties of algebras free in a class.

LEMMA 1.0.2. *Assume* $\boldsymbol{A}_1$ *and* $\boldsymbol{A}_2$ *are free in a class of algebras* $\mathbb{K}$*, respectively over* X_1 *and* X_2*. If* $|X_1| = |X_2|$ *then* $\boldsymbol{A}_1 \cong \boldsymbol{A}_2$*. This fact allows to speak of* $\boldsymbol{A}_1$ *(or* $\boldsymbol{A}_2$*) as* the algebra free in $\mathbb{K}$ over $|X_1|$ many generators.

The free $\mathbb{V}$*-algebra over* κ *many generators exists for any cardinal* κ *and any variety* $\mathbb{V}$ *containing non-trivial members.*

In the sequel of this chapter we shall only consider varieties containing non-trivial members. For any variety $\mathbb{V}$, we shall denote $\boldsymbol{F}_{\mathbb{V}}^{\kappa}$ the free $\mathbb{V}$-algebra over κ generators.

DEFINITION 1.0.3. *Let $\mathbb{K}$ be a class of algebras and $\boldsymbol{A}$ be an algebra of the same signature. We define the congruence $\Theta_{\boldsymbol{A}}(\mathbb{K})$ as*

$$\Theta_{\boldsymbol{A}}(\mathbb{K}) = \bigcap\{\theta \mid \theta \in Con(\boldsymbol{A}),\ \boldsymbol{A}/\theta \in \mathbf{S}(\mathbb{K})\}.$$

LEMMA 1.0.4. *If $\boldsymbol{A}$ is free for $\mathbb{K}$ over X then the algebra $\boldsymbol{A}/\Theta_{\boldsymbol{A}}(\mathbb{K})$ is free for $\mathbb{K}$ over the set $X/\Theta_{\boldsymbol{A}}(\mathbb{K}) = \{x/\Theta_{\boldsymbol{A}}(\mathbb{K}) \mid x \in X\}$. Moreover $\boldsymbol{A}/\Theta_{\boldsymbol{A}}(\mathbb{K}) \in \mathbf{SP}(\mathbb{K})$.*

Lemma 1.0.4 allows to recover free algebras in a variety $\mathbb{V}$ from the *absolutely free algebras* in the class of all algebras of the same signature as $\mathbb{V}$.

DEFINITION 1.0.5. *Let $\mathbb{K}$ be the class of all algebras of some fixed signature. Then an algebra free in $\mathbb{K}$ over a set X is called* absolutely free *over X.*

LEMMA 1.0.6. *Let $\mathbb{K}$ be the class of all algebras of some fixed signature and let $\mathbb{H} \subseteq \mathbb{K}$. If $\boldsymbol{A}$ is absolutely free in $\mathbb{K}$ over X then $\boldsymbol{A}$ is free for $\mathbb{H}$ over X.*

Let $\boldsymbol{A}$ be an algebra, and φ a term on the signature of $\boldsymbol{A}$ whose individual variables are in $\{x_1, \ldots, x_n\}$. Then, throughout the chapter, for any $\langle a_1, \ldots, a_n\rangle \in \boldsymbol{A}^n$ we denote

$$\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n)$$

the element of $\boldsymbol{A}$ obtained by substituting uniformly over φ each occurrence of x_i with a_i for all $i \in \{1, 2, \ldots, n\}$, and interpreting each operation symbol in φ with the corresponding operation of $\boldsymbol{A}$. The function mapping each $\langle a_1, \ldots, a_n\rangle \in \boldsymbol{A}^n$ to $\varphi^{\boldsymbol{A}}(a_1, \ldots, a_n) \in \boldsymbol{A}$ will be denoted $\varphi^{\boldsymbol{A}}$.

DEFINITION 1.0.7. *Let $\mathbb{K}$ be the class of all algebras of some fixed signature, and let $\mathbb{K}_0$ be the set of constants in this signature. Let further X be a set disjoint from $\mathbb{K}_0$. Then the* term algebra *$\boldsymbol{T}_{\mathbb{K}}(X)$ is the algebra whose universe is the smallest set $T(X)$ such that it contains the set of terms $X \cup \mathbb{K}_0$ and for any function symbol f in the signature, if f is n-ary, then for all $t_1, t_2, \ldots, t_n \in T(X)$, the term $f(t_1, t_2, \ldots, t_n)$ is in $T(X)$. The operations in $\boldsymbol{T}_{\mathbb{K}}(X)$ are defined as follows. For any n-ary function symbol f in the signature, and all $t_1, \ldots, t_n \in T(X)$,*

$$f^{\boldsymbol{T}_{\mathbb{K}}(X)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)\,.$$

LEMMA 1.0.8. *Let $\mathbb{K}$ be the class of all algebras of some fixed signature. Then $\boldsymbol{T}_{\mathbb{K}}(X)$ is absolutely free in $\mathbb{K}$ over X. Hence, if $\mathbb{H} \subseteq \mathbb{K}$ then $\boldsymbol{T}_{\mathbb{K}}(X)/\Theta_{\boldsymbol{T}_{\mathbb{K}}(X)}(\mathbb{H})$ is free for $\mathbb{H}$ over $X/\Theta_{\boldsymbol{T}_{\mathbb{K}}(X)}(\mathbb{H})$.*

Definitions 1.0.3, 1.0.5, and 1.0.7 and Lemmas 1.0.4, 1.0.6, and 1.0.8 show that free algebras in a variety over κ generators do not satisfy any additional equations, written on the set of variables $\{v_\alpha\}_{\alpha\leq\kappa}$, but those holding in every algebra in the variety. More precisely, every algebra $\boldsymbol{A}$ in a variety $\mathbb{V}$ is presented by a pair $\langle X, E\rangle$, where X is a set of generators, and E is a set of $\mathbb{V}$-equations whose variables are elements of X. An algebra is free in $\mathbb{V}$ over κ generators if it is presented by

$$\langle \{v_\alpha\}_{\alpha \leq \kappa}, \emptyset \rangle \, .$$

Moreover, the following holds.

LEMMA 1.0.9. *Each algebra in a variety $\mathbb{V}$ belongs to $\mathbf{H}(\boldsymbol{F}_{\mathbb{V}}^{\kappa})$ for some κ.*

We recall that for any algebraizable (in the sense defined in Chapter II) extension L of BL (or, *mutatis mutandis*, of MTL), the Lindenbaum algebra of the logic L over a set of variables V is the algebra obtained by quotienting the absolutely free algebra $\boldsymbol{T}_{\mathbb{K}}(V)$ (where $\mathbb{K}$ is the class of all algebras whose signature is the language of L) by the congruence $\equiv$ given by $\varphi \equiv \psi$ iff L proves both $\varphi \to \psi$ and $\psi \to \varphi$. See [20] for further background on algebraic logic and Lindenbaum algebras. The importance for logic of free algebras resides in their connection with Lindenbaum algebras, as stressed in the following lemma.

LEMMA 1.0.10. *Let* L *be a logic and* $\mathbb{V}$ *be its equivalent algebraic semantics. Then the Lindenbaum algebra of* L *is isomorphic to* $\boldsymbol{F}_{\mathbb{V}}^{\omega}$*, the free* $\mathbb{V}$*-algebra over denumerably many generators. Moreover, the Lindenbaum algebra of the* L*-formulas written over the set* $\{v_i\}_{i=1}^{n}$ *of propositional variables is isomorphic to* $\boldsymbol{F}_{\mathbb{V}}^{n}$*, for each integer* $n \geq 0$.

Thus, knowledge of the fine structure of free algebras in a variety $\mathbb{V}$ often translates to knowledge of the fine structure of the logic whose algebraic semantics is given by $\mathbb{V}$. When available, a concrete, intrinsic representation of free algebras is then a very powerful tool for the study of the associated logic. Unfortunately, finding concrete representations of free algebras is usually a highly non-trivial task. In this chapter we shall deal with such representations, and in particular with those that represent free algebras as algebras of functions over powers of the real unit interval $[0, 1]$. We call this kind of results a *functional representation*. A useful byproduct of concrete representation theorems are normal forms: as a matter of fact a main ingredient in these theorems consists in finding a uniform way to represent elements of the free algebras, that is, a uniform way to pick a representative element from each one of the equivalence classes of formulas which constitute the Lindenbaum algebras of the logic at hand.

A most useful tool for building concrete representation of free algebras is given by the following lemma.

DEFINITION 1.0.11. *An algebra* $\boldsymbol{A}$ *is* generic *for a variety* $\mathbb{V}$ *if*

$$\mathbb{V} = \mathbf{HSP}(\{\boldsymbol{A}\}) \, .$$

Note that $\boldsymbol{F}_{\mathbb{V}}^{\omega}$ is generic for $\mathbb{V}$.

LEMMA 1.0.12. *Let* $\boldsymbol{A}$ *be generic for a variety* $\mathbb{V}$*. Then, for each cardinal* κ *the free* $\mathbb{V}$*-algebra over* κ *generators is isomorphic to the subalgebra of* $\boldsymbol{A}^{\boldsymbol{A}^{\kappa}}$ *generated by the projections* $\langle a_1, a_2, \ldots, a_\alpha, \ldots \rangle \in \boldsymbol{A}^{\kappa} \mapsto a_\alpha$*, for each* $\alpha \leq \kappa$*. Further, let* $\boldsymbol{A}$ *be generic for the subvariety of* $\mathbb{V}$ *generated by all* κ*-generated algebras in* $\mathbb{V}$*. Then again,* $\boldsymbol{F}_{\mathbb{V}}^{\kappa}$ *is isomorphic to the subalgebra of* $\boldsymbol{A}^{\boldsymbol{A}^{\kappa}}$ *generated by the projections.*

Since each algebraic term is a finite string, for each infinite cardinal κ the free $\mathbb{V}$-algebra over κ generators is the algebra of all functions $f\colon \boldsymbol{A}^\kappa \to \boldsymbol{A}$ such that there exist an integer $m \geq 0$ and ordinals $\alpha_1 < \alpha_2 < \cdots < \alpha_m < \kappa$ and a function $g \in \boldsymbol{F}_{\mathbb{V}}^m$ such that for each $\langle a_1, a_2, \ldots, a_\alpha, \ldots\rangle \in \boldsymbol{A}^\kappa$,

$$f(a_1, a_2, \ldots, a_\alpha, \ldots) = g(a_{\alpha_1}, a_{\alpha_2}, \ldots, a_{\alpha_m})\,.$$

This observation justifies the fact that in the rest of the chapter we shall deal only with finitely generated free algebras, unless we explicitly state this is not the case.

Lemma 1.0.12 is really useful in the algebraic study of schematic extensions of BL or MTL, in particular for those extensions L enjoying *canonical* standard completeness, that is, the variety which constitutes the algebraic semantics of L contains, up to isomorphism, a unique standard algebra. In this case, if we single out one such standard algebra $\boldsymbol{A} = \langle [0,1], \odot, \to, \wedge, 0\rangle$, then the free n-generated algebra is the algebra of all functions $f\colon [0,1]^n \to [0,1]$ constructible from the projections by means of pointwise defined operations: that is, the zero of the algebra is the constant function 0 and for each binary operation $* \in \{\odot, \to, \wedge\}$,

$$(f * g)(a_1, \ldots, a_n) = f(a_1, \ldots, a_n) * g(a_1, \ldots, a_n)\,.$$

It is worth to stress that even the availability of the aforementioned form of standard completeness, or of a standard algebra that is generic for the variety under consideration, does not render the task of describing the concrete structure of the free algebras a trivial one. Good examples supporting this statement are the concrete representation of free MV-algebras (see Section 2.1) and of free BL-algebras (see Section 5.2).

As is well known, each variety as a category is complete and co-complete. This fact allows to express each free algebra as a suitable co-power of the singly generated free algebra. As a matter of fact, if $\{\boldsymbol{A}_i\}_{i\in I}$ is a family of algebras in a variety $\mathbb{V}$, such that $\{\boldsymbol{A}_i\}$ is presented by $\langle X_i, E_i\rangle$ for all $i \in I$, then their coproduct $\coprod_{i\in I} \boldsymbol{A}_i$ is the algebra in $\mathbb{V}$ presented by $\langle \biguplus_{i\in I} X_i, \biguplus_{i\in I} E_i\rangle$, where $\uplus$ denotes disjoint union, and each set of equations E_j uses only variables from the image of the injection of X_j into $\biguplus_{i\in I} X_i$. Recall now that $\boldsymbol{F}_{\mathbb{V}}^1$ is presented by $\langle\{v\}, \emptyset\rangle$. Then the following holds.

LEMMA 1.0.13. *In each variety $\mathbb{V}$, the κ-generated free $\mathbb{V}$-algebra is the co-power of κ copies of the singly generated free $\mathbb{V}$-algebra:*

$$\boldsymbol{F}_{\mathbb{V}}^\kappa = \coprod^\kappa \boldsymbol{F}_{\mathbb{V}}^1\,.$$

The preceding Lemma together with Lemma 1.0.12, are the most useful tools to build free algebras that we shall use in this chapter. In particular Lemma 1.0.13 is used in combination with finite spectral dualities. If a dual category $\mathbb{V}^{\mathrm{op}}$ to a variety $\mathbb{V}$ is available, we can try to work on the dual side, and express coproducts, which are notoriously hard to compute, as the dual of products in the dual category. If products in $\mathbb{V}^{\mathrm{op}}$ are easily handled, then what is left to do is just to characterise the singly generated free algebra in $\mathbb{V}$ and its dual. This approach is amazingly simple to apply for some locally finite subvarieties of MTL, as the objects dual to algebras turn out to be combinatorially defined finite partially ordered sets.

We close this section with a few words on free Boolean algebras, and functional (in)completeness. As the algebra $\{0,1\}$ is generic for the variety of Boolean algebras, the free n-generated Boolean algebra is the algebra of functions $f\colon \{0,1\}^n \to \{0,1\}$ generated by the projections, equipped with pointwise defined operations. As is well-known this algebra is the algebra of *all* functions $f\colon \{0,1\}^n \to \{0,1\}$. One way to prove this fact is to realize each such function, also called a *truth table*, either as a disjunction of minterms or a conjunction of maxterms, where a minterm is a conjunction of variables and negated variables, and dually, a maxterm is a disjunction of variables and negated variables. This fact is referred to as the *functional completeness* of Boolean logic. A trivial argument on cardinalities shows that functional completeness is not enjoyed by standard complete logics. However, those logics whose equivalent semantics is a locally finite variety are amenable to the definition of conjunctive and disjunctive normal forms, for suitably defined notions of generalised minterms and maxterms. This is shown to be the case for Gödel logics (see Section 4), and for logics that are schematic extension of WNM (Section 4.5). A similar approach is not conceivable for logics whose equivalent semantics is not locally finite, as it is the case for Łukasiewicz, product, BL, SBL, and MTL logics, among others. In these cases a more sophisticated theory is needed in order to develop normal-form like results. An instance of this kind of approach is given by the theory of *Schauder hats* for MV-algebras, the algebraic semantics of Łukasiewicz logics.

As a last remark we point out that for any schematic extension L of BL, if we let $\mathbb{V}$ denote the variety constituting the algebraic semantics of L then

$$\boldsymbol{F}^0_{\mathbb{V}} \cong \{0,1\}\,.$$

In the sequel we shall then focus on n-generated free algebras for integers $n > 0$, only.

We fix some notation we use throughout this chapter. Apart from the conjunction connective, that we denote $\&$, we shall blur the distinction between a connective and the algebraic operation interpreting it in standard and free MV-algebras and related structures. Then, unless otherwise stated, we use $\odot$ to denote the Łukasiewicz t-norm and the monoidal conjunction operation in free MV-algebras and related structures. For what regards the other operations, unless otherwise stated, we shall use $\oplus$ for the monoidal disjunction, $\to$ for the residuum of $\odot$, $\neg$ for the negation, $\vee$ and $\wedge$ for the lattice operations, and $\ominus$ for truncated difference, that is, $x \ominus y = x \odot \neg y$. Further, for any element a in an MV-algebra we denote ma and a^m, the element $a \oplus a \oplus \cdots \oplus a$ and $a \odot a \odot \cdots \odot a$, respectively, where in both expressions, a occurs m many times. We shall use the same convention for MV-terms φ and write $m\varphi$ and φ^m for, respectively, the m-fold monoidal conjunction and disjunction of φ with itself.

For each set A, each subset $K = \{j_1, j_2, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$, and each $\langle a_1, \ldots, a_n\rangle \in A^n$, we denote $\pi_K(a_1, \ldots, a_n) = \langle a_{j_1}, a_{j_2}, \ldots, a_{j_k}\rangle$ the *K-projection* of $\langle a_1, \ldots, a_n\rangle$. We shall sometimes write π_k instead of $\pi_{\{k\}}$.

2 MV-algebras and related structures

We recall that the variety $\mathbb{MV}$ of MV-algebras constitutes the equivalent algebraic semantics of Łukasiewicz logic. In this section we introduce McNaughton's result about the functional representation of free MV-algebras. Then we deal with representation theorems for related logics, e.g. the logic of Wajsberg hoops, finitely valued Łukasiewicz logics and extensions/expansions of Łukasiewicz logic with division operators.

We start introducing some structures whose elements are polynomials in n indeterminates. As we shall see, these polynomials, the linear ones in particular, play a major role in the functional representation of free MV-algebras and related structures.

DEFINITION 2.0.14. *Let* $\mathsf{Z}[x_1,\ldots,x_n]$ *be the domain of polynomials in* n *indeterminates and integer coefficients,* $\mathsf{Q}(x_1,\ldots,x_n)$ *its field of fractions and* $\mathsf{Z}_1[x_1,\ldots,x_n]$ *its submodule of polynomials of degree at most* 1.

Note that a generic member $p \in \mathsf{Z}_1[x_1,\ldots,x_n]$ has the form

$$p(x_1,\ldots,x_n) = b + \sum_{i=1}^{n} a_i x_i\,, \quad \text{for some } a_1,\ldots,a_n,b \in \mathsf{Z}.$$

2.1 McNaughton Theorem

Łukasiewicz logic enjoys standard completeness. In particular, the variety $\mathbb{MV}$ is generated by the standard MV-algebra

$$\langle [0,1], \oplus, \neg, 0\rangle\,,$$

where $x \oplus y = \min\{1, x+y\}$ and $\neg x = 1 - x$, for all $x,y \in [0,1]$. Hence, by Lemma 1.0.12, the free n-generated MV-algebra $\boldsymbol{F}^n_{\mathbb{MV}}$ is the subalgebra of $[0,1]^{[0,1]^n}$ generated by the projections, equipped with pointwise defined operations, that is, for any $\langle t_1,\ldots,t_n\rangle \in [0,1]^n$:

$$(f \oplus g)(t_1,\ldots,t_n) = \min\{1, f(t_1,\ldots,t_n) + g(t_1,\ldots,t_n)\}\,;$$

$$(\neg f)(t_1,\ldots,t_n) = 1 - f(t_1,\ldots,t_n)\,; \qquad 0(t_1,\ldots,t_n) = 0\,.$$

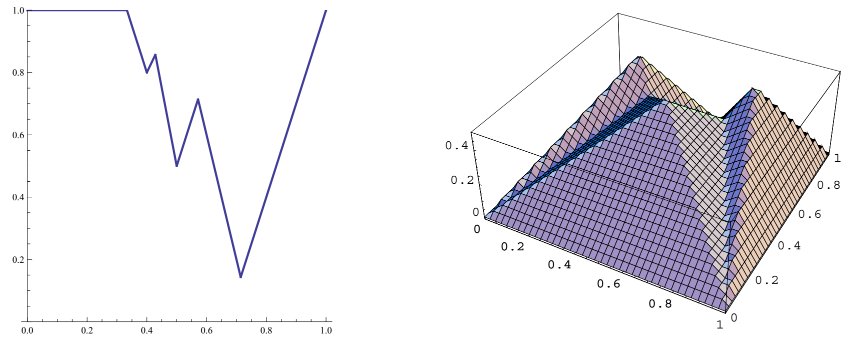
However, this result sheds no light on the intrinsic form of elements in $\boldsymbol{F}^n_{\mathbb{MV}}$. McNaughton representation Theorem states that elements of $\boldsymbol{F}^n_{\mathbb{MV}}$ are a special type of functions.

DEFINITION 2.1.1. *For each integer* $n \geq 0$*, a function* $f\colon [0,1]^n \to [0,1]$ *is called a* McNaughton function *if* f *is continuous (in the usual topology) and it is piecewise linear with integer coefficients, that is, there exists a finite set* $\{p_1,p_2,\ldots,p_u\}$ *of linear polynomials in* $\mathsf{Z}_1[x_1,\ldots,x_n]$ *(called the* linear components *of* f*), with each* p_i *of the form*

$$p_i(t_1,\ldots,t_n) = b_i + \sum_{j=1}^{n} a_{ij} t_j\,, \quad \textit{for all } \langle t_1,\ldots,t_n\rangle \in [0,1]^n\,,$$

for some $a_{i1},\ldots,a_{in},b_i \in \mathsf{Z}$*, such that, for each* $\langle t_1,\ldots,t_n\rangle \in [0,1]^n$*, there exists* $i \in \{1,2,\ldots,u\}$ *with*

$$f(t_1,\ldots,t_n) = p_i(t_1,\ldots,t_n)\,.$$



(a) $f_1 \in \boldsymbol{F}^1_{\mathbb{MV}}$.

(b) $f_2 \in \boldsymbol{F}^2_{\mathbb{MV}}$.

Figure 1. Examples of McNaughton functions.

THEOREM 2.1.2. *For each integer $n \geq 0$, the free n-generated MV-algebra $\boldsymbol{F}^n_{\mathbb{MV}}$ is isomorphic to the algebra of all McNaughton functions $f\colon [0,1]^n \to [0,1]$, equipped with pointwise defined operations.*

The proof of the theorem splits in the proofs of the two containments. Showing that each element of $\boldsymbol{F}^n_{\mathbb{MV}}$ is a McNaughton function is rather straightforward.

LEMMA 2.1.3.
For each integer $n \geq 0$, the elements of $\boldsymbol{F}^n_{\mathbb{MV}}$ are n-variable McNaughton functions.

Proof. By induction on the complexity of terms. □

Proving the other containment, that is that each n-variable McNaughton function belongs to $\boldsymbol{F}^n_{\mathbb{MV}}$, is much harder. In the literature there are several proofs of this fact. The proof we are going to give relies on the machinery of regular triangulations, Farey mediants and Schauder hats.

We shall assume familiarity with the basic concepts of convex geometry, in particular with the notions of compact, convex polyhedron (an intersection of finitely many halfspaces which is bounded, that is, it is included in a ball with finite radius) and with the lattices of faces of such a polyhedron. Throughout the chapter "polyhedron" always stands for compact convex polyhedron. We refer to [44] for background on convex and polyhedral geometry.

A *polyhedral complex* Σ is a finite set of polyhedra closed under taking faces and such that any pair of them intersect in a common face. The *support* $|\Sigma|$ of Σ is the set-theoretic union of all its polyhedra.

By a *rational point* we mean a point $\mathbf{v} \in ([0,1] \cap \mathsf{Q})^n$ for some integer $n \geq 0$. Given a rational point $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$, the *denominator* $\mathrm{den}(\mathbf{v})$ of $\mathbf{v}$ is the least common denominator of the components $v_1, \ldots, v_n$. The *homogeneous coordinates* of $\mathbf{v}$ are the components of the point $\mathrm{den}(\mathbf{v})\langle \mathbf{v}, 1 \rangle \in \mathsf{Z}^{n+1}$.

The following property of McNaughton functions is best formulated in terms of polyhedral complexes.

LEMMA 2.1.4. *For any McNaughton function $f\colon [0,1]^n \to [0,1]$, there is a polyhedral complex Σ such that $|\Sigma| = [0,1]^n$ and f is linear over each polyhedron in Σ. Moreover each vertex of a polyhedron in Σ is a rational point.*

Proof. Let $f\colon [0,1]^n \to [0,1]$ be a McNaughton function. Add to the set of linear components of f all linear polynomials of the form 0, $x_i - 1$ and x_i, for all $i \in \{1,\dots,n\}$. Display $p_1, p_2, \dots, p_k\colon [0,1]^n \to [0,1]$ the set of linear polynomials so obtained. For each pair $\langle i,j\rangle \in \{1,\dots,k\}^2$, the set of points $H_{i,j} = \{\mathbf{x} \in \mathsf{R}^n \mid p_i(\mathbf{x}) = p_j(\mathbf{x})\}$ is a, possibly empty, affine linear space. In particular, if $i \neq j$ and $H_{i,j} \neq \emptyset$ then $H_{i,j}$ is $(n-1)$-dimensional. Each point $\mathbf{x} \in [0,1]^n$ satisfies a finite set $D_{\mathbf{x}}$ of conditions of the form $p_i(\mathbf{x}) \leq p_j(\mathbf{x})$, hence it belongs to an n-dimensional polyhedron $P_{\mathbf{x}} \subseteq [0,1]^n$ arising as the intersection of finitely many half-spaces, each one of them bounded by an hyperplane of the form $H_{i,j}$. It is then clear that $[0,1]^n$ is subdivided into a finite collection of polyhedra $P_1, \dots, P_h$ each one of them of the form $P_{\mathbf{x}}$ for some $\mathbf{x} \in [0,1]^n$. In particular each intersection $P_{r,s} = P_r \cap P_s$ is a (possibly empty) face of both P_r and P_s as each point in $P_{r,s}$ satisfies all conditions $p_i(\mathbf{x}) \leq p_j(\mathbf{x})$ determining P_r and P_s, some of them in the stronger form $p_i(\mathbf{x}) = p_j(\mathbf{x})$. The collection of polyhedra obtained taking all faces of each polyhedron $P_1, \dots, P_h$ is a polyhedral complex Σ such that $|\Sigma| = [0,1]^n$. It remains to show that f is linear over each P_r. Now, fix $\mathbf{x}$ in the interior of some polyhedron P_r. Then there is p_t such that $f(\mathbf{x}) = p_t(\mathbf{x})$, and for any other $\mathbf{y}$ in the interior of P_r, the line segment l connecting $\mathbf{x}$ to $\mathbf{y}$ does not intersect any space $H_{i,j}$. Since f is continuous and piecewise-linear, $f = p_t$ all over the line segment l, and hence, all over the interior of P_r. By continuity, f coincides with p_t on the whole of P_r. Each vertex of Σ is the solution of a system of equations of the form $p_i(\mathbf{x}) = p_j(\mathbf{x})$, hence is trivially a rational point. $\square$

To synthesize the MV-term corresponding to a given n-variable McNaughton function $f\colon [0,1]^n \to [0,1]$ we need to further subdivide $[0,1]^n$.

DEFINITION 2.1.5. *For each integer $n \geq 0$, an* n-simplex *S is the convex hull of $n+1$ many affinely independent points in R^m for some integer $m \geq n$, called the* vertices *of S. That is, displaying the set of vertices of S as* $\mathrm{Vert}(S) = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n+1}\}$,

$$S = \left\{ \lambda_1 \mathbf{v}_1 + \cdots + \lambda_{n+1}\mathbf{v}_{n+1} \mid 0 \leq \lambda_i \in \mathsf{R}, \sum_{i=1}^{n+1} \lambda_i = 1 \right\}.$$

For instance, a (set containing exactly one) point is a 0-simplex, a line segment is a 1-simplex, a triangle is a 2-simplex, and so on. By definition the empty set is the only (-1)-simplex.

A *face* of an n-simplex S is the convex hull of a subset of the vertices of S. Hence $\emptyset$ is face of every simplex.

A *rational n-simplex* is a simplex whose vertices are rational points. Let S be a rational n-simplex S with $\mathrm{Vert}(S) = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n+1}\}$. Its associated matrix M_S is the $(n+1) \times (n+1)$ integer-entry matrix whose ith row is given by the homogeneous coordinates of the vertex $\mathbf{v}_i$. Then S is *regular* (or, *unimodular*) if the absolute value of the determinant of M_S is 1. For any $-1 \leq m \leq n$, a rational m-simplex in $[0,1]^n$ is *regular* if it is a face of a regular n-simplex.

DEFINITION 2.1.6. *A set U of rational simplices is a* rational triangulation *of* $[0,1]^n$ *whenever the following conditions hold:*

1. *if $S, T \in U$ then $S \cap T$ is face of both S and T;*
2. *if $S \in U$ and F is face of S then $F \in U$;*
3. $\bigcup_{S \in U} S = [0,1]^n$.

If, in addition, each simplex in U is regular, then U is a rational regular triangulation.

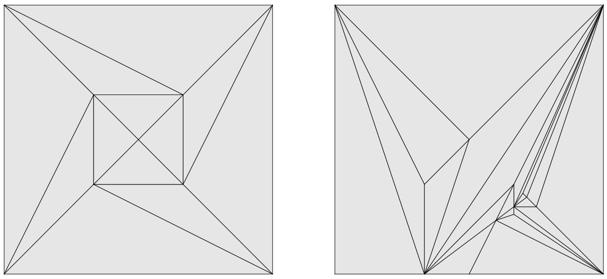


Figure 2. Two examples of rational regular triangulations of $[0,1]^2$.

In the rest of the chapter we shall usually omit the qualification “rational”, as we shall deal with rational simplices and rational regular triangulations only.

We observe the following.

LEMMA 2.1.7. *For each polyhedral complex Σ there exists a triangulation U_Σ such that each polyhedron in Σ is union of simplices in U_Σ.*

Proof. By [66, Prop. 2.9], each polyhedron P in Σ can be subdivided into simplices without introducing new vertices, that is, one can find a finite set S_P of simplices such that $\bigcup_{S \in S_P} S = P$, each two simplices in S_P intersect in a common face, and the set of all vertices of all simplices in S_P coincides with the set of vertices of P. Then $U_\Sigma = \bigcup_{P \in \Sigma} S_P$ is the desired triangulation. □

If U_1, U_2 are triangulations of $[0,1]^n$ and each $S \in U_1$ is contained in some $T \in U_2$, then U_1 is a *refinement* of U_2, in symbols $U_1 \prec U_2$.

Let S be a regular 1-simplex with vertex-set $\mathrm{Vert}(S) = \{\mathbf{v}, \mathbf{w}\}$. Let $\langle v_1, \ldots, v_{n+1}\rangle$, $\langle w_1, \ldots, w_{n+1}\rangle \in \mathsf{Z}^{n+1}$ be the homogeneous coordinates of $\mathbf{v}$ and $\mathbf{w}$, respectively. Then the *Farey mediant* of $\mathbf{v}$ and $\mathbf{w}$ is the point $\mathbf{v} +_F \mathbf{w}$ whose homogeneous coordinates are

$$\langle v_1 + w_1, \ldots, v_{n+1} + w_{n+1}\rangle .$$

LEMMA 2.1.8. *Let S be a regular n-simplex with vertex-set given by* $\mathrm{Vert}(S) = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n+1}\}$. *Let further S_i be the n-simplex with* $\mathrm{Vert}(S_i) = (\mathrm{Vert}(S) \setminus \{\mathbf{v}_i\}) \cup \{\mathbf{v}_1 +_F \mathbf{v}_2\}$*, for each $i \in \{1, 2\}$. Then both S_1 and S_2 are regular.*

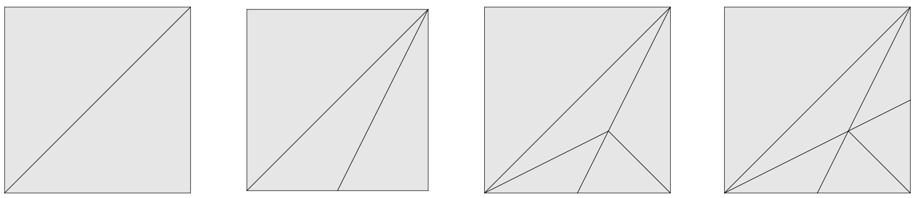
Proof. It follows directly from the definition of Farey mediant and the multilinear properties of the determinant. □

DEFINITION 2.1.9. *Let U be a regular triangulation and $S \in U$ be a 1-simplex with $\mathrm{Vert}(S) = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then the* (edge) starring *of U* along *S, in symbols $U * S$, is the set of simplices obtained as follows:*

1. *put in $U * S$ all simplices of U not containing S;*
2. *replace each simplex $T \in U$ such that S is face of T by the simplices T_1, T_2 with $\mathrm{Vert}(T_i) = (\mathrm{Vert}(T) \setminus \{\mathbf{v}_{3-i}\}) \cup \{\mathbf{v}_1 +_F \mathbf{v}_2\}$ for each $i \in \{1, 2\}$;*
3. *close $U * S$ under taking faces.*

LEMMA 2.1.10. *For each regular triangulation U and 1-simplex $S \in U$, the set of simplices $U * S$ is a regular triangulation refining U.*

Proof. Immediate, from Lemma 2.1.8. □



(a) U (b) $U' = U * S$ (c) $U'' = U' * S'$ (d) $U''' = U'' * S''$

Figure 3. A regular triangulation U of $[0,1]^2$, and three successive starrings along the 1-simplices S, S', S'', where $\mathrm{Vert}(S) = \{\langle 0,0\rangle, \langle 1,0\rangle\}$, $\mathrm{Vert}(S') = \{\langle 1/2,0\rangle, \langle 1,1\rangle\}$, $\mathrm{Vert}(S'') = \{\langle 1,0\rangle, \langle 1,1\rangle\}$.

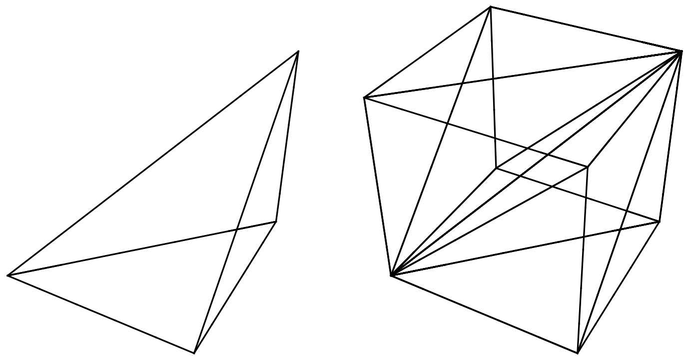
DEFINITION 2.1.11. *Let T be the n-simplex whose vertex-set is $\{\mathbf{v}_j\}_{j=0}^n$, such that $\pi_{\{i\}}(\mathbf{v}_j) = 0$ if $i + j \leq n$, $\pi_{\{i\}}(\mathbf{v}_j) = 1$, otherwise. Let Sym_n be the group of all permutations of the set $\{1, 2, \ldots, n\}$. For each $\sigma \in \mathrm{Sym}_n$ let T_σ be the simplex whose ith vertex is such that its jth component is $\pi_{\{\sigma(j)\}}(\mathbf{v}_i)$. Let F_σ be the set of all faces of T_σ. Then let*

$$U_0^n = \bigcup_{\sigma \in \mathrm{Sym}_n} F_\sigma \, .$$

LEMMA 2.1.12. *Let H be an affine hyperplane of the form $\{\mathbf{x} \in \mathsf{R}^n \mid \sum_{i=1}^n a_i x_i = b\}$ for $a_1, a_2, \ldots, a_n, b \in \mathsf{Z}$. Let further H^- and H^+ be the two closed n-dimensional halfspaces bounded by H. Then there exists a finite sequence*

$$U_0^n = U_0 \succ U_1 \succ \cdots \succ U_u$$

of regular triangulations of $[0,1]^n$ such that for each $i \in \{1, \ldots, u\}$, the triangulation U_i is obtained by starring U_{i-1} along one of its 1-simplices, and each simplex of U_u is entirely contained either in H^- or H^+.



(a) T (b) U_0^3

Figure 4. The 3-simplex T and U_0^3 as in Definition 2.1.11.

Proof. Let U be a regular triangulation of $[0,1]^n$. We display the homogeneous coordinates of each vertex $\mathbf{u}$ of a 1-simplex of U as $\langle u_1, u_2, \dots, u_{n+1}\rangle$. Let $f\colon \mathsf{Z}^{n+1} \to \mathsf{Z}$ be the map $\langle x_1, \dots, x_{n+1}\rangle \mapsto a_1x_1 + \cdots + a_nx_n - bx_{n+1}$. For each regular 1-simplex S with vertex-set $\mathrm{Vert}\,(S) = \{\mathbf{u}, \mathbf{v}\}$, let us define the *discrepancy* $\mathrm{disc}\,(S)$ of S (with respect to H) as

$$\mathrm{disc}\,(S) = \begin{cases} 0 & \text{if } 0 \le f(\mathbf{u}), f(\mathbf{v}) \text{ or } 0 \ge f(\mathbf{u}), f(\mathbf{v}), \\ |f(\mathbf{u})| + |f(\mathbf{v})| & \text{otherwise.} \end{cases}$$

We further set $d = \max\{\mathrm{disc}\,(S) \mid S \in U,\, S \text{ a 1-simplex}\}$ and $s = |\{S \mid S \in U,$ $S \text{ a 1-simplex},\, \mathrm{disc}\,(S) = d\}|$. We proceed by induction on the lexicographically ordered pairs $\langle d, s\rangle$. The base of the induction follows immediately by noting that, by the definition of f, if $d = 0$ then each 1-simplex S of U already lies either in H^- or in H^+. For the inductive step, pick a 1-simplex S of U such that $\mathrm{disc}\,(S) = d$, and let $\mathrm{Vert}\,(S) = \{\mathbf{u}, \mathbf{v}\}$. Let $U' = U * S$. As $S \notin U'$, it suffices to show that each 1-simplex $R \in U' \setminus U$ is such that $\mathrm{disc}\,(R) < d$. If we accomplish this task then by induction we are done, since U' has either a number $s' < s$ of 1-simplices of discrepancy d, or the maximum discrepancy of 1-simplices in U' is $d' < d$, for some d'. Each such 1-simplex R has as vertex-set $\mathrm{Vert}\,(R) = \{\mathbf{u} +_F \mathbf{v}, \mathbf{p}\}$ for some 0-simplex $\mathbf{p}$ of U. Notice that by linearity of f and the definition of Farey mediant the value of $f(\mathbf{u} +_F \mathbf{v})$ is intermediate between $f(\mathbf{u})$ and $f(\mathbf{v})$. Now, if $f(\mathbf{u} +_F \mathbf{v}) = 0$ then $\mathrm{disc}\,(R) = 0$ and we are through. Otherwise, assume without loss of generality that $f(\mathbf{u}) < 0 < f(\mathbf{u} +_F \mathbf{v}) < f(\mathbf{v})$. If $f(\mathbf{p}) \ge 0$ then $\mathrm{disc}\,(R) = 0$ and the lemma is settled. We are left with the case $f(\mathbf{p}) < 0$. To conclude the proof we observe that $\mathrm{disc}\,(R) = |f(\mathbf{u} +_F \mathbf{v})| + |f(\mathbf{p})| < |f(\mathbf{v})| + |f(\mathbf{p})|$ and that $\{\mathbf{v}, \mathbf{p}\}$ is, by Definition 2.1.9, the vertex-set of a 1-simplex $T \in U$. Since $|f(\mathbf{v})| + |f(\mathbf{p})| = \mathrm{disc}\,(T) \le d$, we have shown $\mathrm{disc}\,(R) < d$ as desired. $\square$

LEMMA 2.1.13. *For each integer $n \geq 0$ the set of simplices U_0^n is a regular triangulation of $[0,1]^n$. Further, there exists an infinite sequence*

$$U_0^n = U_0 \succ U_1 \succ U_2 \succ \cdots$$

of regular triangulations such that

1. *for each $i \geq 0$, U_{i+1} is obtained from U_i by starring along a 1-simplex of U_i;*
2. *for each triangulation U there exists an index i such that $U_i \prec U$.*

Proof. For each $\sigma \in \mathrm{Sym}_n$, the n-simplex T_σ is regular by construction; moreover, T_σ is the set of points $\mathbf{x} \in [0,1]^n$ satisfying the following set of inequalities:

$$0 \leq x_{\sigma(1)},\, x_{\sigma(1)} \leq x_{\sigma(2)},\, x_{\sigma(2)} \leq x_{\sigma(3)},\, \ldots,\, x_{\sigma(n-1)} \leq x_{\sigma(n)},\, x_{\sigma(n)} \leq 1\,. \quad (1)$$

It immediately follows that $\bigcup_{\sigma \in \mathrm{Sym}_n} T_\sigma = [0,1]^n$. Notice that $\mathbf{x}$ belongs to the interior of T_σ iff (1) holds with all $\leq$ replaced by $<$. By the same token each m-dimensional face of T_σ is the set of points satisfying (1) with $n+1-m$ many $\leq$ replaced by $=$. It follows that for each $\sigma, \tau \in \mathrm{Sym}_n$, the simplices T_σ and T_τ intersects in a common face. The same applies for the intersection of proper faces of T_σ and T_τ. Then U_0^n is a regular triangulation of $[0,1]^n$. This settles the first statement.

Let us now fix an enumeration $H_1, H_2, \ldots,$ of all affine hyperplanes with integer coefficients. By Lemma 2.1.12 there exists a sequence:

$$\begin{gathered} U_0 \succ U_1 \succ \cdots \succ U_{u_1} \\ U_{u_1} \succ U_{u_1+1} \succ \cdots \succ U_{u_1+u_2} \\ \cdots \\ U_{u_1+\cdots+u_i} \succ U_{u_1+\cdots+u_i+1} \succ \cdots \succ U_{u_1+\cdots+u_{i+1}} \\ \cdots \end{gathered}$$

such that:

1. $U_0 = U_0^n$;
2. U_j is obtained by starring U_{j-1} along one of its 1-simplices;
3. For each $k \geq 1$, each simplex of $U_{u_1+\cdots+u_k}$ is fully contained either in H_k^- or H_k^+.

Now, each simplex of U, being a rational polyhedron, is the intersection of a finite number of halfspaces of the form H_t^- or H_t^+. Hence, the sequence we have constructed eventually refines U. □

U_0^n and the sequence $U_0 \succ U_1 \succ U_2 \succ \cdots$ are respectively called the *fundamental partition* and the *fundamental sequence* of $[0,1]^n$. Note that the set of 0-simplices contained in U_0^n is the set

$$\{\{\mathbf{v}\} \mid \mathbf{v} \in \{0,1\}^n\}\,.$$

We stress that the fundamental partition is by no means canonical for each $n > 1$: the partition is fixed by the choice of the enumeration of rational hyperplanes in the proof of Lemma 2.1.13. On the other hand, the case $n = 1$ is completely canonical, as the fundamental sequence of $[0,1]$ is the one given by Farey sequences.

We are now ready to introduce the notion of *Schauder hat*.

DEFINITION 2.1.14. *Let U be a regular triangulation of $[0,1]^n$, and let $\mathbf{v}$ be a vertex of some simplex in U (note that the 0-simplex $\{\mathbf{v}\}$ belongs to U). Then the Schauder hat with* apex $\mathbf{v}$ *with respect to U is the continuous function $h_{U,\mathbf{v}}\colon [0,1]^n \to [0,1]$ determined by the following constraints:*

1. $h_{U,\mathbf{v}}(\mathbf{v}) = 1/\mathrm{den}(\mathbf{v})$*;*
2. $h_{U,\mathbf{v}}$ *is constantly 0 over each simplex $T \in U$ such that $\mathbf{v} \notin T$;*
3. $h_{U,\mathbf{v}}$ *is linear over each simplex $T \in U$.*

The set $H_U = \{h_{U,\mathbf{v}} \mid \{\mathbf{v}\} \in U\}$ is called the Schauder set *of U.*

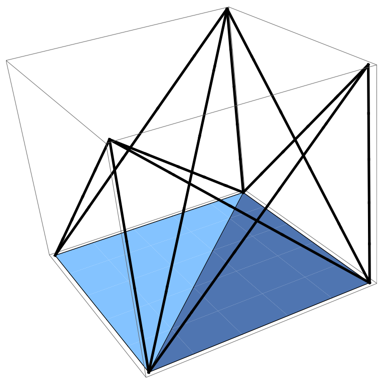


Figure 5. The Schauder hats of apices $\langle 0,0\rangle$, $\langle 1,0\rangle$, and $\langle 1,1\rangle$, with respect to U_0^2.

LEMMA 2.1.15. *For each regular triangulation U of $[0,1]^n$ and any vertex $\mathbf{v}$ of some simplex in U, the Schauder hat $h_{U,\mathbf{v}}$ is a McNaughton function.*

Proof. It is sufficient to prove that the linear function $h \restriction T$, defined as the restriction of $h_{U,\mathbf{v}}$ over an n-simplex T of U having $\mathbf{v}$ among its vertex, has integer coefficients. Display the vertices of T as $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n, \mathbf{v}_{n+1} = \mathbf{v}$, and express $h \restriction T(\mathbf{x})$ as $a_1x_1 + \cdots + a_nx_n + b$. Then conditions (1) and (2) of Definition 2.1.14 are to the effect that the $(n+1)$-tuple of reals $\mathbf{a} = \langle a_1, \ldots, a_n, b\rangle$ is the only solution to the linear system expressed in matricial form by $M\mathbf{a} = \mathbf{d}$, where M is the $(n+1)\times(n+1)$ integer-entry matrix whose ith row is the expression in homogeneous coordinates of $\mathbf{v}_i$, while $d_i = 0$ for all $i \in \{1, \ldots, n\}$, and $d_{n+1} = \mathrm{den}(\mathbf{v})(1/\mathrm{den}(\mathbf{v})) = 1$. Notice that $M = M_T$, and then, by regularity of T, M^{-1} is an $(n+1)\times(n+1)$ integer-entry matrix. It immediately follows that $\mathbf{a} = M^{-1}\mathbf{d}$ is a $(n+1)$-tuple of integers, thus setting the lemma. □

LEMMA 2.1.16. *For each $\mathbf{v} \in \{0,1\}^n$ let $f_{\mathbf{v}}\colon [0,1]^n \to [0,1]$ be the function*

$$\left(\bigwedge_{v_i=0} \neg x_i \;\&\; \bigwedge_{v_i=1} x_i\right)^{\boldsymbol{F}^n_{\mathbb{MV}}}.$$

Then $f_{\mathbf{v}} = h_{U_0^n,\mathbf{v}}$, and hence the set $\{f_{\mathbf{v}} \mid \{\mathbf{v}\} \in U_0^n\}$ is the Schauder set of U_0^n.

Proof. Somewhat lengthy, but straightforward. □

Notice that the formula for the Schauder hat $f_{\mathbf{v}}$ in Lemma 2.1.16 coincides with the Boolean minterm corresponding to the truth-value assignment $x_i \mapsto v_i$.

LEMMA 2.1.17. *Let U be a regular triangulation of $[0,1]^n$ and let $S \in U$ be a 1-simplex with vertices $\mathrm{Vert}(S) = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then the Schauder set of the starring of U along S is*

$$H_{U*S} = \{h_{U*S,\mathbf{v}} \mid \{\mathbf{v}\} \in U\} \cup \{h_{U*S,\mathbf{v}_1+_F\mathbf{v}_2}\},$$

where:

1. $h_{U*S,\mathbf{w}} = h_{U,\mathbf{w}}$ *for each* $\{\mathbf{w}\} \in U$ *such that* $\mathbf{v}_1 \neq \mathbf{w} \neq \mathbf{v}_2$*;*

2. $h_{U*S,\mathbf{v}_1} = h_{U,\mathbf{v}_1} \ominus h_{U,\mathbf{v}_2}$ *and* $h_{U*S,\mathbf{v}_2} = h_{U,\mathbf{v}_2} \ominus h_{U,\mathbf{v}_1}$*;*

3. $h_{U*S,\mathbf{v}_1+_F\mathbf{v}_2} = h_{U,\mathbf{v}_1} \wedge h_{U,\mathbf{v}_2}$.

Proof. Let $\mathbf{w} = \mathbf{v}_1 +_F \mathbf{v}_2$. We first note that $h_{U,\mathbf{v}_1}(\mathbf{w}) = h_{U,\mathbf{v}_2}(\mathbf{w}) = 1/\mathrm{den}(\mathbf{w})$. This is the case since the homogeneous expression of $\mathbf{w}$ is just the sum in Z^{n+1} of the homogeneous expressions of $\mathbf{v}_1$ and $\mathbf{v}_2$. Since for each $i \in \{1,2\}$ we have that $\mathrm{den}(\mathbf{v}_i)$ is the $(n+1)$-component of the homogeneous expression of $\mathbf{v}_i$, it follows that the last component of the sums of the homogeneous expressions of $\mathbf{v}_1$ and $\mathbf{v}_2$ is just $\mathrm{den}(\mathbf{v}_1) + \mathrm{den}(\mathbf{v}_2)$, that is, $\mathrm{den}(\mathbf{w})$. From $h_{U,\mathbf{v}_i}(\mathbf{v}_i) = 1/\mathrm{den}(\mathbf{v}_i)$ we then have $h_{U,\mathbf{v}_1}(\mathbf{w}) = h_{U,\mathbf{v}_2}(\mathbf{w}) = 1/\mathrm{den}(\mathbf{w})$.

Consider $\mathbf{t} \notin \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$. Then $h_{U,\mathbf{t}}(\mathbf{v}_1) = h_{U,\mathbf{t}}(\mathbf{v}_2) = 0$, hence, by linearity, $h_{U*S,\mathbf{t}}(\mathbf{w}) = h_{U,\mathbf{t}}(\mathbf{w}) = 0$. As $U * S$ is a refinement of U, $h_{U*S,\mathbf{t}} = h_{U,\mathbf{t}}$ is linear over each simplex of $U * S$.

Observe now that $h_{U*S,\mathbf{w}}(\mathbf{w}) = \min\{1/\mathrm{den}(\mathbf{w}), 1/\mathrm{den}(\mathbf{w})\} = 1/\mathrm{den}(\mathbf{w})$, and $h_{U*S,\mathbf{w}}(\mathbf{v}_i) = \min\{1/\mathrm{den}(\mathbf{v}_i), 0\} = 0$ for each $i \in \{1,2\}$. Clearly $h_{U*S,\mathbf{w}}(\mathbf{t}) = 0$ for all $\mathbf{t} \notin \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$. Let T be any n-simplex in U. If S is not a face of T, then at least one in $\mathbf{v}_1$,$\mathbf{v}_2$ does not belong to T, hence $h_{U*S,\mathbf{w}}$ is constantly 0 on T. If S is a face of T then $U * S$ contains the two simplices T_1, T_2 replacing T by starring. Note $h_{U,\mathbf{v}_1}$ coincides with $h_{U,\mathbf{v}_2}$ in all vertices of $T_i \setminus \{\mathbf{v}_i\}$, while $h_{U,\mathbf{v}_{2-i}}(\mathbf{v}_i) = 0$ and $h_{U,\mathbf{v}_i}(\mathbf{v}_i) = 1/\mathrm{den}(\mathbf{v}_i)$. Hence, by linearity of both hats $h_{U,\mathbf{v}_i}$ on S, we have $h_{U,\mathbf{v}_{2-i}} \leq h_{U,\mathbf{v}_i}$ on T_i and hence $h_{U*S,\mathbf{w}} = h_{U,\mathbf{v}_{2-i}}$ on T_i.

Note $h_{U*S,\mathbf{v}_1}(\mathbf{v}_1) = h_{U,\mathbf{v}_1}(\mathbf{v}_1) = 1/\mathrm{den}(\mathbf{v}_1)$ while $h_{U*S,\mathbf{v}_1}(\mathbf{w}) = h_{U*S,\mathbf{v}_1}(\mathbf{v}_2) = 0$; moreover $h_{U*S,\mathbf{v}_1}(\mathbf{t}) = 0$ for each vertex $\mathbf{t} \notin \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}\}$. It remains to show that $h_{U*S,\mathbf{v}_1}$ is linear over each n-simplex T of $U * S$ (The case for $h_{U*S,\mathbf{v}_2}$ is dealt with analogously). If S is not a face of T, then at least one in $\mathbf{v}_1$,$\mathbf{v}_2$ does not belong to T. If $\mathbf{v}_1 \notin T$, then $h_{U*S,\mathbf{v}_1} = h_{U,\mathbf{v}_1} = 0$ on T. If $\mathbf{v}_2 \notin T$, then $h_{U,\mathbf{v}_2} = 0$ on T, and hence $h_{U*S,\mathbf{v}_1} = h_{U,\mathbf{v}_1}$ on T. In both cases $h_{U*S,\mathbf{v}_1}$ is linear on T. Assume now $U * S$ contains the two simplices T_1, T_2 that replace T by starring. Recall that $h_{U,\mathbf{v}_{2-i}} \leq h_{U,\mathbf{v}_i}$ over T_i, hence $h_{U*S,\mathbf{v}_i} = h_{U,\mathbf{v}_i} - h_{U,\mathbf{v}_{2-i}}$ over T_i and $h_{U*S,\mathbf{v}_i} = 0$ over T_{2-i}. This concludes the proof. □

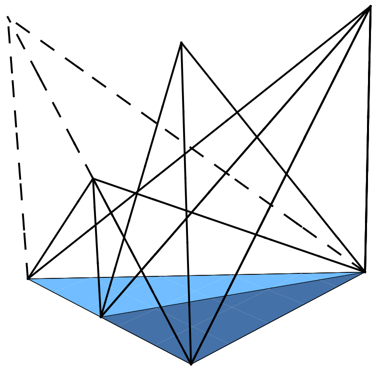


Figure 6. The Schauder hats of apices $\langle 1/2, 0\rangle$, $\langle 1, 0\rangle$, and $\langle 0, 1\rangle$, with respect to $U_0^2 * S$, where S is the 1-simplex of vertices $\mathrm{Vert}(S) = \{\langle 0, 0\rangle, \langle 1, 0\rangle\}$. Note the hat with apex $\langle 1/2, 0\rangle$ is obtained by starring along S.

Lemma 2.1.15, Lemma 2.1.16 and Lemma 2.1.17 show that for each Schauder hat h in the Schauder set of any regular partition in the fundamental sequence of $[0, 1]^n$ one constructs an MV-term φ such that

$$\varphi^{\boldsymbol{F}^n_{\mathrm{MV}}} = h\,.$$

The next step consists in showing that each n-variable McNaughton function f can be realised as a suitable finite combination of Schauder hats.

LEMMA 2.1.18. *For any McNaughton function* $f\colon [0, 1]^n \to [0, 1]$ *there exists an index* i *such that* f *is linear over each simplex in the* i*th member* U_i *of the fundamental sequence of* $[0, 1]^n$.

Proof. By Lemma 2.1.4 there exists a polyhedral complex Σ such that $|\Sigma| = [0, 1]^n$ and f is linear over each polyhedron in Σ. By Lemma 2.1.7 there exists a triangulation U_Σ refining Σ. By Lemma 2.1.13 there exists an index i such that the i-th member U_i of the fundamental sequence refines U_Σ, and hence f is linear over each simplex of U_i. $\square$

We are now ready to prove the hard side of McNaughton representation Theorem.

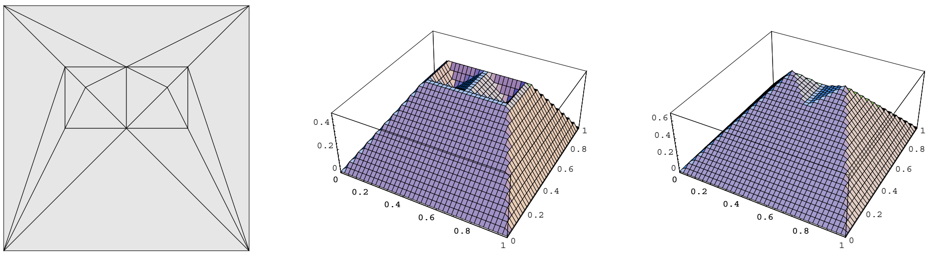
LEMMA 2.1.19. *For any McNaughton function* $f\colon [0, 1]^n \to [0, 1]$ *there exists an* MV*-term* φ *such that*

$$\varphi^{\boldsymbol{F}^n_{\mathrm{MV}}} = f\,.$$

In particular, let the index i *be determined as in Lemma 2.1.18. Then*

$$\varphi = \bigoplus_{\{\mathbf{v}\}\in U_i} m_{\mathbf{v}}\varphi_{\mathbf{v}}\,,$$

where $\varphi_{\mathbf{v}}^{\boldsymbol{F}^n_{\mathrm{MV}}} = h_{U_i,\mathbf{v}}$ *and* $m_{\mathbf{v}} = \mathrm{den}(\mathbf{v})f(\mathbf{v})$ *for each* $\{\mathbf{v}\} \in U_i$.



(a) U (b) f (c) g

Figure 7. A regular triangulation U of $[0,1]^2$ and two McNaughton functions f and g that are linear over each simplex of U.

Proof. First notice that since each linear component of f has integer coefficients then $f(\mathbf{p})$ is an integer multiple $m_{\mathbf{p}}$ of $1/\mathrm{den}(\mathbf{p})$ for each $\mathbf{p} \in ([0,1] \cap \mathsf{Q})^n$.

Now, take U_i as in Lemma 2.1.18 and observe that $f(\mathbf{v}) = m_{\mathbf{v}} h_{U_i,\mathbf{v}}(\mathbf{v})$ while $m_{\mathbf{v}} h_{U_i,\mathbf{v}}(\mathbf{w}) = 0$ for each pair $\mathbf{w} \neq \mathbf{v}$ of U_i. Then $\varphi^{\boldsymbol{F}^n_{\mathbb{MV}}}$ coincides with f at every vertex of U_i. By linearity of both f and of each $h_{U_i,\mathbf{v}}$ over each simplex of U_i, we conclude that $\varphi^{\boldsymbol{F}^n_{\mathbb{MV}}}$ coincides with f over the whole of $[0,1]^n$. □

THEOREM 2.1.20. *For each integer $n \geq 0$, the free n-generated* MV*-algebra $\boldsymbol{F}^n_{\mathbb{MV}}$ is isomorphic to the algebra of all McNaughton functions $f\colon [0,1]^n \to [0,1]$, equipped with pointwise defined operations.*

Proof. By Lemma 2.1.3 and Lemma 2.1.19. □

Lemma 2.1.19 can be seen as a *normal form* lemma for Łukasiewicz logic: each formula φ can be equivalently expressed as a $\oplus$-disjunction of Schauder hat formulas. In this light, Schauder hats plays the role of generalised *semantical minterms*, and the corresponding Schauder hat formula are generalised *syntactical* minterms.

Alternative proofs of Theorem 2.1.20 use different “building blocks”. In particular, they often use the following lemma to find MV-terms expressing truncated linear functions (with integer coefficients).

LEMMA 2.1.21. *Let $f\colon [0,1]^n \to \mathsf{R}$ be an* integer linear polynomial*, that is, a function of the form*

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n} a_i x_i + b\,,$$

for $a_1,\ldots,a_n, b \in \mathsf{Z}$. Then there is an algorithm that outputs an MV*-term φ such that $\varphi^{\boldsymbol{F}^n_{\mathbb{MV}}} = \max\{0,\min\{1,f\}\}$.*

Proof. Let us denote $f^{\#}$ the function $\max\{0,\min\{1,f\}\}$. The proof is by induction on $m = \sum_{i=1}^{n} |a_i|$. If $m = 0$ then $f^{\#}$ is either constantly 0 or 1, then we can take as φ either the term $\perp$ or $\top$, respectively. Assume now $m > 0$ and let a_j be such that $|a_j| = \max_{i=1}^{n} |a_i|$. Suppose first $a_j > 0$. Let $g = f - x_j$. By induction hypothesis we have terms ψ_1 and ψ_2 such that $\psi_1^{\boldsymbol{F}^n_{\mathrm{MV}}} = g^{\#}$ and $\psi_2^{\boldsymbol{F}^n_{\mathrm{MV}}} = (g+1)^{\#}$. We claim that for each $\langle x_1, \ldots, x_n\rangle \in [0,1]^n$ the following identity holds.

$$(g + x_j)^{\#} = (g^{\#} \oplus x_j) \odot (g+1)^{\#} . \tag{2}$$

We run a case analysis on the value of $g(x_1, \ldots, x_n)$. If $g(x_1, \ldots, x_n) > 1$ then both sides of (2) evaluate to 1; similarly if $g(x_1, \ldots, x_n) < -1$ both sides evaluate to 0. If $0 \leq g(x_1, \ldots, x_n) \leq 1$ then $g(x_1, \ldots, x_n) = g^{\#}(x_1, \ldots, x_n)$ and $(g+1)^{\#}$ $(x_1, \ldots, x_n) = 1$. Hence, the right hand side of (2) evaluates to $g(x_1, \ldots, x_n) \oplus x_j = \min\{1, g(x_1, \ldots, x_n) + x_j\}$. As $0 \leq g(x_1, \ldots, x_n)$, this value coincides with $(g(x_1, \ldots, x_n) + x_j)^{\#}$. If $-1 \leq g(x_1, \ldots, x_n) \leq 0$ then $g^{\#}(x_1, \ldots, x_n) = 0$ and $(g+1)^{\#}(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) + 1$. Now, as $g(x_1, \ldots, x_n) \leq 0$, we have $(g(x_1, \ldots, x_n) + x_j)^{\#} = \max\{0, g(x_1, \ldots, x_n) + x_j\} = x_j \odot (g(x_1, \ldots, x_n) + 1)$, that is the value taken by the right hand side of (2). The claim is settled. Then, in our current assumptions, as $f^{\#} = (g + x_j)^{\#}$, we have that the term

$$\varphi = (\psi_1 \oplus x_j) \mathbin{\&} \psi_2$$

is such that $\varphi^{\boldsymbol{F}^n_{\mathrm{MV}}} = f^{\#}$. To conclude the proof it remains to consider the case $a_j < 0$. This is easily settled as we can use the preceding case to find a term ψ such that $\psi^{\boldsymbol{F}^n_{\mathrm{MV}}} = (1-f)^{\#}$, and then conclude that $(\neg\psi)^{\boldsymbol{F}^n_{\mathrm{MV}}} = 1 - (1-f)^{\#} = f^{\#}$. $\square$

And then each McNaughton function $f\colon [0,1]^n \to [0,1]$ is constructed as an arrangement of integer linear polynomials. In particular, one can show that there exists a finite set of truncated integer linear polynomials $\{f_{i,j} \mid i \in I, J \in J_i\}$ such that

$$f = \bigvee_{i\in I} \bigwedge_{j\in J_i} f_{i,j} .$$

In this case, we have a disjunctive normal form whose minterms are finite conjunctions of truncated hyperplanes terms as given by Lemma 2.1.21: each such term can be seen as a generalised *literal*, as in Boolean logic literals are either variables or negated variables.

We give here some details explaining how to construct the $\bigvee$-$\bigwedge$ combination of truncated integer linear polynomials which equals a given n-variable McNaughton function f. First one lists the distinct linear components of f as $p_1, p_2, \ldots, p_k$ and considers the partition of $[0,1]^n$ given by the collection of all full-dimensional polyhedra P_σ indexed by a permutation σ of $\{1, 2, \ldots, k\}$ such that for all $\mathbf{t} \in P_\sigma$ it holds $p_{\sigma(1)}(\mathbf{t}) \geq p_{\sigma(2)}(\mathbf{t}) \geq \cdots \geq p_{\sigma(k)}(\mathbf{t})$. Note f over such a P_σ coincides with $p_{\sigma(i_\sigma)}$ for some index i_σ. Clearly, $\bigwedge_{j=1}^{i_\sigma} p_{\sigma(j)}$ coincides with f all over P_σ. It can be proved, using an analytic-geometric argument that $\left(\bigwedge_{j=1}^{i_\sigma} p_{\sigma(j)}\right)(\mathbf{t}) \leq f(\mathbf{t})$ for all $\mathbf{t} \in [0,1]^n \setminus P_\sigma$. Let Σ denote the set of all permutations σ of $\{1, 2, \ldots, k\}$ such that P_σ is full-dimensional. One then concludes that $\bigwedge_{\sigma\in\Sigma} \bigvee_{j=1}^{i_\sigma} p_{\sigma(j)} = f$, as desired.

As is well known, each MV-algebra $\langle A, \oplus, \neg, 0\rangle$ is isomorphic to its *order dual* $\langle A, \odot, \neg, 1\rangle$, via the map $a \mapsto \neg a$. It follows that both kinds of disjunctive normal forms, that is, $\oplus$-disjunction of Schauder hats, or $\bigvee\bigwedge$-combinations of truncated hyperplanes, can be dualised, obtaining conjunctive normal forms either as $\odot$-conjunctions of Schauder co-hats, or $\bigwedge\bigvee$-combinations of truncated hyperplanes.

When dealing with Wajsberg hoops, which are the topic of the next subsection, and their relations with MV-algebras, the order dual of an MV-algebra turns out to be a useful tool to work with.

2.2 Wajsberg hoops

We recall that the variety of Wajsberg hoops consists of the $\bot$-free subreducts of Wajsberg algebras, which in turns constitutes a variety term-wise equivalent to $\mathbb{MV}$.

The variety $\mathbb{WH}$ of Wajsberg hoops is generated by the standard algebra

$$\langle [0,1], \odot, \to, 1 \rangle ,$$

where $x \odot y = \max\{0, x + y - 1\}$ and $x \to y = \min\{1, 1 - x + y\}$. Notice that $\min\{1, x + y\} = (x \to (x \odot y)) \to y$. Hence the operation $\oplus$ is definable as $x \oplus y = (x \to (x \odot y)) \to y$.

A slight adaptation of McNaughton Theorem yields the functional representation for the n-generated free Wajsberg hoop, as a subalgebra of $[0,1]^{[0,1]^n}$. To accomplish this task we shall work on order duals of MV-algebras, and we have first to dualise some notions of the theory of Schauder hats introduced in the previous subsection. In particular, given a regular triangulation U of $[0,1]^n$ and one of its vertices $\mathbf{v}$, the *Schauder co-hat* $k_{U,\mathbf{v}}$ with apex $\mathbf{v}$ is just the negation of the corresponding Schauder hat.

$$k_{U,\mathbf{v}} = 1 - h_{U,\mathbf{v}} .$$

Schauder co-hats can be defined replacing items 1. and 2. of Definition 2.1.14 with:

1. $k_{U,\mathbf{v}}(\mathbf{v}) = (\mathrm{den}(\mathbf{v}) - 1)/\mathrm{den}(\mathbf{v})$;
2. $k_{U,\mathbf{v}}(\mathbf{v})$ is constantly 1 over each simplex $T \in U$ such that $\mathbf{v} \notin T$.

The set $K_U = \{k_{U,\mathbf{v}} \mid \{\mathbf{v}\} \in U\}$ is the *Schauder co-set* of U. The dual of Lemma 2.1.16 states that $\{g_{\mathbf{v}} \mid \{\mathbf{v}\} \in U_0^n\}$ is the Schauder co-set of U_0^n, where $g_{\mathbf{v}} \colon [0,1]^n \to [0,1]$ is the function

$$\left(\bigvee_{v_i = 0} x_i \oplus \bigvee_{v_i = 1} \neg x_i \right) .$$

We shall also dualise Lemma 2.1.17 as follows.

LEMMA 2.2.1. *Let U be a regular triangulation of $[0,1]^n$ and let $S \in U$ be a 1-simplex with vertices $\mathrm{Vert}(S) = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then the Schauder co-set of the starring of U along S is*

$$K_{U*S} = \{k_{U*S,\mathbf{v}} \mid \{\mathbf{v}\} \in U\} \cup \{k_{U*S,\mathbf{v}_1 +_F \mathbf{v}_2}\} ,$$

where:

1. *$k_{U*S,\mathbf{w}} = k_{U,\mathbf{w}}$ for each $\{\mathbf{w}\} \in U$ such that $\mathbf{v}_1 \neq \mathbf{w} \neq \mathbf{v}_2$;*
2. *$k_{U*S,\mathbf{v}_1} = k_{U,\mathbf{v}_2} \to k_{U,\mathbf{v}_1}$ and $k_{U*S,\mathbf{v}_2} = k_{U,\mathbf{v}_1} \to k_{U,\mathbf{v}_2}$;*
3. *$k_{U*S,\mathbf{v}_1 +_F \mathbf{v}_2} = k_{U,\mathbf{v}_1} \vee k_{U,\mathbf{v}_2}$.*

The proof proceeds straightforwardly once noticed that the dualising map $a \mapsto \neg a$, is such that $0 \mapsto 1$, $a \wedge b \mapsto \neg a \vee \neg b$, $a \ominus b \mapsto \neg b \to \neg a$.

LEMMA 2.2.2. *Let U be a regular triangulation of $[0,1]^n$, and let $\{m_{\mathbf{v}} \mid, \{\mathbf{v}\} \in U\}$ be a collection of integers such that $0 \leq m_{\mathbf{v}} \leq \operatorname{den}(\mathbf{v})$. Then*

$$\bigoplus_{\{\mathbf{v}\}\in U} m_{\mathbf{v}} h_{U,\mathbf{v}} = \bigodot_{\{\mathbf{v}\}\in U} k_{U,\mathbf{v}}^{l_{\mathbf{v}}},$$

for $l_{\mathbf{v}} = \operatorname{den}(\mathbf{v}) - m_{\mathbf{v}}$ for each $\{\mathbf{v}\} \in U$.

Proof. By Lemma 2.1.19, $\bigoplus_{\{\mathbf{v}\}\in U} m_{\mathbf{v}} h_{U,\mathbf{v}}$ is a McNaughton function in $\boldsymbol{F}^n_{\mathbb{MV}}$. By linearity, and the definition of each $l_{\mathbf{v}}$, the real-valued function

$$\bigoplus_{\{\mathbf{v}\}\in U} m_{\mathbf{v}} h_{U,\mathbf{v}} + \bigoplus_{\{\mathbf{v}\}\in U} l_{\mathbf{v}} h_{U,\mathbf{v}}$$

is the constant 1. Hence,

$$\bigoplus_{\{\mathbf{v}\}\in U} m_{\mathbf{v}} h_{U,\mathbf{v}} = \neg \bigoplus_{\{\mathbf{v}\}\in U} l_{\mathbf{v}} h_{U,\mathbf{v}}\,.$$

Now, by De Morgan laws,

$$\neg \bigoplus_{\{\mathbf{v}\}\in U} l_{\mathbf{v}} h_{U,\mathbf{v}} = \bigodot_{\{\mathbf{v}\}\in U} (\neg h_{U,\mathbf{v}})^{l_{\mathbf{v}}} = \bigodot_{\{\mathbf{v}\}\in U} k_{U,\mathbf{v}}^{l_{\mathbf{v}}}\,. \qquad \square$$

THEOREM 2.2.3. *For each integer $n \geq 0$, the free n-generated Wajsberg hoop $\boldsymbol{F}^n_{\mathbb{WH}}$ is the algebra of all McNaughton functions $f\colon [0,1]^n \to [0,1]$ such that*

$$f(1,1,\ldots,1) = 1\,,$$

equipped with pointwise defined operations.

Proof. An easy induction shows that $\varphi^{\boldsymbol{F}^n_{\mathbb{WH}}}(1,1,\ldots,1) = 1$ for each term φ. It remains to prove the converse inclusion. Take a McNaughton functions $f\colon [0,1]^n \to [0,1]$ such that $f(1,1,\ldots,1) = 1$. Lemma 2.1.18 provides an index i such that f is linear over each simplex of U_i. By Lemma 2.1.19 and Lemma 2.2.2,

$$f = \bigoplus_{\{\mathbf{v}\}\in U_i} m_{\mathbf{v}} h_{U_i\mathbf{v}} = \bigodot_{\{\mathbf{v}\}\in U_i} k_{U_i,\mathbf{v}}^{l_{\mathbf{v}}}\,, \tag{3}$$

for a suitable collection $\{l_{\mathbf{v}} \mid \{\mathbf{v}\} \in U_i\}$ of integers. By definition of starring of Schauder co-hats and Lemma 2.2.1, there exists a set of MV-terms $\{\varphi_{\mathbf{v}} \mid \{\mathbf{v}\} \in U_i\}$ such that $k_{U,\mathbf{v}} = (\varphi_{\mathbf{v}})^{\boldsymbol{F}^n_{\mathbb{MV}}}$. That is, $(\varphi_{\mathbf{v}})^{F^n_{\mathbb{MV}}}$ is a Schauder co-hat obtained by finitely many starrings from $K_{U_0^n}$.

It suffices to show that each n-variable Schauder co-hat $k_{U,\mathbf{v}}$, *actually occurring* in the rightmost part of (3) is the interpretation of a Wajsberg hoop term in $F^n_{\mathbb{WH}}$, that is, for each $\mathbf{v}$ such that $l_{\mathbf{v}} > 0$, there exists a Wajsberg hoop term $\psi_{\mathbf{v}}$ such that

$$\psi_{\mathbf{v}}^{\boldsymbol{F}^n_{\mathbb{WH}}} = \varphi_{\mathbf{v}}^{\boldsymbol{F}^n_{\mathbb{MV}}}\,.$$

If this is the case, by (3) it follows that $\left(\&_{\{\mathbf{v}\}\in U} \psi_{\mathbf{v}}^{l_{\mathbf{v}}}\right)^{\boldsymbol{F}^n_{\mathbb{WH}}} = f$, and hence $f \in \boldsymbol{F}^n_{\mathbb{WH}}$.

Notice now that the condition $f(1,1,\dots,1)=1$ implies $l_{\langle 1,1,\dots,1\rangle}=0$, that is the Schauder co-hat with apex $\langle 1,1,\dots,1\rangle$ does not occur in the expression of f.

Then we only need to prove the following claim: for each integer $j>0$ and each vertex $\mathbf{v}\neq\langle 1,1,\dots,1\rangle$, $\{\mathbf{v}\}\in U_j$, there exists a Wajsberg hoop term $\psi_{\mathbf{v}}$ such that $\psi_{\mathbf{v}}^{\boldsymbol{F}^n_{\mathbb{WH}}}=\varphi_{\mathbf{v}}^{\boldsymbol{F}^n_{\mathbb{MV}}}$. The claim is proved by induction on j. It clearly holds for the base case $j=0$, as, for each vertex $\mathbf{v}\neq\langle 1,1,\dots,1\rangle$ of $[0,1]^n$ the following equivalence holds

$$\bigvee_{v_i=0} x_i \oplus \bigvee_{v_i=1} \neg x_i \equiv_{\mathbb{WH}} \bigvee_{v_j=0,v_k=1} x_k \to x_j\,.$$

Let now U_{j+1} be obtained from U_j by starring along a 1-simplex of vertices $\mathbf{v}_1$ and $\mathbf{v}_2$. By induction, each $k_{U_j,\mathbf{v}}$ with $\mathbf{v}\neq\langle 1,1,\dots,1\rangle$ coincides with $\psi_{j,\mathbf{v}}^{F^n_{\mathbb{WH}}}$ for some Wajsberg hoop term $\psi_{j,\mathbf{v}}$. It remains to show that each one of the three Schauder co-hats $k_{U_j,\mathbf{v}_1}\vee k_{U_j,\mathbf{v}_2}$, $k_{U_j,\mathbf{v}_2}\to k_{U_j,\mathbf{v}_1}$, and $k_{U_j,\mathbf{v}_1}\to k_{U_j,\mathbf{v}_2}$ either is a co-hat with apex $\langle 1,1,\dots,1\rangle$ or is expressible as the interpretation in $\boldsymbol{F}^n_{\mathbb{WH}}$ of a Wajsberg hoop term. If $\mathbf{v}_1\neq\langle 1,1,\dots,1\rangle\neq\mathbf{v}_2$ the conclusion trivially follows. Assume, without loss of generality, that $\mathbf{v}_1=\langle 1,1,\dots,1\rangle$ (then $\mathbf{v}_2\neq\langle 1,1,\dots,1\rangle$). Then both $k_{U_j,\mathbf{v}_1}$ and $k_{U_{j+1},\mathbf{v}_1}=k_{U_j,\mathbf{v}_2}\to k_{U_j,\mathbf{v}_1}$ have apex $\langle 1,1,\dots,1\rangle$. Now, $g=\neg k_{U_j,\mathbf{v}_1}$ is an n-variable McNaughton function such that $g(1,1\dots,1)=1$ that is linear over each simplex of U_j and is such that $g(\mathbf{v})=0$ for all vertices $\langle 1,1,\dots,1\rangle\neq\mathbf{v}$ of U_j. Hence, letting $I=\{\mathbf{v}\mid\{\mathbf{v}\}\in U_j\}\setminus\{\langle 1,1,\dots,1\rangle\}$, it holds that

$$g=\bigodot_{\mathbf{v}\in I} k_{U_j,\mathbf{v}}^{\mathrm{den}(\mathbf{v})}\,.$$

By induction, each $k_{U_j,\mathbf{v}}$ with $\mathbf{v}\in I$ is such that there is $\psi_{j,\mathbf{v}}$ with $\psi_{j,\mathbf{v}}^{\boldsymbol{F}^n_{\mathbb{WH}}}=k_{U_j,\mathbf{v}}$. Hence there exists a Wajsberg hoop term ϑ such that $\vartheta^{\boldsymbol{F}^n_{\mathbb{WH}}}=\neg k_{U_j,\mathbf{v}_1}$. We conclude the proof by noting that $k_{U_j,\mathbf{v}_1}\to k_{U_j,\mathbf{v}_2}=\neg k_{U_j,\mathbf{v}_1}\oplus k_{U_j,\mathbf{v}_2}$ and $k_{U_j,\mathbf{v}_1}\vee k_{U_j,\mathbf{v}_2}=(\neg k_{U_j,\mathbf{v}_1}\oplus k_{U_j,\mathbf{v}_2})\to k_{U_j,\mathbf{v}_2}$. □

We can use Theorem 2.2.3 to express free MV-algebras via Wajsberg hoops, attaching one additional bit of information to each element of $\boldsymbol{F}^n_{\mathbb{WH}}$.

THEOREM 2.2.4. *Let WH_n be the universe of $\boldsymbol{F}^n_{\mathbb{WH}}$. Then*

$$\boldsymbol{F}^n_{\mathbb{MV}}\cong\langle WH_n\times\{0,1\},\boxplus,\neg,\bot\rangle\,,$$

where

$$\bot=\langle\top,0\rangle\,,\qquad\neg\langle f,b\rangle=\langle f,1-b\rangle\,,$$

and

$$\langle f,b\rangle\boxplus(g,c)=\begin{cases}\langle f\odot g,0\rangle & \text{if } b=0=c\,,\\ \langle f\to g,1\rangle & \text{if } b=0,c=1\,,\\ \langle g\to f,1\rangle & \text{if } b=1,c=0\,,\\ \langle f\oplus g,1\rangle & \text{if } b=1=c\,.\end{cases}$$

Proof. Let φ be an MV-term. By using the following equivalences:

$$\varphi \,\&\, \neg\psi \equiv_{\mathbb{MV}} \neg(\varphi \to \psi) \qquad \neg\varphi \,\&\, \neg\psi \equiv_{\mathbb{MV}} \neg(\varphi \oplus \psi)$$

and

$$\varphi \to \neg\psi \equiv_{\mathbb{MV}} \neg(\varphi \,\&\, \psi) \qquad \neg\varphi \to \psi \equiv_{\mathbb{MV}} \varphi \oplus \psi \qquad \neg\varphi \to \neg\psi \equiv_{\mathbb{MV}} \psi \to \varphi,$$

one can construct a Wajsberg hoop term ψ such that either $\varphi \equiv_{\mathbb{MV}} \psi$ (if it holds $\varphi^{\boldsymbol{F}^n_{\mathbb{MV}}}(1,1,\ldots,1) = 1$) or $\varphi \equiv_{\mathbb{MV}} \neg\psi$ (if $\varphi^{\boldsymbol{F}^n_{\mathbb{MV}}}(1,1,\ldots,1) = 0$). It is now straightforward to check that the map defined for each $f \in WH_n$, by $\langle f, 0\rangle \mapsto \neg f$, $\langle f, 1\rangle \mapsto f$, is an isomorphism of MV-algebras. □

2.3 Finitely valued Łukasiewicz logics

For each $k > 0$, the subvariety $\mathbb{MV}_k$ that constitutes the equivalent algebraic semantics of the $k+1$-valued Łukasiewicz logic Ł_k is axiomatised via Grigolia's axioms:

- $x^k = x^{k+1}$;
- $(k+1)x^h = (h(x^{h-1}))^{k+1}$, for every integer $h \in \{2,\ldots,k-1\}$ that does not divide k.

The variety $\mathbb{MV}_k$ is generated by the $(k+1)$-element MV-chain $\mathbf{Ł}_k$. By Lemma 1.0.12, $\boldsymbol{F}^n_{\mathbb{MV}_k}$ is the subalgebra of $\mathbf{Ł}_k^{\mathbf{Ł}_k^n}$ generated by the projections. A moment's reflection shows that this algebra coincides with the algebra of restrictions of McNaughton functions in $\boldsymbol{F}^n_{\mathbb{MV}}$ to the set

$$D^n_k = \left\{0, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}, 1\right\}^n .$$

It is easy to see the following.

LEMMA 2.3.1. *For each integer $n > 0$ and each integer $k > 0$,*

$$\boldsymbol{F}^n_{\mathbb{MV}_k} \cong \prod_{\mathbf{v}\in D^n_k} \mathbf{Ł}_{\mathrm{den}(\mathbf{v})} .$$

It remains to compute the denominator of elements in D^n_k. For each integer $d \geq 0$ we denote $\langle \mathrm{Div}(d), |\rangle$ the lattice of (nonnegative) divisors of d, where the partial order is given by $a|b$ if a divides b.

LEMMA 2.3.2. *For each pair of positive integers n, k the set D^n_k contains only rational points whose denominator divides k. More precisely, for every integer d that divides k, the set D^n_k contains exactly $\alpha(n,d)$ many points of denominator d, where:*

- $\alpha(0,1) = 1$ *and* $\alpha(0,d) = 0$ *if* $d > 1$;
- $$\alpha(n,d) = (d+1)^n + \sum_{\emptyset\neq X\subseteq \mathrm{CoAtDiv}(d)} (-1)^{|X|}(\gcd(X)+1)^n ,$$

  where $\mathrm{CoAtDiv}(d)$ *is the set of coatoms in the lattice of divisors of d.*

Proof. Trivially, in $[0,1]^n$ there are exactly $(d+1)^n$ points whose denominator is a divisor of d. We then have to subtract from this set the points whose denominator is a proper divisor of d. We shall use an instance of the combinatorial principle of inclusion/exclusion stating that for a collection $\{A_i\}_{i\in I}$ of sets,

$$\left|\bigcup_{i\in I} A_i\right| = \sum_{S\subseteq I}(-1)^{|S|-1}\left|\bigcap_{i\in S} A_i\right| . \tag{4}$$

For each nonempty subset X of coatoms of $\mathrm{Div}(d)$ we let L_X be the universe of the MV-algebra $\mathbf{Ł}^n_{\mathrm{gcd}(X)}$. Notice that L_X is the set of all rational points in $[0,1]^n$ whose denominator divide each element of X. Then

$$\alpha(n,d) = (d+1)^n - \left|\bigcup_{x\in\mathrm{CoAtDiv}(d)} L_{\{x\}}\right| .$$

Consider now two nonempty subsets X, Y of coatoms in $\mathrm{Div}(d)$. Notice that:

1. $|L_X \cup L_Y| = |L_X| + |L_Y| - |L_X \cap L_Y|$;
2. $L_X \cap L_Y = L_{X\cup Y}$.

Those two statements together with (4) immediately yield

$$\left|\bigcup_{x\in\mathrm{CoAtDiv}(d)} L_{\{x\}}\right| = \sum_{\emptyset\neq X\subseteq\mathrm{CoAtDiv}(d)} (-1)^{|X|-1}|L_X| ,$$

that in turns, since $|L_X| = (\mathrm{gcd}(X)+1)^n$, gives the desired expression for $\alpha(n,d)$. $\square$

We can now improve on the description of Lemma 2.3.1.

THEOREM 2.3.3. *For each integer $n > 0$ and each integer $k > 0$,*

$$\boldsymbol{F}^n_{\mathbb{MV}_k} \cong \prod_{d\in\mathrm{Div}(k)} \mathbf{Ł}_d^{\alpha(n,d)} .$$

As $\mathbb{MV}_k$ is a locally finite variety, we can turn Theorem 2.3.3 into a combinatorial description of the prime spectrum of $\boldsymbol{F}^n_{\mathbb{MV}_k}$, and work on the dual space given by the spectrum. This kind of dual representation is exploited in Section 4.2, to deal with Gödel logic and other schematic extensions of WNM. It is further explored in Section 5.4 for locally finite subvarieties of $\mathbb{BL}$, where, as an application, we obtain explicitly the description of the prime spectrum of $\boldsymbol{F}^n_{\mathbb{MV}_k}$.

2.4 Extensions of MV-algebras

In this section we deal with logics $Ł_\triangle$, $\mathrm{PMV}_\triangle$, ŁΠ, and $Ł\Pi\frac{1}{2}$ (see Section 2.2 of Chapter I). We recall that the corresponding varieties (denoted as $\mathbb{MV}_\triangle$, $\mathbb{PMV}_\triangle$, ŁΠ, and $Ł\Pi\frac{1}{2}$) are generated respectively by the standard algebras:

$$\langle[0,1],\odot,\to_\odot,1,\triangle\rangle \qquad \langle[0,1],\odot,\cdot,\to_\odot,1,\triangle\rangle$$
$$\langle[0,1],\odot,\cdot,\to_\odot,\to_\cdot,1\rangle \qquad \langle[0,1],\odot,\cdot,\to_\odot,\to_\cdot,1,1/2\rangle ,$$

where $\odot$ is Łukasiewicz t-norm, $\cdot$ the product of reals, and $\triangle$ is Baaz's projection operator: $\triangle x = 1$ for $x = 1$ and $\triangle x = 0$ otherwise. Recall the algebras of polynomials introduced in Definition 2.0.14.

DEFINITION 2.4.1. *A subset S of $[0,1]^n$ is* Q-semialgebraic *if it is a Boolean combination of sets of the form*

$$\{\langle x_1,\dots,x_n\rangle \in [0,1]^n \mid P(\langle x_1,\dots,x_n\rangle) > 0\},$$

for $P \in \mathsf{Z}[x_1,\dots,x_n]$. If every $P \in \mathsf{Z}_1[x_1,\dots,x_n]$ then S is linear Q-semialgebraic.

DEFINITION 2.4.2. *Let $\mathbb{E}$ be an equational class among $\mathbb{MV}_\triangle$, $\mathbb{PMV}_\triangle$, and $\text{Ł}\Pi\frac{1}{2}$. An E-*hat over $[0,1]^n$ *is a function $h : [0,1]^n \to [0,1]$ such that there exist a Q-semialgebraic set S and a rational function $P/Q \in \mathsf{Q}(x_1,\dots,x_n)$ such that:*

- *Q has no zeros on S, $h = P/Q$ on S and $h = 0$ on $[0,1]^n \setminus S$;*
- *if $\mathbb{E} = \mathbb{PMV}_\triangle$, then $Q = 1$;*
- *if $\mathbb{E} = \mathbb{MV}_\triangle$, then $Q = 1$, $P \in \mathsf{Z}_1[x_1,\dots,x_n]$ and S is linear Q-semialgebraic.*

If h is the hat determined by S and P/Q as above, then we write $h = \langle S, P/Q\rangle$. An *$E$-function over* $[0,1]^n$ is a finite sum f of E-hats

$$f = \langle S_1, P_1/Q_1\rangle + \cdots + \langle S_r, P_r/Q_r\rangle$$

subject to the condition that $S_i \cap S_j = \emptyset$, for $i \neq j$.

We denote by $B_n(E)$ the set of all E-functions over $[0,1]^n$. A direct inspection shows that $B_n(E)$ equipped with the pointwise operations is an E-algebra that contains the projection functions.

LEMMA 2.4.3. *For any $P \in \mathsf{Z}[x_1,\dots,x_n]$ let $P^\#\colon [0,1]^n \to [0,1]$ be defined by*

$$P^\#(v) = \min\{1, \max\{P(v), 0\}\}\,.$$

Then $P^\# \in \boldsymbol{F}^n_{\mathbb{PMV}_\triangle}$. Further, if $P \in \mathsf{Z}_1[x_1,\dots,x_n]$ then $P^\# \in \boldsymbol{F}^n_{\mathbb{MV}}$.

Proof. The proof is analogous to the proof of Lemma 2.1.21. Indeed let

$$P = k + \sum_{\langle i_0,i_1,\dots,i_n\rangle \in \mathsf{N}^{n+1}} k_{i_0} x_1^{i_1} \cdot \cdots \cdot x_n^{i_n},$$

where $k, k_{i_0} \in \mathsf{Z}$ and $k_{i_0} \neq 0$ only for finitely many i_0's. The proof proceeds by induction on $K = \sum_{\langle i_0,i_1,\dots,i_n\rangle \in \mathsf{N}^{n+1}} |k_{i_0}|$. If $K = 0$ then P is either equal to 0 or to 1 and the claim is settled. If $K \geq 1$ suppose there is $i \in I$ with $k_{i_0} \geq 1$. Then letting $Q = P - x_1^{i_1} \cdot \cdots \cdot x_n^{i_n}$ by induction hypothesis $Q^\#$ and $(Q+1)^\#$ are in $\boldsymbol{F}^n_{\mathbb{PMV}_\triangle}$. It is now easy to check that

$$P^\# = (Q^\# \oplus x_1^{i_1} \cdot \cdots \cdot x_n^{i_n}) \odot (Q+1)^\#\,.$$

The last case is when $K \geq 0$ and for every $i \in I$, $k_{i_0} < 0$. Then the argument above can be applied to $1 - P$ obtaining $(1-P)^\# \in \boldsymbol{F}^n_{\mathbb{PMV}_\triangle}$. The claim follows by noticing that $P^\# = \neg(1-P)^\#$. $\square$

THEOREM 2.4.4. *Let* $\mathbb{E}$ *be any of* $\mathbb{MV}_\triangle$, $\mathbb{PMV}_\triangle$, *and* $\text{Ł}\Pi\frac{1}{2}$. *Then* $\boldsymbol{F}^n_{\mathbb{E}} \cong \boldsymbol{B}_n(E)$.

Proof. Since $\boldsymbol{B}_n(E)$ contains the projections and it is closed under the E-operations, we only have to show that each E-function is contained in $\boldsymbol{F}^n_{\mathbb{E}}$. Each E-function f has the form

$$f = \langle S_1, P_1/Q_1\rangle + \cdots + \langle S_r, P_r/Q_r\rangle,$$

where S_i are pairwise disjoint algebraic sets, hence it can be written as

$$f = \langle S_1, P_1/Q_1\rangle \oplus \cdots \oplus \langle S_r, P_r/Q_r\rangle .$$

Thus in order to prove the theorem, it is enough to prove that each E-hat $\langle S_i, P_i/Q_i\rangle$ is in $\boldsymbol{F}^n_{\mathbb{E}}$. Let $P \in \mathsf{Z}[x_1, \ldots, x_n]$ (resp. in $\mathsf{Z}_1[x_1, \ldots, x_n]$). Then $1 - \triangle(1 - P^{\#})$ is the characteristic function of the set $\{P > 0\} \subseteq [0,1]^n$. Hence the characteristic function of any Q-semialgebraic set (resp. linear Q-semialgebraic set) is an element of $\boldsymbol{F}^n_{\mathbb{PMV}_\triangle}$ (resp. of $\boldsymbol{F}^n_{\mathbb{MV}_\triangle}$).

Let $\langle S, P/Q\rangle$ be a $\mathbb{PMV}_\triangle$-hat. Then $Q = 1$ and letting s be the characteristic function of S, we have $\langle S, P/Q\rangle = s \wedge P^{\#} \in \boldsymbol{F}^n_{\mathbb{PMV}_\triangle}$. The case for $\mathbb{MV}_\triangle$ proceeds analogously.

If $\mathbb{E} = \text{Ł}\Pi\frac{1}{2}$ we can suppose $P \geq 0$ and $Q > 0$ on S. Further, write P and Q as:

$$P = \sum_{i\in I} a_i\lambda_i - \sum_{j\in J} b_j\mu_j;$$
$$Q = \sum_{r\in R} c_r\pi_r - \sum_{t\in T} d_t\rho_t,$$

where $a_i, b_j, c_r, d_t \in \mathsf{N}$ and $\lambda_i, \mu_i, \pi_r, \rho_t$ are monic monomials in $x_1, \ldots, x_n$. Then let m be such that $\sum_{I,J,R,T} a_i + b_j + c_r + d_t \leq 2^m$ and:

$$p = \bigoplus_{i\in I} a_i \cdot (1/2)^m \cdot \lambda_i \ominus \bigoplus_{j\in J} b_j \cdot (1/2)^m \cdot \mu_j =$$
$$= 1/2^m \cdot \left(\bigoplus_{i\in I} a_i \cdot \lambda_i \ominus \bigoplus_{j\in J} b_j \cdot \mu_j \right);$$

$$q = \bigoplus_{r\in R} c_r \cdot (1/2)^m \cdot \pi_i \ominus \bigoplus_{t\in T} d_t \cdot (1/2)^m \cdot \rho_j =$$
$$= 1/2^m \cdot \left(\bigoplus_{r\in R} c_r \cdot \pi_i \ominus \bigoplus_{t\in T} d_t \cdot \rho_j \right).$$

Then $p/q \in \boldsymbol{F}^n_{\text{Ł}\Pi\frac{1}{2}}$ and $\langle S, P/Q\rangle = s \wedge (p/q) \in \boldsymbol{F}^n_{\text{Ł}\Pi\frac{1}{2}}$. $\square$

Let $\boldsymbol{B}_n(\text{Ł}\Pi)$ be the subalgebra of $\boldsymbol{B}_n(\text{Ł}\Pi\frac{1}{2})$ made of functions mapping $\{0,1\}^n$ to $\{0,1\}$.

THEOREM 2.4.5. $\boldsymbol{F}^n_{\text{Ł}\Pi} = \boldsymbol{B}_n(\text{Ł}\Pi)$.

Proof. Let $b \in \boldsymbol{B}_n(\text{Ł}\Pi)$, then by Theorem 2.4.4 there exists a term t in the language of $\text{Ł}\Pi\frac{1}{2}$ that coincides with b when interpreted in $[0,1]$. Consider the term

$$p = \bigvee_{1 \le i \le n} \neg p_i,$$

where

$$p_i = ((x_i \wedge \neg x_i) \oplus (x_i \wedge \neg x_i)) \rightarrow. (x_i \wedge \neg x_i)\,.$$

It is easy to check that for every $v \in [0,1]^n$,

$$p(v) = \begin{cases} 1/2 & \text{if } v \notin \{0,1\}^n, \\ 0 & \text{otherwise.} \end{cases}$$

Then substitute every occurrence in t of the constant $1/2$ with the term p. We obtain a term t' that interpreted in $[0,1]^n$ coincides with t except than on the vertices $\{0,1\}^n$. But each vertex is a Q-semialgebraic set, hence its characteristic function is a term over the language of $\mathbb{PMV}_\triangle$ hence of ŁΠ. The result follows by considering a suitable Boolean combination of t' and the characteristic functions of vertices. □

Note that in this treatment the variety $\mathbb{PMV}$ generated by the standard algebra $\langle [0,1], \odot, \cdot, \rightarrow_\odot, 1\rangle$ is not considered. Indeed, a description of the free algebra in this variety would settle the Pierce-Birkhoff conjecture in semialgebraic geometry [52].

2.5 MV-algebras with division operators

DEFINITION 2.5.1. *A* DMV-algebra $\boldsymbol{A} = \langle A, \oplus, \neg, \{\delta_n\}_{n\in\mathsf{N}}, 0, 1\rangle$ *is an algebraic structure such that* $\boldsymbol{A}^* = \langle A, \oplus, \neg, 0, 1\rangle$ *is an* MV*-algebra and the following hold for every* $x \in A$ *and* $n \in \mathsf{N}$*:*

(D1n) $n\,\delta_n x = x$;

(D2n) $\delta_n x \odot (n-1)\,\delta_n x = 0\,.$

The variety $\mathbb{DMV}$ of DMV-algebras is generated by

$$\langle [0,1], \oplus, \neg, \{\_/n\}_{n\in\mathsf{N}}, 0\rangle\,,$$

hence the free DMV-algebra $\boldsymbol{F}^n_{\mathbb{DMV}}$ over n generators is isomorphic to the DMV-algebra of functions from $[0,1]^n$ to $[0,1]$ generated by the projections.

THEOREM 2.5.2. $\boldsymbol{F}^n_{\mathbb{DMV}}$ *is the algebra of continuous piecewise linear functions with rational coefficients.*

Proof. A direct inspection shows that every element of $\boldsymbol{F}^n_{\mathbb{DMV}}$ is a continuous piecewise linear function, where each piece has rational coefficients.

Let $f\colon [0,1]^n \rightarrow [0,1]$ be a continuous piecewise linear function with rational coefficients. Further, let s be an integer such that $s \cdot f\colon \mathbf{t} \in [0,1]^n \mapsto s \cdot f(\mathbf{t}) \in [0,s]$ is a continuous function with integer coefficients (for example s is the least common multiple of the denominators of the coefficients of pieces of f).

For every $i = 0, \ldots, s-1$, let

$$f_i\colon \mathbf{t} \in [0,1]^n \mapsto ((s \cdot f(\mathbf{t}) - i) \wedge 1) \vee 0 \in [0,1].$$

Since f_i are continuous functions with integer coefficients there exist MV-terms ψ_i such that $f_i = \psi_i^{\boldsymbol{F}^n_{\mathbb{MV}}}$. For any function $g : [0,1]^n \to [0,1]$ we define:

$$Supp(g) = \{\mathbf{t} \in [0,1]^n \mid g(\mathbf{t}) > 0\};$$
$$Supp^{<1}(g) = \{\mathbf{t} \in [0,1]^n \mid 0 < g(\mathbf{t}) < 1\}.$$

We have, for every $i = 1, \ldots, s-1$,

$$Supp^{<1}(f_i) \subseteq Supp(f_i) \subseteq Supp(f_{i-1}).$$

Indeed

$$Supp(f_i) = \{\mathbf{t} \in [0,1]^n \mid s \cdot f(\mathbf{t}) > i\} \subseteq \{\mathbf{t} \in [0,1]^n \mid s \cdot f(\mathbf{t}) > i-1\}.$$

Further, for any $i \neq j$, $Supp^{<1}(f_i) \cap Supp^{<1}(f_j) = \emptyset$. Then

$$f(\mathbf{t}) = \varphi^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}), \qquad \text{where} \quad \varphi = \bigoplus_{i=0}^{s-1} \delta_s \psi_i.$$

Indeed, suppose that $\mathbf{t} \in [0,1]^n$ is such that $f(\mathbf{t}) = 0$. Then for every $i = 0, \ldots, s-1$, $f_i(\mathbf{t}) = 0$ whence $\psi_i^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}) = 0$ and $\varphi^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}) = 0 = f(\mathbf{t})$.

If $f(\mathbf{t}) = 1$ then for every $i = 0, \ldots, s-1$, $\varphi_i^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}) = 1$ whence $\delta_s \varphi_i^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}) = 1/s$ and $\varphi^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}) = 1$.

Suppose now that there exists $i \in \{0, \ldots, s-1\}$ such that $i < s \cdot f(\mathbf{t}) \leq i+1$. Then $\mathbf{t} \in Supp(f_i)$ and $f(\mathbf{t}) = (f_i(\mathbf{t}) + i)/s$. For every $j > i$, we have $s \cdot f(\mathbf{t}) - j \leq s \cdot f(\mathbf{t}) - i - 1 < 0$ whence $f_j(\mathbf{t}) = \psi_j^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}) = 0$. Further, for every $j < i$, we have $s \cdot f(\mathbf{t}) - j \geq s \cdot f(\mathbf{t}) - i + 1 > 1$ whence $f_j(\mathbf{t}) = 1$ and $(\delta_s \psi_j)^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}) = 1/s$. Hence

$$\varphi^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}) = \left(\bigoplus_{j=0}^{i-1} \delta_s \psi_j \oplus \delta_s \psi_i \oplus \bigoplus_{j=i+1}^{s-1} \delta_s \psi_j \right)^{\boldsymbol{F}^n_{\mathbb{DMV}}}(\mathbf{t}) = \sum_{j=0}^{i-1} \frac{1}{s} + \frac{f_i(\mathbf{t})}{s} + 0 = f(\mathbf{t}).$$

□

3 Product algebras

In this section we shall deal with free product algebras and their representations. A first step towards these representation consists in giving a representation of free algebras in a variety deeply related to product algebras, that is, the variety of cancellative hoops.

3.1 Free cancellative hoops

The variety of cancellative hoops is generated by

$$(0,1]^- = \langle (0,1], \cdot, \to, 1 \rangle ,$$

hence by Lemma 1.0.12 the free cancellative hoop over n generators is the cancellative hoop of functions from $(0,1]^n$ to $(0,1]$ that is generated by the projection functions.

