- SET Theory

A collection of well defined objects. Ex→ $A = \{a, e, i, o, u\}$

(i) Empty set – Set having no element. Denoted by ϕ or $\{\}$.

(ii) Singleton set – Set having one element.

(iii) Subset – A set is called subset of set B if for all $x \in A \Rightarrow x \in B$.

Expressed $A \subseteq B$.

(iv) PowerSet – The family of all subsets of A is called power set of A.

- Operations on Sets

Union $A \cup B$, Intersection $A \cap B$, Difference $A - B$

- Cartesian Product of Two sets

$A = \{1, 2\}, \quad B = \{3, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

- Inclusion Exclusion Principles in Set Theory

→ (i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A - B) = n(A) - n(A \cap B)$

(iii) $n(B - A) = n(B) - n(A \cap B)$

(iv) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

- Domain & Range of Relation

$R = \{(1, p), (1, q), (2, q)\}$

∴ Domain $= \{1, 2\}$, Range $= \{p, q\}$

- Reflexive Relation

A relation R is reflexive if for all $x \in A$, $(x, x) \in R$

- Symmetric Relation

A relation R is symmetric if $(a, b) \in R$, then $(b, a) \in R$

Ex→ $A = \{1, 2\}, \quad B = \{1, 2, 3\}$

$R = \{(1,2), (2,1), (2,2)\}$ is symmetric

- Transitive Relation

A relation R is transitive if $(x, y) \in R$ & $(y, z) \in R$, then $(x, z) \in R$

- Equivalence Relation

A relation R is equivalence relation of A, if
1) R is reflexive
2) R is symmetric
3) R is transitive

- Partial Order Relation

A relation R is a partial order relation, if it is
(i) Reflexive (ii) Anti-symmetric (iii) Transitive

Ex (i) Let $A = \{1, 2, 3\}$
$R = \{(1,1), (2,2), (3,3), (1,2) (2,2), (1,3)\}$

(ii) In a set of Real numbers, the relation less than or equal to $(\leq)$ is a partial order relation.

- Partial Order Set (POSET)

The set A, with partial order relation R is called POSET.

- Total Order Relation (~~TOSET~~)

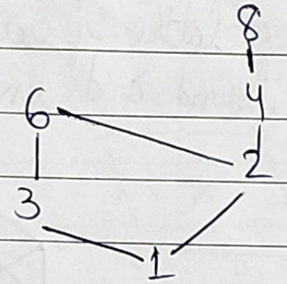
A partial order relation R in set A is called total order relation if for every element $a, b \in A$ such that either aRb or bRa or $a = b$.

- TOSET : A set with total order relation

- Hasse Diagram

A graphical representation of a partial order relation in which all arrowhead are pointing upward is known as Hasse diagram.

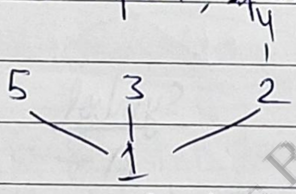
Let $A = \{1, 2, 3, 4, 6, 8\}$ be ordered by relation a divides b.



- Lattice

Maximal element - In a poset, if an element is not related to any other element.

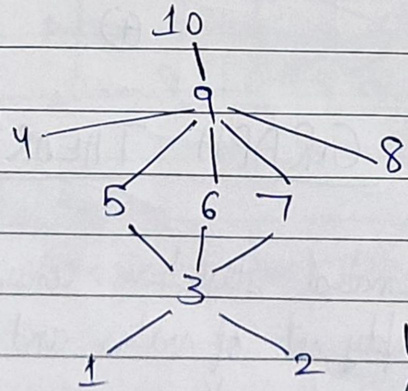
Minimal element - In a poset, if no element is related to an element.



3, 4, 5 are maximal elements
1 is minimal element.

Theorem: A finite non-empty poset (P, R) has atleast one maximal and atleast one minimal element.