A *monomial function* of order n is a function $f\colon (0,1]^n \to (0,1]$ such that there are $m_i \in \mathsf{Z}$ for $i = 1, \dots, n$ and

$$f(x_1, \dots, x_n) = 1 \wedge \prod_{i=1}^{n} x_i^{m_i}.$$

THEOREM 3.1.1. *The free cancellative hoop $\boldsymbol{F}^n_{\mathbb{CH}}$ over n generators is isomorphic to the hoop of piecewise monomial functions, that is functions $f\colon (0,1]^n \to (0,1]$ such that there is a finite family $\{f_{pq}\}$ of monomial functions with*

$$f = \bigvee_p \bigwedge_q f_{pq}\,. \tag{5}$$

Proof. A structural induction shows that every n-variable term over the language of cancellative hoops, when interpreted as a function over $[0,1]^n$, is a piecewise monomial function.

We prove that if f is a function of the form (5) then it can be written starting from projection functions using the operations of cancellative hoops. Indeed, let

$$f(t_1, \dots, t_n) = 1 \wedge \prod_{i=1}^{n} t_i^{m_i},$$

and let $H = \{i \mid m_i > 0\}$ and $K = \{i \mid m_i < 0\}$. If $K = \emptyset$, then $f(t_1, \dots, t_n) = \prod_{i=1}^{n} t_i^{m_i}$ with all $m_i \geq 0$, hence the term τ_f given by

$$\tau_f = \underbrace{x_1 \mathbin{\&} \cdots \mathbin{\&} x_1}_{m_1 \text{ times}} \mathbin{\&} \ \cdots \ \mathbin{\&} \underbrace{x_n \mathbin{\&} \cdots \mathbin{\&} x_n}_{m_n \text{ times}}$$

is such that $\tau_f^{\boldsymbol{F}^n_{\mathbb{CH}}} = f$. (We are denoting by x_i the i-th propositional variable.) If $H = \emptyset$ then $f = 1$ hence we can set $\tau_f = \top$. Otherwise, if $H \neq \emptyset$ and $K \neq \emptyset$, then letting

$$g = \prod_{i \in H} t_i^{m_i} \qquad h = \prod_{i \in K} t_i^{-m_i}\,,$$

we have $\tau_f = \tau_h \to \tau_g$.

If f has the form (5), then it is a Boolean combination of monomial functions and the claim follows from the fact that Boolean operators can be defined in cancellative hoops. □

3.2 Free product algebras

Recall that the variety $\mathbb{P}$ of product algebras is generated by

$$\langle [0,1], \cdot, \to_{\cdot}, 0 \rangle\,,$$

where $x \cdot y = xy$ is the usual product of real numbers, and $x \to_{\cdot} y$ is its residuum: if $x > y$ then $x \to_{\cdot} y = y/x$.

For any product algebra $\boldsymbol{A}$, we let $\mathbf{B}(\boldsymbol{A}) = \{\neg\neg x \mid x \in A\}$. $\mathbf{B}(\boldsymbol{A})$ is the greatest Boolean algebra that is a subalgebra of $\boldsymbol{A}$.

PROPOSITION 3.2.1. *For each cardinal* $\kappa > 0$, $\mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{\kappa})$ *is the free Boolean algebra over* κ *generators. If* Y *is a set of generators of* $\boldsymbol{F}_{\mathbb{P}}^{\kappa}$, *then* $\neg\neg Y$ *is a set of free generators of* $\mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{\kappa}) \cong \boldsymbol{F}_{\mathbb{B}}^{\kappa}$.

Proof. Let $\boldsymbol{B}$ be any Boolean algebra and $f\colon \neg\neg Y \to \boldsymbol{B}$. Since $\boldsymbol{B}$ is a product algebra, then there is a unique $\overline{f}\colon \boldsymbol{F}_{\mathbb{P}}^{n} \to \boldsymbol{B}$ such that for every $y \in Y$, $f(\neg\neg y) = \overline{f}(y)$. Since $\mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{n}) \subseteq \boldsymbol{F}_{\mathbb{P}}^{n}$, then the map $\overline{\overline{f}}$ defined by $\overline{\overline{f}}(t) = \overline{f}(t)$ for each $t \in \mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{n})$ uniquely extend f to a homomorphism of Boolean algebras. $\square$

We can then identify $\mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{n})$ with $\boldsymbol{F}_{\mathbb{B}}^{n}$. It is not difficult to check that for any atom z of $\boldsymbol{F}_{\mathbb{B}}^{n} = \mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{n})$, the interval $[\bot, z] \subseteq \boldsymbol{F}_{\mathbb{P}}^{n}$ equipped with the operations $\cdot$ and

$$x \to_z y = z \wedge (x \to y)\,,$$

is a product algebra.

PROPOSITION 3.2.2. *For each finite cardinal* $n \geq 1$, *let* $\mathrm{at}(\mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{n})) \subseteq \boldsymbol{F}_{\mathbb{P}}^{n}$ *be the set of atoms of* $\mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{n})$. *Then*

$$\boldsymbol{F}_{\mathbb{P}}^{n} \quad \cong \prod_{z \in \mathrm{at}(\mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{n}))} [\bot, z]\,.$$

Proof. By Proposition 3.2.1, $\mathrm{at}(\mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{n})) = \mathrm{at}(\boldsymbol{F}_{\mathbb{B}}^{n})$. Display the set of atoms $\mathrm{at}(\boldsymbol{F}_{\mathbb{B}}^{n})$ as $\{z_1, \ldots, z_k\}$. The map $\varphi : f \in \boldsymbol{F}_{\mathbb{P}}^{n} \to (f \wedge z_1, \ldots, f \wedge z_k) \in [\bot, z_1] \times \cdots \times [\bot, z_k]$ is an isomorphism. Indeed for each $\langle y_1, \ldots, y_k \rangle \in \prod_{z \in \mathrm{at}(\boldsymbol{F}_{\mathbb{B}}^{n})} [\bot, z]$ it holds that $\langle y_1, \ldots, y_k \rangle = \varphi(y_1 \vee \ldots \vee y_k)$, since $z_i \wedge z_j = \bot$ for $i \neq j$. If $\varphi(f) = \varphi(g)$ then for every $i = 1, \ldots, k$, $f \wedge z_i = g \wedge z_i$ hence $\bigvee(f \wedge z_i) = \bigvee(g \wedge z_i)$ and this implies $f = g$ since $\bigvee_{i=1}^{k} z_i = \top$. $\square$

Since the generators of $\boldsymbol{F}_{\mathbb{B}}^{n}$ are $\neg\neg\pi_i$ where π_i are the free generators of $\boldsymbol{F}_{\mathbb{P}}^{n}$, and since $\neg\neg\neg x = \neg\neg x$, then each atom of $\boldsymbol{F}_{\mathbb{B}}^{n}$ has the form

$$p_\epsilon = \bigwedge_{i=1}^{n} \neg^{\epsilon_i} \pi_i\,,$$

where $\epsilon = \langle \epsilon_1, \ldots, \epsilon_k \rangle \in \{1, 2\}^k$ and $\neg^1 = \neg$ while $\neg^2 = \neg\neg$.

Since $\boldsymbol{F}_{\mathbb{B}}^{n} \cong \mathbf{B}(\boldsymbol{F}_{\mathbb{P}}^{n})$, then elements of $\boldsymbol{F}_{\mathbb{B}}^{n}$ can be seen as functions from $[0,1]^n$ to $[0,1]$. Let $\#\epsilon = |\{i \mid \epsilon_i = 2\}|$ and $G_\epsilon = \{\langle t_1, \ldots, t_n \rangle \in [0,1]^n \mid t_i > 0 \text{ iff } \epsilon_i = 2\}$. The function p_ϵ is equal to 1 over G_ϵ and it is 0 otherwise, hence each function $f \in [\bot, p_\epsilon]$ is equal to 0 outside G_ϵ.

We need the following technical lemma that can be shown by direct inspection of the operations on $[0,1]$.

LEMMA 3.2.3. *If* $f\colon [0,1]^n \to [0,1]$ *belongs to* $\boldsymbol{F}_{\mathbb{P}}^{n}$, *then:*

- *if there is* $\langle t_1, \ldots, t_n \rangle \in G_\epsilon$ *such that* $f(t_1, \ldots, t_n) = 0$, *then* $f(s_1, \ldots, s_n) = 0$ *for every* $\langle s_1, \ldots, s_n \rangle \in G_\epsilon$;
- $f(0, \ldots, 0) \in \{0, 1\}$.

PROPOSITION 3.2.4. *For every $\epsilon \in \{1,2\}^n$, $(\bot, p_\epsilon]$ is isomorphic to the free cancellative hoop over $\#\epsilon$ generators.*

Proof. If $\#\epsilon = 0$, then $G_\epsilon = \{\langle 0,\dots,0\rangle\}$ and $(\bot, p_\epsilon] = \{\top\}$ that is the free cancellative hoop over 0 generators. Let $\#\epsilon = k > 0$ and $f \in (\bot, p_\epsilon]$. Then, by Lemma 3.2.3, the function f is never equal to 0 over G_ϵ, hence the function $\widetilde{f}(s_1,\dots,s_k) = = f(t_{i_1}, \dots, t_{i_n})$ where $t_{i_j} = s_j$ for $j = 1,\dots,k$ and $t_i = 0$ otherwise, is a function from $(0,1]^k \to (0,1]$. Let

$$\varphi_\epsilon \colon f \in (\bot, p_\epsilon] \to \widetilde{f} \in (0,1]^{(0,1]^k}.$$

It is easy to check that $\varphi_\epsilon((\bot, p_\epsilon])$ contains the projections and it is closed under hoops operations, hence it contains the free cancellative hoop over k generators. In order to prove the other inclusion, let $f \in (\bot, p_\epsilon]$. Since $f \in \boldsymbol{F}^n_{\mathbb{P}}$ then there exists a term τ in the language of product algebras such that $f = \tau^{\boldsymbol{F}^n_{\mathbb{P}}} \le p_\epsilon$. The claim follows by noticing that there exists a term $\widetilde{\tau}$ with k variables in the language of cancellative hoops, such that $\widetilde{\tau}^{\boldsymbol{F}^n_{\mathbb{CH}}} = \widetilde{f}$. Indeed we can proceed by structural induction: if τ is a variable, then it can only be equal to a x_{i_j} hence $\widetilde{\tau} = x_{i_j}$. If $\tau = \tau_1 \cdot \tau_2$, recalling that $f > 0$, then it can only be $\tau_1^{\boldsymbol{F}^n_{\mathbb{P}}} > 0$ and $\tau_2^{\boldsymbol{F}^n_{\mathbb{P}}} > 0$, hence the claim follows by induction hypothesis. If $\tau = \tau_1 \to \tau_2$ then we have to consider the cases in which either $\tau_1^{\boldsymbol{F}^n_{\mathbb{P}}} = 0$ or $\tau_2^{\boldsymbol{F}^n_{\mathbb{P}}} = 0$. But if $\tau_1^{\boldsymbol{F}^n_{\mathbb{P}}} = 0$ then $\tau^{\boldsymbol{F}^n_{\mathbb{P}}} = 1$ hence we can set $\widetilde{\tau} = \top$. On the other hand, if $\tau_2^{\boldsymbol{F}^n_{\mathbb{P}}} = 0$ since $f > 0$, it can only be $\tau_1^{\boldsymbol{F}^n_{\mathbb{P}}} = 0$. □

Noticing that $\{G_\epsilon \mid \epsilon \in \{1,2\}^n\}$ is a partition of $[0,1]^n$, by Theorem 3.1.1 and Proposition 3.2.4, we have a representation of functions of $\boldsymbol{F}^n_{\mathbb{P}}$.

THEOREM 3.2.5. *The free product algebra $\boldsymbol{F}^n_{\mathbb{P}}$ is isomorphic to the algebra of functions $f \colon [0,1]^n \to [0,1]$ such that for every $\epsilon \in \{1,2\}^n$, with $\#\epsilon = k$, the restriction of f to G_ϵ either is equal to 0 or is a k-variable piecewise monomial function.*

Let $\boldsymbol{G} = \langle G, +, -, \le, 0\rangle$ be any Abelian lattice ordered group (ℓ-group) and let $\bot$ be an element not belonging to $\boldsymbol{G}$. Let G^- be the negative cone of $\boldsymbol{G}$. On the set $G^- \cup \{\bot\}$ we define the operations $\odot$ and $\to$ as follows:

$$x \odot y = \begin{cases} x + y & \text{if } x, y \in G^-, \\ \bot & \text{otherwise;} \end{cases}$$

$$x \to y = \begin{cases} 0 \wedge (y - x) & \text{if } x, y \in G^-, \\ 0 & \text{if } x = \bot, \\ \bot & \text{if } x \in G^- \text{ and } y = \bot. \end{cases}$$

Then $\langle G^- \cup \{\bot\}, \odot, \to, \bot\rangle$ is a product algebra that will be denoted by $\mathfrak{B}(\boldsymbol{G})$. $\mathfrak{B}$ can be extended to a functor from the category of ℓ-groups to the category of product algebras.

The free cancellative hoop $\boldsymbol{F}^n_{\mathbb{CH}}$ is a negatively ordered cancellative monoid, hence there is a unique (up to isomorphism) ℓ-group $\boldsymbol{G}_n$ such that $\boldsymbol{G}_n^- \cong \boldsymbol{F}^n_{\mathbb{CH}}$. Let $\boldsymbol{G}_0 = \{0\}$.

THEOREM 3.2.6. *For each integer $n \ge 1$, $\boldsymbol{F}^n_{\mathbb{P}} \cong \prod_{k=0}^n \mathfrak{B}(\boldsymbol{G}_k)^{\binom{n}{k}}$.*

Proof. For any k there are $\binom{n}{k}$ atoms p_ϵ in $\boldsymbol{F}^n_{\mathbb{B}}$ such that $\#\epsilon = k$. The claim then follows by Theorem 3.2.5, noticing that $\mathfrak{B}(\boldsymbol{G}_k) \cong [0, p_\epsilon]$. □

4 Gödel algebras and related structures

In this section we deal with Gödel algebras. We first introduce a functional representation for free finitely generated Gödel algebras, built up, as we have seen in the previous sections for other varieties, from the generic Gödel algebra $[0,1]$ using Lemma 1.0.12. Then we expound representations of finite Gödel algebras via spectral duality. This last approach is very fruitful for describing free algebras in locally finite varieties. In particular we shall introduce a category of finite combinatorial objects which is dually equivalent to the category of finite Gödel algebras. Then we shall identify the object dual to the free singly generated Gödel algebra, and describe how to effectively compute products in the dual category. An application of Lemma 1.0.13 will provide us with concrete representations of all finitely generated free Gödel algebras.

We end this section by providing another instance of the spectral duality approach, applied to the category of Nilpotent Minimum algebras, the algebraic semantics of Nilpotent Minimum logic. Both Gödel and of Nilpotent Minimum logics are schematic extensions of the logic WNM, which in turns is obtained extending MTL with the axiom $\neg(\varphi \mathbin{\&} \psi) \vee ((\varphi \wedge \psi) \to (\varphi \mathbin{\&} \psi))$. In particular, Gödel logic is given by extending WNM with idempotency of conjunction: $\varphi \to (\varphi \mathbin{\&} \varphi)$, while Nilpotent Minimum logic is WNM plus involutiveness of negation: $\neg\neg\varphi \to \varphi$. These two logics are also extensions of important non-classical logics by way of the same axiom scheme: as is well known, Gödel logic is Intuitionistic logic plus prelinearity: $(\varphi \to \psi) \vee (\psi \to \varphi)$; Nilpotent Minimum logic is obtained by adding prelinearity to Constructive Logic with Strong Negation. On the algebraic side, Gödel algebras are exactly the prelinear Heyting algebras, while Nilpotent Minimum algebras, or NM-algebras, are the prelinear Nelson algebras.

4.1 Functional representation of Gödel algebras

Recall that the variety $\mathbb{G}$ of Gödel algebras is generated by any Gödel chain with infinitely many elements. In particular, the Gödel chain $[0,1]$ is generic for $\mathbb{G}$ (a chain can be equipped with a unique structure of Gödel algebra).

Let $\sim$ be the binary relation on $[0,1]^n$ defined in the following way: given two n-tuples $\mathbf{u} = \langle u_1, \ldots, u_n\rangle, \mathbf{v} = \langle v_1, \ldots, v_n\rangle \in [0,1]^n$ we set $\mathbf{u} \sim \mathbf{v}$ if and only if there is a permutation σ of $\{1, \ldots, n\}$ and a map $\prec_i$: $\{0, \ldots, n\} \to \{<, =\}$ such that

$$0 \prec_0 u_{\sigma(1)} \prec_1 \cdots \prec_{n-1} u_{\sigma(n)} \prec_n 1 \;\text{ iff }\; 0 \prec_0 v_{\sigma(1)} \prec_1 \cdots \prec_{n-1} v_{\sigma(n)} \prec_n 1\,.$$

The relation $\sim$ is an equivalence relation, we denote by $[\mathbf{u}]$ the equivalence class of $\mathbf{u}$. The quotient set $[0,1]^n/\sim$ is hence a partition of $[0,1]^n$.

A routine proof by structural induction shows that

LEMMA 4.1.1. *If φ is a Gödel formula with n variables, then it holds $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{u}) \in \{0, u_1, \ldots, u_n, 1\}$ for every $\mathbf{u} = \langle u_1, \ldots, u_n\rangle \in [0,1]^n$. Further, let $\mathbf{u} \sim \mathbf{v}$. If $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{u}) \in \{0,1\}$, then $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{u}) = \varphi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{v})$, while if $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{u}) = u_i$ then $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{v}) = v_i$.*

Hence the restriction of $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}$ to each equivalence class $[\mathbf{u}]$ is either the constant function 0 or 1, or it coincides with a projection function.

With each class $[\mathbf{u}]$, where

$$0 \prec_0 u_{\sigma(1)} \prec_1 \cdots \prec_{n-1} u_{\sigma(n)} \prec_n 1 \,,$$

we associate a unique *ordered partition* $\rho_{\mathbf{u}} = Q_1 < \cdots < Q_h$ (i.e., a partition equipped with a total order among its blocks) of the set $\{\bot, x_1, \ldots, x_n, \top\}$ in the following way:

- $\bot \in Q_1$; $\top \in Q_h$;
- if $\prec_i$ is $=$, then $x_{\sigma(i)}$ and $x_{\sigma(i+1)}$ belong to the same Q_j;
- if $\prec_i$ is $<$, and $x_{\sigma(i)} \in Q_j$ then $x_{\sigma(i+1)} \in Q_{j+1}$.

Note that if $[\mathbf{u}] \neq [\mathbf{v}]$ then $\rho_{\mathbf{u}} \neq \rho_{\mathbf{v}}$.

By a *Gödel n-partition* we mean an ordered partition $\rho = Q_1 < \cdots < Q_h$ of $\{\bot, x_1, \ldots, x_n, \top\}$ such that $h > 1$, $\bot \in Q_1$, and $\top \in Q_h$. We can hence establish a bijection between Gödel n-partitions and equivalence classes $[\mathbf{u}] \in [0,1]^n/\sim$. If $\rho = \rho_{\mathbf{u}}$ is a Gödel n-partition, we denote by $[\rho]$ the associated equivalence class $[\mathbf{u}]$.

We can define a partial order on Gödel n-partitions, setting $\rho_1 \prec \rho_2$ if $\rho_1 = Q_1 < \cdots < Q_{h-1} < Q_h$ and $\rho_2 = Q_1 < \cdots < Q_{h-1} < R_h < \cdots < R_k$ with $R_h \neq Q_h$. The relation $\prec$, which is irreflexive and transitive, induces a partial order $\preceq$ on the set of all Gödel n-partitions.

Consider a Gödel n-partition $\rho = Q_1 < \cdots < Q_h$. We define the *$G_n$-region* of $[0,1]^n$ associated with ρ as

$$C_\rho = \bigcup_{\rho' \preceq \rho} [\rho'].$$

We denote by P_n the set of Gödel n-partitions and by M_n the set of all Gödel n-partitions that are maximal with respect to the partial order $\preceq$.

EXAMPLE 4.1.2. Consider the Gödel 2-partitions:

- $\pi = \{\bot\} < \{x_1\} < \{x_2\} < \{\top\}$;
- $\sigma = \{\bot\} < \{x_1\} < \{x_2, \top\}$;
- $\tau = \{\bot\} < \{x_1, x_2, \top\}$.

Then $\tau \prec \sigma \prec \pi$ and:

- $[\pi] = \{\langle t_1, t_2\rangle \mid 0 < t_1 < t_2 < 1\}$;
- $[\sigma] = \{\langle t_1, 1\rangle \mid 0 < t_1 < 1\}$;
- $[\tau] = \{\langle 1, 1\rangle\}$.

The corresponding G_2-regions of $[0,1]^2$ are:

- $C_\pi = \{\langle t_1, t_2\rangle \mid 0 < t_1 < t_2 \leq 1\} \cup \{(1,1)\} = [\pi] \cup [\sigma] \cup [\tau]$;
- $C_\sigma = \{\langle t_1, 1\rangle \mid 0 < t_1 \leq 1\} = [\sigma] \cup [\tau]$;
- $C_\tau = \{\langle 1, 1\rangle\} = [\tau]$.

We next discuss minterms for Gödel logic. In the Boolean case, a minterm is a Boolean function that takes value 1 at a prescribed point of $\{0,1\}^n$, and has minimum possible value—namely, zero—at each other point. For Gödel functions, G_n-regions play the rôle of Boolean points, although—as detailed below—the minimum possible value outside of such a region need not be zero.

In order to introduce the next results we need some more definitions. By a *prefix* of an ordered partition $Q_1 < \cdots < Q_k$ we mean a possibly empty (i.e., with no blocks) ordered partition $Q_1 < \cdots < Q_h$ of $Q_1 \cup \cdots \cup Q_h$ for $h \in \{1, \ldots, k\}$. The *longest common prefix* of two distinct ordered partitions $\rho_1 = U_1 < \cdots < U_u$ and $\rho_2 = V_1 < \cdots < V_v$ is an ordered partition $W_1 < \cdots < W_w$ such that for every $i \in \{1, \ldots, w\}$, $U_i = V_i = W_i$ and $U_{w+1} \neq V_{w+1}$ (note that $0 \leq w < \min\{u, v\}$). If, on the other hand, $\rho_1 = \rho_2$, then their longest common prefix is ρ_1 itself.

LEMMA 4.1.3. *Let $W_1 < \cdots < W_w$ be the longest common prefix of $\rho_1 = U_1 < \cdots < U_u$ and $\rho_2 = V_1 < \cdots < V_v$. Then for any formula φ, if $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}$ coincides with a function $g \in W_i$ over $[\rho_1]$ then it coincides with $g \in W_i$ even over $[\rho_2]$.*

Proof. The proof proceeds by structural induction on φ. □

LEMMA 4.1.4. *If $\rho \preceq \tau$ and $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}$ is equal to 1 over $[\tau]$ then $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}$ is equal to 1 over $[\rho]$.*

Proof. Indeed displaying $\rho = Q_1 < \cdots < Q_{h-1} < Q_h$ and $\tau = Q_1 < \cdots < Q_{h-1} < R_h < \cdots < R_k$, if $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}$ were not equal to 1 over $[\rho]$ then $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}} \in Q_i$ with $i < h$ hence even on $[\tau]$ it would be different from 1. □

LEMMA 4.1.5 (1-set property). *If φ and ψ are such that $\{\mathbf{x} \in [0,1]^n \mid \varphi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{x}) = 1\} = \{\mathbf{x} \in [0,1]^n \mid \psi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{x}) = 1\}$ then $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}} = \psi^{\boldsymbol{F}^n_{\mathbb{G}}}$, i.e., functions associated with Gödel formulas are determined by their 1-set.*

Proof. First we prove two auxiliary facts.

1) If there is a Gödel partition ρ such that $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}} = 0$ over $[\rho]$ then, by Lemma 4.1.3, $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}} = 0$ over the uniquely determined minimal partition $\rho_m \preceq \rho$.

2) Let $[\rho] \subseteq \{\mathbf{x} \in [0,1]^n \mid \varphi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{x}) = 1\}$ and display $\rho = U_1 < \cdots < U_u$. Let $\rho \prec \tau$ with $\tau = U_1 < \cdots < U_{n-1} < V < U_n \setminus V$ where $V \subseteq U_n \setminus \{\top\}$ (hence τ covers ρ with respect to the order $\preceq$). If $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}} < 1$ over $[\tau]$ then, by Lemma 4.1.3, $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}} \in V$ over every τ' with $\tau \preceq \tau'$.

Now, if $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}} = 0$ over $[\rho]$, by 1) we can assume ρ is minimal, and then, by Lemma 4.1.4, $\psi^{\boldsymbol{F}^n_{\mathbb{G}}} \neq 1$ over $[\rho]$, and by minimality of ρ, $\psi^{\boldsymbol{F}^n_{\mathbb{G}}} = 0$ over $[\rho]$, too.

There remains to consider those partitions τ such that $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}$ is not constant over $[\tau]$. Then there exists a partition ρ covered by τ such that $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}$ is constantly 1 over $[\rho]$. By our current assumptions $\psi^{\boldsymbol{F}^n_{\mathbb{G}}}$ is constantly 1 over $[\rho]$, too. By 2) $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}$ and $\psi^{\boldsymbol{F}^n_{\mathbb{G}}}$ coincide over $[\pi]$ for each $\pi \succ \rho$.

Hence $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}} = \psi^{\boldsymbol{F}^n_{\mathbb{G}}}$ over every $\pi \in P_n$, i.e. they are the same function over $[0,1]^n$. □

Let us define the following derived connective:

$$x \lhd y = (y \to x) \to y\,.$$

Note that when interpreted in $[0,1]$ we have $x \lhd y = 1$ if and only if $x < y$ or $x = y = 1$ and $x \lhd y = y$ otherwise.

LEMMA 4.1.6. *Let ρ be a Gödel n-partition such that $[\rho] = \{\mathbf{u} \mid \bot \prec_0 u_{\sigma(1)} \prec_1 \cdots \prec_{n-1} u_{\sigma(n)} \prec_n \top\}$. For the sake of conciseness, let $x_{\sigma(0)} = \bot$ and $x_{\sigma(n+1)} = \top$. Set*

$$\chi_\rho = \bigwedge_{i=0}^{n} \delta_i,$$

where

$$\delta_i = \begin{cases} x_{\sigma(i)} \leftrightarrow x_{\sigma(i+1)} & \text{iff } \prec_i \text{ is } =, \\ x_{\sigma(i)} \lhd x_{\sigma(i+1)} & \text{iff } \prec_i \text{ is } <, \end{cases}$$

for every $i \in \{0, \ldots, n\}$.

Then $\chi_\rho^{\boldsymbol{F}^n_{\mathbb{G}}}$ is the smallest term function which coincides with 1 over $[\rho]$ and further it is the unique term function evaluating to 1 exactly over C_ρ.

Proof. From the definition of $\leftrightarrow$ and $\lhd$ it is immediate to check that $\chi_\rho^{\boldsymbol{F}^n_{\mathbb{G}}}$ is constantly 1 over $[\rho]$. By Lemma 4.1.4 it is constantly 1 over C_ρ, too. In order to prove that $\chi_\rho^{\boldsymbol{F}^n_{\mathbb{G}}}$ is the smallest function with such property, pick any $\tau \in P_n$ such that $\tau \not\preceq \rho$. Display ρ as $Q_1 < Q_2 < \cdots < Q_h$. Let π be the longest common prefix of τ and ρ. Notice that if $\rho \prec \tau$ then $\pi = Q_1 < Q_2 < \cdots < Q_{h-1}$, as the hth block of ρ is the union of all blocks of τ of index greater than $h-1$. In this case one can check that one of the conjuncts of χ_ρ evaluates to a function $x_{\sigma(i)} \neq \top$ belonging to the hth block of τ. By Lemma 4.1.3, this is the smallest value a Gödel term function which is 1 over $[\rho]$ can take over $[\tau]$. If ρ and τ are incomparable, then there is $w < h-1$ such that the longest common prefix of ρ and τ has the form $Q_1 < Q_2 < \cdots < Q_w$. In this case one of the conjuncts of χ_ρ evaluates to a function $x_{\sigma(i)} \neq \top$ belonging to the wth block of τ. Again by Lemma 4.1.3, this is the smallest value a Gödel term function which is 1 over $[\rho]$ can take over $[\tau]$. The uniqueness requirement follows at once by Lemma 4.1.5. □

The formula χ_π is called the (*Gödel*) *minterm* associated with π. Just like Boolean minterms afford a disjunctive normal form theorem, one can canonically express each Gödel function as a disjunction of Gödel minterms. Namely,

LEMMA 4.1.7. *Let φ be a term in n variables. Then for every $\pi \in M_n$ there exists a unique $\rho(\varphi,\pi) \in P_n$ with $\rho(\varphi,\pi) \preceq \pi$ such that $\varphi^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{x}) = \chi_{\rho(\varphi,\pi)}^{\boldsymbol{F}^n_{\mathbb{G}}}(\mathbf{x})$, for all $\mathbf{x} \in C_\pi$. Moreover, φ can be equivalently written as*

$$\varphi \equiv_{\mathbb{G}} \bigvee_{\pi \in M_n} \chi_{\rho(\varphi,\pi)}\ .$$

Proof. Immediate, from Lemma 4.1.6. □

DEFINITION 4.1.8. *A function $f\colon [0,1]^n \to [0,1]$ is a* Gödel function *(of n variables) if for every G_n-region C_ρ there exists a function $g \in \{\bot, x_1, \ldots, x_n, \top\}$ such that $f(\mathbf{t}) = g(\mathbf{t})$ for each $\mathbf{t} = \langle t_1, \ldots, t_n \rangle \in C_\rho$. Let $\boldsymbol{G}_n$ be the Gödel algebra of Gödel functions equipped with pointwise operations.*

THEOREM 4.1.9. $\boldsymbol{F}^n_{\mathbb{G}} \cong \boldsymbol{G}_n$.

Proof. Immediate, from Lemma 4.1.7. □

In the next section we shall see that the poset P_n of all Gödel n-partitions can be seen as the object corresponding to the free n-generated Gödel algebra in a category that is dually equivalent to the category of finite Gödel algebras and their homomorphisms. As a matter of fact P_n is order-isomorphic to the *prime spectrum* of $\boldsymbol{F}^n_{\mathbb{G}}$, that is, the set of all the prime filters of this free algebra, ordered by reverse inclusion. The prime spectra of finite Gödel algebras, together with suitable defined maps between them, will constitute the dual category we shall deal with. It is easy to see that a Gödel n-partition encodes both a prime filter of $\boldsymbol{F}^n_{\mathbb{G}}$, and the chain obtained by taking the quotient of $\boldsymbol{F}^n_{\mathbb{G}}$ with respect to the congruence canonically associated with the filter. As a matter of fact, using Theorem 4.1.9, an element ρ of M_n corresponds to the minimal prime filter constituted by all and only those functions constantly 1 over C_ρ. The chain $\boldsymbol{C}_\rho$ obtained from $\boldsymbol{F}^n_{\mathbb{G}}$ by restricting the domain of the Gödel functions to C_ρ is isomorphic to the quotient of $\boldsymbol{F}^n_{\mathbb{G}}$ by the described minimal prime filter. An element $\tau \prec \rho$ of P_n corresponds to the filter constituted by all and only those functions which coincides over C_ρ with one of the functions in the maximum block of τ. Roots of P_n then correspond to maximal filters, and quotienting the free algebra by each one of them gives the two-element Boolean chain.

4.2 Finite spectral duality for Gödel algebras

We recall that two categories $\mathscr{C}$ and $\mathscr{D}$ are *equivalent* if there exist two functors $F\colon \mathscr{C} \to \mathscr{D}$ and $G\colon \mathscr{D} \to \mathscr{C}$ whose compositions FG and GF are naturally isomorphic to the identity functors. That is, for each object C in $\mathscr{C}$ and D in $\mathscr{D}$ there are isomorphisms $\iota_C\colon C \to GF(C)$ and $\kappa_D\colon D \to FG(D)$ such that the following diagrams

$$\begin{array}{ccc} C_1 & \xrightarrow{f} & C_2 \\ \downarrow{\scriptstyle \iota_{C_1}} & & \downarrow{\scriptstyle \iota_{C_2}} \\ GF(C_1) & \xrightarrow{GF(f)} & GF(C_2) \end{array} \qquad \begin{array}{ccc} D_1 & \xrightarrow{g} & D_2 \\ \downarrow{\scriptstyle \kappa_{D_1}} & & \downarrow{\scriptstyle \kappa_{D_2}} \\ FG(D_1) & \xrightarrow{FG(g)} & FG(D_2) \end{array}$$

commute for each $f\colon C_1 \to C_2$ in $\mathscr{C}$ and $g\colon D_1 \to D_2$ in $\mathscr{D}$.

Equivalently, $\mathscr{C}$ and $\mathscr{D}$ are equivalent if there exists a functor

$$F\colon \mathscr{C} \to \mathscr{D}$$

such that F is *full and faithful*, that is

$$\mathrm{Hom}(C_1, C_2) \cong \mathrm{Hom}(F(C_1), F(C_2))$$

for all objects C_1, C_2 in $\mathscr{C}$, and F is *essentially surjective*, that is, for each object D in $\mathscr{D}$ there exists an object C in $\mathscr{C}$ such that

$$D \cong F(C)\,.$$

The *opposite* of a category $\mathscr{D}$ is the category $\mathscr{D}^{\mathrm{op}}$ obtained by reversing all arrows in $\mathscr{D}$. Two categories $\mathscr{C}$ and $\mathscr{D}$ are *dually equivalent* if $\mathscr{C} \equiv \mathscr{D}^{\mathrm{op}}$. In this case, the full and faithful functor $F\colon \mathscr{C} \to \mathscr{D}$ implementing the duality is obviously contravariant, that is

$$h \in \mathrm{Hom}(C_1, C_2) \mapsto F(h) \in \mathrm{Hom}(F(C_2), F(C_1))\,.$$

In this subsection we shall introduce a category that is dual to the category G of finite Gödel algebras and their homomorphisms. Since $\mathbb{G}$ is a locally finite variety, G includes all finitely generated free Gödel algebras.

We shall then show how to compute products in the dual category, and characterise the object dual to the Gödel algebra $\boldsymbol{F}^1_{\mathbb{G}}$. We shall then use Lemma 1.0.13 to obtain a representation of finitely generated free Gödel algebras via finite duality.

Throughout, *poset* is short for partially ordered set. If $\langle P, \leq\rangle$ is a poset and $S \subseteq P$, the *downset* generated by S is

$$\downarrow S = \{p \in P \mid p \leq s \text{ for some } s \in S\}\,.$$

(When S is a singleton $\{s\}$, we shall write $\downarrow s$ for $\downarrow \{s\}$.) A subposet $S \subseteq P$ is a *downset* if $\downarrow S = S$. *Upsets* and $\uparrow S$ are defined analogously. The one-element poset is denoted $\mathbf{1}$. A *forest* is a poset F such that $\downarrow x$ is totally ordered for any $x \in F$. If P is a poset, by $P_\perp$ we denote the poset obtained by adding a new bottom element $\perp$ to P. A *tree* is a forest with a minimum element, hence for each forest F, $F_\perp$ is a tree.

Gödel algebras are Heyting algebras satisfying the prelinearity condition. From any finite poset P one reconstructs a Heyting algebra, as follows. Let $\mathbf{Sub}P$ be the family of all downsets of P. When partially ordered by inclusion, $\mathbf{Sub}P$ is a finite distributive lattice, and thus carries a unique Heyting implication adjoint to the lattice meet operation via *residuation*. Explicitly, if $\boldsymbol{L}$ is a finite distributive lattice, then its Heyting implication is given by

$$x \to y = \bigvee\{z \in \boldsymbol{L} \mid z \wedge x \leq y\}$$

for all $x, y \in L$. Accordingly, we regard $\mathbf{Sub}P$ as a Heyting algebra.

Conversely, one can obtain a finite poset from any finite Heyting algebra $\boldsymbol{H}$. Let $\mathrm{Spec}\boldsymbol{H}$ denote the poset of prime filters of $\boldsymbol{H}$, ordered by reverse inclusion. Equivalently, one can think of $\mathrm{Spec}\boldsymbol{H}$ as the poset of minterms (i.e. join-irreducible elements) of $\boldsymbol{H}$, with the order they inherit from $\boldsymbol{H}$. (Recall that a *filter* of $\boldsymbol{H}$ is an upset of $\boldsymbol{H}$ closed under meets; it is *prime* if it does not contain the bottom element of $\boldsymbol{H}$, and contains either y or z whenever it contains $y \vee z$. We further recall that $x \in \boldsymbol{H}$ is *join-irreducible* if it is not the bottom element of $\boldsymbol{H}$, and whenever $x = y \vee z$ for $y, z \in \boldsymbol{H}$, then either $x = y$ or $x = z$.)

The constructions of the two preceding paragraphs are inverse to each other, in the sense that for any finite Heyting algebra $\boldsymbol{H}$ one has an isomorphism of Heyting algebras

$$\mathbf{Sub}\mathrm{Spec}\boldsymbol{H} \cong \boldsymbol{H}. \tag{6}$$

Let us now restrict attention to finite *Gödel* algebras. Horn proved [50, 2.4] that a Heyting algebra $\boldsymbol{H}$ is a Gödel algebra if and only if its prime filters are a forest under reverse inclusion, i.e. if $\mathrm{Spec}\boldsymbol{H}$ is a forest. Note that a downset of a forest F is itself a forest, and we shall call it a *subforest* of F. Summing up, a combinatorial representation of finite Gödel algebras is obtained as follows. Let $\boldsymbol{G}$ be a finite Gödel algebra. Then $\boldsymbol{G}$ is canonically isomorphic to the Gödel algebra of all subforests of a finite forest $\mathrm{Spec}\boldsymbol{G}$ as in (6), where $\mathrm{Spec}\boldsymbol{G}$ is unique to within a poset isomorphism. Under the representation (6), each element $g \in \boldsymbol{G}$ corresponds to a uniquely determined subforest of $\mathrm{Spec}\boldsymbol{G}$ (and conversely). Such a correspondence can be extended to a duality of categories.

We now give a detailed construction of this dual equivalence. Given two forests F and G, an order preserving map $f\colon F \to G$ is *open* if $x' \leq f(x)$ in G implies that there exists $y \leq x$ in F such that $f(y) = x'$. Open maps carry downsets to downsets.

Let G be the category of finite Gödel algebras where arrows are Gödel homomorphisms and let F be the category whose objects are forests and whose arrows are open maps.

We shall define the functors $\mathbf{Sub}\colon \mathsf{F} \to \mathsf{G}$ and $\mathrm{Spec}\colon \mathsf{G} \to \mathsf{F}$ implementing the dual equivalence.

Let $\mathrm{Sub}\, F$ be the set of all subforests (i.e., downsets) of a forest $F \in \mathsf{F}$. We shall equip $\mathrm{Sub}\, F$ with the structure of a Gödel algebra as follows. For all subforests H, K of F, let

$$H \to K = F \setminus {\uparrow}(H \setminus K)\,. \tag{7}$$

Then, the structure

$$\mathbf{Sub}F = \langle \mathrm{Sub}\, F, \cap, \to, \emptyset \rangle$$

is a Gödel algebra. To prove this, we only need to show that (7) defines the residuum of subforest intersection, that is:

$$H \to K = \bigvee \{L \in \mathrm{Sub}\, F \mid L \cap H \leq K\}\,. \tag{8}$$

Pick $x \in {\uparrow}(H \setminus K)$. Then there exists $y \leq x$ such that $y \in H \setminus K$. Clearly, no such x belongs to a subforest L such that $L \cap H \subseteq K$. An easy check now shows that $F \setminus {\uparrow}(H \setminus K)$ is the greatest subforest L of F such that $L \cap H \subseteq K$, thus settling (8).

To turn $\mathbf{Sub}$ into a functor from F to G we specify how it carries morphisms to the other side. For each order-preserving open map $f\colon F \to G$ we let $\mathbf{Sub}f\colon \mathbf{Sub}G \to \mathbf{Sub}F$ be defined by taking preimages. For any $a \in \mathbf{Sub}G$:

$$(\mathbf{Sub}f)(a) = f^{-1}(a)\,.$$

Let now $\mathrm{Spec}\boldsymbol{A}$ be the set of all prime filters of the Gödel algebra $\boldsymbol{A} \in \mathsf{G}$, ordered by reverse inclusion. For each homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ in G, we let $\mathrm{Spec}h\colon \mathrm{Spec}\boldsymbol{B} \to \mathrm{Spec}\boldsymbol{A}$ be defined again by taking preimages. For any $\mathfrak{p} \in \mathrm{Spec}\boldsymbol{B}$:

$$(\mathrm{Spec}h)(\mathfrak{p}) = h^{-1}(\mathfrak{p})\,.$$

Then, $\mathrm{Spec}\colon \mathsf{G} \to \mathsf{F}$ is a functor from the category of finite Gödel algebras to the category of finite forests.

Note that $\mathrm{Spec}\boldsymbol{A}$ is order-isomorphic to the poset of join-irreducible elements of $\boldsymbol{A}$, with the inherited order. As a matter of fact, each prime filter $\mathfrak{p}$ of $\boldsymbol{A}$ is principal and is generated by a unique join-irreducible element, which we denote $j_{\mathfrak{p}}$.

Once we have defined the functors $\mathbf{Sub}$ and Spec, we proceed in showing they realize the desired dual equivalence. We first show that for each Gödel algebra $\boldsymbol{A} \in \mathsf{G}$ the following holds.

$$\boldsymbol{A} \cong \mathbf{Sub}\,\mathrm{Spec}\,\boldsymbol{A}\,. \tag{9}$$

Let $\varphi_{\boldsymbol{A}}$ be the map $x \in \boldsymbol{A} \mapsto \{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\,\boldsymbol{A},\, j_{\mathfrak{p}} \leq x\}$. Clearly, $\{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\,\boldsymbol{A},$ $j_{\mathfrak{p}} \leq x\}$ is a downset of $\mathrm{Spec}\,\boldsymbol{A}$, hence $\varphi_{\boldsymbol{A}}$ maps to $\mathbf{Sub}\,\mathrm{Spec}\,\boldsymbol{A}$. Observe that if x, y are two distinct elements of $\boldsymbol{A}$ then there exists a join irreducible element $j \in \boldsymbol{A}$ such that, either $j \leq x$ and $j \not\leq y$, or $j \not\leq x$ and $j \leq y$; hence $\varphi_{\boldsymbol{A}}(x) \neq \varphi_{\boldsymbol{A}}(y)$, and the injectivity of $\varphi_{\boldsymbol{A}}$ is proved. Surjectivity of $\varphi_{\boldsymbol{A}}$ follows by noting that $\varphi_{\boldsymbol{A}}(\bigvee_{\mathfrak{p}\in S} j_{\mathfrak{p}}) = S$ for each downset S of $\mathrm{Spec}\,\boldsymbol{A}$. It remains to show that $\varphi_{\boldsymbol{A}}$ is a homomorphism of Gödel algebras. Trivially, $\varphi_{\boldsymbol{A}}(\bot_{\boldsymbol{A}}) = \emptyset$. For what concerns conjunction, $\varphi_{\boldsymbol{A}}(x \wedge y) = \{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\,\boldsymbol{A},\, j_{\mathfrak{p}} \leq x \wedge y\} = \{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\,\boldsymbol{A},\, j_{\mathfrak{p}} \leq x\} \cap \{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\,\boldsymbol{A},\, j_{\mathfrak{p}} \leq y\} = \varphi_{\boldsymbol{A}}(x) \cap \varphi_{\boldsymbol{A}}(y)$. To show that $\varphi_{\boldsymbol{A}}(x \to y) = \mathrm{Spec}\,\boldsymbol{A} \setminus \uparrow(\varphi_{\boldsymbol{A}}(x) \setminus \varphi_{\boldsymbol{A}}(y))$ we first note that $\varphi_{\boldsymbol{A}}(x \to y) = \{\mathfrak{p} \mid j_{\mathfrak{p}} \leq x \to y\} = \{\mathfrak{p} \mid x \wedge j_{\mathfrak{p}} \leq y\}$, by residuation. Pick $\mathfrak{q} \in \mathrm{Spec}\,\boldsymbol{A}$. Then $x \wedge j_{\mathfrak{q}} \leq y$ iff $x \wedge j_{\mathfrak{q}} \leq y \wedge j_{\mathfrak{q}}$. Observe that for each $z \in \boldsymbol{A}$ and each $\mathfrak{r} \in \mathrm{Spec}\,\boldsymbol{A}$, either $z \wedge j_{\mathfrak{r}} = \bot_{\boldsymbol{A}}$, or $z \wedge j_{\mathfrak{r}} = j_{\mathfrak{s}}$ for some $\mathfrak{s} \in \mathrm{Spec}\,\boldsymbol{A}$. A direct computation now shows $x \wedge j_{\mathfrak{q}} > y \wedge j_{\mathfrak{q}}$ iff there is $\mathfrak{t} \in \mathrm{Spec}\,\boldsymbol{A}$ such that $j_{\mathfrak{t}} \leq j_{\mathfrak{q}}$ and $\mathfrak{t} \in \varphi_{\boldsymbol{A}}(x)$ but $\mathfrak{t}_{\boldsymbol{A}} \notin \varphi(y)$, that is, $x \wedge j_{\mathfrak{q}} > y \wedge j_{\mathfrak{q}}$ iff $j_{\mathfrak{q}} \in \uparrow(\varphi_{\boldsymbol{A}}(x) \setminus \varphi_{\boldsymbol{A}}(y))$, and we are done. Thus, $\varphi_{\boldsymbol{A}} \colon \boldsymbol{A} \to \mathbf{Sub}\,\mathrm{Spec}\,\boldsymbol{A}$ is the desired isomorphism.

We next show that for each finite forest $F \in \mathsf{F}$ the following holds.

$$F \cong \mathrm{Spec}\,\mathbf{Sub}\,F\,. \tag{10}$$

Notice that for each $x \in F$, its downset $\downarrow x$ is a join irreducible element of $\mathbf{Sub}\,F$. Let then ψ_F be the map $x \in F \mapsto \langle\downarrow x\rangle$, where $\langle\downarrow x\rangle$ is the prime filter of $\mathbf{Sub}\,F$ generated by $\downarrow x$. Trivially, ψ_F is an order-preserving bijection. It remains to show that ψ_F is open. Pick $\mathfrak{p} \leq \langle\downarrow x\rangle$ in $\mathrm{Spec}\,\mathbf{Sub}\,F$. Let $y = \psi_F^{-1}(\mathfrak{p})$. Now, $\langle\downarrow y\rangle \supseteq \langle\downarrow x\rangle$ iff $\downarrow y \subseteq \downarrow x$ iff $y \leq x$, as it has to be shown. We conclude that ψ_F is an isomorphism in F.

To complete the picture we have to show that the functors $\mathbf{Sub}$ and Spec well behave on morphisms. We first show that for each $h \colon \boldsymbol{A} \to \boldsymbol{B}$ it holds that

$$\varphi_{\boldsymbol{B}} \circ h = \mathbf{Sub}\mathrm{Spec}h \circ \varphi_{\boldsymbol{A}}\,.$$

Pick $x \in \boldsymbol{A}$, then $\varphi_{\boldsymbol{B}}(h(x)) = \{\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\boldsymbol{B}, j_{\mathfrak{q}} \leq h(x)\}$. Now, $\varphi_{\boldsymbol{A}}(x) = \{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\boldsymbol{A}, j_{\mathfrak{p}} \leq x\}$. Hence, we have to show

$$(\mathbf{Sub}\mathrm{Spec}h)(\{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}\boldsymbol{A}, j_{\mathfrak{p}} \leq x\}) = \{\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Spec}\boldsymbol{B}, j_{\mathfrak{q}} \leq h(x)\}\,. \tag{11}$$

By definition of the functors $\mathbf{Sub}$ and Spec, the left hand side of (11) reduces to $\{h(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}\boldsymbol{A}, j_{\mathfrak{p}} \leq x\}$. Since $x = \bigvee_{j_{\mathfrak{p}}\leq x} j_{\mathfrak{p}}$ and $h(x) = \bigvee_{j_{\mathfrak{q}}\leq h(x)} j_{\mathfrak{q}}$, the identity (11) holds.

Lastly, we have to show that for each $f \colon F \to G$ it holds that

$$\psi_G \circ f = \mathrm{Spec}\mathbf{Sub}f \circ \psi_F\,.$$

Pick $x \in F$, then $\psi_G(f(x)) = \langle\downarrow f(x)\rangle$, while $\psi_F(x) = \langle\downarrow x\rangle$. It must hold that

$$(\mathrm{Spec}\mathbf{Sub}f)(\langle\downarrow x\rangle) = \langle\downarrow f(x)\rangle\,, \tag{12}$$

which is obvious from the definitions of Spec and $\mathbf{Sub}$.

We have thus shown the following.

THEOREM 4.2.1. *The categories* G *and* F *are dually equivalent through the functors* $\mathbf{Sub}\colon \mathsf{F} \to \mathsf{G}$ *and* $\mathrm{Spec}\colon \mathsf{G} \to \mathsf{F}$.

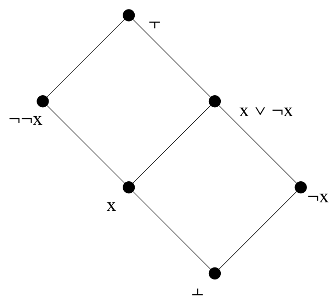
As an application of Theorem 4.2.1, for each integer $n \geq 0$ the free Gödel algebra $\boldsymbol{F}^n_{\mathbb{G}}$ can be characterised through a combinatorial description of the forest $\mathrm{Spec}\boldsymbol{F}^n_{\mathbb{G}}$. Throughout, we write

$$\mathscr{F}_n = \mathrm{Spec}\boldsymbol{F}^n_{\mathbb{G}}\,.$$

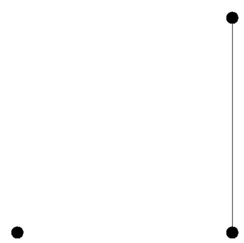
Let us denote $\boldsymbol{G}_i$ the unique (up to isomorphism) Gödel chain of cardinality $i+1$.

LEMMA 4.2.2. $\mathscr{F}_1 = \{a, b, c\}$ *where the order relation is the reflexive closure of* $b < c$.

Proof. By direct inspection, or by applying the functional representation theorem 4.1.9, the singly generated free Gödel algebra is $\boldsymbol{F}^1_{\mathbb{G}} \cong \boldsymbol{G}_1 \times \boldsymbol{G}_2$. Clearly, $\boldsymbol{G}_1 \times \boldsymbol{G}_2$ has exactly three join irreducible elements: denoted x the generator, these elements are $x, \neg x, \neg\neg x$, with $x < \neg\neg x$ as the only pair of elements in covering relation. □



(a) $\boldsymbol{F}^1_{\mathbb{G}}$.



(b) Spec $\boldsymbol{F}^1_{\mathbb{G}}$.

Figure 8. The Hasse diagram of $\boldsymbol{F}^1_{\mathbb{G}}$ and its prime spectrum.

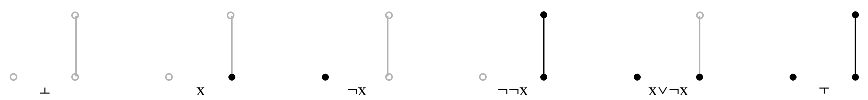


Figure 9. The subforests $F(g)$ of $\mathscr{F}_1$ corresponding to elements $g \in \boldsymbol{F}^1_{\mathbb{G}}$.

By Lemma 1.0.13 we can hence describe the free Gödel algebra through its dual forest. Since coproducts in G correspond to products in F, by Lemma 1.0.13 we have:

THEOREM 4.2.3.

$$\mathrm{Spec}\boldsymbol{F}^n_{\mathbb{G}} = \prod_{i=1}^{n} \mathscr{F}_1\,.$$

We hence need to describe how to compute finite products in the category F. Given two forests $\langle F, \leq_F \rangle$ and $\langle G, \leq_G \rangle$, we denote by $F + G$ the coproduct of F and G, that is the forest $\langle F \uplus G, \leq_{F+G} \rangle$ where $F \uplus G$ is the disjoint set union of F and G and $x \leq_{F+G} y$ if and only if either $x, y \in F$ and $x \leq_F y$ or $x, y \in G$ and $x \leq_G y$. It is easy to check that $\langle F \uplus G, \leq_{F+G} \rangle$ is indeed the coproduct of F and G.

LEMMA 4.2.4. *Let F, G, and H be three forests. Then:*

1. *if $|F| = 1$, then $F \times G \cong G$;*
2. *$(F + G) \times H \cong (F \times H) + (G \times H)$;*
3. *$F_\perp \times G_\perp \cong ((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp$.*

Proof.

1. It immediately follows from the fact that the singleton is the terminal object of F.
2. It follows by a standard argument, since coproducts in F are disjoint unions.
3. Let us display $F_\perp$ isomorphically as $(F_1 + F_2 + \cdots + F_u)_\perp$ for a uniquely determined family $\{F_i\}_{i=1}^u$ of trees. Analogously $G_\perp \cong (G_1 + G_2 + \cdots + G_v)_\perp$, for a uniquely determined family $\{G_j\}_{j=1}^v$ of trees. By applying 2., we can rewrite the forest $(F \times G_\perp) + (F \times G) + (F_\perp \times G)$ isomorphically as the following coproducts of trees

$$\sum_{i=1}^{u}(F_i \times G_\perp) \; + \; \sum_{i=1}^{u}\sum_{j=1}^{v}(F_i \times G_j) \; + \; \sum_{j=1}^{v}(F_\perp \times G_j)\,.$$

We denote r_0 and s_0 the roots of $F_\perp$ and $G_\perp$, respectively. Further, we denote t_0 the root of the tree $((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp$.

We next define the maps $\pi_{F_\perp} \colon ((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp \to F_\perp$ and $\pi_{G_\perp} \colon ((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp \to G_\perp$ as follows. First we set $\pi_{F_\perp}(t_0) = r_0$ and $\pi_{G_\perp}(t_0) = s_0$. Now, each $x \in (((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp) \setminus \{t_0\}$, must belong to a unique tree either of the form $F_i \times G_\perp$ or $F_i \times G_j$ or $F_\perp \times G_j$. Writing F_0 and G_0 for $F_\perp$ and $G_\perp$, respectively, then $x \in F_i \times G_j$, for a uniquely determined pair $\langle i, j \rangle$ with $i + j > 0$. We define $\pi_{F_\perp}(x) = \iota_{F_i}(\pi_{F_i}(x))$, where $\pi_{F_i} \colon F_i \times G_j \to F_i$ is the projection function, while $\iota_{F_i} \colon F_i \to F_\perp$ is the set-theoretic inclusion of the support of F_i into $F_\perp$. Similarly, we define $\pi_{G_\perp}(x) = \iota_{G_j}(\pi_{G_j}(x))$. It is clear that both $\pi_{F_\perp}$ and $\pi_{G_\perp}$ are well-defined morphisms of finite forests.

Now, let us take a forest H and two morphisms $f \colon H \to F_\perp$ and $g \colon H \to G_\perp$. We shall construct a map $\langle f, g \rangle \colon H \to ((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp$ such that $\pi_{F_\perp} \circ \langle f, g \rangle = f$ and $\pi_{G_\perp} \circ \langle f, g \rangle = g$. To accomplish this task we partition the set H as follows. Let $R_0 = f^{-1}(r_0)$ and $R_1 = H \setminus R_0$. Analogously, let $S_0 = g^{-1}(s_0)$ and $S_1 = H \setminus S_0$. Then the following is a partition of the set H:

$$\{R_0 \cap S_0, R_0 \cap S_1, R_1 \cap S_0, R_1 \cap S_1\}\,.$$

We refine this partition by further subdividing $R_1 \cap S_1$, as follows. Let R_2 be the set of all $x \in R_1 \cap S_1$ such that there is $y < x$ in H with $f(y) = r_0$ and $g(y) \neq s_0$. Notice that, since g is an order-preserving open map, if G_j is the unique tree in $\{G_h\}_{h=1}^v$ such that $g(x) \in G_j$, then also $g(y) \in G_j$. Similarly, let S_2 be the set of all $x \in R_1 \cap S_1$ such that there is $y < x$ in H with $f(y) \neq r_0$ and $g(y) = s_0$. Finally let $T_2 = (R_1 \cap S_1) \setminus (R_2 \cup S_2)$. Then

$$\{R_0 \cap S_0, R_0 \cap S_1, R_1 \cap S_0, R_2, S_2, T_2\}$$

is a partition of the set H. For each $x \in R_0 \cap S_0$ we let $\langle f, g\rangle(x) = t_0$; for each $x \in R_0 \cap S_1$ we note that there is a unique tree G_j such that $g(x) \in G_j$: we then let $\langle f, g\rangle(x)$ be the uniquely determined element t of $F_\perp \times G_j$ such that $\pi_{F_\perp}(t) = f(x) = r_0$ and $\pi_{G_j}(t) = g(x)$, where these two maps are the projections of the product $F_\perp \times G_j$; for each $x \in R_1 \cap S_0$ we reason analogously, letting $\langle f, g\rangle(x)$ be the uniquely determined element t of $F_i \times G_\perp$ such that $\pi_{F_i}(t) = f(x)$ and $\pi_{G_\perp}(t) = g(x) = s_0$. Some additional care is needed to deal with the remaining cases. For each $x \in R_2$ we note that there are uniquely determined trees F_i and G_j such that $f(x) \in F_i$ and $g(x) \in G_j$. Since $x \in R_2$, there is $y < x$ in H such that $f(y) = r_0$ and $g(y) \neq s_0$. As morphisms of finite forests must carry downsets to downsets, $\langle f, g\rangle(x)$ must belong to $F_\perp \times G_j$, but now, reasoning as in the preceding cases we let $\langle f, g\rangle(x)$ be the uniquely determined element t of $F_\perp \times G_j$ such that $\pi_{F_\perp}(t) = f(x)$ and $\pi_{G_j}(t) = g(x)$. For each $x \in S_2$ we reason analogously, letting $\langle f, g\rangle(x)$ be the uniquely determined element t of $F_i \times G_\perp$ such that $\pi_{F_i}(t) = f(x)$ and $\pi_{G_\perp}(t) = g(x)$. Finally, the last case $x \in T_2$ is again similarly dealt with, as we let $\langle f, g\rangle(x)$ be the uniquely determined element t of $F_i \times G_j$ such that $\pi_{F_i}(t) = f(x)$ and $\pi_{G_j}(t) = g(x)$. A simple check now shows $\pi_{F_\perp} \circ \langle f, g\rangle = f$ and $\pi_{G_\perp} \circ \langle f, g\rangle = g$ as desired.

There remains to show that $\langle f, g\rangle$ is the only map with this property. Let $h\colon H \to ((F \times G_\perp) + (F \times G) + (F_\perp \times G))_\perp$ be a morphism in F such that $\pi_{F_\perp} \circ h = f$ and $\pi_{G_\perp} \circ h = g$. Clearly h must coincide with $\langle f, g\rangle$ over $R_0 \cap S_0$. Notice that if $x \in R_0 \cap S_1$ then $f(x) = r_0$ and $g(x) \neq s_0$, whence $h(x)$ must belong to $F_\perp \times G_j$ for a uniquely determined G_j. But then $h(x) = \langle f, g\rangle(x)$ as otherwise $F_\perp \times G_j$, together with its projections, would not be the product in F of $F_\perp$ and G_j. A completely analogous argument shows that h must coincide with $\langle f, g\rangle$ over $R_1 \cap S_0$. If $x \in R_1 \cap S_1$ then there are uniquely determined F_i and G_j, $i \neq 0 \neq j$, such that $\pi_{F_i}(h(x)) = f(x)$ and $\pi_{G_j}(h(x)) = g(x)$, where projections are those of the product $F_i \times G_j$. If $x \in R_2 \subseteq R_1 \cap S_1$ then there is $y < x$ in H such that $\pi_{F_\perp}(h(y)) = r_0$ while $\pi_{G_\perp}(h(y)) \neq s_0$, that is $y \in R_0 \cap S_1$, and hence $h(y) \in F_\perp \times G_j$. Since h is order-preserving and open, x must belong to the isomorphic copy of F_i included as a set in $F_\perp$, but then $h(x) = \langle f, g\rangle(x)$ as otherwise $F_\perp \times G_j$, together with its projections, would not be the product in F of $F_\perp$ and G_j. Similar arguments hold for $x \in S_2$ or $x \in T_2$. Hence $h = \langle f, g\rangle$ and the proof is complete. $\square$

Note that, writing $\mathbf{1}$ for the one-element poset, we have $\mathscr{F}_1 \cong \mathbf{1} + \mathbf{1}_\perp$. Then $\mathscr{F}_2 \cong (\mathbf{1}+\mathbf{1}_\perp)\times(\mathbf{1}+\mathbf{1}_\perp) \cong \mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp+(\mathbf{1}_\perp\times\mathbf{1}_\perp) \cong \mathbf{1}+\mathbf{1}_\perp+\mathbf{1}_\perp+(\mathbf{1}_\perp+\mathbf{1}+\mathbf{1}_\perp)_\perp$.

EXAMPLE 4.2.5. As shown in Figure 10, product of forests obviously fails to be the cartesian product of posets, and its underlying set generally has more points than the cartesian product of the underlying sets.

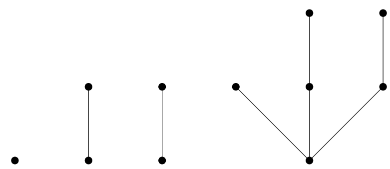


Figure 10. $\mathscr{F}_1 \times \mathscr{F}_1 = \mathscr{F}_2$.

Lemma 4.2.4 allows us to compute all finite products. In particular we can prove the following theorem.

THEOREM 4.2.6.

$$\mathscr{F}_n = H_n + (H_n)_\perp \,,$$

where $H_0 = \emptyset$ *and*

$$H_n = \sum_{i=0}^{n-1} \binom{n}{i} (H_i)_\perp \,. \tag{13}$$

Proof. Clearly $\mathscr{F}_0 = H_0 + (H_0)_\perp \cong \mathbf{1} \cong \mathrm{Spec}\boldsymbol{F}^0_{\mathbb{G}}$. By induction one shows that $H_n + (H_n)_\perp = (H_{n-1} + (H_{n-1})_\perp) \times (H_1 + (H_1)_\perp)$. □

Let $\boldsymbol{G}_\perp$ be the uniquely determined Gödel algebra whose lattice reduct is obtained appending a new bottom element to the lattice reduct of a Gödel algebra $\boldsymbol{G}$. Notice that $\mathbf{Sub}\, H_0$ is the trivial, one-element, algebra.

THEOREM 4.2.7. *For each integer* $n > 0$*,*

$$\boldsymbol{F}^n_{\mathbb{G}} \cong \prod_{i=0}^{n-1} ((\mathbf{Sub}\, H_i)_\perp)^{\binom{n}{i}} \times \left(\prod_{i=0}^{n-1} ((\mathbf{Sub}\, H_i)_\perp)^{\binom{n}{i}} \right)_\perp \,.$$

Proof. Immediate, from Theorem 4.2.6. □

It is now easy to give a recurrence formula for the cardinality of $\boldsymbol{F}^n_{\mathbb{G}}$.

COROLLARY 4.2.8.

$$|\boldsymbol{F}^n_{\mathbb{G}}| = c_n^2 + c_n \,,$$

for $c_0 = 1$ *and*

$$c_n = \prod_{i=0}^{n-1} (c_i + 1)^{\binom{n}{i}} \,.$$

4.3 Gödel hoops

We recall the reader that Gödel hoops are precisely the $\perp$-free subreducts of Gödel algebras. Clearly, they constitute a variety, denoted $\mathbb{GH}$.

For each finite Gödel hoop G, Let $\mathrm{Spec}^*(G) = \mathrm{Spec}(G) \cup \{G\}$ be the map assigning to each finite Gödel hoop G the tree obtained by adding G to the prime spectrum of G, ordered by reverse inclusion. Similarly, let $\mathbf{Sub}^*$ be the map assigning each finite tree T to the Gödel hoop of *nonempty* subforests of T. Define Spec^* and $\mathbf{Sub}^*$ on morphisms by taking preimages, as in Section 4.2. Then $\mathbf{Sub}^*$ and Spec^* realizes a dual equivalence between the category T of finite trees and their morphisms, and the category GH of finite Gödel hoops and their homomorphisms.

T is a subcategory of F which inherits products, while the coproduct $T_1 + T_2 + \cdots + T_k$ is obtained collapsing into one element all the roots of the trees in the forest $T_1 \uplus T_2 \uplus \cdots \uplus T_k$. Note also $\mathbf{Sub}^* F_\perp = \mathbf{Sub}\, F$.

THEOREM 4.3.1.

$$\boldsymbol{F}^n_{\mathbb{GH}} \cong \mathbf{Sub}^*((H_n)_\perp) \cong \mathbf{Sub}(H_n) \cong \prod_{i=0}^{n-1} (\mathbf{Sub}\,(H_i)_\perp)^{\binom{n}{i}} .$$

Moreover,

$$|\boldsymbol{F}^n_{\mathbb{GH}}| = c_n ,$$

where the sequence of integers $\{c_i\}_{i \in \mathsf{N}}$ is defined as in Corollary 4.2.8.

Proof. Immediate, from Theorems 4.2.6, 4.2.7, and Corollary 4.2.8. □

Turning to functional representation, Theorem 4.3.1 is equivalently formulated as follows. Call a function $f\colon S \to [0,1]$ *positive on* S if $f(s) > 0$ for all $s \in S$.

THEOREM 4.3.2. *$\boldsymbol{F}^n_{\mathbb{GH}}$ is the algebra of restrictions to $(0,1]^n$ of Gödel functions positive on $(0,1]^n$, equipped with pointwise defined operations.*

4.4 Finite-valued Gödel algebras

Let $\mathbb{G}_k$ be the subvariety of $\mathbb{G}$ generated by the $k+1$ element Gödel chain $\boldsymbol{G}_k$. The *height* of an element in a forest its the cardinality of its downnset. The height of a forest is the maximum height of its elements. Let F_k be the category formed by all finite forests with height at most k.

The following results are easily derived corollaries of Theorems 4.2.1 and 4.1.9.

THEOREM 4.4.1. *The category G_k of all finite Gödel algebras generated by the $\boldsymbol{G}_k$ is dually equivalent to F_k. Moreover, the dual object to $\boldsymbol{F}^n_{\mathbb{G}_k}$ is obtained from $\mathscr{F}_n$ by deletion of all elements of height greater than k.*

THEOREM 4.4.2. *$\boldsymbol{F}^n_{\mathbb{G}_k}$ is isomorphic to the algebra of restrictions of functions in $\boldsymbol{F}^n_{\mathbb{G}}$ to the set of all rational points in $[0,1]^n$ whose denominator divides k.*

4.5 Nilpotent minimum algebras

In this section we adapt the spectral duality representation given for Gödel algebras in the previous subsection to the case of Nilpotent Minimum algebras. We obtain representations of finitely generated free NM-algebras as the dual objects to suitably defined finite $\{0,1\}$-labelled forests. We then obtain the functional representation theorem for free NM-algebras in the form of algebras of certain $[0,1]$-valued functions over $[0,1]^n$.

Recall that negation in NM-algebras is involutive, that is $\neg\neg x = x$ for any element x. Moreover, choice of the negation equips every chain C with a unique structure of NM-algebra:

$$\boldsymbol{C} = \langle C, \odot, \rightarrow_{\odot}, 0 \rangle ,$$

where

$$x \odot y = \begin{cases} \min\{x, y\} & \text{if } x > \neg y \,, \\ 0 & \text{otherwise;} \end{cases}$$

and the residual implication $\rightarrow_{\odot}$ is given by

$$x \rightarrow_{\odot} y = \begin{cases} 1 & \text{if } x \leq y \,, \\ \max\{\neg x, y\} & \text{otherwise.} \end{cases}$$

The variety $\mathbb{NM}$ of Nilpotent Minimum algebras is generated by the standard algebra $[0,1]$ determined by the standard involutive negation $\neg x = 1 - x$.

Given an NM-chain $\boldsymbol{C}$, one can define its *positive* and *negative parts*, respectively:

$$C^{+} = \{x \in C \mid x > \neg x\} \qquad \text{and} \qquad C^{-} = \{x \in C \mid x < \neg x\} \,.$$

An NM-chain $\boldsymbol{C}$ has at most one fixpoint of negation, that is, an element e such that $\neg e = e$. If $\boldsymbol{C}$ has the negation fixpoint, then the domain of $\boldsymbol{C}$ is $C^{-} \cup \{e\} \cup C^{+}$; if, on the other hand, $\boldsymbol{C}$ lacks the negation fixpoint then its domain is $C^{-} \cup C^{+}$.

For any NM-algebra $\boldsymbol{A}$, we denote $\mathbf{B}(\boldsymbol{A})$ the *Boolean skeleton* of $\boldsymbol{A}$, that is, the largest subalgebra of $\boldsymbol{A}$ that is a Boolean algebra; analogously we denote $\mathbf{MV}(\boldsymbol{A})$ the largest subalgebra of $\boldsymbol{A}$ that is an MV-algebra, or, the MV-*skeleton* of $\boldsymbol{A}$. It can be shown that both $\mathbf{B}(\boldsymbol{A})$ and $\mathbf{MV}(\boldsymbol{A})$ always exist.

One can check that $\mathbf{B}(\boldsymbol{C}) \cong \{0,1\}$ for any NM-chain $\boldsymbol{C}$, and either $\mathbf{MV}(\boldsymbol{C}) \cong \mathbf{Ł}_1$ ($\cong \{0,1\}$), or $\mathbf{MV}(\boldsymbol{C}) \cong \mathbf{Ł}_2 \cong \{0, 1/2, 1\}$: the former case obtains if $\boldsymbol{C}$ lacks the negation fixpoint, the latter if $\boldsymbol{C}$ has it. Moreover $\boldsymbol{C}$ actually retracts onto its MV-skeleton. Due to the subdirect representation theorem, these properties extend from chains to the directly indecomposable NM-algebras, as follows.

PROPOSITION 4.5.1. *An* NM*-algebra* $\boldsymbol{A}$ *is directly indecomposable if and only if* $\mathbf{B}(\boldsymbol{A}) \cong \{0,1\}$. *Moreover,* $\boldsymbol{A}$ *is directly indecomposable if and only if* $\mathbf{MV}(\boldsymbol{A}) \cong \mathbf{Ł}_1$ *or* $\mathbf{MV}(\boldsymbol{A}) \cong \mathbf{Ł}_2$. *Moreover,* $\boldsymbol{A}$ *has the negation fixpoint if and only if* $\mathbf{MV}(\boldsymbol{A}) \cong \mathbf{Ł}_2$.

As another consequence of the subdirect representation by chains, one notes that given a directly indecomposable NM-algebra $\boldsymbol{A}$, the restriction of its operations to its positive part equips A^{+} with a unique structure of Gödel hoop. Vice versa, any Gödel hoop can be seen both as the positive part of a uniquely determined directly indecomposable NM-algebra with negation fixpoint, and as the positive part of a uniquely determined directly indecomposable NM-algebra without negation fixpoint. This allows to strengthen Proposition 4.5.1, as follows.

DEFINITION 4.5.2. *Let* $\boldsymbol{D} = \langle D, \&, \to, \top \rangle$ *be a Gödel hoop. The* disconnected rotation *of* $\boldsymbol{D}$ *is the algebra*

$$\mathbf{DR}(\boldsymbol{D}) = \langle D \times \{1\} \cup D \times \{0\}, \otimes, \Rightarrow, \sqcap, \bot \rangle,$$

with operations defined as follows:

$$\langle x,i\rangle \sqcap \langle y,j\rangle = \langle y,j\rangle \sqcap \langle x,i\rangle = \begin{cases} \langle x \wedge y, 1\rangle & \text{if } i=j=1,\\ \langle x \vee y, 0\rangle & \text{if } i=j=0,\\ \langle x, 0\rangle & \text{if } i<j; \end{cases}$$

$$\langle x,i\rangle \otimes \langle y,j\rangle = \langle y,j\rangle \otimes \langle x,i\rangle = \begin{cases} \langle x \,\&\, y, 1\rangle & \text{if } i=j=1,\\ \langle \top, 0\rangle & \text{if } i=j=0,\\ \langle y \to x, 0\rangle & \text{if } i<j; \end{cases}$$

$$\langle x,i\rangle \Rightarrow \langle y,j\rangle = \begin{cases} \langle x \to y, 1\rangle & \text{if } i=j=1,\\ \langle y \to x, 0\rangle & \text{if } i=j=0,\\ \langle x \,\&\, y, 0\rangle & \text{if } i>j,\\ \langle \top, 1\rangle & \text{if } i<j; \end{cases}$$

$$\bot = \langle \top, 0\rangle.$$

The connected rotation of $\boldsymbol{D}$ *is*

$$\mathbf{CR}(\boldsymbol{D}) = \left\langle D \times \{1\} \cup \left\{\left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle\right\} \cup D \times \{0\}, \otimes, \Rightarrow, \sqcap, \bot \right\rangle$$

where the operations $\otimes$, $\Rightarrow$, $\sqcap$, *and* $\bot$ *are given as in the definition of the disconnected rotation extended by:*

$$\langle x,i\rangle \sqcap \left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle \sqcap \langle x,i\rangle = \begin{cases} \left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle & \text{if } i=1,\\ \langle x,i\rangle & \text{otherwise;} \end{cases}$$

$$\langle x,i\rangle \otimes \left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle \otimes \langle x,i\rangle = \begin{cases} \langle x,i\rangle & \text{if } i=1,\\ \langle \top,0\rangle & \text{otherwise;} \end{cases}$$

$$\langle x,i\rangle \Rightarrow \left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle = \begin{cases} \left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle & \text{if } i=1,\\ \langle \top,1\rangle & \text{otherwise;} \end{cases}$$

$$\left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle \Rightarrow \langle x,i\rangle = \begin{cases} \left\langle \tfrac{1}{2}, \tfrac{1}{2}\right\rangle & \text{if } i=0,\\ \langle \top,1\rangle & \text{otherwise.} \end{cases}$$

A routine verification shows that the disconnected rotation of a Gödel hoop is an NM-algebra without negation fixpoint, while its connected rotation is an NM-algebra with fixpoint.

PROPOSITION 4.5.3. *An* NM*-algebra* $\boldsymbol{A}$ *is directly indecomposable if and only if it is isomorphic to the connected or to the disconnected rotation of a Gödel hoop* $\boldsymbol{G}(\boldsymbol{A})$.

We recall that the prime spectrum $\mathrm{Spec}\boldsymbol{A}$ of an NM-algebra $\boldsymbol{A}$ is the poset of prime filters of $\boldsymbol{A}$ ordered by reverse inclusion.

PROPOSITION 4.5.4. *If $\boldsymbol{A}$ is a directly indecomposable* NM*-algebra, then* $\mathrm{Spec}\boldsymbol{A}$ *is isomorphic to the poset of prime filters of the Gödel algebra* $\boldsymbol{G}(\boldsymbol{A})_\perp$.

Proof. An easy check shows that the set of prime filters of $\boldsymbol{A}$ is the union of the set of prime filters of the Gödel hoop A^+ together with A^+ itself. □

Note that for each Gödel hoop $\boldsymbol{H}$, the rotations $\mathbf{CR}(\boldsymbol{H})$ and $\mathbf{DR}(\boldsymbol{H})$ have the same prime spectrum:

$$\mathrm{Spec}\mathbf{CR}(\boldsymbol{H}) \cong \mathrm{Spec}\boldsymbol{H}_\perp \cong \mathrm{Spec}\mathbf{DR}(\boldsymbol{H})\,.$$

We need an additional bit of information to distinguish these two cases. We start defining a category of trees with a bit attached. Each tree will correspond to a directly indecomposable finite NM-algebra: the bit will specify whether this algebra has the negation fixpoint. Then our duality is extended to a category of forests of labelled trees: each such forest will correspond to a direct product of directly indecomposable finite NM-algebras, and viceversa.

DEFINITION 4.5.5. *A* labelled tree *is a pair* $\langle T, i\rangle$ *where* T *is a tree and* $i \in \{0,1\}$. *A* labelled forest *is a finite set of labelled trees. A morphism of labelled trees* $\langle T, i\rangle$ *and* $\langle T', i'\rangle$ *is an open order-preserving map from* T *to* T' *such that* $i \le i'$. *A morphism of labelled forests* $\phi : F_1 \to F_2$ *is specified by morphisms of labelled trees* $\phi_j : \langle T_j, i_j\rangle \to \langle T'_j, i'_j\rangle$ *where* $\langle T'_j, i'_j\rangle \in F_2$ *for every* $\langle T_j, i_j\rangle \in F_1$.

It is trivial to check that labelled forests and their morphisms form a category that we denote by LF. Let LT be the subcategory of labelled trees and DNM the category of directly indecomposable NM-algebras and their homomorphisms.

For any directly indecomposable NM-algebra A, let

$$\mathrm{Spec}^+\boldsymbol{A} = \langle \mathrm{Spec}\boldsymbol{A}, i\rangle\,,$$

where $i = 0$ if A has a negation fixpoint and $i = 1$ otherwise. For each homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ in DNM, let $\mathrm{Spec}^+h : \mathrm{Spec}^+\boldsymbol{B} \to \mathrm{Spec}^+\boldsymbol{A}$ be given, as usual, by taking counterimages

$$(\mathrm{Spec}^+h)(\mathfrak{p}) = h^{-1}(\mathfrak{p}) = \{x \in \boldsymbol{A} \mid h(x) \in \mathfrak{p}\}\,.$$

Note that since there are no homomorphisms from NM-algebras with negation fixpoint into algebras without fixpoint, the duals of the morphisms in DNM are well defined morphisms in LT.

THEOREM 4.5.6. *The categories* DNM *and* LT *are dually equivalent via* Spec^+.

Proof. A tedious check shows that Spec^+ is full, faithful, and essentially surjective. □

Let NM be the category of finite NM-algebras.

THEOREM 4.5.7. *The categories* NM *and* LF *are dually equivalent via* Spec^+.

Proof. Immediate, from Theorem 4.5.6. □

We can hence give a description of the free NM-algebra in terms of its dual using the same approach we have applied to Gödel algebras in the preceding subsection. In order to do this, we need to describe the product in the category of labelled trees. Coproduct is just disjoint union.

PROPOSITION 4.5.8. *Let $\langle S, i\rangle$ and $\langle T, j\rangle$ be labelled trees, and let $S \times T$, together with its projections π_S, π_T be the product of S and T in the category F of forests. The labelled tree*

$$\langle S \times T, ij\rangle\,,$$

with the projection maps

$$\pi_{\langle S,i\rangle}\colon \langle S \times T, ij\rangle \to \langle S, i\rangle \quad \textit{and} \quad \pi_{\langle T,j\rangle}\colon \langle S \times T, ij\rangle \to \langle T, j\rangle\,,$$

induced by π_S and π_T, is the product of $\langle S, i\rangle$ and $\langle T, j\rangle$ in the category LT.

To compute products in LF it is sufficient to apply the distribution law: Let $\langle R, h\rangle$ be a labelled tree in LT, then the following holds in LF:

$$\langle R, h\rangle \times (\langle S, i\rangle + \langle T, j\rangle) = (\langle R, h\rangle \times \langle S, i\rangle) + (\langle R, h\rangle \times \langle T, j\rangle)\,.$$

Proof. Clearly, $\pi_{\langle S,i\rangle}$ and $\pi_{\langle T,j\rangle}$ are morphisms in LT (and in LF) as $ij \le i, j$. Take two morphisms $f_S\colon \langle R, k\rangle \to \langle S, i\rangle$ and $f_T\colon \langle R, k\rangle \to \langle T, j\rangle$ in LT. By definition, $k \le i, j$, and hence $k \le ij$. Then, by the properties of product in the category F of finite forests, there exists a unique morphism $g\colon \langle R, k\rangle \to \langle S \times T, ij\rangle$ such that $\pi_{\langle S,i\rangle} \circ g = f_S$ and $\pi_{\langle T,j\rangle} \circ g = f_T$.

The distribution law follows immediately from Lemma 4.2.4.2. □

Let $\boldsymbol{F}^n_{\mathbb{NM}}$ be the free NM-algebra over n generators. Let $\mathscr{LF}_n$ be the labelled forest such that

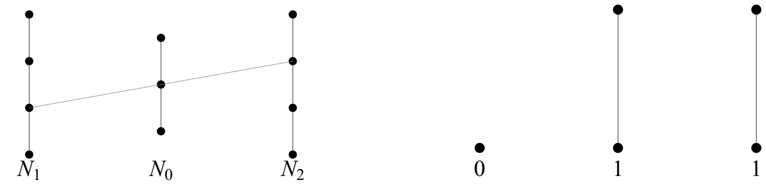
$$\mathrm{Spec}^+ \boldsymbol{F}^n_{\mathbb{NM}} = \mathscr{LF}_n\,.$$

Let C_n denote a chain of $n+1$ elements.

PROPOSITION 4.5.9. *$\mathscr{LF}_1$ is the forest consisting of the labelled tree $\langle C_0, 0\rangle$ and two copies of the labelled tree $\langle C_1, 1\rangle$.*

Proof. All singly generated NM-chains are isomorphic either to the three element chain ordered by $\bot < x < \top$ (where x is the generating element); or to the four element chain $\{\bot, x, \neg x, \top\}$, that can be generated in two ways: $\bot < \neg x < x < \top$ and $\bot < x < \neg x < \top$; or they have less than three elements, and in this case they are homomorphic images of the preceding ones. Let us denote the three element chain as N_0 and the two four element chains by N_1 and N_2. Then $\boldsymbol{F}^1_{\mathbb{NM}}$ embeds in the product $N_0 \times N_1 \times N_2$. By direct computation one finds NM-terms in one variable evaluating to all 48 elements of $N_0 \times N_1 \times N_2$. Hence, $\boldsymbol{F}^1_{\mathbb{NM}} \cong N_0 \times N_1 \times N_2$. The spectrum of N_0 is C_0 and it has the negation fixpoint. The spectrum of N_1 and N_2 is C_1, and they lack the negation fixpoint. □

Recall that, by Lemma 1.0.13, $\mathrm{Spec}^+ \boldsymbol{F}^n_{\mathbb{NM}} = \prod_{i=1}^n \mathscr{LF}_1$.



(a) $F^1_{\mathbb{NM}} \cong N_0 \times N_1 \times N_2$ with generator.

(b) $\mathscr{L}\mathscr{F}_1$.

Figure 11. The free singly generated NM-algebra and its labelled prime spectrum.

THEOREM 4.5.10.

$$\mathscr{L}\mathscr{F}_n \cong 2^n \langle (H_n)_\perp, 1 \rangle + \sum_{i=0}^{n-1} 2^i \binom{n}{i} \langle (H_i)_\perp, 0 \rangle \,,$$

where the poset H_i, for $i = 0, \ldots, n$, is defined by (13).

Proof. Immediate, from Proposition 4.5.8 and Proposition 4.5.9. □

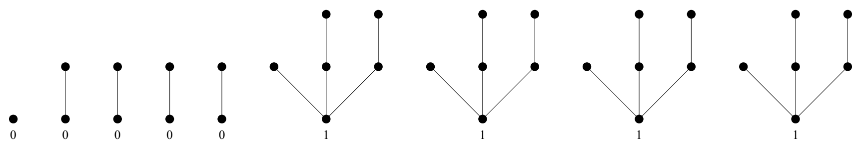


Figure 12. $\mathscr{L}\mathscr{F}_2 = \mathscr{L}\mathscr{F}_1 \times \mathscr{L}\mathscr{F}_1$.

COROLLARY 4.5.11.

$$\boldsymbol{F}^n_{\mathbb{NM}} \cong (\mathbf{DR}(\mathbf{Sub} H_n))^{2^n} \times \prod_{i=0}^{n-1} (\mathbf{CR}(\mathbf{Sub} H_i))^{2^i \binom{n}{i}} \,,$$

and

$$|\boldsymbol{F}^n_{\mathbb{NM}}| = (2c_n)^{2^n} \cdot \prod_{i=0}^{n-1} (2(c_i + 1))^{2^i \binom{n}{i}} \,,$$

where the sequence of integers $\{c_i\}_{i \in \mathsf{N}}$ is defined as in Corollary 4.2.8.

4.6 Functional representation of free NM-algebras

By a *signed ordered partition* of $\{x_1, \ldots, x_n\}$ we mean a pair $\langle \pi, \lambda \rangle$, where π is an ordered partition and λ is a function $\lambda \colon \{1, \ldots, n\} \to \{-1, 0, 1\}$ called the *sign function*, satisfying the following:

1. $\pi = Q_1 <_\pi \cdots <_\pi Q_h$ is an ordered partition of the set
$$\{x_i \mid i \in \{1, \ldots, n\},\ \lambda(i) \neq 0\} \cup \{\top\}\,;$$
2. $\top \in Q_h$.

We impose a partial order on signed ordered partitions of $\{x_1, \ldots, x_n\}$ by the following stipulation. We set $\langle \pi_1, \lambda_1 \rangle \prec \langle \pi_2, \lambda_2 \rangle$ if the following hold:

- $\pi_1 = Q_1 <_{\pi_1} \cdots <_{\pi_1} Q_{h-1} <_{\pi_1} Q_h$ and $\pi_2 = Q_1 <_{\pi_2} \cdots <_{\pi_2} Q_{h-1} <_{\pi_2} R_h <_{\pi_2} \cdots <_{\pi_2} R_k$ with $R_h \neq Q_h$;
- $\lambda_1 = \lambda_2$.

The relation $\prec$ is clearly irreflexive and transitive. The partial order $\preceq$ induced by $\prec$ is the desired partial order on the set of all signed ordered partitions of $\{x_1, \ldots, x_n\}$.

It is easy to see that the poset $\langle F_n, \preceq \rangle$ of all signed ordered partitions of $\{x_1, \ldots, x_n\}$ is a finite forest. In particular $\langle \pi, \lambda_1 \rangle \vee_{\preceq} \langle \rho, \lambda_2 \rangle$ exists if and only if $\langle \pi, \lambda_1 \rangle$ and $\langle \rho, \lambda_2 \rangle$ are $\preceq$-comparable, and hence it must be the case $\lambda_1 = \lambda = \lambda_2$, and then obviously $\langle \pi, \lambda_1 \rangle \vee_{\preceq} \langle \rho, \lambda_2 \rangle = \max_{\preceq}\{\langle \pi, \lambda \rangle, \langle \rho, \lambda \rangle\}$; for what regards $\preceq$-meets, $\langle \pi, \lambda_1 \rangle \wedge_{\preceq} \langle \rho, \lambda_2 \rangle$ exists if and only if $\lambda_1 = \lambda = \lambda_2$, and can be described as follows. To avoid trivialities, assume $\pi \neq \rho$. Display $\pi = B_1 <_\pi \cdots <_\pi B_h <_\pi R_{h+1} \cdots <_\pi R_u$ and $\rho = B_1 <_\rho \cdots <_\rho B_h <_\rho S_{h+1} \cdots <_\rho S_v$, with $h \in \{0, 1, \ldots, \min\{u, v\} - 1\}$, and $R_{h+1} \neq S_{h+1}$ (clearly, if $h = 0$ already the first blocks of π and ρ are distinct). Then $\langle \pi, \lambda_1 \rangle \wedge_{\preceq} \langle \rho, \lambda_2 \rangle = \langle \tau, \lambda \rangle$ for

$$\tau = B_1 <_\tau \cdots <_\tau B_h <_\tau \bigcup_{i=h+1}^{u} R_i \,.$$

Note that $\bigcup_{i=h+1}^{u} R_i = \bigcup_{i=h+1}^{v} S_i$. The sequence of blocks $B_1 <_\tau \cdots <_\tau B_h$ is the *longest common prefix* of $\langle \pi, \lambda \rangle$ and $\langle \rho, \lambda \rangle$.

The set of trees of F_n is in bijection with sign functions: as a matter of fact, for each sign function $\lambda\colon \{1, \ldots, n\} \to \{-1, 0, 1\}$, the signed ordered partition $\langle \pi_\lambda, \lambda \rangle$ where π_λ is composed by the single block $\{x_i \mid \lambda(i) \neq 0\} \cup \{\top\}$, is $\preceq$ than every $\langle \rho, \lambda \rangle \in F_n$, while it is incomparable to every $\langle \rho, \lambda' \rangle$ for $\lambda \neq \lambda'$. We denote T_λ the tree with root $\langle \pi_\lambda, \lambda \rangle$.

For each $t \in [0, 1]$, let $||t|| = \max\{t, 1 - t\}$.

The *basic region* $[\pi, \lambda]$ associated with a signed ordered partition $\langle \pi, \lambda \rangle$, with $\pi = Q_1 <_\pi \cdots <_\pi Q_h$, is the set of all points $\langle t_1, \ldots, t_n \rangle \in [0, 1]^n$ such that for all $i \in \{1, \ldots, n\}$:

1. $$\lambda(i) = \begin{cases} 1 & \text{if } t_i > \frac{1}{2}\,, \\ 0 & \text{if } t_i = \frac{1}{2}\,, \\ -1 & \text{if } t_i < \frac{1}{2}\,; \end{cases}$$
2. $x_i \in Q_h$ if and only if $||t_i|| = 1$;
3. x_i belongs to a block of π that is $<_\pi$ than the block x_j belongs to, if and only if $||t_i|| < ||t_j||$.

A straightforward verifications shows the following.

LEMMA 4.6.1. *The set of all basic regions associated with signed ordered partitions of* $\{x_1, \dots, x_n\}$ *is a partition of* $[0, 1]^n$.

We define the region associated with a signed ordered partition $\langle \pi, \lambda \rangle$ as

$$N(\pi, \lambda) \;=\; \bigcup_{\langle \rho,\lambda\rangle \preceq \langle \pi,\lambda\rangle} [\rho, \lambda]\,.$$

We denote $\mathcal{N}_n$ the set of all regions associated with signed ordered partitions of the set $\{x_1, \dots, x_n\}$. We further denote $N(\lambda) = \bigcup_{\langle \rho,\lambda\rangle \in T_\lambda} [\rho, \lambda]$.

LEMMA 4.6.2. $N(\lambda) = \bigcup_{\langle \rho,\lambda\rangle \in T_\lambda} N(\rho, \lambda)$ *for each* $\lambda \in \{-1, 0, 1\}^n$*. Moreover, the set of all regions* $N(\lambda)$*, for* λ *ranging over all the sign functions (there are* 3^n *of them), is a partition of* $[0, 1]^n$*. Further,* $\langle \mathcal{N}_n, \subseteq \rangle$ *is order-isomorphic to* $\langle F_n, \preceq \rangle$ *via the map* $N(\pi, \lambda) \mapsto \langle \pi, \lambda \rangle$.

Proof. It follows immediately from Lemma 4.6.1 and the definitions of basic region and of $N(\pi, \lambda)$. □

We now introduce normal forms for NM-terms, based on a suitable definition of minterms. First we introduce the following derived connectives:

$$\begin{aligned}
\circ x &= (x \leftrightarrow \neg x) \mathbin{\&} (x \leftrightarrow \neg x)\,;\\
+x &= \neg((\neg(x \mathbin{\&} x)) \mathbin{\&} (\neg(x \mathbin{\&} x)))\,;\\
-x &= +(\neg x)\,.
\end{aligned}$$

The above defined operations have the following properties in the standard algebra $[0, 1]$. $\circ x = 1$ if and only if $x = \neg x$, $\circ x = 0$ otherwise; $+x = 1$ iff $x > \neg x$, $+x = 0$ otherwise; $-x = 1$ iff $x < \neg x$, $-x = 0$ otherwise.

Let $\lambda\colon \{1, \dots, n\} \to \{-1, 0, 1\}$ be a sign function. For each $j \in \{-1, 0, 1\}$, let $T_j = \{i \mid \lambda(i) = j\} \subseteq \{1, \dots, n\}$. We define

$$\sigma_\lambda = \bigwedge_{i \in T_{-1}} -x_i \wedge \bigwedge_{i \in T_0} \circ x_i \wedge \bigwedge_{i \in T_1} +x_i\,.$$

One checks that $\sigma_\lambda^{\boldsymbol{F}^n_{\mathrm{NM}}}(t_1, \dots, t_n) = 1$ if and only if $\langle t_1, \dots, t_n \rangle \in N(\lambda)$, and that $\sigma_\lambda^{\boldsymbol{F}^n_{\mathrm{NM}}}(t_1, \dots, t_n) = 0$ otherwise.

DEFINITION 4.6.3. *Consider a signed ordered partition* $\langle \pi, \lambda \rangle$*. Display* $\pi = Q_1 < \dots < Q_h$*. Denote* $X_\lambda = \{i \mid \lambda(i) \neq 0\}$*. Let* $\sigma\colon X_\lambda \to X_\lambda$ *be a permutation and* $\prec_i\colon \{1, \dots, n\} \to \{<, =\}$ *be a map such that* $[\pi, \lambda]$ *is the set of all points* $\langle t_1, \dots, t_n \rangle \in [0, 1]^n$ *satisfying:*

- $t_i = 1/2$ *if and only if* $\lambda(i) = 0$*;*
- $t^{(\epsilon(1))}_{\sigma(1)} \prec_1 t^{(\epsilon(2))}_{\sigma(2)} \prec_2 \cdots \prec_{n-1} t^{(\epsilon(n))}_{\sigma(n)} \prec_n 1$*, where* $\epsilon(i) = \lambda(\sigma(i))$*, and* $t^{(1)}_i = t_i$ *while* $t^{(-1)}_i = 1 - t_i$*.*

Set $x_{\sigma(|X_\lambda|+1)} = \top$ *and define*

$$\chi^{+}_{\pi,\lambda} = \sigma_\lambda \wedge \bigwedge_{i=1}^{|X_\lambda|} \delta_i\,,$$

where

$$\delta_i = \begin{cases} x^{\epsilon(i)}_{\sigma(i)} \leftrightarrow x^{\epsilon(i+1)}_{\sigma(i+1)} & \text{if } \prec_i \text{ is } =,\\ x^{\epsilon(i)}_{\sigma(i)} \lhd x^{\epsilon(i+1)}_{\sigma(i+1)} & \text{if } \prec_i \text{ is } <, \end{cases}$$

for every $i \in \{1,\ldots,|X_\lambda|\}$ *and where* $x_i^{(1)} = x_i$ *while* $x_i^{(-1)} = \neg x_i$*. The formula* $\chi^{+}_{\langle\pi,\lambda\rangle}$ *is called the* positive minterm *at* $\langle\pi,\lambda\rangle$.

LEMMA 4.6.4. *The function* $(\chi^{+}_{\pi,\lambda})^{\boldsymbol{F}^n_{\mathbb{NM}}}$ *is the smallest term function constantly* 1 *over* $[\pi,\lambda]$*; further, it is constantly* 0 *over* $[0,1]^n \setminus N(\lambda)$ *and is the smallest term function constantly* 1 *over* $N(\pi,\lambda)$.

Proof. Recalling that the restriction of the operations to the positive part $(1/2,1]$ of the standard NM-chain $[0,1]$ makes it into a Gödel hoop, the proof follows from Lemma 4.1.6 and the properties of σ_λ. □

We shall introduce *negative minterms*, too.

LEMMA 4.6.5. *For any* $\langle\pi_\lambda,\lambda\rangle$ *set* $\chi^{-}_{\pi_\lambda,\lambda} = \bot$*. For any* $\langle\pi,\lambda\rangle$ *such that* $\langle\pi,\lambda\rangle \neq \langle\pi_\lambda,\lambda\rangle$*, let* $\langle\rho,\lambda\rangle$ *denote the unique* $\preceq$*-predecessor of* $\langle\pi,\lambda\rangle$*. Let*

$$\chi^{-}_{\pi,\lambda} = \sigma_\lambda \wedge \neg \bigvee_{\langle\tau,\lambda\rangle\,\in\,\max_{\preceq}(T_\lambda\setminus\uparrow\langle\pi,\lambda\rangle)} \chi^{+}_{\tau,\lambda}\,.$$

Then, $(\chi^{-}_{\pi,\lambda})^{\boldsymbol{F}^n_{\mathbb{NM}}}(t_1,\ldots,t_n) =$

$$= \begin{cases} 1-(\chi^{+}_{\rho,\lambda})^{\boldsymbol{F}^n_{\mathbb{NM}}}(t_1,\ldots,t_n) & \text{if } \langle t_1,\ldots,t_n\rangle \in \bigcup_{\langle\tau,\lambda\rangle\,\in\,\uparrow\langle\pi,\lambda\rangle} N(\tau,\lambda),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We note that $(\bigvee_{\langle\tau,\lambda\rangle\,\in\,\max_{\preceq}(T_\lambda\setminus\uparrow\langle\pi,\lambda\rangle)} \chi^{+}_{\tau,\lambda})^{\boldsymbol{F}^n_{\mathbb{NM}}}(t_1,\ldots,t_n) = 1$ if and only if $\langle t_1,\ldots,t_n\rangle \in \bigcup_{\langle\tau,\lambda\rangle\,\in\,T_\lambda\setminus\uparrow\langle\pi,\lambda\rangle} N(\tau,\lambda)$ and it coincides with $(\chi^{+}_{\rho,\lambda})^{\boldsymbol{F}^n_{\mathbb{NM}}}(t_1,\ldots,t_n)$ for all $\langle t_1,\ldots,t_n\rangle \in \bigcup_{\langle\tau,\lambda\rangle\,\in\,\uparrow\langle\pi,\lambda\rangle} N(\tau,\lambda)$. □

We also introduce *fixpoint minterms*.

LEMMA 4.6.6. *Assume there is* $i \in \{1,\ldots,n\}$ *such that* $\lambda(i) = 0$*. Let*

$$\chi^{\circ}_{\pi_\lambda,\lambda} = \sigma_\lambda \wedge x_i\,.$$

Then

$$(\chi^{\circ}_{\pi_\lambda,\lambda})^{\boldsymbol{F}^n_{\mathbb{NM}}}(t_1,\ldots,t_n) = \begin{cases} \frac{1}{2} & \text{if } \langle t_1,\ldots,t_n\rangle \in N(\lambda),\\ 0 & \text{otherwise}\,.\end{cases}$$

Proof. Immediate, from the properties of σ_λ. □

LEMMA 4.6.7. *Let φ be a term in n variables and $\langle \pi, \lambda \rangle$ be a maximal signed ordered partition of $\{x_1, \ldots, x_n\}$. Then there exists a unique signed ordered partition $\langle \rho, \lambda \rangle \preceq \langle \pi, \lambda \rangle$, which we denote $\rho(f, \pi, \lambda)$, and a unique $*(f, \lambda) \in \{-, \circ, +\}$ such that*

$$\varphi^{\boldsymbol{F}^n_{\mathbb{NM}}}(t_1, \ldots, t_n) = (\chi^{*(f,\lambda)}_{\rho(f,\pi,\lambda)})^{\boldsymbol{F}^n_{\mathbb{NM}}}(t_1, \ldots, t_n)$$

for all $\langle t_1, \ldots, t_n \rangle \in N(\pi, \lambda)$.Moreover, φ can be equivalently written as

$$\varphi \equiv_{\mathbb{NM}} \bigvee_{\lambda \in \{-1,0,1\}^n} \; \bigvee_{\langle \pi,\lambda \rangle \in \max_{\preceq} T_\lambda} \chi^{*(f,\lambda)}_{\rho(f,\pi,\lambda)} \,.$$

Proof. From Lemma 4.6.4, Lemma 4.6.5, and Lemma 4.6.6. □

DEFINITION 4.6.8. *A function $f\colon [0,1]^n \to [0,1]$ is an* NM*-function over n variables if for each region $N(\pi, \lambda)$ there is a function $g \in \{\bot, 1 - x_1, \ldots, 1 - x_n, x_1, \ldots, x_n, \top\}$ and $f(t_1, \ldots, t_n) = g(t_1, \ldots, t_n)$ for all $\langle t_1, \ldots, t_n \rangle \in N(\pi, \lambda)$. The* NM*-algebra of all* NM*-functions equipped with pointwise operations will be denoted by $\boldsymbol{NM}_n$.*

THEOREM 4.6.9. $\boldsymbol{F}^n_{\mathbb{NM}} \cong \boldsymbol{NM}_n$.

Proof. Immediate, from Lemma 4.6.7. □

Another interesting property of NM-functions, analogous to Lemma 4.1.5, is the following.

PROPOSITION 4.6.10 ($0, 1$-set property). *Let $f, g \in \boldsymbol{NM}_n$ be such that $f^{-1}(1) = g^{-1}(1)$ and $f^{-1}(0) = g^{-1}(0)$. Then $f = g$.*

Finally, notice that the posets $\langle F_n, \preceq \rangle$ and $\mathrm{Spec}\boldsymbol{F}^n_{\mathbb{NM}}$ are order-isomorphic via the map $\langle \pi, \lambda \rangle \mapsto {\uparrow}\chi^+_{\pi,\lambda}$, as positive minterms are exactly those join-irreducible elements of $\boldsymbol{F}^n_{\mathbb{NM}}$ that singly generate its prime filters.

5 BL-algebras and related structures

In this section we introduce the functional representation theorem for free n-generated BL-algebras. We shall then show how to adapt this construction to obtain the representation of free n-generated SBL-algebras. We shall then deal with locally finite subvarieties of $\mathbb{BL}$. However, we first discuss the case of $\mathrm{BL}_\triangle$-algebras, as it is studied in [59].

5.1 Free BL$_\triangle$-algebras

For each integer $n \geq 0$, the subvariety of $\mathbb{BL}$ generated by the class of n-generated BL-algebras is in turns generated by the algebra

$$(n+1)[0,1]_{\text{Ł}}\,,$$

that is, the ordinal sum of $n + 1$ many copies of the standard MV-algebra. By Lemma 1.0.12 it is thus sufficient, in order to characterise the free n-generated BL-algebra $\boldsymbol{F}^n_{\mathbb{BL}}$, to identify the subalgebra of the algebra

$$(n+1)[0,1]_{\text{Ł}}^{((n+1)[0,1]_{\text{Ł}})^n}$$

generated by the projections.

This has proved not to be an easy task to tackle. The problem lies in the complex interaction of the behaviour of functions in $\boldsymbol{F}^n_{\mathbb{BL}}$ over different regions of the domain $[0, n+1]^n$. The intricacies of this interaction will be dealt with in Section 5.2: as a first step towards the description of elements of $\boldsymbol{F}^n_{\mathbb{BL}}$, we focus on free algebras on a related variety, obtained by expanding the language of BL with Baaz's $\triangle$ unary connective. The resulting variety $\mathbb{BL}_\triangle$ is much easier to deal with, as the operator $\triangle$ allows for definition by cases, killing off most of the interacting effects between regions.

DEFINITION 5.1.1. *A* $\mathrm{BL}_\triangle$*-algebra* $\boldsymbol{A}$ *is a* BL*-algebra with an additional unary operator* $\triangle$ *satisfying, for each* $x, y \in \boldsymbol{A}$*:*

1. $\triangle x \vee \neg\triangle x = 1$*;*
2. $\triangle 1 = 1$*;*
3. $\triangle(x \to y) \leq \triangle x \to \triangle y$*;*
4. $\triangle x \leq x$*;*
5. $\triangle\triangle x = \triangle x$.

Let $\mathbb{BL}_\triangle$ *denote the variety of all* $\mathrm{BL}_\triangle$*-algebras.*

LEMMA 5.1.2. *For every* $\mathrm{BL}_\triangle$ *chain* $\boldsymbol{C}$ *and for each* $x \in \boldsymbol{C}$*,*

$$\triangle x = \begin{cases} 1 & \textit{if } x = 1, \\ 0 & \textit{otherwise}\,. \end{cases}$$

Further, $\mathbb{BL}_\triangle$ *enjoys standard completeness, and in particular, for each integer* $n \geq 0$*, the subvariety of* $\mathbb{BL}_\triangle$ *generated by the* n*-generated* $\mathrm{BL}_\triangle$*-algebras is generated by the* $\mathrm{BL}_\triangle$*-algebra* $(n+1)[0,1]_{\text{Ł}}$.

Note that we can identify the universe of $(n+1)[0,1]_{\text{Ł}}$ with the interval $[0, n+1]$ in the obvious way, and the top element of the algebra is then the integer $n+1$.

Hence again, by Lemma 1.0.12, to characterise $\boldsymbol{F}^n_{\mathbb{BL}_\triangle}$, one has to identify the functions $f\colon ((n+1)[0,1]_{\text{Ł}})^n \to (n+1)[0,1]_{\text{Ł}}$ generated by the projections. The structure of ordinal sum of $(n+1)[0,1]_{\text{Ł}}$ suggests to partition the domain $[0, n+1]^n$ into *cells* as follows.

DEFINITION 5.1.3. *For each integer* $n \geq 0$*, a* cell *is a subset* C *of* $[0, n+1]^n$ *of the form* $C = C_1 \times C_2 \times \cdots \times C_n$*, where each* C_i *is either the singleton* $I_{n+1} = \{n+1\}$ *or is of the form* $I_j = [j, j+1)$ *for some integer* $j \in \{0, 1, \ldots, n\}$.

It turns out that the behaviour of any element $f \in \boldsymbol{F}^n_{\mathbb{BL}_\triangle}$ is redundant over some cells, that is, the restriction of f to a one cell may be completely determined by the restriction of f to another, distinct, cell. The complement of the set of redundant cells are exactly those cells $I_{k_1} \times \cdots \times I_{k_n}$ such that $(\bigcup_{i=1}^n \{k_i\} \cup \{0\}) \setminus \{n+1\}$ is an initial segment of N, that is, it is of the form $\{0, 1, \ldots, h\}$ for some h. Let $I(n)$ be the set-theoretic union of all non-redundant cells. Then the algebra of restrictions of functions in $\boldsymbol{F}^n_{\mathbb{BL}_\triangle}$ to the domain $I(n)$ is a $\mathrm{BL}_\triangle$-algebra isomorphic to $\boldsymbol{F}^n_{\mathbb{BL}_\triangle}$.

The restriction of a function $f \in \boldsymbol{F}^n_{\mathbb{BL}_\triangle}$ to a cell can be seen as a sort of "generalised" McNaughton function.

DEFINITION 5.1.4. *Let $\{y_1, \ldots, y_k\}$ be a subset of the set of variables $\{x_1, \ldots, x_n\}$, and let $f \in \boldsymbol{F}^k_{\mathbb{MV}}$ be a McNaughton function. For each $i \in \{0, \ldots, n+1\}$ we define a function $f_{i,n} \colon I^k_i \to I_i \cup \{0, n+1\}$ as follows:*

1. *if $i = 0$ and $f(y_1, \ldots, y_k) \neq 1$, then $f_{i,n}(y_1, \ldots, y_k) = f(y_1, \ldots, y_k)$;*
2. *if $i = 0$ and $f(y_1, \ldots, y_k) = 1$, then $f_{i,n}(y_1, \ldots, y_k) = n + 1$;*
3. *if $i \in \{1, \ldots, n\}$ and $f(1, 1, \ldots, 1) = 1$ and $f(y_1 - i, \ldots, y_k - i) \neq 1$, then $f_{i,n}(y_1, \ldots, y_k) = f(y_1 - i, \ldots, y_k - i) + i$;*
4. *if $i \in \{1, \ldots, n\}$ and $f(1, 1, \ldots, 1) = 1$ and $f(y_1 - i, \ldots, y_k - i) = 1$, then $f_{i,n}(y_1, \ldots, y_k) = n + 1$;*
5. *if $i = n + 1$ and $f(1, 1, \ldots, 1) = 1$, then $f_{i,n}(y_1, \ldots, y_k) = n + 1$;*
6. *if $i > 0$ and $f(1, 1, \ldots, 1) = 0$, then $f_{i,n}(y_1, \ldots, y_k) = 0$.*

We are now ready to subdivide cells into regions whose characteristic function can be expressed with a suitable $\mathrm{BL}_\triangle$-term. These regions, namely, *linear semialgebraic sets*, play roughly the same role as the polyhedra in the complex Σ of Lemma 2.1.4 play for McNaughton functions.

DEFINITION 5.1.5. *For a set S and a subset K of $\{1, \ldots, n\}$, a function $f \colon S^n \to S$ is* essentially $|K|$-ary *over S^K if, for any two points $\langle x_1, \ldots, x_n\rangle, \langle y_1, \ldots, y_n\rangle$ of S^n such that $x_i = y_i$ for all $i \in K$, it holds that $f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$. If $i \in K$ then f* depends on *x_i. If K is the smallest set such that f is essentially $|K|$-ary over S^K then f* only depends on *$\{x_i \mid i \in K\}$.*

The following definition introduces a slight variant of the Q-semialgebraic sets introduced in Definition 2.4.1.

DEFINITION 5.1.6. *Let C be a cell. A* linear semialgebraic subset *of C is the set of solutions in C of a finite set of linear inequalities E of the form*

$$f(x_1, \ldots, x_n) \lhd g(x_1, \ldots, x_n)\,,$$

for $\lhd \in \{\leq, <\}$, satisfying the following stipulations.

1. *For each inequality $e \in E$ of the form $f(x_1, \ldots, x_n) \lhd g(x_1, \ldots, x_n)$, there is an integer $i(e) \in \{0, \ldots, n+1\}$ such that for all $j \in \{1, \ldots, n\}$, if either f or g depends on the variable x_j, then for all $\langle a_1, \ldots, a_n\rangle \in C$, $\lfloor a_j \rfloor = i(e)$.*
2. *For each inequality $e \in E$ of the form $f(x_1, \ldots, x_n) \lhd g(x_1, \ldots, x_n)$, there is a subset $\{y_1, \ldots, y_k\}$ of variables, and $h, k \in \boldsymbol{F}^k_{\mathbb{MV}}$ such that f and g only depend on $\{y_1, \ldots, y_k\}$, and for all $\langle x_1, \ldots, x_n\rangle \in C$ it holds that*

   $$f(x_1, \ldots, x_n) = h_{i(e),n}(y_1, \ldots, y_k)$$

   and

   $$g(x_1, \ldots, x_n) = k_{i(e),n}(y_1, \ldots, y_k)\,.$$
3. *If $i(e) > 0$ then $h(1, 1, \ldots, 1) = 1 = k(1, 1, \ldots, 1)$.*

The following lemma shows the usefulness of the operator $\triangle$ in the task of isolating the behaviour of a $\mathrm{BL}_\triangle$-function f over a linear semialgebraic subset, that is, for any linear semialgebraic subset Y it is possible to express with a term the characteristic function of Y. When dealing with BL-algebras (without $\triangle$), it is in general not possible to express such characteristic functions.

LEMMA 5.1.7. *For every linear semialgebraic subset Y of a cell C, there is a $\mathrm{BL}_\triangle$-term φ_Y such that for all $\langle t_1, \ldots, t_n\rangle \in [0, n+1]^n$ it holds that $\varphi_Y^{(n+1)[0,1]_{\text{Ł}}}(t_1, \ldots, t_n) \in \{0, n+1\}$ and*

$$\langle t_1, \ldots, t_n\rangle \in Y \quad \textit{if and only if} \quad \varphi_Y^{(n+1)[0,1]_{\text{Ł}}}(t_1, \ldots, t_n) = n+1\,.$$

Lemma 5.1.7 allows to prove the functional representation theorem for $\boldsymbol{F}^n_{\mathbb{BL}_\triangle}$.

DEFINITION 5.1.8. *A* $\mathrm{BL}_\triangle$-partition *of $I(n)$ is a partition of $I(n)$ into linear semialgebraic subsets of cells. An* elementary $\mathrm{BL}_\triangle$-function *is a function f with domain $\mathrm{dom}(f) = Y$, for Y a linear semialgebraic subset of some cell C, and is such that either f is constant on Y, and then this constant value is either 0 or $n+1$, or there exist $i(f) \in \{0, \ldots, n+1\}$ and $g \in \boldsymbol{F}^k_{\mathbb{MV}}$ with $f = g_{i(f),n}$ and for each $j \in \{1, \ldots, n\}$, if g depends on the variable x_j, then $\lfloor a_j \rfloor = i(f)$ for all $\langle a_1, \ldots, a_n\rangle \in C$.*

THEOREM 5.1.9. *For each integer $n \geq 0$, the free n-generated $\mathrm{BL}_\triangle$-algebra $\boldsymbol{F}^n_{\mathbb{BL}_\triangle}$ is the algebra whose universe is the set of all functions $f\colon I(n) \to [0, n+1]$ such that there exist a $\mathrm{BL}_\triangle$-partition $\{P_1, \ldots, P_m\}$ and elementary $BL_\triangle$-functions $f_1, \ldots, f_m$ satisfying, for each index $i \in \{1, \ldots, m\}$:*

1. $\mathrm{dom}(f_i) = P_i$;
2. *f coincides with f_i over P_i.*

The operations are defined pointwise.

5.2 Free BL-algebras

In this section we state the functional representation theorem for the n-generated free BL-algebra $\boldsymbol{F}^n_{\mathbb{BL}}$. Recall that the subvariety generated by the class of all n-generated BL-algebras is generated by the algebra $(n+1)[0,1]_{\text{Ł}}$, hence $\boldsymbol{F}^n_{\mathbb{BL}}$ is (isomorphic to) the subalgebra of the algebra of all the functions $f\colon ((n+1)[0,1]_{\text{Ł}})^n \to (n+1)[0,1]_{\text{Ł}}$ generated by the projections. By Lemma 1.0.12, it is thus sufficient to characterise this set.

Roughly speaking, we characterize elements of $\boldsymbol{F}^n_{\mathbb{BL}}$, which we call *n-ary* BL*-functions*, in terms of a finite collection of suitable elements of the free n-generated Wajsberg hoop $\boldsymbol{F}^n_{\mathbb{WH}}$, suitably arranged over a finite, partially ordered, combinatorial structure, which we call the *Fubini tree* of the set $\{1, \ldots, n\}$ (see Definition 5.2.1). Each node of a Fubini tree corresponds bijectively to a non-redundant cell $\prod_{i=1}^n I_{i_k}$, with each $i_k \neq n+1$, of Definition 5.1.3; the additional partially ordered structure imposed on the set of cells will record the interdependencies between cells. We call these arrangements *(n-ary) encodings*. Each encoding uniquely determines an n-ary real valued function over $[0, n+1]^n$, in fact, a BL-function. The set of n-ary BL-functions, equipped with operations defined pointwise by the fundamental operations of $(n+1)[0,1]_{\text{Ł}}$, is the free n-generated BL-algebra.

Throughout the section let $[n]$ denote the set $\{1, 2, \ldots, n\}$. We start with some basic definitions. Let $K \subseteq [n]$ have cardinality $m = |K|$, and let S be a set. We call any function $\mathbf{x} \colon K \to S$ an *m-tuple indexed by K*, and we safely regard $\mathbf{x}$ as a point in the m-dimensional space S^m whose coordinates are indexed by K. For each point $\mathbf{x} = \langle x_1, \ldots, x_n\rangle \in S^n$ its *projection over S^K*, denoted $\pi_K(\mathbf{x})$, is the m-tuple indexed by K defined as $k \in K \mapsto x_k = (\mathbf{x})_k$. If $R \subseteq S^n$, the *projection of R over S^K* is $\pi_K(R) = \{\pi_K(\mathbf{x}) \mid \mathbf{x} \in R\}$. Hence, $S^{[n]} = \pi_{[n]}(S^n) = S^n$. Even though $S^{[n]} = S^n$ we consistently write $S^{[n]}$ throughout the rest of this section, since we shall very often be dealing with points in S^K for subsets K of $[n]$.

Assume $\mathbf{x}_1 \in S^{K_1}$ and $\mathbf{x}_2 = S^{K_2}$ for $K_1, K_2 \subseteq [n]$ and $K_1 \cap K_2 = \emptyset$. Then we define the point

$$\mathbf{x}_1 \dot{\times} \mathbf{x}_2$$

as the point $\mathbf{x} \in S^{K_1 \cup K_2}$ such that $\pi_{K_1}(\mathbf{x}) = \mathbf{x}_1$ and $\pi_{K_2}(\mathbf{x}) = \mathbf{x}_2$, that is, $\mathbf{x}$ is the point whose coordinate of index $j \in K_1 \cup K_2$ equals coordinate $(\mathbf{x}_1)_j$ if j is in K_1 and coordinate $(\mathbf{x}_2)_j$ if j is in K_2. For subsets R_1, R_2 of S we denote $R_1^{K_1} \dot{\times} R_2^{K_2}$ the set $\{\pi_{K_1}(\mathbf{x}) \dot{\times} \pi_{K_2}(\mathbf{y}) \mid \mathbf{x} \in R_1^{[n]}, \mathbf{y} \in R_2^{[n]}\}$.

DEFINITION 5.2.1. *A* Fubini partition *R of $[n]$ is an ordered partition of $[n]$ into nonempty subsets, that is, $R = \langle\{B_1, \ldots, B_l\}, \leq\rangle$ where the set $\{B_1, \ldots, B_l\}$ is a partition of $[n]$ and $B_i \leq B_j$ if $i \leq j$. In short, we write $R = \langle B_1, \ldots, B_l\rangle$. We denote by $\mathscr{F}_{[n]}$ the set of Fubini partitions of $[n]$.*

Let $R, S \in \mathscr{F}_{[n]}$. We say that R precedes *S, in symbols $R \leq_{\mathscr{F}} S$, if R has the form $\langle B_1, \ldots, B_l, B_{l+1}\rangle$ and $S = \langle B_1, \ldots, B_l, C_1, \ldots, C_m\rangle$ for $m \geq 1$. Say that S* covers *R, or that R is* the predecessor *of S, in symbols $R \prec_{\mathscr{F}} S$, if R has the form $\langle B_1, \ldots, B_l, B_{l+1}\rangle$ and $S = \langle B_1, \ldots, B_l, C_1, C_2\rangle$.*

Note $\langle\mathscr{F}_{[n]}, \leq_{\mathscr{F}}\rangle$ is a tree; by a slight abuse of notation we shall denote $\min(\mathscr{F}_{[n]})$ $= \langle[n]\rangle$ simply by $[n]$.

EXAMPLE 5.2.2. Let $n = 2$. Then, $\langle\mathscr{F}_{[2]}, \leq_{\mathscr{F}}\rangle$ is a tree with root [2], covered by the leaves $\langle\{1\}, \{2\}\rangle$ and $\langle\{2\}, \{1\}\rangle$. Let $n = 3$. Then, $\langle\mathscr{F}_{[3]}, \leq_{\mathscr{F}}\rangle$ is a tree with root [3], covered by the leaves $\langle\{1, 2\}, \{3\}\rangle$, $\langle\{1, 3\}, \{2\}\rangle$, $\langle\{2, 3\}, \{1\}\rangle$, and the nodes $\langle\{1\}, \{2, 3\}\rangle$, $\langle\{2\}, \{1, 3\}\rangle$, and $\langle\{3\}, \{1, 2\}\rangle$; the node $\langle\{1\}, \{2, 3\}\rangle$ is covered by the leaves $\langle\{1\}, \{2\}, \{3\}\rangle$ and $\langle\{1\}, \{3\}, \{2\}\rangle$; the node $\langle\{2\}, \{1, 3\}\rangle$ is covered by the leaves $\langle\{2\}, \{1\}, \{3\}\rangle$ and $\langle\{2\}, \{3\}, \{1\}\rangle$; and finally the node $\langle\{3\}, \{1, 2\}\rangle$ is covered by the leaves $\langle\{3\}, \{1\}, \{2\}\rangle$ and $\langle\{3\}, \{2\}, \{1\}\rangle$.

On the basis of $F_{[n]}$, we define a partition of the domain $[0, n+1]^{[n]}$ of n-ary BL-functions. We let $\{P_0, P_1\}$ be the following partition of $[0, n+1]^{[n]}$: $P_1 = [1, n+1]^{[n]}$ and $P_0 = [0, n+1]^{[n]} \setminus P_1$. For $b \in \{0, 1\}$, each point $\mathbf{x} \in P_b$ is associated with a Fubini partition $R_{\mathbf{x}}$, that reflects the linear order among the integer parts of the components of $\mathbf{x}$, in the following terms.

DEFINITION 5.2.3. *For each $b \in \{0,1\}$ and every point $\mathbf{x} \in P_b$, we let $R_\mathbf{x}$ denote the Fubini partition* of $\mathbf{x}$, *defined as follows. Let $J = \{\lfloor x_j \rfloor \mid j \in [n]\}$. For each $j \in J$ let $B_j = \{k \in [n] \mid \lfloor x_k \rfloor = j\}$. If $\{n, n+1\} \subseteq J$ then let $J' = J \setminus \{n+1\}$, and $C_n = B_n \cup B_{n+1}$; let further $C_j = B_j$ for each $j \in J \setminus \{n, n+1\}$. If $\{n, n+1\} \not\subseteq J$ then let $J' = J$ and $C_j = B_j$ for each $j \in J$. Set $R_\mathbf{x} = \{C_j \mid j \in J'\}$ with $C_i \leq C_j$ iff $i \leq j$. For every $b \in \{0,1\}$ and every Fubini partition R of $[n]$, we let*

$$P_{b,R} = \{\mathbf{x} \in P_b \mid R_\mathbf{x} = R\}.$$

EXAMPLE 5.2.4. Let $n = 3$, and let $\mathbf{x} = \langle x_1, x_2, x_3 \rangle = \langle 1.2, 4, 0.2 \rangle \in [0,4]^{[3]}$. Then $J = \{0,1,4\}$, $B_0 = \{3\}$, $B_1 = \{1\}$, and $B_4 = \{2\}$. Here, $J' = J$, $C_0 = B_0$, $C_1 = B_1$, $C_4 = B_4$, and $R_\mathbf{x} = \langle C_0, C_1, C_4 \rangle$. Also, $P_{0,\langle\{3\},\{1\},\{2\}\rangle} = \{\langle x_1, x_2, x_3 \rangle \in [0,4]^{[3]} \mid 0 = \lfloor x_3 \rfloor < \lfloor x_1 \rfloor < \lfloor x_2 \rfloor\}$, and $P_{1,\langle\{3\},\{1\},\{2\}\rangle} = \{\langle x_1, x_2, x_3 \rangle \in [0,4]^{[3]} \mid 1 = \lfloor x_3 \rfloor < \lfloor x_1 \rfloor < \lfloor x_2 \rfloor\}$. The point $\mathbf{x}$ above is in $P_{0,\langle\{3\},\{1\},\{2\}\rangle}$.

Note that $\{P_{b,R} \mid b \in \{0,1\},\ R \in \mathscr{F}_{[n]}\}$ is a partition of $[0, n+1]^n$. As a matter of fact each set $P_{b,R}$ is the union of a finite number of cells as defined in Definition 5.1.3. In particular, $P_{b,R}$ contains exactly one non-redundant cell: the behaviour of a BL-function f over other cells in $P_{b,R}$ is determined by the behaviour of f over the non-redundant one.

EXAMPLE 5.2.5. Let $n = 2$. Then $\{P_{b,[2]}, P_{b,\langle\{1\},\{2\}\rangle}, P_{b,\langle\{2\},\{1\}\rangle} \mid b = 0,1\}$ forms a partition of $[0,3]^{[2]}$. In particular, $P_{0,[2]} = \{\langle x_1, x_2 \rangle \mid 0 \leq x_1, x_2 < 1\}$, $P_{0,\langle\{1\},\{2\}\rangle} = \{\langle x_1, x_2 \rangle \mid 0 \leq x_1 < 1 \leq x_2 \leq 3\}$, $P_{0,\langle\{2\},\{1\}\rangle} = \{\langle x_1, x_2 \rangle \mid 0 \leq x_2 < 1 \leq x_1 \leq 3\}$, $P_{1,[2]} = \{\langle x_1, x_2 \rangle \mid 1 \leq x_1, x_2 < 2 \text{ or } 2 \leq x_1, x_2 \leq 3\}$, $P_{1,\langle\{1\},\{2\}\rangle} = \{\langle x_1, x_2 \rangle \mid 1 \leq x_1 < 2 \leq x_2 \leq 3\}$, $P_{1,\langle\{2\},\{1\}\rangle} = \{\langle x_1, x_2 \rangle \mid 1 \leq x_2 < 2 \leq x_1 \leq 3\}$.

We now need to adapt the definition of polyhedral complex given in Section 2.1 to the BL-functions setting.

DEFINITION 5.2.6. *An* open polyhedron *is the relative interior of a polyhedron. An* open polyhedral complex *is the set of relative interiors of the polyhedra of a polyhedral complex.*

Note that linear semialgebraic subsets are finite unions of open polyhedra, or, equivalently, supports of open polyhedral complexes. It will turn out that for every BL-function f there exists an open polyhedral complex Σ whose support is $[0, n+1]^n$ such that f is linear over each open polyhedron in Σ.

We shall associate with each Fubini partition $R \in \mathscr{F}_{[n]}$ a partially defined function f over $[0,1]^{[n]}$, which we call a *prismwise* Wajsberg function. The domain of f will be an open polyhedral complex, and over each element of this complex, f will coincide with a Wajsberg function in $\boldsymbol{F}^n_{\mathbb{WH}}$.

Recall Definition 5.1.5 for the notion of a function essentially $|K|$-ary over $[0,1]^K$.

DEFINITION 5.2.7 (Prismwise Wajsberg Function). *Let K be a nonempty subset of $[n]$. Let $\{\Delta_i \mid i \in [l]\}$ be an open polyhedral complex in $[0,1]^{[n]\setminus K}$ such that $\Delta_i \subseteq [0,1)^{[n]\setminus K}$ for every $i \in [l]$. An n-ary* prismwise Wajsberg function *is a function f from*

$$\mathrm{dom}(f) = \bigcup_{i \in [l]} \Delta_i \dot{\times} [0,1]^K$$

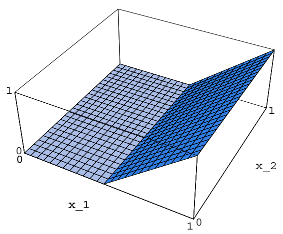
to $[0,1]$ *such that for each* Δ_i*, there exists a Wajsberg function* g *in* $\boldsymbol{F}^n_{\mathbb{WH}}$*, essentially* $|K|$*-ary over* $[0,1]^K$*, such that the restriction of* f *to the* prism $\Delta_i \dot\times [0,1]^K$ *coincides with* g*, namely,*

$$f \upharpoonright \Delta_i \dot\times [0,1]^K = g.$$

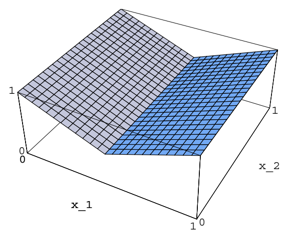
We call $\{\Delta_i\}_{i\in[l]}$ *the* base *of* f *and* g *the* realization *of* f *over* Δ_i*. If* $\mathbf{x} \in \mathrm{dom}(f)$*, then there exists a unique* Δ_i *in the base of* f *such that* $\mathbf{x} \in \Delta_i \dot\times [0,1]^K$*. Therefore, we also say that* g *is the realization of* f *over* $\mathbf{x}$*. We let* $\mathbb{PW}_{[n]}$ *denote the set of* n*-ary prismwise Wajsberg functions.*

We let $\emptyset$ denote the unique n-ary prismwise Wajsberg function with empty domain. Note that in particular, the Wajsberg function $g \in \boldsymbol{F}^n_{\mathbb{WH}}$ is the n-ary prismwise Wajsberg function coinciding with g over the domain $[0,1]^{[n]}$.

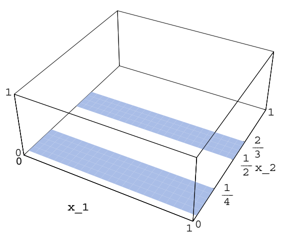
Observe that both $\mathrm{dom}(f)$ and $[0,1]^{[n]} \setminus \mathrm{dom}(f)$ are supports of open polyhedral complexes whose polyhedra are prisms of the form $\Delta \dot\times [0,1]^K$ for suitable open polyhedra Δ. We remark that each prism $\Delta \dot\times [0,1]^K$ is a linear semialgebraic subset of $[0,1]^n$. On the other hand, each linear semialgebraic subset of a cell $C = \prod_{i=1}^n I_{k_i}$, with each $k_i \le n$, has the form $\{\langle k_1,\ldots,k_n\rangle + \mathbf{x} \mid \mathbf{x} \in \bigcup_{i\in I} \Delta_i \dot\times [0,1)^K\}$ for some open polyhedral complex $\{\Delta_i\}_{i\in I}$ and $K \subseteq [n]$.



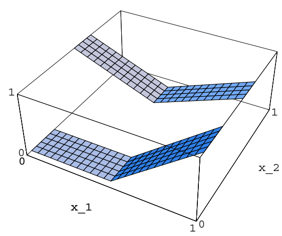
(a) $f_1 \in \mathbb{WH}_{[2]}$.



(b) $f_2 \in \mathbb{WH}_{[2]}$.



(c) $\mathrm{dom}(f)$.



(d) $f \in \mathbb{PW}_{[2]}$.

Figure 13. Sampling Definition 5.2.7 with $n = 2$. Let $K = \{1\}$. (a) and (b) sketch Wajsberg functions f_1 and f_2 in $\mathbb{WH}_{[2]}$. Both f_1 and f_2 are essentially unary over $[0,1]^{\{1\}}$. Let $\Delta_1 = \{\langle x_2\rangle \mid 0 < x_2 < 1/4\} \subseteq [0,1)^{\{2\}}$, and let $\Delta_2 = \{\langle x_2\rangle \mid 1/2 < x_2 < 2/3\} \subseteq [0,1)^{\{2\}}$. The base of f is $\{\langle x_2\rangle \mid 0 < x_2 < 1/4 \text{ or } 1/2 < x_2 < 2/3\} \subseteq [0,1)^{\{2\}}$. (c) and (d) describe the prismwise Wajsberg function f in $\mathbb{PW}_{[2]}$ whose realizations over Δ_1 and Δ_2 are f_1 and f_2 respectively. Namely, f coincides with f_1 over $\Delta_1 \times [0,1]^{\{1\}}$, and with f_2 over $\Delta_2 \times [0,1]^{\{1\}}$.

We are now ready to formally introduce the notion of *encodings* of BL-functions.

DEFINITION 5.2.8 (Encoding). *The set $A_{[n]}$ of n-ary* encodings *is the set of all pairs of functions*

$$\langle L\colon \mathscr{F}_{[n]} \to \mathbb{PW}_{[n]},\, H\colon \mathscr{F}_{[n]} \to \mathbb{PW}_{[n]}\rangle$$

satisfying the following conditions:

(i) $\mathrm{dom}(L([n])) = [0,1]^{[n]}$ *and* $L([n]) \in \boldsymbol{F}^n_{\mathbb{WH}}$.

(ii) $\mathrm{dom}(H([n])) = [0,1]^{[n]}$ *and* $H([n]) \in \boldsymbol{F}^n_{\mathbb{WH}}$, *or* $\mathrm{dom}(H([n])) = \emptyset$.

(iii) *Let $I \in \{L, H\}$, $S \prec_{\mathscr{F}} R \in \mathscr{F}_{[n]}$ and let K be the maximum block of R. Let further F_K be the $([n] \setminus K)$-dimensional* semi-open face

$$F_K = [0,1)^{[n]\setminus K} \dot{\times} \{1\}^K ,$$

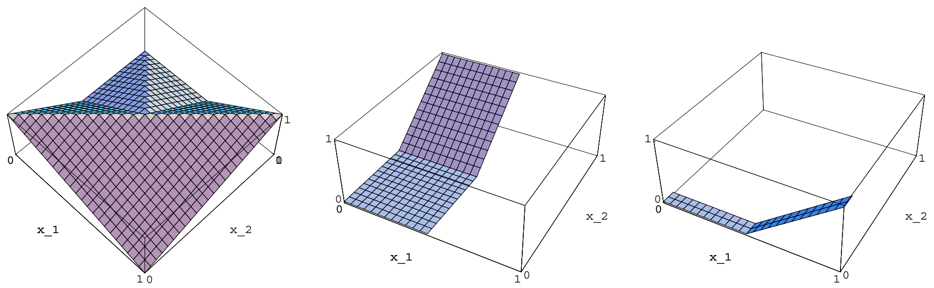
and

$$\Delta_{K,I,S} = \pi_{[n]\setminus K}\left(I(S)^{-1}(b) \cap F_K\right) ,$$

where $b \in \{0,1\}$ is equal to 0 if and only if: $I = L$, $S = [n]$, and $\mathrm{dom}(H(S)) = \emptyset$. *Then, for $I \in \{L, H\}$, the function I is essentially $|K|$-ary over K and its domain is:*

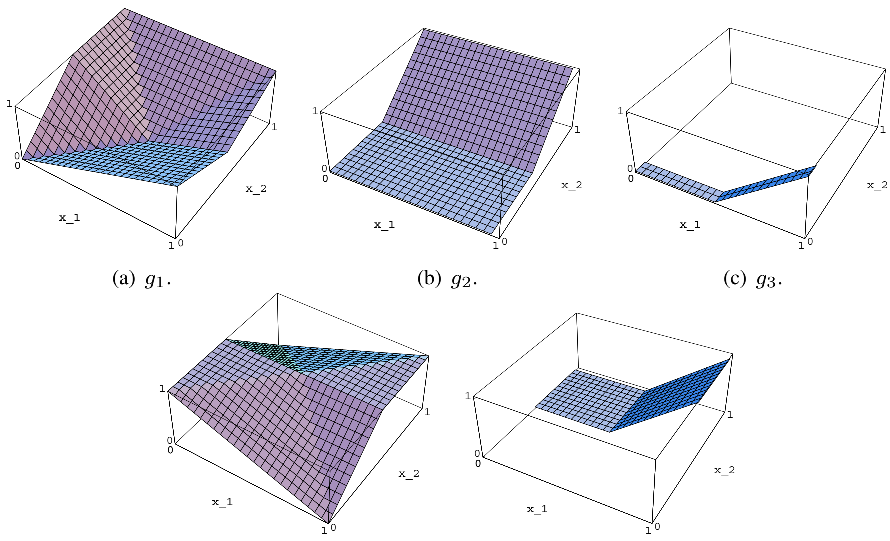
$$\mathrm{dom}(I(R)) = \Delta_{K,I,S} \dot{\times} [0,1]^K.$$

The role of the bit b in Definition 5.2.8 can be understood by comparison with Theorem 2.2.4.



(a) h_1. (b) h_2. (c) h_3.

Figure 14. Sampling Definition 5.2.8 with b=0. E_0=$\langle L_0, H_0\rangle$$\in$$A_{[2]}$. $L_0\colon F_{[2]} \to \mathbb{PW}_{[2]}$ is the map $[2]$$\mapsto$$h_1$, $\langle\{1\},\{2\}\rangle$$\mapsto$$h_2$, $\langle\{2\},\{1\}\rangle$$\mapsto$$h_3$. Here, $\mathrm{dom}(h_1)$=$[0,1]^{[2]}$. As b=0, we have $H_0([2])$=$\mathrm{dom}(H_0([2]))$=$\emptyset$, and since then $H_0([2])^{-1}(1)$=$\emptyset$, we have that the map $H_0\colon F_{[2]} \to \mathbb{PW}_{[2]}$ is identically $\emptyset$. By inspection, $\mathrm{dom}(h_2) = \{\langle x_1\rangle \in [0,1)^{\{1\}} \mid 0 \le x_1 \le 1/2\} \times [0,1]^{\{2\}}$, since $\pi_{\{1\}}\left(h_1^{-1}(0) \cap \{\langle x_1, x_2\rangle \mid 0 \le x_1 < 1, x_2 = 1\}\right) = \{\langle x_1\rangle \in [0,1)^{\{1\}} \mid 0 \le x_1 \le 1/2\}$; $\mathrm{dom}(h_3) = \{\langle x_2\rangle \in [0,1)^{\{2\}} \mid 0 = x_2\} \times [0,1]^{\{1\}}$, since $\pi_{\{2\}}\left(h_1^{-1}(0) \cap \{\langle x_1, x_2\rangle \mid 0 \le x_2 < 1, x_1 = 1\}\right) = \{\langle x_2\rangle \in [0,1)^{\{1\}} \mid 0 = x_2\}$.



(a) g_1. (b) g_2. (c) g_3.

(d) g_4. (e) g_5.

Figure 15. Sampling Definition 5.2.8 with $b = 1$. $E_1 = \langle L_1, H_1\rangle \in A_{[2]}$. $L_1 \colon F_{[2]} \to \mathbb{PW}_{[2]}$ is the map $[2] \mapsto g_1$, $\langle\{1\},\{2\}\rangle \mapsto g_2$, $\langle\{2\},\{1\}\rangle \mapsto g_3$. $H_1 \colon F_{[2]} \to \mathbb{PW}_{[2]}$ is the map $[2] \mapsto g_4$, $\langle\{1\},\{2\}\rangle \mapsto \emptyset$, $\langle\{2\},\{1\}\rangle \mapsto g_5$. g_1 and g_4 are in $\mathbb{WH}_{[2]}$, with $\mathrm{dom}(g_1) = \mathrm{dom}(g_4) = [0,1]^{[2]}$. By inspection, $\mathrm{dom}(g_2) = \{\langle x_1\rangle \in [0,1)^{\{1\}} \mid 0 \le x_1 < 1\} \times [0,1]^{\{2\}}$, since $\pi_{\{1\}}(g_1^{-1}(1) \cap \{\langle x_1, x_2\rangle \mid 0 \le x_1 < 1, x_2 = 1\}) = \{\langle x_1\rangle \in [0,1)^{\{1\}} \mid 0 \le x_1 < 1\}$; $\mathrm{dom}(g_3) = \{\langle x_2\rangle \in [0,1)^{\{2\}} \mid 0 = x_2\} \times [0,1]^{\{1\}}$, since $\pi_{\{2\}}(g_1^{-1}(1) \cap \{\langle x_1, x_2\rangle \mid 0 \le x_2 < 1, x_1 = 1\}) = \{\langle x_2\rangle \in [0,1)^{\{1\}} \mid 0 = x_2\}$; and $\mathrm{dom}(g_5) = \{\langle x_2\rangle \in [0,1)^{\{2\}} \mid 1/2 \le x_2 < 1\} \times [0,1]^{\{1\}}$, since $\pi_{\{2\}}(g_4^{-1}(1) \cap \{\langle x_1, x_2\rangle \mid 0 \le x_2 < 1, x_1 = 1\}) = \{\langle x_2\rangle \in [0,1)^{\{1\}} \mid 1/2 \le x_2 < 1\}$. Notice that $H((\{1\},\{2\})) = \emptyset$, as $\pi_{\{1\}}(g_4^{-1}(1) \cap \{\langle x_1, x_2\rangle \mid 0 \le x_1 < 1, x_2 = 1\}) = \emptyset$.

In most cases, we can get easily rid of the necessity of explicitly dealing with the value of b by using the following simplification.

Let $\langle L, H\rangle \in A_{[n]}$ and $I \in \{L, H\}$. Then we denote by $I_+(R)$ the function

$$I_+(R)(\mathbf{x}) \quad = \begin{cases} 1 - I(R)(\mathbf{x}) & \text{if } I = L,\ R = [n] \text{ and } H([n]) = \emptyset, \\ I(R)(\mathbf{x}) & \text{otherwise.} \end{cases}$$

Analogously, if $g \in \boldsymbol{F}^n_{\mathbb{WH}}$ is realizing $I(R)$ over Δ_i (for $\{\Delta_i\}_{i\in I}$ being the base of $I(R)$) then we let $g_+ = 1 - g$ if $I = L$, $R = [n]$ and $H([n]) = \emptyset$, and $g_+ = g$, otherwise.

As the terminology suggests, the purpose of encodings is that of *specifying* certain real valued functions from $[0, n+1]^{[n]}$ to $[0, n+1]$, in fact, BL-functions. The task now in order is the following. Given an encoding $A \in A_{[n]}$, describe a uniquely determined map from $[0, n+1]^{[n]}$ to $[0, n+1]$ in terms of the information encapsulated in A: the BL-function encoded by A.

For each $x \in [0, n+1]$, we let $\lfloor x \rceil = 1$ if $x = n+1$, and $\lfloor x \rceil = x - \lfloor x \rfloor$ otherwise. For each $\mathbf{x} = \langle x_1, \ldots, x_n \rangle \in [0, n+1]^{[n]}$, we let $\lfloor \mathbf{x} \rceil = \langle \lfloor x_1 \rceil, \ldots, \lfloor x_n \rceil \rangle \in [0,1]^{[n]}$.

To evaluate the encoding $A = \langle L, H \rangle$ at $\mathbf{x}$ we shall visit the branch constituted by the partitions $\leq_{\mathscr{F}} R_{\mathbf{x}}$ in the tree $\mathrm{dom}(L)$ if $\mathbf{x} \in P_0$, and in the tree $\mathrm{dom}(H)$ if $\mathbf{x} \in P_1$. As a matter of fact, if $\lfloor \mathbf{x} \rceil$ does not belong to the domain of $I(R_{\mathbf{x}})$, then the value of the BL-function encoded by A at $\mathbf{x}$ is given by a function $I(S)$ for some $S <_{\mathscr{F}} R_{\mathbf{x}}$ at a point $\mathbf{x}'$ which precedes $\lfloor \mathbf{x} \rceil$ in the so-called *ascending sequence* of $\mathbf{x}$. The ascending sequence of a point starts from the 0th step $\langle 1, 1, \ldots, 1 \rangle = \{1\}^{[n]}$, and then fixes at the ith step only the (fractional part of the) components of $\mathbf{x}$ that appear in the i-th block of $R_{\mathbf{x}}$.

DEFINITION 5.2.9. *Consider $A = \langle I_0, I_1 \rangle \in A_{[n]}$ and let $b \in \{0, 1\}$. For each $R = \langle B_1, \ldots, B_k \rangle \in \mathscr{F}_{[n]}$ let $\mathbf{x} \in P_{b,R}$. Display the totally ordered subset $\{S \in \mathscr{F}_{[n]} \mid S \leq_{\mathscr{F}} R\}$ as*

$$[n] = R_1 \prec_{\mathscr{F}} R_2 \prec_{\mathscr{F}} \cdots \prec_{\mathscr{F}} R_k = R\,.$$

We define the ascending sequence *of* $\mathbf{x}$ *as*

$$\{1\}^{[n]} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k = \lfloor \mathbf{x} \rceil\,,$$

where for each $i \in \{0, 1, \ldots, k-1\}$ and $j \in [n]$, $(\mathbf{x}_i)_j = 1$ if j belongs to the maximum block of R_{i+1} and $(\mathbf{x}_i)_j = x_j - \lfloor x_j \rfloor$ otherwise. Equivalently we can write $\mathbf{x}_i = \pi_{[n] \setminus \bigcup_{j=i+1}^{k} B_j}(\mathbf{x}_k - \lfloor \mathbf{x}_k \rfloor) \dot{\times} \{1\}^{\bigcup_{j=i+1}^{k} B_j}$.

The evaluation depth $d_{\mathbf{x}}^A$ *of* $\mathbf{x}$ *in A is*

$$d_{\mathbf{x}}^A = \max\{i \in [k] \mid \mathbf{x}_i \in \mathrm{dom}(I_b(R_i))\}$$

if this maximum exists, $d_{\mathbf{x}}^A = 0$ otherwise. Notice that $d_{\mathbf{x}}^A = 0$ if and only if $b = 1$ and $\mathrm{dom}(I_1([n])) = \emptyset$. In this case $d_{\mathbf{y}}^A = 0$ for all $\mathbf{y} \in P_1$.

We call

$$\mathbf{f}_{\mathbf{x}}^A = \mathbf{x}_{d_{\mathbf{x}}^A}$$

the fractional part *of* $\mathbf{x}$ *in A. If $d_{\mathbf{x}}^A > 0$ we call the integer $i_{\mathbf{x}}^A$ such that*

$$i_{\mathbf{x}}^A = \min(\{\lfloor x_j \rfloor \mid j \in B_{d_{\mathbf{x}}^A}\} \cup \{n\})\,,$$

the integer part *of* $\mathbf{x}$ *in A. Further, we call the Wajsberg function g realizing $I_b(R_{d_{\mathbf{x}}^A})$ over $\mathbf{x}_{d_{\mathbf{x}}^A}$ the* responsible *for* $\mathbf{x}$ *in A. Finally, if $d_{\mathbf{x}}^A = 0$ then the* responsible *of* $\mathbf{x}$ *in A is the function constantly 0 and the integer part $i_{\mathbf{x}}^A$ is set to 0.*

EXAMPLE 5.2.10. Let $\mathbf{x} = \langle .2, 2.7, 4 \rangle$, $\mathbf{y} = \langle .2, 2.7, 3.5 \rangle$, $\mathbf{z} = \langle .6, 2.4, 3.8 \rangle$ be points in $[0, 4]^{[3]}$. In particular note that $\mathbf{x}, \mathbf{y}, \mathbf{z} \in P_{0, \langle \{1\}, \{2\}, \{3\} \rangle}$. Then, the ascending sequence of $\mathbf{x}$ is $\mathbf{x}_0 = \langle 1, 1, 1 \rangle$, $\mathbf{x}_1 = \langle .2, 1, 1 \rangle$, $\mathbf{x}_2 = \langle .2, .7, 1 \rangle$, $\mathbf{x}_3 = \langle .2, .7, 1 \rangle$; analogously, the ascending sequence of $\mathbf{y}$ is $\mathbf{y}_0 = \langle 1, 1, 1 \rangle$, $\mathbf{y}_1 = \langle .2, 1, 1 \rangle$, $\mathbf{y}_2 = \langle .2, .7, 1 \rangle$, $\mathbf{y}_3 = \langle .2, .7, .5 \rangle$; finally, the ascending sequence of $\mathbf{z}$ is $\mathbf{z}_0 = \langle 1, 1, 1 \rangle$, $\mathbf{z}_1 = \langle .6, 1, 1 \rangle$, $\mathbf{z}_2 = \langle .6, .4, 1 \rangle$, $\mathbf{z}_3 = \langle .6, .4, .8 \rangle$. Assume now that $A = \langle L, H \rangle \in A_{[3]}$ is such that $\mathrm{dom}(L(\langle \{1, 2, 3\} \rangle)) = [0, 1]^{[3]}$, $\mathrm{dom}(L(\langle \{1\}, \{2, 3\} \rangle)) = [.3, .6]^{\{1\}} \times [0, 1]^{\{2,3\}}$, and $\mathrm{dom}(L(\langle \{1\}, \{2\}, \{3\} \rangle)) = [.3, .6]^{\{1\}} \times [.5, .7]^{\{2\}} \times [0, 1]^{\{3\}}$. Then $d_{\mathbf{x}}^A = d_{\mathbf{y}}^A = 1$ and $d_{\mathbf{z}}^A = 2$; $\mathbf{f}_{\mathbf{x}}^A = \mathbf{x}_1 = \langle .2, 1, 1 \rangle = \mathbf{y}_1 = \mathbf{f}_{\mathbf{y}}^A$ and $\mathbf{f}_{\mathbf{z}}^A = \mathbf{z}_2 = \langle .6, .4, 1 \rangle$; finally, $i_{\mathbf{x}}^A = 0 = i_{\mathbf{y}}^A$ and $i_{\mathbf{z}}^A = 2$.

Finally, we define the map F, that sends the encodings in $A_{[n]}$ to functions from $[0, n+1]^{[n]}$ to $[0, n+1]$.

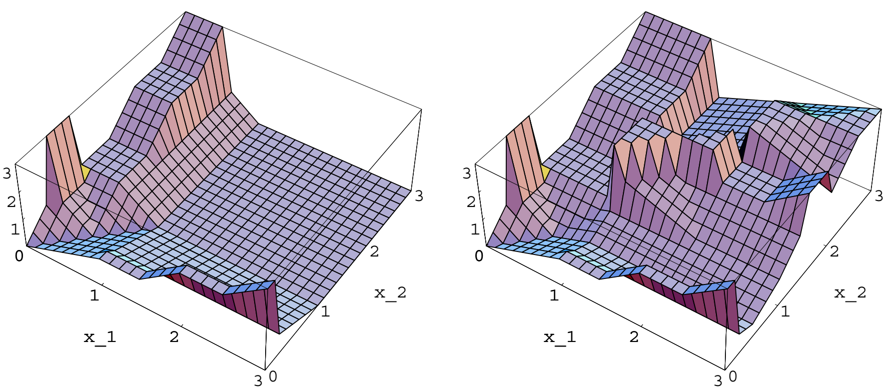
DEFINITION 5.2.11 (BL-functions). *We let F be the map*

$$A = \langle I_0, I_1 \rangle \in A_n \mapsto f_A \in [0, n+1]^{[0,n+1]^{[n]}}$$

defined as follows. Let $\mathbf{x} = \langle x_1, \ldots, x_n \rangle \in [0, n+1]^{[n]}$, let $b \in \{0, 1\}$ be such that $\mathbf{x} \in P_b$, and let $R = R_{d^A_{\mathbf{x}}} \in \mathscr{F}_{[n]}$. Then:

$$f_A(\mathbf{x}) \quad = \begin{cases} I_{b+}(R)(\mathbf{f}^A_{\mathbf{x}}) + i^A_{\mathbf{x}} & \textit{if } I_{b+}(R)(\mathbf{f}^A_{\mathbf{x}}) < 1, \\ n+1 & \textit{otherwise.} \end{cases}$$

We let $F(A_{[n]}) = \{F(A) \mid A \in A_{[n]}\}$ denote the set of n-ary BL*-functions.*



(a) $F(E_0)\colon [0,3]^{[2]} \to [0,3]$. (b) $F(E_1)\colon [0,3]^{[2]} \to [0,3]$.

Figure 16. Sampling Definition 5.2.11, with the encodings E_0 and E_1 given in Figure 14 and Figure 15 respectively.

THEOREM 5.2.12 (Functional Representation). *The algebra*

$$\langle F(A_{[n]}), \odot, \to, \bot \rangle,$$

where the operations are pointwise defined as the corresponding operations of algebra $(n+1)[0,1]_{Ł}$, is the free n-generated BL*-algebra.*

It is possible to directly equip the set $A_{[n]}$ with a structure of an BL-algebra, turning it into the free n-generated BL-algebra. Details are given in [5].

We remark that the n-generated free $\mathbb{BL}_\triangle$-algebra can be represented as a pair of functions $\langle L\colon \mathscr{F}_{[n]} \to \mathbb{PW}_{[n]}, H\colon \mathscr{F}_{[n]} \to \mathbb{PW}_{[n]} \rangle$ satisfying conditions (i) and (ii) of Definition 5.2.8, while in condition (iii) the requirements on $\Delta_{K,I,S}$ are relaxed to $\Delta_{K,I,S} = \pi_{[n]\setminus K}(|\Sigma|)$ for some open polyhedral complex Σ such that $|\Sigma| \subseteq F_K$.

The proof of Theorem 5.2.12 splits in two parts. The proof that each element of $\boldsymbol{F}^n_{\mathbb{BL}}$ is the image under the map F of some encoding amounts to a lengthy verification. The proof that for each encoding $\langle I_0, I_1\rangle \in A_{[n]}$ it is possible to effectively construct a term φ such that

$$\varphi^{\boldsymbol{F}^n_{\mathbb{BL}}} = F(\langle I_0, I_1\rangle)$$

goes through a series of technical lemmas, whose proofs are in [5].

Until the end of this subsection, not to burden notation, we shall write $\varphi(\mathbf{t})$ instead of $\varphi^{\boldsymbol{F}^n_{\mathbb{BL}}}(\mathbf{t})$ for the interpretation of a term φ in $\boldsymbol{F}^n_{\mathbb{BL}}$.

We start introducing the derived connectives $\lhd$ and $\diamond$:

$$\begin{aligned} x \lhd y &= (x \to y) \odot ((y \to x) \to x); \\ x \diamond y &= ((x \lhd y) \to y) \wedge ((y \lhd x) \to x). \end{aligned}$$

LEMMA 5.2.13. *For each $i, j \in [n]$ and $\mathbf{t} = \langle t_1, \ldots, t_n\rangle \in [0, n+1]^{[n]}$:*

$$(x_i \lhd x_j)(\mathbf{t}) = \begin{cases} n+1 & \textit{if } \lfloor t_i \rfloor < \lfloor t_j \rfloor, \\ t_j & \textit{otherwise;} \end{cases}$$

$$(x_i \diamond x_j)(\mathbf{t}) = \begin{cases} n+1 & \textit{if } \lfloor t_i \rfloor = \lfloor t_j \rfloor, \\ \max\{t_i, t_j\} & \textit{otherwise;} \end{cases}$$

$$(\bot \lhd x_i)(\mathbf{t}) = \neg\neg x_i(\mathbf{t}) = \begin{cases} n+1 & \textit{if } 0 < \lfloor t_i \rfloor, \\ t_i & \textit{otherwise;} \end{cases}$$

$$(\bot \diamond x_i)(\mathbf{t}) = (\neg\neg x_i \to x_i)(\mathbf{t}) = \begin{cases} n+1 & \textit{if } 0 = \lfloor t_i \rfloor, \\ t_i & \textit{otherwise.} \end{cases}$$

Let $R = \langle B_1, \ldots, B_k\rangle \in \mathscr{F}_{[n]}$ be such that $i \in B_k$. Let $C = \{\langle j, j'\rangle \in B_m \times B_m \mid m \in [k]\}$, and let $D = \{\langle j, j'\rangle \in B_m \times B_{m+1} \mid m \in [k-1]\}$. We define terms $x_{i,0,R}$ and $x_{i,1,R}$, as follows:

$$x_{i,0,R} = \left(\left(\bigwedge_{j \in B_1} \bot \diamond x_j\right) \wedge \left(\bigwedge_{\langle j,j'\rangle \in C} x_j \diamond x_{j'}\right) \wedge \left(\bigwedge_{\langle j,j'\rangle \in D} x_j \lhd x_{j'}\right)\right) \to x_i\,;$$

$$x_{i,1,R} = \left(\left(\bigwedge_{j \in B_1} \bot \lhd x_j\right) \wedge \left(\bigwedge_{\langle j,j'\rangle \in C} x_j \diamond x_{j'}\right) \wedge \left(\bigwedge_{\langle j,j'\rangle \in D} x_j \lhd x_{j'}\right)\right) \to x_i\,.$$

LEMMA 5.2.14 (Cuboidwise Variable Isolation). *Let $i \in [n]$, let $b \in \{0, 1\}$, and let $R = \langle B_1, \ldots, B_k\rangle \in \mathscr{F}_{[n]}$ be such that $i \in B_k$. For every $\mathbf{t} = \langle t_1, \ldots, t_n\rangle \in [0, n+1]^{[n]}$,*

$$x_{i,b,R}(\mathbf{t}) = \begin{cases} t_i & \textit{if } \mathbf{t} \in P_{b,R}, \\ t_i & \textit{if } \mathbf{t} \in P_{b,R'},\ R \prec_{\mathscr{F}} \langle B_1, \ldots, B'_k, B''_k\rangle \leq_{\mathscr{F}} R',\ i \in B'_k, \\ n+1 & \textit{otherwise.} \end{cases}$$

Let $b \in \{0,1\}$, let $R = \langle B_1, \dots, B_k \rangle \in \mathscr{F}_{[n]}$, and let $g \in \mathbb{WH}_{[n]}$ be essentially $|B_k|$-ary over $[0,1]^{B_k}$. Finally, let ϕ_g be a Wajsberg hoop term such that $\phi_g^{\boldsymbol{F}_{\mathbb{WH}}^{|B_k|}} = g$. For every $i \in B_k$, we let $\phi_{g,b,R}$ denote the term obtained by substituting in ϕ_g, for every $i \in B_k$, the variable x_i by the term $x_{i,b,R}$. In symbols:

$$\phi_{g,b,R} = \phi_g[x_i \leftarrow x_{i,b,R} \mid i \in B_k].$$

Let $b \in \{0,1\}$, let $S, R \in \mathscr{F}_{[n]}$ such that

$$S = \langle B_1, \dots, B_{k-1} \cup B_k \rangle \prec_{\mathscr{F}} \langle B_1, \dots, B_{k-1}, B_k \rangle = R\,,$$

let g be a realization of $I_b(R)$, so that

$$I_b(R)|_{\Delta_1 \times \cdots \times \Delta_{k-1} \times [0,1]^{B_k}} = g\,,$$

where $\Delta_{k-1} \subseteq [0,1)^{B_{k-1}}$ is the relative interior of a polyhedron $\delta \subseteq [0,1]^{B_{k-1}}$. Let γ be any nonempty face of δ, and let $f_\gamma \in \boldsymbol{F}_{\mathbb{WH}}^n$ be a Wajsberg function essentially $|B_{k-1}|$-ary over $[0,1]^{B_{k-1}}$ such that

$$f_\gamma^{-1}(1) = (\gamma \cup \{1\}^{B_{k-1}}) \times [0,1]^{[n] \setminus B_{k-1}}.$$

LEMMA 5.2.15 (Prismwise Masking). *For every $i \in B_k$, and every point $\mathbf{t} \in [0, n+1]^{[n]}$ such that $x_{i,b,R}(\mathbf{x}) = t_i$, if $\pi_{B_{k-1}}(\lfloor \mathbf{t} \rceil) \notin \gamma$, then $\phi_{f_\gamma,b,S}(\mathbf{t}) < t_i$, otherwise, $\phi_{f_\gamma,b,S}(\mathbf{t}) = n+1$.*

For every $b \in \{0,1\}$, every $R \in \mathscr{F}_{[n]}$, every realization g of $I_b(R)$, and every $i \in B_k$, we define terms $x_{i,b,R,g}$ as follows:

$$\begin{aligned} x_{i,b,[n],g} &= x_{i,b,[n]}\,; \\ x_{i,b,R,g} &= \left(\bigwedge_{\gamma \neq \delta} (\phi_{f_\gamma,b,S} \to x_{i,b,R}) \right) \to (\phi_{f_\delta,b,S} \to x_{i,b,R})\,, \end{aligned}$$

where γ ranges over any proper nonempty face of δ. This construction satisfies the following statement.

LEMMA 5.2.16 (Prismwise Variable Isolation). *Let $b \in \{0,1\}$, let $R = \langle B_1, \dots, B_k \rangle \in \mathscr{F}_{[n]}$, let g be a realization of $I_b(R)$, and let $i \in B_k$. Let $\mathbf{t} = \langle t_1, \dots, t_n \rangle \in [0, n+1]^{[n]}$. Then:*

$$x_{i,b,R,g}(\mathbf{t}) = \begin{cases} t_i & \textit{if } \mathbf{t} \in P_{b,R,g}, \\ n+1 & \textit{otherwise.} \end{cases}$$

where $P_{b,R,g} = \{\mathbf{t} \in P_b \mid g \text{ realization of } I_b(R) \text{ responsible for } \mathbf{t} \text{ in } (I_0, I_1)\}$.

Let $b \in \{0,1\}$, $R = \langle B_1, \dots, B_k \rangle \in \mathscr{F}_{[n]}$ with $k \geq 1$, let g be any realization of $I_b(R)$, and let ϕ_g be a Wajsberg hoop term such that $\phi_g^{\boldsymbol{F}_{\mathbb{WH}}^n} = g$. Finally, let $\theta_{b,R,g} = \phi_g[x_i \leftarrow x_{i,b,R,g} \mid i \in B_k]$.

LEMMA 5.2.17 (Prismwise Term Isolation). *Let* $\mathbf{t} = \langle t_1, \ldots, t_n \rangle \in [0, n+1]^{[n]}$. *Then:*

$$\theta_{b,R,g}(\mathbf{t}) = \begin{cases} g(\mathbf{f}_{\mathbf{t}}^{(I_0,I_1)}) + i_{\mathbf{t}}^{(I_0,I_1)} & \text{if } \mathbf{t} \in P_{b,R,g} \text{ and } g(\mathbf{f}_{\mathbf{t}}^{(I_0,I_1)}) < 1, \\ n+1 & \text{otherwise.} \end{cases}$$

Finally we let the term φ be defined as follows. If $I_1([n]) = \emptyset$, then

$$\varphi = \neg\theta_{0,[n],g} \wedge \bigwedge_{R \neq [n]} \bigwedge_{h} \theta_{0,R,h},$$

where g is the realization of $I_0([n])$ and h ranges over the realizations of $I_0(R)$ for $R \in \mathscr{F}_{[n]}$; otherwise, if $I_1([n]) \neq \emptyset$,

$$\varphi = \bigwedge_{b \in \{1,0\}} \bigwedge_{R \in \mathscr{F}_{[n]}} \bigwedge_{h} \theta_{b,R,h},$$

where $h \in I_b(R)$ ranges over the realizations of $I_b(R)$ for $b \in \{0,1\}$ and $R \in \mathscr{F}_{[n]}$.

LEMMA 5.2.18 (Normal Form). *There is an algorithm that for every* $\langle I_0, I_1 \rangle \in A_{[n]}$ *outputs a term* φ *such that*

$$F(\langle I_0, I_1 \rangle) = \varphi^{\boldsymbol{F}^n_{\mathrm{BL}}}.$$

5.3 SBL-algebras and some revisitations

In this section we shall describe the free n-generated SBL-algebra as an algebra of suitable restrictions of the functions in the set $F(A_{[n]})$ of n-ary BL-functions. Analogously, we describe free algebras for some other subvarieties of $\mathbb{BL}$ as algebras of restrictions of $F(A_{[n]})$.

The SBL-algebra given by the ordinal sum $\{0,1\} \oplus n[0,1]_{\text{Ł}}$ is generic for the variety generated by all n-generated SBL-algebras. Since $\{0,1\} \oplus n[0,1]_{\text{Ł}}$ is a BL-subalgebra of $(n+1)[0,1]_{\text{Ł}}$, we have the following.

THEOREM 5.3.1. *For any integer* $n > 0$*, the free* n*-generated* SBL*-algebra* $\boldsymbol{F}^n_{\mathbb{SBL}}$ *is the algebra of restrictions of functions in* $F(A_{[n]})$ *to the set*

$$(\{0\} \cup [1, n+1])^n \,.$$

Analogous results hold for all subvarieties $\mathbb{V}$ of $\mathbb{BL}$ such that the variety generated by all n-generated $\mathbb{V}$-algebras is singly generated by a subalgebra of $(n+1)[0,1]_{\text{Ł}}$. As an example we give isomorphic representations of $\boldsymbol{F}^n_{\mathbb{MV}}$ and $\boldsymbol{F}^n_{\mathbb{G}}$. Note that the ordinal sum $\omega\{0,1\}$ of denumerably many copies of the two-element Boolean algebra is generic for $\mathbb{G}$.

THEOREM 5.3.2. *For any integer* $n > 0$*:*

- $\boldsymbol{F}^n_{\mathbb{MV}}$ *is isomorphic to the algebra of restrictions of functions in* $F(A_{[n]})$ *to the set*

$$([0,1) \cup \{n+1\})^n \,.$$

- $\boldsymbol{F}^n_{\mathbb{G}}$ *is isomorphic to the algebra of restrictions of functions in* $F(A_{[n]})$ *to the set*

$$\{0, 1, \ldots, n+1\}^n \,.$$

Section 5.4 combines this technique with the combinatorial spectral dualities to offer a framework where to deal with locally finite subvarieties of $\mathbb{BL}$.

It must be stressed that there are varieties of BL-algebras, corresponding to prominent logics, which are not amenable to the same treatment of Theorem 5.3.1 and Theorem 5.3.2. As a matter of fact, consider the variety of product algebras $\mathbb{P}$. As is well known it does not exist any product algebra that is generic for the variety generated by n-generated product algebras and that embeds into $(n+1)[0,1]_{Ł}$. Hence we cannot represent $\boldsymbol{F}^n_{\mathbb{P}}$ as an algebra of restrictions of functions in $F(A_{[n]})$. Roughly speaking, to accomplish such a task we should replace each copy of $[0,1]$ with a non-standard unit interval $[0,1]^*$. In this case, for instance, $\boldsymbol{F}^1_{\mathbb{P}}$ would be the algebra of restrictions of BL-functions to the set $\{0\} \cup (2-\epsilon, 2]$, for ϵ being an infinitesimal.

5.4 Locally finite varieties of BL-algebras

In this section we devise a spectral duality for locally finite BL-algebras. This kind of duality affords a combinatorial representation of those algebras, and in particular of finitely generated free algebras. The combinatorial representation provides us a with a powerful tool to investigate these algebras: for instance we can effectively compute products and coproducts and determine the exact cardinalities of free algebras. Compare with the analogous constructions given for Gödel algebras and related structures in Section 4.

Let FBL denote the category of finite BL-algebras and their homomorphisms. Let $\boldsymbol{A}$ be a finite BL-algebra. Recall that the *prime spectrum* of $\boldsymbol{A}$, in symbols $\mathrm{Spec}\boldsymbol{A}$, is the set of all prime filters of $\boldsymbol{A}$ ordered by reverse inclusion.

Given a finite BL-chain $\boldsymbol{C}$, let c denote its largest idempotent distinct from $\top^{\boldsymbol{C}}$. We define its *top part* as

$$T(\boldsymbol{C}) = \{x \in \boldsymbol{C} \mid x > c\}\,.$$

Let N^+ denote $\mathsf{N} \setminus \{0\}$. Further, for each prime filter $\mathfrak{p} \in \mathrm{Spec}\boldsymbol{A}$, we denote $\boldsymbol{A}/\mathfrak{p}$ the quotient of $\boldsymbol{A}$ modulo the congruence determined by $\mathfrak{p}$.

DEFINITION 5.4.1. *The* weighted spectrum *of $\boldsymbol{A}$ is the function*

$$\mathrm{wSpec}\,\boldsymbol{A}\colon \mathrm{Spec}\,\boldsymbol{A} \to \mathsf{N}^+$$

such that

$$(\mathrm{wSpec}\,\boldsymbol{A})(\mathfrak{p}) = |T(\boldsymbol{A}/\mathfrak{p})|\,,$$

for every prime filter $\mathfrak{p} \in \mathrm{Spec}\,\boldsymbol{A}$.

DEFINITION 5.4.2. *A* weighted forest *is a map $w\colon F \to \mathsf{N}^+$, where F is a forest.*

Given two weighted forests $w\colon F \to \mathsf{N}^+$ and $w'\colon F' \to \mathsf{N}^+$ a weighted forest morphism *$g\colon w \to w'$ is a map $g\colon F \to F'$ such that:*

1. *g is* order-preserving, *that is, if $x \leq y \in F$, then $g(x) \leq g(y)$;*
2. *g is* open, *that is, whenever $x' \leq g(x) \in F'$, then there is $y \leq x \in F$ such that $g(y) = x'$;*
3. *g* respects weights, *that is, for each $x \in F$ there is $y \leq x$ such that $g(y) = g(x)$ and $w'(g(y))$ divides $w(y)$.*

Recall that an order-preserving open map carries downsets to downsets. It is easy to check that weighted forests and their morphisms forms a category, denoted WF.

LEMMA 5.4.3. *Let the map* wSpec *be extended to* FBL*-morphisms as follows. For any pair of algebras* $\boldsymbol{A}, \boldsymbol{B}$ *in* FBL*, and for any homomorphism* $h\colon \boldsymbol{A} \to \boldsymbol{B}$*, let the map* $\mathrm{wSpec}\, h\colon \mathrm{wSpec}\, \boldsymbol{B} \to \mathrm{wSpec}\, \boldsymbol{A}$ *be defined by the following stipulation:*

$$\mathfrak{p} \in \mathrm{Spec}\, \boldsymbol{B} \mapsto h^{-1}(\mathfrak{p}) \in \mathrm{Spec}\, \boldsymbol{A}\,.$$

Then wSpec *is a contravariant functor from* FBL *to* WF.

wSpec realises a categorical equivalence between FBL and $\mathsf{WF}^{\mathsf{op}}$. We now describe the contravariant functor $\mathbf{wSub}\colon \mathsf{WF} \to \mathsf{FBL}$ implementing the duality in the opposite direction.

DEFINITION 5.4.4. *Given a weighted forest* $w\colon F \to \mathsf{N}^+$*, a* weighted subforest *of* w *is a map* $w'\colon F' \to \mathsf{N}^+$*, where* F' *is a subforest of* F *(that is, a downset of* F*) such that* $w'(x) \le w(x)$ *for all* $x \in \max F'$*, and* $w'(x) = w(x)$ *otherwise.*

We let $\mathrm{wSub}\, w$ *denote the set of all weighted subforests of* w.

We shall equip $\mathrm{wSub}\, w$ with a structure of BL-algebra. To begin with, writing $\emptyset\colon \emptyset \to \mathsf{N}^+$ for the unique empty weighted forest, we set $\bot = \emptyset$, and $\top = w$. To define $\odot$, consider subforests $u\colon U \to \mathsf{N}^+$ and $v\colon V \to \mathsf{N}^+$ of w. Define a function $a\colon U \cap V \to \mathsf{N}^+ \cup \{0\}$ by

$$a(x) = \begin{cases} \max\{0, u(x) + v(x) - w(x)\} & \text{if } x \in \max U \cap \max V, \\ u(x) & \text{if } x \in \max U \text{ and } x \notin \max V, \\ v(x) & \text{if } x \notin \max U \text{ and } x \in \max V, \\ w(x) & \text{otherwise,} \end{cases}$$

for each $x \in U \cap V$. Let $E = \{x \in U \cap V \mid a(x) > 0\}$, and define $u \odot v\colon E \to \mathsf{N}^+$ by the restriction $u \odot v = a \restriction E$. Turning to implication, we define $u \to v$. First, we set:

$$\begin{aligned} A &= F \setminus {\uparrow}(U \setminus V)\,; \\ B &= \{x \mid x \in \max U \cap \max V \ \text{ and } \ u(x) > v(x)\}\,. \end{aligned}$$

Then we set $E = (A \setminus {\uparrow}B) \cup B$. We define $(u \to v)\colon E \to \mathsf{N}^+$ by

$$(u \to v)(x) = \begin{cases} v(x) + w(x) - u(x) & \text{if } x \in B, \\ v(x) & \text{if } x \in (U \cap V) \setminus (\max U \cap \max V), \\ w(x) - u(x) & \text{if } x \in \min U \setminus V, \\ w(x) & \text{otherwise,} \end{cases}$$

for each $x \in E$.

It can now be proved that for any weighted forest $w\colon F \to \mathsf{N}^+$, the algebra

$$\mathbf{wSub} w = \langle\, \mathrm{wSub}\, w, \odot, \to, \bot \,\rangle$$

is a (finite) BL-algebra. To turn $\mathbf{wSub}$ into a contravariant functor from WF to FBL, we take inverse images again. Namely, if $g\colon w \to w'$ is a morphism between the weighted forests $w\colon F \to \mathsf{N}^+$ and $w'\colon F' \to \mathsf{N}^+$, we define $\mathbf{wSub}g\colon \mathbf{wSub}w' \to \mathbf{wSub}w$ by

$$\mathbf{wSub}g\colon U \in \mathbf{Sub}F' \mapsto g^{-1}(U) \in \mathbf{Sub}F\,,$$

where $\mathbf{Sub}F'$ and $\mathbf{Sub}F$ are applications of the functor $\mathbf{Sub}\colon \mathsf{F} \to \mathsf{G}$ defined in Section 4 (see Theorem 4.2.1). One can prove that $\mathbf{wSub}g$ so defined is a homomorphism of BL-algebras. To sum up, $\mathbf{wSub}$ is a contravariant functor from WF to FBL.

THEOREM 5.4.5 (Finite Duality). *The category of finite* BL*-algebras and their homomorphisms is dually equivalent to the category of weighted forests and their morphisms. That is, the composite functors* $\mathrm{wSpec} \circ \mathbf{wSub}$ *and* $\mathbf{wSub} \circ \mathrm{wSpec}$ *are naturally isomorphic to the identity functors on* WF *and* FBL*, respectively.*

In particular, the functor wSpec is essentially surjective, and this yields the following representation theorem for finite BL-algebras.

COROLLARY 5.4.6. *Any finite* BL*-algebra is isomorphic to* $\mathbf{wSub}\,w$*, for a weighted forest* $w\colon F \to \mathsf{N}^+$ *that is unique to within an isomorphism of weighted forests.*

To manipulate weighted spectra of finite BL-algebras we have to introduce some further notions. In particular we shall define finite products and coproducts in WF.

First, given a weighted forest $w\colon F \to \mathsf{N}^+$, we denote $w_\perp$ the weighted *tree* $w_\perp\colon F_\perp \to \mathsf{N}^+$, where $F_\perp = F \cup \{\perp\}$ with $\perp < x$ for all $x \in F$, and $w_\perp(x) = w(x)$ for all $x \in F$, while $w_\perp(\perp) = 1$. Each weighted tree arises from a weighted forest in this way. Further, for each $e \in \mathsf{N}^+$, we let w_e denote the weighted tree $w_e\colon \emptyset_\perp \to \mathsf{N}^+$ defined by $w_e(\perp) = e$.

LEMMA 5.4.7. *Finite products and coproducts in* WF *have the following properties.*

1. *The coproduct in* WF *of two finite weighted forests* $w\colon F \to \mathsf{N}^+$ *and* $w'\colon F' \to \mathsf{N}^+$ *is (isomorphic to) the weighted forest* $w + w'\colon F \uplus F' \to \mathsf{N}^+$*, where* $\uplus$ *is the disjoint union and* $(w + w')(x) = w(x)$ *if* $x \in F$*, while* $(w + w')(x) = w'(x)$ *if* $x \in F'$*.*

2. *Products distribute over coproducts, that is, for each* $u, v, w \in \mathsf{WF}$*,* $u \times (v + w) \cong (u + v) \times (u + w)$*.*

3. *For each pair of integers* $d, e \in \mathsf{N}^+$*,* $w_d \times w_e \cong w_{\mathrm{lcm}(d,e)}$*, where* lcm *denotes the least common multiple. Moreover, for each pair of weighted trees* $u_\perp$ *and* $v_\perp$*, it holds that* $(w_d \times u_\perp) \times (w_e \times v_\perp) \cong w_{\mathrm{lcm}(d,e)} \times (u_\perp \times v_\perp)$*, and* $w_1 \times u_\perp \cong u_\perp$*.*

4. *For each pair of weighted trees* $u_\perp$ *and* $v_\perp$ *in* WF*,*

$$u_\perp \times v_\perp \cong (u \times v_\perp + u \times v + u_\perp \times v)_\perp\,.$$

As an application of spectral duality we compute the exact structure of the weighted spectra of free algebras in the subvariety of $\mathbb{BL}$ satisfying Grigolia's axioms. Knowledge of this structure easily yields a large amount of information on algebras in this variety: for instance the computation of cardinalities of free algebra follows straightforwardly.

Let us define $x \oplus y$ as

$$((x \to (x \odot y)) \to y) \vee ((y \to (y \odot x)) \to x)\,.$$

We observe that in each MV-algebra this term coincides with $\neg(\neg x \odot \neg y)$, that is, with the Łukasiewicz strong disjunction.

For each integer $k > 0$, let $\mathbb{BL}_k$ be the subvariety of $\mathbb{BL}$ satisfying Grigolia's axioms given in Section 2.3. Let further $\boldsymbol{B}_k^l$ denote the ordinal sum of l copies of the $(k+1)$-element MV-chain $\boldsymbol{Ł}_k$.

LEMMA 5.4.8. *For each integer $k > 0$ the following hold.*

1. *The variety $\mathbb{BL}_k$ is generated by $\{\boldsymbol{B}_k^l \mid l \in \mathsf{N}^+\}$.*
2. *The variety $\mathbb{BL}_k$ is locally finite.*
3. *For a finite BL-algebra $\boldsymbol{A}$, the following are equivalent.*
   (a) $\boldsymbol{A} \in \mathbb{BL}_k$*;*
   (b) $\mathrm{wSpec}\,\boldsymbol{A}$ *has range included in the set of divisors of k.*

Hence the weighted spectrum of a $\mathbb{BL}_k$-algebra is a finite forest the weights of whose elements are divisors of k. By Corollary 5.4.6, the viceversa holds, too: each such forest is the weighted spectrum of a $\mathbb{BL}_k$-algebra.

To compute the structure of $\mathrm{wSpec}\,\boldsymbol{F}^n_{\mathbb{BL}_k}$ we start by introducing a slight variant of the function $\alpha(n,d)$ of Lemma 2.3.2.

We let $\beta(0,d) = 0$ if $d > 1$ and 1 if $d = 1$, and, for $n \geq 1$,

$$\beta(n,d) = d^n + \sum_{\emptyset \neq X \subseteq \mathrm{CoAtDiv}(d)} (-1)^{|X|} (\gcd X)^n\,.$$

LEMMA 5.4.9. *For each integer $k > 0$ let $M_k^0 = w_1$, while for each integer $m > 0$, $M_k^m = \sum_{d|k} \beta(m,d) w_d$. Then for each integer $n > 0$:*

$$\mathrm{wSpec}\,\boldsymbol{F}^n_{\mathbb{BL}_k} \cong (M_k^1 + (M_k^1)_\perp)^n\,.$$

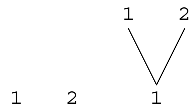
Computing the products in the right hand side of the equation for $\mathrm{wSpec}\,\boldsymbol{F}^n_{\mathbb{BL}_k}$ in Lemma 5.4.9 gives the following recursive formula.

THEOREM 5.4.10. *For each pair of positive integers k, d we define $T^0_{k,d} = w_d$, while for each integer $m > 0$,*

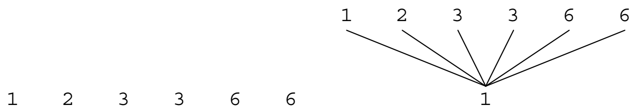
$$T^m_{k,d} = w_d \times \left(\sum_{i=1}^{n} \sum_{e|k} \binom{n}{i} \beta(i,e) T^{m-i}_{k,e} \right)_\perp\,.$$

Then, for each integer $n \geq 0$:

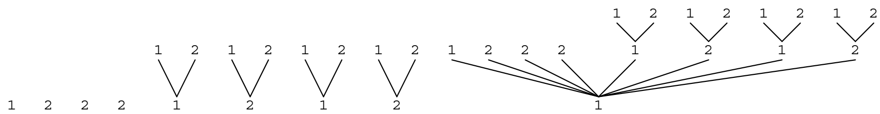
$$\mathrm{wSpec}\,\boldsymbol{F}^n_{\mathbb{BL}_k} \cong \sum_{i=0}^{n} \sum_{d|k} \binom{n}{i} \beta(i,d) T^{n-i}_{k,d}\,.$$



(a) $\operatorname{wSpec} \boldsymbol{F}^1_{\mathbb{BL}_3}$.



(b) $\operatorname{wSpec} \boldsymbol{F}^1_{\mathbb{BL}_7}$.



(c) $\operatorname{wSpec} \boldsymbol{F}^2_{\mathbb{BL}_3}$.

Figure 17. Sampling Lemma 5.4.9.

We write $t(k, n, d)$ for the cardinality of the $\mathbb{BL}_k$-algebra $\operatorname{Sub} T^n_{k,d}$.

COROLLARY 5.4.11. *For each integer* $n \geq 0$ *and* $k, d \in \mathsf{N}^+$, *it holds that* $t(k, 0, d) = d + 1$ *and* $t(k, n, d) = d + \prod_{i=1}^{n} \prod_{e|k} t(k, n-i, e)^{\binom{n}{i}\beta(i,e)}$, *and hence*

$$|\boldsymbol{F}^n_{\mathbb{BL}_k}| = \prod_{i=0}^{n} \prod_{d|k} t(k, n-i, d)^{\binom{n}{i}\beta(i,d)} .$$

We end this section with some observations and some further simple applications.

Clearly, $\mathbb{MV}_k$ is a subvariety of $\mathbb{BL}_k$, hence the weighted spectrum of any algebra $\boldsymbol{A} \in \mathbb{MV}_k$, is a forest of trees of the form w_d for d a divisor of k. Theorem 2.3.3 can then be reformulated in dual terms as $\operatorname{wSpec} \boldsymbol{F}^n_{\mathbb{MV}_k} \cong \sum_{d|k} \alpha(n, d) w_d$.

Analogously, the variety of Gödel algebras coincides with $\mathbb{BL}_1$, and hence the weighted spectrum of a Gödel algebra $\boldsymbol{A}$ is a forest of copies of w_1, and then, forgetting weights, the formula in theorem 5.4.10 specialises to the formula $\operatorname{wSpec} \boldsymbol{F}^n_{\mathbb{G}} \cong \sum_{i=0}^{n-1} \binom{n}{i} (H_i)_\perp + (\sum_{i=0}^{n-1} \binom{n}{i} (H_i)_\perp)_\perp$ of Theorem 4.2.6.

Given a locally finite variety $\mathbb{V}$ of BL-algebras, the subvariety constituted by all $\mathbb{V}$-algebras satisfying $x \wedge \neg x = 0$ is a locally finite variety of SBL-algebras. The corresponding subcategory of finite forests are precisely those weighted spectra of $\mathbb{V}$-algebras such that the weight of the root of every tree is 1.

6 Pierce representation

A very useful construction for characterising free algebras over a set of generators of arbitrary cardinality is the weak Boolean product representation.

In this section we shall explore representation of free algebras in a variety $\mathbb{V}$ as weak Boolean products of the directly indecomposable members of $\mathbb{V}$.

In particular, for some subvarieties $\mathbb{V}$ of $\mathbb{MTL}$ it is possible to build all directly indecomposable algebras applying some construction involving algebras from an auxiliary variety $\mathbb{V}^*$. In this section we explain a construction of this kind and state the conditions of its applicability. As a matter of fact, we have already shown instances of this construction in previous sections of this chapter. Proposition 4.5.3 states that each directly indecomposable NM-algebra without negation fixpoint arises as the disconnected rotation of a Gödel hoop. The construction that yields a product algebra from a cancellative hoop is another example: actually Theorem 3.2.6 shows a decomposition of $\boldsymbol{F}^n_{\mathbb{P}}$ as a direct product of directly indecomposable product algebras.

When the set of directly indecomposable members needed to build an algebra is not finite, then direct products are no longer sufficient. Under some hypotheses that are always satisfied by the structures of our concern, one may consider to generalise direct products to weak Boolean products, as follows.

We recall that a *Stone space* is a totally disconnected compact Hausdorff space.

DEFINITION 6.0.1. *Given a Stone space X, a* weak Boolean product *of a family $\{\boldsymbol{A}_x\}_{x\in X}$ of algebras in a variety $\mathbb{V}$ is a subdirect product $\boldsymbol{A} \hookrightarrow \prod_{x\in X} \boldsymbol{A}_x$ such that, for all $a, b \in \boldsymbol{A}$:*

1. *the set $\{x \in X \mid a(x) = b(x)\}$ is open in X;*
2. *for all clopen sets Z in X, the function $c\colon X \to \prod_{x\in X} \boldsymbol{A}_x$ defined as $c(x) = a(x)$ if $x \in Z$, while $c(x) = b(x)$ otherwise, is such that $c \in \boldsymbol{A}$.*

Clearly, $\boldsymbol{A} \in \mathbb{V}$. An algebra is representable as weak Boolean product *if it is isomorphic to a weak Boolean product.*

The variety $\mathbb{BRL}$ of bounded integral residuated lattices is arithmetical and has the Boolean Factor Congruence property. Hence every non-trivial bounded integral residuated lattice can be represented a weak Boolean product of directly indecomposable members of $\mathbb{BRL}$.

LEMMA 6.0.2. *The directly indecomposable members of $\mathbb{BRL}$ are precisely those lattices $\boldsymbol{A}$ such that their Boolean skeleton $\mathbf{B}(\boldsymbol{A})$ is isomorphic to $\{0, 1\}$.*

In the following, for any filter $\mathfrak{p}$ of a lattice $\boldsymbol{A} \in \mathbb{BRL}$, we write $\boldsymbol{A}/\mathfrak{p}$ for the quotient of $\boldsymbol{A}$ modulo the congruence whose top class is $\mathfrak{p}$. If $\mathfrak{q}$ is a filter of the Boolean skeleton $\mathbf{B}(\boldsymbol{A})$ of $\boldsymbol{A}$, we denote by $\langle\mathfrak{q}\rangle$ the filter of $\boldsymbol{A}$ generated by $\mathfrak{q}$.

THEOREM 6.0.3. *Let $\boldsymbol{A}$ be a lattice in $\mathbb{BRL}$. Let $\mathrm{BSpec}\,\boldsymbol{A}$ be the prime spectrum of the Boolean algebra $\mathbf{B}(\boldsymbol{A})$ equipped with the Stone topology.*

Then for any $\mathfrak{p} \in \mathrm{BSpec}\,\boldsymbol{A}$, the lattice $\boldsymbol{A}/\langle\mathfrak{p}\rangle$ is directly indecomposable, because $\mathbf{B}(\boldsymbol{A}/\langle\mathfrak{p}\rangle) \cong \{0, 1\}$.

The correspondence

$$a \mapsto (a/\langle\mathfrak{p}\rangle)_{\mathfrak{p}\in \mathrm{BSpec}\,\boldsymbol{A}}$$

gives a representation of $\boldsymbol{A}$ *as weak Boolean product of the family of directly indecomposable lattices* $\{\boldsymbol{A}/\langle\mathfrak{p}\rangle\}_{\mathfrak{p}\in \mathrm{BSpec}\,\boldsymbol{A}}$ *over the space* $\mathrm{BSpec}\,\boldsymbol{A}$. *This representation is called the* Pierce Representation *of* $\boldsymbol{A}$.

Obviously, $\mathbb{MTL} \subseteq \mathbb{BRL}$. We shall work with some subvarieties of $\mathbb{MTL}$. An MTL-algebra satisfying the equation

$$\neg\neg(\neg\neg x \to x) = \top \tag{14}$$

is called a *Glivenko* MTL-algebra. It is straightforward to check that $\mathbb{BL}$, $\mathbb{IMTL}$, and $\mathbb{SMTL}$ are all varieties of Glivenko MTL-algebras.

We introduce a condition and a construction, applicable when the condition holds, to produce directly indecomposable Glivenko MTL-algebras starting from basic semihoops (a.k.a. prelinear semihoops) in $\mathbb{MTLH}$.

Pick $\boldsymbol{A} = \langle A, \&, \to, \wedge, \vee, \top\rangle \in \mathbb{MTLH}$. A map $\delta\colon A \to A$ is DL-*admissible* if it satisfies the following, for any $a, b \in A$:

$$\begin{array}{rl}
a \to \delta(a) = \top & \qquad \delta(a \wedge b) = \delta(a) \wedge \delta(b)\\
\delta(\delta(a)) = \delta(a) & \qquad \delta(a \vee b) = \delta(a) \vee \delta(b)\\
\delta(a \to b) = a \to \delta(b) & \qquad \delta(a \,\&\, b) = \delta(\delta(a) \,\&\, \delta(b)).
\end{array}$$

DEFINITION 6.0.4. *Let* δ *be* DL-*admissible for* $\boldsymbol{A} \in \mathbb{MTLH}$. *The following operations are defined on* $S(\boldsymbol{A}, \delta) = (A \times \{1\}) \cup (\delta(A) \times \{0\})$, *for any* $x, y \in \boldsymbol{A}$ *and* $i, j \in \{0, 1\}$*:*

$$\begin{aligned}
\langle x, i\rangle \sqcap \langle y, j\rangle &= \langle y, j\rangle \sqcap \langle x, i\rangle = \begin{cases} \langle x \wedge y, 1\rangle & \textit{if } i = j = 1,\\ \langle x \vee y, 0\rangle & \textit{if } i = j = 0,\\ \langle x, 0\rangle & \textit{if } i < j;\end{cases}\\
\langle x, i\rangle \sqcup \langle y, j\rangle &= \langle y, j\rangle \sqcup \langle x, i\rangle = \begin{cases} \langle x \vee y, 1\rangle & \textit{if } i = j = 1,\\ \langle x \wedge y, 0\rangle & \textit{if } i = j = 0,\\ \langle y, 1\rangle & \textit{if } i < j;\end{cases}\\
\langle x, i\rangle \odot \langle y, j\rangle &= \langle y, j\rangle \odot \langle x, i\rangle = \begin{cases} \langle x \,\&\, y, 1\rangle & \textit{if } i = j = 1,\\ \langle \top, 0\rangle & \textit{if } i = j = 0,\\ \langle y \to x, 0\rangle & \textit{if } i < j;\end{cases}\\
\langle x, i\rangle \Rightarrow \langle y, j\rangle &= \begin{cases} \langle x \to y, 1\rangle & \textit{if } i = j = 1,\\ \langle y \to x, 0\rangle & \textit{if } i = j = 0,\\ \langle \delta(x \,\&\, y), 0\rangle & \textit{if } i > j,\\ \langle \top, 1\rangle & \textit{if } i < j;\end{cases}\\
\top &= \langle \top, 1\rangle;\\
\bot &= \langle \top, 0\rangle.
\end{aligned}$$

Let $\boldsymbol{S}(\boldsymbol{A}, \delta) = \langle S(\boldsymbol{A}, \delta), \odot, \Rightarrow, \sqcap, \sqcup, \bot, \top\rangle$.

Compare Definition 6.0.4 with Definition 4.5.2.

PROPOSITION 6.0.5. *$\boldsymbol{S}(\boldsymbol{A},\delta)$ is a directly indecomposable Glivenko MTL-algebra. Moreover, $\boldsymbol{S}(\boldsymbol{A},\delta) \in \mathbb{IMTL}$ if and only if $\delta = \mathrm{id}_{\boldsymbol{A}}$, while $\boldsymbol{S}(\boldsymbol{A},\delta) \in \mathbb{SMTL}$ if and only if δ is constantly $\top$. Further, the equation*

$$(\neg x \to \neg\neg x)^2 = \neg(x^2) \to \neg\neg(x^2) \tag{15}$$

holds in $\boldsymbol{S}(\boldsymbol{A},\delta)$.

We denote $\mathbb{DL}$ the subvariety of Glivenko MTL-algebras satisfying the Di Nola-Lettieri Equation (15). Notice that $\mathbb{SMTL} \subseteq \mathbb{DL}$, and moreover, the variety generated by the Chang MV-algebra is $\mathbb{MV} \cap \mathbb{DL}$.

Let $\boldsymbol{A} \in \mathbb{DL}$. We denote $\nabla\colon \boldsymbol{A} \to \boldsymbol{A}$ the map $x \mapsto (\neg x \to \neg\neg x)^2$. Further we let $A^\top = \nabla^{-1}(\{\top\})$. It can be checked that the restriction of the hoop operations of $\boldsymbol{A}$ to $A^\top$ turns it into a basic semihoop $\boldsymbol{A}^\top$. An algebra $\boldsymbol{H} = \langle H, \&, \to, \wedge, \vee, \delta, \top\rangle$ is called a *kernel* DL*-algebra* if its hoop reduct $\boldsymbol{H}^-$ is a basic semihoop and $\delta\colon H \to H$ is DL-admissible. By definition of DL-admissible map, the class of kernel DL-algebras forms a variety $\mathbb{KDL}$. For each $\boldsymbol{A}^\top$ let $\delta_{\neg\neg}\colon A^\top \to A^\top$ be the DL-admissible map $x \mapsto \neg\neg x$. Note that $\delta_{\neg\neg}$ coincides with the identity map $\mathrm{id}_{\boldsymbol{A}^\top}$ if $\boldsymbol{A}$ is involutive, while it coincides with the constant function $\top^{\boldsymbol{A}^\top}$ if $\boldsymbol{A} \in \mathbb{SMTL}$. We let $\boldsymbol{P}(\boldsymbol{A})$ denote the KDL-algebra $\langle \boldsymbol{A}^\top, \delta_{\neg\neg}\rangle$.

We associate with each class $\mathbb{K}$ of DL-algebras the class $\mathbb{K}^*$ of KDL-algebras $\langle \boldsymbol{A}, \delta\rangle$ such that $\boldsymbol{S}(\boldsymbol{A},\delta) \in \mathbb{K}$.

THEOREM 6.0.6. *$\mathbb{V}^*$ is a subvariety of $\mathbb{KDL}$ for each subvariety $\mathbb{V}$ of $\mathbb{DL}$.*

The operators $\boldsymbol{S}$ and $\boldsymbol{P}$ can be used, together with the Peirce representation, to obtain description of free algebras.

THEOREM 6.0.7. *For each subvariety $\mathbb{V}$ of DL-algebras, $\mathbf{B}(\boldsymbol{F}_{\mathbb{V}}^{|X|})$ is the free Boolean algebra over the set $\nabla X = \{\nabla(x) \mid x \in X\}$ of generators. Since X and ∇X have the same cardinality, $\mathbf{B}(\boldsymbol{F}_{\mathbb{V}}^{|X|})$ is the free Boolean algebra $\boldsymbol{F}_{\mathbb{B}}^{|X|}$ over $|X|$ generators.*

Recall that the Stone space of $\boldsymbol{F}_{\mathbb{B}}^{|X|}$ is the Cantor space 2^X, as the ultrafilters of $\boldsymbol{F}_{\mathbb{B}}^{|X|}$ are in bijection with the subsets of X.

Assume X is nonempty. Then the map $U \mapsto \{x \in X \mid \nabla(x) \in U\}$ is a bijection from the set of ultrafilters of $\mathbf{B}(\boldsymbol{F}_{\mathbb{V}}^{|X|})$ into 2^X. The inverse map is given by sending S to the ultrafilter U_S generated by $\nabla S \cup \{\neg\nabla(x) \mid x \in X \setminus S\}$.

THEOREM 6.0.8. *Let $\mathbb{V}$ be any subvariety of $\mathbb{DL}$ and let $X \neq \emptyset$ be any set of generating symbols. For each $S \in 2^X$, the algebra $\boldsymbol{F}_{\mathbb{V}}^{|X|}/\langle U_S\rangle$ is directly indecomposable. Further, the free algebra $\boldsymbol{F}_{\mathbb{V}}^{|X|}$ is isomorphic to the weak Boolean product of the family $\{\boldsymbol{S}(\boldsymbol{P}(\boldsymbol{F}_{\mathbb{V}}^{|X|}/\langle U_S\rangle))\}_{S\subseteq X}$ over the space 2^X.*

Usefulness of Theorem 6.0.8 shows itself when the structure of the KDL-algebras $\boldsymbol{P}(\boldsymbol{F}_{\mathbb{V}}^{|X|}/\langle U_S\rangle)$ is known. This has been accomplished for the case when $\mathbb{V}$ is $\mathbb{SMTL}$ or the variety of involutive DL-algebras.

In the following, for any $\boldsymbol{A} \in \mathbb{SMTL}$, let $\boldsymbol{\sigma}(\boldsymbol{A}) = \boldsymbol{S}(\boldsymbol{A}, \delta_\top)$, where $\delta_\top \colon A \to A$ is the constant function $\top$. Analogously, let $\boldsymbol{\iota}(\boldsymbol{A}) = \boldsymbol{S}(\boldsymbol{A}, \mathrm{id}_{\boldsymbol{A}})$ for any involutive DL-algebra $\boldsymbol{A}$.

THEOREM 6.0.9. *Let $\mathbb{V}$ be a nontrivial subvariety of $\mathbb{MTL}$.*

- *If $\mathbb{V}$ is a subvariety of $\mathbb{SMTL}$ then, for each set X the free algebra $\boldsymbol{F}_{\mathbb{V}}^{|X|}$ is isomorphic to the weak Boolean product of the family $\{\boldsymbol{\sigma}(\boldsymbol{F}_{\mathbb{V}^*}^{|S|})\}_{S\subseteq X}$ over the space 2^X.*
- *If $\mathbb{V}$ is a subvariety of involutive DL-algebras then, for each set X the free algebra $\boldsymbol{F}_{\mathbb{V}}^{|X|}$ is isomorphic to the weak Boolean power of the family $\{\boldsymbol{\iota}(\boldsymbol{F}_{\mathbb{V}^*}^{|X|})\}_{S\subseteq X}$ over the space 2^X.*

When X is finite the weak Boolean product reduces to a direct product.

THEOREM 6.0.10. *Let $k \leq 1$ be any integer.*

- *If $\mathbb{V}$ is a nontrivial subvariety of $\mathbb{SMTL}$ then $\boldsymbol{F}_{\mathbb{V}}^k \cong \prod_{i=0}^k \boldsymbol{\sigma}(\boldsymbol{F}_{\mathbb{V}^*}^i)^{\binom{k}{i}}$.*
- *If $\mathbb{V}$ is a nontrivial subvariety of involutive DL-algebras then $\boldsymbol{F}_{\mathbb{V}}^k \cong \boldsymbol{\iota}(\boldsymbol{F}_{\mathbb{V}^*}^k)^{2^k}$.*

We apply Theorem 6.0.9 to the varieties of Gödel algebras and of product algebras. Application of Theorem 6.0.10 to the same varieties gives back the results we already obtained. First we note that $\mathbb{G}^* = \mathbb{GH}$. Hence, for each set X, the free Gödel algebra $\boldsymbol{F}_{\mathbb{G}}^{|X|}$ is isomorphic to the weak Boolean product of the family $\{\boldsymbol{\sigma}(\boldsymbol{F}_{\mathbb{GH}}^{|S|})\}_{S\subseteq X}$ over the space 2^X. Next, consider the variety $\mathbb{CH}$ of cancellative hoops. Note that for $\boldsymbol{A} \in \mathbb{MTLH}$ it holds that

$$\boldsymbol{A} \in \mathbb{CH} \qquad \text{iff} \qquad \boldsymbol{\sigma}(\boldsymbol{A}) \in \mathbb{P} \qquad \text{iff} \qquad \boldsymbol{\iota}(\boldsymbol{A}) \in \mathbb{MV}\,.$$

It follows that if $\mathbb{K}$ is one of $\mathbb{P}$ or $\mathbb{MV}$ then $\mathbb{K}^* = \mathbb{CH}$.

Then, for each set X, the free product algebra $\boldsymbol{F}_{\mathbb{P}}^{|X|}$ is isomorphic to the weak Boolean product of the family $\{\boldsymbol{\sigma}(\boldsymbol{F}_{\mathbb{CH}}^{|S|})\}_{S\subseteq X}$ over the space 2^X. Moreover, for each set X, the free algebra in the variety $\mathbb{CMV} = \mathbb{MV} \cap \mathbb{DL}$ generated by the Chang's MV-algebra, is such that $\boldsymbol{F}_{\mathbb{CMV}}^{|X|}$ is isomorphic to the weak Boolean power of the family $\{\boldsymbol{\iota}(\boldsymbol{F}_{\mathbb{CH}}^{|X|})\}_{S\subseteq X}$ over the space 2^X.

For what concerns $\mathbb{NM}$ and its subvarieties, we note that the subvariety $\mathbb{NM}^-$ of NM-algebras without fixpoint, which is generated by $[0,1] \setminus \{1/2\}$, is such that $(\mathbb{NM}^-)^* = \mathbb{GH}$, and in particular $\boldsymbol{H}$ is a Gödel hoop if and only if $\boldsymbol{\iota}(\boldsymbol{H}) \in \mathbb{NM}^-$. It follows that for each set X, the free algebra $\boldsymbol{F}_{\mathbb{NM}^-}^{|X|}$ is isomorphic to the weak Boolean power of the family $\{\boldsymbol{\iota}(\boldsymbol{F}_{\mathbb{GH}}^{|X|})\}_{S\subseteq X}$ over the space 2^X.

To deal with $\mathbb{NM}$ itself, we first represent $\boldsymbol{F}_{\mathbb{NM}}^{|X|}$ isomorphically as the weak Boolean product of the family $\{\boldsymbol{F}_{\mathbb{NM}}^{|X|}/\langle U\rangle\}_{U\in \mathrm{BSpec}\,\boldsymbol{F}_{\mathbb{NM}}^{|X|}}$, and then for each $U \in \mathrm{BSpec}\,\boldsymbol{F}_{\mathbb{NM}}^{|X|}$, we have that the directly indecomposable NM-algebra $\boldsymbol{F}_{\mathbb{NM}}^{|X|}/\langle U\rangle$ is either isomorphic to the connected rotation (if it has the negation fixpoint) or the disconnected rotation (if it lacks the negation fixpoint) of the free Gödel hoop $\boldsymbol{F}_{\mathbb{GH}}^{|X_U|}$, where $X_U = \{x/\langle U\rangle \mid x \in X\,,\ x/\langle U\rangle > \neg x/\langle U\rangle\} \cup \{\neg x/\langle U\rangle \mid x \in X\,, x/\langle U\rangle < \neg x/\langle U\rangle\}$.

7 Open problems

With the recent characterization of free BL-algebras (see Section 5.2) all the prominent schematic extensions of BL have their functional representation theorems. In stark contrast with BL, the picture is very incomplete for schematic extensions of MTL which are not extensions of BL. As a matter of fact, in the literature one finds concrete representation theorems of free algebras only for certain subvarieties of $\mathbb{WNM}$, in particular for the most prominent of them, Gödel and NM-algebras, and for some minor ones as NMG- and RDP-algebras. A representation of the free WNM-algebras is at present unavailable, but predicting that this gap is soon to be filled constitutes a safe bet. A common feature of the varieties of MTL having a functional representation is the knowledge of a generic chain for the variety. This knowledge is at present not available for the variety of MTL-algebras, and some of its prominent subvarieties, such as IMTL- and $\Pi\mathrm{MTL}$-algebras. The same varieties are still lacking a functional representation theorem for their free algebras. For what concerns extensions/expansions of MTL, the most interesting open problem is the characterization of free PMV-algebras. Such a result will solve the long-standing Birkhoff-Pierce conjecture either in the positive or in the negative.

8 Historical remarks and further reading

Standard references for Section 1 are the Universal Algebra texts: [23, 36, 56].