Upper Bound & Lower Bound

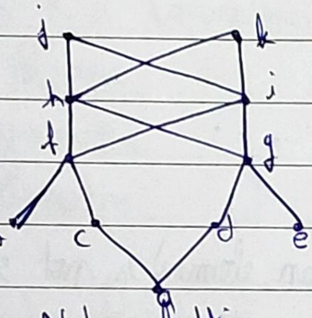


Upper Bound of (5, 6, 7) = 9, 10
Lower Bound of (5, 6, 7) = 1, 3, 2
Least UB of (5, 6, 7) = 9
Greatest UB of (5, 6, 7) = 10
Least LB of (5, 6, 7) = 1, 2
Greatest LB of (5, 6, 7) = 3

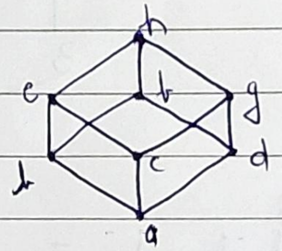
- **Lattice**

A poset (P, R) is said to be a lattice if every two elements in the set L has a unique least upper bound and unique greatest lower bound.

Ex→



Not a lattice



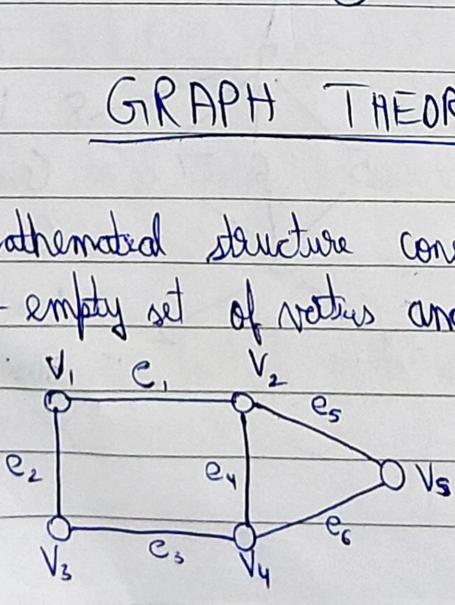
Lattice

- **Logical Connectives**

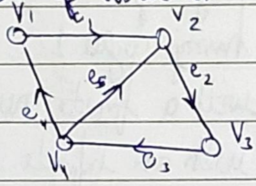
Logical Connective Words	Symbol	Uses
And/ Conjunction/ Join	$\wedge$	$p \wedge q$
Or/ Disjunction/ meet	$\vee$	$p \vee q$
Negation	$-$ or $\sim$	$\sim p$
Equivalent	$\leftrightarrow$	$p \leftrightarrow q$
Condition if .. then ...	$\Rightarrow$	$p \Rightarrow q$
Biconditional if and only if (iff)	$\Leftrightarrow$	$p \Leftrightarrow q$
NAND (NOT + AND)	$\uparrow$	$p \uparrow q$
NOR (NOT + OR)	$\downarrow$	$p \downarrow q$
XOR	$\oplus$	$p \oplus q$

GRAPH THEORY

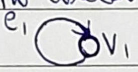
- Graph G is a mathematical structure consisting of two sets V and E where V is a non-empty set of vertices and E is a non empty set of edges



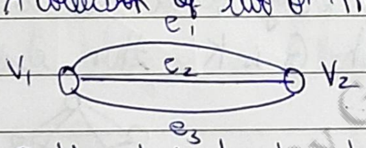
- (i) Trivial Graph : Consisting only one vertex and no edge. Ex - O
- (ii) NULL Graph : A graph consisting n vertices and no edge. Ex→ O O O O
- (iii) Directed graph : consists of direction of edges.



- (iv) Undirected graph : no direction given on edges. Opposite of directed graph.
- (v) Self loop in graph :

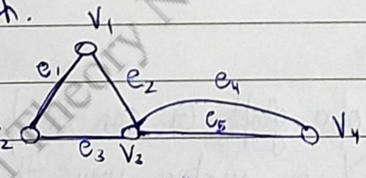


- (vi) Proper edge : Edge which is not self loop is called proper edge.
- (vii) Multi edge : A collection of two or more edges having identical end pt.

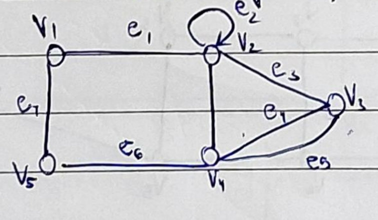


e_1, e_2, e_3 are multiedge

- (viii) Simple graph : Graph which doesnt contain any self loop and multiedge.
- (ix) Multigraph : Graph which doesnt contain any self loop but contain multiedge is called multigraph.



- (x) Pseudo Graph : Contains both self loop and multiedge.



- (xi) <u>Incidency and Adjacency</u>
  - a) Let e_k be an edge joining two vertices V_i & V_j, then e_k is said to be incident of V_i & V_j.
  - b) Two vertices are said to be adjacent if there exist an edge joining the vertices.

(xii) <u>Degree of Vertex</u> → Number of edges incident on a vertex. Self loops are counted twice.

(xiii) <u>Isolated Vertex</u> → Vertex having degree 0.

(xiv) Pendant Vertex → Vertex having degree 1.

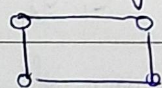
(xv) Finite graph → Graph with a finite number of vertices

(xvi) Infinite graph → Graph with an infinite number of vertices.

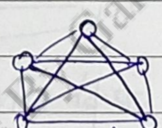
- <u>Some Important Graphs</u>

1) <u>Complete Graph</u> → each vertex connected to every other vertex.

2) Regular graph → if every vertex has same degree. If degree of each vertex of graph G is K, then it is called k-regular graph.



2-Regular Graph

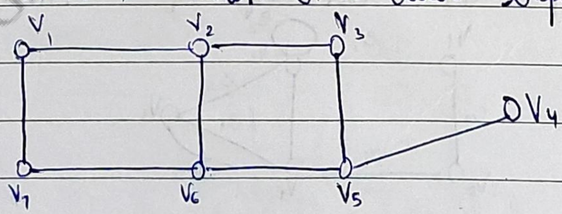


4-Regular Graph

3) <u>Bipartite (or Bigraph)</u>

If the vertex set V of a graph G can be partitioned in two non-empty disjoint subsets X and Y in such a way that edge of G has one end in X and one end in Y. Then G is called bipartite graph.

Ex)



$V = \{V_1, V_2, V_3, V_4, V_5, V_6, V_7\}$

$X = \{V_1, V_6, V_3, V_4\}$ & $Y = \{V_7, V_2, V_5\}$

4) <u>Connected graph</u>

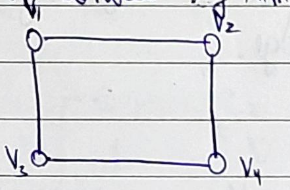
An undirected graph is said to be connected if there is a path between every two vertices.

- Any bipartite graph consisting of 'n' vertices can have at most $1/4\ n^2$ edges.
- Minimum possible number of edges in a bipartite graph of 'n' vertices $= \frac{1}{4} n^2$

5) <u>Complete Bipartite Graph</u>

If every vertex in X is disjoint in every vertex in Y, then it is called a complete bipartite graph. If X and Y condition m & n vertices, then this graph is denoted by $K_{m,n}$.

Ex→



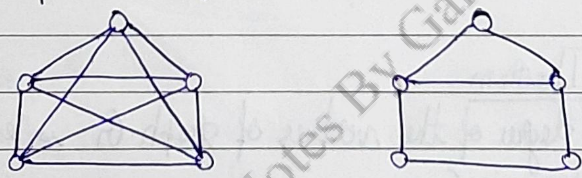
$X = \{V_1, V_4\}$, $Y = \{V_2, V_3\}$

Denoted by $K_{2,2}$

6) <u>Subgraph</u>

Let $G(V, E)$ be a graph. Let V' be a subset of V and let E' be a subset of E whose end points belong to V'. Then $G(V', E')$ is a subgraph of $G(V,E)$.