For what concerns Section 2.1, McNaughton originally proved Theorem 2.1.2 in [57], where he used a version of Lemma 2.1.21. McNaughton's proof is not constructive, since at a certain stage it involves an unbounded search. The first constructive proof of Theorem 2.1.2 is found in [61]. See also [29]. This proof uses Lemma 2.1.21 to express Schauder hats formulas. The proof presented in the chapter is taken from Panti's paper [65] on the geometric proof of the completeness of Łukasiewicz calculus. The techniques used in the proof rely on convex geometry [44], on piecewise linear topology [66] and on De Concini-Procesi lemma on the elimination of points of indeterminacy in toric varieties [37] (Lemma 2.1.12 can be thought of as a non-homogeneous version of the De Concini-Procesi result). A different proof is given in [64]: the idea underlying this proof is to use an equivalent version of Lemma 2.1.21 to compute for each domain of linearity D of a given McNaughton function f an MV-term φ_D that in the free algebra evaluates to 1 exactly over D, together with an MV-term ψ_D whose evaluation coincides with f over D. It is then shown that there exists an integer $k(f,D)$ such that the evaluation of $\varphi_D^{k(f,D)}$ goes to 0 outside D with slopes that are sufficiently high to allow to patch together all terms $(\psi_D \mathbin{\&} \varphi_D^{k(f,D)})$ with a $\bigvee$-combination which yields the desired function f. This proof can be extended to represent free algebras in the variety $\mathbb{MV}(\mathsf{Q})$ obtained expanding the MV signature with a denumerable set of constants $\{c_\delta\}_{\delta\in[0,1]\cap\mathsf{Q}}$ which in the standard algebra are forced to be interpreted in the corresponding rational numbers in $[0,1]$ by way of suitable book-keeping axioms. Unsurprisingly, $\boldsymbol{F}^n_{\mathbb{MV}(\mathsf{Q})}$ is the class of continuous piecewise linear functions where each linear piece has the form $p(\mathbf{x}) = b_i + \sum_{i=1}^n a_i x_i$, where each a_i is an integer, while b_i may be rational. In this chapter we have preferred to expound the proof of McNaughton's Theorem based on regular triangulations and Schauder bases because the notion of normal form as $\oplus$-sum of Schauder hats makes the latter concept the best known approximation to the notion of semantical minterm as an element of the free algebra that, as a function, takes on a prescribed value at a prescribed point, and is the smallest possible elsewhere. Moreover, the notion of Schauder bases constitutes a first step towards the development of the theory of the MV-partitions of unity and the related theory of abstract bases for MV-algebras, which are now lines of research very actively pursued, as witnessed by several chapters of the recent book [63]. See also [15, 53, 54, 62] for some applications. In particular, the paper [62] gives a representation-free characterization of the free MV-algebras using the notion of MV-partition of unity. The analytic-geometric argument needed to develop the sup-inf combination of truncated hyperplanes expounded after Lemma 2.1.21 can be found in [29] or [3]. As is well known, Mundici's functor Γ (see [29]) shows that MV-algebras are categorically equivalent to Abelian ℓ-groups with a distinguished strong unit. It follows that Theorem 2.1.2 can be seen as the specialisation to unital ℓ-groups of Baker and Beynon results on the representation of finitely generated free Abelian ℓ-groups [17, 18, 55]. Jakubík in [51] shows how to derive free MV-algebras from free Abelian ℓ-groups. Finally, it is worth noting that in the recent paper [43] Dubuc and Poveda provide a categorical duality for MV-algebras based on suitably defined sheaves of MV-chains, and show how to derive McNaughton Theorem as a natural consequence of their categorical constructions when they are applied to free algebras. The problem of normal forms for Łukasiewicz logic is strictly connected to the representation of free MV-algebras. See [2, 4, 42] for analysis of normal forms based on truncated hyperplanes and Lemma 2.1.21; see [9] for a normal form for finitely valued Łukasiewicz logics based on Schauder hats. The functional representation of free Wajsberg hoops is found in [1], while the results on finitely valued Łukasiewicz logics can be found in [29]: there, the study of free algebras in the corresponding varieties $\mathbb{MV}_n$ is independent of McNaughton Theorem. Free algebras for the algebraic semantics of extensions/expansions of Łukasiewicz logics are dealt with in [60], while MV-algebras with divisors are studied in [47].

The main sources for free product algebras and free cancellative hoops are the papers [30, 31]. The functional representation theorem was partially developed in [11], and fully in [34].

Gödel algebras have been studied, under the name L-algebras, in two pioneering papers by Horn [49, 50], which characterized the prime spectrum and computed the recursive formula for the cardinality of free finitely generated algebras. It is interesting to note that the category of finite forests shown in this chapter to be dually equivalent to finitely generated Gödel algebras does not constitute a natural duality in the sense of [35]: as we have shown the product of forests has generally more points than the cartesian product of the underlying sets. A description of finitely generated free Gödel algebras and free Gödel hoops is found in [39]. The functional representation of Gödel algebras is given in [11, 46, 47]. The paper [38] is focused on computing coproducts of finitely generated Gödel algebras via their dual spectral representation as products of the corresponding finite forests. The paper [10] states the 1-set property of Lemma 4.1.5. In this chapter we have followed the section on normal forms of [10] for our presentation of the functional representation theorem. Normal forms for Gödel logic have been studied in [16] and in [13]. In this paper free algebras for some subvarieties of $\mathbb{MTL}$ are represented as algebras of maximal-length antichains over suitably constructed finite posets. Here normal forms are defined as lattice disjunctions of minterms, where a semantical minterm is the minimum (in the pointwise order of the poset) maximal-length antichain containing a prescribed element. Free n-generated algebras are then described as embeddings into products of a sufficiently large collection of n-generated chains in the variety: normal forms are used to construct the required embeddings. These poset representations allow to easily find recursive formulas for the cardinalities of free algebras. The fact that Nilpotent Minimum algebras coincides with prelinear Nelson algebras is proved in [28]. Free Nilpotent Minimum algebras with generator set of any cardinality are studied in [26], and in [13] for the finitely generated case. The first functional representation of free NM-algebras is found in [68], while Theorem 4.6.9 is stated as in [14]. The same paper contains the statement of the $0, 1$-set property of Proposition 4.6.10. The spectral duality for finite NM-algebras is developed in [8]. Poset representation and spectral duality for finite RDP-algebras are studied in [22, 67].

The first works on free BL-algebras are by Montagna. In [58] he gives the functional representation for the singly generated case, and in [59] he characterizes the free $\mathrm{BL}_\triangle$-algebras. Recently, the papers [5, 21] give the first concrete representation of free BL-algebras. In this chapter we closely follow [5]. The paper [7] introduces spectral duality representations for locally finite subvarieties of $\mathbb{BL}$. The paper [27] describe free $\mathbb{MV}_k$-algebras with generator set of any cardinality in terms of functions from a Stone space into MV-chains. A description of the prime spectrum, which in these algebras coincides with the maximal one, is provided. When the generator set is finite this construction reduces to the spectral duality described in the chapter. Schauder-hat-like normal forms for BL have been developed in [6, 12].

Pierce representations and weak Boolean products have been applied to determine the structure of free algebras in several subvarieties of $\mathbb{MTL}$ in a series of papers by Cignoli, Torrens, and Busaniche [25, 26, 32, 33]. Our presentation is mainly based on [33]. Weak Boolean products are presented as global sections of sheaves of algebras over Boolean spaces in [24]. The universal algebraic background on directly indecomposable bounded integral residuated lattices and the Boolean Factor Congruence property can be found in [19, 45]. The Di Nola-Lettieri Equation is introduced in [40, 41]. The variety of Nilpotent Minimum algebras without fixpoint is described, along with a classification of the subvarieties of $\mathbb{NM}$, by Gispert in [48].

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Chapter X: Computational Complexity of Propositional Fuzzy Logics

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1 Introduction

This chapter is about computational complexity of decision problems in propositional fuzzy logics and also in algebras which constitute their algebraic semantics. We investigate sets of formulas and relations thereon, with an aim to determine their complexity by ranking them alongside well-known decision problems, such as SAT and TAUT in classical propositional logic. A key problem is, for a given logic, to determine the complexity of the set of its theorems and of the relation of provability of a formula from a finite theory. We rely on completeness theorems and work in a suitably chosen class of algebras, so we are also interested in complexity of appropriate fragments of the algebraic theory. Owing to the multitude of fuzzy logics under investigation, the general framework yields many particular problems, some of which are open.

Naturally, many patterns of thinking familiar from classical logic are not applicable in the many-valued case. For example, in classical propositional logic, one can reduce the problem of provability from finite theories to the problem of theoremhood, using the deduction theorem. The classical deduction theorem is however not generally available in fuzzy logic, and that is why the provability relation is, in general, an interesting complexity problem. To give another example, the duality of satisfiability and tautologousness, known from classical logic and occasioned by its dichotomy, is not valid for algebras corresponding to fuzzy logics. The fact that not only the classical dichotomy is absent, but there are typically infinitely many truth values, makes it actually nontrivial to find upper bounds on complexity of sets of formulas such as SAT. Indeed, a major part of our efforts in this chapter will be targeted to showing, for various existential problems, that if there is a solution, there is a succinct one.

On the other hand, all decision problems considered in this chapter share common lower bounds (not necessarily tight): for each consistent axiomatic extension of the logic $\mathrm{FL_{ew}}$, the SAT problem for the corresponding class of algebras is $\mathbf{NP}$-hard, whereas the TAUT problem is $\mathbf{coNP}$-hard. The word 'hard' is ominous here: while the problems are algorithmically solvable, this chapter does a poor job on attempting to solve them. Rather, it is intent on *classifying* the problems, using polynomial equivalence; we are not concerned about polynomial differences in performance. Throughout, we investigate the *worst-case complexity* of problems, an approach that is preferable for its elegance and robustness as long as one is aware of its limitations.

There is a pattern in results presented in this chapter: for those decision problems whose complexity has been settled (the problems have been proved complete in some complexity class), the situation is analogous to the classical case: satisfiability is $\mathbf{NP}$-complete, while tautologousness and consequence (hence, theoremhood and provability) are $\mathbf{coNP}$-complete. One might ask why consequence relation comes out no more difficult than tautologousness. This chapter tries to answer this question by showing $\mathbf{coNP}$-containment (hence, $\mathbf{coNP}$-completeness) for the universal fragment of the theory of these algebras. Thus we are able to avail ourselves of the classical dichotomy after all, albeit on a metamathematical level: the universal fragment of the theory is $\mathbf{coNP}$-complete if and only if the existential fragment is $\mathbf{NP}$-complete. SAT can be viewed as a fragment of the existential theory and TAUT and CONS as fragments of the universal theory, and that is why complexity results come out as rather flat. It is of course a major question whether this might be the case for those problems that are, so far, open.

Complexity-wise, as well as otherwise, a territory well conquered is propositional Hájek's BL and its extensions. It is not an oversimplification to say that complexity results for the BL family rest on the results for particular MV-algebras, mainly the standard one, and the latter in turn can be derived from well-known results in linear algebra. However, the complexity picture is much less complete for fragments and expansions of BL and of its extensions: here, results are fragmentary despite some considerable effort, while on the other hand, many problems have not been addressed. Shifting from BL to MTL, one moves into an area where open problems outnumber existing results. Decidability results are available for MTL and some of its extensions, and computational complexity has been settled for *particular examples* of left-continuous t-norms. However, a suitable general methodology for tackling complexity problems in semilinear logics weaker than BL is still to be found. Lack of results also prevents us from even mentioning some even weaker semilinear systems; we usually assume our logics are axiomatic extensions of MTL or expansions thereof.

This chapter cannot lay claim to a proper introduction of the investigated logics. For a comprehensive presentation, the reader may wish to consult earlier chapters of this book. Indeed, this chapter will be indigestible to a reader who has not, at the very least, come across the logic BL, its extensions Ł, G, Π, and standard BL-algebras for these logics. Likewise, our treatment of basic computational complexity notions is not intended as an introduction to the topic, but rather as a condensed reference guide. Some skill in algorithmization might also come in useful, as algorithms, where needed, are presented informally within this chapter, and the verification of polynomial nature of some transactions is left to the reader.

The text is organized as follows. Section 2 gives definitions, important notions and results, and notational conventions. Section 3 collects general results, applicable to many particular logics, and some technical statements. Section 4 is dedicated to results on Łukasiewicz logic and its extensions; it contains prototypical complexity results and explains in detail some techniques. Section 5 presents results on (the remaining) extensions of BL given by standard BL-algebras. Section 6 is an overview of available results for fragments and expansions of BL or its extensions. Section 7 gives a flavour of results available for (extensions of) MTL. Section 8 offers an overview of results and Section 9 is an account of achievements in the field, giving references and credits.

2 Notions and problems

This section is a brief exposition of elements of logic, algebra and computational complexity theory. This is accompanied by definitions and discussion of the decision problems that form the subject matter of our investigation. Some other decision problems are pointed out whose already established complexity bounds are relevant.

2.1 Logics and algebras

Logics investigated in this chapter are *algebraizable*; the notion of algebraizability was introduced in [4]. This property amounts to the fact that, under a natural translation between propositional formulas and algebraic identities, provability in a particular propositional logic corresponds to the consequence relation in the (unique) class of algebras that forms its *equivalent algebraic semantics*. In particular, one gets strong completeness w.r.t. this class, which extends also to axiomatic extensions and many language expansions. See Chapter IV for a comprehensive exposition.

Languages and expressions. A *language* $\mathcal{L}$ is a countable set of connectives, each with a given arity in N. The connectives with arity 0 are called *constants*. This chapter only considers languages with finitely many connectives of a positive arity (while there can be infinitely many constants). Given a countably infinite set of variables Var, using the connectives of $\mathcal{L}$ and parentheses one can build in the usual way the set $Fm_{\mathcal{L}}$ of $\mathcal{L}$*-expressions*. These can be viewed as propositional formulas (usually denoted with lowercase Greek characters φ, ψ, etc.) or as algebraic terms (usually denoted with lowercase Latin characters t, s, etc.); first-order $\mathcal{L}$-formulas feature $\mathcal{L}$ as the set of function symbols and $=$ as the predicate symbol.

For $\mathcal{L}$-expressions φ, ψ, we write $\psi \preceq \varphi$ to denote the fact that ψ is a subexpression[1] of φ. For $\mathcal{L}$-expressions φ, ψ, χ, we write $\varphi(\psi/\chi)$ for a substitutional instance of φ where all occurrences of ψ have been replaced with χ. If $X \subseteq Var$, we denote $Fm^X_{\mathcal{L}}$ the $\mathcal{L}$-expressions in variables from X.

If $\mathcal{L}_1 \subseteq \mathcal{L}_2$ are languages and T is a set of $\mathcal{L}_2$-expressions, then the $\mathcal{L}_1$*-fragment* of T is the set $T' \subseteq T$ containing all $\mathcal{L}_1$-expressions in T.

Some languages will be particularly important in this chapter. The logic $\mathrm{FL_{ew}}$ (full Lambek calculus with exchange and weakening) has binary connectives $\&$ (conjunction), $\to$ (implication), $\wedge$, $\vee$ (lattice conjunction/disjunction), and the constant $\overline{0}$. One further defines $\overline{1}$ as $\overline{0} \to \overline{0}$, $\neg\varphi$ as $\varphi \to \overline{0}$, and $\varphi \leftrightarrow \psi$ as $(\varphi \to \psi) \,\&\, (\psi \to \varphi)$. In logics stronger than $\mathrm{FL_{ew}}$, some of the connectives are definable;[2] in particular, $\vee$ is definable in MTL ($\varphi \vee \psi$ is defined as $((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$), and both $\vee$ and $\wedge$ are definable in BL ($\varphi \wedge \psi$ is defined as $\varphi \,\&\, (\varphi \to \psi)$). In Łukasiewicz logic, one can define all the above connectives using $\to$ and $\overline{0}$ (but one can also equivalently start with different sets of connectives). In superintuitionistic logics, $\wedge$ and $\&$ coincide. In classical logic, connectives become interdefinable in the familiar manner.

[1] A connected substring belonging to $Fm_{\mathcal{L}}$.

[2] If L is a logic (extending or expanding $\mathrm{FL_{ew}}$) in a language $\mathcal{L}$, we say that an n-ary connective $c \in \mathcal{L}$ is definable in L iff there is an $\mathcal{L}\setminus\{c\}$-formula $\varphi(x_1, \dots, x_n)$ s.t. $\vdash_{\mathrm{L}} c(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n)$. Analogously, we say c is (term-)definable in a class $\mathbb{K}$ of $\mathcal{L}$-algebras iff $\mathbb{K} \models c(x_1, \dots, x_n) = \varphi(x_1, \dots, x_n)$ for φ as above.

The logic BL (focal to this chapter), together with many of its extensions, is usually considered in the language $\{\&, \to, \overline{0}\}$. This is also the case here; this language will be referred to as the *language of* BL. The definable connectives mentioned above are regarded as abbreviations, and one uses the defining formulas to translate any formula containing the definable connectives to the language of BL. Analogously for MTL with respect to the language $\{\&, \to, \wedge, \overline{0}\}$ (the *language of* MTL). Definable connectives (in particular, $\wedge$ and $\vee$) are used quite freely in many places, and some general results are given for the $\mathrm{FL_{ew}}$-language (in particular, Theorem 3.4.1). Yet a straightforward application of the translations given above, to eliminate definable connectives and thus pass from one language to another, may lead to an exponential blowup in formula size. We argue in Theorem 3.3.3 that there is a translation that preserves satisfiability and tautologousness and that can be performed polynomially.

Propositional logic. A logic L in a language $\mathcal{L}$ is a structural consequence relation $\vdash_{\mathrm{L}} \subseteq \mathcal{P}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$. Often $\vdash_{\mathrm{L}}$ is given by a deductive system, i.e., axioms and deduction rules; cf. Chapter II for a detailed exposition. In a logical setting, we often speak of (propositional) $\mathcal{L}$-formulas rather than $\mathcal{L}$-expressions. An $\mathcal{L}$-theory is a set of $\mathcal{L}$-formulas. If L is a logic in $\mathcal{L}$, $T \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$, $T \vdash_{\mathrm{L}} \varphi$ reads 'φ is provable from T in L', and $\vdash_{\mathrm{L}} \varphi$ is a case of the former with $T = \emptyset$, meaning 'φ is a theorem of L'.

DEFINITION 2.1.1. *Let $\mathcal{L}$ be a language and* L *a logic in the language $\mathcal{L}$. We denote*

$$\mathrm{THM}(\mathrm{L}) = \{\varphi \in Fm_{\mathcal{L}} \mid\ \vdash_{\mathrm{L}} \varphi\} \qquad \text{(theorems of L)}$$

$$\mathrm{CONS}(\mathrm{L}) = \{\langle T, \varphi\rangle \in \mathcal{P}_{\mathrm{fin}}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}} \mid T \vdash_{\mathrm{L}} \varphi\} \qquad \text{(provability from finite theories in L)}$$

The two above notions—theoremhood (for a formula) and provability (for a formula from a finite theory)—will be in the focus of our attention throughout this chapter. For various logics, it will be our objective to classify the set of theorems and the relation of provability from finite theories as to their computational complexity. The restriction to finite theories is necessitated by the need to work with finite objects.

If L_1 is a logic in a language $\mathcal{L}_1$ and L_2 is a logic in a language $\mathcal{L}_2$, we say that L_2 is an *expansion* of L_1 iff $\mathrm{L}_1 \subseteq \mathrm{L}_2$ (entailing $\mathcal{L}_1 \subseteq \mathcal{L}_2$); if $\mathcal{L}_1 = \mathcal{L}_2$ we say 'extension' rather than 'expansion'. If a logic L_2 in a language $\mathcal{L}_2$ expands a logic L_1 in a language $\mathcal{L}_1$, we say the expansion is *conservative* iff, for each $\mathcal{L}_1$-theory $T \cup \{\varphi\}$, $T \vdash_{\mathrm{L}_2} \varphi$ implies $T \vdash_{\mathrm{L}_1} \varphi$; in such a case, we say that L_1 is the $\mathcal{L}_1$*-fragment* of L_2.

DEFINITION 2.1.2. *Basic logic* BL *in the language* $\{\&, \to, \overline{0}\}$ *has axioms:*

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$

(A2) $\varphi \,\&\, \psi \to \varphi$

(A3) $\varphi \,\&\, \psi \to \psi \,\&\, \varphi$

(A4) $\varphi \,\&\, (\varphi \to \psi) \to \psi \,\&\, (\psi \to \varphi)$

(A5a) $(\varphi \to (\psi \to \chi)) \to (\varphi \,\&\, \psi \to \chi)$

(A5b) $(\varphi \,\&\, \psi \to \chi) \to (\varphi \to (\psi \to \chi))$

(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$

(A7) $\overline{0} \to \varphi$

The deduction rule of BL *is modus ponens. Moreover, monoidal t-norm logic* MTL *in the language* $\{\&, \to, \wedge, \overline{0}\}$ *has axioms* (A1)–(A3),

(A4a) $\varphi \wedge \psi \to \varphi$

(A4b) $\varphi \wedge \psi \to \psi \wedge \varphi$

(A4c) $\varphi \,\&\, (\varphi \to \psi) \to \varphi \wedge \psi$

(A5)–(A7)*, and deduction rule modus ponens.*

Uppercase Latin characters are used for logics: MTL, BL, SBL, Ł, G, Π stand for monoidal t-norm logic, basic logic, strict basic logic, Łukasiewicz logic, Gödel logic, product logic respectively. These and other logics are discussed in previous chapters.

Algebraic semantics. Let $\mathcal{L}$ be a language. In an algebraic setting, the elements of $\mathcal{L}$ are thought of as function symbols; $=$ is the predicate symbol. Variables in *Var* are usually denoted with $x, y, z, \ldots$. $\mathcal{L}$*-terms* are $\mathcal{L}$-expressions, denoted with $s, t, \ldots$. For a given language $\mathcal{L}$, an *identity* is a formula $t = s$, where t, s are terms. A *quasiidentity* is a formula $\bigwedge_{i \leq n}(t_i = s_i) \to t = s$ for $n \in \mathsf{N}$, where t, t_i, s, s_i, $i \leq n$ are terms. An *open formula* is a formula without quantifiers. A closed formula (or *sentence*) is a formula without free variables. We use uppercase Greek characters (Φ, Ψ, $\ldots$) for first-order formulas.

An $\mathcal{L}$-algebra is a structure $\boldsymbol{A} = \langle A, \langle c^{\boldsymbol{A}} \mid c \in \mathcal{L}\rangle\rangle$; the functions in $\boldsymbol{A}$ are indexed with the function symbols of $\mathcal{L}$ of matching arities. If $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and t is an $\mathcal{L}$-term, $t^{\boldsymbol{A}}$ denotes the function given by t in $\boldsymbol{A}$. If $\boldsymbol{A}$ is an algebra, A stands for its domain. If $\mathcal{L}' \subseteq \mathcal{L}$ are languages and $\boldsymbol{A} = \langle A, \langle c^{\boldsymbol{A}} \mid c \in \mathcal{L}\rangle\rangle$ is an $\mathcal{L}$-algebra, the $\mathcal{L}'$*-reduct of* $\boldsymbol{A}$ is the algebra $\langle A, \langle c^{\boldsymbol{A}} \mid c \in \mathcal{L}'\rangle\rangle$. If $\mathbb{K}$ is a class of $\mathcal{L}$-algebras, the theory of $\mathbb{K}$ is the set of first-order $\mathcal{L}$-formulas valid in each member of $\mathbb{K}$. We are particularly interested in the equational and quasiequational fragments of first-order algebraic theories, as these (for suitably chosen algebras) correspond to theoremhood and provability in our propositional logics via completeness theorems.

The following notation is used for function symbols of the language of $\mathrm{FL_{ew}}$-algebras: $\{*, \to, \wedge, \vee, 0\}$.

While $=$ is the only predicate symbol, our algebras are lattice-ordered, hence we take the liberty of using predicate symbols $\leq$ and $<$, where for any terms t_1 and t_2, $t_1 \leq t_2$ stands for $t_1 \wedge t_2 = t_1$, while $t_1 < t_2$ stands for $(t_1 \leq t_2) \wedge \neg(t_1 = t_2)$; so, by slight abuse, atomic formulas are of the form $t_1 = t_2$, $t_1 \leq t_2$, $t_1 < t_2$ for some terms t_1, t_2 (naturally, under this convention we may no longer claim that atomic formulas are just identities). Moreover, 0 is the least and 1 the greatest element of the lattice order.

If $\mathcal{L}$ is a language and $\boldsymbol{A}$ is an $\mathcal{L}$-algebra, an $\boldsymbol{A}$*-evaluation* on $Fm_{\mathcal{L}}$ is any homomorphism from $\boldsymbol{Fm}_{\mathcal{L}}$ (i.e., the free algebra on *Var*) to $\boldsymbol{A}$. Each mapping $e\colon\ Var \to A$ can then be uniquely extended to an $\boldsymbol{A}$-evaluation on $Fm_{\mathcal{L}}$. We denote $Val(\boldsymbol{A})$ the set of all $\boldsymbol{A}$-evaluations on $Fm_{\mathcal{L}}$. Further, if $X \subseteq Var$, we denote $Val^X(\boldsymbol{A}) = \{e \restriction Fm_{\mathcal{L}}^X \mid e \in Val(\boldsymbol{A})\}$ (the set of all evaluations on $Fm_{\mathcal{L}}^X$).

For $T \cup \{\varphi\}$ a set of $\mathcal{L}$-expressions, $T \models_{\boldsymbol{A}} \varphi$ iff, for all $\boldsymbol{A}$-evaluations e, we have $e(\varphi) = 1^{\boldsymbol{A}}$ whenever for all $\psi \in T$ we have $e(\psi) = 1^{\boldsymbol{A}}$. For $\mathbb{K}$ a class of $\mathcal{L}$-algebras, $T \models_{\mathbb{K}} \varphi$ iff $T \models_{\boldsymbol{A}} \varphi$ for all $\boldsymbol{A} \in \mathbb{K}$. The relation $\models_{\mathbb{K}}$ is referred to as the *consequence relation in* $\mathbb{K}$. The *finite consequence relation* in $\mathbb{K}$ is the restriction of $\models_{\mathbb{K}}$ to finite theories. We often use the notion of *logic given by* $\mathbb{K}$ (or simply 'logic of $\mathbb{K}$'): within this chapter, the logic given by $\mathbb{K}$ is identified with the finite consequence relation of $\mathbb{K}$.[3] We write $\models_{\mathbb{K}} \varphi$ for $\emptyset \models_{\mathbb{K}} \varphi$ and we speak of *tautologies* of $\mathbb{K}$.

We now define some of these, and other, familiar notions as operators on (classes of) algebras, generalizing the cases from classical logic.

DEFINITION 2.1.3. *Let $\mathcal{L}$ be a language subsuming the language of* $\mathrm{FL_{ew}}$. *Let* $\mathbb{K} \cup \{\boldsymbol{A}\}$ *be a class of $\mathcal{L}$-algebras whose reducts to the* $\mathrm{FL_{ew}}$*-language are* $\mathrm{FL_{ew}}$*-algebras, and let* $\mathbf{1}$ *denote the trivial $\mathcal{L}$-algebra. We denote*

$$\mathrm{TAUT}(\boldsymbol{A}) = \{\varphi \in Fm_{\mathcal{L}} \mid \forall e \in \mathit{Val}(\boldsymbol{A})(e(\varphi) = 1^{\boldsymbol{A}})\} \qquad \text{(tautologies of } \boldsymbol{A})$$

$$\mathrm{TAUT_{pos}}(\boldsymbol{A}) = \{\varphi \in Fm_{\mathcal{L}} \mid \forall e \in \mathit{Val}(\boldsymbol{A})(e(\varphi) > 0^{\boldsymbol{A}})\} \qquad \text{(positive tautologies of } \boldsymbol{A})$$

$$\mathrm{SAT}(\boldsymbol{A}) = \{\varphi \in Fm_{\mathcal{L}} \mid \exists e \in \mathit{Val}(\boldsymbol{A})(e(\varphi) = 1^{\boldsymbol{A}})\} \qquad \text{(satisfiable formulas of } \boldsymbol{A})$$

$$\mathrm{SAT_{pos}}(\boldsymbol{A}) = \{\varphi \in Fm_{\mathcal{L}} \mid \exists e \in \mathit{Val}(\boldsymbol{A})(e(\varphi) > 0^{\boldsymbol{A}})\} \qquad \text{(positively satisfiable formulas of } \boldsymbol{A})$$

$$\mathrm{CONS}(\boldsymbol{A}) = \{\langle T, \varphi\rangle \in \mathcal{P}_{\mathrm{fin}}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}} \mid T \models_{\boldsymbol{A}} \varphi\} \qquad \text{(finite consequence in } \boldsymbol{A})$$

$$\mathrm{TAUT}(\mathbb{K}) = \bigcap_{\boldsymbol{A}\in\mathbb{K}} \mathrm{TAUT}(\boldsymbol{A}) \qquad \text{(tautologies of } \mathbb{K})$$

$$\mathrm{TAUT_{pos}}(\mathbb{K}) = \bigcap_{\boldsymbol{A}\in(\mathbb{K}\setminus\{\mathbf{1}\})} \mathrm{TAUT_{pos}}(\boldsymbol{A}) \qquad \text{(positive tautologies of } \mathbb{K})$$

$$\mathrm{SAT}(\mathbb{K}) = \bigcup_{\boldsymbol{A}\in(\mathbb{K}\setminus\{\mathbf{1}\})} \mathrm{SAT}(\boldsymbol{A}) \qquad \text{(satisfiable formulas of } \mathbb{K})$$

$$\mathrm{SAT_{pos}}(\mathbb{K}) = \bigcup_{\boldsymbol{A}\in\mathbb{K}} \mathrm{SAT_{pos}}(\boldsymbol{A}) \qquad \text{(positively satisfiable formulas of } \mathbb{K})$$

$$\mathrm{CONS}(\mathbb{K}) = \bigcap_{\boldsymbol{A}\in\mathbb{K}} \mathrm{CONS}(\boldsymbol{A}) \qquad \text{(finite consequence in } \mathbb{K})$$

In the above definition, the trivial algebra $\mathbf{1}$ is omitted from consideration for the $\mathrm{TAUT_{pos}}$ and SAT operators, because $\mathrm{TAUT_{pos}}(\mathbf{1}) = \emptyset$ and $\mathrm{SAT}(\mathbf{1}) = Fm_{\mathcal{L}}$. This seems more convenient than handling the omission separately for each case.

NOTATION 2.1.4. For $\mathbb{K}$ a class of algebras, the term $\mathrm{SAT_{(pos)}}(\mathbb{K})$ stands for either of the problems $\mathrm{SAT}(\mathbb{K})$ and $\mathrm{SAT_{pos}}(\mathbb{K})$. Similarly for $\mathrm{TAUT_{(pos)}}(\mathbb{K})$.

We now come to the notion of the (first-order) *theory of* $\mathbb{K}$, where $\mathbb{K}$ is a class of algebras in a given language $\mathcal{L}$. The theory of $\mathbb{K}$ is the set of first-order $\mathcal{L}$-sentences Φ valid in $\mathbb{K}$ (i.e., valid in each $\boldsymbol{A} \in \mathbb{K}$; write $\mathbb{K} \models \Phi$). We use 'equational theory of $\mathbb{K}$' for 'equational fragment of the theory of $\mathbb{K}$'; analogously for the other fragments.

[3] This is not quite consistent with the conception of logics as consequence relations, because the latter involve infinite sets of formulas. However, the finite consequence relation captures full information about the consequence relation in case the latter is finitary, or about its finitary companion in case it is not.

DEFINITION 2.1.5. *Let $\mathcal{L}$ be a language, $\mathbb{K}$ a class of $\mathcal{L}$-algebras. We write*

(i) $\mathrm{Th}_{\mathrm{Eq}}(\mathbb{K})$ *for the equational theory of* $\mathbb{K}$*, i.e., the set of universally quantified* $\mathcal{L}$*-identities valid in* $\mathbb{K}$;

(ii) $\mathrm{Th}_{\mathrm{QEq}}(\mathbb{K})$ *for the quasiequational theory of* $\mathbb{K}$*, i.e., the set of universally quantified* $\mathcal{L}$*-quasiidentities valid in* $\mathbb{K}$;

(iii) $\mathrm{Th}_{\forall}(\mathbb{K})$ *for the universal theory of* $\mathbb{K}$*, i.e., the set of universally quantified open* $\mathcal{L}$*-formulas valid in* $\mathbb{K}$;

(iv) $\mathrm{Th}_{\exists}(\mathbb{K})$ *for the existential theory of* $\mathbb{K}$*, i.e., the set of existentially quantified open* $\mathcal{L}$*-formulas valid in some* $\boldsymbol{A} \in \mathbb{K}$;

(v) $\mathrm{Th}(\mathbb{K})$ *for the (full, first-order) theory of* $\mathbb{K}$.

It is obvious from the definition that $\mathrm{Th}_{\mathrm{Eq}}(\mathbb{K}) \subseteq \mathrm{Th}_{\mathrm{QEq}}(\mathbb{K}) \subseteq \mathrm{Th}_{\forall}(\mathbb{K}) \subseteq \mathrm{Th}(\mathbb{K})$ and $\mathrm{Th}_{\exists}(\mathbb{K}) \subseteq \mathrm{Th}(\mathbb{K})$. It is important to observe that all the inclusions in fact stand for fragments given by conditions that are easy to verify.

The link between the concepts introduced in the last two definitions is clear: for any language $\mathcal{L}$, any $\mathcal{L}$-algebra $\boldsymbol{A}$, any $\boldsymbol{A}$-evaluation e and any $\mathcal{L}$-expression φ, we have $e(\varphi) = 1^{\boldsymbol{A}}$ iff $\boldsymbol{A} \models (\varphi = 1)[e]$. Hence, e.g., $\varphi \in \mathrm{TAUT}(\boldsymbol{A})$ iff $(\varphi = 1) \in \mathrm{Th}_{\mathrm{Eq}}(\boldsymbol{A})$. This yields some straightforward reducibilities, which are collected in Lemma 3.1.1.

If L is a logic in a language $\mathcal{L}$, we usually write $\mathbb{L}$ for the class of $\mathcal{L}$-algebras that forms its equivalent algebraic semantics. The elements of $\mathbb{L}$ are referred to as L-algebras; the linearly ordered elements are L-*chains*.

If $\mathcal{L}$ is a language and $\mathbb{K}$ a class of $\mathcal{L}$-algebras, we write $\mathbf{V}(\mathbb{K})$ for the variety and $\mathbf{Q}(\mathbb{K})$ for the quasivariety generated by $\mathbb{K}$; further, we write $\mathbf{I}(\mathbb{K})$, $\mathbf{H}(\mathbb{K})$, $\mathbf{S}(\mathbb{K})$, $\mathbf{P}(\mathbb{K})$, $\mathbf{P}_{\mathrm{U}}(\mathbb{K})$ for the classes of isomorphic images of $\mathbb{K}$, homomorphic images of $\mathbb{K}$, subalgebras of $\mathbb{K}$, direct products of $\mathbb{K}$, ultraproducts of $\mathbb{K}$, respectively. For $\boldsymbol{A}, \boldsymbol{B}$ two $\mathcal{L}$-algebras, $\boldsymbol{A}$ is *partially embeddable* into $\boldsymbol{B}$ iff every finite partial subalgebra of $\boldsymbol{A}$ is embeddable into $\boldsymbol{B}$, that is, for each finite set $A_0 \subseteq A$ there is a one-one mapping $f\colon A_0 \to B$ such that for each n-ary function symbol g in $\mathcal{L}$, if for $a_1, \ldots, a_n \in A_0$ we have $g^{\boldsymbol{A}}(a_1, \ldots, a_n) \in A_0$, then $f(g^{\boldsymbol{A}}(a_1, \ldots, a_n)) = g^{\boldsymbol{B}}(f(a_1), \ldots, f(a_n))$. For $\mathbb{K}$, $\mathbb{L}$ two classes of $\mathcal{L}$-algebras, $\mathbb{K}$ is partially embeddable into $\mathbb{L}$ iff each finite partial subalgebra of a member of $\mathbb{K}$ is embeddable into a member of $\mathbb{L}$.

Structure of BL-chains. We review a few important facts about decomposition of BL-chains as ordinal sums. This decomposition is an essential part of standard completeness results for BL and also—as we shall see—of the results on its computational complexity. We remark that, while MTL (unlike BL) actually enjoys strong standard completeness, an analogously lucid result about the structure of MTL-chains is not available. Within this book, BL-algebras are studied in detail in Chapter V.

A t-norm $*$ on $[0, 1]$ is a binary operation that is associative, commutative, nondecreasing, satisfying boundary conditions $x * 0 = 0$ and $x * 1 = x$. If $*$ is left continuous, then its residuum $\to$ is uniquely given by $x \to y = \max\{z \mid x * z \leq y\}$ and $[0, 1]_* = \langle [0, 1], *, \to, \wedge, \vee, 0, 1\rangle$ is a standard MTL-algebra. $[0, 1]_*$ is a standard BL-algebra iff $*$ is continuous.

There are three outstanding examples of continuous t-norms; together with their residua, they are listed in the following table (note that $x \to y = 1$ whenever $x \leq y$):

	$x * y$	$x \to y$ for $x > y$
Łukasiewicz	$\max\{x + y - 1, 0\}$	$1 - x + y$
Gödel	$\min\{x, y\}$	y
product	xy	y/x

The next proposition justifies the importance of the three examples above. For a continuous t-norm $*$, the set of its idempotents is a closed subset of $[0, 1]$, its complement is a union of countably many pairwise disjoint open intervals; denote this set of intervals $\mathcal{I}_{\mathrm{o}}$. Let $\mathcal{I}$ be the set of closures of the elements of $\mathcal{I}_{\mathrm{o}}$.

PROPOSITION 2.1.6 (Mostert–Shields Theorem [36]).
Let $*$ be a continuous t-norm on $[0, 1]$.

(i) For each $[a, b] \in \mathcal{I}$, the restriction of $*$ to $[a, b]$ is isomorphic either to the product t-norm on $[0, 1]$ or to the Łukasiewicz t-norm on $[0, 1]$.

(ii) If there are no a, b such that $x, y \in [a, b] \in \mathcal{I}$, then $x * y = \min\{x, y\}$.

For each standard BL-algebra $[0, 1]_*$, the maximal, nontrivial, closed intervals on which $*$ is isomorphic to the Łukasiewicz, Gödel, or product t-norm are referred to as Ł-*components*, G-*components*, and Π-*components* of the t-norm, hence of the algebra $[0, 1]_*$. Not every element of $[0, 1]_*$ belongs to an Ł, G, or Π-component; one also considers trivial, one-element algebras as possible components. If $\boldsymbol{A}$ is a (standard) BL-algebra, one can write $\boldsymbol{A} = \bigoplus_{i \in I} \boldsymbol{A}_i$ for some linearly ordered index set I and for each $\boldsymbol{A}_i$ among copies of $[0, 1]_{\text{Ł}}$, $[0, 1]_{\mathrm{G}}$, $[0, 1]_{\Pi}$, and the trivial algebra $\mathbf{1}$.

We remark that one can prove an analogous decomposition result for saturated BL-chains, and also that BL-chains can be decomposed as ordinal sums of Wajsberg hoops. We will not need these results in this chapter.

Completeness. If $\mathcal{L}$ is a language, L is a logic in the language $\mathcal{L}$ and $\mathbb{K}$ is a class of $\mathcal{L}$-algebras, we say that L is:

(i) *complete* w.r.t. $\mathbb{K}$ iff, for each $\mathcal{L}$-formula, we have $\vdash_{\mathrm{L}} \varphi$ iff $\models_{\mathbb{K}} \varphi$ (i.e., THM(L) = TAUT($\mathbb{K}$));

(ii) *finitely strongly complete* w.r.t. $\mathbb{K}$ iff, for each finite set $T \cup \{\varphi\}$ of $\mathcal{L}$-formulas, we have $T \vdash_{\mathrm{L}} \varphi$ iff $T \models_{\mathbb{K}} \varphi$ (i.e., CONS(L) = CONS($\mathbb{K}$));

(iii) *strongly complete* w.r.t. $\mathbb{K}$ iff, for each set $T \cup \{\varphi\}$ of $\mathcal{L}$-formulas, we have $T \vdash_{\mathrm{L}} \varphi$ iff $T \models_{\mathbb{K}} \varphi$.

Obviously (iii) implies (ii) and (ii) implies (i) for any choice of $\mathcal{L}$, L and $\mathbb{K}$. Algebraizability of a logic L implies strong completeness w.r.t. the class of algebras $\mathbb{L}$ forming its equivalent algebraic semantics; logics investigated within this chapter are semilinear, and hence, strongly complete w.r.t. the chains in $\mathbb{L}$.

In this chapter we are mainly interested in *standard*[4] algebras for each logic L: completeness results, where available, are then formulated in terms of *standard completeness* (SC), *finite strong standard completeness* (FSSC), or *strong standard completeness*

[4]For many logics/classes of algebras (such as BL or MTL), the notion 'standard algebra' has a clear and established meaning. In other cases (and particularly for some expanded languages), it is better to state explicitly what is meant by 'standard'.

(SSC), where in all cases, the term 'standard' means that in the above definitions, the role of the class $\mathbb{K}$ is played by standard algebras in $\mathbb{L}$.

PROPOSITION 2.1.7 (Standard completeness). *The logic* BL *enjoys finite strong standard completeness. The logic* MTL *enjoys strong standard completeness.*

2.2 A visit to complexity theory

Computational model. *Turing machines* capture essential notions of algorithmization, such as computations and their resources, notably *time* and *space*; algorithms are formally identified with Turing machines. A Turing machine has a finite sequence of *tapes* for data storage, each tape consisting of infinitely many fields, with a cursor indicating the current field. One of the tapes is the input tape, and there may also be an output tape; the input (output) tape is assumed to be read-only (write-only). Tape fields may be blank or may contain symbols out of a given finite alphabet. A particular Turing machine is fully determined by a finite alphabet Σ, a finite set of states Q (with the initial state $q_0 \in Q$), and a finite set of instructions Δ.

Each computation starts with all tapes blank except the input tape, which includes the input—a finite string of symbols from Σ, with the cursor on its leftmost symbol; the machine's state is the initial state $q_0 \in Q$. The computation runs in steps, each step processing one instruction from Δ. The next instruction is chosen on basis of the current state and the content of current fields on the sequence of tapes. Each instruction consists of a current state of the machine, a sequence of symbols on current fields of all tapes (some of which may be blank), the next state of the machine out of Q (which may be one of its halting states), a sequence of symbols to be written down to the current fields of all tapes (some of which may be blank), and a sequence out of $\{-1, 0, 1\}$ indicating the move of cursors on all tapes by at most one field. A computation may terminate or not depending on whether a halting state is reached. If it does, output may be written on an output tape. Among halting states, some states may be indicated as accepting or rejecting. A computation will not continue from a halting state, as there is no instruction available; for all other states and sequences of symbols read from tapes, there are one or more instructions available in Δ; if the former is the case for all possible combinations of states and read sequences—or in other words, if the transition relation induced by Δ is a *function*—then the Turing machine in question is *deterministic*; otherwise, it is *nondeterministic*.

Decision problems. Let Σ be a finite alphabet; Σ^* is the set of finite strings out of Σ; a *word* is a finite string $x \in \Sigma^*$ and $|x|$ denotes the *size* of x (the number of symbols on tape). A *decision problem* (or just 'problem') is a set of words $P \subseteq \Sigma^*$. Words in Σ^* are often called inputs or instances. The complement of a problem P is $\overline{P} = \Sigma^* \setminus P$. A Turing machine M with alphabet Σ *accepts* a problem P iff, for each word $x \in \Sigma^*$, we have $x \in P$ iff there is a computation of M with input x that terminates in an accepting state. A problem $P \subseteq \Sigma^*$ is *recursively enumerable* iff it is accepted by a Turing machine. A problem $P \subseteq \Sigma^*$ is *recursive* (or *decidable*) iff both P and $\overline{P}$ are accepted by a Turing machine. This entails there is a Turing machine with alphabet Σ which terminates on any word x in the given alphabet—in an accepting state if $x \in P$, and in a rejecting state if $x \in \overline{P}$; such a machine is said to *decide* P.

Complexity classes. Consider functions $f, g\colon \mathsf{N} \to \mathsf{N}$. Then $f \in O(g)$ ('f is of the order of g') iff there are $c, n_0 \in \mathsf{N}$ such that for each $n \geq n_0$ we have $f(n) \leq c\, g(n)$.

Let $f\colon \mathsf{N} \to \mathsf{N}$ be a function. A Turing machine M (deterministic or not) operates in time f iff, for any input x in the alphabet Σ of M, any computation with input x takes at most $f(|x|)$ steps. $\mathbf{TIME}(f)$ is the class of problems P such that there is a deterministic Turing machine M that accepts P and operates in time $O(f)$; analogously for $\mathbf{NTIME}(f)$ and nondeterministic Turing machines. A Turing machine M (deterministic or not) operates in space f iff, for any input x in the alphabet Σ of M, any computation with input x (terminates and) writes to at most $f(|x|)$ fields on all its tapes together except the input and the output tapes. $\mathbf{SPACE}(f)$ is the class of problems P such that there is a deterministic Turing machine M that accepts P and operates in space $O(f)$; analogously for $\mathbf{NSPACE}(f)$ and nondeterministic Turing machines.
Particular complexity classes important in this chapter are defined as follows:

$$\mathbf{P} = \bigcup_{k \in \mathsf{N}} \mathbf{TIME}(n^k)$$

$$\mathbf{NP} = \bigcup_{k \in \mathsf{N}} \mathbf{NTIME}(n^k)$$

$$\mathbf{PSPACE} = \bigcup_{k \in \mathsf{N}} \mathbf{SPACE}(n^k)$$

If $\mathbf{C}$ is a complexity class, we denote $\mathbf{coC} = \{P \mid \overline{P} \in \mathbf{C}\}$, the class of complements of problems in $\mathbf{C}$. Each deterministic complexity class $\mathbf{C}$ is closed under complementation: if $P \in \mathbf{C}$, then also $\overline{P} \in \mathbf{C}$. It is widely believed, but not known, not to be the case for the class $\mathbf{NP}$. By definition, $\mathbf{P} \subseteq \mathbf{NP}$ and hence $\mathbf{P} \subseteq \mathbf{coNP}$, and it is easy to see that $\mathbf{NP} \subseteq \mathbf{PSPACE}$. It is an important open problem whether any of the inclusions $\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{PSPACE}$ are proper. Each of the classes $\mathbf{P}$, $\mathbf{NP}$, $\mathbf{coNP}$, and $\mathbf{PSPACE}$ is closed under finite unions and intersections.

The following is an equivalent definition of the class $\mathbf{NP}$: a problem $P \subseteq \Sigma^*$ is in $\mathbf{NP}$ iff there is a polynomially balanced[5] binary relation $R \subseteq \Sigma^* \times \Sigma^*$ in $\mathbf{P}$, such that $P = \{x \in \Sigma^* \mid \exists y \in \Sigma^*(\langle x, y\rangle \in R)\}$. Any such word y is called a *witness* for $x \in P$. It is easy to see that any problem P that satisfies this definition is in $\mathbf{NP}$: given x, first "guess" y and then continue (deterministically) to check $\langle x, y\rangle \in R$. This definition of the class $\mathbf{NP}$ constitutes the basis for many proofs of containment in $\mathbf{NP}$ within this chapter. It also hints at how to transform each nondeterministic polynomial-time algorithm into a deterministic one; given x, one searches through all possible witnesses y (up to a polynomial bounded size) and for each such y, checks whether $\langle x, y\rangle \in R$.

Many nondeterministic algorithms in this chapter follow the guess-and-check pattern described above. In the guessing stage, an information of size polynomial in the input size is guessed. The checking stage may come in several steps. In each step, it is understood (though not stated explicitly each time the construction is used), that if a check is unsuccessful, then the given computation terminates in a rejecting state. Likewise, the formulation 'guess an X such that $C(X)$' (for some condition C) is to be understood as 'guess X and check that $C(X)$ holds'.

[5] A relation R is polynomially balanced iff there is a polynomial p s.t. $\langle x, y\rangle \in R$ implies $|y| \leq p(|x|)$.

Reductions and completeness. Throughout this chapter we use the polynomial-time many-one reducibility, also known as Karp reducibility. To define a reduction between two problems P_1 (in an alphabet Σ_1) and P_2 (in an alphabet Σ_2), it is convenient to consider a Turing machine with input tape alphabet Σ_1 and output tape alphabet Σ_2. A problem P_1 is (many-one, polynomial-time) *reducible* to a problem P_2 (write $P_1 \preceq_{\mathbf{P}} P_2$)[6] iff there is a deterministic Turing machine with input tape alphabet Σ_1 and output tape alphabet Σ_2, operating in time n^k for some $k \in \mathsf{N}$ and all $n \geq n_0 \in \mathsf{N}$, and such that, for any pair of input $x \in \Sigma_1^*$ and its output $y \in \Sigma_2^*$ on M, we have $x \in P_1$ iff $y \in P_2$. In other words, there is a polynomial-time-computable function $f\colon \Sigma_1^* \to \Sigma_2^*$ such that $P_1 = \{x \in \Sigma_1^* \mid f(x) \in P_2\}$; if that is the case, then also $\overline{P_1} = \{x \in \Sigma_1^* \mid f(x) \in \overline{P_2}\}$. Reducibility is a preorder, inducing its corresponding equivalence: two decision problems P_1 and P_2 are *polynomially equivalent* (write $P_1 \approx_{\mathbf{P}} P_2$) iff $P_1 \preceq_{\mathbf{P}} P_2$ and $P_2 \preceq_{\mathbf{P}} P_1$. The equivalence $\approx_{\mathbf{P}}$ provides a classification of decision problems of roughly the same complexity. Even though we are currently unable to tell how equivalence classes of $\approx_{\mathbf{P}}$ span over complexity classes defined above, we can prove positive results on $\approx_{\mathbf{P}}$ for particular decision problems.

A decision problem P is said to be *hard* for a complexity class $\mathbf{C}$ (shortly, $\mathbf{C}$-hard) iff any decision problem P' in $\mathbf{C}$ is reducible to P. A decision problem P is *complete* in $\mathbf{C}$ (shortly, $\mathbf{C}$-complete) iff P is $\mathbf{C}$-hard and $P \in \mathbf{C}$. Thanks to transitivity of $\preceq_{\mathbf{P}}$, hardness of a problem P for $\mathbf{C}$ is typically demonstrated by reducing to P *one* problem already known to be $\mathbf{C}$-hard. Showing that a problem is $\mathbf{C}$-hard can be viewed as setting a *lower bound* on its complexity: it is no easier to solve than the problems in $\mathbf{C}$.

Complexity classes in the focus of our attention—$\mathbf{P}$, $\mathbf{NP}$, and $\mathbf{PSPACE}$—are closed under $\preceq_{\mathbf{P}}$: if $\mathbf{C}$ is one of the above classes, $P_1 \preceq_{\mathbf{P}} P_2$ and $P_2 \in \mathbf{C}$, then also $P_1 \in \mathbf{C}$. This provides a way to demonstrate *containment* of a problem in a class $\mathbf{C}$. Containment in $\mathbf{C}$ can of course be proved in a direct way, by designing an algorithm that works within resource bounds given by $\mathbf{C}$. The algorithm may use subroutines that also satisfy the bounds for $\mathbf{C}$. Showing containment of a problem in a complexity class sets an *upper bound* on its complexity: given the computation mode and the bounds, the problem is algorithmically solvable.

Following the nature of decision problems investigated in this chapter, we shall be mainly interested in the classes $\mathbf{NP}$, $\mathbf{coNP}$, and $\mathbf{PSPACE}$, namely in the respective subclasses of problems that are complete for each of them. With each problem, we seek to find a match between its upper and its lower bound; then the problem is ranked alongside other problems already known to be in the particular $\approx_{\mathbf{P}}$-class.

As a matter of fact, a classification in the above sense for many decision problems in fuzzy logic is missing. There are lower bounds that predetermine the problems investigated in this chapter to be computationally hard (cf. Theorem 3.4.1). In particular, theorems of a consistent fuzzy logic extending $\mathrm{FL_{ew}}$ are always $\mathbf{coNP}$-hard. However, some problems may be much harder than that. As for upper bounds, some decision problems are known to be recursive, but no more than that. Important examples include theoremhood and provability from finite theories in MTL and some of its axiomatic

[6] It would be more docile to write $\preceq^{\mathbf{P}}_{\mathrm{m}}$, since Karp reducibility is exactly the polynomial-time analogue of many-one reducibility $\preceq_{\mathrm{m}}$ provided by recursive functions, to be introduced later.

extensions, such as IMTL, SMTL, or $\Pi\mathrm{MTL}$. Another example is provided by theoremhood in the logics $\mathrm{Ł}\Pi$ and $\mathrm{Ł}\Pi\frac{1}{2}$: the problem is known to be in $\mathbf{PSPACE}$, but apparently not known to be complete for that class.

Arithmetical hierarchy. Let N be the standard model of arithmetic. Let $\Phi(x)$ be an arithmetical formula with one free variable; we say $\Phi(x)$ defines a set $A \subseteq \mathsf{N}$ iff for any $n \in \mathsf{N}$, we have $n \in A$ iff $\mathsf{N} \models \Phi(n)$; we say A is definable in N iff there is a Φ that defines it in N. Analogously, one can introduce definable relations in N^k for each natural number k. Via coding, one can consider words over finite alphabets.

An arithmetical formula is *bounded* iff all its quantifiers are bounded (i.e., are of the form $\forall x \leq t$ or $\exists x \leq t$ for some term t). An arithmetical formula is a Σ_1-formula (Π_1-formula) iff it has the form $\exists x\Phi$ ($\forall x\Phi$ respectively) where Φ is a bounded formula. A formula is Σ_2 (Π_2) iff it has the form $\exists x\Phi$ ($\forall x\Phi$ respectively) where Φ is a Π_1-formula (Σ_1-formula respectively). Inductively, one defines Σ_n- and Π_n-formulas for any natural number $n \geq 1$.

A set $A \subseteq \mathsf{N}$ is in the class $\mathbf{\Sigma_n}$ iff there is a Σ_n-formula that defines A in N; analogously for the class $\mathbf{\Pi_n}$. The definition extends to k-tuples and to words over finite alphabets in the obvious fashion. Trivially, any set that is in $\mathbf{\Sigma_n}$ is also in $\mathbf{\Sigma_m}$ and $\mathbf{\Pi_m}$ for $m > n$. If $A \subseteq \mathsf{N}$ is a $\mathbf{\Sigma_n}$-set, then $\overline{A}$ is a $\mathbf{\Pi_n}$-set. $\mathbf{\Sigma_1}$-sets are exactly recursively enumerable sets, while recursive sets are $\mathbf{\Sigma_1} \cap \mathbf{\Pi_1}$. The hierarchy of classes of sets thus defined is called the *arithmetical hierarchy*, and the (complete sets in) classes of sets in the arithmetical hierarchy represent *degrees of undecidability*. The hierarchy is noncollapsing, as it can be shown that for each $n \geq 1$, $\mathbf{\Sigma_{n+1}} \setminus \mathbf{\Sigma_n}$ is nonempty, and so is $\mathbf{\Sigma_1} \setminus (\mathbf{\Sigma_1} \cap \mathbf{\Pi_1})$. A set $A \subseteq \mathsf{N}$ is *arithmetical* iff it is definable by an arithmetical formula in N and so it belongs to the arithmetical hierarchy; otherwise it is *nonarithmetical*.

A suitable notion of reduction is provided by recursive functions. A problem P_1 in an alphabet Σ_1 is *m-reducible* to a problem P_2 in an alphabet Σ_2 (write $P_1 \preceq_{\mathrm{m}} P_2$) iff there is a deterministic Turing machine with input tape alphabet Σ_1 and output tape alphabet Σ_2, halting on all inputs, and such that, for any pair of input x and its output y, we have $x \in P_1$ iff $y \in P_2$. Each of the classes $\mathbf{\Sigma_n}$, $\mathbf{\Pi_n}$ ($n \geq 1$) is closed under m-reducibility. A problem P is $\mathbf{\Sigma_n}$*-hard* (w.r.t. m-reducibility) iff $P' \preceq_{\mathrm{m}} P$ for any $\mathbf{\Sigma_n}$-problem P'. A problem P is $\mathbf{\Sigma_n}$-complete iff it is $\mathbf{\Sigma_n}$-hard and at the same time it is a $\mathbf{\Sigma_n}$-problem. Analogously for $\mathbf{\Pi_n}$.

2.3 Formulas as inputs

If $\mathcal{L}$ is a language, $\mathcal{L}$-expressions $Fm_{\mathcal{L}}$ are well-formed[7] strings, consisting of variables in Var, connectives in $\mathcal{L}$, and and auxiliary symbols (parentheses). First-order algebraic formulas moreover feature the identity symbol $=$, Boolean connectives, quantifiers $\forall$, $\exists$. Again these formulas are well-formed strings.

An essential question is how resources used by an algorithm depend on the size of an input. Inputs are formulas (propositional or first-order), viewed as words in a finite alphabet. We assume a fixed enumeration of the sets of variables and of connectives. Integers are represented in binary, $|n| = \lceil \log(n+1) \rceil$ for $n \geq 1$, so $|n| \in O(\log(n))$.[8]

[7] I.e., they satisfy the usual inductive definition of a propositional formula in the given language.

[8] Throughout we use base 2 logarithm. We write $O(\log(n))$ for $O(\max\{\lceil \log(n) \rceil, 0\}$.

The size of an $\mathcal{L}$-expression φ is the number of tape fields needed to represent it, denoted $|\varphi|$. Given φ, the value $|\varphi|$ is obtained by adding up the sizes of representations of all occurrences of connectives, all occurrences of variables, and all occurrences of auxiliary symbols. Moreover, if φ is an $\mathcal{L}$-expression with n pairwise distinct variables, it is convenient (and equivalent for our purpose) to consider its substitution instance whose variables are indexed with integers up to n; this brings the space needed to represent each of the variables down to $O(\log(n))$.

It is preferable to work with more versatile measures than the actual formula size: in particular, for $\mathcal{L}$-expressions, the number of occurrences of subexpressions, or the overall size of constants in the expression; if we show an algorithm to be polynomial in a measure bounded by $|\varphi|$, then we may conclude it is also polynomial in $|\varphi|$.

On the other hand, $|\varphi|$ is polynomial in the measures mentioned above. Indeed, for an expression φ, denote m the number of occurrences of subexpressions of φ. Each variable takes $O(\log(m))$ tape fields. In a language with finitely many connectives, each connective takes a constant number of fields, so $|\varphi| \in O(m\log(m))$. As for languages with infinitely many connectives, we attend the case of constants for $\mathsf{Q} \cap [0,1]$: for each such q, if $q = \frac{a}{b}$ for some $a, b \in \mathsf{N}$ where $a \leq b$, we have $|q| \in O(|b|)$. Hence, for an expression φ with constants from $\mathsf{Q} \cap [0,1]$, with m occurrences of subformulas, and whose constants have the largest denominator k, we get $|\varphi| \in O(m(\log(m) + \log(k)))$.

For first-order formulas the above considerations are analogous. Validity is a meaningful concept for *sentences*; if a formula Φ is not a sentence, then we consider its universal closure, whose size is polynomial in the size of Φ. Moreover, it is convenient to only consider sentences in prenex form; bringing a given sentence into the prenex form is a routine polynomial-time transformation.

Not all words in the given alphabet are desirable inputs. The assumption of well-formedness in a given language $\mathcal{L}$ (propositional or first-order) is always present. In many cases there are more restrictive assumptions, like the words being universally quantified $\mathcal{L}$-quasiidentities, existential $\mathcal{L}$-sentences, etc. These assumptions are made explicitly for each decision problem. A common trait of these assumptions is that the assumed condition is easy to verify: the class of words $C \subseteq \Sigma^*$ that satisfy the condition is a decision problem in $\mathbf{P}$ (and of limited interest to us).

To illustrate the difference these assumptions make, consider an algorithm accepting $\mathrm{SAT}(\boldsymbol{A})$—the set of satisfiable $\mathcal{L}$-expressions in an $\mathcal{L}$-algebra $\boldsymbol{A}$. The algorithm accepts satisfiable expressions and rejects unsatisfiable ones; what about words that are not $\mathcal{L}$-expressions? In view of definitions presented earlier, the algorithm should reject them. Then—inconveniently—the set of rejected words would consist of a) words that are not $\mathcal{L}$-expressions, and b) $\mathcal{L}$-expressions that are not satisfiable in $\boldsymbol{A}$. Now a) is not at all interesting, while one might be interested in b); indeed b) is the desired complement to $\mathrm{SAT}(\boldsymbol{A})$. This preference can be met by allowing only $\mathcal{L}$-expressions as inputs to the algorithm. (One may think of an auxiliary algorithm that test all input strings for being well-formed $\mathcal{L}$-expressions.) Continuing the given example, suppose we have indeed shown that $\mathrm{SAT}(\boldsymbol{A})$ is in $\mathbf{NP}$, and, for some $\mathcal{L}' \subset \mathcal{L}$, we further want to investigate satisfiability for $\mathcal{L}'$-expressions in $\boldsymbol{A}'$, the $\mathcal{L}'$-reduct of $\boldsymbol{A}$. Can it be argued that $\mathrm{SAT}(\boldsymbol{A}') \preceq_{\mathbf{P}} \mathrm{SAT}(\boldsymbol{A})$ via an identity function (or in other words, can the algorithm deciding $\mathrm{SAT}(\boldsymbol{A})$ be used also for deciding $\mathrm{SAT}(\boldsymbol{A}')$)? Not quite, because the algorithm for $\mathrm{SAT}(\boldsymbol{A})$ accepts all satisfiable $\mathcal{L}$-formulas, some of whom are not (satisfiable) $\mathcal{L}'$-formulas. However, the identity function can be used for reduction if one makes sure that all inputs to the new algorithm are among $\mathcal{L}'$-expressions.

The informal considerations above have a formal counterpart called *promise problems*. A promise problem in an alphabet Σ is a pair (Y, N) where $Y, N \subseteq \Sigma^*$ and $Y \cap N = \emptyset$; the set $Y \cup N$ is called the *promise*. A Turing machine decides the problem (Y, N) iff it accepts all words in Y and rejects all words in N; on inputs outside $Y \cup N$, its behaviour is not specified. Intuitively, an algorithm solving a promise problem is promised that inputs belong to $Y \cup N$; on this condition, it distinguishes the two sets (given computation mode and bounds). Any decision problem P is a promise problem under $Y = P$ and $N = \overline{P}$.

All problems addressed in this chapter come with a promise in $\mathbf{P}$. Consider (Y, N) a promise problem where $Y \cup N$ is in $\mathbf{P}$. If Y is in $\mathbf{NP}$, then $\overline{Y}$ is in $\mathbf{coNP}$, and so is $N = (Y \cup N) \cap \overline{Y}$. If for two promise problems (Y_1, N_1) and (Y_2, N_2) we have $Y_1 \subseteq Y_2$ and $N_1 \subseteq N_2$, then $(Y_1, N_1) \preceq_{\mathbf{P}} (Y_2, N_2)$ via the identity function.

It is useful to generalize the notion of 'fragment' in the following way. If $A \subseteq \Sigma^*$ is any decision problem, the problem $A \cap C$ is called the *C-fragment of A*. Then, if B is a decision problem and A is the C-fragment of B for a condition $C \in \mathbf{P}$, then $A \preceq_{\mathbf{P}} B$.

2.4 Classical logic and Boolean algebras

The usual language of classical logic is $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \overline{0}, \overline{1}\}$; we refer to these connectives as the *full language of classical logic*. In classical context, either $\wedge$ or $\&$ is used for conjunction and the two are interchangeable. It is convenient to start with some functionally complete subset[9] of the above set and to define the remaining connectives.

Classical propositional logic can be introduced in a lot of ways, e.g., via its well-known Hilbert- and Gentzen-style proof systems; we will be using neither, but we remark that one can obtain classical logic by adding the axiom $\varphi \vee \neg\varphi$ to the axioms of the logic $\mathrm{FL_{ew}}$ or some of its consistent axiomatic extensions, so classical propositional logic is one of the axiomatic extensions of $\mathrm{FL_{ew}}$. Hence the two-element Boolean algebra $\{0, 1\}_{\mathrm{B}}$ is a $\mathrm{FL_{ew}}$-algebra; classical propositional logic is just the logic of $\{0, 1\}_{\mathrm{B}}$.

The following sets of formulas are important decision problems in classical propositional logic:

DEFINITION 2.4.1. *Let $\mathcal{L}$ be the full language of classical logic.*

$$\mathrm{SAT}(\{0,1\}_{\mathrm{B}}) = \{\varphi \in \mathit{Fm}_{\mathcal{L}} \mid \exists e \in \mathit{Val}(\{0,1\}_{\mathrm{B}})(e(\varphi) = 1^{\{0,1\}_{\mathrm{B}}})\}$$
$$\mathrm{TAUT}(\{0,1\}_{\mathrm{B}}) = \{\varphi \in \mathit{Fm}_{\mathcal{L}} \mid \forall e \in \mathit{Val}(\{0,1\}_{\mathrm{B}})(e(\varphi) = 1^{\{0,1\}_{\mathrm{B}}})\}$$

It is easy to see that $\mathrm{SAT}(\{0,1\}_{\mathrm{B}})$ is in $\mathbf{NP}$: if a propositional formula φ is classically satisfiable, then a simple proof of the fact is a satisfying evaluation of its propositional variables; this is a piece of information of size polynomial in $|\varphi|$, and the ver-

[9] A set $C \subseteq \mathcal{L}$ of connectives is functionally complete w.r.t. an $\mathcal{L}$-algebra $\boldsymbol{A}$ iff for each $n \in \mathsf{N}$, each n-ary function $f\colon A^n \rightarrow A$ is definable by a C-formula. We remark that, unlike in the classical case, no set of the above connectives is functionally complete for the algebras in the focus of our attention, i.e., algebras given by (left-)continuous t-norms. The interesting question *which* functions in these algebras are definable by formulas is addressed in Chapter IX of this book.

ification process is clearly a polynomial affair. Moreover, $\varphi \in \mathrm{TAUT}(\{0,1\}_{\mathrm{B}})$ iff $\neg\varphi \in \overline{\mathrm{SAT}}(\{0,1\}_{\mathrm{B}})$, so $\mathrm{TAUT}(\{0,1\}_{\mathrm{B}})$ is in $\mathbf{coNP}$.

Denote $\mathrm{SAT}^{\mathrm{CNF}}(\{0,1\}_{\mathrm{B}})$ a fragment of the above SAT problem for formulas in conjunctive normal form, and $\mathrm{TAUT}^{\mathrm{DNF}}(\{0,1\}_{\mathrm{B}})$ a fragment of the above TAUT problem for formulas in disjunctive normal form. (So for both problems, the propositional language is $\{\neg, \wedge, \vee\}$.) In [9], S.A. Cook established a link between propositional logic and computational complexity theory by presenting $\mathrm{SAT}^{\mathrm{CNF}}(\{0,1\}_{\mathrm{B}})$ as a first example of an $\mathbf{NP}$-complete problem:

PROPOSITION 2.4.2 (Cook Theorem).
The $\mathrm{SAT}^{\mathrm{CNF}}(\{0,1\}_{\mathrm{B}})$ *problem is* $\mathbf{NP}$*-complete.*

To obtain $\mathbf{NP}$-hardness, Cook considered an arbitrary but fixed set of words $S \subseteq \Sigma^*$ accepted by a (nondeterministic) Turing machine in time polynomial in the input size, and presented a polynomial-time procedure which constructed, for each string $s \in \Sigma^*$, a propositional formula φ^s in conjunctive normal form in such a way that $s \in S$ iff $\varphi^s \in \mathrm{SAT}^{\mathrm{CNF}}(\{0,1\}_{\mathrm{B}})$. Hence, $S \preceq_{\mathbf{P}} \mathrm{SAT}^{\mathrm{CNF}}(\{0,1\}_{\mathrm{B}})$, and $\mathrm{SAT}^{\mathrm{CNF}}(\{0,1\}_{\mathrm{B}})$ is $\mathbf{NP}$-complete.

$\mathrm{SAT}^{\mathrm{CNF}}(\{0,1\}_{\mathrm{B}})$ is a fragment of $\mathrm{SAT}(\{0,1\}_{\mathrm{B}})$, therefore $\mathrm{SAT}(\{0,1\}_{\mathrm{B}})$ is $\mathbf{NP}$-complete and $\mathrm{TAUT}(\{0,1\}_{\mathrm{B}})$ is $\mathbf{coNP}$-complete.

These complexity results extend immediately to $\mathrm{Th}_{\forall}(\{0,1\}_{\mathrm{B}})$, which is $\mathbf{coNP}$-complete, and to $\mathrm{Th}_{\exists}(\{0,1\}_{\mathrm{B}})$, which is $\mathbf{NP}$-complete; this is observed by realizing that an identity $t = s$ on Boolean expressions can be replaced by the equivalence $t \leftrightarrow s$. By this argument, $\mathrm{Th}(\{0,1\}_{\mathrm{B}})$ is polynomially equivalent to the QBF problem, hence $\mathbf{PSPACE}$-complete.

2.5 Decision problems in the reals

We review some decision problems in the reals, and also in the integers, that are relevant to our purpose.[10]

Consider a system $\mathbf{Ax} \leq \mathbf{b}$ of linear inequalities, where $\mathbf{b}$ is a rational m-vector and $\mathbf{A}$ is a rational $m \times n$-matrix. Assume every rational is represented as a pair of coprime integers and denote k the greatest absolute value of an integer occurring in the representations of $\mathbf{A}$ and $\mathbf{b}$.

The problem of solvability of $\mathbf{Ax} \leq \mathbf{b}$ in the reals is in $\mathbf{P}$. Within this chapter though, we will rely on its $\mathbf{NP}$-containment, which can be observed as follows.

Let $P = \{x \in \mathsf{R}^n \mid \mathbf{Ax} \leq \mathbf{b}\}$ be a nonempty polyhedron in R^n. Each nonempty, inclusion-wise minimal face[11] of P is a solution to $\mathbf{A}'\mathbf{x} = \mathbf{b}'$, where $\mathbf{A}'\mathbf{x} \leq \mathbf{b}'$ is a subsystem of the system $\mathbf{Ax} \leq \mathbf{b}$, i.e., $\mathbf{A}'$ is an $m' \times n$-matrix and $\mathbf{b}'$ an m'-vector for some $m' \leq m$. Fix a nonempty, minimal face of P; then its corresponding system of equations $\mathbf{A}'\mathbf{x} = \mathbf{b}'$ is solvable in R. Let $m'' \leq m'$ denote the rank of $\mathbf{A}'$, and let $\mathbf{A}''\mathbf{x} \leq \mathbf{b}''$ be a subsystem of $\mathbf{A}'\mathbf{x} \leq \mathbf{b}'$ with m'' linearly independent rows. Then

[10] References for the material presented in this subsection are [40] and [16].

[11] A *face* of P is any set $\{\mathbf{x} \in P \mid \mathbf{c}^T\mathbf{x} = d\}$ for $\mathbf{c} \in \mathsf{R}^n, d \in \mathsf{R}$ chosen in such a way that $\mathbf{c}^T\mathbf{x} \leq d$ holds for all $\mathbf{x} \in P$. A face is *minimal* if it does not contain any other face; a minimal face is an affine subspace of R^n. Since the list of all nonempty faces of (nonempty) P is finite, at least one nonempty, inclusion-wise minimal face exists.

$\mathbf{b}''$ is a linear combination of m'' columns of $\mathbf{A}''$, and hence, there is a solution $\mathbf{x} = \langle x_1, \ldots, x_n \rangle \in \mathsf{R}^n$ to $\mathbf{A}''\mathbf{x} = \mathbf{b}''$ where at most m'' of the x_i's are nonzero. Use Cramer's rule to compute the nonzero values of $\mathbf{x}$ (write m instead of m''). For each determinant, its denominator is at most the product of all denominators in $\mathbf{A}''$; if the largest one is k, then the product is at most k^{m^2}, i.e., of size at most $m^2 \log(k)$. The numerator, being a sum of $m!$ numbers bounded by k^{m^2}, is of size at most $m \log(m) + m^2 \log(k)$. Hence, any of the x_i's, as a fraction of two determinants, is of size at most $O(m \log(m) + m^2 \log(k))$.

Summing up, we arrive at the following statement (where instead of 'model' one might say 'solution' or 'evaluation'):

PROPOSITION 2.5.1 (Small-Model Theorem). *Let $\mathbf{A}$ be a rational $m \times n$-matrix and $\mathbf{b}$ a rational m-vector. Let k be the greatest integer occurring in the representations of $\mathbf{A}$ and $\mathbf{b}$. If the system $\mathbf{Ax} \leq \mathbf{b}$ is solvable in R, then it has a rational solution $\mathbf{x}_0$ with the following properties:*

(i) at most m values in $\mathbf{x}_0$ are nonzero;

(ii) any value in $\mathbf{x}_0$ has size polynomial in $|\mathbf{A}|$, $|\mathbf{b}|$; in particular, for $i = 1, \ldots, n$, $|(\mathbf{x}_0)_i|$ is in $O(m \log(m) + m^2 \log(k))$.

Linear Programming Problem. The linear programming (LP) problem[12] is defined as follows: given a rational $m \times n$-matrix $\mathbf{A}$, a rational m-vector $\mathbf{b}$, a rational n-vector $\mathbf{c}$, and a rational number d, does the system $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{c}^T\mathbf{x} < d$ have a solution in R? Again, while the problem is in $\mathbf{P}$, we need its $\mathbf{NP}$-containment. It is not difficult to see that the added strict inequality does not violate the validity of the small-model theorem above, where of course now k relates also to the integers in the representation of $\mathbf{c}$ and d as well. Indeed, given a solvable LP problem in the above notation, the halfspace $\mathbf{c}^T\mathbf{x} < d$ either contains a minimal face of $\mathbf{Ax} \leq \mathbf{b}$; or it intersects one, and then (by minimality) this face is unbounded and it contains some points that are bounded in size by the coefficients in $\mathbf{Ax} \leq \mathbf{b}$, $\mathbf{c}^T\mathbf{x} < d$ in the manner of the small-model theorem.

Modifications of the LP problem are obtained by posing various restrictive conditions; these modifications need not be feasible. In particular, the *integer programming problem*, here referred to as the ILP problem, is obtained by demanding that all variables and coefficients assume integer values. This problem is $\mathbf{NP}$-complete. Containment in $\mathbf{NP}$ can be derived from a small-model theorem for Diophantine equations and inequalities: let $\mathbf{Ax} \leq \mathbf{b}$ be a system of inequalities, where $\mathbf{A}$ is an integral $m \times n$-matrix and $\mathbf{b}$ is an integral m-vector, where the largest absolute value of an integer is k. If the system is solvable in Z, then it has a solution $\mathbf{x}_0$, where for any $1 \leq i \leq n$, $|(\mathbf{x}_0)_i|$ is in $O(m \log(m) + m \log(k))$. The *mixed integer programming* (MIP) problem is a modification of the LP problem demanding that a subset of the variables assume integer values. Particular bounded version of MIP poses the restrictive condition that the variables $x_k, \ldots, x_n$ only take the values 0 or 1. This is also in $\mathbf{NP}$: guess a random assignment of 0's and 1's to $x_k, \ldots, x_n$, then check solvability of the remaining system.

[12] Quite often, the phrase 'linear programming problem' denotes the *task* to either find a maximum of a function $\mathbf{c}^T\mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}$, or to say that none exists. We take the standpoint that a 'problem' is always a decision problem; the optimization task will not be considered in this chapter, so no confusion can arise.

Boolean combinations of linear inequalities. The linear programming problem comes as a conjunction of linear inequalities. Arbitrary Boolean combinations[13] of linear inequalities are apparently more difficult to solve. Consider basic inequalities of the form $\mathbf{a}^T\mathbf{x} \le b$ for a rational n-vector $\mathbf{a}$ and a rational number b; an *inequality formula* is a Boolean combination of basic inequalities. The INEQ problem is: given an inequality formula, is it solvable in the reals? The fact that the LP problem is in $\mathbf{NP}$ entails $\mathbf{NP}$-containment for the INEQ problem: each inequality formula has a logically equivalent disjunctive normal form, which is solvable in the reals iff so is at least one of its disjuncts. Each of the disjuncts can be equivalently transformed into an LP problem (negative literals will use the strict inequality in the LP problem). Because the solvability of the LP problem can be witnessed by a small evaluation, so can the solvability of an INEQ problem. On the other hand, INEQ is $\mathbf{NP}$-hard because classical SAT can be reduced to it, so it is $\mathbf{NP}$-complete.

Universal theory of RCF. Now let us consider the language of ordered fields, i.e., $\{+, \cdot, 0, 1, =, \le\}$. Recall that real numbers R with addition, multiplication, and the usual ordering, are an example of a real closed field (RCF). The first-order theory of real closed fields ($\mathrm{Th}(\mathrm{RCF})$) is complete; hence, each two real closed fields are elementarily equivalent. Moreover, $\mathrm{Th}(\mathrm{RCF})$ is decidable. Both results were proved by A. Tarski by quantifier elimination. J.F. Canny has shown that the existential fragment of the RCF theory is in $\mathbf{PSPACE}$ [7]. As $\mathbf{PSPACE}$ is closed under complementation, also the universal fragment of the RCF theory is in $\mathbf{PSPACE}$. We denote these fragments $\mathrm{Th}_\exists(\mathrm{RCF})$ and $\mathrm{Th}_\forall(\mathrm{RCF})$, respectively.

3 General results and methods

This section presents general statements, applicable for particular examples of logics; while all of the statements are, to a degree, a prerequisite to reading the following sections, this is perhaps especially true about Subsection 3.3, where some technical statements and notation are introduced without which the following sections might be incomprehensible.

Languages in this section are assumed to subsume the language of $\mathrm{FL_{ew}}$, logics are assumed to be at least as strong as $\mathrm{FL_{ew}}$ (with maybe additional connectives, and some connectives definable) and algebras are assumed to be $\mathrm{FL_{ew}}$-algebras (with maybe additional operations). Classes of algebras are assumed nonempty.

3.1 Basic inclusions and reductions

For the next two statements, the discussion of fragments in Subsection 2.3 is relevant.

LEMMA 3.1.1. *Let $\mathcal{L}$ be a language,* L *a logic in the language $\mathcal{L}$, and $\mathbb{K}$ a class of $\mathcal{L}$-algebras. Then*

(i) $\mathrm{THM(L)} \preceq_{\mathbf{P}} \mathrm{CONS(L)}$; *if* L *enjoys the classical (or the $\triangle$-) deduction theorem, then* $\mathrm{THM(L)} \approx_{\mathbf{P}} \mathrm{CONS(L)}$;

(ii) $\mathrm{Th_{Eq}}(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th_{QEq}}(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th}_\forall(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th}(\mathbb{K})$ *and* $\mathrm{Th}_\exists(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th}(\mathbb{K})$;

[13] I.e., a formula with any connectives of the full language of classical logic.

(iii) $\mathrm{TAUT}(\mathbb{K}) \approx_{\mathbf{P}} \mathrm{Th}_{\mathrm{Eq}}(\mathbb{K})$ *and* $\mathrm{CONS}(\mathbb{K}) \approx_{\mathbf{P}} \mathrm{Th}_{\mathrm{QEq}}(\mathbb{K})$;

(iv) $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th}_{\mathrm{QEq}}(\mathbb{K})$;

(v) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) \preceq_{\mathbf{P}} \mathrm{Th}_{\exists}(\mathbb{K})$;

(vi) $\mathrm{Th}_{\forall}(\mathbb{K}) \approx_{\mathbf{P}} \overline{\mathrm{Th}_{\exists}}(\mathbb{K})$.

Proof. In the following, consider an $\mathcal{L}$-expression φ with n variables $x_1, \ldots, x_n$.
(i) $\mathrm{THM}(\mathrm{L})$ is the fragment of $\mathrm{CONS}(\mathrm{L})$ obtained by considering only empty theories. If L enjoys the classical (or the $\triangle$-) deduction theorem, then $\{\psi_1, \ldots, \psi_n\} \vdash_{\mathrm{L}} \varphi$ iff $\vdash_{\mathrm{L}} \psi_1 \mathbin{\&} \ldots \mathbin{\&} \psi_n \to \varphi$ ($\vdash_{\mathrm{L}} \triangle(\psi_1 \mathbin{\&} \ldots \mathbin{\&} \psi_n) \to \varphi$ respectively).
(ii) In all cases, we are dealing with fragments defined by polynomial-time conditions.
(iii) $\varphi \in \mathrm{TAUT}(\mathbb{K})$ iff $(\varphi = 1) \in \mathrm{Th}_{\mathrm{Eq}}(\mathbb{K})$; on the other hand, $(\varphi = \psi) \in \mathrm{Th}_{\mathrm{Eq}}(\mathbb{K})$ iff $(\varphi \leftrightarrow \psi) \in \mathrm{TAUT}(\mathbb{K})$. Analogously for CONS and quasiidentities.
(iv) $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ iff $\forall x_1 \ldots \forall x_n(\varphi = 0 \to 0 = 1) \in \mathrm{Th}_{\mathrm{QEq}}(\mathbb{K})$.
(v) $\varphi \in \mathrm{SAT}(\mathbb{K})$ iff $\exists x_1 \ldots \exists x_n(\varphi = 1) \in \mathrm{Th}_{\exists}(\mathbb{K})$; analogously, $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ iff $\exists x_1 \ldots \exists x_n(\varphi > 0) \in \mathrm{Th}_{\exists}(\mathbb{K})$.
(vi) A consequence of classical duality of quantifiers. □

LEMMA 3.1.2. *Let $\mathcal{L}_1 \subseteq \mathcal{L}_2$ be languages.*

(i) Assume a logic L_2 *in language* $\mathcal{L}_2$ *expands conservatively a logic* L_1 *in language* $\mathcal{L}_1$ *(so* L_1 *is the* $\mathcal{L}_1$*-fragment of* L_2*). Then* $\mathrm{THM}(\mathrm{L}_1) \preceq_{\mathbf{P}} \mathrm{THM}(\mathrm{L}_2)$ *and* $\mathrm{CONS}(\mathrm{L}_1) \preceq_{\mathbf{P}} \mathrm{CONS}(\mathrm{L}_2)$.

(ii) Assume $\mathbb{K}_2$ *is a class of* $\mathcal{L}_2$*-algebras and* $\mathbb{K}_1$ *is the class of* $\mathcal{L}_1$*-reducts of elements of* $\mathbb{K}_2$*. Then* $\mathrm{Th}(\mathbb{K}_1) \preceq_{\mathbf{P}} \mathrm{Th}(\mathbb{K}_2)$*, and analogously for the equational, quasiequational, universal and existential fragments of the two theories.*

LEMMA 3.1.3. *Let $\mathcal{L}$ be a language and $\mathbb{K}$, $\mathbb{L}$ classes of $\mathcal{L}$-algebras.*

(i) Assume $\mathbb{K} \subseteq \mathbb{L}$*. Then*

(a) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) \subseteq \mathrm{SAT}_{(\mathrm{pos})}(\mathbb{L})$;

(b) $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{L}) \subseteq \mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K})$ *and* $\mathrm{CONS}(\mathbb{L}) \subseteq \mathrm{CONS}(\mathbb{K})$.

(ii) Assume $\mathbb{K}$ *is (partially) embeddable into* $\mathbb{L}$*. Then*

(a) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) \subseteq \mathrm{SAT}_{(\mathrm{pos})}(\mathbb{L})$;

(b) $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{L}) \subseteq \mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K})$ *and* $\mathrm{CONS}(\mathbb{L}) \subseteq \mathrm{CONS}(\mathbb{K})$.

Proof. (i) holds by definition. For (ii), it suffices to recall that 0 is an element of $\mathcal{L}$ and hence preserved by morphisms, and the same is true for $1 = 0 \to 0$. □

LEMMA 3.1.4. *Let $\mathcal{L}$ be a language and $\mathbb{K}$ a class of $\mathcal{L}$-algebras containing a nontrivial algebra. Then*

(i) $\mathrm{SAT}(\{0,1\}_{\mathrm{B}}) \subseteq \mathrm{SAT}(\mathbb{K}) \subseteq \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$;

(ii) $\mathrm{TAUT}(\mathbb{K}) \subseteq \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}) \subseteq \mathrm{TAUT}(\{0,1\}_{\mathrm{B}})$;

(iii) $\mathrm{CONS}(\mathbb{K}) \subseteq \mathrm{CONS}(\{0,1\}_{\mathrm{B}})$.

Proof. For any nontrivial $\mathcal{L}$-algebra $\boldsymbol{A} \in \mathbb{K}$, its subalgebra $\{0^{\boldsymbol{A}}, 1^{\boldsymbol{A}}\}$ is a two-element Boolean algebra. So $\{0,1\}_{\mathrm{B}}$ is embeddable into $\boldsymbol{A}$, the class $\{\{0,1\}_{\mathrm{B}}\}$ is embeddable into $\mathbb{K}$, and Lemma 3.1.3 applies. □

LEMMA 3.1.5. *Let $\mathcal{L}$ be a language and $\mathbb{K}$ a class of $\mathcal{L}$-algebras. Then*

(i) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{SAT}_{(\mathrm{pos})}(\mathbf{I}(\mathbb{K}))$ *and* $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{TAUT}_{(\mathrm{pos})}(\mathbf{I}(\mathbb{K}))$;

(ii) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{SAT}_{(\mathrm{pos})}(\mathbf{S}(\mathbb{K}))$ *and* $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{TAUT}_{(\mathrm{pos})}(\mathbf{S}(\mathbb{K}))$;

(iii) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{SAT}_{(\mathrm{pos})}(\mathbf{P}(\mathbb{K}))$ *and* $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{TAUT}_{(\mathrm{pos})}(\mathbf{P}(\mathbb{K}))$;

(iv) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{SAT}_{(\mathrm{pos})}(\mathbf{P}_{\mathrm{U}}(\mathbb{K}))$ *and* $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{TAUT}_{(\mathrm{pos})}(\mathbf{P}_{\mathrm{U}}(\mathbb{K}))$;

(v) $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}) = \mathrm{SAT}_{\mathrm{pos}}(\mathbf{H}(\mathbb{K}))$ *and* $\mathrm{TAUT}(\mathbb{K}) = \mathrm{TAUT}(\mathbf{H}(\mathbb{K}))$.

Proof. For the SAT operators, the left-to-right inclusions are obtained by virtue of $\mathbf{I}$, $\mathbf{S}$, $\mathbf{P}$, $\mathbf{P}_{\mathrm{U}}$, $\mathbf{H}$ being closure operators (in view of Lemma 3.1.3); for the TAUT operators, the converse inclusions hold by the same argument. Indeed for the TAUT operator, equality for all cases is well known. In the following, we set to show the remaining inclusions.
(i) Immediate.
(ii) Let $\boldsymbol{B}$ be a subalgebra of $\boldsymbol{A} \in \mathbb{K}$. Then $\boldsymbol{B}$ is embeddable into $\boldsymbol{A}$ (via identity mapping). Therefore $\mathbf{S}(\mathbb{K})$ is embeddable into $\mathbb{K}$. An application of Lemma 3.1.3 yields the desired inclusions.
(iii) Let $\boldsymbol{B} = \prod_{i\in I} \boldsymbol{A}_i$, where $I \neq \emptyset$ and $\boldsymbol{A}_i \in \mathbb{K}$ for each $i \in I$. If $e_{\boldsymbol{B}}(\varphi) = 1^{\boldsymbol{B}}$ ($e_{\boldsymbol{B}}(\varphi) > 0^{\boldsymbol{B}}$), define $e_{\boldsymbol{A}_i}(x) = \pi_i(e_{\boldsymbol{B}}(x))$ for each $i \in I$ (where π_i is the i-th projection); then $e_{\boldsymbol{A}_i}$ is an evaluation in $\boldsymbol{A}_i$ for each $i \in I$. Clearly $e_{\boldsymbol{A}_i}(\varphi) = 1^{\boldsymbol{A}_i}$ for each $i \in I$ ($e_{\boldsymbol{A}_i}(\varphi) > 0$ for some $i \in I$ respectively), so $\varphi \in \mathrm{SAT}(\boldsymbol{A}_i)$ for each $i \in I$ ($\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A}_i)$ for some $i \in I$ respectively). Likewise, if $e_{\boldsymbol{A}_i}(\varphi) > 0^{\boldsymbol{A}_i}$ for each $i \in I$ and each evaluation $e_{\boldsymbol{A}_i}$, then in particular, for an arbitrary evaluation $e_{\boldsymbol{B}}$ in $\boldsymbol{B}$, we have $\pi_i(e_{\boldsymbol{B}}(\varphi)) > 0^{\boldsymbol{A}_i}$, hence $e_{\boldsymbol{B}}(\varphi) > 0^{\boldsymbol{B}}$.
(iv) Let $\boldsymbol{B} = \prod_{i\in I}^{\mathcal{F}} \boldsymbol{A}_i$, where $I \neq \emptyset$, $\mathcal{F}$ is an ultrafilter on I, and $\boldsymbol{A}_i \in \mathbb{K}$ for each $i \in I$. If $e_{\boldsymbol{B}}(\varphi) = 1^{\boldsymbol{B}}$, let, for each $x \in \mathit{Var}$, $e(x)$ be any $f \in [e_{\boldsymbol{B}}(x)]_{\mathcal{F}}$, and for each $i \in I$, define $e_{\boldsymbol{A}_i}(x) = \pi_i(e(x))$; then $e_{\boldsymbol{A}_i}$ is an evaluation in $\boldsymbol{A}_i$ for each $i \in I$. We have $\{i \mid e_{\boldsymbol{A}_i}(\varphi) = 1^{\boldsymbol{A}_i}\} \in \mathcal{F}$, hence for some $i \in I$ we have $\varphi \in \mathrm{SAT}(\boldsymbol{A}_i)$. Assuming $e_{\boldsymbol{B}}(\varphi) > 0^{\boldsymbol{B}}$ ($e_{\boldsymbol{B}}(\varphi) < 1^{\boldsymbol{B}}$, $e_{\boldsymbol{B}}(\varphi) = 0^{\boldsymbol{B}}$), one gets in the same manner $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A}_i)$ ($\varphi \notin \mathrm{TAUT}(\boldsymbol{A}_i)$, $\varphi \notin \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A}_i)$ respectively) for some $i \in I$.
(v) Let $\boldsymbol{B}$ be a nontrivial homomorphic image of $\boldsymbol{A} \in \mathbb{K}$ via an f. Assume $e_{\boldsymbol{B}}(\varphi) > 0^{\boldsymbol{B}}$. Take $e_{\boldsymbol{A}}(x) \in f^{-1}(e_{\boldsymbol{B}}(x))$ for each variable x; then $e_{\boldsymbol{A}}(\varphi) > 0^{\boldsymbol{A}}$. □

On this basis, we may conclude:

THEOREM 3.1.6. *Let $\mathcal{L}$ be a language and $\mathbb{K}$ a class of $\mathcal{L}$-algebras. Then*

(i) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{SAT}_{(\mathrm{pos})}(\mathbf{Q}(\mathbb{K}))$;

(ii) $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}) = \mathrm{SAT}_{\mathrm{pos}}(\mathbf{V}(\mathbb{K}))$;

(iii) $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}) = \mathrm{TAUT}_{\mathrm{pos}}(\mathbf{Q}(\mathbb{K}))$;

(iv) $\mathrm{TAUT}(\mathbb{K}) = \mathrm{TAUT}(\mathbf{V}(\mathbb{K})) = \mathrm{TAUT}(\mathbf{Q}(\mathbb{K}))$.

The previous theorem says that, in contrast to the case of first-order fuzzy logics, one need not worry about the general/standard semantics distinction in the propositional case. For important examples of logics L considered in this chapter, their equivalent algebraic semantics is a variety that is generated by its standard members (as a quasivariety); then one can apply the above theorem to relate the results obtained on standard algebras also to the general semantics.

3.2 Negations

In classical logic, there is a duality between the SAT and TAUT problems, in the manner of Lemma 3.1.1 (vi). We inspect the conditions under which as much, or at least some of that, may be claimed in a many-valued setting.

A negation $\sim$ in L is *involutive* iff $\sim\sim\varphi \leftrightarrow \varphi$ is a theorem of L; the semantics of $\sim$ is an order-reversing involution. A negation $\neg$ in L is *strict* iff $\neg(\varphi \wedge \neg\varphi)$ is a theorem of L. If both the involutive negation $\sim$ and the strict negation $\neg$ are available in L, then, defining $\triangle\varphi$ as $\neg\sim\varphi$, one can prove the usual axioms for the $\triangle$ connective.

LEMMA 3.2.1. *Let $\mathcal{L}$ be a language and $\mathbb{K}$ a class of $\mathcal{L}$-algebras containing a nontrivial algebra. Then*

(i) $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\neg\varphi \in \overline{\mathrm{SAT}}(\mathbb{K})$;

(ii) $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\neg\varphi \in \overline{\mathrm{TAUT}}(\mathbb{K})$.

If, additionally, $\sim$ is an involutive negation in $\mathbb{K}$, then

(iii) $\varphi \in \mathrm{TAUT}(\mathbb{K})$ *iff* $\sim\varphi \in \overline{\mathrm{SAT}_{\mathrm{pos}}}(\mathbb{K})$;

(iv) $\varphi \in \mathrm{SAT}(\mathbb{K})$ *iff* $\sim\varphi \in \overline{\mathrm{TAUT}_{\mathrm{pos}}}(\mathbb{K})$.

Proof. Items (i), (ii) are easily obtained by observing that in any nontrivial $\mathrm{FL}_{\mathrm{ew}}$-algebra $\boldsymbol{A}$, the equation $\neg x = 1$ has a unique solution, $x = 0^{\boldsymbol{A}}$.
Items (iii), (iv) follow from (i), (ii) by substituting $\sim\varphi$ for φ and using $\sim\sim\varphi \leftrightarrow \varphi$. □

COROLLARY 3.2.2. *Let $\mathcal{L}$ be a language and $\mathbb{K}$ be a class of involutive $\mathcal{L}$-algebras containing a nontrivial algebra. Then*

(i) $\mathrm{TAUT}(\mathbb{K}) \approx_{\mathbf{P}} \overline{\mathrm{SAT}_{\mathrm{pos}}}(\mathbb{K})$;

(ii) $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}) \approx_{\mathbf{P}} \overline{\mathrm{SAT}}(\mathbb{K})$.

Now we explore logics with strict negations.

LEMMA 3.2.3. *Let $\boldsymbol{A}$ be a nontrivial* SMTL-*chain. Then $\{0,1\}_{\mathrm{B}}$ is a homomorphic image of $\boldsymbol{A}$.*

Proof. The mapping $h\colon \boldsymbol{A} \to \boldsymbol{A}$ sending $0^{\boldsymbol{A}}$ to $0^{\boldsymbol{A}}$ and all nonzero elements to $1^{\boldsymbol{A}}$ is a homomorphism of $\boldsymbol{A}$ onto the two-element Boolean subalgebra $\{0^{\boldsymbol{A}}, 1^{\boldsymbol{A}}\}$ of $\boldsymbol{A}$. □

THEOREM 3.2.4. *Let $\mathbb{K}$ be a class of* SMTL-*chains containing a nontrivial one. Then*

(i) $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}) = \mathrm{SAT}(\mathbb{K}) = \mathrm{SAT}(\{0,1\}_{\mathrm{B}})$;

(ii) $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}) = \mathrm{TAUT}(\{0,1\}_{\mathrm{B}})$.

Proof. (i) By Lemma 3.1.4, it is sufficient to show that any formula φ positively satisfiable in $\mathbb{K}$ is classically satisfiable. Let for some $\boldsymbol{A} \in \mathbb{K}$ and some $e_{\boldsymbol{A}}$ be $e_{\boldsymbol{A}}(\varphi) > 0^{\boldsymbol{A}}$. Define e' on $\{0,1\}_{\mathrm{B}}$ using the homomorphism from Lemma 3.2.3: for any formula ψ, let $e'_{\boldsymbol{A}}(\psi) = h(e_{\boldsymbol{A}}(\psi))$. Clearly e' is a well-defined evaluation in $\{0,1\}_{\mathrm{B}}$ and $e'(\varphi) = 1^{\{0,1\}_{\mathrm{B}}}$ iff $e_{\boldsymbol{A}}(\varphi) > 0^{\boldsymbol{A}}$.
(ii) Again it is sufficient to show $\mathrm{TAUT}^{\mathrm{Boole}} \subseteq \mathrm{TAUT}^{\boldsymbol{A}}_{\mathrm{pos}}$ for any nontrivial $\boldsymbol{A} \in \mathbb{K}$. For φ a classical tautology, assume $e_{\boldsymbol{A}}(\varphi) = 0^{\boldsymbol{A}}$ in $\boldsymbol{A}$; but then $e'(\varphi) = 0$ in $\{0,1\}_{\mathrm{B}}$ (e' as above), a contradiction; hence $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A})$. □

3.3 Eliminating compound terms

We start with feasible translations of expressions that preserve satisfiability or tautologousness but eliminate (some) nested connectives in a particular way, at the expense of adding new variables.

Consider a finite language $\mathcal{L}$ and an $\mathcal{L}$-expression $\varphi(x_1, \ldots, x_n)$, where $n \geq 1$. To each subexpression ψ of φ, assign a variable y_ψ in the following manner:

– if ψ is a variable x_i, then let y_ψ be the variable x_i;
– otherwise, let y_ψ be a new variable.

For each $\psi \preceq \varphi$, let S_ψ denote the set of all subexpressions of ψ that are not variables. For each $\psi \in S_\varphi$, if ψ is $c(\psi_1, \ldots, \psi_k)$ for some $k \in \mathsf{N}$, some $\psi_1, \ldots, \psi_k \preceq \varphi$, and some $c \in \mathcal{L}$, let C_ψ stand for $y_\psi \leftrightarrow c(y_{\psi_1}, \ldots, y_{\psi_k})$. For each $\psi \preceq \varphi$, let ψ' be the expression $\&_{\chi \in S_\psi} C_\chi$. Then for each $\psi \preceq \varphi$, if ψ is $c(\psi_1, \ldots, \psi_k)$ for some $k \in \mathsf{N}$, some $\psi_1, \ldots, \psi_k \preceq \psi$, and some $c \in \mathcal{L}$, then ψ' is $\psi'_1 \mathbin{\&} \ldots \mathbin{\&} \psi'_k \mathbin{\&} C_\psi$; if ψ is a variable, then ψ' is $\overline{1}$.

Observe that φ' can be obtained from φ in time polynomial in $|\varphi|$: indeed, for each $\psi \preceq \varphi$, $|C_\psi|$ is in $O(\log(|\varphi|))$, and the number of C_ψ's in φ' is bounded by the number of subexpressions in φ, i.e., by $|\varphi|$.

LEMMA 3.3.1. *For each* $\mathrm{FL_{ew}}$*-expression* φ *we have* $\vdash_{\mathrm{FL_{ew}}} \varphi' \to (y_\varphi \leftrightarrow \varphi)$.

Proof. By induction on formula structure. The cases of ψ being a variable or the constant $\overline{0}$ are simple. For the induction step, let $\psi \preceq \varphi$ be $\psi_1 \circ \psi_2$ for $\circ$ one of $\{\&, \to, \wedge, \vee\}$; the induction assumption is $\psi'_i \to (y_{\psi_i} \leftrightarrow \psi_i)$ for $i = 1, 2$. We obtain $\psi'_1 \mathbin{\&} \psi'_2 \mathbin{\&} C_\psi \to (y_{\psi_1} \leftrightarrow \psi_1) \mathbin{\&} (y_{\psi_2} \leftrightarrow \psi_2) \mathbin{\&} C_\psi$. The antecedent of this implication is ψ', while the succedent implies[14] $((y_{\psi_1} \circ y_{\psi_2}) \leftrightarrow (\psi_1 \circ \psi_2)) \mathbin{\&} (y_\psi \leftrightarrow (y_{\psi_1} \circ y_{\psi_2}))$, whence $y_\psi \leftrightarrow \psi$. □

LEMMA 3.3.2. *Let* $\mathbb{K}$ *a class of* $\mathrm{FL_{ew}}$*-algebras,* φ *a* $\mathrm{FL_{ew}}$*-expression, and* φ' *as above. Then*

(i) $\varphi \in \mathrm{TAUT}(\mathbb{K})$ *iff* $\varphi' \to y_\varphi \in \mathrm{TAUT}(\mathbb{K})$;

(ii) $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\varphi' \to y_\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$;

(iii) $\varphi \in \mathrm{SAT}(\mathbb{K})$ *iff* $\varphi' \mathbin{\&} y_\varphi \in \mathrm{SAT}(\mathbb{K})$;

(iv) $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\varphi' \mathbin{\&} y_\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

[14] Using $((\alpha_1 \leftrightarrow \beta_1) \mathbin{\&} (\alpha_2 \leftrightarrow \beta_2)) \to ((\alpha_1 \circ \alpha_2) \leftrightarrow (\beta_1 \circ \beta_2))$ for $\circ$ one of $\{\&, \to, \wedge, \vee\}$; if this congruence is provable for other connectives, then the statement extends to formulas with these connectives.