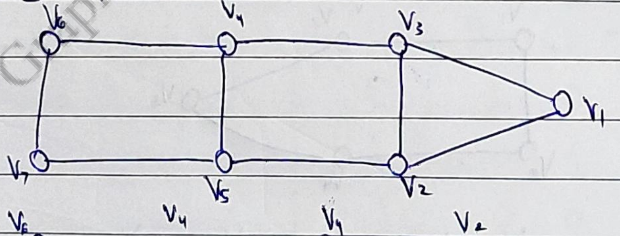
Ex→

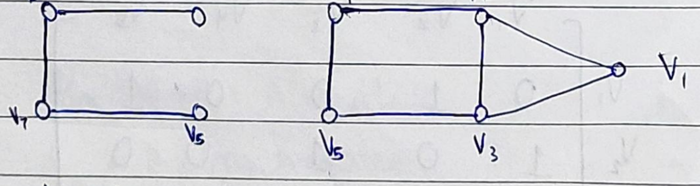


- <u>Decomposition of Graph</u>

A graph is said to be decomposed in two subgraph G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2 =$ null.

Ex→



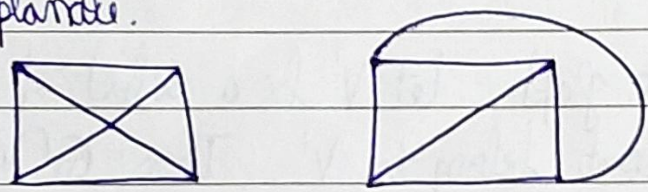


- <u>Complement of Graph</u>

The complement of graph G is defined as a simple graph with the same vertex set as G and where two vertices u & v are adjacent only when they are not adjacent in G.

- **Planar Graph**

A graph which can be drawn in a plane such that its edges do not cross is called planar.



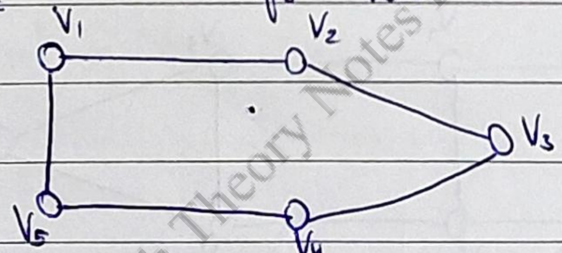
- **Handshaking Theorem**

The sum of degree of the vertices of graph G is equal to twice the number of edges in G.

$$\sum \text{degrees} = 2 \times \text{edges}$$

- **Matrix Representation Of Graph**

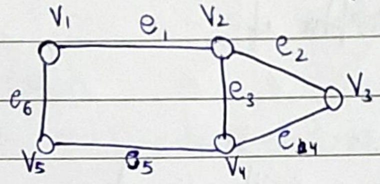
1) Adjacency Matrix → drawn for vertices



$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array}
\begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\
\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

If the graph is directed, then adjacency will not be symmetric.

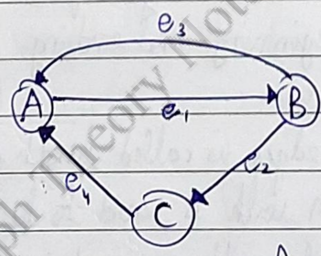
2) <u>Incidence Matrix</u> → drawn for vertices incident with edges



$$\begin{array}{c|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline V_1 & 1 & 0 & 0 & 0 & 0 & 1 \\ V_2 & 1 & 1 & 1 & 0 & 0 & 0 \\ V_3 & 0 & 1 & 0 & 1 & 0 & 0 \\ V_4 & 0 & 0 & 1 & 1 & 1 & 0 \\ V_5 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}$$