Proof. (i) If $\varphi \in \mathrm{TAUT}(\mathbb{K})$, then by Lemma 3.3.1 $\varphi' \to (y_\varphi \leftrightarrow \overline{1})$ is a tautology of $\mathbb{K}$. On the other hand, if $\varphi(x_1, \dots, x_n)$ is not a tautology of $\mathbb{K}$, there is an $\boldsymbol{A} \in \mathbb{K}$ and an evaluation $e_{\boldsymbol{A}}$ such that $e_{\boldsymbol{A}}(\varphi) < 1^{\boldsymbol{A}}$; define $e'_{\boldsymbol{A}}(x_i) = e_{\boldsymbol{A}}(x_i)$ for $1 \leq i \leq n$, and if $\psi \preceq \varphi$, set $e'_{\boldsymbol{A}}(y_\psi) = e_{\boldsymbol{A}}(\psi)$. Then clearly $e'_{\boldsymbol{A}}(\varphi') = 1^{\boldsymbol{A}}$, while $e'_{\boldsymbol{A}}(y_\varphi) < 1^{\boldsymbol{A}}$.
(ii) If $\varphi \in \overline{\mathrm{TAUT}_{\mathrm{pos}}}(\mathbb{K})$, clearly $\varphi' \to y_\varphi \in \overline{\mathrm{TAUT}_{\mathrm{pos}}}(\mathbb{K})$. Conversely, let $\varphi' \to y_\varphi \in \overline{\mathrm{TAUT}_{\mathrm{pos}}}(\mathbb{K})$, so $e_{\boldsymbol{A}}(\varphi' \to y_\varphi) = 0^{\boldsymbol{A}}$ for some $\boldsymbol{A} \in \mathbb{K}$ and some $e_{\boldsymbol{A}}$. By Lemma 3.3.1, $e_{\boldsymbol{A}}(\varphi' \to (\varphi \to y_\varphi)) = 1^{\boldsymbol{A}}$, so $e_{\boldsymbol{A}}(\varphi \to (\varphi' \to y_\varphi)) = 1^{\boldsymbol{A}}$, and hence $e_{\boldsymbol{A}}(\varphi) = 0^{\boldsymbol{A}}$, so $\varphi \in \overline{\mathrm{TAUT}_{\mathrm{pos}}}(\mathbb{K})$.
(iii) If $\varphi \in \mathrm{SAT}(\mathbb{K})$, clearly $\varphi' \,\&\, y_\varphi \in \mathrm{SAT}(\mathbb{K})$. Conversely, for $\boldsymbol{A} \in \mathbb{K}$, if $e_{\boldsymbol{A}}(\varphi') = 1^{\boldsymbol{A}}$, then in particular $e_{\boldsymbol{A}}(y_\psi) = c^{\boldsymbol{A}}(e_{\boldsymbol{A}}(y_{\psi_1}), \dots, e_{\boldsymbol{A}}(y_{\psi_k})) = e_{\boldsymbol{A}}(\psi)$ whenever $\psi \preceq \varphi$ is $c(\psi_1, \dots, \psi_k)$. We have $e_{\boldsymbol{A}}(y_\varphi) = e_{\boldsymbol{A}}(\varphi) = 1^{\boldsymbol{A}}$, hence $\varphi \in \mathrm{SAT}(\mathbb{K})$.
(iv) If $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$, clearly $\varphi' \,\&\, y_\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$. Conversely, if $\varphi \in \overline{\mathrm{SAT}_{\mathrm{pos}}}(\mathbb{K})$, then $\varphi \leftrightarrow \overline{0}$ is a tautology of $\mathbb{K}$ and hence so is $\varphi' \to (y_\varphi \leftrightarrow \overline{0})$, using Lemma 3.3.1. Hence $\varphi' \,\&\, y_\varphi \to \overline{0}$ is a tautology of $\mathbb{K}$, so $\varphi' \,\&\, y_\varphi$ is unsatisfiable in $\mathbb{K}$. □

THEOREM 3.3.3. *Let $\mathbb{K}$ be a class of $\mathrm{FL}_{\mathrm{ew}}$-algebras. Assume $c \in \mathcal{L}$ is term-definable in $\mathbb{K}$, c is not among $\{\&, \to\}$, and let $\mathbb{K}^{c^-}$ be the class of $\mathcal{L} \setminus \{c\}$-reducts of $\mathbb{K}$. Then*

(i) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) \approx_{\mathbf{P}} \mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}^{c^-})$;

(ii) $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K}) \approx_{\mathbf{P}} \mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K}^{c^-})$.

Proof. We give the proof for TAUT; for the other operators it is analogous. Clearly $\mathrm{TAUT}(\mathbb{K}^{c^-}) \preceq_{\mathbf{P}} \mathrm{TAUT}(\mathbb{K})$ (cf. Lemma 3.1.2). Conversely, if φ is an $\mathcal{L}$-formula, then $\varphi \in \mathrm{TAUT}(\mathbb{K})$ iff $\varphi' \to y_\varphi \in \mathrm{TAUT}(\mathbb{K})$; apply to φ' the desired translation, i.e., if the identity $c(x_1, \dots, x_k) = \chi(x_1, \dots, x_k)$ holds in $\mathbb{K}$ for some $\mathcal{L} \setminus \{c\}$-term χ, replace each occurrence of $y_\psi \leftrightarrow c(y_{\psi_1}, \dots, y_{\psi_k})$ with $y_\psi \leftrightarrow \chi(y_{\psi_1}, \dots, y_{\psi_k})$. Replace each occurrence of $\leftrightarrow$ using $\&$ and $\to$; denote the resulting formula φ''. Then $\varphi' \to y_\varphi \in \mathrm{TAUT}(\mathbb{K})$ iff $\varphi'' \to y_\varphi \in \mathrm{TAUT}(\mathbb{K}^{c^-})$, and moreover φ'' can be obtained from φ' by a procedure operating in time polynomial in $|\varphi'|$. □

Further we eliminate compound terms from first-order formulas; in particular, we work with existential sentences (using the technique, one can also eliminate compound terms from universal sentences, whose negations are existential sentences).

DEFINITION 3.3.4. *Let Φ be a first-order formula in a language $\mathcal{L}$. We say that Φ is* without compound terms *iff each atomic formula in Φ is either $t_0 = t_1$ or $t_0 \leq t_1$ or $t_0 < t_1$, there being an $i \in \{0, 1\}$ such that t_i is a variable while t_{1-i} is a variable or a term $f(x_1, \dots, x_n)$ for some n-ary function symbol $f \in \mathcal{L}$ and some variables $x_1, \dots, x_n$.*

LEMMA 3.3.5. *Let $\mathcal{L}$ be a language, let Φ be an existential $\mathcal{L}$-sentence. Then there is an existential $\mathcal{L}$-sentence Φ', such that:*

(i) *Φ' is of the form $\exists x_1 \dots \exists x_k(\Phi_1 \wedge \Phi_2)$, where $\Phi_1 \wedge \Phi_2$ is an open formula without compound terms*;

(ii) *$\boldsymbol{A} \models \Phi$ iff $\boldsymbol{A} \models \Phi'$ for each $\mathcal{L}$-algebra $\boldsymbol{A}$*;

(iii) *Φ' can be computed from Φ by an algorithm working in time polynomial in $|\Phi|$.*

Proof. (i) Let Φ be an existential sentence. One may assume Φ is in prenex form (in particular, there is a polynomial-time algorithm which brings a sentence into an equivalent sentence in prenex form).

Let $T = \{t_1, \ldots, t_m\}$ be the collection of all terms in Φ_0. Let S be the collection of T-subterms, i.e., for each $s \in Fm_{\mathcal{L}}$, we have $s \in S$ iff $s \preceq t_i$ for some $i \in \{1, \ldots, m\}$. To each term $s \in S$, assign a variable x_s as follows:

– if s is a variable, let x_s be the variable s
– otherwise, let x_s be a new variable.

Denote S' the terms in S that are not variables.

Now let Φ_1 result from Φ_0 by replacing each atomic formula $t_1 = t_2$ ($t_1 \leq t_2$, $t_1 < t_2$) in Φ_0 with the atomic formula $x_{t_1} = x_{t_2}$ ($x_{t_1} \leq x_{t_2}$, $x_{t_1} < x_{t_2}$ respectively). Then all terms in Φ_1 are variables.

Moreover, let Φ_2 be

$$\bigwedge_{\substack{s \in S' \\ s \text{ is } f(s_1,\ldots,s_n)}} (x_s = f(x_{s_1}, \ldots, x_{s_n})).$$

Then Φ_2 is without compound terms. Finally, let Φ' be the existential closure of $\Phi_1 \wedge \Phi_2$.
(ii) It is elementary to check that Φ and Φ' are equivalent in any $\mathcal{L}$-algebra.
(iii) Identifying all (sub)terms in Φ, introducing new variables for subterms where necessary, building Φ_1 out of variables standing for terms in Φ, and listing all identities obtained for variables from the structure of subterms, is clearly polynomial in $|\Phi|$. $\square$

In the following sections, we will not only be eliminating compound terms, but we will be using the particular translation of existential sentences given in the previous proof. For lack of a better term, we refer to the result of such a translation of an existential sentence Φ as the *existential normal form* of Φ.

3.4 Lower bounds

THEOREM 3.4.1. *Let $\mathcal{L}$ be a language and $\mathbb{K}$ a class of $\mathcal{L}$-algebras containing a non-trivial algebra. Then*

(i) $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K})$ *is* $\mathbf{coNP}$*-hard*;
(ii) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K})$ *is* $\mathbf{NP}$*-hard.*

Proof. Consider formulas of classical propositional logic in the language $\{\neg, \&, \vee\}$. Recall Proposition 2.4.2: satisfiability in $\{0,1\}_{\mathrm{B}}$ for CNF-formulas is $\mathbf{NP}$-complete, hence tautologousness in $\{0,1\}_{\mathrm{B}}$ for DNF-formulas is $\mathbf{coNP}$-complete. The two problems are denoted $\mathrm{SAT}^{\mathrm{CNF}}(\{0,1\}_{\mathrm{B}})$ and $\mathrm{TAUT}^{\mathrm{DNF}}(\{0,1\}_{\mathrm{B}})$, respectively.

(i) We show $\mathrm{TAUT}^{\mathrm{DNF}}(\{0,1\}_{\mathrm{B}}) \preceq_{\mathbf{P}} \mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K})$. If $\varphi(x_1, \ldots, x_n)$ is a formula of classical propositional logic in DNF, define $\varphi^\star$ as

$$((x_1 \vee \neg x_1) \mathbin{\&} \ldots \mathbin{\&} (x_n \vee \neg x_n)) \to \varphi(x_1, \ldots, x_n).$$

We claim $\varphi \in \mathrm{TAUT}^{\mathrm{DNF}}(\{0,1\}_{\mathrm{B}})$ iff $\varphi^\star \in \mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K})$.

Assume first that φ is a classical tautology in DNF: a formula $D_1 \vee \ldots \vee D_m$, where each D_j, $1 \leq j \leq m$, is a conjunction of literals in variables among $x_1, \ldots, x_n$ and $m \in \mathsf{N}$. Without loss of generality we may assume each D_j contains each of its literals at most once.[15] We will show that $\varphi^\star$ is a theorem of $\mathrm{FL_{ew}}$ (hence a tautology of each $\mathrm{FL_{ew}}$-algebra, and a positive tautology of each nontrivial $\mathrm{FL_{ew}}$-algebra). Recall that

$$\psi \mathbin{\&} \bigvee_{j<k} \chi_j \leftrightarrow \bigvee_{j<k} (\psi \mathbin{\&} \chi_j) \tag{aux}$$

is a theorem of $\mathrm{FL_{ew}}$ for any choice of $\mathrm{FL_{ew}}$-formulas $\{\psi\} \cup \{\chi_j\}_{j<k}$. Assume $e \in Val^{\{x_1,\ldots,x_n\}}(\{0,1\}_\mathrm{B})$ is a (restriction of) Boolean evaluation; for each $i \in \{1, \ldots, n\}$, let x_i^e be the literal x_i if $e(x_i) = 1^{\{0,1\}_\mathrm{B}}$, and the literal $\neg x_i$ otherwise; let E^e be the formula $x_1^e \mathbin{\&} \ldots \mathbin{\&} x_n^e$. Then $e(E^e) = 1^{\{0,1\}_\mathrm{B}}$, and if $e' \in Val^{\{x_1,\ldots,x_n\}}(\{0,1\}_\mathrm{B})$, we have $e'(E^e) = 1^{\{0,1\}_\mathrm{B}}$ iff $e = e'$. Using (aux), the formula $(x_1 \vee \neg x_1) \mathbin{\&} \ldots \mathbin{\&} (x_n \vee \neg x_n)$ is $\mathrm{FL_{ew}}$-equivalent to the formula $\bigvee_{e \in Val^{\{x_1,\ldots,x_n\}}(\{0,1\}_\mathrm{B})} E^e$; we aim at showing

$$\vdash_{\mathrm{FL_{ew}}} \Big(\bigvee_{e \in Val^{\{p_1,\ldots,p_n\}}(\{0,1\}_\mathrm{B})} E^e \Big) \to \bigvee_{j=1}^{m} D_j \,. \tag{1}$$

As φ is a classical tautology, for each Boolean evaluation e we reason as follows. There is a $j_e \in \{1, \ldots, m\}$ such that $e(D_{j_e}) = 1^{\{0,1\}_\mathrm{B}}$; this implies that the literals in D_{j_e} are among the literals in E^e, and weakening gives the sequent $E^e \Rightarrow D_{j_e}$ in $\mathrm{FL_{ew}}$. Now, for each Boolean evaluation e, if the last sequent is provable, then (introducing $\vee$ to the right) so is $E^e \Rightarrow \varphi$. Finally (introducing $\vee$ to the left repeatedly), we get $\vdash_{\mathrm{FL_{ew}}} \bigvee_{e \in Val^{\{x_1,\ldots,x_n\}}(\{0,1\}_\mathrm{B})} E^e \Rightarrow \bigvee_{j=1}^m D_j$, whence (1) follows.

On the other hand, if φ is not a classical tautology, there is a Boolean evaluation e such that $e(\varphi) = 0^{\{0,1\}_\mathrm{B}}$. For each $\boldsymbol{A} \in \mathbb{K}$, define $e_{\boldsymbol{A}}$ in such a way that $e_{\boldsymbol{A}}(x_i) = e(x_i)$ for $1 \leq i \leq n$. Then $e_{\boldsymbol{A}}((x_1 \vee \neg x_1) \mathbin{\&} \ldots \mathbin{\&} (x_n \vee \neg x_n)) = 1^{\boldsymbol{A}}$ (because $e_{\boldsymbol{A}}$ only takes classical values on x_i's), so $e_{\boldsymbol{A}}(\varphi^\star) = 0^{\boldsymbol{A}}$. Hence, $\varphi^\star \notin \mathrm{TAUT_{(pos)}}(\mathbb{K})$.

(ii) We show $\mathrm{SAT^{CNF}}(\{0,1\}_\mathrm{B}) \preceq_\mathbf{P} \mathrm{SAT_{(pos)}}(\mathbb{K})$. If $\varphi(x_1, \ldots, x_n)$ is a formula of classical propositional logic in CNF, define φ° as

$$(x_1 \vee \neg x_1) \mathbin{\&} \ldots \mathbin{\&} (x_n \vee \neg x_n) \mathbin{\&} \varphi(x_1, \ldots, x_n).$$

We claim $\varphi \in \mathrm{SAT^{CNF}}(\{0,1\}_\mathrm{B})$ iff $\varphi^\circ \in \mathrm{SAT_{(pos)}}(\mathbb{K})$.

If φ is classically satisfiable, there is a Boolean evaluation e such that $e(\varphi) = 1^{\{0,1\}_\mathrm{B}}$. For each $\boldsymbol{A} \in \mathbb{K}$, define $e_{\boldsymbol{A}}$ in such a way that $e_{\boldsymbol{A}}(x_i) = e(x_i)$ for $1 \leq i \leq n$. Then $e_{\boldsymbol{A}}((x_1 \vee \neg x_1) \mathbin{\&} \ldots \mathbin{\&} (x_n \vee \neg x_n)) = 1^{\boldsymbol{A}}$, and $e_{\boldsymbol{A}}(\varphi) = 1^{\boldsymbol{A}}$ by assumption. We get $\varphi^\circ \in \mathrm{SAT_{(pos)}}(\mathbb{K})$.

On the other hand, assume φ is a formula in CNF which is not classically satisfiable. Using (aux), the formula φ° is $\mathrm{FL_{ew}}$-equivalent to

$$\bigvee_{e \in Val^{\{x_1,\ldots,x_n\}}(\{0,1\}_\mathrm{B})} (E^e \mathbin{\&} \varphi). \tag{2}$$

[15] More precisely, the problem $\mathrm{TAUT^{DNF}}(\{0,1\}_\mathrm{B})$ is polynomially equivalent to its modification where no conjunction may contain identical literals.

Now φ is $C_1 \mathbin{\&} \ldots \mathbin{\&} C_m$, where each C_j, $1 \leq j \leq m$ is a disjunction of literals in variables among $x_1, \ldots, x_n$ and $m \in \mathsf{N}$. For each (restriction of) Boolean evaluation e, it is sufficient to show that $E^e \mathbin{\&} \varphi$ in (2) is not (positively) satisfiable in any nontrivial $\boldsymbol{A} \in \mathbb{K}$ (the trivial algebra is omitted from consideration for $\mathrm{SAT}(\mathbb{K})$, and no formula is positively satisfiable in a trivial algebra). For a given e, we may reason as follows. If φ is unsatisfiable, there is a $j_e \in \{1, \ldots, m\}$ such that $e(C_{j_e}) = 0^{\{0,1\}_{\mathrm{B}}}$; that is, $e(l) = 0^{\{0,1\}_{\mathrm{B}}}$ for each literal l in C_{j_e}. The formula $E^e \mathbin{\&} \varphi$ is $\mathrm{FL_{ew}}$-equivalent to

$$E^e \mathbin{\&} C_{j_e} \mathbin{\&} (C_1 \mathbin{\&} \ldots \mathbin{\&} C_{j_e - 1} \mathbin{\&} C_{j_e + 1} \mathbin{\&} \ldots \mathbin{\&} C_m). \tag{3}$$

If C_{j_e} is $l_1 \vee \ldots \vee l_q$ for some $q \geq 1$, then, using (aux) again, $E^e \mathbin{\&} C_{j_e}$ is $\mathrm{FL_{ew}}$-equivalent to $\bigvee_{1 \leq k \leq q}(E^e \mathbin{\&} l_k)$. For each $1 \leq k \leq q$, if l_k is an x_i for some i, then $e(x_i) = e(l_k) = 0^{\{0,1\}_{\mathrm{B}}}$, hence x_i^e is $\neg x_i$, and the latter occurs in E^e; so the formula $E^e \mathbin{\&} l_k$ is a conjunction of literals where both x_i and $\neg x_i$ occur. A dual argument applies when l_k is a $\neg x_i$ for some i. Recall $\varphi \mathbin{\&} \neg\varphi \leftrightarrow \overline{0}$ is a theorem of $\mathrm{FL_{ew}}$, in particular, $\varphi \mathbin{\&} \neg\varphi$ is unsatisfiable in a nontrivial $\mathrm{FL_{ew}}$-algebra. So $\bigvee_{1 \leq k \leq q}(E^e \mathbin{\&} l_k)$ is unsatisfiable in a nontrivial $\mathrm{FL_{ew}}$-algebra and hence, so is (3). Hence, φ° is not (positively) satisfiable in $\mathbb{K}$. □

Part (i) of the above theorem was proved in a stronger way in [29]: theoremhood in a consistent substructural logic is **coNP**-hard, if moreover the logic has the disjunction property, then it is **PSPACE**-hard. We will not be able to use this stronger result as semilinear logics do not have the disjunction property.[16]

4 Łukasiewicz logic

Łukasiewicz logic merits particular attention when studying fuzzy logic, and computational complexity of its propositional fragment is no exception. **NP**-completeness of satisfiability in the standard MV-algebra $[0,1]_{Ł}$ was proved in 1987 by D. Mundici; many other complexity results in propositional fuzzy logic refer to this result.

Łukasiewicz logic Ł can be viewed as the axiomatic extension of BL with axiom $\neg\neg\varphi \to \varphi$. The equivalent algebraic semantics of Ł is the variety $\mathbb{MV}$ of MV-algebras. Propositional Łukasiewicz logic is strongly complete w.r.t. MV-chains, and finitely strongly complete w.r.t. the standard algebra $[0,1]_{Ł}$ given by the Łukasiewicz t-norm; these results are due to C.C. Chang. Therefore, the section starts with investigating complexity of decision problems in the standard MV-algebra $[0,1]_{Ł}$; this happens in Subsection 4.1. Next, relying on Y. Komori's characterization of subvarieties of $\mathbb{MV}$, Subsection 4.2 addresses complexity of decision problems in these subvarieties.

Within this section we work with the language $\{\&, \to, \overline{0}\}$.

4.1 The standard MV-algebra

Our aim is to investigate complexity of the SAT, TAUT and CONS problems in the standard MV-algebra $[0,1]_{Ł}$, given by the Łukasiewicz t-norm $*_{Ł}$ and its residuum $\to_{Ł}$ on $[0,1]$. Thanks to finite strong standard completeness of Ł, the obtained results will apply also to theoremhood and provability in Ł.

[16] However, the logic $\mathrm{FL_{ew}}$, whose language we borrow and which acts as a basis for many of our considerations, does have the disjunction property and thus, theoremhood in $\mathrm{FL_{ew}}$ is **PSPACE**-hard.

LEMMA 4.1.1. *For each $x, y, z \in [0,1]$ the following hold in the reals:*

(i) $x *_{\text{Ł}} y = z$ *iff* $((x+y-1 \geq 0) \wedge (z = x+y-1)) \vee ((x+y-1 < 0) \wedge (z = 0))$;

(ii) $x \rightarrow_{\text{Ł}} y = z$ *iff* $((x \leq y) \wedge (z = 1)) \vee ((x > y) \wedge (z = 1 - x + y))$.

We make a statement about complexity of the universal fragment of $\mathrm{Th}([0,1]_{\text{Ł}})$, and obtain result for SAT, TAUT, and CONS in $[0,1]_{\text{Ł}}$ as a corollary.

THEOREM 4.1.2. $\mathrm{Th}_\forall([0,1]_{\text{Ł}})$ *is* $\mathbf{coNP}$*-complete.*

Proof. Hardness follows from Theorem 3.4.1, in view of Lemma 3.1.1. The latter also says it suffices to address $\mathrm{Th}_\exists([0,1]_{\text{Ł}})$, a problem polynomially equivalent to the complement of $\mathrm{Th}_\forall([0,1]_{\text{Ł}})$. In the rest of the proof, we show containment of $\mathrm{Th}_\exists([0,1]_{\text{Ł}})$ in $\mathbf{NP}$. We present a nondeterministic algorithm, working in time polynomial in the input size, which uses a subroutine deciding the INEQ problem. (See also discussion below.)

ALGORITHM EX-L `// accepts` $\mathrm{Th}_\exists([0,1]_{\text{Ł}})$

`input:` Φ `// existential sentence in the language of BL`

`begin`

`normalForm()` Using Lemma 3.3.5, transform Φ into a (logically equivalent) sentence in existential normal form, $\exists x_1 \dots \exists x_n(\Phi_1 \wedge \Phi_2)$. Remove the quantifier prefix, and consider the formula $\Phi_1 \wedge \Phi_2$ as a Boolean combination of equations and inequalities in $[0,1]_{\text{Ł}}$.[17]

`guessOrder()` Guess a linear ordering $\leq_0$ of the set $V = \{0, 1, x_1, \dots, x_n\}$, such that $0 \leq_0 x_i \leq_0 1$ for $1 \leq i \leq n$, and $0 <_0 1$; henceforth we take this ordering as fixed, and exploit this piece of information (without assigning exact values to the variables). Let Ψ denote the conjunction of conditions expressing the ordering $\leq_0$, i.e., Ψ is

$$\bigwedge_{\substack{x,y \in V \\ x =_0 y}} (x = y) \wedge \bigwedge_{\substack{x,y \in V \\ x <_0 y}} (x < y)$$

`checkOrder()` Check that Φ_1 is consistent with $\leq_0$. Recall that Φ_1 is a Boolean combination of equations and inequalities between pairs of variables in V. Since $\leq_0$ gives a full information about ordering of all variables in Φ_1, it is easy to perform the check; first assess the validity of the atomic conditions in Φ_1 against $\leq_0$, then compute the validity of Φ_1, using Boolean operations.

`checkInR()` Use Lemma 4.1.1 to replace each equation in Φ_2 of type $x *_{\text{Ł}} y = z$ or $x \rightarrow_{\text{Ł}} y = z$ with an equivalent condition in the language of linear inequalities in R. Other equations in Φ_2 (i.e., those of the form $x = c$ for c a constant) remain intact. Let Φ_2' denote the conjunction of thus obtained equations. Pass $\Phi_2' \wedge \Psi$ to a subroutine deciding the INEQ problem.[18]

`end`

[17] The sentence $\exists x_1 \dots \exists x_n(\Phi_1 \wedge \Phi_2)$ is valid in $[0,1]_{\text{Ł}}$ iff there is an n-tuple $a_1, \dots, a_n \in [0,1]$ that satisfies the Boolean combination of equations and inequalities given by Φ_1 and Φ_2 in $[0,1]_{\text{Ł}}$.

[18] A bit more work is needed to rewrite the atomic conditions in $\Phi_2' \wedge \Psi$ as basic inequalities; we leave this as an exercise to the reader.

We claim the algorithm operates in time polynomial in $|\Phi|$, relying in case of the first step on Lemma 3.3.5, for the next two steps the claim is obvious. If the step `checkInR()` is reached, the subroutine for INEQ is called, which (as explained in Subsection 2.5) operates in polynomial time.

Correctness of the algorithm is argued as follows. By Lemma 3.3.5, the formula $\exists x_1 \ldots \exists x_n(\Phi_1 \wedge \Phi_2)$, the existential normal form of Φ, is logically equivalent to Φ itself. Assume the existential normal form is true in $[0,1]_Ł$. Then there are values $a_1, \ldots, a_n \in [0,1]$ such that both Φ_1 and Φ_2 are satisfied under any evaluation e such that $e(x_i) = a_i$. Then one of the computations that guess such an ordering of the x_i's that mirrors the actual ordering of the a_i's within $[0,1]$ will be an accepting computation:[19] Φ_1 will be consistent with the guessed ordering, and by Lemma 4.1.1 and a correctness argument for the algorithm for INEQ (knowing that the conditions in Φ_2' and in Ψ are satisfied by $a_1, \ldots, a_n$), we may conclude that the computation will terminate in an accepting state. Conversely, it is clear that any solution found by the algorithm yields a satisfying evaluation of $\Phi_1 \wedge \Phi_2$. $\square$

Discussion. One can simplify the above algorithm to actually show $\mathrm{Th}_\exists([0,1]_Ł) \preceq_\mathbf{P}$ INEQ (also obtaining $\mathbf{NP}$-containment for $\mathrm{Th}_\exists([0,1]_Ł)$). This is done as follows: take the existential sentence Φ; transform it into an existential normal form; remove quantifiers, add boundary conditions; replace equations involving $*_Ł$, $\rightarrow_Ł$ with their equivalents in the language of linear inequalities; pass to INEQ algorithm. We have preferred the version given in the proof because working with an explicit ordering enables us to incorporate, later on, some additional connectives, such as $\triangle$, whose semantics is order-determined.

On the other hand, one could do without the subroutine for the INEQ problem and use a subroutine for the LP problem instead, involving more nondeterminism. We refrain from going into detail, but refer the reader to the proof of Theorem 4.2.5, where this modified version of the algorithm is used for Komori algebras, relying on a subroutine deciding the ILP problem. The algorithm for Komori algebras relies on the discrete order of integers; here, one would have to feed the formula Ψ—with its strict inequalities—to the LP algorithm, which can be done taking a new variable ϵ and replacing each strict inequality $x <_0 y$ in Ψ with $x + \epsilon \leq y$, finally adding $\epsilon > 0$; this yields a formula Ψ' conforming to the criteria on a LP problem.

COROLLARY 4.1.3.

(i) $\mathrm{SAT}_{(\mathrm{pos})}([0,1]_Ł)$ *is* $\mathbf{NP}$*-complete.*

(ii) $\mathrm{TAUT}_{(\mathrm{pos})}([0,1]_Ł)$ *and* $\mathrm{CONS}([0,1]_Ł)$ *are* $\mathbf{coNP}$*-complete.*

(iii) $\mathrm{THM}(Ł)$ *and* $\mathrm{CONS}(Ł)$ *are* $\mathbf{coNP}$*-complete.*

Proof. (i) Hardness stated in Theorem 3.4.1; containment follows from $\mathbf{NP}$-containment of $\mathrm{Th}_\exists([0,1]_Ł)$ using Lemma 3.1.1.
(ii) Analogous to (i), with respect to $\mathrm{Th}_\forall([0,1]_Ł)$.
(iii) Using (finite strong) standard completeness result for Ł. $\square$

[19] One cannot say '*the* computation that guesses the ordering' because another nondeterministic step is still ahead in the INEQ subroutine.

It still remains to show that the decision problems are nontrivial in the sense of being distinct from each other and from the classical case.

LEMMA 4.1.4.

(i) $\mathrm{SAT}(\{0,1\}_{\mathrm{B}}) \subsetneq \mathrm{SAT}([0,1]_{Ł}) \subsetneq \mathrm{SAT}_{\mathrm{pos}}([0,1]_{Ł})$;

(ii) $\mathrm{TAUT}([0,1]_{Ł}) \subsetneq \mathrm{TAUT}_{\mathrm{pos}}([0,1]_{Ł} \subsetneq \mathrm{TAUT}(\{0,1\}_{\mathrm{B}})$.

Proof. For a variable p:

$$p \wedge \neg p \in \mathrm{SAT}_{\mathrm{pos}}([0,1]_{Ł}) \setminus \mathrm{SAT}([0,1]_{Ł})$$
$$p \leftrightarrow \neg p \in \mathrm{SAT}([0,1]_{Ł}) \setminus \mathrm{SAT}(\{0,1\}_{\mathrm{B}})$$
$$p \vee \neg p \in \mathrm{TAUT}_{\mathrm{pos}}([0,1]_{Ł}) \setminus \mathrm{TAUT}([0,1]_{Ł})$$
$$(p \vee \neg p) \,\&\, (p \vee \neg p) \in \mathrm{TAUT}(\{0,1\}_{\mathrm{B}}) \setminus \mathrm{TAUT}_{\mathrm{pos}}([0,1]_{Ł}).$$ □

4.2 Axiomatic extensions of Łukasiewicz logic

A result of Y. Komori characterizing subvarieties of $\mathbb{MV}$ makes it possible to extend the complexity results obtained for Łukasiewicz logic also to its axiomatic extensions. For L a consistent axiomatic extension of Łukasiewicz logic, let $\mathbb{MV}^{\mathrm{L}}$ be the subvariety of $\mathbb{MV}$ forming its equivalent algebraic semantics.

We start by introducing notation for finite MV-chains and Komori chains. Denote $\mathsf{N}_0 = \mathsf{N} \setminus \{0\}$ and $\mathsf{N}_1 = \mathsf{N} \setminus \{0,1\}$.

DEFINITION 4.2.1. *For $n \in \mathsf{N}_0$, denote:*

(i) $\mathbf{Ł}_{n+1}$ *the subalgebra of* $[0,1]_{Ł}$ *with the domain* $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$;

(ii) $\boldsymbol{K}_{n+1}$ *the algebra*

$$\langle \{\langle i,a\rangle \in \mathsf{Z} \times \mathsf{Z} \mid \langle 0,0\rangle \leq_{\mathrm{lex}} \langle i,a\rangle \leq_{\mathrm{lex}} \langle n,0\rangle\}, *_{\boldsymbol{K}_{n+1}}, \to_{\boldsymbol{K}_{n+1}}, \langle 0,0\rangle\rangle,$$

where $\leq_{\mathrm{lex}}$ *is the lexicographic order on* $\mathsf{Z} \times \mathsf{Z}$ *and* $*_{\boldsymbol{K}_{n+1}}$, $\to_{\boldsymbol{K}_{n+1}}$ *are given by* $\langle i,x\rangle *_{\boldsymbol{K}_{n+1}} \langle j,y\rangle = \max_{\mathrm{lex}}(\langle 0,0\rangle, \langle i+j-n, x+y\rangle)$ *and* $\langle i,x\rangle \to_{\boldsymbol{K}_{n+1}} \langle j,y\rangle = \min_{\mathrm{lex}}(\langle n,0\rangle, \langle n-i+j, y-x\rangle)$.

Observe that for each $n \in \mathsf{N}_0$, the algebra $\mathbf{Ł}_{n+1}$ is isomorphic to a subalgebra of $\boldsymbol{K}_{n+1}$ (obtained by considering only elements $\langle x,0\rangle$).

DEFINITION 4.2.2. *For $A, B \subseteq \mathsf{N}_1$, denote*

(i) $\mathbb{K}_A = \{\boldsymbol{K}_a \mid a \in A\}$;

(ii) $\mathbb{L}_B = \{\boldsymbol{L}_b \mid b \in B\}$.

PROPOSITION 4.2.3 ([10, 19, 31]). *For* L *a consistent axiomatic extension of* Ł:

(i) $\mathbb{MV}^{\mathrm{L}} = \mathbf{V}(\mathbb{K}_A \cup \mathbb{L}_B)$ *for some* $A, B \subseteq \mathsf{N}_1$;

(ii) if (i) is true for L, *A, and B, then also* $\mathbb{MV}^{\mathrm{L}} = \mathbf{Q}(\mathbb{K}_A \cup \mathbb{L}_B)$;

(iii) $\mathbf{V}(\mathbb{K}_A \cup \mathbb{L}_B) = \mathbb{MV}$ *iff either of A, B is infinite*;

(iv) L *is finitely axiomatizable.*

These results are crucial for us: for L a consistent axiomatic extension of Ł, if $\mathbb{MV}^{\mathrm{L}}$ is a proper subvariety of $\mathbb{MV}$, it is generated by a pair of finite lists of algebras of known structure; it is generated by these algebras as a quasivariety; and, using strong completeness theorem for axiomatic extensions of Ł (or, of BL), results on complexity of TAUT and CONS for $\mathbb{MV}^{\mathrm{L}}$ apply also to THM and CONS in L. The fact that, for given $\mathbb{MV}^{\mathrm{L}}$, the pair of lists need not be unique is of no material importance here.

Now we are interested in complexity of SAT, TAUT, and CONS problems in the algebras defined above, $\boldsymbol{Ł}_{n+1}$ and $\boldsymbol{K}_{n+1}$. We work with the algebra $\boldsymbol{K}_{n+1}$ for a fixed $n \in \mathsf{N}_0$. For $\boldsymbol{Ł}_{n+1}$, the upper bound is obvious as the algebra is finite (one can also argue that $\boldsymbol{Ł}_{n+1}$ is a subalgebra of $\boldsymbol{K}_{n+1}$, as above).

LEMMA 4.2.4. *Let* $n \in \mathsf{N}_1$. *Let* $\langle i_1, x_1\rangle, \langle i_2, x_2\rangle, \langle i_3, x_3\rangle \in [\langle 0,0\rangle, \langle n,0\rangle]_{\mathrm{lex}}$ *in* $\mathsf{Z} \times \mathsf{Z}$. *Then the following holds in the integers:*

(i) $\langle i_1, x_1\rangle *_{\boldsymbol{K}_{n+1}} \langle i_2, x_2\rangle = \langle i_3, x_3\rangle$ *iff either*

$i_1 + i_2 - n \leq -1$ *and* $i_3 = 0$ *and* $x_3 = 0$*, or*
$i_1 + i_2 - n = 0$ *and* $x_1 + x_2 \leq -1$ *and* $i_3 = 0$ *and* $x_3 = 0$*, or*
$i_1 + i_2 - n = 0$ *and* $x_1 + x_2 \geq 0$ *and* $i_3 = 0$ *and* $x_3 = x_1 + x_2$*, or*
$i_1 + i_2 - n \geq 1$ *and* $i_3 = i_1 + i_2 - n$ *and* $x_3 = x_1 + x_2$.

(ii) $\langle i_1, x_1\rangle \to_{\boldsymbol{K}_{n+1}} \langle i_2, x_2\rangle = \langle i_3, x_3\rangle$ *iff either*

$i_1 < i_2$ *and* $i_3 = n$ *and* $x_3 = 0$*, or*
$i_1 = i_2$ *and* $x_1 \leq x_2$ *and* $i_3 = n$ *and* $x_3 = 0$*, or*
$i_1 = i_2$ *and* $x_1 \geq x_2 + 1$ *and* $i_3 = n$ *and* $x_3 = x_2 - x_1$*, or*
$i_1 \geq i_2 + 1$ *and* $i_3 = n - i_1 + i_2$ *and* $x_3 = x_2 - x_1$.

THEOREM 4.2.5. $\mathrm{Th}_\forall(\boldsymbol{K}_{n+1})$ *is* $\mathbf{coNP}$*-complete for each* $n \in \mathsf{N}_0$.

Proof. Hardness follows from Theorem 3.4.1. We argue $\mathbf{NP}$-containment for existential sentences valid in $\boldsymbol{K}_{n+1}$ for an arbitrary but fixed $n \in \mathsf{N}_0$.

ALGORITHM EX-K `// accepts` $\mathrm{Th}_\exists(\boldsymbol{K}_{n+1})$

`input:` Φ `// existential sentence in the language of BL`

`begin`

`normalForm()` Using Lemma 3.3.5, transform Φ into an existential normal form $\exists x_1 \ldots \exists x_n(\Phi_1 \wedge \Phi_2)$. Remove quantifiers. In the open formula $\Phi_1 \wedge \Phi_2$, for $j = 1, \ldots, n$ replace each occurrence of the variable x_j with a pair $\langle i_j, z_j\rangle$ where for $j = 1, \ldots, n$, the pair i_j, z_j are new variables. Consider $\Phi_1 \wedge \Phi_2$ as a Boolean combination of equations and inequalities in $\boldsymbol{K}_{n+1}$.

`guessOrder()` Guess a linear ordering $\leq_0$ of the set $\{\langle i_j, z_j\rangle\}_{j=1}^n \cup \{\langle 0,0\rangle, \langle n,0\rangle\}$, in such a way that $\langle 0,0\rangle \leq_0 \langle i_j, z_j\rangle \leq_0 \langle n,0\rangle$ for $1 \leq j \leq n$, and $\langle 0,0\rangle <_0 \langle n,0\rangle$. Denote Ψ the set of conditions expressing the ordering $\leq_0$.

`checkOrder()` As in the proof of Theorem 4.1.2.

`checkInZ()` Rewrite the conditions in Ψ and in Φ_2 into the language of the o-group of Z and check their solvability, in the following (nondeterministic) manner. Define a new empty system $\mathcal{S}$. Processing the conditions in Ψ one by one, in case of:

– $\langle i_j, z_j\rangle = \langle i_k, z_k\rangle$ in Ψ, put $i_j = i_k$, $x_j = x_k$ into $\mathcal{S}$;
– $\langle i_j, z_j\rangle < \langle i_k, x_k\rangle$ in Ψ, put either $i_j = i_k$ and $z_j + 1 \leq z_k$, or $i_j + 1 \leq i_k$, into $\mathcal{S}$.

Note that this process imposes boundary appropriate conditions on the pairs of variables. Then process the conditions in Φ_2 one by one, for each equation $x *_{\boldsymbol{K}_{n+1}} y = z$, choose exactly one of the four mutually exclusive cases from Lemma 4.2.4 (i), and add it into $\mathcal{S}$; analogously for each atomic formula of type $x \rightarrow_{\boldsymbol{K}_{n+1}} y = z$, again using a case from among its equivalents posed by Lemma 4.2.4 (ii). Then process each atomic formula of type $\langle i_j, x_j\rangle = \langle 0, 0\rangle$ as above. Finally, pass $\mathcal{S}$ to an algorithm for the ILP problem.
`end` $\square$

THEOREM 4.2.6. *Let* L *be a consistent axiomatic extension of Łukasiewicz logic. Then* $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{MV}^{\mathrm{L}})$ *is* $\mathbf{NP}$*-complete, whereas* $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{MV}^{\mathrm{L}})$ *and* $\mathrm{CONS}(\mathbb{MV}^{\mathrm{L}})$ *are* $\mathbf{coNP}$*-complete.*

Proof. Hardness by Theorem 3.4.1. Consider L, $\mathbb{MV}^{\mathrm{L}}$ as stated. If L is Ł, then results in the previous subsection apply. Otherwise, using Theorem 4.2.3 (i), we have $\mathbb{MV}^{\mathrm{L}} = \mathbf{V}(\mathbb{K}_A \cup \mathbb{L}_B)$ for some finite $A, B \subseteq \mathsf{N}_1$. Fix such a pair A and B. Recall $\mathbf{V}(\mathbb{K}_A \cup \mathbb{L}_B) = \mathbf{Q}(\mathbb{K}_A \cup \mathbb{L}_B)$ (cf. Theorem 4.2.3 (ii)). Then by Theorem 3.1.6: $\mathrm{SAT}_{(\mathrm{pos})}(\mathbf{V}(\mathbb{K}_A \cup \mathbb{L}_B)) = \mathrm{SAT}_{(\mathrm{pos})}(\mathbf{Q}(\mathbb{K}_A \cup \mathbb{L}_B)) = \mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}_A \cup \mathbb{L}_B) = \bigcup_{a\in A}\mathrm{SAT}_{(\mathrm{pos})}(\boldsymbol{K}_a) \cup \bigcup_{b\in B}\mathrm{SAT}_{(\mathrm{pos})}(\boldsymbol{L}_b)$ (last equality by definition of $\mathrm{SAT}_{(\mathrm{pos})}$). Since $\mathbf{NP}$ is closed under finite unions, we may conclude, on the basis of A, B being finite, Theorem 4.2.5 and the remark about finite MV-chains, that $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{MV}^{\mathrm{L}})$ is in $\mathbf{NP}$. Hence, $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{MV}^{\mathrm{L}})$ is in $\mathbf{coNP}$ by Corollary 3.2.2. By an analogous argument we get that $\overline{\mathrm{CONS}}(\mathbb{MV}^{\mathrm{L}})$ is in $\mathbf{NP}$, and hence $\mathrm{CONS}(\mathbb{MV}^{\mathrm{L}})$ is in $\mathbf{coNP}$. $\square$

Using strong completeness for extensions $\mathrm{L} \supseteq$ Ł with respect to the corresponding varieties $\mathbb{MV}^{\mathrm{L}}$, we may conclude

COROLLARY 4.2.7. *Let* L *be a consistent axiomatic extension of Łukasiewicz logic. Then* $\mathrm{THM}(\mathrm{L})$ *and* $\mathrm{CONS}(\mathrm{L})$ *are* $\mathbf{coNP}$*-complete.*

5 Logics of standard BL-algebras

This section studies decision problems in propositional BL and extensions given by continuous t-norms (distinct from Łukasiewicz). We start with Gödel and product logics, proceed to BL and SBL, and then we discuss logics given by single standard BL-algebras (and thus, by single continuous t-norms; hence the section title). Within this section, we work with the language $\{\&, \rightarrow, \overline{0}\}$.

5.1 Gödel and product logics

Gödel logic G can be obtained as an axiomatic extension of BL with the axiom $\varphi \rightarrow \varphi \,\&\, \varphi$. It is discussed in detail in Chapter VII. The logic is strongly complete w.r.t. its standard algebra $[0,1]_{\mathrm{G}}$, given by the continuous t-norm $x *_{\mathrm{G}} y = \min\{x, y\}$ and its residuum $x \rightarrow_{\mathrm{G}} y = y$ for $x > y$.

Product logic extends BL with the axiom $(\varphi \to \chi) \vee ((\varphi \to (\varphi \,\&\, \psi)) \to \psi)$. Product logic is finitely strongly complete w.r.t. its standard algebra $[0,1]_\Pi$, given by the continuous t-norm $x *_\Pi y = xy$; the residuum is $x \to_\Pi y = \frac{y}{x}$ for $x > y$.

Both logics extend SMTL (and SBL), therefore Theorem 3.2.4 applies for the SAT, $\mathrm{SAT_{pos}}$ and $\mathrm{TAUT_{pos}}$ operators on the respective standard algebra; it is therefore sufficient to address the set of tautologies and the consequence relation. Moreover, recall that in Gödel logic, we have the classical deduction theorem, i.e., for $\psi_1, \ldots, \psi_n, \varphi$ formulas, we have $\{\psi_1, \ldots, \psi_n\} \vdash_{\mathrm{G}} \varphi$ iff $\vdash_{\mathrm{G}} \psi_1 \,\&\, \ldots \,\&\, \psi_n \to \varphi$. In view of Lemma 3.1.1, it is therefore sufficient to investigate theoremhood for Gödel logic.

THEOREM 5.1.1. *Let $\boldsymbol{L}$ be a Gödel chain. Then $\mathrm{Th}_\forall(\boldsymbol{L})$ is* $\mathbf{coNP}$*-complete.*

Proof. Hardness follows from Theorem 3.4.1. We show $\mathbf{NP}$-containment of $\mathrm{Th}_\exists(\boldsymbol{L})$. First, in a Gödel chain $\boldsymbol{L} = \langle *, \to, \wedge, \vee, 0, 1\rangle$, we have $x * y = x \wedge y = \min\{x, y\}$ and $x \to y = 1$ iff $x \leq y$, otherwise $x \to y = y$. Hence for any term $t(x_1, \ldots, x_n)$ and for any evaluation e in $\boldsymbol{L}$, the value $e(t')$ for any subterm $t' \preceq t$ will be among $V = \{0^{\boldsymbol{L}}, e(x_1), \ldots, e(x_n), 1^{\boldsymbol{L}}\}$, and moreover, operations are order-determined, i.e., the value $e(t')$ is fully determined by the ordering of V.

So, determining the validity of an existential sentence $\exists x_1 \ldots \exists x_n \Phi(x_1, \ldots, x_n)$, one can replace the existential quantification over all n-tuples of values in $[0,1]$ by an existential quantification over all such orderings of variables occurring in Φ that are possible in $\boldsymbol{L}$, with respect to its bottom and top elements.

We describe a nondeterministic ALGORITHM EX-G which accepts existential sentences valid in $\boldsymbol{L}$. A sentence $\exists x_1 \ldots \exists x_n \Phi(x_1, \ldots, x_n)$ is given, where Φ is a Boolean combination of identities. Guess a linear ordering $\leq_0$ of the set $V = \{0, x_1, \ldots, x_n, 1\}$, such that $0 \leq_0 x_i \leq_0 1$ for $1 \leq i \leq n$, and in such a way that the length of any strictly increasing $\leq_0$-chain does not exceed the cardinality of $\boldsymbol{L}$; this information is polynomial in the input size. Then compute the Boolean value of each identity in Φ: on the basis of $\leq_0$, evaluate all terms, then determine whether they are $=_0$-equal. Then compute the Boolean value of Φ, accept iff this value is 1.

It is easy to see that the algorithm runs in time polynomial in the input size. It is equally easy to see that it accepts just the set $\mathrm{Th}_\exists(\boldsymbol{L})$. □

COROLLARY 5.1.2.

(i) $\mathrm{TAUT}([0,1]_{\mathrm{G}})$ *is* $\mathbf{coNP}$*-complete.*

(ii) $\mathrm{THM}(\mathrm{G})$ *is* $\mathbf{coNP}$*-complete.*

Now we address product logic, via its standard algebra. We start with a lemma.

LEMMA 5.1.3. *For each $c \in (0,1)$, the standard* MV*-algebra $[0,1]_{Ł}$ is isomorphic to the* cut product algebra *$\langle [c,1], *_c, \to_c, c, 1\rangle$ where*

$$x *_c y = \max\{c, x *_\Pi y\} \qquad\qquad x \to_c y = x \to_\Pi y$$

The element c is called the cut.

Proof. For a fixed c, the isomorphism is given by $f(x) = c^{1-x}$, its inverse by $f^{-1}(y) = 1 - \log_c(y)$. □

THEOREM 5.1.4. $\mathrm{Th}_\forall([0,1]_\Pi)$ *is* $\mathbf{coNP}$*-complete.*

Proof. For hardness, see Theorem 3.4.1. We show $\mathbf{NP}$-containment of $\mathrm{Th}_\exists([0,1]_\Pi)$ by presenting a nondeterministic algorithm, operating on existential sentences, accepting $\mathrm{Th}_\exists([0,1]_\Pi)$, and working in time polynomial in the input formula size. The algorithm uses a subroutine deciding the existential theory of $[0,1]_Ł$.

ALGORITHM EX-PRODUCT `// accepts` $\mathrm{Th}_\exists([0,1]_\Pi)$

`input:` Φ `// existential sentence in the language of BL`

`begin`

`normalForm()` As in the proof of Theorem 4.1.2.

`guessOrder()` As in the proof of Theorem 4.1.2.

`checkOrder()` As in the proof of Theorem 4.1.2.

`eliminateZero()` Partition the equalities in Φ_2 into two classes: let Φ_2^0 contain those equalities in Φ_2 which contain at least one $=_0$-equal variable, let $\Phi_2^{>0}$ contain the remaining equalities. Then check that all equalities in Φ_2^0 are consistent with $\leq_0$, as follows. Processing them one by one, in case of:

– an equality $x = 0$, check $x =_0 0$;
– an equality $x *_\Pi y = z$, if $x =_0 0$, check $z =_0 0$, analogously for y; if $z =_0 0$, check $x =_0 0$ or $y =_0 0$;
– an equality $x \to_\Pi y = z$, if $x =_0 0$, check $z =_0 1$; if $y =_0 0$, check that either $z =_0 0$ and $x \neq_0 0$, or $x =_0 0$ and $z =_0 1$; if $z =_0 0$, check $x \neq_0 0$ and $y =_0 0$.

In the checking process, we have made sure that all atomic formulas in Φ_2^0 are valid under $\leq_0$. Finally, omit from Ψ the variables that are $=_0$-equal to 0, obtaining a $\Psi^{>0}$.

`positiveL()` Test whether the conditions in $\Phi_2^{>0}$ and the conditions in $\Psi^{>0}$ are satisfiable by positive values in $[0,1]_\Pi$; by Lemma 5.1.3, this is iff they are satisfiable by positive values in $[0,1]_Ł$. Use the subroutine `checkInR()` in the proof of Theorem 4.1.2 to test the positive satisfiability of $\Phi_2^{>0} \wedge \Psi^{>0}$ in $[0,1]_Ł$. It follows from that proof that the subroutine works in polynomial time.

`end` □

COROLLARY 5.1.5.

(i) $\mathrm{TAUT}([0,1]_\Pi)$ *and* $\mathrm{CONS}([0,1]_\Pi)$ *are* $\mathbf{coNP}$*-complete.*

(ii) $\mathrm{THM}(\Pi)$ *and* $\mathrm{CONS}(\Pi)$ *are* $\mathbf{coNP}$*-complete.*

5.2 BL and SBL

In this subsection we address theorems and provability in the logic BL and in its axiomatic extension SBL with the axiom $\neg(\varphi \wedge \neg\varphi)$. We denote $\mathbb{BL}^{\mathrm{st}}$ and $\mathbb{SBL}^{\mathrm{st}}$ the classes of all standard BL-algebras and all standard SBL-algebras, respectively. We work with fragments of the respective algebraic theories, relying on finite strong standard completeness results: propositional BL (SBL) is finitely strongly complete with respect to the class $\mathbb{BL}^{\mathrm{st}}$ ($\mathbb{SBL}^{\mathrm{st}}$ respectively), hence, $\mathrm{THM(BL)} = \mathrm{TAUT}(\mathbb{BL}^{\mathrm{st}})$ and $\mathrm{CONS(BL)} = \mathrm{CONS}(\mathbb{BL}^{\mathrm{st}})$, and analogously for SBL.

Apart from that, both BL and SBL are also complete w.r.t. some particular standard BL-algebras. Using the partial embedding technique, it is not difficult to see that any standard BL-algebra which is an ordinal sum with a first component that is an Ł-component and with infinitely many Ł-components generates the full variety $\mathbb{BL}$; the converse also holds.

PROPOSITION 5.2.1 ([1]). *Let $\boldsymbol{A} = \bigoplus_{i\in I} \boldsymbol{A}_i$ be a standard BL-algebra. Then $\mathbf{V}(\boldsymbol{A}) = \mathbb{BL}$ iff there is a first component $\boldsymbol{A}_{i_0}$, which is an Ł-component, and for infinitely many $i \in I$, $\boldsymbol{A}_i$ is an Ł-component. If that is the case, then also $\mathbf{Q}(\boldsymbol{A}) = \mathbb{BL}$. Hence,* $\mathrm{THM}(\mathrm{BL}) = \mathrm{TAUT}(\boldsymbol{A})$ *and* $\mathrm{CONS}(\mathrm{BL}) = \mathrm{CONS}(\boldsymbol{A})$.

Analogous results hold for SBL and each standard BL-algebra which is an ordinal sum with infinitely many Ł-components, *not* starting with an Ł-component.

PROPOSITION 5.2.2. *Let $\boldsymbol{A} = \bigoplus_{i\in I} \boldsymbol{A}_i$ be a standard BL-algebra. Then $\mathbf{V}(\boldsymbol{A}) = \mathbb{SBL}$ iff for infinitely many $i \in I$, $\boldsymbol{A}_i$ is an Ł-component, and either there is a first component $\boldsymbol{A}_{i_0}$, which is* not *an Ł-component, or there is no first component. If that is the case, then also $\mathbf{Q}(\boldsymbol{A}) = \mathbb{SBL}$. Hence,* $\mathrm{THM}(\mathrm{SBL}) = \mathrm{TAUT}(\boldsymbol{A})$ *and* $\mathrm{CONS}(\mathrm{SBL}) = \mathrm{CONS}(\boldsymbol{A})$.

These facts will be used later when dealing with complexity of individual standard BL-algebras: the two above types of sums will not be considered, as they generate $\mathbb{BL}$ or $\mathbb{SBL}$ and hence the results for BL and SBL (obtained in this subsection) apply.

Let us now investigate TAUT and CONS in $\mathbb{BL}^{\mathrm{st}}$ and $\mathbb{SBL}^{\mathrm{st}}$. The results for SAT, $\mathrm{SAT}_{\mathrm{pos}}$, and $\mathrm{TAUT}_{\mathrm{pos}}$ for the class $\mathbb{BL}^{\mathrm{st}}$ will be obtained as Corollary 5.3.3; for $\mathbb{SBL}^{\mathrm{st}}$, use Theorem 3.2.4. For TAUT and CONS, we rely on the following lemma.

LEMMA 5.2.3. *Let $\Phi(x_1, \ldots, x_k)$ be an open formula in the language of* BL. *Then $\mathbb{BL}^{\mathrm{st}} \models \Phi$ iff $\boldsymbol{A} \models \Phi$ for each standard BL-algebra $\boldsymbol{A}$ with at most $k+1$ components.*

Proof. The left-to-right implication holds by definition. For the converse one, we give a partial embedding argument. If $\mathbb{BL}^{\mathrm{st}} \not\models \Phi$, there is a standard BL-algebra $\boldsymbol{A}$ and an evaluation $e_{\boldsymbol{A}}$ such that $\boldsymbol{A} \not\models \Phi[e_{\boldsymbol{A}}]$. Write $a_j = e_{\boldsymbol{A}}(x_j)$ for $1 \leq j \leq k$. Possibly re-enumerating, w.l.o.g. assume $a_1 \leq a_2 \leq \cdots \leq a_k$. For $1 \leq j \leq k$, let $\boldsymbol{A}_j$ be the component of $\boldsymbol{A}$ s.t. $a_j \in \boldsymbol{A}_j$; if $a_j \in \boldsymbol{A}_{i_1} \cap \boldsymbol{A}_{i_2}$ for $i_1 < i_2$, let $j = i_2$. It follows from Theorem 2.1.6 that $\{0^{\boldsymbol{A}}\} \cup \bigcup_{1\leq j\leq k} \boldsymbol{A}_j \cup \{1^{\boldsymbol{A}}\}$ is a BL-subchain of $\boldsymbol{A}$, so for any term t occurring in Φ, we have $e_{\boldsymbol{A}}(t) \in \{0^{\boldsymbol{A}}\} \cup \bigcup_{1\leq j\leq k} \boldsymbol{A}_j \cup \{1^{\boldsymbol{A}}\}$. Let $\boldsymbol{A}_0$ be the first component of the ordinal sum $\boldsymbol{A}$ if there is one, if not, then let $\boldsymbol{A}_0$ be any component. If $\boldsymbol{A}_j$ is a trivial component for $1 \leq j \leq k$, replace it with a G-component. Define $\boldsymbol{A}' = \bigoplus_{j\leq k} \boldsymbol{A}_j$. If $e(x_j) = a_j$ in $\boldsymbol{A}'$, then $\boldsymbol{A}' \not\models \Phi[e]$. The BL-chain $\boldsymbol{A}'$ is isomorphic to a standard BL-algebra $\boldsymbol{B}$ with at most $k+1$ components via some f, and $\{0^{\boldsymbol{A}}\} \cup \bigcup_{1\leq j\leq k} \boldsymbol{A}_j \cup \{1^{\boldsymbol{A}}\}$ is isomorphic to a subchain of $\boldsymbol{B}$. Define $e_{\boldsymbol{B}}$ in $\boldsymbol{B}$ s.t. $e_{\boldsymbol{B}}(x_j) = f^{-1}(a_j)$ for $x_1, \ldots, x_k$; then $\boldsymbol{B} \not\models \Phi[e_{\boldsymbol{B}}]$, hence $\boldsymbol{B} \not\models \Phi$. $\square$

Combining Lemma 5.1.3 with the above proof, one can replace each copy of $[0,1]_\Pi$ with two copies of $[0,1]_{\text{Ł}}$ (and it is easy to replace copies of $[0,1]_{\mathrm{G}}$ by a suitable number of $[0,1]_{\text{Ł}}$ copies as well). Then the partial embedding gives us the easy implication from

Proposition 5.2.1. For SBL, reasoning in full analogy (with the proviso that the first component—if any—is not Ł), we get $\mathbb{SBL}^{\mathrm{st}} \models \Phi$ iff $\boldsymbol{A} \models \Phi$ for each standard SBL-algebra $\boldsymbol{A}$ with at most $k+1$ components.

THEOREM 5.2.4. $\mathrm{Th}_\forall(\mathbb{BL}^{\mathrm{st}})$ *and* $\mathrm{Th}_\forall(\mathbb{SBL}^{\mathrm{st}})$ *are* $\mathbf{coNP}$*-complete.*

Proof. Hardness follows from Theorem 3.4.1; we address $\mathbf{coNP}$-containment for the case of BL, with comments on modifications for the SBL case.

We work with the complement of $\mathrm{Th}_\forall(\mathbb{BL}^{\mathrm{st}})$. The complement consists of universal sentences Φ that do not hold in at least one standard BL-algebra; by Lemma 5.2.3, one can limit oneself to a class $\mathbb{C}$ of finite sums of cardinality bounded polynomially by $|\Phi|$. Equivalently, one can consider the set of existential sentences Φ that are valid in at least one standard BL-algebra in the class $\mathbb{C}$. We give a nondeterministic algorithm that accepts $\mathrm{Th}_\exists(\mathbb{BL}^{\mathrm{st}})$ (its modification accepts $\mathrm{Th}_\exists(\mathbb{SBL}^{\mathrm{st}})$) and works in time polynomial in the input size.

ALGORITHM EX-BL `//accepts` $\mathrm{Th}_\exists(\mathbb{BL}^{\mathrm{st}})$

`input:` Φ `// existential sentence in the language of BL`

`begin`

`normalForm()` Using Lemma 3.3.5, transform Φ into a (logically equivalent) sentence in existential normal form, $\exists x_1 \ldots \exists x_n(\Phi_1 \wedge \Phi_2)$. Note that n is bounded polynomially by $|\Phi|$, in particular, $n \in O(|\Phi|)$.

`guessOrdinalSum()` Guess a $k \in \mathsf{N}$, $k \leq n+1$. Guess an ordinal sum $\boldsymbol{A}$ of k components (i.e., a sequence of k symbols out of Ł, G, Π).
`// For SBL, the first component is` Π.

`componentDelimiters()` Introduce constants $\frac{1}{k}, \ldots, \frac{k-1}{k}, 1$ for the idempotent elements of **A** that delimit its components, in their real order, (0 we already have). Set $V = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\} \cup \{x_1, \ldots, x_n\}$.

`guessOrder()` Guess a linear ordering $\leq_0$ of elements of V, in such a way that $\leq_0$ preserves the strict ordering of the constants imposed by the delimiters they represent.

`For` $i < k$, we say that any variable $x \in V$ such that $\frac{i}{k} \leq_0 x \leq_0 \frac{i+1}{k}$ *belongs to* i.

`checkOrder()` Check that $\leq_0$ is consistent with Ψ_1, as in the proof of Theorem 4.1.2.

`checkExternal()` Check that $\leq_0$ is compatible with all identities in Φ_2, as far as the operations are order-determined, as follows. Consider each atomic formula in Φ_2. In case of:

- $x = 0$ for some variable x, check $x =_0 0$ ($x =_0 1$ respectively).
- $x * y = z$ for some variables x, y, z, then if, for some $l \leq k$, we have $x \leq_0 \frac{l}{k} \leq_0 y$, then check that $z =_0 x$; analogously for $y \leq_0 \frac{l}{k} \leq_0 x$. If, on the other hand, for some $l < k$, we have $\frac{l}{k} \leq x, y \leq \frac{l+1}{k}$, then check $\frac{l}{k} \leq z \leq \frac{l+1}{k}$.
- $x \to y = z$ for some variables x, y, z, then if $x \leq_0 y$, check $z =_0 1$. If, $x >_0 y$, then if, for some $l \leq k$, we have $x \geq_0 \frac{l}{k} >_0 y$, then check $z =_0 y$; if, on the other hand, for some $l < k$ we have $\frac{l}{k} \leq x, y \leq \frac{l+1}{k}$, then check $\frac{l}{k} \leq z \leq \frac{l+1}{k}$.

`checkInternal()` For each $\leq_0$-interval $[\frac{l}{k}, \frac{l+1}{k}]$, $l = 0, \ldots, k-1$, check that $\leq_0$ is compatible with all identities in Φ_2 for the variables belonging to l. Working for a fixed l, consider Φ_2 restricted to those identities where at least one variable belongs to l. Construct a system $\mathcal{S}_l$ of identities, as follows. $\mathcal{S}_l$ is initially empty. Consider each identity in Φ_2. In case of:

– $x = 0$ for some variable x, add $x = 0$ ($x = 1$ respectively) into $\mathcal{S}_l$.
– $x * y = z$ for some variables x, y, z, then if x and y belong to l, add $x * y = z$ into $\mathcal{S}_l$.
– $x * y = z$ for some variables x, y, z, then if x and y belong to l, add $x \to y = z$ into $\mathcal{S}_l$.

Further, add a conjunction of atomic formulas defining $\leq_0$ for $\frac{l}{k}$, $\frac{l+1}{k}$, and the variables in l, into $\mathcal{S}_l$; replace $\frac{l}{k}$ with 0 and $\frac{l+1}{k}$ with 1. In case the l-th component of $\boldsymbol{A}$ is:

– an Ł-component, use ALGORITHM EX-L to check satisfiability of $\mathcal{S}_l$ in $[0,1]_Ł$.
– a Π-component, use ALGORITHM EX-PRODUCT to check satisfiability of $\mathcal{S}_l$ in $[0,1]_\Pi$.
– a G-component, use the external check based on $\leq_0$.

`end` □

Because both BL and SBL enjoy finite strong standard completeness, we may conclude:

COROLLARY 5.2.5. *The sets* THM(BL), CONS(BL), THM(SBL), CONS(SBL) *are* **coNP***-complete.*

5.3 Other logics of standard BL-algebras

The term ‘logics of standard BL-algebras’ may be ambiguous. Suppose that, given a class $\mathbb{K}$ of standard BL-algebras, we do have a clear idea what is meant by a ‘logic of $\mathbb{K}$’. Then one reading of the term is that for each standard BL-algebra $\boldsymbol{A}$, we consider the logic of $\boldsymbol{A}$; another reading is that we take arbitrary nonempty classes of standard BL-algebras and for each, we consider the logic of $\mathbb{K}$. Most of this section is dedicated to discussing the former meaning; at the end, we give some remarks on the latter. All of this is going to happen in view of previously presented results that addressed particularly important choices of standard BL-algebras; here, we cater for the “remaining cases”.

PROPOSITION 5.3.1 ([14]). *Let $\boldsymbol{A}$ be a standard* BL*-algebra. Then the logic of $\boldsymbol{A}$ is an axiomatic extension of* BL *obtained by adding finitely many axioms.*

We investigate each given standard BL-algebra as to the complexity of its SAT, TAUT, and CONS problems. We start with some easy results; recall that in the ordinal-sum decomposition of a standard BL-algebra $\boldsymbol{A}$, either $\boldsymbol{A}$ has a first component Ł, or $\boldsymbol{A}$ is an SBL-algebra.

THEOREM 5.3.2. *Let $\mathbb{K}$ be a nonempty class of standard* BL*-algebras and let each $\boldsymbol{A} \in \mathbb{K}$ be of the type* Ł $\oplus$ X *for some ordinal sum X (possibly void). Then*

(i) $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}) = \mathrm{TAUT}_{\mathrm{pos}}([0,1]_Ł)$;

(ii) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) = \mathrm{SAT}_{(\mathrm{pos})}([0,1]_Ł)$.

Proof. $[0,1]_Ł$ is isomorphic to a subalgebra of each $\boldsymbol{A} \in \mathbb{K}$; use Lemma 3.1.3 to obtain the left-to-right inclusion in (i) and the right-to-left inclusion in (ii). Moreover, for each $\boldsymbol{A} \in \mathbb{K}$, a mapping f sending x to $\neg\neg x$ in $\boldsymbol{A}$ is a homomorphism of $\boldsymbol{A}$ onto (an isomorphic copy of) $[0,1]_Ł$; we have $f(x) = 0$ iff $x = 0$ and $f(x) = 1$ if $x = 1$, which yields the converse inclusions. □

COROLLARY 5.3.3.

(i) $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{BL}^{\mathrm{st}}) = \mathrm{TAUT}_{\mathrm{pos}}([0,1]_Ł)$;

(ii) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{BL}^{\mathrm{st}}) = \mathrm{SAT}_{(\mathrm{pos})}([0,1]_Ł)$.

Proof. We have $\mathbb{BL}^{\mathrm{st}} = \mathbb{SBL}^{\mathrm{st}} \cup \mathbb{L}$, where $\mathbb{L}$ denotes the class of standard BL-algebras with a first component Ł.
(i) $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{BL}^{\mathrm{st}}) = \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{SBL}^{\mathrm{st}}) \cap \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{L}) =$
$\mathrm{TAUT}(\{0,1\}_{\mathrm{B}}) \cap \mathrm{TAUT}_{\mathrm{pos}}([0,1]_Ł)$, where the last equality holds by combining Theorem 3.2.4 with the theorem above. The statement then follows from Lemma 3.1.4.
(ii) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{BL}^{\mathrm{st}}) = \mathrm{SAT}_{(\mathrm{pos})}(\mathbb{SBL}^{\mathrm{st}}) \cup \mathrm{SAT}_{(\mathrm{pos})}(\mathbb{L}) =$
$\mathrm{SAT}(\{0,1\}_{\mathrm{B}}) \cup \mathrm{SAT}_{(\mathrm{pos})}([0,1]_Ł)$, analogously to the above case. □

Apply Lemma 4.1.4 to show that the SAT, $\mathrm{SAT}_{\mathrm{pos}}$, and $\mathrm{TAUT}_{\mathrm{pos}}$ problems for the class of standard BL-algebras are distinct from each other and from the classical case. Compare this to Theorem 3.2.4 for classes of standard SBL-algebras (i.e., not starting with an Ł-component).

5.3.1 Finite sums

THEOREM 5.3.4. *Let $\boldsymbol{A}$ be a standard BL-algebra which is a finite ordinal sum of Ł, G, and Π-components. Then $\mathrm{Th}_\forall(\boldsymbol{A})$ is $\mathbf{coNP}$-complete.*

Proof. Hardness follows from Theorem 3.4.1. We prove $\mathbf{NP}$-containment for $\mathrm{Th}_\exists(\boldsymbol{A})$; the type and cardinality of the sum of $\boldsymbol{A}$ is used as a built-in information. This is the only difference from Algorithm EX-BL, which guesses a finite ordinal sum $\boldsymbol{B}$ and tests whether the given formula is in $\mathrm{Th}_\exists(\boldsymbol{B})$. Here, we fix a standard BL-algebra $\boldsymbol{A}$ which is a finite ordered sum of k components, so the `guessOrdinalSum()` step is omitted.

ALGORITHM EX-FIN `//accepts` $\mathrm{Th}_\exists(\boldsymbol{A})$

`input:` Φ `// existential sentence in the language of BL`

`begin`

`normalForm()` As in the proof of Theorem 5.2.4.

`componentDelimiters()` As in the proof of Theorem 5.2.4.

`guessOrder()` As in the proof of Theorem 5.2.4.

`checkOrder()` As in the proof of Theorem 5.2.4.

`checkExternal()` As in the proof of Theorem 5.2.4.

`checkInternal()` As in the proof of Theorem 5.2.4.

`end` □

COROLLARY 5.3.5. *Let $\boldsymbol{A}$ be a standard* BL*-algebra that is a finite ordinal sum and let* $\mathrm{L}(\boldsymbol{A})$ *be the logic of* $\boldsymbol{A}$*. Then*

(i) $\mathrm{SAT}_{(\mathrm{pos})}(\boldsymbol{A})$ *is* $\mathbf{coNP}$*-complete*;

(ii) $\mathrm{TAUT}_{(\mathrm{pos})}(\boldsymbol{A})$ *and* $\mathrm{CONS}(\boldsymbol{A})$ *are* $\mathbf{coNP}$*-complete*;

(iii) $\mathrm{THM}(\mathrm{L}(\boldsymbol{A}))$ *and* $\mathrm{CONS}(\mathrm{L}(\boldsymbol{A}))$ *are* $\mathbf{coNP}$*-complete.*

Proof. For SAT, $\mathrm{SAT}_{\mathrm{pos}}$, $\mathrm{TAUT}_{\mathrm{pos}}$, consider that if $\boldsymbol{A}$ has a first component Ł, and then one can use Theorem 5.3.2, otherwise one can use Theorem 3.2.4. □

5.3.2 Infinite sums

Now we consider those standard BL-algebras that are ordinal sums of infinitely many components. As stated in Propositions 5.2.1 and 5.2.2, a standard BL-algebra $\boldsymbol{A}$ that is an ordinal sum with infinitely many Ł-components generates either the variety $\mathbb{BL}$ (in case the sum has a first component Ł) or the variety $\mathbb{SBL}$ (otherwise). In both cases, the respective variety is generated as a quasivariety by $\boldsymbol{A}$. Hence, it is sufficient to deal with standard BL-algebras with *finitely many* Ł*-components*.

A standard BL-algebra with finitely many, say n, Ł-components seems much more tangible than the general case: one can think about it in terms of $n+1$ ordinal subsums that are without Ł-components and sit inbetween the n Ł-components. Some of these sums may be finite ordinal sums of G- and Π-components, some others may be infinite sums thereof, and some may be void. The infinite sums might be a problem, because there are too many such infinite sums for a finite description. Fortunately, from the point of view of the (quasi)equational theory of the algebra, we need not describe the infinite sums of G- and Π-components exactly, as the following analysis shows.

Consider two algebras $\boldsymbol{X}$, $\boldsymbol{Y}$, where either is a standard BL-algebra or the trivial one-element algebra. Let us take standard BL-algebras $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ that are two arbitrary infinite sums of G- and Π-components. It is easy to see that $\boldsymbol{Z}_1$ is partially embeddable into $\boldsymbol{Z}_2$ and vice versa, and therefore the standard BL-algebras $\boldsymbol{X} \oplus \boldsymbol{Z}_1 \oplus \boldsymbol{Y}$ and $\boldsymbol{X} \oplus \boldsymbol{Z}_2 \oplus \boldsymbol{Y}$ have the same universal theory. So the universal theory of any standard BL-algebra can be encoded by a finite string in the alphabet $\text{Ł}, \mathrm{G}, \Pi, \infty\text{Ł}, \infty\Pi$.

DEFINITION 5.3.6 (Canonical Standard BL-algebra). *A standard* BL*-algebra is canonical iff it is an ordinal sum that is either* ∞Ł, $\Pi \oplus \infty$Ł*, or a finite sum of* Ł, G, Π, *and* $\infty\Pi$*, where no* G *is preceded or followed by another* G*, and no* $\infty\Pi$ *is preceded or followed by a* G*, a* Π*, or another* $\infty\Pi$*.*

PROPOSITION 5.3.7. *For each standard* BL*-algebra* $\boldsymbol{A}$ *there is a canonical standard* BL*-algebra* $\boldsymbol{A}'$ *such that* $\mathrm{Th}_\forall(\boldsymbol{A}) = \mathrm{Th}_\forall(\boldsymbol{A}')$.

Not only canonical standard BL-algebras generate all possible subvarieties of $\mathbb{BL}^{\mathrm{st}}$ that are generated by any single standard BL-algebra, but, as shown in [14], nonisomorphic canonical standard BL-algebras generate distinct subvarieties. Hence, the strings in the alphabet $\{\text{Ł}, \infty\text{Ł}, \mathrm{G}, \Pi, \infty\Pi\}$ give a finite-string representation of all varieties (and all sets of propositional tautologies) given by a single standard BL-algebra. Moreover each of these varieties is generated as a quasivariety by the standard BL-algebra, so this consideration extends also to the respective quasivarieties (finite consequence relations).

As in previous cases, we address the universal theory of each of the canonical standard BL-algebras with finitely many L-components (and thus, no ∞Ł-components).

THEOREM 5.3.8. *Let $\boldsymbol{A}$ be a canonical standard BL-algebra which is an infinite ordinal sum with finitely many L-components. Then $\mathrm{Th}_\forall(\boldsymbol{A})$ is $\mathbf{coNP}$-complete.*

Proof. Hardness follows from Theorem 3.4.1. We prove $\mathbf{NP}$-containment of $\mathrm{Th}_\exists(\boldsymbol{A})$, using an algorithm that (accepts the given set and) relies on a built-in information about $\boldsymbol{A}$ in terms of a string of symbols. It works as follows: first, it guesses another finite sum $\boldsymbol{B}$ of Ł, G, and Π, whose cardinality is linear in the input size. Then, it checks that the ordinal sum $\boldsymbol{B}$ is a subsum of the sum of $\boldsymbol{A}$, in such a way that a first Ł-component in $\boldsymbol{B}$ is also a first component in $\boldsymbol{A}$; if this is true, then $\boldsymbol{B}$ is a subalgebra of $\boldsymbol{A}$ and thus, any solution found in $\boldsymbol{B}$ will be a solution in $\boldsymbol{A}$ also. Finally, it tests the input existential formula for validity in $\boldsymbol{B}$.

ALGORITHM EX-INF `//accepts` $\mathrm{Th}_\exists(\boldsymbol{A})$

`input:` Φ `// existential sentence in the language of BL`

`begin`

`normalForm()` As in the proof of Theorem 5.2.4 (in particular, n is the number of variables in the formula $\Phi_1 \wedge \Phi_2$).

`guessOrdinalSum()` Guess a $k \in \mathsf{N}$, $k \leq n+1$.
Guess an ordinal sum $\boldsymbol{B}$ of k components (a sequence of k symbols out of Ł, G, Π).

`checkEmbedding()` Check whether the sum of $\boldsymbol{B}$ is embeddable into the sum of $\boldsymbol{A}$ (as a sequence of symbols into a sequence of symbols), in such a way that an initial Ł of $\boldsymbol{B}$ (if any) is mapped to an initial Ł in $\boldsymbol{A}$.
From now on, work with $\boldsymbol{B}$ instead of $\boldsymbol{A}$.

`componentDelimiters()` As in the proof of Theorem 5.2.4.

`guessOrder()` As in the proof of Theorem 5.2.4.

`checkOrder()` As in the proof of Theorem 5.2.4.

`checkExternal()` As in the proof of Theorem 5.2.4.

`checkInternal()` As in the proof of Theorem 5.2.4.

`end`

Let us look at the `checkEmbedding()` step. First of all, we discuss this is a subroutine working in time polynomial in k, hence in $|\Phi|$. As a matter of fact, the check can be done deterministically in polynomial time; but it is simpler and sufficient to present the nondeterministic check. Take the finite-string representation of $\boldsymbol{B}$ (in the alphabet Ł, G, Π), for each element of the sum of $\boldsymbol{B}$, guesses a natural number points into the finite-string representation of $\boldsymbol{A}$ (in the alphabet Ł, G, Π, $\infty\Pi$), then checks that this assignment is a one-one embedding in terms of components (more than one Π-component in $\boldsymbol{B}$ can be mapped onto a single $\infty\Pi$-component of $\boldsymbol{A}$), and that it satisfies the condition that if Ł is initial in $\boldsymbol{B}$, then it is mapped onto an initial L-component in $\boldsymbol{A}$. This is a polynomial-time procedure: the cardinality of $\boldsymbol{B}$ is bounded polynomially by $|\Phi|$ and the cardinality of $\boldsymbol{A}$ (as a finite string in the alphabet Ł, G, Π, $\infty\Pi$) is fixed.

If there is a satisfying evaluation in $\boldsymbol{A}$, then one can find a finite subsum harbouring it; we know by Lemma 5.2.3 it is enough to search all finite subsums up to length $k+1$. The algorithm works with each such subsum as finite sum and works in exactly the same way as in the case for finite sums.

It is perhaps worth remarking that under this construction, the algorithm "throws away" some of the constructed sums which actually could supply a satisfying evaluation embeddable into $\boldsymbol{A}$; for example, if we permitted G-components in $\boldsymbol{B}$ to map onto $\infty\Pi$ components, the algorithm would still be correct because, if later the algorithm guesses an ordering that assigns finitely many idempotent values into this G-component, then these could map onto some delimiting idempotent values in the $\infty\Pi$-segment. We prefer, however, to work solely with the string representations and not to go back to the structure of the standard BL-algebras. $\square$

COROLLARY 5.3.9. *Let $\boldsymbol{A}$ be a standard* BL*-algebra that is an infinite ordinal sum and let* $\mathrm{L}(\boldsymbol{A})$ *be the logic of* $\boldsymbol{A}$*. Then*

- *(i)* $\mathrm{SAT}_{(\mathrm{pos})}(\boldsymbol{A})$ *is* $\mathbf{coNP}$*-complete*;
- *(ii)* $\mathrm{TAUT}_{(\mathrm{pos})}(\boldsymbol{A})$ *and* $\mathrm{CONS}(\boldsymbol{A})$ *are* $\mathbf{coNP}$*-complete*;
- *(iii)* $\mathrm{THM}(\mathrm{L}(\boldsymbol{A}))$ *and* $\mathrm{CONS}(\mathrm{L}(\boldsymbol{A}))$ *are* $\mathbf{coNP}$*-complete.*

5.3.3 Logics given by classes of standard BL-algebras

Let us first discuss the case that $\mathbb{K}$ is finite. The following is a consequence of a more general result of [17].

PROPOSITION 5.3.10. *If* $\mathbb{K}$ *is a finite, nonempty class of standard* BL*-algebras, the logic of* $\mathbb{K}$ *is an axiomatic extension of* BL *obtained by adding finitely many axioms.*

Recall that $\mathbf{NP}$ is closed under finite unions, and consider that $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K}) = \bigcup_{\boldsymbol{A}\in\mathbb{K}} \mathrm{SAT}_{(\mathrm{pos})}(\boldsymbol{A})$; $\mathbf{coNP}$ is closed under finite intersections, and $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K}) = \bigcap_{\boldsymbol{A}\in\mathbb{K}} \mathrm{TAUT}_{(\mathrm{pos})}(\boldsymbol{A})$ and $\mathrm{CONS}(\mathbb{K}) = \bigcap_{\boldsymbol{A}\in\mathbb{K}} \mathrm{CONS}(\boldsymbol{A})$. On that basis, and in view of Theorem 3.4.1, we may conclude:

THEOREM 5.3.11. *If* $\mathbb{K}$ *is a finite, nonempty class of standard* BL*-algebras and* $\mathrm{L}(\mathbb{K})$ *is the logic of* $\mathbb{K}$*, then*

- *(i)* $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K})$ *is* $\mathbf{NP}$*-complete*;
- *(ii)* $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K})$ *and* $\mathrm{CONS}(\mathbb{K})$ *are* $\mathbf{coNP}$*-complete*;
- *(iii)* $\mathrm{THM}(\mathrm{L}(\mathbb{K}))$ *and* $\mathrm{CONS}(\mathrm{L}(\mathbb{K}))$ *are* $\mathbf{coNP}$*-complete.*

Now let us address the case when $\mathbb{K}$ is an arbitrary (possibly infinite) class of standard BL-algebras. We rely on the following statement.

PROPOSITION 5.3.12 ([24]). *Let* $\mathbb{K}$ *be a class of standard* BL*-algebras. Then there is a finite class* $\mathbb{L}$ *of standard* BL*-algebras such that* $\mathbf{V}(\mathbb{K}) = \mathbf{V}(\mathbb{L})$.

COROLLARY 5.3.13. *Let* $\mathbb{K}$ *be a nonempty class of standard* BL*-algebras. Then*

- *(i)* $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{K})$ *is* $\mathbf{NP}$*-complete*;
- *(ii)* $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{K})$ *is* $\mathbf{coNP}$*-complete.*

Proof. For TAUT the statement follows from the previous one. For SAT, $\mathrm{SAT_{pos}}$, and $\mathrm{TAUT_{pos}}$, recall that we can partition $\mathbb{K}$ into a class $\mathbb{K}_{Ł}$ of those algebras in $\mathbb{K}$ that have a first component Ł, and a class $\mathbb{K}_{\bar{Ł}}$ of the remaining algebras in $\mathbb{K}$. Then use Theorem 5.3.2 and Theorem 3.2.4 for the two classes, and Lemma 3.1.4 on their union/intersection. □

6 Logics in modified languages

In this section we discuss some fragments and expansions of BL and MTL and their extensions. As regards fragments, we limit ourselves to dropping $\overline{0}$, i.e., we consider the falsehood-free language of hoops. For expansions, we consider in particular logics with $\triangle$, logics with new propositional constants, logics with an involutive negation as an independent connective, and the logics $Ł\Pi$ and $Ł\Pi\frac{1}{2}$.

6.1 Falsehood-free fragments

MTLH (monoidal t-norm hoop logic) is obtained from MTL by dropping $\overline{0}$ from the language and dropping the axiom $\overline{0} \to \varphi$. BLH (basic hoop logic) is obtained from BL in the same manner. BLH can be extended to $Ł\mathrm{H}$, GH, and $\Pi\mathrm{H}$ in exactly the same way as BL extends to Ł, G, and Π. Besides, *cancellative hoop logic* CHL is obtained from BLH by adding the axiom $(\varphi \to (\varphi \mathbin{\&} \psi)) \to \psi$.

PROPOSITION 6.1.1 (Conservativeness [13]).

- *(i)* MTL, IMTL, *and* SMTL *are conservative expansions of* MTLH;
- *(ii)* BL *and* SBL *are conservative expansions of* BLH;
- *(iii)* Ł *is a conservative expansion of* $Ł\mathrm{H}$;
- *(iv)* G *is a conservative expansion of* GH;
- *(v)* Π *is a conservative expansion of* $\Pi\mathrm{H}$.

So each of the falsehood-free fragments inherits the complexity class of its counterpart in the full language (cf. Lemma 3.1.2). For CHL, we use the following fact:

PROPOSITION 6.1.2. *Let $\mathcal{L}$ be the language of hoops and $T \cup \{\varphi\} \subseteq Fm_{\mathcal{L}}$. Then $T \vdash_{\mathrm{CHL}} \varphi(p_1, \ldots, p_n)$ iff $T \vdash_{\Pi} (\bigwedge_{i=1}^{n} \neg\neg p_i) \to \varphi$.*

COROLLARY 6.1.3. $\mathrm{CONS(CHL)} \preceq_{\mathbf{P}} \mathrm{CONS}(\Pi)$*, so* $\mathrm{CONS(CHL)}$ *is in* $\mathbf{coNP}$.

In the following we prove $\mathbf{coNP}$-hardness for theoremhood in BLH, $Ł\mathrm{H}$, GH, $\Pi\mathrm{H}$, CHL. Let p be a new variable, and consider the following translation function, operating on BL-formulas:

- if φ is $\overline{0}$ then φ° is p
- if φ is atomic then φ° is $\varphi \vee p$
- if φ is $\psi \mathbin{\&} \chi$ then φ° is $(\psi^\circ \mathbin{\&} \chi^\circ) \vee p$
- if φ is $\psi \to \chi$ then φ° is $\psi^\circ \to \chi^\circ$

PROPOSITION 6.1.4 (Interpretation in falsehood-free fragments [13]). *Let* L *be* MTL, BL, $\mathrm{Ł}$*, or* G, LH *its falsehood-free fragment, and* φ *a formula in the language of* MTL. *Then*

(i) $\vdash_{\mathrm{L}} \varphi$ *iff* $\vdash_{\mathrm{LH}} \varphi^{\circ}$;

(ii) $\vdash_{\Pi\mathrm{H}} \varphi^{\circ}$ *iff* $\vdash_{\Pi} \varphi$ *and* $\vdash_{\mathrm{Ł}} \varphi$;

(iii) $\vdash_{\mathrm{CHL}} \varphi^{\circ}$ *iff* $\vdash_{\mathrm{Ł}} \varphi$.

COROLLARY 6.1.5. *Theoremhood (and hence, provability from finite theories) in the logics* BLH, $\mathrm{ŁH}$, GH, $\Pi\mathrm{H}$*, and* CHL *is* $\mathbf{coNP}$*-hard, hence* $\mathbf{coNP}$*-complete. Moreover,* $\mathrm{MTL} \approx_{\mathbf{P}} \mathrm{MTLH}$.

Proof. The set $\mathrm{THM}(\mathrm{Ł}) \cap \mathrm{THM}(\Pi) = \mathrm{TAUT}([0,1]_{\mathrm{Ł}}) \cap \mathrm{TAUT}([0,1]_{\Pi})$ can be observed to be $\mathbf{coNP}$-hard by Theorem 3.4.1; hence $\mathbf{coNP}$-hardness for $\Pi\mathrm{H}$. The other cases are straightforward. □

Satisfiability for (classes of) hoops is not an interesting problem. If $\mathcal{L}$ is the language of hoops and $\mathbb{K}$ is a nonempty class of $\mathcal{L}$-hoops, then $\mathrm{SAT}(\mathbb{K}) = \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}) = Fm_{\mathcal{L}}$ (clearly in any hoop, each formula is satisfiable by evaluating all of its variables with the value 1).

6.2 Logics with $\triangle$

Let L be MTL or its extension. The $\triangle$-expansion $\mathrm{L}_{\triangle}$ of L is obtained by adding the rule of $\triangle$-generalization: from φ derive $\triangle\varphi$, and the following axioms:

($\triangle$1) $\triangle\varphi \vee \neg\triangle\varphi$

($\triangle$2) $\triangle(\varphi \vee \psi) \to (\triangle\varphi \vee \triangle\psi)$

($\triangle$3) $\triangle\varphi \to \varphi$

($\triangle$4) $\triangle\varphi \to \triangle\triangle\varphi$

($\triangle$5) $\triangle(\varphi \to \psi) \to (\triangle\varphi \to \triangle\psi)$

Recall that the deduction theorem for a logic $\mathrm{L}_{\triangle}$ reads as follows: $T \cup \{\varphi\} \vdash_{\mathrm{L}_{\triangle}} \psi$ iff $T \vdash_{\mathrm{L}_{\triangle}} \triangle\varphi \to \psi$. Hence $\mathrm{CONS}(\mathrm{L}_{\triangle}) \approx_{\mathbf{P}} \mathrm{THM}(\mathrm{L}_{\triangle})$ and it is sufficient to investigate complexity of the set $\mathrm{THM}(\mathrm{L}_{\triangle})$ (cf. Lemma 3.1.1).

Adding $\triangle$ in the above manner expands the logic L conservatively; in particular, we can expand each standard L-algebra into a $\mathrm{L}_{\triangle}$-algebra, and then prove standard completeness results.

We remark that for any logic L, in any $\mathrm{L}_{\triangle}$-chain $\boldsymbol{A}$, the semantic counterpart of $\triangle$ is the function given by $\triangle(1) = 1$, $\triangle(x) = 0$ for $x \neq 1$.

THEOREM 6.2.1. *Let* L *be the logic of a standard* BL*-algebra. Then* $\mathrm{THM}(\mathrm{L}_{\triangle})$ *is* $\mathbf{coNP}$*-complete.*

Proof. For hardness, combine Lemma 3.1.2 with results on standard L-algebras. Containment by an inspection of containment proof for L; it is sufficient to note that the $\triangle$-operation is order-determined. In this way, we actually obtain $\mathbf{coNP}$-completeness of the universal fragment of the theory of the appropriate standard $\mathrm{L}_{\triangle}$-algebra. □

6.3 Logics with constants

Expanding the language with constants is addressed in a comprehensive manner in Chapter VIII. We rely on [11] for the general framework and on [22] for results. Admittedly, the results are rather fragmentary and there are some open and many unattempted problems. We discuss three examples: Łukasiewicz logic, Gödel logic, and product logic, each expanded with new propositional constants as explained below.

In a general setting, one expands the language with names for some elements of a chosen algebra, often a standard one. In that case, one takes an arbitrary $C \subseteq [0,1]$, countable and closed under all operations, to be the canonical semantics of new constants. But in order to be able to reason about complexity, we need much more: elements of C should be representable by finite words, C should be decidable (preferably in $\mathbf{P}$), and the C-words should admit feasible evaluation of operations. Thus we restrict our attention to the case $C = \mathsf{Q} \cap [0,1]$.

Let $[0,1]_*$ be a standard BL-algebra. Let $\mathcal{Q} = \{\overline{q} \mid q \in \mathsf{Q} \cap (0,1)\}$ be a set of new propositional constants. If $\mathcal{L}$ is the language of BL, define $\mathcal{L}^{\mathcal{Q}} = \mathcal{L} \cup \mathcal{Q}$. If L is the logic of $[0,1]_*$, then $\mathrm{L}(\mathcal{Q})$ in the language $\mathcal{L}^{\mathcal{Q}}$ expands L with the *bookkeeping axioms*

$$\overline{r} \mathbin{\&} \overline{s} \leftrightarrow \overline{r * s} \quad \text{and} \quad \overline{r} \to \overline{s} \leftrightarrow \overline{r \to s}$$

for each $r, s \in \mathsf{Q} \cap [0,1]$. Each logic $\mathrm{L}(\mathcal{Q})$ has its equivalent algebraic semantics (L-algebras enriched with constants, satisfying the axioms), it has standard semantics (standard L-algebras, ditto), but we are interested primarily in its *canonical* semantics, which is given by the initial standard algebra $[0,1]_*$ and the canonical interpretation of $\overline{q}$ with q for each $q \in \mathsf{Q} \cap [0,1]$. If $[0,1]_*$ is a standard BL-algebra, L is the logic of $[0,1]_*$, then $[0,1]_*^{\mathcal{Q}}$ denotes the canonical $\mathrm{L}(\mathcal{Q})$-algebra. Now let us see where bookkeeping axioms can take us, completeness-wise.

PROPOSITION 6.3.1 ([21, 39]). *Let $[0,1]_*$ and L be as above. Then*

(i) $\mathrm{L}(\mathcal{Q})$ *has the canonical FSSC iff $*$ is the Łukasiewicz t-norm*;

(ii) $\mathrm{L}(\mathcal{Q})$ *has the canonical SC if $*$ is the Gödel or the product t-norm.*

Note also that finite strong *standard* completeness (i.e., completeness with respect to the class of all standard algebras) holds for any $\mathrm{L}(\mathcal{Q})$ obtained in this manner, and consequently, each $\mathrm{L}(\mathcal{Q})$ is conservative over L and L is the $\mathcal{L}$-fragment of $\mathrm{L}(\mathcal{Q})$.

Now for complexity: we work in the canonical standard algebras, starting with the expansion of Łukasiewicz logic.

THEOREM 6.3.2. $\mathrm{Th}_\forall([0,1]_{\text{Ł}}^{\mathcal{Q}})$ *is* $\mathbf{coNP}$*-complete.*

Proof. Hardness follows from the result for $[0,1]_{\text{Ł}}$ without constants (Theorem 4.1.2): we have $\mathrm{Th}_\forall([0,1]_{\text{Ł}}) \preceq_{\mathbf{P}} \mathrm{Th}_\forall([0,1]_{\text{Ł}}^{\mathcal{Q}})$ since the former is the fragment of the latter given by restriction to the BL-language; cf. also Lemma 3.1.2.

We show $\mathbf{NP}$-containment of $\mathrm{Th}_\exists([0,1]_{\text{Ł}}^{\mathcal{Q}})$ by a discussion of ALGORITHM EX-L accepting $\mathrm{Th}_\exists([0,1]_{\text{Ł}})$. In the case with constants, the input is an existential sentence Φ with constants, represented as two integers in binary. As before, we first transform Φ into an existential normal form (see Lemma 3.3.5). In the normal form for Φ we have, in addition to the former, also identities of the form $x = c$ for x a variable, c a constant. Then we guess an ordering $\leq_0$ of $V = \{x_1, \ldots, x_n, c_1, \ldots, c_k\}$, where $c_1, \ldots, c_k$ are those constants in Φ, and 0 and 1. Then we check that Φ_1 is compatible with $\leq_0$. Then we need to translate each equation $x = c$ into the language of linear programming. But that is easy, since for c being $\frac{p}{q}$, we may translate the equation $x = \frac{p}{q}$ with $xq = p$. The rest is as before. □

COROLLARY 6.3.3.

(i) $\mathrm{SAT}_{(\mathrm{pos})}([0,1]^{\mathcal{Q}}_{\text{Ł}})$ *is* $\mathbf{NP}$*-complete.*

(ii) $\mathrm{TAUT}_{(\mathrm{pos})}([0,1]^{\mathcal{Q}}_{\text{Ł}})$ *and* $\mathrm{CONS}([0,1]^{\mathcal{Q}}_{\text{Ł}})$*, are* $\mathbf{coNP}$*-complete.*

(iii) $\mathrm{THM}(\text{Ł}(\mathcal{Q}))$ *and* $\mathrm{CONS}(\text{Ł}(\mathcal{Q}))$ *are* $\mathbf{coNP}$*-complete.*

Proof. For hardness, use Lemma 3.1.2. For (i), (ii), consider Lemma 3.1.1. For (iii), use finite strong canonical completeness of Ł$(\mathcal{Q})$. □

THEOREM 6.3.4. $\mathrm{Th}_{\forall}([0,1]^{\mathcal{Q}}_{\mathrm{G}})$ *is* $\mathbf{coNP}$*-complete.*

Proof. Hardness follows from follows from Theorem 5.1.1 and Lemma 3.1.2. We show $\mathbf{NP}$-containment of $\mathrm{Th}_{\exists}([0,1]^{\mathcal{Q}}_{\mathrm{G}})$, modifying ALGORITHM EX-G (we are only considering the standard algebra) to cater for truth constants. Let Φ be an existential sentence $\exists x_1 \ldots \exists x_n \Phi(x_1, \ldots, x_n)$ in the language of $([0,1]^{\mathcal{Q}}_{\mathrm{G}})$, where Φ is an open Boolean combination of identities. Let $\frac{k_1}{l_1}, \ldots, \frac{k_m}{l_m}$ be a list of truth constants in Φ distinct from 0, 1. Guess a linear ordering $\leq_0$ of the set $V = \{0, 1, x_1, \ldots, x_n, \frac{k_1}{l_1}, \ldots, \frac{k_m}{l_m}\}$, such that 0 is at the bottom, 1 is at the top, and the natural order of the constants is preserved. Compute the value of each atomic formula in Φ: on the basis of $\leq_0$, evaluate all atomic formulas. Evaluate the Boolean combination Φ, accept iff the result is 1. □

THEOREM 6.3.5.

(i) $\mathrm{SAT}_{(\mathrm{pos})}([0,1]^{\mathcal{Q}}_{\mathrm{G}})$ *is* $\mathbf{NP}$*-complete.*

(ii) $\mathrm{TAUT}_{(\mathrm{pos})}([0,1]^{\mathcal{Q}}_{\mathrm{G}})$ *and* $\mathrm{CONS}([0,1]^{\mathcal{Q}}_{\mathrm{G}})$ *are* $\mathbf{coNP}$*-complete.*

(iii) $\mathrm{THM}(\mathrm{G}(\mathcal{Q}))$ *is* $\mathbf{coNP}$*-complete*

(iv) $\mathrm{CONS}(\mathrm{G}(\mathcal{Q}))$ *is* $\mathbf{coNP}$*-complete.*

Proof. (i) to (iii) are clear, but we prove (iv), because CONS in the canonical algebra need not correspond to provability (due to lack of canonical FSSC). For $\mathrm{CONS}(\mathrm{G}(\mathcal{Q}))$, recall that $\mathrm{G}(\mathcal{Q})$ enjoys the FSSC, and observe that for each standard $\mathrm{G}(\mathcal{Q})$-algebra $\boldsymbol{A}$, there is a filter F on the G-algebra of constants $\mathsf{Q} \cap [0,1]$ such that $\boldsymbol{A}$ interprets the elements of F with 1, whereas other constants are interpreted with pairwise distinct elements. If $\{\varphi_1, \ldots, \varphi_n\}, \psi \in \overline{\mathrm{CONS}}(\mathrm{G}(\mathcal{Q}))$, then this shows in some standard $\mathrm{G}(\mathcal{Q})$-algebra given by a filter F on the algebra of constants, and the only information needed about F is which constants *occurring in the input* belong to it. So we may guess a rational (inbetween two constants occurring in the input) and use a variant of the algorithm given above on $\boldsymbol{A}$ obtained in this manner. □

THEOREM 6.3.6. $\mathrm{Th}_\forall([0,1]^{\mathcal{Q}}_\Pi)$ *is in* $\mathbf{PSPACE}$.

Proof. We show $\mathrm{Th}_\exists([0,1]^{\mathcal{Q}}_\Pi) \preceq_{\mathbf{P}} \mathrm{Th}_\exists(\mathrm{RCF})$. Consider, for $k \in \mathsf{N}$, its binary representation $(c_{\lfloor\log(k)\rfloor}\dots c_0)$. Then $k = \sum_{i\leq\lfloor\log(k)\rfloor} c_i.2^i$; let this be the *term* $\overline{k}$ *corresponding to* k in the language of RCF.

Let Φ be an existential $\Pi(\mathcal{Q})$-sentence. Transform it into an existential normal form $\exists x_1\dots\exists x_n(\Phi_1 \wedge \Phi_2)$. Define Φ_2' as follows. Process the identities in Φ_2 one by one, for $i_1, i_2, i_3 \leq n$, in case of:

- $x_{i_1} = \frac{c}{d}$, replace with $x_{i_1}\cdot\overline{d} = \overline{c}$;
- $x_{i_1} * x_{i_2} = x_{i_3}$, replace with $x_{i_1}\cdot x_{i_2} = x_{i_3}$;
- $x_{i_1} \to x_{i_2} = x_{i_3}$, replace with $((x_{i_1} \leq x_{i_2}) \wedge (x_{i_3} = 1)) \vee$ $((x_{i_1} > x_{i_2}) \wedge (x_{i_1}\cdot x_{i_3} = x_{i_2}))$.

Let Φ_3 denote the formula $\bigwedge_{i=1}^n 0 \leq x_i \wedge x_i \leq 1$ for $1 \leq i \leq n$ (boundary conditions).

Let Φ' denote the formula $\Phi_1 \wedge \Phi_2' \wedge \Phi_3$. Then $\Phi \in \mathrm{Th}_\exists[0,1]^{\mathcal{Q}}_\Pi)$ if and only if $\exists x_1\dots\exists x_n\Phi' \in \mathrm{Th}_\exists(\mathrm{RCF})$. Moreover, the latter formula can be computed from Φ in time polynomial in $|\Phi|$. $\square$

EXAMPLE 6.3.7. There is a standard BL-algebra $[0,1]_*$ which is an infinite sum of Ł-components such that both $\mathrm{TAUT}([0,1]^{\mathcal{Q}}_*)$ and $\mathrm{SAT}([0,1]^{\mathcal{Q}}_*)$ are nonarithmetical.