Count 2 for self loop

3) <u>Path Matrix</u>



Adjacency Matrix $A = \begin{array}{c|ccc} & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 1 & 0 & 1 \\ C & 1 & 0 & 0 \end{array}$

$$B_m = A + A^2 + A^3 + \dots A^m$$, where A is adjacency matrix, m is vertices

$\therefore A^2 = A \cdot A = \begin{array}{c|ccc} & A & B & C \\ \hline A & 1 & 0 & 1 \\ B & 1 & 1 & 0 \\ C & 0 & 1 & 0 \end{array}$ $\quad A^3 = A^2 \cdot A = \begin{array}{c|ccc} & A & B & C \\ \hline A & 1 & 1 & 0 \\ B & 1 & 1 & 1 \\ C & 1 & 0 & 1 \end{array}$

$\therefore B_3 = A + A^2 + A^3$

$\therefore B_3 = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}$ $\quad \Rightarrow P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Eulers formula

If G is a planar graph, Then, $v - e + f = 2$

v = number of vertices

e = number of edges

f = number of faces including exterior face

- **Isomorphism of Graph**

Two graph G_1 & G_2 are isomorphic if

(i) No of vertices are same.

(ii) No of edges are same.

(iii) An equal number of vertices with given degree.

(iv) Vertex correspondence & edge correspondence valid.

- **Homeomorphic Graph**

Two graphs G and G' are said to be homeomorphic if they can be obtained from the same graph. If G & G' are homeomorphic, they need not be isomorphic.

- **Walk, Trail & Path**

1) Walk: A walk is a finite alternating sequence $v_1 e_1 v_2 e_2 v_3 e_3 \dots v_n e_n$ of vertices and edges, beginning and ending with same or different vertices.

2) Length of Walk – Number of edges is called length of the walk.

3) Closed and Open Walk – A walk is said to be closed if its origin & terminus (ending) vertex ($v_0 = v_n$) is equal, otherwise it is open walk.

4) Trail: Any walk having different edges is called trail.

5) Circuit – A closed trail is called circuit.

6) Path – A walk is called a path if vertices aren't repeated.

7) Cycle – A closed path is called cycle.

- **Eulerian Path**

A path in graph is said to be an eulerian path if it traverses each edge in the graph once and only once.

- **Eulerian Circuit**

A circuit in a graph is said to be eulerian if it traverses each edge in the graph once and only once. A closed eulerian path is called an eulerian circuit.

- <u>Eulerian Graph</u>

A connected graph which contains a Eulerian circuit is called Eulerian graph.

- <u>Hamiltonian Path</u>

A path which contains every vertex of a graph exactly once is a hamiltonian path.

- <u>Hamiltonian Circuit</u>

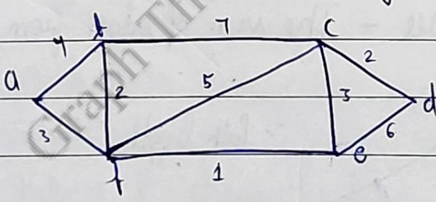
A circuit that passes through each of the vertices in a graph G exactly once except the starting vertex & end vertex is called hamiltonian circuit.

- <u>Hamiltonian Graph</u>

Graph containing a hamiltonian circuit is called hamiltonian graph.

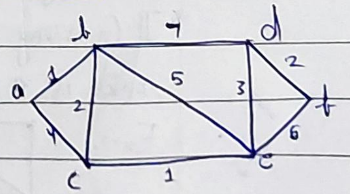
- <u>Weighted Graph</u>

A graph is called a weighted graph if a non-negative integer W(e) is associated to each edge and this W(e) is a weight of corresponding edge.