Proof. Let $A \subseteq \mathsf{N}$ be any set. Let $[0,1]_*$ be a standard BL-algebra with idempotents 0, 1, and $\frac{1}{n}$ for $n \in A$, and all components isomorphic to $[0,1]_{\text{Ł}}$. Let us introduce rational constants into $[0,1]_*$. Now observe the following: for each $n \in \mathsf{N}$, we have $n \in A$ iff the formula

$$\overline{\tfrac{1}{n}} \mathbin{\&} \overline{\tfrac{1}{n}} \leftrightarrow \overline{\tfrac{1}{n}}$$

is in $\mathrm{TAUT}([0,1]^{\mathcal{Q}}_*)$ (or $\mathrm{SAT}([0,1]^{\mathcal{Q}}_*)$). Therefore, $A \preceq_{\mathrm{m}} \mathrm{TAUT}([0,1]^{\mathcal{Q}}_*)$ and $A \preceq_{\mathrm{m}} \mathrm{SAT}([0,1]^{\mathcal{Q}}_*)$. Fixing A as a nonarithmetical set closes the proof. $\square$

6.4 Logics with an involutive negation

We discuss expansions of a given logic L with a new unary connective $\sim$, which behaves as a decreasing involution. The resulting logic $\mathrm{L}_\sim$ is particularly interesting when the definable negation $\neg$ in the logic is the strict negation, because of the two negations' interplay. This means the cases when L is an extension of SMTL. Then $\triangle\varphi$ is defined as $\neg{\sim}\varphi$. In particular, if L extends SMTL, then $\mathrm{L}_\sim$ results from L by adding the rule $\varphi/\triangle\varphi$ and the axioms

$$\sim\sim\varphi \leftrightarrow \varphi$$
$$\triangle(\varphi \to \psi) \to \triangle({\sim}\psi \to {\sim}\varphi)$$
$$\neg\varphi \to {\sim}\varphi$$

The semantics of $\sim$ on $[0,1]$ is given by decreasing involutions; a prominent example is the function $1-x$ on $[0,1]$, the *canonical* involutive negation. We say that $\mathrm{L}_\sim$-algebra is standard iff its $\mathrm{FL_{ew}}$-reduct is a standard MTL-algebra, no matter what the $\sim$-operation is. If $\boldsymbol{A}$ is a standard MTL-algebra and $\sim$ is an involutive negation, denote $\boldsymbol{A}^\sim$ the algebra that is an expansion of $\boldsymbol{A}$ with the involutive negation $\sim$. Further, denote $\mathbb{A}^\sim$ the class of all algebras that expand $\boldsymbol{A}$ with some involutive negation $\sim$.

Let A be a set totally ordered by $\leq$. Let $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \ldots, \langle a_n, b_n\rangle$ be a finite number of pairs from A. We say that these pairs are nested (w.r.t. $\leq$) iff there is a permutation σ on $\{1, \ldots, n\}$ such that $a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)} \leq b_{\sigma(n)} \leq \cdots \leq a_{\sigma(2)} \leq a_{\sigma(1)}$ and for each $i = 1, \ldots, n-1$ we have $a_{\sigma(i)} = a_{\sigma(i+1)}$ iff $b_{\sigma(i)} = b_{\sigma(i+1)}$.

LEMMA 6.4.1. *Let* $0 < a_0 < \cdots < a_k < 1$ *be real numbers. Then there is a decreasing involution* $\sim$ *on* $[0, 1]$ *such that* $\sim(a_i) = a_{k-i}$ *for* $i = 0, \ldots, k$.

PROPOSITION 6.4.2. *If* L *is the logic of some standard* BL*-algebra, then* $\mathrm{L}_\sim$ *enjoys finite strong standard completeness.*

This entails that, if L is the logic of some standard BL-algebra, $\mathrm{L}_\sim$ is a conservative expansion of L.

THEOREM 6.4.3. *Let* $\boldsymbol{A}$ *be a standard* SBL*-algebra which is a finite ordinal sum. Then* $\mathrm{Th}_\forall(\mathbb{A}^\sim)$ *is* $\mathbf{coNP}$*-complete.*

Proof. Hardness follows from Lemma 3.1.2, considering that $\boldsymbol{A}$ is the $\sim$-free reduct of each $\boldsymbol{A}^\sim \in \mathbb{A}^\sim$. We prove $\mathbf{NP}$-containment of $\mathrm{Th}_\exists(\mathbb{A}^\sim)$. Since $\boldsymbol{A}$ is a standard SBL-algebra, we modify ALGORITHM EX-FIN to cater for the involutive negation. We rely on Lemma 6.4.1.

ALGORITHM EX-INV `//accepts` $\mathrm{Th}_\exists(\mathbb{A}^\sim)$

`input:` Φ `// ex. sentence in the language of BL with` $\sim$

`begin`

`normalForm()`

`componentDelimiters()`

`guessOrder()`

`checkOrder()`

`checkInvolution()` Let k be the number of variables in the normal form of Φ. For each x_i, x_j, $i, j \leq k'$, if $x_i = \sim x_j$ occurs in Φ_2, put down a pair $\{x_i, x_j\}$. Do this for every occurrence of $\sim$ in Eq. Check that the pairs thus created are nested w.r.t. $\leq_0$.

`checkExternal()`

`checkInternal()`

`end`

We discuss why this works correctly: assume for an algebra $\boldsymbol{A}^\sim$ obtained from $\boldsymbol{A}$ by adding some involutive involution, we have $\models_{\boldsymbol{A}} \Phi$. Then one guess of $\leq_0$ will be the real ordering of $\boldsymbol{A}$ on all values of subformulas. The properties of the involution on $\mathbf{A}$ warrant that the conditions in the step `checkInvolution()` will be satisfied. (And the equations in the remaining operations will be solvable.) Hence, there is an accepting computation. On the other hand, if there is an accepting computation on Φ, then the values of all variables determine a complete evaluation of Φ; by Lemma 6.4.1, there is a decreasing involution on $[0, 1]$ which satisfies all the identities prescribed by φ on the given values. Note that the check of soundness of the ordering w.r.t. involution is independent of the other steps and can be performed at any stage (after the ordering $\leq_0$ is established). $\square$

The algorithm can be modified to work also for $\mathrm{SBL}_\sim$: instead of working with a fixed algebra, the algorithm first (transforms the input formula into a normal form, and then) guesses an ordinal sum whose first component is Π and whose number of components is bounded by $2m+1$, where m is the number of variables in the normal form (for each variable, we consider its component and the component of its $\sim$-negation).

Finally, for infinite ordinal sums, we may restrict our attention to the canonical ones; if $\boldsymbol{A}$ is any standard BL-algebra and $\boldsymbol{A}'$ is the canonical standard BL-algebra with the same universal (equational, quasiequational) theory, then by completeness, the logic of $\mathbb{A}^\sim$ coincides with the logic of $\mathbb{A}'^\sim$.

COROLLARY 6.4.4. *Let* L *be the logic of a standard* SBL*-algebra. Then* $\mathrm{THM}(\mathrm{L}_\sim)$ *and* $\mathrm{CONS}(\mathrm{L}_\sim)$ *are* $\mathbf{coNP}$*-complete.*

One might further ask about complexity of fragments of the theory of *particular* (standard) $\mathrm{SBL}_\sim$ algebras (or $\mathrm{SMTL}_\sim$-algebras); this is an interesting open problem. Research in this area will be framed by available results on the structure of the lattice of subvarieties of the variety of $\mathrm{SBL}_\sim$-algebras. Chapter VIII gives details. We mention an important example, the case of standard $\Pi_\sim$-algebras given by a combination of the standard product algebra $[0,1]_\Pi$ and an arbitrary involutive negation $\sim$ on $[0,1]$; such an algebra will be denoted $[0,1]_\Pi^\sim$.

For $[0,1]_\Pi$ and $\sim_1$, $\sim_2$ two involutive negations on $[0,1]$, we say $\sim_1$ and $\sim_2$ are *isomorphic* w.r.t. $[0,1]_\Pi$ iff there is an isomorphism of $[0,1]_\Pi^{\sim_1}$ onto $[0,1]_\Pi^{\sim_2}$. Any such isomorphism must obviously be an automorphism of the product t-norm. It is not difficult to see that the cardinality of the class of isomorphism classes of algebras $[0,1]_\Pi^\sim$, for all choices of $\sim$ on $[0,1]$, is that of the continuum. The following result can be obtained:

PROPOSITION 6.4.5 ([18]). *Let* $\sim_1$, $\sim_2$ *be two involutive negations on* $[0,1]$*. Then we have* $\mathrm{TAUT}([0,1]_\Pi^{\sim_1}) = \mathrm{TAUT}([0,1]_\Pi^{\sim_2})$ *iff* $[0,1]_\Pi^{\sim_1}$ *is isomorphic to* $[0,1]_\Pi^{\sim_2}$.

Hence, nice complexity results can only be obtained for a minority of such algebras. This result concerning the standard product algebra can be generalized to particular ordinal sums of Ł- and Π-components, as shown in [27]. In the next section, we give an upper bound for the particular case of $\sim$ being the function $1-x$.

6.5 The logics ŁΠ and ŁΠ$\frac{1}{2}$

The logics ŁΠ and ŁΠ$\frac{1}{2}$ are discussed in detail in Chapter VIII. The logic ŁΠ is a result of combining the Łukasiewicz and the product connectives in a single logical system. We use subscripts to distinguish between the two sets of connectives where necessary, writing $\&_{Ł}$, $\&_\Pi$, etc.; $\sim$ denotes the involutive negation in Ł. The logic ŁΠ$\frac{1}{2}$ has, in addition, a constant $\frac{1}{2}$ in the language. All the connectives of the three important schematic extensions of BL are available in both ŁΠ and ŁΠ$\frac{1}{2}$. Moreover, the $\triangle$ connective is defined by $\triangle\varphi$ being $\neg{\sim}\varphi$. Then the $\triangle$-deduction theorem entails polynomial equivalence of THM and CONS for both ŁΠ and ŁΠ$\frac{1}{2}$ (cf. Lemma 3.1.1).

The logics ŁΠ and ŁΠ$\frac{1}{2}$ can be presented as expansions of Łukasiewicz logic; another approach to axiomatizing ŁΠ is the following one. Take the logic $\Pi_\sim$ and define $\varphi \to_{Ł} \psi$ as ${\sim}(\varphi \,\&_\Pi {\sim}(\varphi \to_\Pi \psi))$. Add $(\varphi \to_{Ł} \psi) \to ((\psi \to_{Ł} \chi) \to (\varphi \to_{Ł} \chi))$ to the axioms of $\Pi_\sim$. Then one can define all the other connectives of ŁΠ, in particular, $\&_Ł$. For the logic $Ł\Pi\frac{1}{2}$, introduce a new constant $\frac{1}{2}$ and add the axiom $\frac{1}{2} \leftrightarrow_Ł \sim\frac{1}{2}$.

The standard semantics of ŁΠ and $Ł\Pi\frac{1}{2}$ on $[0,1]$ is obtained by combining the standard semantics for both sets of connectives: $[0,1]_{Ł\Pi} = \langle *_Ł, \to_Ł, *_\Pi, \to_\Pi, 0\rangle$. The standard algebra $[0,1]_{Ł\Pi\frac{1}{2}}$ has additionally a constant operation $\frac{1}{2}$. Completeness of ŁΠ with respect to $[0,1]_{Ł\Pi}$ and of $Ł\Pi\frac{1}{2}$ with respect to $[0,1]_{Ł\Pi\frac{1}{2}}$ can be proved; it follows that ŁΠ expands its Ł-, G-, and Π-fragment conservatively, and that $Ł\Pi\frac{1}{2}$ is a conservative expansion of ŁΠ. This also shows what has not been mentioned in the previous subsection, that the logic ŁΠ is a complete axiomatization of the standard $\Pi_\sim$-algebra given by the product t-norm and the involutive negation $1-x$.

Owing to the presence of an involutive negation, Lemma 3.2.1 holds in full, establishing usual reductions between tautologousness and satisfiability. Summing up, we only need to investigate TAUT and SAT for ŁΠ and $Ł\Pi\frac{1}{2}$.

LEMMA 6.5.1.

(i) $\mathrm{TAUT}([0,1]_{Ł\Pi}) \approx_{\mathbf{P}} \mathrm{TAUT}([0,1]_{Ł\Pi\frac{1}{2}})$.

(ii) $\mathrm{SAT}([0,1]_{Ł\Pi}) \approx_{\mathbf{P}} \mathrm{SAT}([0,1]_{Ł\Pi\frac{1}{2}})$.

Proof. (i) It is sufficient to show $\mathrm{TAUT}([0,1]_{Ł\Pi\frac{1}{2}}) \preceq_{\mathbf{P}} \mathrm{TAUT}([0,1]_{Ł\Pi})$, the other reduction follows from the fact that the latter set is the $\frac{1}{2}$-free fragment of the former one (cf. Lemma 3.1.2). Let φ be an $Ł\Pi\frac{1}{2}$-formula, p be a new variable; then

$$\varphi \in \mathrm{TAUT}([0,1]_{Ł\Pi\frac{1}{2}}) \text{ iff } \triangle(p \leftrightarrow_Ł \sim p) \to_Ł \varphi(\tfrac{1}{2}/p) \in \mathrm{TAUT}([0,1]_{Ł\Pi}).$$

Observe that $\varphi(\frac{1}{2}/p)$ is an ŁΠ-formula. Now assume $\varphi \in \mathrm{TAUT}([0,1]_{Ł\Pi\frac{1}{2}})$; for any evaluation e in $[0,1]_{Ł\Pi}$, either $e(p \leftrightarrow_Ł \sim p) = 1$, then $e(p) = 0.5$, and $e(\varphi(\frac{1}{2}/p)) = 1$ by assumption, or $e(p \leftrightarrow_Ł \sim p) < 1$, so $e(\triangle(p \leftrightarrow_Ł \sim p)) = 0$, and hence $e(\triangle(p \leftrightarrow_Ł \sim p) \to_Ł \varphi(\frac{1}{2}/p)) = 1$. Conversely, if $\triangle(p \leftrightarrow_Ł \sim p) \to_Ł \varphi(\frac{1}{2}/p) \in \mathrm{TAUT}([0,1]_{Ł\Pi})$, then in particular all evaluations e in $[0,1]_{Ł\Pi}$ such that $e(p) = 0.5$ give $e(\varphi(\frac{1}{2}/p)) = 1$, so $\varphi \in \mathrm{TAUT}([0,1]_{Ł\Pi\frac{1}{2}})$.
(ii) Similarly, for p new variable, $\varphi \in \mathrm{SAT}([0,1]_{Ł\Pi\frac{1}{2}})$ iff $(p \leftrightarrow_Ł \sim p) \&_Ł \varphi(\frac{1}{2}/p) \in \mathrm{SAT}([0,1]_{Ł\Pi})$. □

Tautologousness and satisfiability in $[0,1]_{Ł\Pi}$ and $[0,1]_{Ł\Pi\frac{1}{2}}$ can be shown to be in $\mathbf{PSPACE}$ using a polynomial reduction to $\mathrm{Th}_\exists(\mathrm{RCF})$. Since the TAUT problems for both algebras are polynomially equivalent (and the same goes for SAT), we henceforth work only with $[0,1]_{Ł\Pi\frac{1}{2}}$.

THEOREM 6.5.2. $\mathrm{Th}_\forall([0,1]_{Ł\Pi\frac{1}{2}})$ *and* $\mathrm{Th}_\exists([0,1]_{Ł\Pi\frac{1}{2}})$ *are in* $\mathbf{PSPACE}$.

Proof. Both proofs are conducted by replacing atomic formulas in the language of $Ł\Pi\frac{1}{2}$-algebras with their equivalents in the ordered field of reals. We show that

$$\mathrm{Th}_\exists([0,1]_{Ł\Pi\frac{1}{2}}) \preceq_{\mathbf{P}} \mathrm{Th}_\exists(\mathrm{RCF}).$$

Let Φ be an existential sentence in the language of $Ł\Pi\frac{1}{2}$. Using Lemma 3.3.5, transform Φ into a formula Φ' of the form $\exists x_1 \dots \exists x_n(\Phi_1 \wedge \Phi_2)$, where Φ_1 is a Boolean

combination of atomic formulas $x_i = x_j, x_i \leq x_j, x_i < x_j$ for some pairs $1 \leq i, j \leq n$, while Φ_2 is a conjunction of identities without compound terms.
Define a formula Φ'' as $\exists x_1 \dots \exists x_n(\Phi_1 \wedge \Phi_2' \wedge \Phi_3)$, where

- Φ_1 is as before,
- Φ_3 is $0 \leq x_i \leq 1$ for each $1 \leq i \leq n$ (boundary conditions), and
- Φ_2' results from Φ_2 by processing all identities in Φ_2 $(i, j, k \in \{1, \dots, n\})$ in the following way:

  (i) keep any identity $x_i = c$, where c is a constant;

  (ii) replace any $x_i *_{Ł} x_j = x_k$ with $(x_i + x_j - 1 \leq 0 \wedge x_k = 0) \vee (x_i + x_j - 1 > 0 \wedge x_k = x_i + x_j - 1)$;

  (iii) replace any $x_i *_{\Pi} x_j = x_k$ with $x_i.x_j = x_k$;

  (iv) replace any $x_i \rightarrow_{Ł} x_j = x_k$ with $(x_i \leq x_j \wedge x_k = 1) \vee (x_i > x_j \wedge 1 - x_i + x_j = x_k)$;

  (v) replace any $x_i \rightarrow_{\Pi} x_j = x_k$ with $(x_i \leq x_j \wedge x_k = 1) \vee (x_i > x_j \wedge x_i * x_k = x_j)$.

Clearly Φ holds in $[0,1]_{Ł\Pi\frac{1}{2}}$ iff Φ'' holds in R; for replacement of Łukasiewicz connectives, see Lemma 4.1.1. Moreover, it is obvious that the translation process can be performed in time polynomial in $|\Phi|$. $\square$

Interestingly, a converse reducibility also holds in a stronger way; namely, we are going to show next that $\mathrm{Th}_\forall(\mathrm{RCF}) \preceq_{\mathbf{P}} \mathrm{TAUT}([0,1]_{Ł\Pi\frac{1}{2}})$.

Let us start with defining a bijection f of $(0,1)$ onto R. Take

$$f_{\mathrm{neg}}(x) = \frac{4x}{2x-1} \quad f_{\mathrm{pos}}(x) = \frac{4-4x}{2x-1}$$

Then the inverse functions to $f_{\mathrm{neg}}, f_{\mathrm{pos}}$ are

$$f_{\mathrm{neg}}^{-1}(x) = \frac{x}{2x-4} \quad f_{\mathrm{pos}}^{-1}(x) = \frac{x+4}{2x+4}$$

The function f is defined from $f_{\mathrm{neg}}, f_{\mathrm{pos}}$ as follows:

$$f(x) = \begin{cases} f_{\mathrm{neg}}(x) & \text{if } 0 < x < \frac{1}{2} \\ 0 & \text{if } x = \frac{1}{2} \\ f_{\mathrm{pos}}(x) & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

and the inverse function to f is

$$f^{-1}(x) = \begin{cases} f_{\mathrm{neg}}^{-1}(x) & \text{if } x < 0 \\ \frac{1}{2} & \text{if } x = 0 \\ f_{\mathrm{pos}}^{-1}(x) & \text{if } x > 0 \end{cases}$$

Using f, define an isomorphic copy of the ordered field of reals on $(0,1)$: let $\mathsf{R}^0 = \langle (0,1), +^0, \cdot^0, 0^0, 1^0 \rangle$, where for $x, y \in (0,1)$ the operations of R^0 are as follows:

$$\begin{aligned} x +^0 y &= f^{-1}(f(x) + f(y)) \\ x \cdot^0 y &= f^{-1}(f(x) \cdot f(y)) \\ 0^0 &= \tfrac{1}{2} \\ 1^0 &= \tfrac{5}{6} \end{aligned}$$

and $x \leq^0 y$ is strictly order-reversing on $(0, \frac{1}{2})$ and on $(\frac{1}{2}, 1)$ and order-preserving otherwise.

As R^0 is an isomorphic copy of the ordered field of reals, their theories coincide. We now define a function that assigns to each open RCF-formula Φ a term in the language of $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$. The function is defined by induction on formula structure.

A first observation to be made is that comparisons $=$, $\leq$ and $<$ on $[0,1]$ can be expressed by equations in the language of ŁΠ$\frac{1}{2}$-algebras. In particular, $x = y$ corresponds to $\triangle(x \leftrightarrow_{\text{Ł}} y) = 1$ (denote $t^{=}(x,y)$ the term on the left-hand side), $x \leq y$ corresponds to $\triangle(x \rightarrow_{\text{Ł}} y) = 1$ and $x < y$ corresponds to $\triangle(x \rightarrow_{\text{Ł}} y) \mathbin{\&_{\text{Ł}}} {\sim}\triangle(x \leftrightarrow_{\text{Ł}} y) = 1$.

Recall that for $x, y \in (0,1)$ we have $x \leq^0 y$ iff either $x \leq \frac{1}{2} \leq y$ or $y \leq x$ and either $x, y < \frac{1}{2}$ or $\frac{1}{2} < x, y$. So $x \leq^0 y$ on $(0,1)$ can be expressed as $t^{\leq_0}(x,y) = 1$, where $t^{\leq_0}(x,y)$ is

$$\triangle(x \vee {\sim}y \rightarrow_{\text{Ł}} \tfrac{1}{2}) \vee (\triangle(y \rightarrow_{\text{Ł}} x) \wedge ({\sim}\triangle(\tfrac{1}{2} \rightarrow_{\text{Ł}} x \vee y) \vee {\sim}\triangle(x \wedge y \rightarrow_{\text{Ł}} \tfrac{1}{2})))$$

Suppose Φ is an open RCF-formula. We will need to assume that Φ is without compound terms; Subsection 3.3 shows how to polynomially eliminate compound terms from a given formula (the statement there is given for existential sentences; apply it to a negation of the universal closure of Φ).

LEMMA 6.5.3. *The functions* 0^0, 1^0, $+^0$, $\cdot^0$ *are term-definable in* $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$.

Proof. Clearly $0^\star$ is $\frac{1}{2}$ and $1^\star$ is $\frac{5}{6}$.[20] For $+^0$, $\cdot^0$, it is enough to recall that these functions are piecewise rational; their definability in $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$ follows. □

So for each term t in the language of RCF, we have a defining term $t_\star$ in $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$; if Φ has no compound terms, then in particular for each term t in Φ, $|t_\star|$ is polynomial in $|t|$. Now assume Φ is an open RCF-formula without compound terms. We define t^Φ as follows:

- if Φ is $s = u$, then t^Φ is $t^{=}(s_\star, u_\star)$;
- if Φ is $s \leq u$, then t^Φ is $t^{\leq^0}(s_\star, u_\star)$;
- if Φ is $\neg\Theta$, then t^Φ is ${\sim}t^\Theta$;
- if Φ is $\Theta \wedge \Psi$, then t^Φ is $t^\Theta \wedge t^\Psi$.

LEMMA 6.5.4. *Let* Φ *be an open* RCF*-formula. Then* Φ *holds in the ordered field of reals iff* $t^\Phi = 1$ *holds in* $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$.

[20] See the proof of Lemma 6.5.7 for the definition of rationals in $[0,1]_{\text{Ł}\Pi\frac{1}{2}}$.

We have shown:

THEOREM 6.5.5. $\mathrm{Th}_\forall(\mathsf{R}) \preceq_{\mathbf{P}} \mathrm{THM}([0,1]_{\text{ŁΠ}\frac{1}{2}})$.

Combining the results, we get

THEOREM 6.5.6. $\mathrm{THM}(\text{ŁΠ}\tfrac{1}{2}) \approx_{\mathbf{P}} \mathrm{Th}_\forall(\mathsf{R})$.

We close this subsection by pointing out how logics with rational constants can be interpreted in the logic ŁΠ$\frac{1}{2}$ (and hence also ŁΠ).

LEMMA 6.5.7. $\mathsf{Q} \cap [0,1]$ *is polynomially term-definable in* $[0,1]_{\text{ŁΠ}\frac{1}{2}}$.

Proof. For each $q = \frac{k}{l} \in \mathsf{Q} \cap [0,1]$ we seek to find an ŁΠ$\frac{1}{2}$-expression φ such that for each evaluation e in $[0,1]_{\text{ŁΠ}\frac{1}{2}}$ we have $e(\varphi) = q$. Moreover, we demand that $|\varphi|$ be polynomial in $|q| \leq 2\log(l)$.

Choose $n \in \mathsf{N}$ the least such that $l < 2^n$; then $2^{n-1} \leq l < 2^n \leq 2l$, therefore $\log(l) < n \leq \log(l) + 1$ and $n \in O(\log(l))$.

First, for each $n \in \mathsf{N}$: $\frac{1}{2^n} = \frac{1}{2} *_\Pi \cdots *_\Pi \frac{1}{2}$ (n times); the number of factors is n.

To define $\frac{k}{2^n}$ for $k > 1$, we cannot use $\frac{k}{2^n} = \frac{1}{2^n} \oplus \cdots \oplus \frac{1}{2^n}$ (k times), because then the cardinality of the sum would be the value of k (which is exponential in $|k|$). Instead, consider that $k = \sum_{i \leq \lfloor \log(k) \rfloor} c_i.2^i$, where $(c_{\lfloor \log(k) \rfloor} \dots c_0)$ is the binary representation of k. For $k \leq 2^n$, define $\frac{k}{2^n} = \bigoplus_{i \leq \lfloor \log(k) \rfloor} (c_i *_\Pi \frac{1}{2^{n-i}})$. Here, the cardinality of the sum is $|k|$ and each summand is in $O(n)$.

Finally, put $\frac{k}{l} = \frac{k}{2^n} \rightarrow_\Pi \frac{l}{2^n}$. $\square$

The following corollary entails Theorem 6.3.6 concerning the logic $\Pi(\mathcal{Q})$; however, for Ł$(\mathcal{Q})$ and $\mathrm{G}(\mathcal{Q})$, we were able to obtain a better upper bound in Subsection 6.3 than the one implied by the following corollary.

COROLLARY 6.5.8. *Let* L *be one of* Ł, G, Π. *Then* $\mathrm{THM}(\mathrm{L}(\mathcal{Q})) \preceq_{\mathbf{P}} \mathrm{THM}(\text{ŁΠ}\tfrac{1}{2})$.

7 MTL and its axiomatic extensions

Despite the tumultuous research in the family of MTL and its extensions, complexity results leave much to be desired for those logics that do not also extend BL. It follows from [5] that MTL and its axiomatic extensions SMTL and IMTL are decidable since the corresponding varieties of algebras enjoy the finite embeddability property (FEP); in fact this implies that the universal theories of the respective varieties of algebras are decidable. Hence, due to (strong) completeness of MTL (IMTL, SMTL) w.r.t. the class of MTL-algebras (IMTL-algebras, SMTL-algebras respectively), both theorems and provability from finite theories for each of the three logics are decidable. Apart from that, theoremhood and provability in axiomatic extension ΠMTL of MTL is decidable (though ΠMTL-algebras do not have the FEP), as shown in [28].

The problem with improving this upper bound seems to be our insufficient knowledge of the structure of MTL-algebras (or even standard MTL-algebras). However, for particular (classes of) MTL-algebras, whose structure is known, usual complexity results can be obtained, as shown below.

Within this section, we work with the language $\{\&, \rightarrow, \wedge, \overline{0}\}$.

7.1 (Weak) nilpotent minimum logic

MTL was introduced in [12], along with its axiomatic extensions WNM (weak nilpotent minimum) and NM (nilpotent minimum). WNM is an axiomatic extension of MTL with the axiom

$$(\varphi \mathbin{\&} \psi \to \overline{0}) \vee (\varphi \wedge \psi \to \varphi \mathbin{\&} \psi)$$

and NM is the involutive extension of WNM, obtained by adding to WNM the axiom

$$\neg\neg\varphi \leftrightarrow \varphi.$$

Let us look at the semantics of these logics. A unary function $n\colon [0,1] \to [0,1]$ is a weak negation iff it is order-reversing, $n(0) = 1$, $n(1) = 0$, and $x \leq n(n(x))$ for all $x \in [0,1]$). Given a weak negation n, one defines

$$x *_n y = \begin{cases} 0 & \text{if } x \leq n(y) \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

Then $*_n$ is a left-continuous t-norm and the (MTL-)algebra $[0,1]_{*_n}$ is a standard WNM-algebra. If one starts with the function $1-x$ in the role of n, the resulting t-norm is

$$x *_{\mathrm{NM}} y = \begin{cases} 0 & \text{if } x + y \leq 1 \\ \min\{x,y\} & \text{otherwise} \end{cases}$$

and the corresponding standard algebra is denoted $[0,1]_{\mathrm{NM}}$. If one starts from a different involutive negation, then the whole algebra determined by it is isomorphic to $[0,1]_{\mathrm{NM}}$. The logic NM is strongly complete w.r.t. the algebra $[0,1]_{\mathrm{NM}}$.

Also the logic WNM enjoys strong completeness w.r.t. the above standard algebras, but this class is not tangible enough for our purpose. It turns out that we can prove completeness with respect to a narrower class.

For $k \in \mathsf{N} \setminus \{0\}$, define $I_k = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$. Let $S \subseteq I_k$ be arbitrary such that $\frac{i}{k} \in S$ iff $\frac{k-i-1}{k} \in S$ for $i \leq \frac{k}{2}$. Define on $[0,1]$ the following function n_k^S:

$$n_k^S(x) = \begin{cases} 1 - x & \text{if } x \in [\frac{i}{k}, \frac{i+1}{k}] \text{ for } \frac{i}{k} \in S \\ \frac{k-i-1}{k} & \text{otherwise.} \end{cases}$$

It is easy to check that n_k^S is a weak negation, which determines a corresponding standard WNM-algebra $[0,1]_{\mathrm{WNM}_k^S}$.

PROPOSITION 7.1.1 ([12]). *The class of* WNM*-chains is partially embeddable into the class of standard* WNM*-algebras* $[0,1]_{\mathrm{WNM}_k^S}$. *In particular, each n-element partial subalgebra of a* WNM*-chain is embeddable into a* $[0,1]_{\mathrm{WNM}_{n'}^S}$ *for some $n' \leq 2n+2$ and some choice of S.*

COROLLARY 7.1.2. *The variety* $\mathbb{WNM}$ *of* WNM*-algebras is generated by the class of algebras* $[0,1]_{\mathrm{WNM}_k^S}$ *(where $k \in \mathsf{N} \setminus 0$, S is an arbitrary subset of* $\{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$*) as a quasivariety, and the logic* WNM *enjoys finite strong standard completeness with respect to this class of algebras.*

We now explore complexity of the two logics; the results first appeared in [32].

THEOREM 7.1.3. $\mathrm{Th}_\forall([0,1]_{\mathrm{NM}})$ *is* $\mathbf{coNP}$*-complete.*

Proof. Hardness follows from Theorem 3.4.1. We show $\mathbf{NP}$-containment of the set $\mathrm{Th}_\exists([0,1]_{\mathrm{NM}})$.[21]

For any term $t(x_1,\dots,x_n)$ and for any evaluation e in $[0,1]_{\mathrm{NM}}$, the value $e(t')$ for any subterm $t' \preceq t$ will be among

$$V = \{0, e(x_1), \dots, e(x_n), \neg e(x_1), \dots, \neg e(x_n), 1\}$$

and that operations on V are order-determined, i.e., the value $e(t')$ is fully determined by the ordering of V. Note that the negation $\neg$ is involutive, and so $e(x) < e(y)$ iff $\neg e(y) < \neg e(x)$.

It follows that, given an existential sentence in the language of MTL, we may replace the existential quantification over all evaluations by an existential quantification over all orderings of variables occurring in the input formula and of their negations (with respect to the bottom and top elements of the algebra). The ordering must respect the involutive negation, i.e., the pairs $\langle e(x_1), \neg e(x_1)\rangle$, $\langle e(x_2), \neg e(x_2)\rangle$, …, $\langle e(x_n), \neg e(x_n)\rangle$ must be nested.

We describe a nondeterministic ALGORITHM EX-NM which accepts existential sentences valid in $[0,1]_{\mathrm{NM}}$. An existential sentence $\exists x_1 \dots \exists x_n \Phi(x_1,\dots,x_n)$ is given, where Φ is a Boolean combination of identities. Guess a linear ordering $\leq_0$ of the set $\{0, x_1, \dots, x_n, \neg x_1, \dots, \neg x_n, 1\}$, such that $0 \leq_0 x_i \leq_0 1$ for $1 \leq i \leq n$, and in such a way that the pairs induced by $\neg$ are nested. This is clearly polynomial in the input size. Then compute the value of each identity in Φ: on the basis of $\leq_0$, evaluate all terms and subsequently also all atomic formulas. Then compute the value of the Boolean combination in Φ, accept iff this value is 1. □

Using finite strong standard completeness theorem for NM, proved in [12], we get

COROLLARY 7.1.4.

(i) $\mathrm{SAT}_{(\mathrm{pos})}([0,1]_{\mathrm{NM}})$ *is* $\mathbf{NP}$*-complete.*

(ii) $\mathrm{TAUT}_{(\mathrm{pos})}([0,1]_{\mathrm{NM}})$ *and* $\mathrm{CONS}([0,1]_{\mathrm{NM}})$ *are* $\mathbf{coNP}$*-complete.*

(iii) $\mathrm{THM}(\mathrm{NM})$ *and* $\mathrm{CONS}(\mathrm{NM})$ *are* $\mathbf{coNP}$*-complete.*

THEOREM 7.1.5. *For each choice of k and S,* $\mathrm{Th}_\forall([0,1]_{\mathrm{WNM}_k^S})$ *is* $\mathbf{coNP}$*-complete.*

Proof. Let k and S be fixed and let $[0,1]_{\mathrm{WNM}_k^S}$ be the standard WNM-algebra given by these parameters. Hardness follows from Theorem 3.4.1. We show $\mathbf{NP}$-containment of $\mathrm{Th}_\exists([0,1]_{\mathrm{WNM}_k^S})$, in a manner similar to the proof of Theorem 7.1.3.

Clearly for any term $t(x_1,\dots,x_n)$ and for any evaluation e in $[0,1]_{\mathrm{WNM}_k^S}$, the value $e(t')$ for any subterm $t' \preceq t$ will be among

$$V = \{0, e(x_1), \dots, e(x_n), \neg e(x_1), \dots, \neg e(x_n), \neg\neg e(x_1), \dots, \neg\neg e(x_n), 1\}$$

and operations on V are order-determined, i.e., the value $e(t')$ is fully determined by the ordering of V. Therefore again, we take an existential quantification over particular orderings.

[21] Cf. also the proof of Theorem 5.1.1.

We describe a nondeterministic ALGORITHM EX-WNM which accepts existential sentences valid in $[0,1]_{\mathrm{WNM}_k^S}$. Let an existential sentence $\exists x_1 \ldots \exists x_n \Phi(x_1, \ldots, x_n)$ be given, where Φ is a Boolean combination of identities. Guess a linear ordering $\leq_0$ of a set consisting of 0, 1 and terms x_i, $\neg x_i$ and $\neg\neg x_i$ for $1 \leq i \leq n$ such that $0 \leq_0 x_i \leq_0 1$ for $1 \leq i \leq n$, such that the pairs $\langle \neg x_1, \{x_1, \neg\neg x_1\}\rangle$, $\langle \neg x_2, \{x_2, \neg\neg x_2\}\rangle$, $\ldots$, $\langle \neg x_n, \{x_n, \neg\neg x_n\}\rangle$ are nested, satisfy $x_i \leq \neg\neg x_i$ and conform to functionality of $\neg$, namely, if for any expressions a, b we have $a =_0 b$, then $\neg a =_0 \neg b$. This can be done in time polynomial in $|\Phi|$. Then compute the value of each atomic formula in Φ: on the basis of $\leq_0$, evaluate all terms and subsequently also all atomic formulas. Then compute the value of the Boolean formula Φ, accept iff the value is 1. $\square$

THEOREM 7.1.6.

(i) $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{WNM})$ *is* $\mathbf{NP}$*-complete.*

(ii) $\mathrm{TAUT}_{(\mathrm{pos})}(\mathbb{WNM})$ *and* $\mathrm{CONS}(\mathbb{WNM})$ *are* $\mathbf{coNP}$*-complete.*

(iii) $\mathrm{THM}(\mathrm{WNM})$ *and* $\mathrm{CONS}(\mathrm{WNM})$ *are* $\mathbf{coNP}$*-complete.*

Proof. Hardness follows from Theorem 3.4.1. Due to Proposition 7.1.1, we may limit ourselves to algebras $[0,1]_{\mathrm{WNM}_k^S}$ for k bounded polynomially by the length of the input, as follows. (i) Given a formula φ with n variables as an instance of $\mathrm{SAT}_{(\mathrm{pos})}(\mathbb{WNM})$, guess a $k \leq 2n+2$ and ask whether $\varphi \in \mathrm{SAT}_{(\mathrm{pos})}([0,1]_{\mathrm{WNM}_k^S})$; the latter is in $\mathbf{NP}$. This algorithm solves the problem correctly: If a formula φ is (positively) satisfiable in any WNM-algebra, then it is also (positively) satisfiable in some WNM-chain. The evaluation of subterms forms a partial subalgebra, which is embeddable into an algebra $[0,1]_{\mathrm{WNM}_k^S}$ for some $k \leq 2n+2$ and some choice of S by Proposition 7.1.1. Hence $\varphi \in \mathrm{SAT}_{(\mathrm{pos})}(\mathbb{WNM})$ iff there are k and S (both polynomial in $|\varphi|$) such that $\varphi \in \mathrm{SAT}_{(\mathrm{pos})}([0,1]_{\mathrm{WNM}_k^S})$. (ii) If a formula is not a tautology, then this shows in some WNM-chain. The rest is as in (i). The case for CONS is analogous. $\square$

We remark that one can obtain these complexity bounds also for tautologies of each of $[0,1]_{\mathrm{NM}}$ and of $[0,1]_{\mathrm{WNM}_k^s}$ for $k \in \mathsf{N} \setminus 0$ and a choice of S enriched with constants for Q; hence, e.g., the logic $\mathrm{NM}(\mathcal{Q})$ (expansion of NM with rational constants, axiomatized by adding bookkeeping axioms as valid in the standard NM-algebra) is $\mathbf{coNP}$-complete. Details can be found in [15].

8 Overview of results and open problems

Table 1 gives a summary of results obtained for logics (i.e., theoremhood and provability from finite theories); it omits results obtained for the corresponding classes of algebras (i.e., satisfiability, tautologousness, and finite consequence relation), both for spatial reasons and because these results do not add much new information.

The presentation is rather condensed, therefore it merits some explanation. The rows of the table represent logics considered in this chapter. For each of these, the column entries specify the complexity result for theoremhood/provability (these are, for all cases, identical). The '–' character means the logic has not been considered within this chapter. However, one can still use Lemma 3.1.2 to obtain bounds on complexity of fragments/expansions.

Logic L	THM(L)	CONS(L)	0-free fragment	$\triangle$ expansion	Q expansion	$\sim$ expansion
BL	**coNP**-c.	**coNP**-c.	**coNP**-c.	**coNP**-c.	–	–
SBL	**coNP**-c.	**coNP**-c.	**coNP**-c	**coNP**-c.	–	**coNP**-c.
Ł	**coNP**-c.	**coNP**-c.	**coNP**-c.	**coNP**-c.	**coNP**-c.	–
L $\supset$Ł	**coNP**-c.	**coNP**-c.	–	–	–	–
G	**coNP**-c.	**coNP**-c.	**coNP**-c.	**coNP**-c.	**coNP**-c.	**coNP**-c.
Π	**coNP**-c.	**coNP**-c.	**coNP**-c.	**coNP**-c.	$\in$ **PSPACE**	**coNP**-c.
L$(*) \supset$ BL	**coNP**-c.	**coNP**-c.	–	**coNP**-c.	–	**coNP**-c.
Ł$\Pi\frac{1}{2}$	$\in$ **PSPACE**	$\in$ **PSPACE**	–	–	–	–
MTL	decidable	decidable	decidable	–	–	–
IMTL	decidable	decidable	decidable	–	–	–
SMTL	decidable	decidable	decidable	–	–	–
ΠMTL	decidable	decidable	–	–	–	–
NM	**coNP**-c.	**coNP**-c.	–	–	**coNP**-c.	–
WNM	**coNP**-c.	**coNP**-c.	–	–	–	–

Table 1. Complexity of Propositional Logics

Open problems and directions. As already observed, BL and its extensions have received quite a thorough treatment as to basic questions of computational complexity of theoremhood and provability in the propositional case. (Chapter XI addresses arithmetical complexity of first-order calculi.) From these basic results, roughly speaking one can move in several directions.

The greatest itch, no doubt, is the lack of complexity results for MTL and its extensions (that are not at the same time extensions of BL). The same is true for semilinear extensions of other substructural logics introduced in this book, some of whom are not known to be decidable; this is the case of Uninorm Logic UL. Another track of research can be taken by modifying the language. This has been done in a degree, but the picture is still very incomplete both ways—shifting to fragments of the basic language, or to its expansions, or both. Moreover, one can pass from tautologousness and provability to more intricate problems involving propositional or first-order formulas. This includes admissible rules, quantified propositional formulas, etc. One can also look at (fragments of) algebraic theories for classes of algebras among the equivalent algebraic semantics for the logics. This is what has been, to a degree, presented in this chapter, but our approach has been a utilitarian one, while the topic is of independent interest.

9 Historical remarks and further reading

Earlier chapters of this book present fuzzy logics within the broader framework of substructural logics. In particular, the concept of *semilinearity* is introduced as an essential trait of fuzzy logics, and it is shown how some prototypical fuzzy logics can be obtained as semilinear extensions of well-known substructural logics, or as axiomatic extensions thereof. To exemplify, MTL is the semilinear extension of the logic $\mathrm{FL_{ew}}$ and G is the semilinear extension of intuitionistic logic INT.

Semilinearity put aside, there is a substantial amount of complexity results on substructural logics (out of the scope of this chapter). Remarkably, the full Lambek calculus FL, and logics obtained by adding the structural rules of exchange, weakening and contraction to FL, possess streamlined Gentzen-style calculi, some of whom admit proofsearch in $\mathbf{PSPACE}$ (due to the *subformula property:* each formula occurring in the proof of φ is a subformula of φ). This is the case of FL itself, as well as $\mathrm{FL_{ew}}$; for both these logics, theoremhood is in fact $\mathbf{PSPACE}$-complete. Unfortunately, the virtue of polynomial-space proof search seems to be lost when semilinearity is assumed. Instead, we have relied on completeness with respect to a suitable class of algebras and on sufficient understanding of the structure of these algebras. It has hopefully been demonstrated why, for many logics discussed in this chapter, one can work with the universal fragment of the full algebraic theory of standard algebras to obtain the same complexity result as for the equational, or the quasiequational, fragment. We stress that in a general case, provability from finite theories may be computationally much harder than theoremhood. For example, provability from finite theories in FL is undecidable.

Interestingly, the paper [6] also shows $\mathbf{PSPACE}$-containment for theoremhood and provability in the logic $\mathrm{GBL_{ewf}}$. This logic is obtained from $\mathrm{FL_{ew}}$ by adding the divisibility axiom; adding semilinearity to $\mathrm{GBL_{ewf}}$, one obtains BL. It is shown in the paper that provability in $\mathrm{GBL_{ewf}}$ is $\mathbf{PSPACE}$-hard.

Finally, let us mention a classic result of R. Statman [41]: theoremhood in intuitionistic logic INT (which is exactly $\mathrm{FL}_{\mathrm{ewc}}$) is $\mathbf{PSPACE}$-complete (and, because this logic enjoys the deduction theorem, so is provability).

We now turn to the story of unravelling computational complexity of propositional fuzzy logics presented in this chapter. It begins with D. Mundici's result establishing $\mathbf{NP}$-completeness of satisfiability of propositional formulas in the standard Lukasiewicz algebra, presented in [37] in 1987. Mundici's method was proving an upper bound by reasoning about the functions represented by formulas. Our method is a rather straightforward reduction to the INEQ problem; however, we try to recover the geometry behind the small-model theorem that gives upper bounds for both LP and INEQ. Thus our approach rather resembles the proof presented in [20], where a reduction to the bounded version of MIP problem is used. Mundici's pioneering work laid a basis for results in other logics given by continuous t-norms, to be obtained much later; at the time of Mundici's result, mathematical fuzzy logic, as a homogeneous branch of formal nonclassical logics, had yet to be developed.

It was not until almost ten years later that the fundamentals of Hájek's basic logic were laid, which was the starting point of a focused and tumultuous research involving also computational complexity issues. Theoremhood was addressed via tautologousness in the corresponding (class of) standard algebras indicated by completeness theorems. In our presentation, we use the ideas of the proofs in the respective papers to obtain a more general result, shifting from identities in the algebraic theory to quasiidentities and universal theories.

In 1998, the paper [2] determined the complexity of theoremhood in product logic, showing it to be $\mathbf{coNP}$-complete using a reduction to Łukasiewicz logic. This paper also mentions Gödel logic, but disclaims the result as common knowledge. As to the fact that (positively) satisfiable formulas and positive tautologies in the two standard algebras coincide with classical satisfiability and tautologousness respectively, this step was taken in [21]. The last reference gives a good overview of complexity results for Łukasiewicz, Gödel, and product logics.

In 2002, the paper [3] settled the complexity of theoremhood for propositional BL, relying on its standard completeness, Mostert–Shields theorem, and a simple but crucial observation on the number of components needed to represent potential counterexamples to tautologousness (incarnated here as Lemma 5.2.3). Later in the same year, a similar method was adapted to work also for any propositional logic given by a standard BL-algebra, in the paper [25]. The latter result, which implies there are only countably many subvarieties of $\mathbb{BL}$ given by a single standard BL-algebra, was anticipated in the strong result of [1] which characterizes those standard BL-algebras that generate the full variety $\mathbb{BL}$.

Another complexity result in the family of BL and its extensions concerned axiomatic extensions of Łukasiewicz logic and was presented in [8]; it relies on Komori's characterization of subvarieties of $\mathbb{MV}$, presented in [31]. The paper [30] also belongs to this family, showing that admissibility of rules in Łukasiewicz logic is in $\mathbf{PSPACE}$.

For logics in expanded or restricted languages, results are rather fragmentary and some are negative. The computational complexity results for falsehood-free logics, as presented here, are covered by the comprehensive paper [13]. For involutive negations added as an independent new connective, the result [26] shows that theoremhood and provability in a logic given by some standard SBL-algebra and the class of all involutive negations is $\mathbf{coNP}$-complete. However, this result says nothing about individual combinations of standard algebras and involutive negations, most notably, it says nothing about $[0,1]_{\text{Ł}\Pi}$. As to the logics $\text{Ł}\Pi$ and $\text{Ł}\Pi\frac{1}{2}$, containment in $\mathbf{PSPACE}$ for theoremhood was shown in [23] using the universal/existential fragment of the RCF-theory; the latter was shown to be in $\mathbf{PSPACE}$ in [7]. The result of polynomial equivalence of theorems of $\text{Ł}\Pi\frac{1}{2}$ and the universal fragment of the RCF theory comes from [33]. Intriguingly, these problems are apparently not known to be $\mathbf{PSPACE}$-complete. Computational complexity for logics with rational constants has been addressed in [22]. In fact, the scope of the paper is slightly broader than presented here, dealing also with finite sums of particular properties. Moreover, rational expansions of standard WNM-algebras have been studied in [15].

For MTL and its extensions (expansions), results are mostly limited to decidability, obtained as in [5] for MTL and its axiomatic extensions SMTL and IMTL via finite embeddability property, and in [28] for $\Pi\mathrm{MTL}$. Results given here on NM and WNM are from [32].

We also mention Abelian logic, the logic of Abelian ℓ-groups (see [35]). This logic not only belongs to the family of semilinear substructural logics, but relates both to Łukasiewicz logic (see [34]) and to product logic (because of its semantics). This logic proves exactly all expressions φ in the ℓ-group language for which $\varphi \geq 0$ holds in all ℓ-groups, or equivalently, in Z. Using a variant of the small-model theorem for integers, as given in Section 2.5, one can show that the universal theory of the additive group on Z (and hence, due to a classic result of Y.S. Gurevich and A.I. Kokorin, of all Abelian o-groups) is in $\mathbf{coNP}$. V. Weispfenning ([42]) extended this result to ℓ-groups by extending the small-model theorem to incorporate also the width of the lattice ordering: if a universal formula Φ is not valid in the class of all ℓ-groups, then this can be shown in an ℓ-group with of width polynomially bounded by $|\Phi|$. Hence, universal theory of Abelian ℓ-groups is $\mathbf{coNP}$-complete.

To complement the account, we relied on [38] as a reference text on computational complexity.

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Chapter XI: Arithmetical Complexity of First-Order Fuzzy Logics

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1 Introduction

This chapter presents several results on the complexity of predicate fuzzy logics, understood as first-order versions of ($\triangle$-)core propositional fuzzy logics (see the previous chapter). We will discuss several semantics for them, and for each semantics we will try to classify the complexity (in the sense of arithmetical hierarchy) of the sets of tautologies (formulae which are always evaluated into 1), of the positive tautologies (formulae which are always evaluated into a strictly positive value), of satisfiable formulae (formulae which are evaluated into 1 by some evaluation) and of positively satisfiable formulae (formulae which are evaluated into a strictly positive number by some evaluation).

The most important semantics we will discuss are the *general semantics* (given by all chains for the logic), the *real semantics* (given by all the chains for the logic having as lattice reduct $[0, 1]$) and the *standard semantics* (that is, the intended semantics for the logic, in some case coinciding with the real semantics, but in general a proper subclass of the real semantics; we will be more precise in Section 3). We will also consider the *rational semantics* (given by the rational valued chains for the logic), the *finite semantics* (given by all finite chains for the logic), the *complete chain semantics* (given by all complete chains for the logic), and the *witnessed semantics* (given by all models in which the truth value of each universally quantified formula is the minimum of truth values of instances and analogously for existential quantifier and maximum). Finally we will also discuss fragments of predicate logics, like the falsum-free fragment, the fragment with negation, implication and quantifiers and the monadic fragment.

The results, collected in tables present throughout the chapter, show that our predicate logics, with a very few exceptions (like the monadic fragment of classical logic), turn out to be undecidable (we will prove a quite general undecidability result in Section 2). Hence, the main problem we will address in this chapter is not whether a given predicate logic is decidable or not, but rather how undecidable it is, i.e. what is its undecidability degree.

For the general semantics, the undecidability degrees are low ($\mathbf{\Sigma_1}$ for tautologicity and $\mathbf{\Pi_1}$ for satisfiability). For the standard semantics, it depends: in the cases where we have standard completeness, like MTL or IMTL, the undecidability degrees are trivially as in the general semantics, in other cases, like Łukasiewicz first-order logic, the undecidability degrees are higher but still in the arithmetical hierarchy, while in product logic or in BL logic both tautologicity and satisfiability for the standard semantics fall outside the arithmetical hierarchy.

In this chapter, a basic knowledge of first-order fuzzy logics and arithmetic is assumed. One can find the necessary background in the former chapters of this handbook. In particular, for the arithmetical hierarchy, see the notes in Chapter X, Section 2.2. Recall classical logic with its deductive system and its models: a formula is provable if and only if it is a tautology (true in all models—completeness of classical logic); the set of all such formulae is recursively enumerable (i.e. in $\mathbf{\Sigma_1}$ in the sense of the arithmetical hierarchy) and, if its language has at least one predicate whose arity is at least binary, then it is $\mathbf{\Sigma_1}$-complete.[1] Similarly for theories like Peano arithmetic PA: the set of its provable formulae (equal to the set of formulae true in all its models) is $\mathbf{\Sigma_1}$-complete. PA has its *standard model:* the structure $\mathbf{N}$ of natural numbers with addition and multiplication. And the set of all formulae true in this standard model is extremely undecidable, it is outside the arithmetical hierarchy.

As regards to axiomatic systems for arithmetic, we will mainly use Robinson's arithmetic Q^+. Its axioms are those of equality plus the following ones:

$$
\begin{aligned}
&(\forall x)(S(x) \neq 0)\\
&(\forall x)(\forall y)(S(x) = S(y) \to (x = y))\\
&(\forall x)(\neg x = 0 \to (\exists y)(S(y) = x))\\
&(\forall x)(x + 0 = x)\\
&(\forall x)(\forall y)(x + S(y) = S(x + y))\\
&(\forall x)(x \cdot 0 = 0)\\
&(\forall x)(\forall y)(x \cdot S(y) = x \cdot y + x)\\
&(\forall x)(\forall y)(x \leq y \leftrightarrow \exists z(y = x + z))\\
&(\forall x)(\forall y)(x \leq y \vee y \leq x)\\
&(\forall x)(\forall y)(\forall z)((x \leq y \wedge y \leq z) \to x \leq z)\\
&(\forall x)(\forall y)((x \leq y \wedge y \leq x) \to x = y)\\
&(\forall x)(\forall y)((x \leq y \wedge (x \neq y)) \leftrightarrow S(x) \leq y).
\end{aligned}
$$

For every natural number n, $\overline{n}$ is defined by $\overline{0} = 0$ and $\overline{n+1} = S(\overline{n})$. Peano arithmetic PA is obtained from Q^+ by adding for any formula $\phi(x_1, \ldots, x_n, y)$ the following induction schema: $\forall x_1 \ldots \forall x_n((\phi(x_1, \ldots, x_n, 0) \wedge \forall y(\phi(x_1, \ldots, x_n, y) \to \phi(x_1, \ldots, x_n, S(y)))) \to \forall y \phi(x_1, \ldots, x_n, y))$.

This chapter is organized as follows: Section 2 contains abstract results on any semantics given by a class of linearly ordered algebras and in particular results on the general semantics of fuzzy predicate logics. Section 3 is devoted to the complexity of standard semantics of logics extending the basic fuzzy logic $\mathrm{BL}\forall$, particularly Łukasiewicz, Gödel and product logic, $\mathrm{SBL}\forall$ and $\mathrm{BL}\forall$ itself. Section 4 deals with the complexity of semantics given by finite and rational chains and their relations to the real-valued semantics. Section 5 completes the picture by presenting several further results on arithmetical hierarchy of first-order fuzzy logics: complexity of semantics of witnessed models, complexity of semantics of completely ordered models, results on some fragments of fuzzy

[1] Please do not confuse the completeness of a theory in a logic with the $\mathbf{\Sigma_n}$-completeness or $\mathbf{\Pi_n}$-completeness of a set of formulae in the sense of arithmetical hierarchy. Such set is $\mathbf{\Sigma_n}$-complete if it is in $\mathbf{\Sigma_n}$ and is $\mathbf{\Sigma_n}$-hard; similarly for $\mathbf{\Pi_n}$.

logics (with restricted set of connectives or only with monadic predicates), and results on axiomatic extensions of Łukasiewicz. We conclude the section with a list of open problems. Finally, Section 6 completes the chapter with some historical and bibliographical notes for further reading.

2 General results and general semantics

This section considers equality-free first-order fuzzy logics in the full vocabulary $\mathcal{P}$, i.e. containing functional and relational symbols of all arities (monadic fragments are addressed in Section 5). Moreover, we will work with arbitrary classes of linearly ordered MTL-algebras or their expansions corresponding to ($\triangle$-)core fuzzy logics in richer languages. These classes will be usually denoted by $\mathbb{K}$, and we will always assume (to avoid dealing with non-interesting trivial cases) that they are not empty and do not contain the trivial algebra. When $\mathbb{K}$ is a class of (expansions of) MTL-chains and no further condition is assumed, we just say for simplicity that it is *a class of chains*.

DEFINITION 2.0.1. *Given a class* $\mathbb{K}$ *of chains we define the following sets of sentences:*

$$\begin{aligned}
\mathrm{TAUT}(\mathbb{K}) = & \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \textit{ for every } \boldsymbol{A} \in \mathbb{K} \textit{ and every safe } \boldsymbol{A}\textit{-structure } \mathbf{M}, \\
& \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}\}. \\
\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}) = & \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \textit{ for every } \boldsymbol{A} \in \mathbb{K} \textit{ and every safe } \boldsymbol{A}\textit{-structure } \mathbf{M}, \\
& \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \overline{0}^{\boldsymbol{A}}\}. \\
\mathrm{SAT}(\mathbb{K}) = & \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \textit{ there exist } \boldsymbol{A} \in \mathbb{K} \textit{ and a safe } \boldsymbol{A}\textit{-structure } \mathbf{M} \textit{ such that} \\
& \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}\}. \\
\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}) = & \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \textit{ there exist } \boldsymbol{A} \in \mathbb{K} \textit{ and a safe } \boldsymbol{A}\textit{-structure } \mathbf{M} \textit{ such that} \\
& \|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \overline{0}^{\boldsymbol{A}}\}.
\end{aligned}$$

DEFINITION 2.0.2. *Let* L *be a ($\triangle$-)core fuzzy logic. Instead of* $\mathrm{TAUT}(\mathbb{K})$ *we write*

- $\mathrm{genTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is the class of all* L*-chains (the* general semantics*).*
- $\mathrm{realTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is the class of all* real L*-chains, i.e. whose lattice reduct is the real unit interval* $[0,1]$.
- $\mathrm{stTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is a subclass of real* L*-chains which are considered the* intended real semantics *(or* standard semantics*) of* L.
- $\mathrm{ratTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is the class of all* rational L*-chains, i.e. whose lattice reduct is the rational unit interval* $[0,1]^{\mathbb{Q}}$.
- $\mathrm{intratTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *consists of a single rational* L*-chain which is considered the* intended rational L*-chain.*
- $\mathrm{finTAUT}(\mathrm{L}\forall)$ *if* $\mathbb{K}$ *is the class of all* finite L*-chains.*

We define analogous notations for the sets $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$*,* $\mathrm{SAT}(\mathbb{K})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *in all the cases.*

Given a ($\triangle$-)core fuzzy logic L, we write $\Sigma \models_{\mathrm{real}(\mathrm{L}\forall)} \varphi$ meaning that $\Sigma \models_{\mathbb{K}} \varphi$ when $\mathbb{K}$ is the class consisting of all real L-chains; moreover $\mathrm{realCons}(\mathrm{L}\forall, \Sigma)$ denotes the set $\{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid \Sigma \models_{\mathrm{real}(\mathrm{L}\forall)} \varphi\}$, and the analogous definitions for the other semantics.

LEMMA 2.0.3 ([5]). *Let* L *be a ($\triangle$-)core fuzzy logic. If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are* L*-chains such that there is a* σ*-embedding (i.e. an embedding preserving all existing infima and suprema) from* $\boldsymbol{A}$ *into* $\boldsymbol{B}$*, then:*

1. $\mathrm{TAUT}(\boldsymbol{B}) \subseteq \mathrm{TAUT}(\boldsymbol{A})$,
2. $\mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{B}) \subseteq \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A})$,
3. $\mathrm{SAT}(\boldsymbol{A}) \subseteq \mathrm{SAT}(\boldsymbol{B})$,
4. $\mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A}) \subseteq \mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{B})$.

The negation operation $\neg a = a \to \overline{0}$ allows us to obtain several easy but useful relations between sets of tautologies and satisfiable sentences. We choose a rather general formulation to cope with other possible negations in logics expanded with extra connectives as those presented in Chapter VIII.

LEMMA 2.0.4. *Let* $\mathbb{K}$ *be a class of chains and let* $\sim$ *be an operation present in all members of* $\mathbb{K}$ *such that for every* x*,* $\sim x = 1$ *iff* $x = 0$*. Then for every* $\varphi \in \mathrm{Sent}_{\mathcal{P}}$*:*

1. $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\sim\varphi \notin \mathrm{SAT}(\mathbb{K})$,
2. $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\sim\varphi \notin \mathrm{TAUT}(\mathbb{K})$.

LEMMA 2.0.5. *Let* $\mathbb{K}$ *be a class and let* $\sim$ *be an operation present in all members of* $\mathbb{K}$ *such that for every* x*,* $\sim x = 0$ *iff* $x = 1$*. Then for every* $\varphi \in \mathrm{Sent}_{\mathcal{P}}$*:*

1. $\varphi \in \mathrm{SAT}(\mathbb{K})$ *iff* $\sim\varphi \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$,
2. $\varphi \in \mathrm{TAUT}(\mathbb{K})$ *iff* $\sim\varphi \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

The previous lemma applies, in particular, when $\sim$ is an involutive negation (i.e. $\sim\sim x = x$ for every x), e.g. when $\mathbb{K}$ can is a class of expansions of IMTL-chains.

LEMMA 2.0.6. *Let* $\mathbb{K}$ *be a class of chains and let* $\sim$ *be an operation present in all members of* $\mathbb{K}$ *such that for every* x*,* $\sim\sim x = 1$ *iff* $x > 0$*. Then for every* $\varphi \in \mathrm{Sent}_{\mathcal{P}}$*:*

1. $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\sim\sim\varphi \in \mathrm{TAUT}(\mathbb{K})$,
2. $\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *iff* $\sim\sim\varphi \in \mathrm{SAT}(\mathbb{K})$.

The previous lemma applies, in particular, when $\sim$ is a strict negation (i.e. $\sim x = \overline{0}$ for every $x \neq \overline{0}$), e.g. when $\mathbb{K}$ can is a class of expansions of SMTL-chains. For chains with $\triangle$ we have the following:

LEMMA 2.0.7. *Let $\mathbb{K}$ be a class of chains with the $\triangle$ operation. Then for every sentence φ we have the following relations:*

1. $\varphi \in \mathrm{SAT}(\mathbb{K})$ *iff* $\neg\triangle\varphi \notin \mathrm{TAUT}(\mathbb{K})$,
2. $\varphi \in \mathrm{TAUT}(\mathbb{K})$ *iff* $\neg\triangle\varphi \notin \mathrm{SAT}(\mathbb{K})$ *iff* $\neg\triangle\varphi \notin \mathrm{SAT_{pos}}(\mathbb{K})$,
3. $\varphi \in \mathrm{TAUT_{pos}}(\mathbb{K})$ *iff* $\neg\triangle(\neg\varphi) \in \mathrm{TAUT}(\mathbb{K})$,
4. $\varphi \in \mathrm{SAT_{pos}}(\mathbb{K})$ *iff* $\neg\triangle(\neg\varphi) \in \mathrm{SAT}(\mathbb{K})$.

We can obtain some lower bounds for the complexity of some of these problems. In case of the sets of satisfiable sentences, it is easy that they are always $\mathbf{\Pi_1}$-hard.

PROPOSITION 2.0.8. *For every class $\mathbb{K}$ of chains, $\mathrm{SAT}(\mathbb{K})$ is $\mathbf{\Pi_1}$-hard.*

Proof. Recall that we assume $\mathbb{K}$ to be non-empty. If φ is a sentence and $\{P_i \mid 1 \le i \le n\}$ are the predicate symbols from $\mathcal{P}$ appearing in φ, we define the sentence $Crisp(\varphi) = \bigwedge_{1\le i\le n}(\forall\overrightarrow{x})(P_i(\overrightarrow{x}) \vee \neg P_i(\overrightarrow{x}))$. Recall that $\boldsymbol{B}_2$ denotes the Boolean algebra of two elements. Now just observe that for every $\varphi \in \mathrm{Sent}_{\mathcal{P}}$, $\varphi \in \mathrm{SAT}(\boldsymbol{B}_2)$ iff $Crisp(\varphi)\&\varphi \in \mathrm{SAT}(\mathbb{K})$, and since the satisfiability problem in classical logic is $\mathbf{\Pi_1}$-hard so it must be $\mathrm{SAT}(\mathbb{K})$. □

Now we consider the TAUT problems. In the sequel, for every sentence φ, 2φ denotes the sentence $\neg((\neg\varphi)\ \&\ (\neg\varphi))$.

LEMMA 2.0.9. *Let* L *be any ($\triangle$-)core fuzzy logic. For every sentence φ, we have that* $2\varphi \vee 2(\neg\varphi) \in \mathrm{genTAUT}(\mathrm{L}\forall)$.

Proof. Let $\boldsymbol{A}$ be an L-chain and $\mathbf{M}$ an $\boldsymbol{A}$-model. If $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \le \|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$, then we have $\|(\neg\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. If $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$, then $\|(\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. (Observe that $\neg x \le x$ implies $\neg x \le \neg\neg x$, thus $\neg x \to (\neg x \to \overline{0}) = \overline{1}$, hence $(\neg x)^2 \to \overline{0} = \overline{1}$.) In either case we have $\|\neg(\neg\varphi)^2 \vee \neg(\neg\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. □

DEFINITION 2.0.10. *Let φ be a classical sentence. Consider its prenex normal form in classical logic, $Q_1x_1 \dots Q_nx_n\ \psi(x_1,\dots,x_n)$, where ψ is a lattice combination of literals. We define a formula $\varphi^\star$ by induction as follows: if φ is a literal, then $\varphi^\star = 2\varphi$; * commutes with quantifiers, $\wedge$ and $\vee$.*

LEMMA 2.0.11. *Let φ be a lattice combination of literals,* L *be a ($\triangle$-)core fuzzy logic and $\mathbb{K}$ a class of* L*-chains. The following are equivalent:*

(1) φ is a classical propositional tautology,

(2) $\varphi^\star$ is an L*-tautology,*

(3) $\varphi^\star$ is a tautology for every chain in $\mathbb{K}$,

(4) $\varphi^\star$ is a positive tautology for every chain in $\mathbb{K}$.

Proof. Recall that by our standing assumption $\mathbb{K}$ is non-empty. $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are obvious. We prove $(1) \Rightarrow (2)$. By distributivity, φ can be equivalently written as $\bigwedge_{i=1}^{n} \bigvee_{j=1}^{n_i} \alpha_{i,j}$, where $\alpha_{i,j}$ are literals. Thus, φ is a classical tautology iff for every $i \in \{1, \ldots, n\}$, $\bigvee_{j=1}^{n_i} \alpha_{i,j}$ is a classical tautology. This is the case if for every $i \in \{1, \ldots, n\}$ there are $j_1, j_2 \in \{1, \ldots, n_i\}$ such that $\alpha_{i,j_1} = \neg\alpha_{i,j_2}$. Hence, $2\alpha_{i,j_1} \vee 2\alpha_{i,j_2}$ is an L-tautology by previous lemma and, since this formula implies $\bigvee_{j=1}^{n_i} 2\alpha_{i,j}$, we have that $\varphi^\star$ is an L-tautology. We finally prove $(4) \Rightarrow (1)$ by contraposition. If φ is not a classical propositional tautology, then there is an evaluation e on $\boldsymbol{B}_2$ such that $e(\varphi) = 0$. Since $\varphi^\star$ and φ are equivalent in classical logic, we also have $e(\varphi^\star) = 0$. Now, given any $\boldsymbol{A} \in \mathbb{K}$, it is clear that e can also be seen as an evaluation on $\boldsymbol{A}$ and $e(\varphi^\star) = \overline{0}^{\boldsymbol{A}}$. $\square$

LEMMA 2.0.12. *Let $\varphi = (\exists x_1) \ldots (\exists x_n)\ \psi(x_1, \ldots, x_n)$, where ψ is a lattice combination of literals, be a purely existential formula,* L *be a ($\triangle$-)core fuzzy logic and $\mathbb{K}$ a class of* L*-chains. The following are equivalent:*

(1) $\varphi \in \mathrm{TAUT}(\boldsymbol{B}_2)$,

(2) $\varphi^\star \in \mathrm{genTAUT}(\mathrm{L}\forall)$,

(3) $\varphi^\star \in \mathrm{TAUT}(\mathbb{K})$,

(4) $\varphi^\star \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$.

Proof. Again, $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are obvious. $(4) \Rightarrow (1)$ is proved as in the previous lemma. We prove $(1) \Rightarrow (2)$. Suppose that φ is a classical tautology. By Herbrand's Theorem, there is a classical propositional tautology of the form $\bigvee_{i=1}^{m} \psi(t_1^i, \ldots, t_n^i)$, where the t_j^i's are closed terms. By the previous lemma, recalling that * commutes with $\vee$, we have that $\bigvee_{i=1}^{m} \psi^\star(t_1^i, \ldots, t_n^i) \in \mathrm{genTAUT}(\mathrm{L}\forall)$. By an easy proof in L$\forall$, we can derive $\varphi^\star = (\exists x_1) \ldots (\exists x_n)\ \psi^\star(x_1, \ldots, x_n)$, and hence we have proved (2). $\square$

THEOREM 2.0.13. *For every class $\mathbb{K}$ of chains, the sets* $\mathrm{TAUT}(\mathbb{K})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *are $\boldsymbol{\Sigma_1}$-hard.*

Proof. The set of provable existential formulae of first-order classical logic is $\boldsymbol{\Sigma_1}$-hard (observe that here we are using our general hypothesis that assumes that our first-order logics have a full vocabulary). Indeed, the Herbrand form φ^H of any sentence φ is purely existential, and φ is provable iff φ^H is provable. The claim now follows from the previous lemma. $\square$

In order to prove that the $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ problems are $\boldsymbol{\Pi_1}$-hard, we will deal with their complement, the sets of $\mathbb{K}$-contradictions:

- $\mathrm{TAUT}_0(\mathbb{K}) = \{\varphi \in \mathrm{Sent}_{\mathcal{P}} \mid$ for every $\boldsymbol{A} \in \mathbb{K}$ and every safe $\boldsymbol{A}$-structure $\mathbf{M}$, $\|\varphi\|_{\mathbf{M}}^{\boldsymbol{A}} = \overline{0}^{\boldsymbol{A}}\}$.

An adaptation of the proof of the $\boldsymbol{\Sigma_1}$-hardness of the TAUT and $\mathrm{TAUT}_{\mathrm{pos}}$ problems allows to obtain the same result for the newly defined set. We do it in the following lemmata and their consequences.

LEMMA 2.0.14. *Let* L *be any (*$\triangle$*-)core fuzzy logic. For every sentence* φ *we have* $\varphi^2 \wedge (\neg\varphi)^2 \in \mathrm{genTAUT}_0(\mathrm{L}\forall)$.

Proof. Let $\boldsymbol{A}$ be an L-chain and $\mathbf{M}$ an $\boldsymbol{A}$-model. Otherwise, if $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} \leq \|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$, then $\|\varphi^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. If $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > \|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}}$, then $\|(\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. In either case we obtain $\|\varphi^2 \wedge (\neg\varphi)^2\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$. $\square$

This yields a definition analogous to the previous one:

DEFINITION 2.0.15. *Let* φ *be a classical sentence. Consider its prenex normal form in classical logic,* $Q_1x_1 \dots Q_nx_n\ \psi(x_1,\dots,x_n)$*, where* ψ *is a lattice combination of literals. We define a formula* φ° *by induction as follows: if* φ *is a literal, then* $\varphi^\circ = \varphi^2$*;* $^\circ$ *commutes with quantifiers,* $\wedge$ *and* $\vee$.

LEMMA 2.0.16. *Let* φ *be a lattice combination of literals,* L *be a (*$\triangle$*-)core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. The following are equivalent:*

(1) φ *is a classical propositional contradiction,*

(2) φ° *is an* L*-contradiction,*

(3) φ° *is a contradiction for every chain in* $\mathbb{K}$.

Proof. $(2) \Rightarrow (3)$ is trivial. We show $(1) \Rightarrow (2)$. By distributivity, φ can be equivalently written as $\bigvee_{i=1}^{n} \bigwedge_{j=1}^{n_i} \alpha_{i,j}$, where $\alpha_{i,j}$ are literals. Thus, φ is a classical contradiction iff for every $i \in \{1,\dots,n\}$, $\bigwedge_{j=1}^{n_i} \alpha_{i,j}$ is a classical contradiction. Hence, for every $i \in \{1,\dots,n\}$ there are $j_1, j_2 \in \{1,\dots,n_i\}$ such that $\alpha_{i,j_1} = \neg\alpha_{i,j_2}$. Therefore, $\alpha^2_{i,j_1} \wedge \alpha^2_{i,j_2}$ is an L-contradiction by the previous lemma and, since this formula is implied by $\bigwedge_{j=1}^{n_i} \alpha^2_{i,j}$, we have that φ° is an L-contradiction too. $(3) \Rightarrow (1)$ can be easily proved by contraposition. If φ is not a classical propositional contradiction, then there is an evaluation e on $\mathcal{B}_2$ such that $e(\varphi) = 1$. Since φ° and φ are equivalent in classical logic, we also have $e(\varphi^\circ) = 1$. Now, given any $\boldsymbol{A} \in \mathbb{K}$, it is clear that e can also be seen as an evaluation on $\boldsymbol{A}$ and $e(\varphi^\circ) = \overline{1}^{\boldsymbol{A}}$. $\square$

LEMMA 2.0.17 (Dual Herbrand's Theorem). *Let* $\varphi = (\forall x_1)\dots(\forall x_n)\ \psi(x_1,\dots,x_n)$ *be a purely universal sentence.* φ *is a classical contradiction if, and only if, there exists* m *and closed terms* $\{t^i_1,\dots,t^i_n \mid i = 1,\dots,m\}$ *such that* $\bigwedge_{i=1}^{m} \psi(t^i_1,\dots,t^i_n)$ *is a classical propositional contradiction.*

Proof. It is a trivial consequence of the usual Herbrand's Theorem. $\square$

LEMMA 2.0.18. *Let* $\varphi = (\forall x_1)\dots(\forall x_n)\ \psi(x_1,\dots,x_n)$*, where* ψ *is a lattice combination of literals, be a purely universal formula,* L *be a (*$\triangle$*-)core fuzzy logic and* $\mathbb{K}$ *a class of* L*-chains. The following are equivalent:*

(1) $\varphi \in \mathrm{TAUT}_0(\mathcal{B}_2)$*,*

(2) $\varphi^\circ \in \mathrm{genTAUT}_0(\mathrm{L}\forall)$*,*

(3) $\varphi^\circ \in \mathrm{TAUT}_0(\mathbb{K})$.

Proof. Again, $(2) \Rightarrow (3)$ is trivial and $(3) \Rightarrow (1)$ is proved as in Lemma 2.0.16. Let us show $(1) \Rightarrow (2)$. Suppose that φ is a classical contradiction. By the dual Herbrand's Theorem, there are closed terms t_j^i such that $\bigwedge_{i=1}^m \psi(t_1^i, \ldots, t_n^i)$ is a classical propositional contradiction. By Lemma 2.0.16, recalling that $^\circ$ commutes with $\wedge$, we have that $\bigwedge_{i=1}^m \psi^\circ(t_1^i, \ldots, t_n^i) \in \mathrm{genTAUT}_0(\mathrm{L}\forall)$. Therefore, we obtain $\varphi^\circ = (\forall x_1) \ldots (\forall x_n)\, \psi^\circ(x_1, \ldots, x_n) \in \mathrm{genTAUT}_0(\mathrm{L}\forall)$. □

LEMMA 2.0.19. *The set of classical purely universal first-order contradictions is $\mathbf{\Sigma_1}$-hard.*

Proof. First observe that the set all contradictions is $\mathbf{\Sigma_1}$-hard (again because we are in the full vocabulary). Indeed, the set of all tautologies is $\mathbf{\Sigma_1}$-hard and we have that for any sentence φ, φ is a contradiction iff $\neg\varphi$ is a tautology. Now given any sentence φ we can write the following chain of equivalencies: φ is a contradiction iff $\neg\varphi$ is a tautology iff its Herbrand form (purely existential) $(\neg\varphi)^H$ is a tautology iff $\neg(\neg\varphi)^H$ is a contradiction. The latter is a purely universal formula, so we are done. □

THEOREM 2.0.20. *For every (non-empty) class $\mathbb{K}$ of chains, the set $\mathrm{TAUT}_0(\mathbb{K})$ is $\mathbf{\Sigma_1}$-hard and thus $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ is $\mathbf{\Pi_1}$-hard.*

Proof. It follows from the previous two lemmata and the fact that $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ is the complementary set of $\mathrm{TAUT}_0(\mathbb{K})$. □

On the other hand, completeness with respect to a Hilbert-style calculus gives upper bounds for the complexity:

PROPOSITION 2.0.21. *Let* L *be a recursively axiomatizable ($\triangle$-)core fuzzy logic and $\mathbb{K}$ be a class of* L*-chains. If* L$\forall$ *has the* FS$\mathbb{K}$C*, then* $\mathrm{TAUT}(\mathbb{K})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *are* $\mathbf{\Sigma_1}$*, while* $\mathrm{SAT}(\mathbb{K})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *are* $\mathbf{\Pi_1}$.

Proof. $\mathrm{TAUT}(\mathbb{K})$ is $\mathbf{\Sigma_1}$ because it is the set of theorems of a recursively axiomatizable logic. Using Lemma 2.0.4 ($\varphi \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ iff $\neg\varphi \notin \mathrm{TAUT}(\mathbb{K})$) we obtain that $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ is $\mathbf{\Pi_1}$. As regards to $\mathrm{SAT}(\mathbb{K})$, notice that for every $\varphi \in \mathrm{Sent}_{\mathcal{P}}$ we have: $\varphi \in \mathrm{SAT}(\mathbb{K})$ iff $\varphi \not\models_{\mathbb{K}} \overline{0}$ iff $\varphi \nvdash_{\mathrm{L}\forall} \overline{0}$. Using again Lemma 2.0.4 (now $\varphi \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ iff $\neg\varphi \notin \mathrm{SAT}(\mathbb{K})$) we obtain that $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ is $\mathbf{\Sigma_1}$. □

In particular, since a first-order logic is always complete with respect to the semantics of all chains, we obtain:

COROLLARY 2.0.22. *For every recursively axiomatizable first-order ($\triangle$-)core fuzzy logic* L$\forall$*,* $\mathrm{genTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{genTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* $\mathbf{\Sigma_1}$*-complete,* $\mathrm{genSAT}(\mathrm{L}\forall)$ *and* $\mathrm{genSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* $\mathbf{\Pi_1}$*-complete.*

Moreover, it yields the following general undecidability result:

COROLLARY 2.0.23. *For every ($\triangle$-)core fuzzy logic, the first-order logic* L$\forall$ *is undecidable.*

See all the results for the general semantics in Table 1.

Problem	Complexity
$\mathrm{genTAUT}(\mathrm{L}\forall)$	$\mathbf{\Sigma_1}$-complete
$\mathrm{genSAT}(\mathrm{L}\forall)$	$\mathbf{\Pi_1}$-complete
$\mathrm{genTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$	$\mathbf{\Sigma_1}$-complete
$\mathrm{genSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$	$\mathbf{\Pi_1}$-complete

Table 1. Complexity results for the general semantics.

3 Complexity of standard semantics

Let L be a ($\triangle$-)core fuzzy logic. Recall that the *standard semantics* of L is given by a subclass class $\mathbb{K}$ of the class of all real L-chains which are considered to be the *intended real* L*-chains*. What are the intended real L-chains? This can possibly have several meanings but for t-norm based logics we may understand it as follows: if the L is introduced as the logic of a (left-)continuous t-norm $*$ then the intended chain is just $[0,1]_*$ (like Łukasiewicz logic, etc.); if it is introduced as the logic of a set of t-norms (like BL) then $\mathbb{K}$ is the set of the corresponding algebras $[0,1]_*$. Note that each real BL-chain is given by a continuous t-norm, thus for BL all real BL-chains are intended, and hence in the case of BL standard chains coincide with real chains.

We shall discuss particular prominent logics: first logics extending $\mathrm{BL}\forall$ and then logics extending $\mathrm{MTL}\forall$ (but not $\mathrm{BL}\forall$). The main results we will obtain are collected in Table 2, where $(Ł\oplus)\forall$ stands for any logic given by a continuous t-norm which is an ordinal sum of Łukasiewicz t-norm with any continuous t-norm (i.e. Łukasiewicz t-norm is its first component in the ordinal sum representation with possibly infinitely many components); analogously for $(\mathrm{G}\oplus)\forall$ and $(\Pi\oplus)\forall$.

We will often identify a standard BL-algebra $\boldsymbol{A}$ with its corresponding continuous t-norm, and hence we identify the components of the t-norm with the corresponding algebras, which will be called *t-norm components of* $\boldsymbol{A}$. Thus, when speaking e.g. of a product component of a t-norm we may indifferently mean either the isomorphic copy of the product t-norm or the isomorphic copy of the standard product algebra.

We start with Gödel logic $\mathrm{G}\forall$. It is clear that each countable G-chain $\boldsymbol{A}$ embeds into the standard G-chain $[0,1]_{\mathrm{G}}$ by an isomorphism preserving all infinite suprema and infima existing in $\boldsymbol{A}$. This gives standard completeness. Therefore:

$$\mathrm{stTAUT}(\mathrm{G}\forall) = \mathrm{genTAUT}(\mathrm{G}\forall) \qquad \mathrm{stTAUT}_{\mathrm{pos}}(\mathrm{G}\forall) = \mathrm{genTAUT}_{\mathrm{pos}}(\mathrm{G}\forall)$$
$$\mathrm{stSAT}(\mathrm{G}\forall) = \mathrm{genSAT}(\mathrm{G}\forall) \qquad \mathrm{stSAT}_{\mathrm{pos}}(\mathrm{G}\forall) = \mathrm{genSAT}_{\mathrm{pos}}(\mathrm{G}\forall).$$

LEMMA 3.0.1. *The sets* $\mathrm{stTAUT}(\mathrm{G}\forall)$ *and* $\mathrm{stTAUT}_{\mathrm{pos}}(\mathrm{G}\forall)$ *are* $\mathbf{\Sigma_1}$*-complete, while the sets* $\mathrm{stSAT}_{\mathrm{pos}}(\mathrm{G}\forall)$ *and* $\mathrm{stSAT}(\mathrm{G}\forall)$ *are* $\mathbf{\Pi_1}$*-complete.*

Now let us turn to first-order Łukasiewicz logic. First, it is easy to show that $\mathrm{stTAUT}(Ł\forall)$ is in $\mathbf{\Pi_2}$. This follows immediately from the Pavelka-style completeness of $\mathrm{RPL}\forall$ discussed in Chapter VIII: $\varphi \in \mathrm{stTAUT}(Ł\forall)$ iff the provability degree $|\varphi|_{\mathrm{RPL}\forall}$ of φ equals 1, i.e. $(\forall r < 1$ rational$)(\exists d)(d$ is an RPL-proof of $(\overline{r} \to \varphi))$. Clearly, this condition is Π_2. The proof of $\mathbf{\Pi_2}$-hardness is much harder. From now on, our logic is $Ł\forall$.

Logic	stTAUT	stSAT	$\mathrm{stTAUT}_{\mathrm{pos}}$	$\mathrm{stSAT}_{\mathrm{pos}}$
MTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
IMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
SMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
WCMTL∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard
ΠMTL∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard
BL∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic
SBL∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic
Ł∀	$\mathbf{\Pi_2}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Sigma_2}$-complete
G∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
Π∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic
(Ł⊕)∀	$\mathbf{\Pi_2}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Sigma_2}$-complete
(G⊕)∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
(Π⊕)∀	Non-arithmetic	Non-arithmetic	Non-arithmetic	Non-arithmetic
C_nMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
C_nIMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
WNM∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
NM∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete

Table 2. Complexity results for the standard semantics of prominent first-order fuzzy logics.

DEFINITION 3.0.2.

(1) Call a formula classical *if all connectives it contains are among* $\wedge, \vee, \neg$.

(2) A standard model $\mathbf{M}$ *(of* Ł∀*) is* predefinite *if for each classical formula* φ *and each evaluation* v, $\|\varphi\|_{\mathbf{M},\mathrm{v}} \neq \frac{1}{2}$.

(3) For an n*-ary predicate* P, $\delta(P)$ *is the formula* $[(\forall x_1, \ldots, x_n)(P(x_1, \ldots, x_n) \vee \neg P(x_1, \ldots, x_n)]^2$.

It is easy to verify that a model $\mathbf{M}$ is predefinite if, and only if, $\|\delta(P)\|_{\mathbf{M}} > 0$ for each predicate P.

DEFINITION 3.0.3. *For each predefinite structure* $\mathbf{M} = \langle M, \langle P_{\mathbf{M}}\rangle_P, \langle f_{\mathbf{M}}\rangle_f\rangle$ *the corresponding Boolean structure is* $\mathbf{M}_{/01} = \langle M, \langle P'_{\mathbf{M}}\rangle_P, \langle f_{\mathbf{M}}\rangle_f\rangle$ *where* $P'_{\mathbf{M}}(\overrightarrow{a}) = 1$ *iff* $P_{\mathbf{M}}(\overrightarrow{a}) > \frac{1}{2}$*, otherwise* $P'_{\mathbf{M}}(\overrightarrow{a}) = 0$.

LEMMA 3.0.4. *Let* $\mathbf{M}$ *be predefinite and* φ *a classical formula. Then*

$$\|\varphi\|_{\mathbf{M}_{/01}} = 1 \textit{ iff } \|\varphi\|_{\mathbf{M}} > \tfrac{1}{2} \textit{ iff } \|\varphi^2\|_{\mathbf{M}} > 0.$$

Proof. This is clear for atomic formulae and follows by an easy induction for all classical formulae (elaborate the induction step for $\neg, \wedge, \forall$ using the preceding lemma). □

We shall need the following fact from recursion theory:

LEMMA 3.0.5. *There is a $\mathbf{\Sigma_1}$-relation $C \subseteq \mathsf{N}^2$ such that, if we define $C_m = \{n \mid \langle m, n\rangle \in C\}$ then the set $Fin = \{m \mid C_m$ is finite$\}$ is a $\mathbf{\Sigma_2}$-complete set (thus $\mathsf{N} \setminus Fin$ is a $\mathbf{\Pi_2}$-complete set).*

DEFINITION 3.0.6. *Assume that $\gamma(x, y)$ is a $\mathbf{\Sigma_1}$-formula defining a binary relation C in the standard model of arithmetics $\mathbf{N}$, i.e. such that for each $m, n \in \mathsf{N}$,*

$$\langle m, n\rangle \in C \quad \textit{iff} \quad \|\gamma(\overline{m}, \overline{n})\|_{\mathbf{N}} = 1.$$

Let Predef *stand for $\delta(=) \wedge \delta(\leq)$ and let, for each m, γ^*_m stand for the formula* $\mathrm{Predef} \wedge (Q^+)^2 \wedge (\forall x, y)(2\gamma(\overline{m}, x) \wedge 2\gamma(\overline{m}, y) \wedge 2(x \neq y) \to \neg(U(x) \leftrightarrow U(y)))$ *where U is a new unary predicate (see page 854 for a presentation of Q^+).*

LEMMA 3.0.7. *Under the above relation, C_m is finite iff $\gamma^*_m \in \mathrm{stSAT_{pos}}(\text{Ł}\forall)$.*

Proof. First assume that C_m is finite, say $C_m = \{n_1, \dots, n_k\}$. Take $\mathbf{N}$ and expand it to a model $\mathbf{M}$ by defining $U_{\mathbf{M}}(n_i) = i/k$, $U_{\mathbf{M}}(j) = 0$ for j distinct from $n_1, \dots, n_k$. Verify easily that $\|\gamma^*_m\|_{\mathbf{M}} \geq \frac{1}{k}$. Indeed, $\|\mathrm{Predef}\|_{\mathbf{M}} = 1,\ \|Q^+\|_{\mathbf{M}} = 1$. Take $a, b \in \mathsf{N}$ and assume that the value $\|2\gamma(m, a) \wedge 2\gamma(m, b) \wedge 2(a \neq b)\|_{\mathbf{M}}$ is positive (otherwise there is nothing to prove). This means that the values of $\gamma(m, a), \gamma(m, b), a \neq b$ are $> \frac{1}{2}$ and hence $= 1$ (since in $\mathbf{M}$ everything except $U_{\mathbf{M}}$ is crisp). But then $a \neq b$ and $a, b \in C_m$; thus for some $i, j \leq k$ we have $a = n_i, b = n_j, i \neq j$ and $\|\neg(U(a) \leftrightarrow U(b))\|_{\mathbf{M}} = |\frac{i}{k} - \frac{j}{k}| \geq \frac{1}{k}$. Thus $\|\gamma^*_m\|_{\mathbf{M}} \geq \frac{1}{k}$.

Conversely, let C_m be infinite, $C_m = \{n_i\}_{i=1}^{\infty}$. We show that γ^*_m is not positively satisfiable. Assume it is, let $\|\gamma^*_m\|_{\mathbf{M}} = t > 0$. Delete $U_{\mathbf{M}}$ from $\mathbf{M}$; we obtain a predefinite model $\mathbf{M}'$ of the language of Q^+, hence $\mathbf{M}'' = \mathbf{M}'_{/01}$ is a model of Q^+. We may assume that $\mathbf{N}$ is an initial segment of $\mathbf{M}''$. Since $\|\gamma(m, n_i)\|_{\mathbf{N}} = 1$ for $i = 1, 2, \dots$, we have $\|\gamma(m, n_i)\|_{\mathbf{M}''} = 1$, thus $\|\gamma(m, n_i)\|_{\mathbf{M}'} > \frac{1}{2}$. Hence $\|2\gamma(m, n_i)\|_{\mathbf{M}'} = 1$. For $i \neq j$ we obtain $\|2(n_i \neq n_j)\|_{\mathbf{M}'} = 1$. Come back to $\mathbf{M}$ (returning $U_{\mathbf{M}}$). Since $\|\gamma^*_m\|_{\mathbf{M}} = t$ we obtain $\|\neg(U(n_i) \leftrightarrow U(n_j))\|_{\mathbf{M}} \geq t$, thus putting $\|U(n_i)\|_{\mathbf{M}} = t_i$ we obtain $|t_i - t_j| \geq t$ for $i \neq j$. But this is a clear contradiction, because i, j run over all natural numbers and t_i, t_j, t are positive reals. This completes the proof. □

THEOREM 3.0.8. *The set* $\mathrm{stSAT_{pos}}(\text{Ł}\forall)$ *is $\mathbf{\Sigma_2}$-complete and the set* $\mathrm{stTAUT}(\text{Ł}\forall)$ *is $\mathbf{\Pi_2}$-complete.*

Proof. The mapping associating to each natural m the formula γ^*_m reduces the $\mathbf{\Sigma_2}$-complete set Fin from the Lemma 3.0.5 to the set $\mathrm{stSAT_{pos}}(\text{Ł}\forall)$. □

Now let us discuss $\mathrm{stTAUT_{pos}}(\text{Ł}\forall)$ and $\mathrm{stSAT}(\text{Ł}\forall)$ (it is easy to check the same result could be proved for $\mathrm{RPL}\forall$ as well).

THEOREM 3.0.9. $\mathrm{stTAUT_{pos}}(\text{Ł}\forall)$ *is a $\mathbf{\Sigma_1}$-complete set and hence* $\mathrm{stSAT}(\text{Ł}\forall)$ *is a $\mathbf{\Pi_1}$-complete set.*

Proof. If $\varphi \in \mathrm{stTAUT_{pos}}(\text{Ł}\forall)$ then $\neg\varphi$ has no model over $[0, 1]_{\text{Ł}}$ (no standard model) and therefore is contradictory; recall that each consistent theory over Ł∀ has a standard model (see e.g. [12, 13]); thus $\neg\varphi \vdash_{\text{Ł}\forall} \overline{0}$. Conversely, if $\neg\varphi \vdash_{\text{Ł}\forall} \overline{0}$ then $\neg\varphi$ has no model and hence $\varphi \in \mathrm{stTAUT_{pos}}(\text{Ł}\forall)$. Clearly $\neg\varphi \vdash_{\text{Ł}\forall} \overline{0}$ is a $\mathbf{\Sigma_1}$ condition; thus $\mathrm{stTAUT_{pos}}(\text{Ł}\forall)$ is in $\mathbf{\Sigma_1}$. It is $\mathbf{\Sigma_1}$-hard by Theorem 2.0.13. □

Now let us consider the logics $(Ł\oplus)\forall$ and $(G\oplus)\forall$ and their standard semantics. Take an arbitrary continuous t-norm which is an ordinal sum whose first summand is $\boldsymbol{C}$, $\boldsymbol{C}$ being Łukasiewicz, Gödel or product t-norm. Denote this t-norm by $\boldsymbol{C}\oplus$ and for simplicity assume that the first positive idempotent is $\frac{1}{2}$. This can always be always be achieved up to an isomorphism. Furthermore, we may assume without loss of generality that the isomorphism from the restriction of $\boldsymbol{C}\oplus$ to $[0, \frac{1}{2}]$ to $\boldsymbol{C}$ defined on $[0, 1]$ is just the mapping sending x to $2x$. Let us say that our t-norm *begins well* with $\boldsymbol{C}$. $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}}$ denotes the truth value of a sentence φ in a $\boldsymbol{C}$-model $\mathbf{M}$; similarly for $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}}$. $(\boldsymbol{C}\oplus)\forall$ is the predicate logic given by $\boldsymbol{C}\oplus$.

DEFINITION 3.0.10. *Let h be the following mapping of $[0, 1]$ onto itself: $h(x) = 2x$ for $x \leq \frac{1}{2}$, $h(x) = 1$ for $x \in [\frac{1}{2}, 1]$. Let $\mathbf{M} = \langle M, \langle P_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ be a $[0, 1]$-structure of the language in question. We define a structure $h(\mathbf{M})$ of the form $\langle M, \langle P'_{\mathbf{M}} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ where for each P (n-ary) and each tuple $a_1, \ldots, a_n \in M$, $P'_{\mathbf{M}}(a_1, \ldots, a_n) = h(P_{\mathbf{M}}(a_1, \ldots, a_n))$. Furthermore, we define another structure $\mathbf{M}/2$ as $\langle M, \langle P_{\mathbf{M}/2} \rangle_{P \in \mathbf{P}}, \langle f_{\mathbf{M}} \rangle_{f \in \mathbf{F}} \rangle$ where $(P_{\mathbf{M}/2})(a_1, \ldots, a_n) = P_{\mathbf{M}}(a_1, \ldots, a_n)/2$.*

LEMMA 3.0.11. *Let $\boldsymbol{C}\oplus$ begin well with $\boldsymbol{C}$ and let h be the mapping in the previous definition. Then h is an algebraic homomorphism of $\langle [0, 1], \boldsymbol{C}\oplus, \rightarrow^{\boldsymbol{C}\oplus}, 0, 1 \rangle$ onto $\langle [0, 1], \boldsymbol{C}, \rightarrow^{\boldsymbol{C}}, 0, 1 \rangle$ preserving infinite joins and meets. Consequently, for each sentence φ,*

(1) $h(\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}}) = \|\varphi\|^{\boldsymbol{C}}_{h(\mathbf{M})}$,

(2) $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} = h(\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}/2})$.

The proof is easy.

THEOREM 3.0.12. *A sentence φ is a standard positive tautology of $\boldsymbol{C}\oplus$ iff it is a standard positive tautology of $\boldsymbol{C}$. Moreover, φ is standardly positively $\boldsymbol{C}\oplus$-satisfiable iff φ is standardly positively $\boldsymbol{C}$-satisfiable. In symbols we have:* $\mathrm{stTAUT}_{\mathrm{pos}}((\boldsymbol{C}\oplus)\forall) = \mathrm{stTAUT}_{\mathrm{pos}}(\boldsymbol{C}\forall)$ *and* $\mathrm{stSAT}_{\mathrm{pos}}((\boldsymbol{C}\oplus)\forall) = \mathrm{stSAT}_{\mathrm{pos}}(\boldsymbol{C}\forall)$.

Proof. This is immediate from the preceding lemma which shows that there is an $\mathbf{M}$ with $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}} = 0$ iff there is an $\mathbf{M}$ with $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} = 0$, and the same for $\neq 0$. □

LEMMA 3.0.13. *Let $\boldsymbol{C}$ be* Ł *or* G *and let φ be a sentence such that $\varphi \in \mathrm{stSAT}(\boldsymbol{C}\forall)$. Then there exists a Henkin theory T proving φ. For $\boldsymbol{C}$ being* Ł, *T can be assumed to be maximally consistent.*

Proof. It follows from the properties of Henkin theories, see Section 4 of Chapter II. □

Call a model $\mathbf{M}$ *fully named* if each element of $\mathbf{M}$ is the interpretation of a constant. Recall that the canonical model $\mathbf{CM}(T)$ of a Henkin theory T is fully named (see Chapter II). In particular, for each formula $(\exists x)\alpha(x, y, \ldots)$ and each $a, \ldots \in \mathbf{CM}(T)$, if $(\exists x)\alpha(x, a, \ldots)$ is 1-true in $\mathbf{CM}(T)$ then there is a $b \in \mathbf{CM}(T)$ such that $\alpha(b, a, \ldots)$ is 1-true there. This is used in the proof of the next theorem.

LEMMA 3.0.14. *Let $\boldsymbol{C}$ be* Ł *or* G *and let $\boldsymbol{C}\oplus$ be a t-norm beginning by $\boldsymbol{C}$. For an arbitrary sentence φ, if $\varphi \in \mathrm{stSAT}(\boldsymbol{C})$ then $\varphi \in \mathrm{stSAT}(\boldsymbol{C}\oplus)$.*

Proof. Given φ, let $T_0 = \{\varphi\}$, let T be its Henkin extension over the logic $\boldsymbol{C}\forall$ and let $\mathbf{M} = \mathbf{CM}(T)$. For $\boldsymbol{C}$ being Gödel use the fact that each (countable) model over G$\forall$ can be understood as a standard model by embedding the G-algebra of truth functions into [0,1] by a 1-1 mapping of truth values preserving all infinite joins and meets (easy). Thus $\mathbf{M}$ is a standard fully named Henkin model of φ. Now let $\boldsymbol{C}$ be Łukasiewicz. One can show that the Lindenbaum algebra $Lind$ used to construct the canonical model $\mathbf{CM}(T)$ is Archimedean, i.e. for each its element $x < 1$ there is a natural n such that $x^n = 0$. Furthermore, each (countable) Archimedean MV-chain embeds to the standard algebra $[0,1]_{Ł}$ by an isomorphic embedding i preserving all infinite joins and meets existing in this chain. Hence the Ł-model $\mathbf{M} = \mathbf{CM}(T)$ is made standard by replacing each value from $Lind$ by its i-image. (If necessary consult [12] and some references thereof.)

Thus in both cases we have a standard fully named model of Henkin theory T. Let $f(x) = \frac{x}{2}$ for $x < 1$, $f(1) = 1$. Make $\mathbf{M}$ to a $\boldsymbol{C}\oplus$-structure $\mathbf{M}'$ (with the $\boldsymbol{C}$-component on $[0, \frac{1}{2}]$) and with the same domain as $\mathbf{M}$ by defining $P_{\mathbf{M}'}(a_1, \dots) = f(P_{\mathbf{M}}(a_1, \dots))$ for all P and $a_1, \dots$ We show by induction on the complexity of closed $\overline{T}$-formulae α, $\|\alpha\|^{\boldsymbol{C}\oplus}_{\mathbf{M}'} = f(\|\alpha\|^{\boldsymbol{C}}_{\mathbf{M}})$.

This is evident for atoms and connectives (since $[0, \frac{1}{2}) \cup \{1\}$ is a $\boldsymbol{C}$-subalgebra of $[0,1]_*$) and for $\forall$ (since f preserves infinite meets); similarly for $\|(\exists x)\beta\|^{\boldsymbol{C}}_{\mathbf{M}_1} < 1$. For $\|(\exists x)\beta\|^{\boldsymbol{C}}_{\mathbf{M}_1} = 1$ use witnessing: there is a c such that $\|\beta(c)\|^{\boldsymbol{C}}_{\mathbf{M}_1} = 1$. In particular, since $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} = 1$ we obtain $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}_1} = 1$ and $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}'} = 1$. □

THEOREM 3.0.15.

(1) $\mathrm{stSAT_{pos}}(\mathrm{G}\forall) = \mathrm{stSAT}(\mathrm{G}\forall)$.

(2) $\mathrm{stSAT}((\mathrm{G}\oplus)\forall) = \mathrm{stSAT}(\mathrm{G}\forall)$.

(3) $\mathrm{stSAT}((\text{Ł}\oplus)\forall) = \mathrm{stSAT}(\text{Ł}\forall)$.

Proof. (1) Clearly each sentence standardly satisfiable in G$\forall$ is positively satisfiable there. Conversely if $0 < r = \|\varphi\|^{\mathrm{G}}_{\mathbf{M}} < 1$ for some standard $\mathbf{M}$ then taking a one-one increasing mapping of $[0,1]$ onto itself produce an isomorphic copy $\mathbf{M}'$ of $\mathbf{M}$ such that $\|\varphi\|^{\mathrm{G}}_{\mathbf{M}'} = \frac{1}{2}$. Then apply the homomorphism h from Definition 3.0.10 and observe that it is a homomorphism of the G-structure $\mathbf{M}'$ to the G-structure $h(\mathbf{M}')$ sending $\frac{1}{2}$ to 1. Thus $\|\varphi\|^{\mathrm{G}}_{h(\mathbf{M}')} = 1$.

(2) Each sentence standardly satisfiable in $(\mathrm{G}\oplus)\forall$ is standardly satisfiable in G$\forall$ by Lemma 3.0.11; the converse inclusion follows from Lemma 3.0.14. A similar line of reasoning proves (3). □

COROLLARY 3.0.16.

$$\mathrm{stSAT_{pos}}((\mathrm{G}\oplus)\forall) = \mathrm{stSAT_{pos}}(\mathrm{G}\forall) = \mathrm{stSAT}(\mathrm{G}\forall) = \mathrm{stSAT}((\mathrm{G}\oplus)\forall).$$

LEMMA 3.0.17. *If $*$ begins with Ł then for each φ, φ is a standard tautology of* Ł$\forall$ *iff $\neg\neg\varphi$ is a tautology of $*$; similarly for satisfiability.*

Proof. By Lemma 3.0.11, φ is a standard tautology of Ł$\forall$ iff φ is a $[\frac{1}{2}, 1]$-tautology of $*$ (for each $\mathbf{M}$, $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}} \in [\frac{1}{2}, 1]$) iff $\neg\neg\varphi$ is a tautology of $*$. Similarly for satisfiability. □

The following theorem collects results of arithmetical complexity not stated till now.

THEOREM 3.0.18.

(1) $\mathrm{stSAT}((\mathrm{G}\oplus)\forall) = \mathrm{stSAT_{pos}}((\mathrm{G}\oplus)\forall)$ *is* $\mathbf{\Pi_1}$*-complete,* $\mathrm{stTAUT_{pos}}((\mathrm{G}\oplus)\forall)$ *is* $\mathbf{\Sigma_1}$*-complete and* $\mathrm{stTAUT}((\mathrm{G}\oplus)\forall)$ *is* $\mathbf{\Sigma_1}$*-hard.*

(2) $\mathrm{stTAUT_{pos}}((\text{Ł}\oplus)\forall)$ *and* $\mathrm{stSAT}((\text{Ł}\oplus)\forall)$ *are* $\mathbf{\Sigma_1}$*-complete,* $\mathrm{stSAT_{pos}}((\text{Ł}\oplus)\forall)$ *is* $\mathbf{\Sigma_2}$*-complete, and* $\mathrm{stTAUT}((\text{Ł}\oplus)\forall)$ *is* $\mathbf{\Pi_2}$*-hard.*

Proof. (1) By Corollary 3.0.16, and Theorems 3.0.12 and 2.0.20. (2) By Theorem 3.0.12, Lemma 3.0.13, Theorem 3.0.15, Lemma 3.0.17 and Theorem 3.0.9. □

We now characterize the classes $\mathbb{K}$ of standard BL-algebras such that $\mathrm{TAUT}(\mathbb{K})$ is recursively axiomatizable. More precisely, we will prove that $\mathrm{TAUT}(\mathbb{K})$ is recursively axiomatizable iff $\mathbb{K}$ is the singleton of $[0,1]_{\mathrm{G}}$.

In the sequel, if the free variables of ψ are among $x_1,\ldots,x_n$ and $a_1,\ldots,a_n \in M$, we write $\|\psi(a_1,\ldots,a_n)\|^{\boldsymbol{A}}_{\mathbf{M}}$ to mean $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M},\mathrm{v}}$ where $\mathrm{v}(x_i) = a_i$ $(i = 1,\ldots,n)$.

We warn the reader that even if we identify the t-norm components with their associated algebras, t-norm components of a standard BL-algebra are in general different from its Wajsberg components. Here below we list some basic differences.

(1) The top of a Wajsberg component is 1, and its supremum or its infimum may fail to be in the component. To the contrary, 1 need not belong to a Łukasiewicz or product or Gödel component, and such components contain their supremum and their infimum.

(2) An infinite Wajsberg component can never be a product algebra, and product components are ordinal sums of a two-element Wajsberg algebra and a cancellative hoop (possibly with 1 replaced by the supremum of the component).

(3) A Gödel component is considered as the ordinal sum of uncountably many two-element Wajsberg hoops, again, possibly with 1 replaced by the supremum of the component.

(4) In an ordinal sum of Wajsberg components there must be a first component, whereas the same is not true of an ordinal sum of t-norm components.

(5) In an ordinal sum of Wajsberg components, each component is a subhoop of the whole algebra $\boldsymbol{A}$, whereas the same is not true of t-norm components (considered as BL-algebras), because if $x \leq y$ are in a t-norm component $\boldsymbol{A}_i$ of $\boldsymbol{A}$ such that $\max(A_i) < 1$ and if $\to$ and $\to_i$ denote the residual in $\boldsymbol{A}$ and in $\boldsymbol{A}_i$ respectively, then $x \to_i y = \max(A_i)$, and $x \to y = 1$.

For any two formulae λ and υ, we write $\lambda \uparrow \upsilon$ for $(\lambda \to \upsilon) \to \upsilon$, and we adopt a similar notation for elements of a BL-chain.

LEMMA 3.0.19. *Let* $\boldsymbol{A}$ *be a* BL*-chain,* $\mathbf{M}$ *a first-order* $\boldsymbol{A}$*-structure and* λ, υ *sentences, possibly with parameters from* M*. Then,* $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ *if either* $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ *or* $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$*, or* $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} > \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$ *and* $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}}$ *and* $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$ *are not in the same Wajsberg component. Otherwise,* $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \max\{\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}}, \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}\}$.

Proof. The claim is trivial if either $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ or $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. Thus, suppose $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} < \overline{1}^{\boldsymbol{A}}$ and $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} < \overline{1}^{\boldsymbol{A}}$. If $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}}$ and $\|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$ are in the same Wajsberg component, then clearly $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\lambda \vee \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$, and the claim follows.

If $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} \leq \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$, then $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$, and again the claim follows. Finally, if $\|\lambda\|^{\boldsymbol{A}}_{\mathbf{M}} > \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$ and these two elements are not in the same Wajsberg component, then $\|\lambda \to \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\upsilon\|^{\boldsymbol{A}}_{\mathbf{M}}$, and $\|\lambda \uparrow \upsilon\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. □

Let ϕ be a sentence. We define for every sentence ψ, possibly with parameters from M, the sentence ψ^{ϕ} in the following inductive way:

- $\psi^{\phi} = \psi \vee \phi$ if ψ is atomic and different from $\overline{0}$ and $\overline{1}$; $\overline{0}^{\phi} = \phi$ and $\overline{1}^{\phi} = \overline{1}$.
- ${}^{\phi}$ commutes with $\to$, $\exists$ and $\forall$, i.e. $(\psi \to \gamma)^{\phi} = \psi^{\phi} \to \gamma^{\phi}$, $((\exists x)\psi)^{\phi} = (\exists x)(\psi^{\phi})$ and $((\forall x)\psi)^{\phi} = (\forall x)(\psi^{\phi})$.
- $(\psi \,\&\, \gamma)^{\phi} = (\psi^{\phi} \,\&\, \gamma^{\phi}) \vee \phi$.

Note that $\vdash_{\mathrm{BL}\forall} \phi \to \psi^{\phi}$ for every formula ψ.

In the sequel, given a Łukasiewicz or Gödel or product component $\boldsymbol{C}$ of a standard BL-algebra $\boldsymbol{A}$, given a formula ϕ and an $\boldsymbol{A}$-structure $\mathbf{M}$ such that $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \in \boldsymbol{C}$, but $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \neq \max(\boldsymbol{C})$, $\boldsymbol{C}^{\phi}$ denotes the algebra whose domain is $\{c \in C \mid \|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \leq c\}$, whose bottom is $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}}$ and whose operations are $x \star^{\phi} y = \max\{(x \star y), \|\phi\|^{\boldsymbol{A}}_{\mathbf{M}}\}$, and $x \to^{\phi} y = \min\{(x \to y), \max(\boldsymbol{C})\}$. Note that $\boldsymbol{C}^{\phi}$ is isomorphic to $\boldsymbol{C}$ if $\boldsymbol{C}$ is either a Łukasiewicz or a Gödel component, or if $\boldsymbol{C}$ is a product component and $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} = \min(\boldsymbol{C})$. On the other hand, if $\boldsymbol{C}$ is a product component and $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} > \min(\boldsymbol{C})$, then $\boldsymbol{C}^{\phi}$ is isomorphic to $[0,1]_{Ł}$.

Suppose now that $\boldsymbol{C}$ is a Łukasiewicz or Gödel or product component of a standard BL-algebra $\boldsymbol{A}$, let m be its maximum, and let $\mathbf{M}$ be an $\boldsymbol{A}$-structure. Let ϕ be a sentence such that $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \in \boldsymbol{C} \setminus \{m\}$, and let $\boldsymbol{C}^{\phi}$ be as shown above. Let $S = \{a \in A \mid \forall c \in C (c \leq a)\}$. Note that $m \in S$, and that S is closed under $\star$. Indeed, if $a, b \in S$, then assuming without loss of generality $a \leq b$, either a is an idempotent, and then $a \star b = a \in S$, or a belongs to a Wajsberg component $\boldsymbol{W}$ whose intersection with $\boldsymbol{C}$ is either empty or reduced to the minimum of S. In any case $W \subseteq S$, therefore $a \star b \in W \subseteq S$. Now define a $\boldsymbol{C}^{\phi}$-structure $\mathbf{M}^{\phi}$ such that constants and function symbols are interpreted as in $\mathbf{M}$ and for every n-ary predicate P and every $a_1, \ldots, a_n \in M$, $P_{\mathbf{M}^{\phi}}(a_1, \ldots, a_n) = (P_{\mathbf{M}}(a_1, \ldots, a_n) \vee \|\phi\|^{\boldsymbol{A}}_{\mathbf{M}}) \wedge m$. The interpretation is then extended to a map $\| \ldots \|^{\boldsymbol{C}^{\phi}}_{\mathbf{M}^{\phi}}$ from the set of all formulae into $\boldsymbol{C}^{\phi}$ in the usual way. For simplicity, let us write $\|\psi\|_{\phi}$ instead of $\|\psi\|^{\boldsymbol{C}^{\phi}}_{\mathbf{M}^{\phi}}$, and $\|\psi\|$ instead of $\|\psi\|^{\boldsymbol{A}}_{\mathbf{M}}$.

LEMMA 3.0.20. *For every formula ξ one has:*

(i) $\|\xi^{\phi}\| \in S$ *iff* $\|\xi\|_{\phi} = m$.

(ii) *If* $\|\xi^{\phi}\| \notin S$, *then* $\|\xi\|_{\phi} = \|\xi^{\phi}\|$.

Proof. First of all, note that if $\|\xi^\phi\| \notin S$, then $\|\xi^\phi\| \in C^\phi \setminus \{m\}$, because $\|\xi^\phi\| \geq \|\phi\| = \min(\boldsymbol{C}^\phi)$. We prove the first claim by induction on ξ.

The claim is obvious if ξ is atomic (in particular, if $\xi = \overline{0}$ or $\xi = \overline{1}$). If $\xi = \lambda \mathbin{\&} \upsilon$, then (i) follows immediately from the induction hypothesis and from the fact that S is closed under $\star$. As regards to (ii), if $\|\xi^\phi\| \notin S$, then at least one of $\|\xi^\phi\|$, $\|\upsilon^\phi\|$ is not in S, since S is closed under $\star$. If they are both in $C^\phi \setminus \{m\}$, then by the induction hypothesis, $\|\lambda \mathbin{\&} \upsilon\|_\phi = \|\lambda\|_\phi \star^\phi \|\upsilon\|_\phi = \|\lambda^\phi\| \star^\phi \|\upsilon^\phi\| = \|(\lambda^\phi \mathbin{\&} \upsilon^\phi) \vee \phi\| = \|\xi^\phi\|$. If, say, $\|\lambda^\phi\| \in S$ and $\|\upsilon^\phi\| \in C^\phi \setminus \{m\}$, then $\|\xi^\phi\| = \|\upsilon^\phi\|$, and by the induction hypothesis, $\|\xi^\phi\| = \|\upsilon^\phi\| = \|\upsilon\|_\phi = m \star_\phi \|\upsilon\|_\phi = \|\lambda\|_\phi \star_\phi \|\upsilon\|_\phi = \|\xi\|_\phi$.

Suppose now that $\xi = \lambda \to \upsilon$; as regards to (i), $\|\xi^\phi\| \in S$ iff either $\|\upsilon^\phi\| \in S$ or $\|\lambda^\phi\| \leq \|\upsilon^\phi\|$ and $\|\upsilon^\phi\| \in C^\phi \setminus \{m\}$. In the former case, $\|\upsilon\|_\phi = m$, and $\|\xi^\phi\| \in S$. In the latter case, by the induction hypothesis, $\|\xi\|_\phi = \|\lambda\|_\phi \to^\phi \|\upsilon\|_\phi = \|\lambda^\phi\| \to^\phi \|\upsilon^\phi\| = m$.

Conversely, if $\|\xi\|_\phi = m$, then $\|\lambda\|_\phi \leq \|\upsilon\|_\phi$, therefore either $\|\upsilon\|_\phi = m$ and then, by the induction hypothesis, $\|\upsilon^\phi\| \in S$ and $\|\xi^\phi\| \in S$, or $\|\lambda^\phi\| \leq \|\upsilon^\phi\|$, and then $\|\xi^\phi\| = 1$. In any case, $\|\xi^\phi\| \in S$.

As regards to (ii), if $\|\xi^\phi\| \in C^\phi \setminus \{m\}$, then we must have $\|\upsilon^\phi\| \in C^\phi \setminus \{m\}$, and $\|\upsilon^\phi\| < \|\lambda^\phi\|$. If $\|\lambda^\phi\| \in S$, then $\|\xi^\phi\| = \|\upsilon^\phi\|$, and by the induction hypothesis $\|\xi\|_\phi = \|\upsilon\|_\phi = \|\upsilon^\phi\| = \|\xi^\phi\|$. If $\|\lambda^\phi\| \in C^\phi \setminus \{m\}$, then $\|\xi\|_\phi = \|\lambda\|_\phi \to^\phi \|\upsilon\|_\phi = \|\lambda^\phi\| \to \|\upsilon^\phi\| = \|\xi^\phi\|$.

Next, suppose $\xi = (\forall x)\lambda(x)$. If $\|\xi^\phi\| \in S$, then for all $d \in M$, $\|\lambda(d)^\phi\| \in S$. Hence claim (i) follows from the induction hypothesis. If $\|\xi^\phi\| \in C^\phi \setminus \{m\}$, then since $\inf(S) = m \in S$, there is a $d \in M$ such that $\|\lambda(d)^\phi\| \in C^\phi \setminus \{m\}$, and claim (ii) follows from the induction hypothesis.

Finally, suppose that $\xi = (\exists x)\lambda(x)$. If for some $d \in M$, $\|\lambda(d)^\phi\| \in S$, then claim (i) follows from the induction hypothesis. If $\|\xi^\phi\| \in S$, but for all $d \in M$, $\|\lambda(d)^\phi\| \in C^F \setminus \{m\}$, then $\|\xi^\phi\| = m$. Thus, by the induction hypothesis, $\|\xi\|_\phi = \sup\{\|\lambda(d)\|_\phi \mid d \in M\} = \sup\{\|\lambda(d)^\phi\| \mid d \in M\} = m$, as desired. $\square$

THEOREM 3.0.21. *Let* $\mathbb{K}$ *be a class of standard* BL*-algebras that contains a* BL*-algebra* $\boldsymbol{A}$ *which is not isomorphic to* $[0,1]_{\mathrm{G}}$. *Then* $\mathrm{TAUT}(\mathbb{K})$ *is* $\boldsymbol{\Pi_2}$*-hard. Therefore, the only logic* L *which is complete with respect to a class* $\mathbb{K}$ *of standard* BL*-algebras such that* $\mathrm{realTAUT}(\mathrm{L}\forall)$ *is recursively axiomatizable is Gödel logic.*

Proof. We will recursively reduce the set $\mathrm{stTAUT}(\text{Ł}\forall)$, which is known to be $\boldsymbol{\Pi_2}$-complete, to $\mathrm{TAUT}(\mathbb{K})$. Let us extend the language by adding of a new unary predicate symbol U. Let $\phi = (\forall x)U(x)$, let $\gamma = (\forall x)((U(x) \to \phi) \vee (U(x) \uparrow \phi)$, and let, for every sentence ψ of L, $\psi^\star$ be the sentence

$$\psi^\star = \psi^\phi \vee (\exists x)(\psi^\phi \uparrow U(x)) \vee \gamma.$$

LEMMA 3.0.22. *Let* $\boldsymbol{A} \in \mathbb{K}$*, let* $\star$ *and* $\to$ *be its monoid operation and its residual respectively, and let* $\mathbf{M}$ *be an* $\boldsymbol{A}$*-structure such that* $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$*. Then:*

(i) There is $d \in M$ *such that* $\|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$ *and* $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}}$ *are in the same Wajsberg component* $\boldsymbol{W}$*, and* $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} < \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$.

(ii) $\|\psi^\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \in \boldsymbol{W} \setminus \{1\}$.

Proof. Throughout the whole proof we write $\|\dots\|$ for $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}}$.

(i) If for all $d \in M$ either $\|\phi\| = \|U(d)\|$, or $\|U(d)\| = 1$, or $\|\phi\|$ and $\|U(d)\|$ do not belong to the same component, then by Lemma 3.0.19 for all $d \in M$ we would have either $\|U(d) \to \phi\| = 1$, or $\|U(d) \uparrow \phi\| = 1$. Hence, $\|\gamma\| = 1$, and $\|\psi^\star\| = 1$, a contradiction.

(ii) Let $\boldsymbol{W}$ be the Wajsberg component which $\|\phi\|$ belongs to. If $\|\psi^\phi\| \notin W \setminus \{1\}$, then since $\|\psi^\phi\| \geq \|\phi\| = \inf\{\|U(d)\| \mid d \in M\}$, we have that for some $d \in M$, $\|U(d)\| < \|\psi^\phi\|$, and either $\|\psi^\phi\| = 1$ or $\|U(d)\|$ and $\|\psi^\phi\|$ are not in the same component. In the first case $\|\psi^\star\| = 1$, and in the second one by Lemma 3.0.19 we would have $\|(\exists x)(\psi^\phi \uparrow U(x))\| \geq \|\psi^\phi \uparrow U(d)\| = 1$, and once again $\|\psi^\star\| = 1$, which is impossible. □

The rest of the proof of Theorem 3.0.21 Now let $\boldsymbol{A}$, $\mathbf{M}$, ϕ, $\boldsymbol{W}$, etc. be as in the lemma, and let $m = \sup(W)$ and $c = \inf(W)$. Let us write once again $\|\dots\|$ for $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}}$. By Lemma 3.0.22 (i), $\boldsymbol{W}$ is a Wajsberg component of $\boldsymbol{A}$ with more than two elements. Let $C = (W \setminus \{1\}) \cup \{c, m\}$. Then since $\boldsymbol{A}$ is a standard BL-algebra and since $\boldsymbol{W}$ has more than two elements, C is the domain of a t-norm component $\boldsymbol{C}$ of $\boldsymbol{A}$, which is either a Łukasiewicz or a product component (remember that a Gödel t-norm component is the ordinal sum of Wajsberg components of two elements, and hence if it has more than two elements it is not a Wajsberg component). Moreover, by Lemma 3.0.22 (i), $\|\phi\| < m$. Finally, if $\boldsymbol{C}$ is a product component, then $c < \|\phi\|$, because $\|\phi\| \in W$, $\boldsymbol{W}$ is cancellative (hence unbounded), and $c = \inf(\boldsymbol{W}) \notin W$. Now let $\boldsymbol{C}^\phi$ be defined from $\boldsymbol{C}$ as in Lemma 3.0.20. Then $\boldsymbol{C}^\phi$ is an isomorphic copy of $[0,1]_{Ł}$. Let S and $\|\dots\|_\phi$ be defined as in Lemma 3.0.20. Then by Lemma 3.0.20 for every sentence λ we have:

(i) If $\|\lambda^\phi\| \in S$, then $\|\lambda\|_\phi = m$.

(ii) If $\|\lambda^\phi\| \notin S$, then $\|\lambda\|_\phi = \|\lambda^\phi\|$.

We conclude the proof of Theorem 3.0.21 by demonstrating that $\psi \in \mathrm{stTAUT}(Ł\forall)$ iff $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$. Suppose $\psi^\star \notin \mathrm{TAUT}(\mathbb{K})$. Let $\boldsymbol{A}$ be a standard BL-chain and $\mathbf{M}$ be an $\boldsymbol{A}$-structure such that $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$. Let $\boldsymbol{W}$, $\boldsymbol{C}$ and $\boldsymbol{C}^\phi$ be as in Lemma 3.0.22. Then by Lemma 3.0.22 $\boldsymbol{C}^\phi$ is isomorphic to $[0,1]_{Ł}$, and by Lemma 3.0.20 we obtain an interpretation $\|\dots\|_\phi$ into $\boldsymbol{C}^\phi$ such that $\|\psi\|_\phi < 1$. Hence $\psi \notin \mathrm{stTAUT}(Ł\forall)$.

Conversely, suppose $\psi \notin \mathrm{stTAUT}(Ł\forall)$. Let $\boldsymbol{A} \in \mathbb{K}$ be not isomorphic to $[0,1]_{\mathrm{G}}$ and $\mathbf{M}$ be a $[0,1]_{Ł}$-structure such that $\|\psi\|^{[0,1]_{Ł}}_{\mathbf{M}} < 1$. Then $\boldsymbol{A}$ has either a Łukasiewicz component or a product component, $\boldsymbol{C}$ say. We define an $\boldsymbol{A}$-structure $\mathbf{M}'$ as follows:

- The domain of $\mathbf{M}'$ is the domain M of $\mathbf{M}$, and the interpretation of all constant symbols and function symbols is as in $\mathbf{M}$.
- For all $d_0 \in M$, let $U_{\mathbf{M}'}(d_0)$ be such that $U_{\mathbf{M}'}(d_0) \in C$, and $\overline{0}^{\boldsymbol{C}} < \inf\{U_{\mathbf{M}'}(d) \mid d \in M\} < U_{\mathbf{M}'}(d_0) < \sup\{U_{\mathbf{M}'}(d) \mid d \in M\} < \overline{1}^{\boldsymbol{C}}$.
- Before defining the interpretation of the other predicate symbols, we note that for all $d \in M$, $\overline{0}^{\boldsymbol{C}} < \|\phi\|^{\boldsymbol{A}}_{\mathbf{M}'} < \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}'} < \overline{1}^{\boldsymbol{C}}$, and hence the algebra $\boldsymbol{C}^\phi$ defined from $\boldsymbol{C}$ as in Lemma 3.0.20, is isomorphic to $[0,1]_{Ł}$ via an isomorphism h.

Then define for every n-ary predicate P and for every $d_1, \ldots, d_n \in M$ we have: $P_{\mathbf{M}'}(d_1, \ldots, d_n) = h(P_{\mathbf{M}}(d_1, \ldots, d_m))$.

Now let us write $\|\ldots\|$ for $\|\ldots\|^{\boldsymbol{A}}_{\mathbf{M}'}$, and let us define an interpretation $\|\ldots\|_\phi$ from $\|\ldots\|$ as in Lemma 3.0.20. Then by Lemma 3.0.20 we obtain $1^{\boldsymbol{C}} > \|\psi\|_\phi = \|\psi^\phi\|$. Moreover, since for all $d \in M$, $\|\phi\| < \|U(d)\|$ and $\|\phi\|, \|U(d)\| \in \boldsymbol{C}^\phi \setminus \{1\}$, by Lemma 3.0.19, $\|\gamma\| < \overline{1}^{\boldsymbol{C}}$. Finally, again by Lemma 3.0.19, $\|(\exists x)(\psi^\phi \uparrow U(x))\| = \max\{\sup\{\|U(d)\| \mid d \in M\}, \|\psi^\phi\|\} < \overline{1}^{\boldsymbol{C}}$. Thus, we have $\|\psi^\star\| < \overline{1}^{\boldsymbol{C}} \leq 1$, and $\psi^\star \notin \mathrm{TAUT}(\mathbb{K})$. This concludes the proof of Theorem 3.0.21. □

We now prove the non-arithmeticity of sets of the form $\mathrm{TAUT}(\mathbb{K})$, $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$, $\mathrm{SAT}(\mathbb{K})$, and $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$, where $\mathbb{K}$ is a set of standard BL-algebras which contains a BL-algebra which is either isomorphic to $[0,1]_\Pi$ or begins with Π. See e.g. [22]. We start with the SAT classes.

THEOREM 3.0.23. *Suppose that* $\mathbb{K}$ *contains a standard* BL*-algebra which begins with* Π *or is isomorphic to* $[0,1]_\Pi$. *Then* $\mathrm{SAT}(\mathbb{K})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *are not arithmetical.*

Proof. We work in a language containing the language of Peano Arithmetic PA, including a binary predicate symbol $\leq$ for order, plus an additional unary predicate U. For every formula ψ of PA, we denote by $\psi^{\neg\neg}$ the formula obtained replacing every atomic subformula γ of ψ by $\neg\neg\gamma$. Note that if a BL-chain either begins with Π or is isomorphic to $[0,1]_\Pi$, then $\psi^{\neg\neg}$ is interpreted either as 0 or as 1. We now consider the following formulae:

- $\theta_1 = (\forall x)\neg\neg U(x) \,\&\, \neg(\forall x)U(x)$.
- $\theta_2 = (\forall x)(\forall y)((U(x) \to (U(y) \,\&\, U(x))) \to U(y))$.
- The conjunction of all formulae $\sigma^{\neg\neg}$ such that σ is an axiom of Q^+. We denote such conjunction by θ_3.[2]
- The formula $\theta_4 = (\forall x)(U(S(x)) \leftrightarrow ((\forall y)((y \leq x)^{\neg\neg} \to U(y)))^3$.

LEMMA 3.0.24. *If* $\boldsymbol{A}$ *is a standard* BL*-algebra and* $\mathbf{M}$ *is an* $\boldsymbol{A}$*-structure such that* $\|\theta_1 \,\&\, \theta_2\|^{\boldsymbol{A}}_{\mathbf{M}} > 0$*, then* $\boldsymbol{A}$ *begins with* Π *or is isomorphic to* $[0,1]_\Pi$.

Proof. Suppose first that $\boldsymbol{A}$ begins with a Łukasiewicz component (or is isomorphic to $[0,1]_{\text{Ł}}$). If for some b, $\|U(b)\|^{\boldsymbol{A}}_{\mathbf{M}}$ is in the first component, then $\|(\forall x)U(x)\|^{\boldsymbol{A}}_{\mathbf{M}} = \|(\forall x)\neg\neg U(x)\|^{\boldsymbol{A}}_{\mathbf{M}}$, and $\|\theta_1\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$. If either $\boldsymbol{A}$ begins with a Gödel component or has no first t-norm component, then we must have $\inf\{\|U(d)\| \mid d \in M\} = 0$ and $\|U(d)\| > 0$ for all $d \in M$, otherwise $\|\theta_1\| = 0$. Hence, if either $\boldsymbol{A}$ begins with a Gödel component or has no first t-norm component, then for all $d \in M$ there must be $d' \in M$ such that $\|U(d')\|^{\boldsymbol{A}}_{\mathbf{M}} < \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$ and $\|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$ and $\|U(d')\|^{\boldsymbol{A}}_{\mathbf{M}}$ are not in the same Wajsberg component. This implies $\|(U(d') \to ((U(d') \,\&\, U(d))) \to U(d)\|^{\boldsymbol{A}}_{\mathbf{M}} = \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$, and $\|\theta_2\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$. □

[2] We assume that θ_3 includes $\gamma^{\neg\neg}$ when γ is an axiom of equality or the crispness axiom $x = y \vee \neg(x = y)$ for $=$.

LEMMA 3.0.25.
If $\boldsymbol{A}$ is a standard BL*-algebra,* $\mathbf{M}$ *is an* $\boldsymbol{A}$*-model, and* $\|\theta_1 \,\&\, \theta_2 \,\&\, \theta_3 \,\&\, \theta_4\|^{\boldsymbol{A}}_{\mathbf{M}} > 0$*, then:*

(1) For every formula $\gamma(x_1,\ldots,x_n)$ *in the language of PA whose free variables are among* $x_1,\ldots,x_n$ *and for all* $d_1,\ldots,d_n \in M$*,* $\|\gamma^{\neg\neg}(d_1,\ldots,d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} \in \{0,1\}$*.*

(2) Let $\mathbf{M}^{\neg\neg}$ *be the classical structure whose domain*[3] *is* M*, in which function and constant symbols are interpreted as in* $\mathbf{M}$ *and such that for every* n*-ary predicate symbol* P *and for all* $d_1,\ldots,d_n \in M$*, one has* $\mathbf{M}^{\neg\neg} \models P(d_1,\ldots,d_n)$ *iff* $\|P^{\neg\neg}(d_1,\ldots,d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$*. Then* $\mathbf{M}^{\neg\neg}$ *is a (classical) model of* Q^+*.*

(3) $\mathbf{M}^{\neg\neg}$ *is isomorphic to the standard model* $\mathbf{N}$ *of natural numbers.*

Proof. (1) Let us write $\|\ldots\|$ instead of $\|\ldots\|^{\boldsymbol{A}}_{\mathbf{M}}$. By Lemma 3.0.24, $\boldsymbol{A}$ begins with Π or is isomorphic to $[0,1]_\Pi$, and hence it is an SBL-chain. Therefore, for every sentence ψ, $\|\psi^{\neg\neg}\| \in \{0,1\}$.

(2) By an easy induction, we see that $\mathbf{M}^{\neg\neg} \models \psi$ iff $\|\psi^{\neg\neg}\| = 1$ for every sentence ψ. Moreover by (1), $\|\theta_3\| \in \{0,1\}$, and by our assumptions, $\|\theta_3\| > 0$. Hence, $\|\theta_3\| = 1$. It follows that every axiom of Q^+ is true in $\mathbf{M}^{\neg\neg}$.

(3) Suppose, by the way of contradiction, that $\mathbf{M}^{\neg\neg}$ is a non-standard model of Q^+. First of all, note that

(+) $\inf\{\|U(d)\| \mid d \in M\} = 0$ and for all $d \in M$, $\|U(d)\| > 0$,

(otherwise $\|\theta_1\| = 0$). We claim that there is a $c \in M$ such that for all $b \in M$,

(++) if $\mathbf{M}^{\neg\neg} \models c \le b$, then $\|U(S(b))\| \le (\|(\forall x)((x \le^{\neg\neg} b) \to U(x))\|)^2$.

Indeed, if

(*) $\|U(S(b))\| > (\|(\forall x)((x \le^{\neg\neg} b) \to U(x))\|)^2$,

then $\|U(S(b)) \to ((\forall x)(x \le^{\neg\neg} b \to U(x)))^3\| \le \|(\forall x)(x \le^{\neg\neg} b \to U(x))\|$, and if (*) holds for unboundedly many b, (that is, if for all $c \in M$ there is a $b \in M$ such that $\mathbf{M}^{\neg\neg} \models c \le b$ and (*) holds), then $\|\theta_4\| = 0$, against our assumption.

Since $\inf\{\|U(d) \mid d \in M\} = 0$, there is a $d \in M$ such that $\|U(d)\| < 1$, and since if condition (++) holds for $c \in M$, then it holds for all $c' \in M$ such that $\mathbf{M}^{\neg\neg} \models c \le c'$, we can suppose, without loss of generality, $\mathbf{M}^{\neg\neg} \models d < c$, and by (++), $\|U(c)\| \le \|U(d)\|^2 < 1$. Moreover, by the previous observation (that is, condition (*) is upward preserved), we may assume without loss of generality that c is non-standard. Hence, by an iterated use of (++) we see that for every natural number n, $\|U(c+n)\| \le \|U(c)\|^{2^n}$, and $\inf\{\|U(c+n)\| \mid n \in \mathsf{N}\} = 0$. But since c is non-standard, then $\mathbf{M}^{\neg\neg} \models c + c > c + n$ for every $n \in \mathsf{N}$, and hence, by (*), $\|U(c+c)\| \le (\|U(c+n)\|)^2$ for every natural number n. It follows $\|U(c+c)\| = 0$, contradicting condition (+).

We have derived a contradiction from our assumption that $\mathbf{M}^{\neg\neg}$ was not standard, and hence $\mathbf{M}^{\neg\neg}$ is isomorphic to $\mathbf{N}$. □

[3] If the predicate $=$ for equality is not interpreted as crisp equality, then instead of M we have to take as domain the set of all equivalence classes of elements of M modulo the equivalence $\equiv$ defined by $a \equiv b$ iff $\|a =^{\neg\neg} b\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$.

We conclude the proof of Theorem 3.0.23. Since the set of sentences of PA which are true in $\mathbf{N}$ is not arithmetical, it suffices to prove:

LEMMA 3.0.26. *Let ψ be any PA-sentence. Then,* $\mathbf{N} \models \psi$ *iff* $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K})$ *iff* $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

Proof. If $\mathbf{N} \not\models \psi$, then we have seen that if $\boldsymbol{A} \in \mathbb{K}$ and $\mathbf{M}$ is an $\boldsymbol{A}$-structure such that $\|\theta_1 \& \theta_2 \& \theta_3 \& \theta_4\|^{\boldsymbol{A}}_{\mathbf{M}} > 0$, then $\mathbf{M}^{\neg\neg}$ is isomorphic to $\mathbf{N}$. Moreover, an easy induction shows that for every formula $\gamma(x_1, \ldots, x_n)$ of T and for all $d_1, \ldots, d_n \in M$, one has $\mathbf{M}^{\neg\neg} \models \gamma(d_1, \ldots, d_n)$ iff $\|\gamma^{\neg\neg}(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Hence, $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$, and $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$. *A fortiori*, $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \notin \mathrm{SAT}(\mathbb{K})$.

Conversely, assume $\mathbf{N} \models \psi$. Take $\boldsymbol{A} \in \mathbb{K}$ which either begins with Π or is isomorphic to $[0,1]_\Pi$, and define an $\boldsymbol{A}$-structure $\mathbf{M}$ as follows: the domain M of $\mathbf{M}$ is N and the constants and the function symbols of PA are interpreted as in $\mathbf{N}$. Moreover, for $d_1, \ldots, d_n \in M$ and for every n-ary predicate of the language of PA, we set $P_{\mathbf{M}}(d_1, \ldots, d_n) = 1$ if $\mathbf{N} \models P(d_1, \ldots, d_n)$ and $P_{\mathbf{M}}(d_1, \ldots, d_n) = 0$ otherwise. Finally, $U_{\mathbf{M}}(n) = (\frac{1}{2})^{3^n}$, where each real number is thought of as an element of the first component $[0,1]_\Pi$ of $\boldsymbol{A}$.

It is easily seen that $\|\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Hence, $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K})$, and *a fortiori* $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \& \psi^{\neg\neg} \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$. This ends the proof of Lemma 3.0.26. ☐

At this point the proof of Theorem 3.0.23 is immediate. ☐

COROLLARY 3.0.27. *For every* $\mathrm{L} \in \{\Pi, \mathrm{BL}, \mathrm{SBL}, \Pi\oplus\}$*, the sets* $\mathrm{stSAT}(\mathrm{L}\forall)$ *and* $\mathrm{stSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are not arithmetical.*

We now consider the complexity of $\mathrm{TAUT}(\mathbb{K})$ and $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ where $\mathbb{K}$ is a set of standard BL-chains such that at least one of them begins with Π or is isomorphic to $[0,1]_\Pi$. Let T, θ_1, θ_2, θ_3 and θ_4 be as in the proof of Theorem 3.0.23.

THEOREM 3.0.28. *Let ψ be a sentence of PA and $\psi^\star$ the sentence* $(\theta_1 \& \theta_2 \& \theta_3 \& \theta_4) \to \psi^{\neg\neg}$. *Then,* $\mathbf{N} \models \psi$ *iff* $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$ *iff* $\psi^\star \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$. *Hence,* $\mathrm{TAUT}(\mathbb{K})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *are not arithmetical.*

Proof. Suppose first $\mathbf{N} \not\models \psi$. Let $\boldsymbol{A} \in \mathbb{K}$ be such that $\boldsymbol{A}$ begins with Π or is isomorphic to $[0,1]_\Pi$, and consider an $\boldsymbol{A}$-structure $\mathbf{M}$ as follows: the domain M of $\mathbf{M}$ is N and the constants and the function symbols of T are interpreted as in $\mathbf{N}$. Moreover, for every n-ary predicate of PA and for all $k_1, \ldots, k_n \in \mathsf{N}$, we stipulate that $P_{\mathbf{M}}(k_1, \ldots, k_n) = 1$ if $\mathbf{N} \models P(k_1, \ldots, k_n)$ and $P_{\mathbf{M}}(k_1, \ldots, k_n) = 0$ otherwise. Finally, $U_{\mathbf{M}}(n) = (\frac{1}{2})^{3^n}$, where each real number is thought of as an element of the first component $[0,1]_\Pi$ of $\boldsymbol{A}$. It is easily seen that $\|\theta_1 \& \theta_2 \& \theta_3 \& \theta_4\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, and $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$. Hence, $\psi^\star \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$, and *a fortiori* $\psi^\star \notin \mathrm{TAUT}(\mathbb{K})$.

Now suppose that $\mathbf{N} \models \psi$, and let us prove that $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$ (hence, *a fortiori*, $\psi^\star \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$). Thus, let $\boldsymbol{A} \in \mathbb{K}$ and $\mathbf{M}$ be any $\boldsymbol{A}$-structure, and let us prove that $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Once again,we will write $\|\ldots\|$ instead of $\|\ldots\|^{\boldsymbol{A}}_{\mathbf{M}}$. Clearly, $\|\psi^\star\| = 1$ if $\|\theta_1 \& \theta_2 \& \theta_3 \& \theta_4\| = 0$. Hence, we may assume, without loss of generality, that $\|\theta_1 \& \theta_2 \& \theta_3 \& \theta_4\| > 0$. By Lemma 3.0.24 we have that either $\boldsymbol{A}$ begins with Π or is isomorphic to $[0,1]_\Pi$, and by Lemma 3.0.25 we can construct a classical model $\mathbf{M}^{\neg\neg}$ which is isomorphic to $\mathbf{N}$ and such that for every sentence γ in the language of PA, one has $\mathbf{M}^{\neg\neg} \models \gamma$ iff $\|\gamma^{\neg\neg}\| = 1$. Hence, if $\mathbf{N} \models \psi$, then $\|\psi^{\neg\neg}\| = 1$, and finally $\|\psi^\star\| = 1$. Summing up, if $\mathbf{N} \models \psi$, then $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$, and the claim is proved. $\square$

COROLLARY 3.0.29. *For every* $\mathrm{L} \in \{\Pi, \mathrm{BL}, \mathrm{SBL}, \Pi\oplus\}$*, the sets* $\mathrm{stTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{stTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are not arithmetical.*

We now prove that with a finite number of exceptions, for all classes $\mathbb{K}$ of standard BL-algebras, $\mathrm{TAUT}(\mathbb{K})$ is not arithmetical. We first prove that $\mathrm{TAUT}(\mathbb{K})$ is not arithmetical when $\mathbb{K}$ contains a standard BL-algebra with a product component (not necessarily the first component).

From the proof of Theorem 3.0.28 it follows that the set of standard tautologies of $\Pi\forall$ of the form $\psi = \theta_3 \to \gamma$ (where θ_3 is as in the proof of Theorem 3.0.28) is not arithmetical. Hence, it suffices to find an algorithm that associates to every sentence ψ of the form shown above a sentence $\psi^\star$ such that $\psi \in \mathrm{stTAUT}(\Pi\forall)$ if, only if, $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$. Once again, U denotes a unary predicate symbol not in the language of $\Pi\forall$, and we define $\phi = (\forall x)U(x)$. Moreover, let us define:

$$\theta = (\forall x)(U(x) \uparrow \phi),$$

$$\sigma = (\forall x)(\forall y)((U(x) \uparrow U(y)) \leftrightarrow (U(y) \uparrow U(x))).$$

For every sentence ψ in the language of $\Pi\forall$ of the form $\theta_3 \to \gamma$, we define $\psi^\star = ((\theta \,\&\, \sigma) \to \psi^\phi) \vee (\exists x)(\psi^\phi \uparrow U(x))$.

THEOREM 3.0.30. *Let* $\mathbb{K}$ *a set of standard* BL*-algebras containing one with a product component (not necessarily the first component). Then for every sentence* ψ *of the form* $\theta_3 \to \gamma$ *and not containing the symbol* U*, one has:* $\psi \in \mathrm{stTAUT}(\Pi\forall)$ *iff* $\psi^\star \in \mathrm{TAUT}(\mathbb{K})$*. Thus (by Theorem 3.0.28)* $\mathrm{TAUT}(\mathbb{K})$ *is not arithmetical.*

Proof. We start with one lemma.

LEMMA 3.0.31. *Let* $\boldsymbol{A}$ *be a standard* BL*-algebra, let* $\star$ *and* $\to$ *be its monoid operation and its residual respectively, and let* $\mathbf{M}$ *be an* $\boldsymbol{A}$*-structure such that* $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$*. Let us write* $\|\dots\|$ *for* $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}}$*. Then:*

- *(i) For all* $a \in M$*,* $\|U(a)\|$ *and* $\|\phi\|$ *do not belong to the same Wajsberg component.*
- *(ii) The set* $\{\|U(a)\| \mid a \in M\}$ *has no minimum (hence it is infinite).*
- *(iii) There is an* $a \in M$ *such that the set* $\{\|U(b))\| \mid \|U(b))\| \leq \|U(a)\|\}$ *is included in a single Wajsberg component* $\boldsymbol{W}$ *of* $\boldsymbol{A}$*, which is necessarily a cancellative component. Moreover,* $\|\psi^\phi\| \in (W \cup \{\|\phi\|\}) \setminus \{1\}$*.*

Proof. (i) Suppose that $\|U(a)\|$ and $\|\phi\|$ are in the same Wajsberg component. Then since $\|U(b)\| \geq \|\phi\|$ for all $b \in M$, by Lemma 3.0.19 $\|U(a) \uparrow \phi\| = \|U(a)\|$, and $\|(\forall x)(U(x) \uparrow \phi)\| = \inf\{\|U(a)\| \mid a \in M\} = \|\phi\|$. Since $\|\phi\| \leq \|\psi^\phi\|$, one would have $\|\psi^\star\| = 1$, a contradiction.

(ii) If $\{\|U(a)\| \mid a \in M\}$ has a minimum, then this minimum is equal to $\|\phi\|$, which is in contradiction with (i).

(iii) Suppose, by way of contradiction, that for every $a \in M$ there is a $b \in M$ such that $\|U(b)\| < \|U(a)\|$ and $\|U(a)\|, \|U(b)\|$ do not belong to the same Wajsberg component. Then, by Lemma 3.0.19, we have $\|U(b) \uparrow U(a)\| = \|U(a)\|$, and $\|U(a) \uparrow U(b)\| = 1$. Hence $\|\sigma\| = \inf\{\|U(a)\| \mid a \in M\} = \|\phi\|$. It follows that $\|\sigma\| \leq \|\psi^{\phi}\|$, and $\|\psi^{\star}\| = 1$, a contradiction. Thus there are $a \in M$ and a Wajsberg component $\boldsymbol{W}$ of $\boldsymbol{A}$ such that for all $b \in M$, if $\|U(b)\| \leq \|U(a)\|$, then $\|U(b)\| \in W$. Now $\boldsymbol{W}$ is either a cancellative hoop or (the reduct of) a Wajsberg algebra. In the latter case $\|\phi\|$, being the infimum of a subset of W (namely, of $\{\|U(b)\| \mid \|U(b)\| \in W\}$) would be in $\boldsymbol{W}$. This contradicts (i). Hence $\boldsymbol{W}$ is a cancellative component. Finally, suppose $\|\psi^{\phi}\| \notin (W \cup \{\|\phi\|\}) \setminus \{1\}$. Then clearly $\|\psi^{\phi}\| \neq 1$, otherwise $\|\psi^{\star}\| = 1$. Thus $\|\psi^{\phi}\| \notin W \cup \{\|\phi\|\}$. Now $\|\psi^{\phi}\| \geq \|\phi\|$. Thus if $b \in M$ is such that $\|U(b)\| \in W \setminus \{1\}$, then $\|\psi^{\phi}\| > \|U(b)\|$, and $\|\psi^{\phi}\|$ and $\|U(b)\|$ do not belong to the same component. Therefore, $\|\psi^{\phi} \uparrow U(b)\| = 1$, and $\|(\exists x)(\psi^{\phi} \uparrow U(x))\| = 1$. It follows that $\|\psi^{\star}\| = 1$, a contradiction. □

The rest of the proof of Theorem 3.0.30 If $\psi^{\star} \notin \mathrm{TAUT}(\mathbb{K})$, then by Lemma 3.0.31, there is an $a \in M$ such that the set $\{\|U(b))\| \mid \|U(b))\| \leq \|U(a)\|\}$ is included in a single cancellative component $\boldsymbol{W}$ of $\boldsymbol{A}$. Moreover, $\|\psi^{\phi}\| \in (W \cup \{\|\phi\|\}) \setminus \{1\}$. Let $m = \sup(W)$, and let $C = (W \setminus \{1\}) \cup \{\|\phi\|, m\}$. Then C is the domain of a product component $\boldsymbol{C}$ of $\boldsymbol{A}$. Let $\boldsymbol{C}^{\phi}$ be constructed from $\boldsymbol{C}$ as in the proof of Lemma 3.0.20. Then, since $\|\phi\| = \min(\boldsymbol{C})$, $\boldsymbol{C}^{\phi} = \boldsymbol{C}$. Now let $S = \{a \in A \mid \forall b \in C(b \leq a)\}$, and let $\| \dots \|_{\phi}$ be defined as in Lemma 3.0.20, taking into account that $\boldsymbol{C}^{\phi} = \boldsymbol{C}$. Then by Lemma 3.0.31 (iii), $\|\psi^{\phi}\| \notin S$, and by Lemma 3.0.20 we obtain $\|\psi\|_{\phi} \neq 1$. Hence $\psi \notin \mathrm{stTAUT}(\Pi\forall)$.

Conversely, suppose that $\psi \notin \mathrm{stTAUT}(\Pi\forall)$. Then there is a $[0,1]_{\Pi}$-structure $\mathbf{M}$ such that $\|\psi\|^{[0,1]_{\Pi}}_{\mathbf{M}} \neq 1$. Then we must have $\|\theta_3\|^{[0,1]_{\Pi}}_{\mathbf{M}} = 1$, and hence the domain M of $\mathbf{M}$ must be infinite (its domain is a model of Q^{+}).

Now take an element $\boldsymbol{A} \in \mathbb{K}$ with a product t-norm component $\boldsymbol{C}$. Up to isomorphism, we may assume that $\mathbf{M}$ is a $\boldsymbol{C}$-structure. We define an $\boldsymbol{A}$-structure $\mathbf{M}'$ such that $\|\psi^{\star}\|^{\boldsymbol{A}}_{\mathbf{M}'} \neq 1$ as follows. The domain M' of $\mathbf{M}'$ is the domain M of $\mathbf{M}$. Moreover, for all $d \in M$, let $U_{\mathbf{M}'}(d)$ be such that $\overline{0}^{\boldsymbol{C}} < U_{\mathbf{M}'}(d)$, $\sup\{U_{\mathbf{M}'}(d)) \mid d \in M\} < \overline{1}^{\boldsymbol{C}}$, and $\inf\{U_{\mathbf{M}'}(d) \mid d \in M\} = \overline{0}^{\boldsymbol{C}}$. Then $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}'} = \overline{0}^{\boldsymbol{C}}$, and $\boldsymbol{C}^{\phi} = \boldsymbol{C}$ (cf. the proof of Theorem 3.0.28 for the construction of $\boldsymbol{C}^{\phi}$ from $\boldsymbol{C}$). Moreover, for every n-ary predicate P different from U, and for every $d_1, \dots, d_n \in M$, we define $P_{\mathbf{M}'}(d_1, \dots, d_n) = h(P_{\mathbf{M}}(d_1, \dots, d_n))$, where h is an isomorphism between $[0,1]_{\Pi}$ and $\boldsymbol{C}$. Thus if $P_{\mathbf{M}}(d_1, \dots, d_n) = 1$, then $P_{\mathbf{M}'}(d_1, \dots, d_n) = \overline{1}^{\boldsymbol{C}} = \sup(C)$. Note that the interpretation $\| \dots \|_{\phi}$ constructed from $\| \dots \| = \| \dots \|^{\boldsymbol{A}}_{\mathbf{M}'}$ according to Lemma 3.0.20 is just $\| \dots \|^{\boldsymbol{C}}_{\mathbf{M}}$. Hence, by Lemma 3.0.20, we obtain that $\|\psi^{\phi}\|^{\boldsymbol{A}}_{\mathbf{M}'} = \|\psi\|^{\boldsymbol{C}}_{\mathbf{M}} \in \boldsymbol{C} \setminus \{\overline{1}^{\boldsymbol{C}}\}$. Moreover, for all $d \in M$, $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} < \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$, and $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}}$ is not in the same Wajsberg component as $\|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$. Thus, by Lemma 3.0.19, $\|\sigma\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Also, all elements of the form $\|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}$ with $d \in M$ are in the same Wajsberg component. Hence, by Lemma 3.0.19 again, $\|\theta\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, and $\|(\sigma \mathbin{\&} \theta) \to \psi^{\phi}\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$. Finally,

$$\begin{aligned}\|(\exists x)(\psi^{\phi} \uparrow U(x))\|^{\boldsymbol{A}}_{\mathbf{M}} &= \sup\{\|\psi^{\phi} \uparrow U(d)\|^{\boldsymbol{A}}_{\mathbf{M}} \mid d \in M\} = \\ &= \sup\{\max\{\|\psi^{\phi}\|^{\boldsymbol{A}}_{\mathbf{M}}, \|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}}\} \mid d \in M\} = \\ &= \max\{\|\psi^{\phi}\|^{\boldsymbol{A}}_{\mathbf{M}}, \sup\{\|U(d)\|^{\boldsymbol{A}}_{\mathbf{M}} \mid d \in M\}\} < 1.\end{aligned}$$

It follows that $\|\psi^{\star}\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$, and $\psi^{\star} \notin \mathrm{TAUT}(\mathbb{K})$. This concludes the proof of Theorem 3.0.30. $\square$

Now we extend this non-arithmeticity result to the case where $\mathbb{K}$ contains a standard algebra with at least one Łukasiewicz t-norm component which is neither the first nor the last component. Clearly, we may assume that there is no algebra in $\mathbb{K}$ with a product component, otherwise we already know that $\mathrm{TAUT}(\mathbb{K})$ is not arithmetical. In order to prove our claim, we will directly reduce the set of sentences which are valid in the standard model $\mathbf{N}$ of natural numbers to $\mathrm{TAUT}(\mathbb{K})$.

Let θ_3, U and $^{\neg\neg}$ be as in the proof of Theorem 3.0.23, let $\phi = (\forall x)U(x)$ and let $^{\phi}$ be as in Theorem 3.0.21. Consider the following formulae:

- $\gamma_1 = (\forall x)\neg\neg U(x)$.
- γ_2 is defined to be the conjunction of all formulae of the form
$$\forall x_1 \ldots \forall x_n(\neg P(x_1, \ldots, x_n) \vee \neg\neg P(x_1, \ldots, x_n)),$$
where P is a predicate symbol of PA, including equality if it is not assumed to be crisp.
- $\delta_1 = (\exists x)((U(x) \uparrow \phi) \vee (U(x) \to \phi^2))$.
- $\delta_2 = (\exists x)(U(S(x))^2 \to (\exists y)(y \leq^{\neg\neg} x \mathbin{\&} U(y)))$.

Now let for every formula ψ in the language of PA,

$$\psi^{\star} = (\gamma_1 \mathbin{\&} \gamma_2 \mathbin{\&} \theta_3) \to (\psi^{\neg\neg} \vee \phi \vee \delta_1 \vee \delta_2).$$

THEOREM 3.0.32. *Let $\mathbb{K}$ be a class of standard* BL*-algebras containing an element with a Łukasiewicz component which is neither the first one nor the last one. Then for every formula ψ of the language of arithmetic, one has: $\mathbf{N} \models \psi$ iff $\psi^{\star} \in \mathrm{TAUT}(\mathbb{K})$.*

Proof. ($\Rightarrow$). We argue contrapositively. Suppose that $\psi^{\star} \notin \mathrm{TAUT}(\mathbb{K})$. Let $\boldsymbol{A} \in \mathbb{K}$ and let $\mathbf{M}$ be an $\boldsymbol{A}$-structure such that $\|\psi^{\star}\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. Once again, let us write $\|\ldots\|$ instead of $\|\ldots\|^{\boldsymbol{A}}_{\mathbf{M}}$.

LEMMA 3.0.33.

(i) There is a Wajsberg component $\boldsymbol{W}$ of $\boldsymbol{A}$, isomorphic to $[0,1]_{Ł}$, which is not the first component of $\boldsymbol{A}$, and for all $d \in M$, $\|U(d)\| \in W \backslash \{1\}$. Moreover, $\|\phi\| \in W$.

(ii) For every PA-sentence with parameters in M, $\|\lambda^{\neg\neg}\| \in \{0,1\}$.

(iii) Define a model $\mathbf{M}^{\neg\neg}$ from $\mathbf{M}$ as in the proof of Theorem 3.0.23. Then $\mathbf{M}^{\neg\neg}$ is isomorphic to the standard model of natural numbers.

Proof. (i) For all $d \in M$, $\|U(d)\| \neq 1$, otherwise $\|\psi^\star\| \geq \|\delta_1\| = 1$. Moreover all the elements of the form $\|U(d)\|$ with $d \in M$ belong to the same Wajsberg component $\boldsymbol{W}$, because if $\|U(a)\| < \|U(b)\|$ and $\|U(a)\|$, $\|U(b)\|$ do not belong to the same component, then since for all $a \in M$, $\|\phi\| \leq \|U(a)\|$, by Lemma 3.0.19 we obtain that $\|U(b) \uparrow \phi\| = 1$, and $\|\psi^\star\| \geq \|\delta_1\| = 1$. For the same reason, $\|\phi\| \in W$. Now $\boldsymbol{W}$ cannot be a cancellative component, because by our assumptions $\boldsymbol{A}$ contains no product component. Moreover it cannot be isomorphic to $\boldsymbol{B}_2$, otherwise for all $d \in M$ we would have $\|U(d) \to \phi^2\| = 1$, and $\|\psi^\star\| \geq \|\delta_1\| = 1$. Hence $\boldsymbol{W}$ must be an isomorphic copy of $[0,1]_{\text{Ł}}$. Now if $\boldsymbol{W}$ were the first component, then for all $d \in M$, $\|\neg\neg U(d)\| = \|U(d)\|$, $\|\gamma_1\| = \|\phi\|$, and $\|\psi^\star\| \geq \|\gamma_1 \to (\psi^{\neg\neg} \vee \phi)\| = 1$, which is impossible.

(ii) The proof is by induction on the formula λ. Actually, the induction steps are immediate, so it suffices to prove the claim for atomic formulae $P(d_1, \dots, d_n)$. If $P^{\neg\neg}(d_1, \dots, d_n) \notin \{0,1\}$, then $P^{\neg\neg}(d_1, \dots, d_n)$ is in the first Wajsberg component, otherwise by Lemma 3.0.19 $\|\neg\neg P(d_1, \dots, d_n)\| = 1$. It would follow that $\|\gamma_2\|$ is in the first Wajsberg component and is different from 1. Since $\|\phi\|$ is the infimum of a Wajsberg component which is not the first component, $\|\gamma_2\| \leq \|\phi\| \leq \|\psi^{\neg\neg} \vee \phi\|$, and $\|\psi^\star\| = 1$, which is impossible.

(iii) First of all, we must have $\|\theta_3\| > 0$, otherwise $\|\psi^\star\| = 1$. Since $\|\theta_3\| \in \{0,1\}$, $\|\theta_3\| = 1$, and hence $\mathbf{M}^{\neg\neg}$ is a model of Q^+ (recall that if λ is a sentence of *PA*, then $\mathbf{M}^{\neg\neg} \models \lambda$ iff $\|\lambda^{\neg\neg}\| = 1$). Now $\|\delta_2\| < 1$, otherwise $\|\psi^\star\| = 1$. Hence, for all $d \in M$ we must have $\|U(S(d))\|^2 > \sup\{\|U(b)\| \mid \mathbf{M}^{\neg\neg} \models b \leq d\}$. Now let for $n \in \mathsf{N}$, d_n be the realization of n in $\mathbf{M}^{\neg\neg}$. Then $\|U(d_1)\|^2 > \|U(d_0)\|$, $\|U(d_2)\|^2 > \|U(d_1)\|$, etc. Continuing, since by (i), for every n, $\|U(d_n)\| \in W \setminus \{1\}$ and since $\boldsymbol{W}$ is isomorphic to $[0,1]_{\text{Ł}}$, we easily obtain that $\sup\{\|U(d_n)\| \mid n \in \mathsf{N}\} = \sup(W \setminus \{1\})$. Now suppose that d is a non-standard element in $\mathbf{M}^{\neg\neg}$. Then we should have that $\|U(S(d))\|^2 > \sup(W \setminus \{1\})$, which is impossible, because by (i) we have $\|U(S(d))\| \in \boldsymbol{W} \setminus \{1\}$. □

The rest of the proof of Theorem 3.0.32 At this point, the proof of ($\Rightarrow$) is immediate: if $\|\psi^\star\| \neq 1$, then $\|\psi^{\neg\neg}\| = 0$, and $\mathbf{M}^{\neg\neg} \not\models \psi$. Since $\mathbf{M}^{\neg\neg}$ is isomorphic to $\mathbf{N}$, we conclude that $\mathbf{N} \not\models \psi$.

($\Leftarrow$) Suppose that $\mathbf{N} \not\models \psi$. Take $\boldsymbol{A} \in \mathbb{K}$ with a Łukasiewicz t-norm component $\boldsymbol{C}$ which is neither the first nor the last component. Clearly $\boldsymbol{C}$ is isomorphic to the Łukasiewicz t-norm. Let for every $q \in \mathsf{Q} \cap [0,1]$, $q^{\boldsymbol{C}}$ denote the isomorphic copy of q in $\boldsymbol{C}$. Define an $\boldsymbol{A}$-structure $\mathbf{M}$ as follows: the domain, M, of $\mathbf{M}$ is N, the function symbols and the constants of *PA* are interpreted as in $\mathbf{N}$, $=$ is interpreted as crisp equality (that is, $n =_{\mathbf{M}} m = 1$ if $n = m$ and $n =_{\mathbf{M}} m = 0$ otherwise, and $\leq$ is interpreted similarly as the usual order in $\mathbf{N}$. Then clearly $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$.

Now define $U_{\mathbf{M}}(n)$ recursively by $U_{\mathbf{M}}(0) = (\frac{1}{2})^{\boldsymbol{C}}$, and $U_{\mathbf{M}}(n+1)) = \left(\frac{U_{\mathbf{M}}(n)+3}{4}\right)^{\boldsymbol{C}}$. For all $n \in \mathsf{N}$ we have $(\frac{1}{2})^{\boldsymbol{C}} \leq \|U(n)\| < \|U(n+1)\| < \overline{1}^{\boldsymbol{C}}$. Thus $\|U(n)\| \in W \setminus \{1\}$. Moreover $\|U(n+1)^2\| = \frac{\|U(n)\|+1}{2} > \|U(n)\|$ and therefore $\|U(S(n))^2 \to (\exists x)(x \leq^{\neg\neg} n \,\&\, U(x))\| < \overline{1}^{\boldsymbol{C}}$. It follows that $\|\delta_2\| = \overline{1}^{\boldsymbol{C}} < 1$, because $\boldsymbol{C}$ is not the last component. Also, it is easily seen that $\|\gamma_1 \,\&\, \gamma_2\| = 1$ and that $\|(\psi^{\neg\neg} \vee \phi)\| = (\frac{1}{2})^{\boldsymbol{C}}$. Finally, for every $n \in \mathsf{N}$, $\|U(n) \to \phi^2\| \leq (\frac{1}{2})^{\boldsymbol{C}}$, and $\|U(n) \uparrow \phi\| = \|U(n)\|$. Thus $\|\delta_1\| = \sup\{\|U(n)\| \mid n \in \mathsf{N}\} = \overline{1}^{\boldsymbol{C}} < 1$. Hence, $\|\psi^\star\| = \overline{1}^{\boldsymbol{C}} < 1$. □

It follows:

THEOREM 3.0.34. *Let* $\mathbb{K}$ *be a class of standard* BL*-algebras containing an element* not *isomorphic to any of* $[0,1]_{\mathrm{G}}$, $[0,1]_{\text{Ł}}$, $[0,1]_{\text{Ł}} \oplus [0,1]_{\mathrm{G}}$, $[0,1]_{\mathrm{G}} \oplus [0,1]_{\text{Ł}}$, $[0,1]_{\text{Ł}} \oplus [0,1]_{\text{Ł}}$ *and* $[0,1]_{\text{Ł}} \oplus [0,1]_{\mathrm{G}} \oplus [0,1]_{\text{Ł}}$. *Then* $\mathrm{TAUT}(\mathbb{K})$ *is not arithmetical.*

Proof. If a standard BL-algebra is not among the ones shown above, then either it contains a product t-norm component or it contains a Łukasiewicz t-norm component which is neither the first component nor the last component. The claim follows from Theorems 3.0.30 and 3.0.32. $\square$

In the final part of this section we consider the arithmetical complexity of the standard semantics of logics of left-continuous t-norms in proper sense, i.e. logics extending $\mathrm{MTL}\forall$ but not $\mathrm{BL}\forall$, such as $\mathrm{MTL}\forall$ itself, $\mathrm{IMTL}\forall$ or $\mathrm{SMTL}\forall$. Since these logics are typically introduced as the logics of certain semantics of t-norms all their real chains are actually intended models and hence standard chains.

THEOREM 3.0.35. *Let* $\mathrm{L}\forall$ *be a logic enjoying the* $\mathrm{FS}\mathbb{K}\mathrm{C}$ *for* $\mathbb{K}$ *being the class of all real* L*-chains. Then* $\mathrm{realTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{realTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* $\mathbf{\Sigma_1}$*-complete, while* $\mathrm{realSAT}(\mathrm{L}\forall)$ *and* $\mathrm{realSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* $\mathbf{\Pi_1}$*-complete.*

Proof. It is a consequence of Propositions 2.0.8 and 2.0.21, and Theorems 2.0.13 and 2.0.20. $\square$

The prominent logics of left-continuous t-norms collected in Table 2 fall under the scope of this theorem and, thus, the complexities thereof are justified.

4 Complexity of finite and rational-chain semantics

Let $\boldsymbol{A}$ be any finite chain and let $\overline{0} = a_1 < \cdots < a_n = \overline{1}$ be the elements of A in increasing order. $\mathrm{L}_{\boldsymbol{A}}$, the first-order many-valued logic based on $\boldsymbol{A}$, is defined semantically as follows: $\mathrm{L}_{\boldsymbol{A}}$ has a language $\mathcal{P}_{\boldsymbol{A}}$ containing, besides parentheses, variables, predicate symbols, function symbols, a k-ary connective F for each k-ary operation $F^{\boldsymbol{A}}$ on $\boldsymbol{A}$ (different symbols for different operations), plus the quantifiers $\exists$ and $\forall$. For each connective F introduced in this way, we refer to $F^{\boldsymbol{A}}$ as *the realization of* F *in* $\boldsymbol{A}$. Since $\boldsymbol{A}$ is finite, each $\boldsymbol{A}$-structure $\mathbf{M} = \langle M, \langle P_{\mathbf{M}}\rangle_{P\in\mathcal{P}_{\boldsymbol{A}}}, \langle f_{\mathbf{M}}\rangle_{f\in\mathcal{P}_{\boldsymbol{A}}}\rangle$ is safe, because universal quantifiers are interpreted by taking the minimum value of instances, and existential quantifiers by taking the maximum value of instances.

For every set $T \cup \{\phi\}$ of sentences of $\mathrm{L}_{\boldsymbol{A}}$, the consequence relation $\models_{\mathrm{L}_{\boldsymbol{A}}}$ in $\mathrm{L}_{\boldsymbol{A}}$ is defined as follows: $T \models_{\mathrm{L}_{\boldsymbol{A}}} \phi$ iff for every $\boldsymbol{A}$-structure $\mathbf{M}$, one has that if $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \psi$ for all $\psi \in T$, then $\langle \boldsymbol{A}, \mathbf{M}\rangle \models \phi$. If $\emptyset \models_{\mathrm{L}_{\boldsymbol{A}}} \phi$, then we say that ϕ is an $\boldsymbol{A}$*-tautology* and we write $\models_{\mathrm{L}_{\boldsymbol{A}}} \phi$.

THEOREM 4.0.1. *For every finite chain* $\boldsymbol{A}$*, the set of* $\boldsymbol{A}$*-tautologies is* $\mathbf{\Sigma_1}$.

Proof. We will associate to $\boldsymbol{A}$ a recursively axiomatized classical first-order theory $T_{\boldsymbol{A}}$ and to every sentence ϕ of $\mathrm{L}_{\boldsymbol{A}}$ a formula ϕ^T in the language of $T_{\boldsymbol{A}}$ such that the map $\phi \mapsto \phi^T$ is computable and ϕ is an $\boldsymbol{A}$-tautology iff ϕ^T is a theorem of $T_{\boldsymbol{A}}$. This will clearly suffice to prove the theorem. First of all, the theory $T_{\boldsymbol{A}}$ has all function symbols in $\mathrm{L}_{\boldsymbol{A}}$. Moreover $T_{\boldsymbol{A}}$ has a constant symbol c^T for each element c of A, plus an additional constant u (for undefined) and an additional k-ary functional symbol f_ϕ for each formula ϕ with k free variables (the intended meaning is that $f_\phi(d_1, \ldots, d_k) = \|\phi(d_1, \ldots, d_k)\|^{\boldsymbol{A}}_{\mathbf{M}}$; in particular, if ϕ is a sentence, then f_ϕ is a constant). $T_{\boldsymbol{A}}$ has two binary predicate symbols $=$ and $<$. The intended meaning of $x = y$ is that x is equal to y, and the intended meaning of $x < y$ is that $x, y \in A$ and x is less than y in the order of $\boldsymbol{A}$. Finally $T_{\boldsymbol{A}}$ has two unary predicate symbols M and A. The intended meanings of $M(v)$ and of $A(v)$ are: v is in the domain M of individuals of the first-order structure we are referring to, and v is an element of the algebra $\boldsymbol{A}$, respectively.

It is a little bit boring to write all the axioms of $T_{\boldsymbol{A}}$, therefore we only describe them informally and we leave the obvious formal translation to the reader.

(0) Identity axioms for $=$.

(1) A group of axioms which say that the domain M of individuals is disjoint from A and u is neither in M nor in A.

(2) An axiom saying that every element is either in A or in M or u.

(3) Axioms describing the structure of $\boldsymbol{A}$, that is:
   - (3a) $c_i^T < c_j^T$ for each $1 \leq i < j \leq n$, and $\neg(c_i^T < c_j^T)$ for each $j \leq i$;
   - (3b) axioms of the form $\neg(c_i^T = c_j^T)$ for each $i \neq j$;
   - (3c) axioms of the form $F(e_1^T, \ldots, e_k^T) = e^T$ for each k-ary connective F and for all $e_1, \ldots, e_k, e \in A$ such that $F^{\boldsymbol{A}}(e_1, \ldots, e_k) = e$;
   - (3d) $(\forall v)(A(v) \leftrightarrow (v = c_1^T \vee \cdots \vee v = c_n^T)$ saying that $A = \{c_1, \ldots, c_n\}$;
   - (3e) axioms saying that for every connective F corresponding to an operation $F^{\boldsymbol{A}}$, $F(x_1, \ldots, x_k)$ is undefined (i.e. it is equal to u) if some of the x_i is not in A;
   - (3f) an axiom saying that if $x < y$ then $x, y \in A$.

(4) Axioms describing the structure $\mathbf{M}$, that is for every:
   - (4a) constant symbol d of $\mathrm{L}_{\boldsymbol{A}}$, an axiom saying that $d \in M$;
   - (4b) k-ary function symbol g of $\mathrm{L}_{\boldsymbol{A}}$, an axiom saying that for all $x_1, \ldots, x_k$, $g(x_1, \ldots, x_k) \in M$ if $x_1, \ldots, x_k \in M$ and $g(x_1, \ldots, x_k) = u$ otherwise.

(5) Axioms describing the behavior of $\|\phi(v_1, \ldots, v_k)\|^{\boldsymbol{A}}_{\mathbf{M}}$, that is:
   - (5a) if $v_1, \ldots, v_k$ are all in M, then $f_\phi(v_1, \ldots, v_k)$ is in M, otherwise we define $f_\phi(v_1, \ldots, v_k) = u$;
   - (5b) for every k-ary connective F of $\mathrm{L}_{\boldsymbol{A}}$ we define $f_{F(\phi_1,\ldots,\phi_k)}(v_1, \ldots, v_l) = F(f_{\phi_1}(v_1, \ldots, v_l), \ldots, f_{\phi_k}(v_1, \ldots, v_l))$ (thus $f_{F(\phi_1,\ldots,\phi_k)}(v_1, \ldots, v_l) = u$ if for some i, $f_{\phi_i}(v_1, \ldots, v_l) = u$, otherwise $f_{F(\phi_1,\ldots,\phi_k)}(v_1, \ldots, v_l) \in A$);
   - (5c) an axiom saying that for $j = 1, \ldots, n$, $f_{(\forall v)\phi}(v_1, \ldots, v_k) = c_j$ iff (i) $v_1, \ldots, v_k \in M$, (ii) for all $v \in M$, $f_\phi(v, v_1, \ldots, v_k) \geq c_j$ and (iii) for some $v \in M$, $f_\phi(v, v_1, \ldots, v_k) = c_j$;
   - (5d) an axiom saying that for $j = 1, \ldots, n$, $f_{(\exists v)\phi}(v_1, \ldots, v_k) = c_j$ iff (i) $v_1, \ldots, v_k \in M$, (ii) for all $v \in M$, $f_\phi(v, v_1, \ldots, v_k) \leq c_j$ and (iii) for some $v \in M$, $f_\phi(v, v_1, \ldots, v_k) = c_j$.

LEMMA 4.0.2.

(a) Let $\mathbf{M}$ *be an* $\boldsymbol{A}$*-structure for* $\mathrm{L}_{\boldsymbol{A}}$*. Then there is a model* $\mathbf{M}^*$ *of* $T_{\boldsymbol{A}}$ *(in the sense of classical logic) such that for every sentence* ϕ *of* $\mathrm{L}_{\boldsymbol{A}}$ *and for every* $c_i \in A$ *one has:* $\mathbf{M}^* \models f_\phi = c_i^T$ *iff* $\|\phi\|_{\mathbf{M}}^{\boldsymbol{A}} = c_i$.

(b) Let $\mathbf{H}$ *be a model of* $T_{\boldsymbol{A}}$ *(again, in the sense of classical logic). Then there is an* $\boldsymbol{A}$*-structure* $\mathbf{H}^+$ *for* $\mathrm{L}_{\boldsymbol{A}}$ *such that for every sentence* ϕ *of* $\mathrm{L}_{\boldsymbol{A}}$ *and for every* $c_i \in A$ *one has:* $\mathbf{H} \models f_\phi = c_i^T$ *iff* $\|\phi\|_{\mathbf{H}^+}^{\boldsymbol{A}} = c_i$.

Proof. (a) Given $\mathbf{M}$, we can assume without loss of generality that $M \cap A = \emptyset$. Let $u^* \notin M \cup A$, and consider the model $\mathbf{M}^*$ whose universe is $M^* = M \cup A \cup \{u^*\}$ and whose constants, operations and predicates are as follows:

(i) If c_i^T is a constant for an element of A, then $(c_i^T)^{\mathbf{M}^*} = c_i$; if c is a constant of $\mathrm{L}_{\boldsymbol{A}}$, then $c^{\mathbf{M}^*} = c^{\mathbf{M}}$; if c is a constant of the form f_ϕ, ϕ a sentence of $\mathrm{L}_{\boldsymbol{A}}$, then $c^{\mathbf{M}^*} = \|\phi\|_{\mathbf{M}}^{\boldsymbol{A}}$. Finally, u is interpreted as u^*.

(ii) If f is a k-ary function symbol in $\mathrm{L}_{\boldsymbol{A}}$, then $f^{\mathbf{M}^*}$ is defined by $f^{\mathbf{M}^*}(d_1,\dots,d_k) = f^{\mathbf{M}}(d_1,\dots,d_k)$ if $d_{1,\dots,}d_k \in M$, and $f^{\mathbf{M}^*}(d_1,\dots,d_k) = u^*$ otherwise; if F is a k-ary connective of $\mathrm{L}_{\boldsymbol{A}}$, then $F^{\mathbf{M}^*}(d_1,\dots,d_k) = F^{\boldsymbol{A}}(d_1,\dots,d_k)$ if $d_1,\dots,d_k \in A$, and $F^{\mathbf{M}^*}(d_1,\dots,d_k) = u^*$ otherwise; if $\phi(v_1,\dots,v_k)$ is a formula of $\mathrm{L}_{\boldsymbol{A}}$ with free variables $v_1,\dots,v_k$, then $f_\phi^{\mathbf{M}^*}(d_1,\dots,d_k) = \|\phi(d_1,\dots,d_k)\|_{\mathbf{M}}^{\boldsymbol{A}}$ if $d_1,\dots,d_k \in M$, and $f_\phi^{\mathbf{M}^*}(d_1,\dots,d_k) = u^*$ otherwise.

(iii) $\mathbf{M}^* \models d = e$ iff d is equal to e; $\mathbf{M}^* \models d < e$ iff $d, e \in A$ and $d < e$ in the order of $\boldsymbol{A}$; $\mathbf{M}^* \models M(d)$ iff $d \in M$ and $\mathbf{M}^* \models A(d)$ iff $d \in A$.

It is clear that for every formula $\phi(v_1,\dots,v_k)$, for every $c_i \in A$ and for every $d_1,\dots,d_k \in M$: $\mathbf{M}^* \models f_\phi(d_1,\dots,d_k) = c_i^T$ iff $\|\phi(d_1,\dots,d_k)\|_{\mathbf{M}}^{\boldsymbol{A}} = c_i$, and (a) follows.

(b) Let $\mathbf{H}$ be a model of $T_{\boldsymbol{A}}$; we define an algebra $\boldsymbol{A}^+$ and an $\boldsymbol{A}^+$-structure $\mathbf{H}^+$ based on $\boldsymbol{A}^+$ as follows:

(i) The domain A^+ of $\boldsymbol{A}^+$ is the set $\{d \in H \mid \mathbf{H} \models A(d)\}$ and the operations of $\boldsymbol{A}^+$ are the restrictions to A^+ of the operations $F^{\mathbf{H}}$ of $\mathbf{H}$ such that F is a connective of $\mathrm{L}_{\boldsymbol{A}}$. Trivially, $\boldsymbol{A}^+$ is isomorphic to $\boldsymbol{A}$ (here we use in a crucial way the fact that $\boldsymbol{A}$ is finite).

(ii) $H^+ = \{d \in H \mid \mathbf{H} \models M(d)\}$; for every constant c of $\mathrm{L}_{\boldsymbol{A}}$, $c^{\mathbf{H}^+} = c^{\mathbf{H}}$; for every k-ary function symbol g of $\mathrm{L}_{\boldsymbol{A}}$, $g^{\mathbf{H}^+}$ is the function from $(H^+)^k$ into H^+ defined for all $d_1,\dots,d_k \in H^+$, by $g^{\mathbf{H}^+}(d_1,\dots,d_k) = g^{\mathbf{H}}(d_1,\dots,d_k)$ (i.e. $g^{\mathbf{H}^+}$ is the restriction of $g^{\mathbf{H}}$ to $(H^+)^k$).

(iii) For every k-ary predicate P and every $d_1,\dots,d_k \in H^+$, $\|P(d_1,\dots,d_k)\|_{\mathbf{H}^+}^{\boldsymbol{A}} = f_P^{\mathbf{H}}(d_1,\dots,d_k)$.

Then $\|.\|_{\mathbf{H}^+}^{\boldsymbol{A}}$ uniquely extends to all formulae in such a way that for every formula $\phi(v_1,\dots,v_k)$, for every $c_i \in A$ and for every $d_1,\dots,d_k \in M$: $\mathbf{H} \models f_\phi(d_1,\dots,d_k) = c_i^T$ iff $\|\phi(d_1,\dots,d_k)\|_{\mathbf{H}^+}^{\boldsymbol{A}} = c_i$, and (b) follows. □

The rest of the proof of Theorem 4.0.1 It suffices to associate to every sentence ϕ of $\mathrm{L}_{\boldsymbol{A}}$ the formula $f_\phi = \overline{1}^T$ (remind that $\overline{1}$ is the top element of $\boldsymbol{A}$). Then by Lemma 4.0.2 we have that the following are equivalent:

(i) There is an $\boldsymbol{A}$-structure $\mathbf{M}$ such that $\|\phi\|^{\boldsymbol{A}}_{\mathbf{M}} \neq \overline{1}$.

(ii) There is a model $\mathbf{H}$ of $T_{\boldsymbol{A}}$ such that $f_\phi = \overline{1}^T$ is not valid in $\mathbf{H}$.

Thus we conclude that ϕ is an $\boldsymbol{A}$-tautology iff $T_{\boldsymbol{A}} \vdash f_\phi = \overline{1}^T$, and the set of $\boldsymbol{A}$-tautologies is Σ_1. □

We have seen that $\mathrm{TAUT}(\boldsymbol{A})$ is $\boldsymbol{\Sigma_1}$. Similar arguments show that for every sentence ϕ of $\mathrm{L}_{\boldsymbol{A}}$ we have:

- $\phi \in \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A})$ iff $T_{\boldsymbol{A}} \vdash \overline{0}^T < f_\phi$,
- $\phi \in \mathrm{SAT}(\boldsymbol{A})$ iff $T_{\boldsymbol{A}}$ plus $f_\phi = \overline{1}^T$ is consistent,
- $\phi \in \mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A})$ iff $T_{\boldsymbol{A}}$ plus $f_\phi > \overline{0}^T$ is consistent.

THEOREM 4.0.3. *Let $\boldsymbol{A}$ be a finite chain.* $\mathrm{TAUT}(\boldsymbol{A})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A})$ *are in* $\boldsymbol{\Sigma_1}$. *Moreover,* $\mathrm{SAT}(\boldsymbol{A})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A})$ *are in* $\boldsymbol{\Pi_1}$.

Observe that the proof of this theorem would be completely analogous if instead of a linearly ordered algebra $\boldsymbol{A}$ would be an arbitrary finite algebra (in a finite language), as this was the essential requirement to build the classical first-order theory $T_{\boldsymbol{A}}$.

By the general hardness results from Section 2 we obtain:

COROLLARY 4.0.4. *For every finite chain $\boldsymbol{A}$,*

1. $\mathrm{TAUT}(\boldsymbol{A})$ *and* $\mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A})$ *are* $\boldsymbol{\Sigma_1}$*-complete,*

2. $\mathrm{SAT}(\boldsymbol{A})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A})$ *are* $\boldsymbol{\Pi_1}$*-complete.*

From these results, we can obtain some upper bounds for the arithmetical complexities with respect to the finite-chain semantics, when the class of finite chains is recursively enumerable:

THEOREM 4.0.5. *Suppose that* L *is a ($\triangle$-)core fuzzy logic such that there is a computable enumeration of all (up to isomorphism) finite* L*-chains. Then:*

(a) $\mathrm{finTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{finTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are in* $\boldsymbol{\Pi_2}$.

(b) $\mathrm{finSAT}(\mathrm{L}\forall)$ *and* $\mathrm{finSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are in* $\boldsymbol{\Sigma_2}$.

Proof. Let $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_n, \ldots$ be a computable enumeration of all finite L-chains. Then $\phi \in \mathrm{finTAUT}(\mathrm{L}\forall)$ iff $(\forall n)(\phi \in \mathrm{TAUT}(\boldsymbol{A}_n))$ and $\phi \in \mathrm{finTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ iff $(\forall n)(\phi \in \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A}_n))$. Since the sequence $\langle \boldsymbol{A}_n \mid n \in \mathsf{N}\rangle$ is computable, by Theorem 4.0.3, $\{\langle \phi, n\rangle \mid \phi \in \mathrm{TAUT}(\boldsymbol{A}_n)\}$ and $\{\langle \phi, n\rangle \mid \phi \in \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A}_n)\}$ are in Σ_1, and claim (a) follows.

Regarding claim (b), we have that $\phi \in \mathrm{finSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ iff $(\exists n)(\phi \in \mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A}_n))$, and $\phi \in \mathrm{finSAT}(\mathrm{L}\forall)$ iff $(\exists n)(\phi \in \mathrm{SAT}(\boldsymbol{A}_n))$, and the claim follows from the computability of the sequence $\langle \boldsymbol{A}_n \mid n \in \mathsf{N}\rangle$ and from Theorem 4.0.3 (note that if $R(n, x)$ is Π_1, then $(\exists n)R(n, x)$ is in turn Σ_2). □

Problem	Complexity
$\mathrm{finTAUT}(\mathrm{L}\forall)$	$\mathbf{\Sigma_1}$-hard, $\mathbf{\Pi_2}$
$\mathrm{finSAT}(\mathrm{L}\forall)$	$\mathbf{\Pi_1}$-hard, $\mathbf{\Sigma_2}$
$\mathrm{finTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$	$\mathbf{\Sigma_1}$-hard, $\mathbf{\Pi_2}$
$\mathrm{finSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$	$\mathbf{\Pi_1}$-hard, $\mathbf{\Sigma_2}$

Table 3. Arithmetical complexity bounds for the finite-chain semantics when L is recursively axiomatizable.

THEOREM 4.0.6. *If* L *is a finitely axiomatizable ($\triangle$-)core fuzzy logic, then there is a computable enumeration of all (up to isomorphism) finite* L*-chains.*

Proof. We can obtain a computable enumeration of all finite L-chains as follows: clearly there is a computable enumeration of all the finite algebras of the signature of L (first put the trivial algebra in the list, then enumerate all the (finitely many) structures with two elements, 0 and 1, then all the (finitely many) structures with three elements, $0, 1$ and 2, etc.). Let $\boldsymbol{C}_1, \boldsymbol{C}_2, \ldots, \boldsymbol{C}_n, \ldots$ be the computable list of structures obtained in this way, and assume without loss of generality that $\boldsymbol{C}_1$ is the trivial algebra. Now let $\boldsymbol{A}_1 = \boldsymbol{C}_1$ (note that the trivial algebra is a totally ordered algebraic model of any logic with that signature); then for every n, check whether $\boldsymbol{C}_n$ is a chain and whether it satisfies the finite axiomatization of L. This can be done with a finite computation. If so, let $\boldsymbol{A}_n = \boldsymbol{C}_n$; otherwise, let $\boldsymbol{A}_n = \boldsymbol{C}_1$. □

From the last two theorems, together with the general results in Section 2, we can obtain uniform bounds for the complexity of finite-chain semantics in recursively axiomatizable ($\triangle$-)core fuzzy logics; see the results in Table 3. It applies, in particular, to all the prominent fuzzy logics, for instance if L is Ł, G, Π, BL, SBL, MTL, SMTL, IMTL, ΠMTL, WCMTL, C$_n$MTL, C$_n$IMTL, WNM, or NM, $\mathrm{finTAUT}(\mathrm{L}\forall)$ is $\mathbf{\Pi_2}$, etc. Note that the sets $\mathrm{finTAUT}(\mathrm{L}\forall)$ for these logics may have repetitions, e.g. if L has only finitely many totally ordered algebraic models (this is the case for $\mathrm{L} = \Pi$ or for $\mathrm{L} = \Pi\mathrm{MTL}$). In this case, $\mathrm{finTAUT}(\mathrm{L}\forall)$ is not only $\mathbf{\Pi_2}$, but even $\mathbf{\Sigma_1}$: for instance $\mathrm{finTAUT}(\Pi\forall)$ and $\mathrm{finTAUT}(\Pi\mathrm{MTL}\forall)$ coincide with the set of classical first-order tautologies, which is $\mathbf{\Sigma_1}$-complete. Next will show that in some cases the upper bounds are reached as well.

THEOREM 4.0.7. *Let* L *be a recursively axiomatizable ($\triangle$-)core fuzzy logic such that the following conditions hold:*

(1) For every finite cardinal m*, there is a finite* L*-chain with at least* m *elements.*

(2) There is an L*-formula* $\phi(p)$ *such that for every* L*-chain* $\boldsymbol{A}$ *and for every* $\boldsymbol{A}$*-evaluation* v*,* $v(\phi(p)) \in \{\overline{0}^{\boldsymbol{A}}, \overline{1}^{\boldsymbol{A}}\}$*, and there are evaluations* v_0 *and* v_1 *such that* $v_0(\phi(p)) = \overline{0}^{\boldsymbol{A}}$ *and* $v_1(\phi(p)) = \overline{1}^{\boldsymbol{A}}$*.*

Then the set $\mathrm{finTAUT}(\mathrm{L}\forall)$ *is* $\mathbf{\Pi_2}$*-complete.*

Proof. Let Φ denote the conjunction of all axioms of Q^+ and let for every formula γ, γ^+ be the result of replacing in γ every atomic formula δ by $\phi(\delta)$. Notice that for every model $\langle \boldsymbol{A}, \mathbf{M}\rangle$ the value $\|\gamma^+\|^{\boldsymbol{A}}_{\mathbf{M}}$ is crisp.

Let $\mathbf{M}$ be a first-order safe structure over an L-chain $\boldsymbol{A}$ such that $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. We define a classical model $\mathbf{M}^{Q^+}$ for the language of Q^+ as follows. The domain of $\mathbf{M}^{Q^+}$ is the domain M of $\mathbf{M}$ modulo the equivalence $\sim$ defined as: $d \sim d'$ if, and only if, $\|\phi(d = d')\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. For every n-ary function symbol f of Q^+ and for every $d_1, \ldots, d_n \in M$, $f^{\mathbf{M}^{Q^+}}([d_1], \ldots, [d_n]) = [f^{\mathbf{M}}(d_1, \ldots, d_n)]$, where for each $d \in M$, $[d]$ denotes its equivalence class modulo $\sim$. Finally, for every n-ary predicate symbol P of Q^+ and for every $d_1, \ldots, d_n \in M$, we stipulate that $\langle [d_1], \ldots, [d_n]\rangle \in P^{\mathbf{M}^{Q^+}}$ if, and only if, $\|\phi(P(d_1, \ldots, d_n))\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. By induction on δ we can easily prove:

Claim 1: For each formula $\delta(x_1, \ldots, x_n)$ and any elements $d_1, \ldots, d_n \in M$ we have: $\|\delta(d_1, \ldots, d_n)^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ iff $\mathbf{M}^{Q^+} \models \delta([d_1], \ldots, [d_n])$.

Conversely, given a model $\mathbf{H}$ of Q^+ and an L-chain $\boldsymbol{A}$ we define a first-order structure $\mathbf{H}^{\boldsymbol{A}}$ on $\boldsymbol{A}$ (restricted to the language of Q^+) as follows: the domain of $\mathbf{H}^{\boldsymbol{A}}$ coincides with the domain H of $\mathbf{H}$ and the function symbols and the constants are interpreted as in $\mathbf{H}$; moreover, let z, o be elements of A such that for every evaluation v, we have $v(\phi(p)) = \overline{0}^{\boldsymbol{A}}$ if $v(p) = z$ and $v(\phi(p)) = \overline{1}^{\boldsymbol{A}}$ if $v(p) = o$. Then for every n-ary predicate symbol P and for every $d_1, \ldots, d_n \in H$, we define $\|\delta(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{H}^{\boldsymbol{A}}} = o$ if $\mathbf{H} \models \delta(d_1, \ldots, d_n)$ and $\|\delta(d_1, \ldots, d_n)\|^{\boldsymbol{A}}_{\mathbf{H}^{\boldsymbol{A}}} = z$ otherwise. Then, again by induction on δ, we can easily prove:

Claim 2: For every formula $\delta(x_1, \ldots, x_n)$ of Q^+ and for every $d_1, \ldots, d_n \in H$ one has: $\|\delta(d_1, \ldots, d_n)^+\|^{\boldsymbol{A}}_{\mathbf{H}^{\boldsymbol{A}}} = \overline{1}^{\boldsymbol{A}}$ iff $\mathbf{H} \models \delta(d_1, \ldots, d_n)$.

Now let $X = \{n \mid (\forall m)(\exists k)R(n, m, k)\}$, with R recursive, be a $\mathbf{\Pi_2}$-complete set. Let $R'(x, y, z)$ be a formula of Q^+ representing R in Q^+, that is, for all n, m, k, if $R(n, m, k)$ is true, then $R'(\overline{n}, \overline{m}, \overline{k})$ is provable in Q^+ and if $R(n, m, k)$ is false, then $\neg R'(\overline{n}, \overline{m}, \overline{k})$ is provable in Q^+. Let R^+ be the formula obtained from R' by replacing every atomic subformula ψ by $\phi(\psi)$. Then R^+ behaves as a crisp formula. Finally, let P be a new unary predicate, and let $\Psi(x)$ be the formula

$$\Phi^+ \to (\forall y)((\exists u)((u \leq y)^+ \wedge (P(S(u)) \to P(u))) \vee (\exists z)R^+(x, y, z)).$$

We claim that for every n, $n \in X$ iff $\Psi(\overline{n})$ is true in every first-order model over a finite L-chain. Indeed, suppose $n \in X$. Let $\boldsymbol{A}$ be an L-chain with m elements, and let $\mathbf{M}$ be a first-order model over $\boldsymbol{A}$. If $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{0}^{\boldsymbol{A}}$, then $\|\Psi(\overline{n})\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. Otherwise, $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, that is, the translation of every axiom of Q^+ is true in $\langle \boldsymbol{A}, \mathbf{M}\rangle$.

Claim 3: If $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, then for every theorem ψ of Q^+, $\|\psi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$.

Proof of Claim 3: Suppose not. Then, by Claim 1, $\mathbf{M}^{Q^+}$ would be a model of Q^+ which does not satisfy ψ, a contradiction.

Now let y be an element of the universe of $\mathbf{M}$. If $\|(y \leq \overline{m})^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, then by Claim 3, $\|(y = \overline{0} \vee y = \overline{1} \vee \cdots \vee y = \overline{m})^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, because $Q^+ \vdash (\forall x)(x \leq \overline{m} \leftrightarrow (x = \overline{0} \vee x = \overline{1} \vee \cdots \vee x = \overline{m}))$.

Moreover since $n \in X$, for $y = 0, 1, \ldots, m$, there is a k_y such that $R(n, y, k_y)$ is true. Then for such k_y, $R'(\overline{n}, \overline{y}, \overline{k_y})$ is provable in Q^+ and $\|R^+(\overline{n}, \overline{y}, \overline{k_y})\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, again by Claim 3. If $\|(y > \overline{m})^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, then for some i such that $\|(\bar{i} \leq y)^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, we must have $\|P(S(\bar{i})) \to P(\bar{i})\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$, otherwise $\|P(\overline{0})\|^{\boldsymbol{A}}_{\mathbf{M}} < \|P(\overline{1})\|^{\boldsymbol{A}}_{\mathbf{M}} < \cdots < \|P(\overline{m+1})\|^{\boldsymbol{A}}_{\mathbf{M}}$ and A would have more than m elements. Thus in this case $(\exists u)((u \leq y)^+ \wedge (P(S(u)) \to P(u)))$. In any case, if $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$ and $n \in X$ then $\|(\forall y)((\exists u)((u \leq y)^+ \wedge (P(S(u)) \to P(u))) \vee (\exists z)R^+(x, y, z))\|^{\boldsymbol{A}}_{\mathbf{M}} = \overline{1}^{\boldsymbol{A}}$. Thus $\Psi(\overline{n})$ has truth value $\overline{1}^{\boldsymbol{A}}$.

Now suppose that $n \notin X$. Then for some m there is no k such that $R(n, m, k)$. Let $\boldsymbol{A}$ be an L-chain with more than m elements. Let $\overline{0}^{\boldsymbol{A}} = a_0 < a_1 < \cdots < a_h = \overline{1}^{\boldsymbol{A}}$ with $h \geq m$, be the elements of A. Consider the first-order structure $\mathbf{N}^{\boldsymbol{A}}$ over $\boldsymbol{A}$ obtained from the standard model $\mathbf{N}$ of natural numbers according to Claim 2. Moreover, let us set, for $i = 0, \ldots, h$, $P^{\mathbf{N}^{\boldsymbol{A}}}(i) = a_i$ and for $i > h$, $P^{\mathbf{N}^{\boldsymbol{A}}}(i) = \overline{1}^{\boldsymbol{A}}$. Then by Claim 2, $\|\Phi^+\|^{\boldsymbol{A}}_{\mathbf{N}^{\boldsymbol{A}}} = \overline{1}^{\boldsymbol{A}}$, $\|(\exists u)((u \leq \overline{m})^+ \wedge (P(S(u)) \to P(u)))\|^{\boldsymbol{A}}_{\mathbf{N}^{\boldsymbol{A}}} = a_{h-1} < \overline{1}^{\boldsymbol{A}}$ and $\|(\exists z)R^+(\overline{n}, \overline{m}, z))\|^{\boldsymbol{A}}_{\mathbf{N}^{\boldsymbol{A}}} = \overline{0}^{\boldsymbol{A}}$. It follows that $\|\Psi(\overline{n})\|^{\boldsymbol{A}}_{\mathbf{N}^{\boldsymbol{A}}} = a_{h-1} < \overline{1}^{\boldsymbol{A}}$. □

COROLLARY 4.0.8. *Let* L *be a recursively axiomatizable ($\triangle$-)core fuzzy logic such that for every finite cardinal* m*, there is a finite* L*-chain with at least* m *elements. Then* $\mathrm{finTAUT}(\mathrm{L}\forall)$ *is* $\boldsymbol{\Pi_2}$*-complete if one of the following sufficient conditions is satisfied:*

1. L *has a strict negation* $\sim$.
2. L *expands* WNM.
3. L *is a* $\triangle$*-core fuzzy logic.*

Proof. By the hypothesis, all logics above satisfy condition (1) of Theorem 4.0.7. As regards to condition (2), for logics with a strict negation $\sim$, take $\phi(p) = \sim\sim p$, and note that for every evaluation v in an L-chain, if $v(p) = \overline{0}$, then $v(\phi(p)) = \overline{0}$, otherwise $v(\phi(p)) = \overline{1}$. For logics expanding WNM, take $\phi(p) = \neg((\neg((\neg\neg p)^2))^2)$ and note that for any evaluation v in an L-chain, if $v(\neg\neg p) \leq v(\neg p)$, then $v(\phi(p)) = \overline{0}$, otherwise $v(\phi(p)) = \overline{1}$. As regards to $\triangle$-core fuzzy logics, it is clear that $\phi(p) = \triangle(p)$ satisfies condition (2) of Theorem 4.0.7. □

COROLLARY 4.0.9. *For every* $\mathrm{L} \in \{\mathrm{SMTL}, \mathrm{NM}, \mathrm{WNM}, \mathrm{SBL}, \mathrm{G}\}$*, we have that* $\mathrm{finTAUT}(\mathrm{L}\forall)$ *is* $\boldsymbol{\Pi_2}$*-complete.*

COROLLARY 4.0.10. *Let* L *be a* $\triangle$*-core fuzzy logic such that for every finite cardinal* m*, there is a finite* L*-chain with at least* m *elements. Then* $\mathrm{finTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *is* $\boldsymbol{\Pi_2}$*-complete, and* $\mathrm{finSAT}(\mathrm{L}\forall)$ *and* $\mathrm{finSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ *are* $\boldsymbol{\Sigma_2}$*-complete.*

Proof. It follows from Corollary 4.0.8 by using some relations in Lemma 2.0.7:

- $\varphi \in \mathrm{finTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$ iff $\neg\triangle(\neg\varphi) \in \mathrm{TAUT}(\mathrm{L}\forall)$,
- $\varphi \in \mathrm{finTAUT}(\mathrm{L}\forall)$ iff $\neg\triangle\varphi \notin \mathrm{finSAT}(\mathrm{L}\forall)$ iff $\neg\triangle\varphi \notin \mathrm{finSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$. □

From the general results in Section 2 and the real and rational completeness properties we obtain many arithmetical complexity results with respect to the rational semantics for prominent logics as collected in Table 4. In the case of $\mathrm{BL}\forall$ and $\mathrm{SBL}\forall$ we need an additional result:

PROPOSITION 4.0.11. *The sets* $\mathrm{ratTAUT}(\mathrm{BL}\forall)$ *and* $\mathrm{ratTAUT}(\mathrm{SBL}\forall)$ *are in* $\mathbf{\Sigma_1}$*, and hence* $\mathrm{ratSAT_{pos}}(\mathrm{BL}\forall)$ *and* $\mathrm{ratSAT_{pos}}(\mathrm{SBL}\forall)$ *are in* $\mathbf{\Pi_1}$.

Proof. Consider the extensions of $\mathrm{BL}\forall$ and $\mathrm{SBL}\forall$ by the schema $\Phi = (\forall x)(\chi \mathbin{\&} \varphi) \to (\chi \mathbin{\&} (\forall x)\varphi)$, where x is not free in χ. Call them $\mathrm{BL}\forall^+$ and $\mathrm{SBL}\forall^+$, respectively. Φ is valid in every model on a densely ordered BL-chain, but it is not a tautology for all BL-chains (see [21]). It is easy to see that $\mathrm{BL}\forall^+$ (resp. $\mathrm{SBL}\forall^+$) enjoys strong completeness with respect to models over rational BL-chains (resp. SBL-chains) (see Chapter V) and it is not necessary to require that those models satisfy the additional schema because their chains are densely ordered. Therefore, $\mathrm{ratTAUT}(\mathrm{BL}\forall)$ turns out to be the set of theorems of the logic $\mathrm{BL}\forall^+$, and analogously for $\mathrm{ratTAUT}(\mathrm{SBL}\forall)$; this proves the result. $\square$

On the other hand, as we prove later in this section, $\mathrm{finTAUT}(Ł\forall)$ is $\mathbf{\Pi_2}$-complete. This allows to prove the following result:

PROPOSITION 4.0.12. $\mathrm{finTAUT}(\mathrm{BL}\forall)$ *is* $\mathbf{\Pi_2}$*-complete.*

Proof. For every sentence φ we consider the formula $\varphi^{\neg\neg}$ resulting from φ by adding double negation $\neg\neg$ to all atoms. Then for every $\varphi \in \mathrm{Sent}_{\mathcal{P}}$: $\varphi^{\neg\neg} \in \mathrm{finTAUT}(\mathrm{BL}\forall)$ iff $\varphi \in \mathrm{finTAUT}(Ł\forall)$. Indeed, the left-to-right implication is obvious because the negation is involutive in Łukasiewicz logic; as for the converse one let us assume that $\varphi \in \mathrm{finTAUT}(Ł\forall)$ and consider any model $\mathbf{M}$ over a finite BL-chain $\boldsymbol{A}$. Taking into account the structure of BL-chains described in previous chapters, it is enough to distinguish two cases:

(1) Assume that $\boldsymbol{A}$ is an ordinal sum $\boldsymbol{C}_1 \oplus \boldsymbol{C}_2$ where $\boldsymbol{C}_1$ is a finite MV-chain. Then we define a model $\mathbf{M}'$ over $\boldsymbol{C}_1$ from $\mathbf{M}$ in the following way: take the same domain, the same interpretation of constants and functionals, and for every n-ary predicate symbol P and elements $a_1, \dots, a_n$ in the domain set $P_{\mathbf{M}'}(a_1, \dots, a_n) = P_{\mathbf{M}}(a_1, \dots, a_n)$ if $P_{\mathbf{M}}(a_1, \dots, a_n) \in C_1$ and $P_{\mathbf{M}'}(a_1, \dots, a_n) = \overline{1}^{\boldsymbol{A}}$ otherwise. Now it is easy to prove by induction that for every formula α and every evaluation v: $\|\alpha^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M},\mathrm{v}} = \|\alpha\|^{\boldsymbol{C}_1}_{\mathbf{M}',\mathrm{v}}$. Hence $\|\varphi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi\|^{\boldsymbol{C}_1}_{\mathbf{M}'} = \overline{1}^{\boldsymbol{A}}$.

(2) Assume that $\boldsymbol{A}$ is an SBL-chain (i.e. its negation is strict). Then we define a model $\mathbf{M}'$ over $\boldsymbol{B}_2$ from $\mathbf{M}$ in the following way: take the same domain, the same interpretation of constants and functionals, and for every n-ary predicate symbol P and elements $a_1, \dots, a_n$ in the domain set $P_{\mathbf{M}'}(a_1, \dots, a_n) = 0$ if $P_{\mathbf{M}}(a_1, \dots, a_n) = \overline{0}^{\boldsymbol{A}}$ and $P_{\mathbf{M}'}(a_1, \dots, a_n) = 1$ otherwise. Now we have: $\|\varphi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\varphi^{\neg\neg}\|^{\boldsymbol{B}_2}_{\mathbf{M}'} = \|\varphi\|^{\boldsymbol{B}_2}_{\mathbf{M}'} = 1$. Therefore, we have proved that $\mathrm{finTAUT}(\mathrm{BL}\forall)$ is $\mathbf{\Pi_2}$-hard. The $\mathbf{\Pi_2}$ containment follows from Theorems 4.0.5 and 4.0.6. $\square$

Some more results on complexity of finite-chain semantics will be obtained soon when comparing such semantics with the real and rational ones.