- <u>Shortest Path in a Weighted Graph : Dijkstra's Algorithm</u>

a	b	c	d	e	f
$\boxed{0}$	∞	∞	∞	∞	∞
$\boxed{0}$	$\boxed{1}$	4	∞	∞	∞
$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	8	6	∞
$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	8	$\boxed{4}$	∞
$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	$\boxed{7}$	$\boxed{4}$	10
$\boxed{0}$	$\boxed{1}$	$\boxed{3}$	$\boxed{7}$	$\boxed{4}$	$\boxed{9}$



∴ Shortest dist between a & f = 9

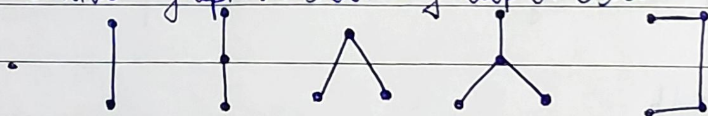
- Every tree with n vertices has exactly n-1 edges.

__/__/__

TREES

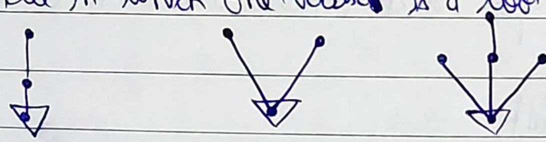
- Tree - A connected graph without any loop or circuits.

Ex->



- Rooted Tree : Tree in which one vertex is a root

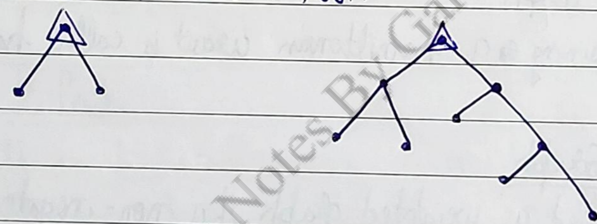
Ex->



- Binary Tree

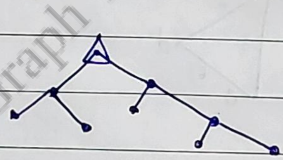
A binary tree is defined as a tree in which there is exactly one vertex of degree two and each of the remaining vertices is of degree one or three and vertex of degree two serves as a root.

Ex->



- Pendant Vertex of tree - vertex with degree 1.
- Path length of tree - The sum of edges from the root of all pendant vertices.

Ex->

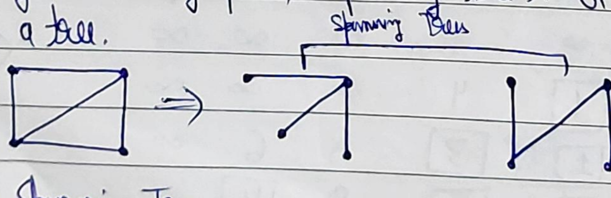


∴ Path length = 2 + 2 + 2 + 3 + 3
= 14

- Spanning Tree

If G is any connected graph, a spanning tree in G is a subgroup of T of G, which is a tree.

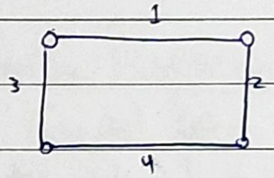
Ex->



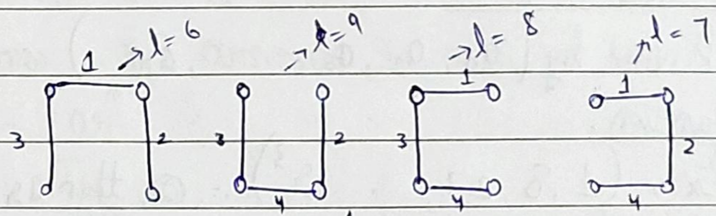
- Minimal Spanning Tree

Let G be a weighted graph. A minimal spanning tree of G is a spanning tree of G with minimum weight.