Logic	ratTAUT	ratSAT	$\mathrm{ratTAUT_{pos}}$	$\mathrm{ratSAT_{pos}}$
MTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
IMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
SMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
WCMTL∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard
ΠMTL∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-hard
BL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-complete
SBL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-hard	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-complete
Ł∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
Π∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
G∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
C_nMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
C_nIMTL∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
WNM∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
NM∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete

Table 4. Complexity results for the rational semantics.

Observe that the completeness properties imply that for some prominent logics we have $\mathrm{genTAUT}(\mathrm{L}\forall) = \mathrm{realTAUT}(\mathrm{L}\forall) = \mathrm{ratTAUT}(\mathrm{L}\forall)$. Now, in addition, we will consider the semantics given by intended rational chains. Of course, this can only be done for those logics where it makes sense to have an intended semantics over the rational unit interval, i.e. logics L_* given by a left-continuous t-norm $*$ such that its restriction to $[0,1]^{\mathbb{Q}}$ is well-defined. We denote the corresponding algebra as $[0,1]^{\mathbb{Q}}_*$. This can be done, for instance, for the logic NM corresponding to the nilpotent minimum t-norm. By inspecting the usual the proof of the fact that every countable NM-chain can be σ-embedded into $[0,1]_{\mathrm{NM}}$ one realizes that the embedding can be in fact defined into the rationals and thus we have $\mathrm{genTAUT}(\mathrm{NM}\forall) = \mathrm{stTAUT}(\mathrm{NM}\forall) = \mathrm{intratTAUT}(\mathrm{NM}\forall)$ and they are $\mathbf{\Sigma_1}$-complete. The three main continuous t-norms satisfy the required property as well, i.e. we have well-defined algebras over the rationals $[0,1]^{\mathbb{Q}}_{\text{Ł}}$, $[0,1]^{\mathbb{Q}}_{\Pi}$ and $[0,1]^{\mathbb{Q}}_{\mathrm{G}}$; the same goes for their ordinal sums. Let $\mathcal{Q}$ be the set of ordinal sums of these three rational BL-chains. Given $\mathbb{K} \subseteq \mathcal{Q}$, $\overline{\mathbb{K}}$ will denote the subset of $\mathcal{R}$ given by the substitution in the elements of $\mathbb{K}$ of each component for its corresponding basic real chain. We start with the case of $[0,1]^{\mathbb{Q}}_{\text{Ł}}$.

LEMMA 4.0.13. *Let $\mathbf{M}$ and $\mathbf{M}'$ be two first-order structures with the same domain M over $[0,1]_{\text{Ł}}$, and let ϕ, ψ be first-order sentences with parameters from M and $\delta(x)$ be a first-order formula with parameters from M and with x as its only free variable. Let for any two real numbers α, β, $d(\alpha,\beta)$ denote the distance between α and β, that is, $d(\alpha,\beta) = \max\{\alpha-\beta, \beta-\alpha\}$. Then for every positive real number γ we have:*

(i) If $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq \gamma$, then $d(\|\neg\phi\|_{\mathbf{M}}, \|\neg\phi\|_{\mathbf{M}'}) \leq \gamma$.

(ii) If $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq \gamma$ and $d(\|\psi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}'}) \leq \gamma$, then $d(\|\phi \mathbin{\&} \psi\|_{\mathbf{M}}, \|\phi \mathbin{\&} \psi\|_{\mathbf{M}'}) \leq 2\gamma$.

*(iii) If for all $a \in M$, $d(\|\delta(a)\|_{\mathbf{M}}, \|\delta(a)\|_{\mathbf{M}'}) \leq \gamma$,
then $d(\|(\forall x)\delta(x)\|_{\mathbf{M}}, \|(\forall x)\delta(x)\|_{\mathbf{M}'}) \leq \gamma$.*

Proof. Almost trivial. □

COROLLARY 4.0.14. *If ϕ is a sentence of complexity k and $\mathbf{M}$ and $\mathbf{M}'$ are first-order structures with the same domain M over $[0,1]_{Ł}$ such that for every closed atomic subformula ψ of ϕ, $d(\|\psi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}'}) \leq \gamma$, then $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq 2^k\gamma$.*

LEMMA 4.0.15. *Let ϕ be a first-order sentence of complexity k and let for every n, $\mathbf{Ł}_n$ denote the finite MV-chain with $n+1$ elements. Let $\mathbf{M}$ be a first-order structure over $[0,1]_{Ł}$ with domain M such that $\|\phi\|_{\mathbf{M}} < 1$ and let n be such that $2^{-n} < 1 - \|\phi\|_{\mathbf{M}}$. Then there is a first-order structure $\mathbf{M}'$ over $\mathbf{Ł}_{2^{k+n}}$ such $\|\phi\|_{\mathbf{M}'} < 1$.*

Proof. Let for every atomic formula ψ with parameters in M, $m(\psi)$ denote the maximum natural number such that $\frac{m(\psi)}{2^{n+k}} \leq \|\psi\|_{\mathbf{M}}$. Define a new first-order structure $\mathbf{M}'$ with domain M letting for every atomic formula ψ with parameters in M, $\|\psi\|_{\mathbf{M}'} = \frac{m(\psi)}{2^{n+k}}$. Then $d(\|\psi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}'}) < \frac{1}{2^{n+k}}$ and by Corollary 4.0.14, $d(\|\phi\|_{\mathbf{M}}, \|\phi\|_{\mathbf{M}'}) \leq 2^k \frac{1}{2^{n+k}} = \frac{1}{2^n} < 1 - \|\phi\|_{\mathbf{M}}$. It follows that $\|\phi\|_{\mathbf{M}'} < 1$. Now $\mathbf{M}'$ has been defined as a first-order structure over $[0,1]_{Ł}$, but since for every sentence δ, $\|\delta\|_{\mathbf{M}'} \in \mathbf{Ł}_{2^{k+n}}$, $\mathbf{M}'$ can be also regarded as a first-order structure over $\mathbf{Ł}_{2^{k+n}}$. □

THEOREM 4.0.16. $\mathrm{stTAUT}(Ł\forall) = \mathrm{intratTAUT}(Ł\forall) = \mathrm{finTAUT}(Ł\forall)$ *and they are* $\mathbf{\Pi_2}$*-complete.*

Proof. The Inclusions from left to right follow from Lemma 2.0.3, therefore it suffices to prove that $\mathrm{finTAUT}(Ł\forall) \subseteq \mathrm{stTAUT}(Ł\forall)$. But this is immediate from Lemma 4.0.15. □

Recall that if $\mathbb{K}$ is a class of MV-chains, then $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}) = \{\phi \mid \neg\phi \notin \mathrm{TAUT}(\mathbb{K})\}$. Thus we obtain:

THEOREM 4.0.17. $\mathrm{stSAT}_{\mathrm{pos}}(Ł\forall) = \mathrm{intratSAT}_{\mathrm{pos}}(Ł\forall) = \mathrm{finSAT}_{\mathrm{pos}}(Ł\forall)$ *and they are* $\mathbf{\Sigma_2}$*-complete.*

In the SAT case the situation is different:

THEOREM 4.0.18. $\mathrm{finSAT}(Ł\forall) \subsetneq \mathrm{intratSAT}(Ł\forall)$.

Proof. The inclusion is obvious; let us show the difference with an example. Let S be a unary function symbol, P be a unary predicate symbol and $\mathbf{0}$ be a constant symbol. Consider the following sentence:

$$\Phi = (P(\mathbf{0}) \leftrightarrow \neg P(\mathbf{0})) \,\&\, (\forall x)(P(x) \leftrightarrow ((P(S(x)) \oplus P(S(x))))$$

On the one hand, Φ is satisfiable in $[0,1]_{Ł}^{\mathbf{Q}}$. Indeed, take the structure $\mathbf{M}$ on $[0,1]_{Ł}$ whose domain is the set of natural numbers, $\mathbf{0}$ and S are respectively interpreted as 0 and the successor function, and $\|P(n)\|_{\mathbf{M}} = \frac{1}{2^{n+1}}$. On the other hand Φ is not satisfiable in models over finite MV-chains. Indeed, let $\mathbf{M}$ be a first-order structure over a finite chain $\mathbf{Ł}_n$. The first conjunct of Φ is satisfiable if the chain has a negation fixpoint (i.e. it has an odd number of truth-values); assume it. A contradiction will be reached when combining it with the satisfiability of the second conjunct. Thus, assume, in addition, that $\|(\forall x)(P(x) \leftrightarrow ((P(S(x)) \oplus P(S(x)))\|_{\mathbf{M}} = 1$. This means that for every $a \in M$, $P_{\mathbf{M}}(a) = P_{\mathbf{M}}(S_{\mathbf{M}}(a)) \oplus P_{\mathbf{M}}(S_{\mathbf{M}}(a))$; therefore, if $P_{\mathbf{M}}(a) \neq 1$, then $P_{\mathbf{M}}(a) > P_{\mathbf{M}}(S_{\mathbf{M}}(a))$. So we obtain $P_{\mathbf{M}}(0) > P_{\mathbf{M}}(S_{\mathbf{M}}(0)) > P_{\mathbf{M}}(S_{\mathbf{M}}(S_{\mathbf{M}}(0))) > \cdots$, a contradiction with the finiteness of $\mathbf{Ł}_n$. □

Similarly, Theorems 4.0.16 and 4.0.17 do not extend to finite consequence relation:

COROLLARY 4.0.19. *There are formulae ϕ and ψ of Łukasiewicz logic such that* $\phi \in \mathrm{finCons}(\text{Ł}, \psi)$ *and* $\phi \notin \mathrm{intratCons}(\text{Ł}, \psi)$.

Proof. It is immediate from the last theorem. Indeed, take any $\varphi \in \mathrm{intratSAT}(\text{Ł}\forall) \setminus \mathrm{finSAT}(\text{Ł}\forall)$, and then it is clear than $\overline{0} \in \mathrm{finCons}(\text{Ł}, \varphi)$ and $\overline{0} \notin \mathrm{intratCons}(\text{Ł}, \varphi)$. □

We consider now other intended rational chains.

THEOREM 4.0.20. *Let $\mathbb{K} \subseteq \mathcal{Q}$ such that there exists $\boldsymbol{A} \in \mathbb{K}$ whose first component is product. Then* $\mathrm{TAUT}(\mathbb{K})$, $\mathrm{SAT}(\mathbb{K})$, $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$ *are non-arithmetical.*

Proof. Let T, θ_1, θ_2, θ_3 and θ_4 be as in the proof of Theorem 3.0.23. Then one can prove the following claims for every Φ in the language of PA:

$$\begin{aligned} \mathbf{N} \models \Phi \ \ &\text{iff } \theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K}) \\ &\text{iff } \theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K}). \\ \mathbf{N} \models \Phi \ \ &\text{iff } \theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \mathbin{\&} \Phi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K}) \\ &\text{iff } \theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \mathbin{\&} \Phi^{\neg\neg} \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K}). \end{aligned}$$

We justify the first claim (the second one is proved analogously). Assume that $\mathbf{N} \models \Phi$. By Lemma 3.0.26, this implies that $\theta_1 \& \theta_2 \& \theta_3 \& \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}(\overline{\mathbb{K}})$, and hence $\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$ and $\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$. Assume now that $\mathbf{N} \not\models \Phi$. Then we construct a countermodel $\mathbf{M}$ over $\boldsymbol{A}$ as follows:

(a) The domain of $\mathbf{M}$ is the set of natural numbers, and the constant 0 and the function symbols of Q^+ are interpreted as in the standard model $\mathbf{N}$ of natural numbers.

(b) If P is an n-ary predicate symbol of Q^+ and $k_1, \ldots, k_n$ are natural numbers, then $P_{\mathbf{M}}(k_1, \ldots, k_n) = 1$ if $P(k_1, \ldots, k_n)$ is true in $\mathbf{N}$ and $P_{\mathbf{M}}(k_1, \ldots, k_n) = 0$ otherwise.

(c) Let f be the affine bijective transformation from $[0, 1]^{\mathsf{Q}}$ to the interval where the first component of $\boldsymbol{A}$ is defined. For every natural number n, $U_{\mathbf{M}}(n) = f(2^{-3^n})$.

It is readily seen that $\|\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ and $\|\Phi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$. Therefore, $\|\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$ and hence $\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \notin \mathrm{TAUT}(\mathbb{K})$ and $\theta_1 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4 \to \Phi^{\neg\neg} \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$. □

THEOREM 4.0.21.

1. *If* $\boldsymbol{C}$ *is the first component of a chain* $\boldsymbol{A} \in \mathcal{Q}$*, then* $\mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{A}) = \mathrm{TAUT}_{\mathrm{pos}}(\boldsymbol{C})$ *and* $\mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{A}) = \mathrm{SAT}_{\mathrm{pos}}(\boldsymbol{C})$*.*
2. $\mathrm{SAT}_{\mathrm{pos}}([0,1]^{\mathrm{Q}}_{\mathrm{G}}) = \mathrm{SAT}([0,1]^{\mathrm{Q}}_{\mathrm{G}})$ *and it is* $\mathbf{\Pi_1}$*-complete.*
3. *If* $\boldsymbol{A} \in \mathcal{Q}$ *begins with a Gödel component, then* $\mathrm{SAT}(\boldsymbol{A}) = \mathrm{SAT}([0,1]^{\mathrm{Q}}_{\mathrm{G}})$*.*
4. $\mathrm{TAUT}_{\mathrm{pos}}([0,1]^{\mathrm{Q}}_{\mathrm{G}})$ *is* $\mathbf{\Sigma_1}$*-complete.*
5. $\mathrm{TAUT}([0,1]_{\Pi}) = \mathrm{TAUT}([0,1]^{\mathrm{Q}}_{\Pi})$*.*
6. *If* $\mathbb{K} \subseteq \mathcal{Q}$ *contains some component non-isomorphic to* $[0,1]^{\mathrm{Q}}_{\mathrm{G}}$*, then* $\mathrm{TAUT}(\mathbb{K})$ *is* $\mathbf{\Pi_2}$*-hard.*
7. *If* $\mathbb{K} \subseteq \mathcal{Q}$ *contains at least one algebra non-isomorphic to any of* $[0,1]^{\mathrm{Q}}_{\mathrm{G}}$*,* $[0,1]^{\mathrm{Q}}_{\text{Ł}}$*,* $[0,1]^{\mathrm{Q}}_{\text{Ł}} \oplus [0,1]^{\mathrm{Q}}_{\mathrm{G}}$*,* $[0,1]^{\mathrm{Q}}_{\mathrm{G}} \oplus [0,1]^{\mathrm{Q}}_{\text{Ł}}$*,* $[0,1]^{\mathrm{Q}}_{\text{Ł}} \oplus [0,1]^{\mathrm{Q}}_{\text{Ł}}$ *and* $[0,1]^{\mathrm{Q}}_{\text{Ł}} \oplus [0,1]^{\mathrm{Q}}_{\mathrm{G}} \oplus [0,1]^{\mathrm{Q}}_{\text{Ł}}$*, then* $\mathrm{TAUT}(\mathbb{K})$ *is non-arithmetical.*

Proof. The first four claims are proved by checking that the proofs of the corresponding results for the standard semantics actually work as well for intended rational semantics. 5 is proved in the appendix of [4]. Point 6 is shown by reducing the problem to $\mathrm{TAUT}([0,1]^{\mathrm{Q}}_{\text{Ł}})$, which we know is $\mathbf{\Pi_2}$-hard, as in the proof of Theorem 3.0.21. Similarly, for the last point we use the fact, proved in the previous theorem, that the set $\mathrm{TAUT}([0,1]^{\mathrm{Q}}_{\Pi})$ is non-arithmetical and perform the analogous reduction to that problem as in Theorem 3.0.30. □

As for the relation between tautologies over real and finite chains, we have the following result:

THEOREM 4.0.22. *Let* L *be a consistent (*$\triangle$*-)core fuzzy logic. If there exist* L*-chains over* $[0,1]$ *whose t-norm is not isomorphic to Łukasiewicz, then* $\mathrm{realTAUT}(\mathrm{L}\forall) \neq \mathrm{finTAUT}(\mathrm{L}\forall)$ *and* $\mathrm{genTAUT}(\mathrm{L}\forall) \neq \mathrm{finTAUT}(\mathrm{L}\forall)$*.*

Proof. Let $[0,1]_*$ be a real L-chain defined by a left-continuous t-norm $*$ non-isomorphic to the Łukasiewicz t-norm. Then:

(i) If $*$ is continuous, then the formula

$$(C\forall) \quad (\exists x)(P(x) \to (\forall y)P(y))$$

is an $\boldsymbol{A}$-tautology for any finite L-chain $\boldsymbol{A}$, but it is not a $[0,1]_*$-tautology.

(ii) If $*$ is not continuous, then the formula Φ

$$(\forall x)(\chi \,\&\, \psi) \to (\chi \,\&\, (\forall x)\psi), \text{ where } x \text{ is not free in } \chi$$

is an $\boldsymbol{A}$-tautology for any finite L-chain $\boldsymbol{A}$, but it is not a $[0,1]_*$-tautology.

Indeed:

(i) It is well known that, for any continuous t-norm $*$ which is not isomorphic to Łukasiewicz t-norm, the corresponding negation $n_*(x) = x \Rightarrow_* 0$ is not (right) continuous at $x = 0$. Let $b = lim_{x\to 0^+} n_*(x)$. We know that $b < 1$. Take an infinite decreasing sequence $1 > a_1 > a_2 > \cdots > a_n \cdots > 0$ with limit 0. Consider the $[0,1]_*$-model $\mathbf{M} = \langle \mathsf{N}, P_{\mathbf{M}} \rangle$ where $P_{\mathbf{M}}(n) = a_n$. Then we have $\|(\exists x)(P(x)\to(\forall y)P(y))\|^{[0,1]_*}_{\mathbf{M},e} = \sup_n\{a_n \Rightarrow_* (\inf_n a_n)\} = \sup_n\{a_n \Rightarrow_* 0\} = \sup_n n_*(a_n) = b < 1$. On the other hand, it is clear that the formula has value $\overline{1}^{\boldsymbol{A}}$ in any structure over a finite L-chain $\boldsymbol{A}$.

(ii) For simplicity, let us take the following instance of Φ:

$$(\forall x)(P(c) \,\&\, Q(x)) \to P(c) \,\&\, (\forall x)Q(x)$$

where c is a 0-ary functional symbol. If the t-norm is not right-continuous there is a sequence $\langle a_n \mid n \geq 1\rangle$ and an element b such that $b * \inf\{a_n \mid n \geq 1\} < \inf\{b * a_n \mid n \geq 1\}$. Consider the $[0,1]_*$-model $\mathbf{M} = \langle \mathsf{N}, P_{\mathbf{M}}, Q_{\mathbf{M}}\rangle$ and an evaluation of variables e such that $P_{\mathbf{M}}(c_{\mathbf{M}}) = b$ and $Q_{\mathbf{M}}(n) = a_n$ for every n. Then $\|(\forall x)(P(c) \,\&\, Q(x)) \to P(c) \,\&\, (\forall x)Q(x)\|^{[0,1]_*}_{\mathbf{M},e} = \inf\{b * a_n, n \geq 1\} \Rightarrow_* (b * \inf\{a_n, n \geq 1\}) < 1$. But an easy computation shows that for any finite chain the formula is a tautology (take into account that the $\inf$ becomes a $\min$). $\square$

5 Further topics on arithmetical hierarchy in first-order logics

5.1 Semantics of witnessed models

The semantics of witnessed models has been introduced in Chapter II; we quickly recall a few necessary notions for the reader's convenience. For each continuous t-norm $*$, the propositional logic given by it (Definition 1.1.19 of Chapter I) is denoted L_* and the corresponding predicate logic is denoted $\mathrm{L}_*\forall$. Let $\boldsymbol{A}$ be a BL-chain. A model $\langle \boldsymbol{A}, \mathbf{M}\rangle$ is *witnessed* if for each formula $\varphi(x, y, \dots)$ and for each $b, \dots \in M$,

$$\|(\forall x)\varphi(x, b, \dots)\|^{\boldsymbol{A}}_{\mathbf{M}} = \min_a \|\varphi(a, b, \dots)\|^{\boldsymbol{A}}_{\mathbf{M}},$$
$$\|(\exists x)\varphi(x, b, \dots)\|^{\boldsymbol{A}}_{\mathbf{M}} = \max_a \|\varphi(a, b, \dots)\|^{\boldsymbol{A}}_{\mathbf{M}},$$

(i.e. there is an a with minimal (maximal) value of $\|\varphi(a, b, \dots)\|$). Alternatively we say that $\mathbf{M}$ is $\boldsymbol{A}$-witnessed.

FACT 5.1.1 ([20]). *Over the Łukasiewicz logic* Ł$\forall$, *each countable standard model* $\mathbf{M}$ *is an elementary submodel of a witnessed standard model* $\mathbf{M}'$.

Given a logic L, we denote by $\mathrm{L}\forall^w$ the logic $\mathrm{L}\forall$ extended by the following axioms:

$(C\forall)$ $\quad(\exists x)(\varphi(x) \to (\forall y)\varphi(y))$,
$(C\exists)$ $\quad(\exists x)((\exists y)\varphi(y) \to \varphi(x))$.

FACT 5.1.2 (Chapter II and [20]).

(1) For our logics, the logic $\mathrm{L}\forall^w$ *is strongly complete w.r.t. witnessed models.*

(2) A model $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *is elementarily embeddable into a witnessed model iff* $(C\forall)$ *and* $(C\exists)$ *are true in* $\langle \boldsymbol{A}, \mathbf{M}\rangle$.

(3) The following are equivalent:

- $(C\forall), (C\exists) \in \mathrm{genTAUT}(\mathrm{L}_*\forall)$,
- $(C\forall), (C\exists) \in \mathrm{stTAUT}(\mathrm{L}_*\forall)$,
- $*$ *is the Łukasiewicz t-norm.*

For each L_*-chain $\boldsymbol{A}$ let $\boldsymbol{A}^{\neg\neg}$ be the homomorphic image of $\boldsymbol{A}$ defined by the mapping $f(x) = \neg\neg x$. For each L_*-model $\langle \boldsymbol{A}, \mathbf{M}\rangle$, $\mathbf{M} = \langle M, \langle P_{\mathbf{M}}\rangle_P, \langle f_{\mathbf{M}}\rangle_f\rangle$, let $\mathbf{M}^{\neg\neg}$ be the $\boldsymbol{A}^{\neg\neg}$-model $\mathbf{M}^{\neg\neg} = \langle M, \langle P^{\neg\neg}_{\mathbf{M}}\rangle_P, \langle f_{\mathbf{M}}\rangle_f\rangle$, where, for each P and $\mathbf{a}$, $P^{\neg\neg}_{\mathbf{M}}(\mathbf{a}) = f(P_{\mathbf{M}}(\mathbf{a})) = \neg\neg P_{\mathbf{M}}(\mathbf{a})$. For a t-norm with Gödel negation $\boldsymbol{A}^{\neg\neg}$ is the two-element Boolean algebra; for a t-norm beginning by a Łukasiewicz component $\boldsymbol{A}^{\neg\neg}$ is Łukasiewicz.

FACT 5.1.3. *Let* $\boldsymbol{A}$ *be an* L_**-chain and* $\langle \boldsymbol{A}, \mathbf{M}\rangle$ *a witnessed model. Then* $\langle \boldsymbol{A}^{\neg\neg}, \mathbf{M}^{\neg\neg}\rangle$ *is a witnessed model such that for each formula* φ *and each tuple* $\mathbf{a}$ *of elements of* M, $\|\varphi(\mathbf{a})\|^{\boldsymbol{A}^{\neg\neg}}_{\mathbf{M}^{\neg\neg}} = \|\neg\neg\varphi(\mathbf{a})\|^{\boldsymbol{A}}_{\mathbf{M}}$.

Now we start to discuss the arithmetical complexity of general and standard semantics of the predicate logics given by continuous t-norms with semantics restricted to (standard and general) witnessed models. We write $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$ for the set of formulae true in all witnessed general models, $\mathrm{wstTAUT}$, $\mathrm{wgenSAT}$ and wstSAT in the obvious analogous sense.

THEOREM 5.1.4. *For each* $*$, $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$ *is* $\boldsymbol{\Sigma_1}$*-complete and* $\mathrm{wgenSAT}(\mathrm{L}_*\forall)$ *is* $\boldsymbol{\Pi_1}$*-complete.*

Proof. The fact that $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$ is in $\boldsymbol{\Sigma_1}$ follows from the completeness theorems for the corresponding logic. Similarly, $\mathrm{wgenSAT}(\mathrm{L}_*\forall)$ is in $\boldsymbol{\Pi_1}$ since satisfiability is equivalent to consistency.

Concerning the $\boldsymbol{\Sigma_1}$-completeness of $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$: For Łukasiewicz it follows from the fact that tautologies are the same as witnessed tautologies. For $*$ with strict negation observe that φ is a Boolean tautology iff $\varphi^{\neg\neg}$ is a witnessed general tautology of $\mathrm{L}_*\forall$. Finally for a t-norm $*$ beginning by Łukasiewicz, φ is a (witnessed) general tautology of Łukasiewicz logic iff $\varphi^{\neg\neg}$ is a witnessed general tautology of $\mathrm{L}_*\forall$. This reduces a $\boldsymbol{\Sigma_1}$-complete set to our set $\mathrm{wgenTAUT}(\mathrm{L}_*\forall)$. □

THEOREM 5.1.5. *For every* $*$ *with strict negation:*

(1) The following five sets of formulae are equal: $\mathrm{wgenSAT}$, $\mathrm{wgenSAT_{pos}}$, wstSAT, $\mathrm{wstSAT_{pos}}$, $\mathrm{SAT}(\boldsymbol{B}_2)$.

(2) The following sets are equal: $\mathrm{wgenTAUT_{pos}}$, $\mathrm{wstTAUT_{pos}}$, $\mathrm{TAUT}(\boldsymbol{B}_2)$ *(but different from* $\mathrm{wstTAUT}$, $\mathrm{wgenTAUT}$*).*

Proof. (1) All sets include $\mathrm{SAT}(\boldsymbol{B}_2)$ and are included in $\mathrm{wgenSAT}_{\mathrm{pos}}$. Thus if a formula is Boolean satisfiable it is in all sets in question. Conversely, if $\varphi \in \mathrm{wgenSAT}_{\mathrm{pos}}$, thus for some witnessed $\langle \mathbf{M}, \boldsymbol{A}\rangle$ it holds $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} > 0$, then $\|\neg\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, hence $\|\varphi\|^{\boldsymbol{B}_2}_{\mathbf{M}^{\neg\neg}} = 1$, i.e. φ is Boolean satisfiable.

(2) Clearly, $\mathrm{wgenTAUT}_{\mathrm{pos}} \subseteq \mathrm{wstTAUT}_{\mathrm{pos}} \subseteq \mathrm{TAUT}(\boldsymbol{B}_2)$. Conversely, if φ is not in $\mathrm{wgenTAUT}_{\mathrm{pos}}$, thus for some witnessed $\langle \boldsymbol{A}, \mathbf{M}\rangle$ it holds $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$, then $\|\neg\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$ hence $\|\varphi\|^{\boldsymbol{B}_2}_{\mathbf{M}^{\neg\neg}} = 0$, i.e. φ is not a Boolean tautology. Finally, the *tertium non datur* formula $(\forall x)(P(x) \vee \neg P(x))$ is a formula in $\mathrm{TAUT}(\boldsymbol{B}_2)$ but not in $\mathrm{wstTAUT}$, hence not in $\mathrm{wgenTAUT}$. □

Now recall Definition 3.0.10 of the mapping h (for all $x \in [0,1]$, let $h(x) = 2x$ for $x \leq 1/2$ and $h(x) = 1$ for $x \geq 1/2$), the model $h(\mathbf{M})$ for each standard model $\mathbf{M}$ and the notation $\boldsymbol{C}\oplus$ for a continuous t-norm whose first component is $\boldsymbol{C}$ on $[0, \frac{1}{2}]$. Also recall the mapping f from (the proof of) 3.0.14 ($f(x) = \frac{x}{2}$ for $x < 1$, $f(1) = 1$).

LEMMA 5.1.6. *Let $\mathbf{M}$ be a standard model, $\varphi(a_1, \ldots, a_n)$ a formula with $a_1, \ldots, a_n \in M$ substituted for free variables. Write φ for $\varphi(a_1, \ldots, a_n)$ for brevity.*

(1) $h(\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}}) = \|\varphi\|^{\boldsymbol{C}}_{h(\mathbf{M})}$.

(2) If $\mathbf{M}$ is $\boldsymbol{C}$-witnessed then $\|\varphi\|^{\boldsymbol{C}\oplus}_{f(\mathbf{M})} = f(\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}})$.

(3) Consequently, if $\mathbf{M}$ is $\boldsymbol{C}\oplus$-witnessed then $h(\mathbf{M})$ is $\boldsymbol{C}$-witnessed; if $\mathbf{M}$ is $\boldsymbol{C}$-witnessed then $f(\mathbf{M})$ is $\boldsymbol{C}\oplus$-witnessed.

Proof. Easy; compare it with the proofs of 3.0.10 and 3.0.14. □

THEOREM 5.1.7. *For a continuous t-norm $\boldsymbol{C}\oplus$:*

(1) $\mathrm{wstSAT}_{\mathrm{pos}}(\boldsymbol{C}\oplus) = \mathrm{wstSAT}_{\mathrm{pos}}(\boldsymbol{C})$,

(2) $\mathrm{wstTAUT}_{\mathrm{pos}}(\boldsymbol{C}\oplus) = \mathrm{wstTAUT}_{\mathrm{pos}}(\boldsymbol{C})$,

(3) $\mathrm{wstSAT}(\boldsymbol{C}\oplus) = \mathrm{wstSAT}(\boldsymbol{C})$.

Proof. For (1) observe that, for $\mathbf{M}$ $[\boldsymbol{C}]$-witnessed, $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} > 0$ implies $\|\varphi\|^{\boldsymbol{C}\oplus}_{j(\mathbf{M})} > 0$ and for $\mathbf{M}$ $[\boldsymbol{C}\oplus]$-witnessed, $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}} > 0$ implies $\|\varphi\|^{\boldsymbol{C}}_{h(\mathbf{M})} > 0$.

For (2) replace $>$ by $=$. For (3) observe that, similarly to the above and under the respective witnessedness, $\|\varphi\|^{\boldsymbol{C}}_{\mathbf{M}} = 1$ implies $\|\varphi\|^{\boldsymbol{C}\oplus}_{j(\mathbf{M})} = 1$ and $\|\varphi\|^{\boldsymbol{C}\oplus}_{\mathbf{M}} = 1$ implies $\|\varphi\|^{\boldsymbol{C}}_{h(\mathbf{M})} = 1$, □

THEOREM 5.1.8. *For $\bullet$ being* 'wst' *or* 'wgen'*, and each continuous t-norm $*$,*

(1) $\varphi \in \bullet\mathrm{TAUT}_{\mathrm{pos}}(\mathrm{L}_*\forall)$ *iff* $\neg\varphi \notin \bullet\mathrm{SAT}(\mathrm{L}_*\forall)$,

(2) $\varphi \in \bullet\mathrm{SAT}_{\mathrm{pos}}(\mathrm{L}_*\forall)$ *iff* $\neg\varphi \notin \bullet\mathrm{TAUT}(\mathrm{L}_*\forall)$.

Proof. This is true since $\|\neg\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ iff $\|\varphi\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$, for any BL-chain $\boldsymbol{A}$. □

	wstTAUT	wstSAT	$\mathrm{wstTAUT_{pos}}$	$\mathrm{wstSAT_{pos}}$
Ł∀	$\mathbf{\Pi_2}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Sigma_2}$-complete
G∀	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
Π∀	$\mathbf{\Pi_2}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
(Ł⊕)∀	$\mathbf{\Pi_2}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Sigma_2}$-complete
(G⊕)∀	$\mathbf{\Sigma_1}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete
(Π⊕)∀	$\mathbf{\Sigma_2}$-hard	$\mathbf{\Pi_1}$-complete	$\mathbf{\Sigma_1}$-complete	$\mathbf{\Pi_1}$-complete

Table 5. Complexity of standard witnessed semantics.

COROLLARY 5.1.9. *For each continuous t-norm* $*$,

(1) $\mathrm{wgenTAUT_{pos}}(\mathrm{L}_*\forall)$ *is* $\mathbf{\Sigma_1}$*-complete.*

(2) $\mathrm{wgenSAT_{pos}}(\mathrm{L}_*\forall)$ *is* $\mathbf{\Pi_1}$*-complete.*

Proof. By the preceding theorem and Theorem 5.1.4. □

COROLLARY 5.1.10.

(1) If $\mathrm{wstSAT}(\mathrm{L}_*\forall) = \mathrm{wgenSAT}(\mathrm{L}_*\forall)$, *then*
$$\mathrm{wstTAUT_{pos}}(\mathrm{L}_*\forall) = \mathrm{wgenTAUT_{pos}}(\mathrm{L}_*\forall).$$

(2) If $\mathrm{wstTAUT}(\mathrm{L}_*\forall) = \mathrm{wgenTAUT}(\mathrm{L}_*\forall)$, *then*
$$\mathrm{wstSAT_{pos}}(\mathrm{L}_*\forall) = \mathrm{wgenSAT_{pos}}(\mathrm{L}_*\forall).$$

THEOREM 5.1.11. *The set* $\mathrm{wstTAUT}(\text{Ł}\forall)$ *recursively reduces to* $\mathrm{wstTAUT}((\text{Ł}\oplus)\forall)$.

Proof. Let φ be an arbitrary sentence. If $\mathbf{M}$ is a witnessed Ł-model with $\|\varphi\|^{\text{Ł}}_{\mathbf{M}} < 1$ then for the mapping f as above, $f(\mathbf{M})$ is a witnessed (Ł⊕)-model with $\|\neg\neg\varphi\|^{\text{Ł}\oplus}_{f(\mathbf{M})} < 1$. On the other hand, $\mathbf{M}$ is a witnessed (Ł⊕)-model with $\|\neg\neg\varphi\|^{\text{Ł}\oplus}_{\mathbf{M}} < 1$ then $h(\mathbf{M}^{\neg\neg})$ is a witnessed Ł-model in which the value of φ is < 1. Thus the mapping assigning to each φ its double negation is the claimed reduction. □

The complexity results for the witnessed semantics we can obtain are collected in Table 5. Let us justify them. First, the values for witnessed Ł∀ (first row) are the same as for Ł∀ (without assuming witnessed) due to Fact 1 above. Now for the first column: For G∀ see Theorem 5.1.4; for Π∀ see [16]. For (Ł⊕)∀ see Theorem 5.1.11. For (G⊕)∀ and (Π⊕)∀ the only thing we know is by 2.0.13; it is a problem what more can be shown. Theorem 5.1.11 gives also the rest for Ł⊕; for wstSAT and $\mathrm{wstSAT_{pos}}$ the results follows by Theorem 5.1.5 (1) and for $\mathrm{wstTAUT_{pos}}$ by Theorem 5.1.5 (2).

5.2 Semantics of models over complete chains

In this subsection we present an overview of results about complexity of the semantics over complete signs (for more details see [21]). We start from a very simple observation. Let L be a (recursively axiomatizable) axiomatic extension of MTL such that the MacNeille completion of every L-chain is an L-chain. Then every L-chain can be σ-embedded into a complete L-chain. Hence, $\mathrm{L}\forall$ is strongly complete with respect to the class of complete L-chains. Let $\mathrm{complTAUT}(\mathrm{L}\forall)$ denote the set of all sentences which are valid in every complete L-chain, and let $\mathrm{complTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$, $\mathrm{complSAT}(\mathrm{L}\forall)$, $\mathrm{complSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$ be defined analogously.

THEOREM 5.2.1. *Let* L *be a (recursively axiomatizable) axiomatic extension of* MTL *such that the MacNeille completion of every* L*-chain is an* L*-chain. Then:*

(1) $\mathrm{complTAUT}(\mathrm{L}\forall) = \mathrm{genTAUT}(\mathrm{L}\forall)$*, and hence it is* $\mathbf{\Sigma_1}$*-complete.*

(2) $\mathrm{complTAUT}_{\mathrm{pos}}(\mathrm{L}\forall) = \mathrm{genTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$*, and hence it is* $\mathbf{\Sigma_1}$*-complete.*

(3) $\mathrm{complSAT}(\mathrm{L}\forall) = \mathrm{genSAT}(\mathrm{L}\forall)$*, and hence it is* $\mathbf{\Pi_1}$*-complete.*

(4) $\mathrm{complSAT}_{\mathrm{pos}}(\mathrm{L}\forall) = \mathrm{genSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$*, and hence it is in* $\mathbf{\Pi_1}$.

The theorem applies to several prominent fuzzy logics, including the following: MTL, IMTL, SMTL, NM, or $\mathrm{C}_n\mathrm{BL}$ for $n > 0$ (i.e. BL plus the n-contraction schema $\phi^n \to \phi^{n+1}$).

We now investigate the first-order logics of complete chains of extensions of BL. First, we prove the non-arithmeticity of sets of the form $\mathrm{TAUT}(\mathbb{K})$ or $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$ or $\mathrm{SAT}(\mathbb{K})$ or $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$, where $\mathbb{K}$ is a set of complete BL-chains which either contains an infinite product chain or contains a BL-chain of the form $\boldsymbol{B}\oplus\boldsymbol{C}$ where $\boldsymbol{B}$ is an infinite product chain and $\boldsymbol{C}$ is a BL-chain. In this case, we will say that $\mathbb{K}$ *does not exclude* Π. Moreover, we will give a characterization of recursively axiomatizable extensions L of BL such that set of first-order sentences valid in all complete L-chains is recursively axiomatizable. Namely, we will prove that if L is a recursively axiomatizable schematic extensions of BL and $\mathbb{K}$ is the class of complete L-chains, then $\mathrm{TAUT}(\mathbb{K})$ is recursively axiomatizable iff L proves the n-contraction schema $\phi^n \to \phi^{n+1}$ for some n. We start from the non-arithmeticity results.

THEOREM 5.2.2. *Suppose that* $\mathbb{K}$ *is a class of complete* BL*-chains which does not exclude* Π*. Then the followings sets are non-arithmetical:* $\mathrm{TAUT}(\mathbb{K})$*,* $\mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$*,* $\mathrm{SAT}(\mathbb{K})$*, and* $\mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

Proof. The proof is similar to the proofs of Theorem 3.0.23 and of Theorem 3.0.28, and hence we will only point out the parts where the proofs diverge. Let U, $^{\neg\neg}$, θ_1, θ_2, θ_3 and θ_4 be as in the proof of Theorem 3.0.23. As in the proofs of Theorems 3.0.23 and 3.0.28, it suffices to prove:

Claim: Let ψ be any *PA*-sentence, and let $\theta = (\theta_1)^2 \mathbin{\&} \theta_2 \mathbin{\&} \theta_3 \mathbin{\&} \theta_4$. (Note that we have replaced θ_1 by θ_1^2 for reasons that will become clear later). Then:

(1) If $\mathbf{N} \models \psi$, then $\theta \to \psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$, and if $\mathbf{N} \not\models \psi$, then $\theta \mathbin{\&} \psi^{\neg\neg} \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

(2) If $\mathbf{N} \models \psi$, then $\theta \mathbin{\&} \psi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K})$ and if $\mathbf{N} \not\models \psi$, then $\theta \to \psi^{\neg\neg} \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$.

Proof of the claim: (1) Let $\boldsymbol{A} \in \mathbb{K}$ and $\mathbf{M}$ be any $\boldsymbol{A}$-structure. Let us write $\|\dots\|$ instead of $\|\dots\|^{\boldsymbol{A}}_{\mathbf{M}}$. The claim is clear if $\|\theta\| = 0$, and hence assume $\|\theta\| > 0$. We first prove that $\boldsymbol{A}$ is either an infinite product chain or is the ordinal sum of an infinite product chain and a BL-chain. Consider $a = \|(\forall x)U(x)\|$. Then a must belong to the first Wajsberg component, $\boldsymbol{W}$, of $\boldsymbol{A}$, otherwise $\|\theta_1\| = 0$.

We now prove that W has only two elements. Suppose not. If for some $d \in M$, $\|U(d)\| \in W \setminus \{0,1\}$, then $\|(\forall x)\neg\neg U(x)\| = \|(\forall x)U(x)\|$. Hence, $\|\theta_1\| = 0$, and $\|\theta\| = 0$. The same is true if for some $d \in M$, $\|U(d)\| = 0$. It remains to consider the case where for all $d \in M$, either $\|U(d)\| \notin W$ or $\|U(d)\| = 1$, but $\|(\forall x)U(x)\| \in W \setminus \{1\}$. But in this case, $\|(\forall x)U(x)\|$ is the unique coatom, a, of $\boldsymbol{W}$, $\|\neg(\forall x)U(x)\|$ is the unique atom of $\boldsymbol{W}$, and $(\neg a)^2 = 0$. Hence $\|\theta_1^2\| = 0$ (here is the place where we need θ_1^2). Hence, $\boldsymbol{A}$ is an SBL-algebra.

It follows that $\inf\{\|U(d)\| \mid d \in M\} = 0$ and for all $d \in M$, $\|U(d)\| > 0$ (otherwise, $\|\theta_1\| = 0$). Using $\|\theta_2\| > 0$, we can derive that $\boldsymbol{A}$ is either an infinite product chain or the ordinal sum of an infinite product chain and a BL-chain, by an argument which is very similar to the one used in the proof of Lemma 3.0.24.

We now can prove an analogue of Lemma 3.0.25. That is:

(a) For every sentence γ of PA, $\|\gamma^{\neg\neg}\| \in \{0,1\}$.

(b) Since $\|\theta_3\| > 0$, by (1) we have $\|\theta_3\| = 1$. Moreover we can obtain as in Lemma 3.0.25 a model $\mathbf{M}^{\neg\neg}$ from $\mathbf{M}$ such that for every sentence γ of PA, one has $\mathbf{M}^{\neg\neg} \models \gamma$ iff $\|\gamma^{\neg\neg}\| = 1$. Hence, $\mathbf{M}^{\neg\neg} \models Q^+$.

(c) We can prove as in Lemma 3.0.25 that $\mathbf{M}^{\neg\neg}$ is isomorphic to $\mathbf{N}$. Analogously to Lemma 3.0.25, we can see that there is a $c \in M$ such that $\|U(c)\| < 1$ and for all $d \in M$, if $\mathbf{M}^{\neg\neg} \models c \leq d$, then $\|U(S(d))\| \leq \|(\forall x)(x \leq^{\neg\neg} d \to U(x))\|^2$. It follows $\|U(c+n)\| \leq \|U(c)\|^{2^n}$. Now the first component of $\boldsymbol{A}$ is a complete product chain, and in a complete product chain, if $a < 1$, then $\inf\{a^{2^n} \mid n \in \mathsf{N}\} = 0$. Hence, the proof proceeds as in Lemma 3.0.25.

At this point, since $\mathbf{N} \models \psi$ and $\|\theta\| > 0$, we must have $\|\psi^{\neg\neg}\| = 1$, and also $\|\theta \to \psi^{\neg\neg}\| = 1$. Therefore, if $\mathbf{N} \models \psi$, then $\theta \to \psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$, and hence $\theta \to \psi^{\neg\neg} \in \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$. Moreover, if $\mathbf{N} \not\models \psi$, then $\|\theta\| > 0$ implies $\|\psi^{\neg\neg}\| = 0$, and $\theta \,\&\, \psi^{\neg\neg} \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$. *A fortiori*, $\theta \,\&\, \psi^{\neg\neg} \notin \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$.

(2) Let $\boldsymbol{A} \in \mathbb{K}$ be either an infinite product chain or the ordinal sum of an infinite product chain and a BL-chain. Let $a \in A$ be such that $0 < a < 1$ and a is in the first product component of $\boldsymbol{A}$. Take an $\boldsymbol{A}$-structure $\mathbf{M}$ whose universe is N and whose constants and function symbols are interpreted as in $\mathbf{N}$ (if we do not want function symbols, we may replace them by predicates as usual). Moreover, for every n-ary predicate P and for $d_1, \dots, d_n \in \mathsf{N}$ we define $P_{\mathbf{M}}(d_1, \dots, d_n) = 1$ if $\mathbf{N} \models P(d_1, \dots, d_n)$ and $P_{\mathbf{M}}(d_1, \dots, d_n) = 0$ otherwise. Moreover let for all $n \in \mathsf{N}$, $U_{\mathbf{M}}(n) = a^{3^n}$. It is readily seen that $\|\theta\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ and $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ if $\mathbf{N} \models \psi$ and $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 0$ otherwise. Hence, if $\mathbf{N} \models \psi$, then $\theta \,\&\, \psi^{\neg\neg} \in \mathrm{SAT}(\mathbb{K})$, and *a fortiori* $\theta \,\&\, \psi^{\neg\neg} \in \mathrm{SAT}_{\mathrm{pos}}(\mathbb{K})$. Moreover if $\mathbf{N} \not\models \psi$, then $\theta \to \psi \notin \mathrm{TAUT}_{\mathrm{pos}}(\mathbb{K})$, and *a fortiori* $\theta \to \psi \notin \mathrm{TAUT}(\mathbb{K})$. $\square$

COROLLARY 5.2.3. *For each logic* $\mathrm{L} \in \{\Pi, \mathrm{BL}, \mathrm{SBL}\}$*, the following sets of sentences are not arithmetical:* $\mathrm{complSAT}(\mathrm{L}\forall)$*,* $\mathrm{complSAT}_{\mathrm{pos}}(\mathrm{L}\forall)$*,* $\mathrm{complTAUT}(\mathrm{L}\forall)$*, and* $\mathrm{complTAUT}_{\mathrm{pos}}(\mathrm{L}\forall)$.

The next theorem, which describes the complexity of satisfiability and of tautologicity problems for Łukasiewicz logic with respect to the class of complete MV-chains, is easy to prove:

THEOREM 5.2.4.

(i) $\mathrm{complTAUT}(Ł\forall) = \mathrm{stTAUT}(Ł\forall)$*, and so it is* $\mathbf{\Pi_2}$*-complete.*

(ii) $\mathrm{complSAT}(Ł\forall) = \mathrm{stSAT}(Ł\forall)$*, and so it is* $\mathbf{\Pi_1}$*-complete.*

(iii) $\mathrm{complTAUT}_{\mathrm{pos}}(Ł\forall) = \mathrm{stTAUT}_{\mathrm{pos}}(Ł\forall)$*, and so it is* $\mathbf{\Sigma_1}$*-complete.*

(iv) $\mathrm{complSAT}_{\mathrm{pos}}(Ł\forall) = \mathrm{stSAT}_{\mathrm{pos}}(Ł\forall)$*, and so it is* $\mathbf{\Sigma_2}$*-complete.*

Proof. Every complete MV-chain is either isomorphic to $[0,1]_Ł$ or a finite MV-chain, and hence it is a complete subalgebra of $[0,1]_Ł$. Hence, the semantics based on the class of complete MV-chains is equivalent to the standard semantics for predicate Łukasiewicz logic. □

We now characterize the schematic extensions L of BL such that $\mathrm{complTAUT}(\mathrm{L}\forall)$ is recursively axiomatizable.

THEOREM 5.2.5. *Let* L *be a recursively axiomatizable schematic extension of* BL*, and assume that* L*, as a propositional logic, is complete with respect to the class of complete* L*-chains. Then the following are equivalent:*

(1) For some natural number $n \geq 1$*,* L *proves the* n*-contraction schema* $\phi^n \to \phi^{n+1}$.

(2) $\mathrm{complTAUT}(\mathrm{L}\forall)$ *is recursively axiomatizable.*

Proof. $(1) \Rightarrow (2)$ Follows from Theorem 5.2.1, since in Chapter V it is proved that for any variety $\mathbb{V}$ of $\mathrm{C}_n\mathrm{BL}$-chains, the class of all chains in $\mathbb{V}$ is closed under MacNeille completions.

$(2) \Rightarrow (1)$ Let $\mathbb{V}$ be the variety equivalent to L, and assume that for no natural number n, L proves the n-contraction schema. Then (see Chapter V) either $\mathbb{V}$ contains all finite MV-chains, and hence it contains $[0,1]_Ł$, or it contains an infinite product chain. In the latter case, it contains $[0,1]_\Pi$, and by Theorem 5.2.2, $\mathrm{complTAUT}(\mathrm{L}\forall)$ is not arithmetical. In the former case, (2) follows from the following claim.

Claim: Let $\mathbb{K}$ be a class of complete BL-chains. If $[0,1]_Ł \in \mathbb{K}$, then $\mathrm{TAUT}(\mathbb{K})$ is $\mathbf{\Pi_2}$-hard.

Proof of the claim: Let, for every formula ψ, $\psi^{\neg\neg}$ be the formula obtained by replacing in ψ every atomic subformula γ by $\neg\neg\gamma$. We claim that for every sentence ψ we have $\psi \in \mathrm{stTAUT}(Ł\forall)$ iff $\neg\neg\psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$. This clearly implies the claim of the lemma and hence of Theorem 5.2.5.

If $\neg\neg\psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$, then since $[0,1]_Ł \in \mathbb{K}$, we have $\neg\neg\psi^{\neg\neg} \in \mathrm{stTAUT}(Ł\forall)$, and $\psi \in \mathrm{stTAUT}(Ł\forall)$, as $\vdash_Ł \psi \leftrightarrow \neg\neg\psi^{\neg\neg}$.

Conversely, consider any complete BL-chain $\boldsymbol{A}$. Then $\boldsymbol{A}$ has a complete first Wajsberg component $\boldsymbol{W}$, which is necessarily a complete Wajsberg algebra, and hence it can be σ-embedded into $[0,1]_{Ł}$. Moreover for every $\boldsymbol{A}$-structure $\mathbf{M}$, let us define $\mathbf{M}'$ as follows: $\mathbf{M}'$ has the same domain as $\mathbf{M}$ and the constants and the function symbols are interpreted as in $\mathbf{M}$; moreover for every n-ary predicate P and for every $a_1, \dots, a_n \in M$, $P_{\mathbf{M}'}(a_1, \dots, a_n) = \|\neg\neg P(a_1, \dots, a_n)\|$. Then we can prove by induction on ψ, that for every sentence ψ, the following conditions hold:

(1) If $\sup(W \setminus \{1\}) \in W$, then $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'}$.

(2) If $\sup(W \setminus \{1\}) \notin W$ and $\|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'} < 1$, then $\|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'} = \|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}}$.

(3) If $\sup(W \setminus \{1\}) \notin W$ and $\|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'} = 1$, then either $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$ or $\|\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \sup(W \setminus \{1\})$.

In any case, we have $\|\neg\neg\psi^{\neg\neg}\|^{\boldsymbol{A}}_{\mathbf{M}} = \|\psi\|^{\boldsymbol{W}}_{\mathbf{M}'}$. Hence, if $\psi \in \mathrm{stTAUT}(Ł\forall)$, then $\neg\neg\psi^{\neg\neg} \in \mathrm{TAUT}(\mathbb{K})$. □

5.3 Fragments with implication and negation

Here we shall understand that a logic $\mathrm{L}_1\forall$ is a fragment of the logic $\mathrm{L}_2\forall$ given by the subset C of connectives of $\mathrm{L}_2\forall$ if $\mathrm{L}_1\forall$-formulae are $\mathrm{L}_2\forall$-formulae not containing any connective which is not in C, and each $\mathrm{L}_1\forall$-formula φ is $\mathrm{L}_1\forall$-provable iff it is $\mathrm{L}_2\forall$-provable. If this is the case then for each $\mathrm{L}_1\forall$-theory T and formula φ it follows that T proves φ over $\mathrm{L}_1\forall$ iff T proves φ over $\mathrm{L}_2\forall$; moreover each $\mathrm{L}_1\forall$-algebra (chain) is an $\mathrm{L}_2\forall$-algebra (chain). We are going to discuss fragments of some extensions of $\mathrm{BL}\forall$ given by the connectives $\to, \neg$ (equivalently, $\to, \overline{0}$) and by all connectives except $\overline{0}$ (and of course except $\neg$)—the hoop logics.

We shall discuss logics extending $\mathrm{BL}\forall$, particularly Łukasiewicz, Gödel, product logic, $\mathrm{SBL}\forall$ and $\mathrm{BL}\forall$ itself. For such a logic $\mathrm{L}\forall$ we denote its fragment given by the connectives $\to, \neg$ as $\mathrm{L}\forall\restriction(\to,\neg)$. These logics are recursively axiomatized, i.e. the set $\mathrm{genTAUT}(\mathrm{L}\forall\restriction(\to,\neg))$ of general tautologies is $\mathbf{\Sigma_1}$ and the set $\mathrm{genSAT}(\mathrm{L}\forall\restriction(\to,\neg))$ of general satisfiable formulae (= consistent formulae) is $\mathbf{\Pi_1}$.

To prove hardness, recall that each $Ł\forall$-formula is equivalent to some $Ł\forall\restriction(\to,\neg)$-formula; thus the set $\mathrm{genTAUT}(Ł\forall)$ recursively reduces to $\mathrm{genTAUT}(Ł\forall\restriction(\to,\neg))$ and similarly for genSAT (as well as for stTAUT, stSAT, as we shall need later).

For Gödel logic the set $\mathrm{SAT}(\boldsymbol{B}_2\restriction(\to,\neg))$ of classically satisfiable formulae with connectives only $\to, \neg$ recursively reduces to $\mathrm{genSAT}(\mathrm{G}\forall\restriction(\to,\neg))$ by the mapping associating to each formula φ the formula $\varphi^{\neg\neg}$ resulting from φ by adding double negation to each atomic subformula; analogously for $\mathrm{genTAUT}$.

This works also for any logic with strict negation. On the other hand, for a logic $\mathrm{L}_*\forall$ given by any continuous t-norm $*$ whose first component is Łukasiewicz observe that the mapping sending any φ to $\varphi^{\neg\neg}$ recursively reduces $\mathrm{genSAT}(Ł\forall\restriction(\to,\neg))$ to $\mathrm{genSAT}(\mathrm{L}_*\forall\restriction(\to,\neg))$ and analogously for $\mathrm{genTAUT}$.

LEMMA 5.3.1.
The set $\mathrm{genTAUT}(\mathrm{BL}\forall\restriction(\to,\neg))$ *is* $\mathbf{\Sigma_1}$*-hard and* $\mathrm{genSAT}(\mathrm{BL}\forall\restriction(\to,\neg))$ *is* $\mathbf{\Pi_1}$*-hard.*

Proof. Let $\varphi^{\neg}$ result from φ by replacing each atom by its negation. Recall that for each sentence φ not containing the existential quantifier, φ is a general tautology of

$\text{Ł}\forall$ iff $\varphi^{\neg}$ is a general tautology of $\mathrm{BL}\forall$ [12]. One can prove in the same way that φ is generally satisfiable in $\text{Ł}\forall$ iff $\varphi^{\neg}$ is generally satisfiable in $\mathrm{BL}\forall$. Clearly if φ is a $(\to, \overline{0})$-formula then so is $\varphi^{\neg}$. The set of all $(\to, \overline{0})$-sentences not containing $\exists$ that are general $\text{Ł}\forall$ tautologies is $\mathbf{\Sigma_1}$-complete and the set of all such formulae that are generally $\text{Ł}\forall$-satisfiable is $\mathbf{\Pi_1}$-complete (since in $\text{Ł}\forall$ each sentence is equivalent to a $(\to, \overline{0})$-sentence not containing $\exists$). This gives the result. □

We focus now on the standard semantics. In the case of first-order Łukasiewicz logic we have that $\mathrm{stTAUT}(\text{Ł}\forall)$ recursively reduces to $\mathrm{stTAUT}(\text{Ł}\forall\!\restriction\!(\to, \neg))$ and conversely; thus the set $\mathrm{stTAUT}(\text{Ł}\forall\!\restriction\!(\to, \neg))$ is $\mathbf{\Pi_2}$-complete. Similarly to genSAT, we see that the set $\mathrm{stSAT}(\text{Ł}\forall\!\restriction\!(\to, \neg))$ is $\mathbf{\Pi_1}$-complete.

Finally we survey some results on non-arithmeticity. We shall only sketch their proofs since they are rather laborious; for details see [18] and references thereof.

DEFINITION 5.3.2. *For a set $\mathbb{K}$ of continuous t-norms, we denote the set of all finite sets Σ of predicate $(\to, \overline{0})$-formulae that are standardly $\mathbb{K}$-satisfiable (i.e. for some t-norm $*$ from $\mathbb{K}$ there is a standard interpretation $\mathbf{M}$ with $\|\alpha\|^*_{\mathbf{M}} = 1$ for each $\alpha \in \Sigma$) by* $\mathrm{stSAT}_{(f)}(\mathbb{K}\forall\!\restriction\!(\to, \overline{0}))$. *By $\|\Sigma\|^*_{\mathbf{M}}$ we denote the minimum of the values $\|\alpha\|^*_{\mathbf{M}}$ for $\alpha \in \Sigma$.*

THEOREM 5.3.3. *For each set $\mathbb{K}$ of continuous t-norms containing the product t-norm, the set* $\mathrm{stSAT}_{(f)}(\mathbb{K}\forall\!\restriction\!(\to, \overline{0}))$ *is not arithmetical.*

Let us sketch the proof. Recall the notation $\varphi^{\neg\neg}$ for the formula resulting from φ by putting double negation to all atomic subformulae of φ; for a set Ψ of formulae, $\Psi^{\neg\neg}$ is the set $\{\varphi^{\neg\neg} | \varphi \in \Psi\}$. For a standard interpretation $\mathbf{M}$, let $\mathbf{M}^{\neg\neg}$ result from $\mathbf{M}$ by replacing each positive value by 1. Clearly, if $*$ is a continuous t-norm with Gödel negation then for each standard interpretation $\mathbf{M}$ and each formula φ we have $\|\varphi^{\neg\neg}\|^*_{\mathbf{M}} = \|\varphi\|^{\boldsymbol{B}_2}_{\mathbf{M}^{\neg\neg}}$.

Now recall the finite axiomatic system Q^+ (see Introduction). Since it is a system in classical logic we may assume that all axioms are $(\to, \overline{0})$-formulae (other connectives being classically definable). Evidently, if $*$ has Gödel negation then for each (standard) interpretation $\mathbf{M}$, $\|(Q^+)^{\neg\neg}\|^*_{\mathbf{M}} > 0$ iff $\mathbf{M}^{\neg\neg}$ is a classical model of Q^+ (possibly with non-absolute equality—you may factorize).

DEFINITION 5.3.4. *Let U be a unary predicate distinct from the symbols of Q^+. We introduce the following axioms:*

Ψ_0 $\quad \neg(\forall x)U(x)$

Ψ_1 $\quad (\forall x)(\forall y)(\forall z)[((U(x) \to U(y)) \to U(y)) \to$
$\quad [(U(y) \to U(z)) \to ((U(y) \to U(x)) \to U(x)) \to ((U(y) \to U(x)) \to U(x))]]$

Ψ_2 $\quad (\forall x)\neg\neg U(x)$

Ψ_3 $\quad (\forall x)((U(x) \to U(S(x))) \to U(x))$ $\qquad$ (*S is the successor from Q^+*)

Ψ_4 $\quad (\forall x)(\forall y)(\forall z)(\neg\neg(x \leq y) \to (U(y) \to U(x)).$ $\qquad$ (*$<$ from Q^+*))

Finally, Ψ will stand for the set $\{\Psi_i\ i = 0, \ldots, 4\} \cup (Q^+)^{\neg\neg}$

Note that all formulae in Ψ are $(\to, \overline{0})$-formulae.

LEMMA 5.3.5. *Let Φ be a $(\to, \overline{0})$-formula of arithmetic. Φ is true in the standard model $\mathbf{N}$ of arithmetic iff the finite set $\Psi \cup \{\Phi^{\neg\neg}\}$ is standardly $\mathrm{L}_{\mathbb{K}}\forall\!\restriction\!(\to, \neg)$-satisfiable.*

	Ł	G	Π	SBL	BL
genTAUT	$\mathbf{\Sigma_1}$-compl.	$\mathbf{\Sigma_1}$-compl.	$\mathbf{\Sigma_1}$-compl.	$\mathbf{\Sigma_1}$-compl.	$\mathbf{\Sigma_1}$-compl.
genSAT	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.
stTAUT	$\mathbf{\Pi_2}$-compl.	$\mathbf{\Sigma_1}$-compl.	NA	NA	NA
stSAT	$\mathbf{\Pi_1}$-compl.	$\mathbf{\Pi_1}$-compl.	NA	NA	NA

Table 6. Arithmetical complexity of $(\to, \neg)$-fragments.

The theorem follows: the preceding lemma gives a recursive reduction of the set of $(\to, \neg)$-formulae true in the standard model to our set $\mathrm{stSAT}_{(f)}(\mathrm{L}_{\mathbb{K}}\forall\!\restriction\!(\to, \overline{0}))$.

THEOREM 5.3.6. *The set of standard tautologies of* $\Pi\forall\!\restriction\!(\to, \overline{0})$ *is not arithmetical.*

We again sketch a proof. Here we shall use Ψ_0 and Ψ_2 as well as Q^+ defined above. Let Ψ_3' be the axiom

$$(\forall x))(U(S(x)) \lhd (\forall z)(\neg\neg(z < x) \to U(z)))$$

where $\alpha \lhd \beta$ is $(\beta \to (\beta \to \alpha)) \to \beta$. Ψ will be the finite set $\{\Psi_0, \Psi_2, \Psi_3'\} \cup (Q^+)^{\neg\neg}$. And let us agree that if T is a finite set $\{\alpha_1, \dots, \alpha_n\}$ of formulae and β is a formula then $T \to \beta$ means $(\alpha_1 \to (\alpha_2 \to \dots (\alpha_n \to \beta) \dots))$.

LEMMA 5.3.7. *Let* Φ *be a sentence of arithmetic.* $\mathbf{N} \models \Phi$ *iff the formula* $\Psi \to \Phi^{\neg\neg}$ *is a standard tautology of* $\Pi\forall\!\restriction\!(\to, \neg)$.

This shows that the function assigning to each $(\to, \overline{0})$-sentence Φ of arithmetic the sentence $\Psi \to \Phi^{\neg\neg}$ recursively reduces the non-arithmetical set of $(\to, \overline{0})$-sentences true in $\mathbf{N}$ to the set of standard tautologies of $\Pi\forall\!\restriction\!(\to, \neg)$. This completes the (sketch of a) proof of Theorem 5.3.6.

THEOREM 5.3.8. *For each set* $\mathbb{K}$ *of continuous t-norms containing the product t-norm, the set of standard tautologies of the logic* $\mathrm{L}_{\mathbb{K}}\forall\!\restriction\!(\to, \neg)$ *is non-arithmetical.*

For the proof sketch, we use the notation: $\varphi \uparrow \psi$ stands for $(\varphi \to \psi) \to \psi$. Θ is the set $\{\Psi_0, \Psi_2, \Sigma\}$ of formulae where Ψ_0, Ψ_2 is as above (i.e. $(\forall x)\neg\neg U(x)$, $\neg(\forall x)U(x)$) and Σ is the formula $(\forall x, y)((U(x) \uparrow U(y)) \to (U(y) \uparrow U(x)))$. Now, for each $(\to, \overline{0})$-formula Φ not containing the predicate U take the pair $\{(\Theta \to \Phi), (\exists x)(\Phi \uparrow U(x))\}$. Observe that both formulae are $(\to, \overline{0})$-formulae. Their disjunction will be denoted by $\Phi^{\#}$. This is not a $(\to, \overline{0})$-formula, but to say for some α, β that $\alpha \vee \beta$ is a tautology is the same as to say that $(\alpha \uparrow \beta) \wedge (\beta \uparrow \alpha)$ is a tautology and this is to same as say that both $\alpha \uparrow \beta$ and $\beta \uparrow \alpha$ are tautologies. One shows that under the assumptions of the theorem a $(\to, \overline{0})$-formula Φ not containing the predicate U is a $\Pi\forall$-tautology iff $\Phi^{\#}$ is an $\mathrm{L}_{\mathbb{K}}\forall$-tautology hence if the corresponding two $(\to, \overline{0})$-formulae are $\mathrm{L}_{\mathbb{K}}\forall\!\restriction\!(\to, \overline{0})$-tautologies. This recursively reduces the (non-arithmetical) set $\mathrm{stTAUT}(\Pi\forall\!\restriction\!(\to, \overline{0}))$ to the set $\mathrm{stTAUT}(\mathrm{L}_{\mathbb{K}}\forall\!\restriction\!(\to, \overline{0}))$ which gives a proof of Theorem 5.3.8.

The results are summarized in the Table 6. One may also consult [7].

5.4 Complexity of hoop logics

This section deals with falsity-free first-order fuzzy logics. They can be obtained in two ways: given a ($\triangle$-)core fuzzy logic L, (1) one can consider $\mathrm{L}\forall^-$, the falsity-free fragment of $\mathrm{L}\forall$, or (2) one can first take L^-, the falsity-free fragment of L, and then consider its first-order version $\mathrm{L}^-\forall$. We are going to prove that in many important cases both logics coincide.

DEFINITION 5.4.1. *Let $\boldsymbol{A}$ be an L^--chain and let $a \in A$. By $\boldsymbol{A} \searrow [a, 1]$ we mean the algebra whose domain is $[a, 1]$, whose lattice operations and whose implication are the restrictions to $[a, 1]$ of the corresponding operations of $\boldsymbol{A}$, and whose monoid operation $*$ is $x * y = (x \cdot y) \vee a$. Moreover by $\boldsymbol{A}^+$ we denote the ordinal sum of the two element MV-chain and $\boldsymbol{A}$. By $\boldsymbol{A}'$ we denote $\boldsymbol{A}^+$ if $\boldsymbol{A}$ has no minimum, and $\boldsymbol{A}$ with 0 interpreted as the minimum of $\boldsymbol{A}$ otherwise.*

The following lemma is easy to demonstrate.

LEMMA 5.4.2.

1. *Let L be one of MTL, BL, Ł, or G, let $\boldsymbol{A}$ be an L^--chain and $a \in A$. Then $\boldsymbol{A} \searrow [a, 1]$ is an L-chain.*
2. *Let L be one of SMTL or SBL, and let $\boldsymbol{A}$ be an L^--chain. Then $\boldsymbol{A}^+$ is an L-chain.*
3. *Let L be one of Π, ΠMTL, and let $\boldsymbol{A}$ be an L^--chain. Then $\boldsymbol{A}'$ is an L-chain.*

PROPOSITION 5.4.3. *Let L be an axiomatic extension of MTL such that at least one of the following conditions hold:*

1. *For every L^--chain $\boldsymbol{A}$, and for every $a \in A$, $\boldsymbol{A} \searrow [a, 1]$ is an L-chain.*
2. *For every L^--chain $\boldsymbol{A}$, $\boldsymbol{A}^+$ is an L-chain.*
3. *L extends SMTL and for every L^--chain $\boldsymbol{A}$, $\boldsymbol{A}'$ is an L-chain.*

Then $\mathrm{L}\forall^- = \mathrm{L}^-\forall$.

Proof. Assume that 1 holds. Since both logics are finitary it is enough to prove that for every finite set $T \cup \{\phi\}$ of formulae of, $T \vdash_{\mathrm{L}\forall^-} \phi$ iff $T \vdash_{\mathrm{L}^-\forall} \phi$.

For the non-trivial direction, suppose that ϕ is invalidated in a model $\mathbf{M}$ of T on a totally ordered L-semihoop $\boldsymbol{H}$. Now let γ be the conjunction of all universal closures of subformulae of $T \cup \{\phi\}$ and let $a = \|\gamma\|^{\boldsymbol{H}}_{\mathbf{M}}$. Consider $\boldsymbol{H} \searrow [a, 1]$ and define a new interpretation $\mathbf{M} \searrow [a, 1]$ letting for every n-ary predicate P, $P^{\mathbf{M}\searrow[a,1]}(a_1, \ldots, a_n) = P^{\mathbf{M}}(a_1, \ldots, a_n) \vee a$. It is enough to prove that $\mathbf{M} \searrow [a, 1]$ is a safe structure on $\boldsymbol{H} \searrow [a, 1]$ which is a model of T but not of φ. Thus, we want to see that for any first-order formula χ in the language with $\overline{0}$, the value $\|\chi\|^{\boldsymbol{H}\searrow[a,1]}_{\mathbf{M}\searrow[a,1]}$ exists for any evaluation. To this end, we define a translation ^{0} by induction as:

$$(P(x_1, \ldots, x_n))^0 = P(x_1, \ldots, x_n) \vee \gamma$$

$$(\varphi \,\&\, \psi)^0 = (\varphi^0 \,\&\, \psi^0) \vee \gamma \qquad (\varphi \to \psi)^0 = \varphi^0 \to \psi^0$$

$$(\varphi \wedge \psi)^0 = \varphi^0 \wedge \psi^0 \qquad (\varphi \vee \psi)^0 = \varphi^0 \vee \psi^0$$

$$\overline{0}^0 = \gamma \qquad ((\forall x)\varphi)^0 = (\forall x)\varphi^0 \qquad ((\exists x)\varphi)^0 = (\exists x)\varphi^0.$$

One can check, by an easy induction, that for every first-order formula χ in the language with $\overline{0}$ and any evaluation v, $\|\chi\|^{\boldsymbol{H}\searrow[a,1]}_{\mathbf{M}\searrow[a,1],v} = \|\chi^0\|^{\boldsymbol{H}}_{\mathbf{M},v}$. Therefore, we have a safe structure. On the other hand, another easy induction shows that for every ψ subformula of $T \cup \{\phi\}$, $\|\psi^0\|^{\boldsymbol{H}}_{\mathbf{M},v} = \|\psi\|^{\boldsymbol{H}}_{\mathbf{M},v}$. Thus, for every $\psi \in T$, $\|\psi\|^{\boldsymbol{H}\searrow[a,1]}_{\mathbf{M}\searrow[a,1],v} = \|\psi^0\|^{\boldsymbol{H}}_{\mathbf{M},v} = \|\psi\|^{\boldsymbol{H}}_{\mathbf{M},v} = 1$ and $\|\phi\|^{\boldsymbol{H}\searrow[a,1]}_{\mathbf{M}\searrow[a,1],v} = \|\phi^0\|^{\boldsymbol{H}}_{\mathbf{M},v} = \|\phi\|^{\boldsymbol{H}}_{\mathbf{M},v} < 1$.

The proof with assumptions 2 and 3 is rather similar, and we only point out the differences with the proof from assumption 1. In the case of 2, we must take $\boldsymbol{H}^+$ and in the case of 3 we must take $\boldsymbol{H}'$. Moreover in both cases we need not change $\mathbf{M}$ and we need not use the translation. Safety follows from the fact that negation is strict and hence negated formulae are evaluated either in 0 or in 1. □

COROLLARY 5.4.4. *Let* L *be any of* MTL, BL, $\mathrm{Ł}$, G, SMTL, SBL, Π *or* $\Pi\mathrm{MTL}$. *Then* $\mathrm{L}\forall^- = \mathrm{L}^-\forall$.

We focus on the falsity-free fragments of the four most important fuzzy logics—$\mathrm{BL}\forall$, $\mathrm{Ł}\forall$, $\mathrm{G}\forall$, and $\Pi\forall$; they are denoted by $\mathrm{BLH}\forall$, $\mathrm{ŁH}\forall$, $\mathrm{GH}\forall$, $\Pi\mathrm{H}\forall$ respectively. According to the previous corollary, their general semantics is given by bounded hoops, Wajsberg hoops, Gödel hoops and product hoops respectively. The *standard semantics* are the $\overline{0}$-free reducts of the corresponding standard semantics of the original logic, i.e. for $\mathrm{BLH}\forall$ all t-norm algebras, for $\mathrm{ŁH}\forall$, $\mathrm{GH}\forall$, $\Pi\mathrm{H}\forall$ the corresponding t-norm algebra with the same name.

As proved above, $\mathrm{BL}\forall$ is a conservative expansion of $\mathrm{BLH}\forall$ and the similarly for the other logics. Thus general tautologies of $\mathrm{BLH}\forall$ are just falsity-free general tautologies of $\mathrm{BL}\forall$ and the same for Ł, G, Π. For trivial reasons, the same holds for standard tautologies: standard tautologies of $\mathrm{BLH}\forall$ are, by definition, just falsity-free standard tautologies of $\mathrm{BLH}\forall$ and similarly for Ł, G, Π.

For any logic $\mathrm{L}\forall$ of our logics, let $\mathrm{stTAUT}(\mathrm{L}\forall)$ denote the set of all its standard tautologies; for $\mathrm{L}\forall$ having $\overline{0}$ let $\mathrm{stTAUT}^+(\mathrm{L}\forall)$ denote the set of all its standard falsity-free tautologies (i.e. not containing $\overline{0}$). Recall that the sets $\mathrm{stTAUT}(\mathrm{G}\forall)$, $\mathrm{stTAUT}(\mathrm{Ł}\forall)$, $\mathrm{stTAUT}(\Pi\forall)$, $\mathrm{stTAUT}(\mathrm{BL}\forall)$ are respectively $\boldsymbol{\Sigma_1}$-complete, $\boldsymbol{\Pi_2}$-complete, not arithmetical, not arithmetical. Our question is what is the arithmetical complexity of standard tautologies of the hoop logic, or equivalently the complexity of $\mathrm{stTAUT}^+(\mathrm{G}\forall)$, $\mathrm{stTAUT}^+(\mathrm{Ł}\forall)$, $\mathrm{stTAUT}^+(\Pi\forall)$, $\mathrm{stTAUT}^+(\mathrm{BL}\forall)$. We will show that it is the same as the complexity of the corresponding set stTAUT.

To this end, we need to consider again, now for real-valued chains, the constructions used above. Given a t-norm algebra $\boldsymbol{B}$ and $0 \leq a < 1$, we consider the BL-algebra $\boldsymbol{B} \searrow [a,1]$. Clearly there is a monotone $1-1$ mapping of $[a,1]$ onto $[0,1]$ transforms $B \searrow [a,1]$ into a t-norm algebra. $\boldsymbol{B} \searrow [1,1]$ is the degenerated one element algebra. On the other hand, given a predicate language $\mathcal{P}$ and $Q(d)$, a closed atomic formula with Q, d not belonging to $\mathcal{P}$, we define $\gamma = Q(d)$ and for each formula φ of the language

$\mathcal{P}$ define the formula φ^0 as in the proof of Proposition 5.4.3. If $\mathbf{M}$ is a safe standard interpretation of $\mathcal{P}$ and $a \in [0,1]$, then $\langle \mathbf{M}, a\rangle$ is any expansion of $\mathbf{M}$ with $\|\gamma\| = a$; $\mathbf{M} \searrow [a,1]$ results from $\mathbf{M}$ by replacing the interpretation $P_{\mathbf{M}}$ of each predicate P (of $\mathcal{P}$) by $P'_{\mathbf{M}}(u_1,\ldots,u_n) = \max\{P_{\mathbf{M}}(u_1,\ldots,u_n), a\}$. The following lemma is easily verified by induction:

LEMMA 5.4.5. *For each formula φ of $\mathcal{P}$, $\mathbf{M}$-evaluation* v *of variables, and $a \in [0,1]$,*

$$\|\varphi^0\|^{\boldsymbol{B}}_{\langle \mathbf{M},a\rangle,v} = \|\varphi\|^{\boldsymbol{B}\searrow[a,1]}_{\mathbf{M}\searrow[a,1],v}.$$

THEOREM 5.4.6. *The sets* $\mathrm{stTAUT(BLH\forall)}$, $\mathrm{stTAUT^+(BL\forall)}$ *and* $\mathrm{stTAUT(BL\forall)}$ *are mutually recursively reducible (and hence have the same Turing degree). Similarly for* Ł∀, ŁH∀ *and* G∀, GH∀.

Proof. Our lemma recursively reduces $\mathrm{stTAUT(BL\forall)}$ to $\mathrm{stTAUT^+(BL\forall)}$ in the following way: $\varphi \in \mathrm{stTAUT(BL\forall)}$ iff $\varphi^0 \in \mathrm{stTAUT^+(BL\forall)}$. Conversely, a falsity-free formula φ belongs to $\mathrm{stTAUT^+(BL\forall)}$ iff it belongs to $\mathrm{stTAUT(BL\forall)}$, thus the identity mapping of falsity-free formulae reduces $\mathrm{stTAUT^+(BL\forall)}$ to $\mathrm{stTAUT(BL\forall)}$. Further recall that $\mathrm{stTAUT(BLH\forall)} = \mathrm{stTAUT^+(BL\forall)}$. The same for Ł, G. Since if $\boldsymbol{B}$ is the Łukasiewicz or Gödel t-norm algebra and $a < 1$ then $\boldsymbol{B} \searrow [a,1]$ is isomorphic to $\boldsymbol{B}$. □

THEOREM 5.4.7.

(1) $\mathrm{stTAUT(\Pi H\forall)}$, $\mathrm{stTAUT^+(\Pi\forall)}$, $\mathrm{stTAUT(\Pi\forall)} \cap \mathrm{stTAUT(Ł\forall)}$ *are mutually recursively reducible (have the same Turing degree).*

(2) The set $\mathrm{stTAUT(\Pi\forall)} \cap \mathrm{stTAUT(Ł\forall)}$ *is not arithmetical.*

Proof. (1) The mapping sending each φ to φ^0 reduces $\mathrm{stTAUT(\Pi\forall)} \cap \mathrm{stTAUT(Ł\forall)}$ to $\mathrm{stTAUT^+(\Pi\forall)}$ since for $a = 0$ and $\boldsymbol{B}$ being the standard product algebra, the algebra $\boldsymbol{B} \searrow [a,1]$ is just $\boldsymbol{B}$ and for $0 < a < 1$ we know that $\boldsymbol{B} \searrow [a,1]$ is isomorphic to the standard Łukasiewicz algebra.

(2) Consult the proof of non-arithmeticity of $\mathrm{TAUT(\Pi\forall)}$ in above: there is a closed formula Ψ and for each formula Φ of Peano arithmetic a formula Φ' of $\Pi\forall$ such that Φ is true in the standard model of arithmetic iff $\Psi \to \Phi'$ is in $\mathrm{stTAUT(\Pi\forall)}$. Observe that Ψ implies $\neg(\forall x)U(x)\ \&\ (\forall x)\neg\neg U(x)$ and this formula has in each interpretation the value 0 when computed under standard Łukasiewicz semantics; thus $\Psi \to \Phi'$ is in $\mathrm{stTAUT(Ł\forall)}$ for any Φ. Hence the mapping sending each Φ to $\Psi \to \Phi'$ reduces truth in $\mathbf{N}$ to $\mathrm{stTAUT(\Pi\forall)} \cap \mathrm{stTAUT(Ł\forall)}$, which shows that the latter set is not arithmetical. □

COROLLARY 5.4.8.

(1) The set $\mathrm{stTAUT^+(G\forall)} = \mathrm{stTAUT(GH\forall)}$ *is $\mathbf{\Sigma_1}$-complete.*

(2) The set $\mathrm{stTAUT^+(Ł\forall)} = \mathrm{stTAUT(ŁH\forall)}$ *is $\mathbf{\Pi_2}$-complete.*

(3) $\mathrm{stTAUT}^{+}(\Pi\forall) = \mathrm{stTAUT}(\Pi\mathrm{H}\forall)$ *and* $\mathrm{stTAUT}^{+}(\mathrm{BL}\forall) = \mathrm{stTAUT}(\mathrm{BLH}\forall)$ *are not arithmetical.*

REMARK 5.4.9. Every *formula in the language of hoops is satisfiable and hence positively satisfiable (interpret every predicate into* 1*).*

5.5 Monadic BL logics

It is well-known that monadic classical predicate logic, i.e. classical predicate logic without function symbols and with only unary predicate symbols, is decidable. It is a natural question to ask for which sets $\mathbb{K}$ of standard BL-algebras the set $\mathrm{TAUTm}(\mathbb{K})$ of monadic predicate formulae valid in all algebras in $\mathbb{K}$ is decidable. An almost complete answer to this question is provided by the next theorem.

THEOREM 5.5.1. *If* $\mathbb{K}$ *is a class of standard* BL*-chains containing an algebra not isomorphic to* $[0,1]_{Ł}$ *or to* $[0,1]_{\Pi}$*, then* $\mathrm{TAUTm}(\mathbb{K})$ *is undecidable.*

Proof. We will recursively reduce the classical theory T of two equivalence relations to $\mathrm{TAUTm}(\mathbb{K})$. Since T is undecidable (see e.g. [28]) the claim will follow. Let P, Q and H be unary predicate symbols, and let $R = (\exists x)H(x)$. Let E and S denote the binary predicate symbols of T representing the two equivalence relations. We define for every monadic formula ξ (possibly with parameters) of T, a formula ξ^{+} of monadic BL logic in the following inductive way:

If $\xi = E(a,b)$ (where a and b are either variables or parameters) then $\xi^{+} = (P(a) \leftrightarrow P(b)) \vee R$.

If $\xi = S(a,b)$, then $\xi^{+} = (Q(a) \leftrightarrow Q(b)) \vee R$.

If $\xi = \overline{0}$, then $\xi^{+} = R$.

If $\xi = \sigma \mathbin{\&} \gamma$, then $\xi^{+} = (\sigma^{+} \mathbin{\&} \gamma^{+}) \vee R$.

If $\xi = \sigma \to \gamma$, then $\xi^{+} = \sigma^{+} \to \gamma^{+}$.

If $\xi = (\exists x)\sigma$, then $\xi^{+} = (\exists x)\sigma^{+}$.

If $\xi = (\forall x)\sigma$, then $\xi^{+} = (\forall x)\sigma^{+}$.

Now for every ψ of T, we consider the formula $\psi^{\star} = ((\forall x)(R \uparrow (P(x) \vee Q(x))) \to (\psi^{+} \vee (\exists x)(P(x) \vee Q(x))$. We are going to prove that for every sentence ψ of T one has: $T \vdash \psi$ iff $\psi^{\star} \in \mathrm{TAUTm}(\mathbb{K})$. Since * is recursive, this will give the desired result.

LEMMA 5.5.2. *Let* $\boldsymbol{A}$ *be any* BL*-chain and* $\mathbf{M}$ *an* $\boldsymbol{A}$*-structure such that* $\|\psi^{\star}\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$*. Let us write* $\|\ldots\|$ *instead of* $\|\ldots\|^{\boldsymbol{A}}_{\mathbf{M}}$*. Then for all* $d \in M$*,* $\|P(d) \vee Q(d)\| < \|R\| < 1$*, and* $\|P(d) \vee Q(d)\|$ *and* $\|R\|$ *do not belong to the same Wajsberg component.*

Proof. If for some $d \in M$, $\|P(d) \vee Q(d)\| \geq \|R\|$, then by Lemma 3.0.19 we obtain $\|R \uparrow (P(d) \vee Q(d))\| = \|P(d) \vee Q(d)\|$. Hence, $\|(\forall x)(R \uparrow (P(x) \vee Q(x)))\| \leq \|(\exists x)(P(x) \vee Q(x))\|$, and finally $\|\psi^{\star}\| = 1$, a contradiction. If $\|P(d) \vee Q(d)\| < \|R\|$, but $\|P(d) \vee Q(d)\|$ and $\|R\|$ are in the same Wajsberg component, so again by

Lemma 3.0.19 $\|R \uparrow (P(d) \vee Q(d))\| = \|R\|$. Thus $\|(\forall x)(R \uparrow (P(x) \vee Q(x)))\| \leq \|R\| \leq \|\psi^+\|$, and $\|\psi^\star\| = 1$, which is a contradiction. Finally, if $\|R\| = 1$, then $\|\psi^+\| = 1$ and $\|\psi^\star\| = 1$, which is impossible. □

LEMMA 5.5.3. *With reference to the notation of Lemma 5.5.2, let for all $d \in M$, $\|P(d) \vee Q(d)\| < \|R\|$, and $\|P(d) \vee Q(d)\|$ and $\|R\|$ are not in the same Wajsberg component of $\boldsymbol{A}$. Then for every formula ξ of T, either $\|\xi^+\| = 1$ or $\|\xi^+\| = \|R\|$.*

Proof. Clearly, $\|\overline{0}^+\| = \|R\|$. Now let $a, b \in M$, and let $\boldsymbol{W}$ and $\boldsymbol{U}$ be the Wajsberg components of $\boldsymbol{A}$ such that $\|P(a)\| \in W$ and $\|P(b)\| \in U$ (possibly, $\boldsymbol{W} = \boldsymbol{U}$). If $\|P(a)\| = \|P(b)\|$, then $\|E(a,b)^+\| = \|(P(a) \leftrightarrow P(b)) \vee R\| = 1$. Otherwise, using the definition of ordinal sum, it is readily seen that $\|P(a) \leftrightarrow P(b)\| \in (W \cup U) \setminus \{1\}$. Hence $\|P(a) \leftrightarrow P(b)\| < \|R\|$, and $\|E(a,b)^+\| = \|(P(a) \leftrightarrow P(b)) \vee R\| = \|R\|$. Similarly, either $\|S(a,b)^+\| = 1$ or $\|S(a,b)^+\| = \|R\|$. Hence, the claim holds if ξ is atomic. The induction steps corresponding to $\to$, $\exists$ and $\forall$ are immediate. Finally, suppose $\xi = \sigma \,\&\, \gamma$. Then $\|\xi^+\| = \|(\sigma^+ \,\&\, \gamma^+) \vee R\|$, and the claim follows from the induction hypothesis. □

Suppose that the hypotheses of Lemma 5.5.3 are satisfied. Define a model $\mathbf{M}^{\neg\neg}$ of T from $\mathbf{M}$ as follows: the domain of $\mathbf{M}^{\neg\neg}$ is M, and for $a, b \in M$, let $\mathbf{M}^{\neg\neg} \models E(a,b)$ if $\|P(a) \leftrightarrow P(b)\| = 1$, and $\mathbf{M}^{\neg\neg} \models S(a,b)$ iff $\|Q(a) \leftrightarrow Q(b)\| = 1$. Clearly, $\mathbf{M}^{\neg\neg} \models T$.

LEMMA 5.5.4. *Under the same assumptions as in Lemma 5.5.3, for every sentence ξ of T with parameters in M, one has: $\mathbf{M}^{\neg\neg} \models \xi$ iff $\|\xi^+\| = 1$.*

Proof. We proceed by induction on ξ. If $\xi = E(a,b)$, then $\mathbf{M}^{\neg\neg} \models \xi$ iff $\|P(a) \leftrightarrow P(b)\| = 1$ iff $\|(P(a) \leftrightarrow P(b)) \vee R\| = 1$ (because by Lemma 5.5.4 $\|R\| < 1$). The case where $\xi = S(a,b)$ is treated similarly. If $\xi = \overline{0}$, then $\mathbf{M}^{\neg\neg} \not\models \xi$ and $\|\xi^+\| = \|R\| < 1$. Thus the claim holds for ψ atomic.

Suppose $\xi = \lambda \,\&\, \sigma$. Then $\mathbf{M}^{\neg\neg} \models \xi$ iff $\mathbf{M}^{\neg\neg} \models \lambda$ and $\mathbf{M}^{\neg\neg} \models \sigma$ iff (by the induction hypothesis) $\|\lambda^+\| = \|\sigma^+\| = 1$ iff $\|(\lambda^+ \,\&\, \sigma^+) \vee R\| = 1$ iff $\|\xi^+\| = 1$.

For the induction steps corresponding to $\to$, to $\exists$ and to $\forall$ we use the fact that by Lemmata 5.5.2 and 5.5.4 for every sentence γ of T, $\|\gamma^+\| \in \{\|R\|, 1\}$, i.e., $\|\gamma^+\|$ can assume only two truth values. Thus the semantic interpretations of $\to$, $\exists$ and $\forall$ are the same as in classical logic (with 0 replaced by $\|R\|$), and the proof proceeds in a straightforward way. □

We conclude the proof of Theorem 5.5.1. Suppose that $\psi^\star \notin \mathrm{TAUTm}(\mathbb{K})$. Let $\boldsymbol{A} \in \mathbb{K}$ and $\mathbf{M}$ an $\boldsymbol{A}$-structure such that $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. Then, *a fortiori*, $\|\psi^+\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. By Lemmata 5.5.2, 5.5.3 and 5.5.4, we have a model $\mathbf{M}^{\neg\neg}$ of T such that for every sentence γ of T, $\mathbf{M}^{\neg\neg} \models \gamma$ iff $\|\gamma^+\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Since $\|\psi^+\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$, $\mathbf{M}^{\neg\neg} \not\models \psi$, and $T \nvdash \psi$.

Conversely, suppose $T \nvdash \psi$. Let $\mathbf{M}_T$ be a finite or countable model of T such that $\mathbf{M}_T \not\models \psi$. Take $\boldsymbol{A} \in \mathbb{K}$ not isomorphic to any of $[0,1]_{\text{Ł}}$ and $[0,1]_\Pi$. Then $\boldsymbol{A}$ must have an idempotent element (i.e., an element a such that $a \star a = a$) which is different from 0 and from 1. Hence if $0 \leq x < a < y < 1$, then x and y do not belong to the same

Wajsberg component. Now we can obtain an $\boldsymbol{A}$-structure $\mathbf{M}$ such that the following conditions hold:

(i) The domain M of $\mathbf{M}$ coincides with the domain M_T of $\mathbf{M}_T$.

(ii) $a < R_{\mathbf{M}} < 1$ (since $R = (\exists x)H(x)$, it suffices to take $H_{\mathbf{M}}(d) = \frac{a+1}{2}$ for all $d \in M$).

(ii) For all $d \in M$, $P_{\mathbf{M}}(d) < a$ and $Q_{\mathbf{M}}(d) < a$.

(iii) For $a, b \in M$, $P_{\mathbf{M}}(a) = P_{\mathbf{M}}(b)$ iff $\mathbf{M}_T \models E(a, b)$.

(iv) For $a, b \in M$, $Q_{\mathbf{M}}(a) = Q_{\mathbf{M}}(b)$ iff $\mathbf{M}_T \models S(a, b)$.

Then we can easily check that the model $\mathbf{M}^{\neg\neg}$ built from $\mathbf{M}$ as in Lemma 5.5.4 is isomorphic to $\mathbf{M}_T$. Thus, by Lemma 5.5.4, for every sentence γ of T, one has: $\mathbf{M}_T \models \gamma$ iff $\|\gamma^+\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Hence, $\|\psi^+\|^{\boldsymbol{A}}_{\mathbf{M}} \neq 1$. Moreover by conditions (i) and (ii) and by Lemma 3.0.19, for all $d \in M$, $\|R \uparrow (P(d) \vee Q(d))\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$, hence $\|(\forall x)(R \uparrow (P(x) \vee Q(x)))\|^{\boldsymbol{A}}_{\mathbf{M}} = 1$. Finally, $\|(\exists x)(P(x) \vee Q(x))\|^{\boldsymbol{A}}_{\mathbf{M}} \leq a < 1$. Hence, $\|\psi^\star\|^{\boldsymbol{A}}_{\mathbf{M}} < 1$. □

An inspection on the proof of Theorem 5.5.1 yields that if $\psi^\star$ can be invalidated in *some arbitrarily given* linearly ordered BL-algebra $\boldsymbol{A}$, then $T \nvdash \psi$. On the other hand, if $T \nvdash \psi$, then by the proof of Theorem 5.5.1 we obtain a standard BL-algebra (hence a fortiori a linearly ordered BL-algebra) $\boldsymbol{A}$ such that $\psi^\star$ is not valid in $\boldsymbol{A}$. In other words, $T \vdash \psi$ iff $\psi^\star$ is valid in all linearly ordered BL-algebras. Since BL$\forall$ is complete with respect to the interpretations in linearly ordered BL-algebras, and since of course the same property holds for its monadic version BLm$\forall$, we have that for every sentence ψ of the language of T, T $\vdash \psi$ iff BLm$\forall \vdash \psi^\star$. It follows:

THEOREM 5.5.5. *The monadic predicate logic* BLm$\forall$ *is undecidable.*

5.6 Extensions of Łukasiewicz logic

We survey the main results on complexity of logics properly extending Łukasiewicz predicate logic Ł$\forall$ (for details see [6]). Recall the definitions of finite Ł-algebras and Komori algebras (see Chapter VI):

- $\mathbf{Ł}_{n+1}$ is the subalgebra of $[0,1]_{Ł}$ with the domain $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$,
- $\mathbf{K}_{n+1} = \langle\{\langle i, a\rangle \in \mathsf{N} \times_{\mathrm{lex}} \mathsf{Z} \mid \langle 0,0\rangle \leq \langle i, a\rangle \leq \langle n, 0\rangle\}, \oplus_{\mathbf{K}_{n+1}}, \neg_{\mathbf{K}_{n+1}}, \langle 0,0\rangle\rangle$, $\langle i, x\rangle \oplus_{\mathbf{K}_{n+1}} \langle j, y\rangle = \min\{\langle n, 0\rangle, \langle i+j, x+y\rangle\}$ and $\neg_{\mathbf{K}_{n+1}}\langle i, x\rangle = \langle n-i, -x\rangle$.

Recall Komori's result saying that for each consistent propositional logic which is a proper axiomatic extension of Łukasiewicz propositional logic there exist finite subsets A, B of the set of natural numbers bigger than 1 such that $A \cup B \neq \emptyset$ and the set of standard tautologies of this logic is defined as $\bigcap_{i\in A} \mathrm{stTAUT}(\mathbf{K}_i) \cap \bigcap_{j \in B} \mathrm{stTAUT}(\mathbf{Ł}_j)$. ($\mathbf{Ł}_j$ is the finite Łukasiewicz algebra with j-elements). Thus the logic can be denoted by Ł$_{A,B}$; its predicate version is denoted Ł$_{A,B}\forall$.

The algebras $\mathbf{K}_i$ for $i \in A$ and $\mathbf{Ł}_j$ for $j \in B$ are the standard algebras of $Ł_{A,B}$; they generate the variety of general $Ł_{A,B}$-algebras. It can be shown that any Komori algebra and any finite Łukasiewicz algebra is in the variety of general $Ł_{A,B}$-algebras iff it is a subalgebra of a generic $Ł_{A,B}$-algebra. Each safe $\mathbf{K}_n$-structure is witnessed.

THEOREM 5.6.1. *Let* $Ł_{A,B}$ *be any consistent axiomatic extension of* Ł. *Then:*

(1) $\mathrm{genTAUT}(Ł_{A,B}\forall)$ *is* $\mathbf{\Sigma_1}$*-complete.*

(2) $\mathrm{stTAUT}(Ł_{A,B}\forall)$ *is* $\mathbf{\Sigma_1}$*-complete iff* A *is empty.*

(3) $\mathrm{stTAUT}(Ł_{A,B}\forall)$ *is* $\mathbf{\Pi_2}$*-complete iff* A *is non-empty.*

(4) $\mathrm{genSAT}(Ł_{A,B}\forall) = \mathrm{stSAT}(Ł_{A,B}\forall)$ *and this set is* $\mathbf{\Pi_1}$*-complete.*

COROLLARY 5.6.2. *Let* $Ł_{A,B}$ *be any consistent axiomatic extension of* Ł. *Then:*

(1) $\mathrm{genSAT}_{\mathrm{pos}}(Ł_{A,B}\forall)$ *is* $\mathbf{\Pi_1}$*-complete.*

(2) $\mathrm{stSAT}_{\mathrm{pos}}(Ł_{A,B}\forall)$ *is* $\mathbf{\Pi_1}$*-complete iff* A *is empty.*

(3) $\mathrm{stSAT}_{\mathrm{pos}}(Ł_{A,B}\forall)$ *is* $\mathbf{\Sigma_2}$*-complete iff* A *is non-empty.*

(4) $\mathrm{genTAUT}_{\mathrm{pos}}(Ł_{A,B}\forall) = \mathrm{stTAUT}_{\mathrm{pos}}(Ł_{A,B}\forall)$ *and this set is* $\mathbf{\Sigma_1}$*-complete.*

5.7 Open problems

Some unsolved problems in complexity of predicate fuzzy logics:

- What is the exact complexity of $\mathrm{TAUTm}(\mathbb{K})$, when $\mathbb{K}$ is the class of *all* standard BL-algebras?
- Is any of $\mathrm{TAUTm}([0,1]_Ł)$ or $\mathrm{TAUTm}([0,1]_\Pi)$ decidable?[4]
- What is the complexity (outside the arithmetical hierarchy) of the non arithmetic sets, such as standard tautologies of product logic, contained in this chapter?[5]
- If instead of the full vocabulary we consider one which has a countable number of relational symbols of all arities (but not functional symbols), is the general semantics of first-order fuzzy logics still undecidable?
- What are the complexities of sets of standard (positive) tautologies and satisfiable sentences when $\triangle$ is added to the language?
- For which ($\triangle$-)core fuzzy logics L, does $\mathrm{L}\forall^- = \mathrm{L}^-\forall$ hold?
- What is the complexity of the monadic fragments in the case of the general semantics?
- What are the complexities of first-order fuzzy logics when one considers the supersound semantics in the sense of [2]?

[4]Very recently Félix Bou has presented a proof of the undecidability of the monadic fragments of Łukasiewicz and product predicate logics at a conference and at some seminars. As far as the authors know, these results are still not in a written form at the moment of the publication of the present handbook.

[5]The paper [24] gives a rather tight bound to for the complexity of standard tautologies of $\mathrm{BL}\forall$ and $\Pi\forall$, showing that in both cases it is between $\emptyset^\omega$ (the degree of true arithmetic) and $\emptyset^{\omega+1}$ (the degree of the halting problem with oracle on $\emptyset^\omega$).

6 Historical remarks and further reading

For a general reference on recursion theory and arithmetical hierarchy see e.g. [28]. Arguably, the first work considering the arithmetical complexity of a particular first-order fuzzy logic is Scarpellini's paper [29] published on 1962 which shows that the set of (standard) tautologies of Łukasiewicz predicate logic is not recursively axiomatizable. Ragaz proved in his PhD Thesis [27] from 1981 that this set is actually $\mathbf{\Pi_2}$-complete (alternative proofs of the same fact have been obtained by Goldstern in [8] and by Hájek in [11]; we have presented here the latter). When first-order versions (in full language) for other propositional fuzzy logics started being systematically studied by Hájek in his works during the nineties, their undecidability appeared as a general problem. Actually, Montagna and Ono proved in [26] that all first-order versions of consistent axiomatic extensions of MTL are undecidable and thus the issue of their arithmetical complexity became a crucial item in the agenda of Mathematical Fuzzy Logic. Various results concerning the position in the arithmetical hierarchy of the sets of tautologies, positive tautologies, satisfiable sentences, and positively satisfiable sentences w.r.t. the standard semantics of the three main fuzzy logics are in Hájek's papers [9, 10] and in Chapter 6 of his monograph [11]. The next natural step was the study of complexity problems for first-order logics based on other continuous t-norms: this was done again by Hájek in [12, 13] and by Montagna in [22, 23]. Note that [12] contains the first result of non-arithmeticity in the present context (non-arithmeticity of $\mathrm{stSAT}(\Pi\forall)$). In 2005, the survey paper [14], besides collecting the mentioned results, provides a new study where the standard semantics is replaced by the general semantics, i.e. the one given by models over arbitrary linearly ordered BL-algebras. Recent works have extended the scope of the studies on arithmetical complexity issues to: witnessed semantics over product logic [16], logics of complete BL-chains [21], monadic fragments [23], logics with less propositional connectives [17] and extensions of Łukasiewicz logic [6]. Moreover, the recent survey [18] collects information (with references) on sets of standard/general tautologies, satisfiable sentences, also for positive tautologies and satisfiable sentences of important logics (extending $\mathrm{BL}\forall$), logics with Baaz's $\triangle$, with truth constants, logics extending Łukasiewicz logic and Baaz-Gödel logics $\mathrm{G}(V)$ whose set of truth values V is a subset of the real unit interval. The latter are also studied in [1, 15]. Recently, in the paper [25] Montagna and Noguera have presented a general approach to complexity problems extending the scope to core and $\triangle$-core fuzzy logics and to arbitrary semantics (Sections 2 and 4 of the present chapter are based on this paper). Theorem 2.0.20 has been obtained by Bou and Noguera in [3]. Finally, it is worth mentioning the recent paper [19] which gives new important insight into undecidability issues in fuzzy logics with an example of a decidable theory T over Łukasiewicz (propositional) logic and a formula φ such that the theory $T \cup \{\varphi\}$ is undecidable.

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