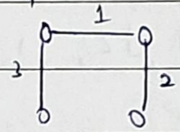
Ex →



Possible spanning trees :



the minimal spanning tree :



, Length = 6

- ## Algorithms for Minimal Spanning Tree

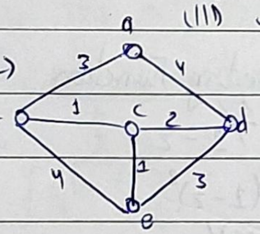
1) **Kruskal's Algorithm**

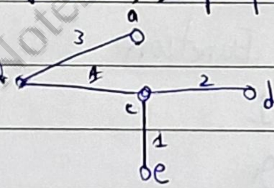
Working Rule : (i) Choose an edge of minimal weight.

(ii) At each step, choose the edge whose inclusion will not create a circuit.

(iii) If G has n vertices, stop after (n − 1) edges.

Ex →





, Weight = 7

2) **Prim's Algorithm**

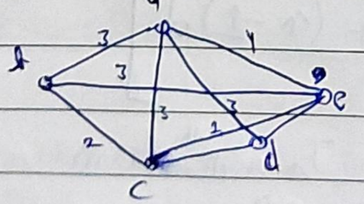
Working Rule :

(i) Select any vertex & choose the edge and smallest weight from G.

(ii) At each stage, choose the edge of smallest weight joining a vertex already included to vertex not yet included.

(iii) Continue until all vertices are included.

Ex →



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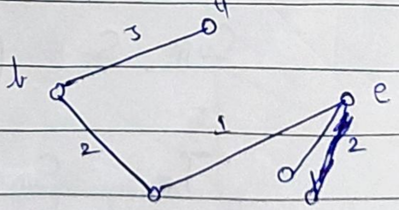

2 + 1 + 2 + 3
= 8

- **Numeric & Generating Function**

1) **Numeric Function**

Denoted by $(a_0, a_1, a_2, \ldots, a_r \ldots)$ and a_r is the r^{th} term of this function.

Ex → $(1, 8, 27, \ldots r^3)$ then $a_r = r^3$

Ex → $(0, 3, 6, 9, 12, 31 \ldots 63, \ldots)$ then $a_r = \begin{cases} 3r & 0 \le r \le 4 \\ 2^r - 1 & r > 4 \end{cases}$

- **Generating Function of Numeric function**

Let $(a_0, a_1, a_2, \ldots a^r)$ be a numeric function, then the infinite series in terms of z. $a_0 + a_1 z + a_2 z^2 + \ldots a_r z^r + \ldots$ is called generating function of numeric function.

$$\sum_{i=0}^{\infty} a_i z^i$$

- **Some standard functions**

Numeric Function	Generating Function
$a_r = 1$	$1/1-z$
$a_r = r$	$z/(1-z)^2$
$a_r = r^2$	$z(z+1)/(1-z)^2$
$a_r = {}^nC_r$	$(1+z)^n$
$a_r = \alpha^r$	$1/1-\alpha z$

- **Arithmetic Progression**

a = first term, T_n = n^{th} term, d = Common difference

General form of AP : $a, a+d, a+2d, a+3d, \ldots$

$$T_n = a + (n-1)d$$

$$\text{Sum}, S = \left(\frac{n}{2}\right)\left[2a + (n-1)d\right]$$

$T_n = S_n - S_{n-1}$, T_n is n^{th} term

If a, b, c are three terms in AP,

then $b = \dfrac{a+c}{2}$

b is the arithmetic mean

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- **Geometric Progression**

General form $\Rightarrow a, ar, ar^2, ar^3 \ldots$

$a \rightarrow$ first term, $r \rightarrow$ common ratio, $T_n \rightarrow n^{th}$ term

$$T_n = ar^{n-1}$$

$$r = \frac{T_n}{T_{n-1}}$$

Sum of first n terms of GP : $S_n = \dfrac{a(r^n - 1)}{(r-1)}$

(i) If a, b, c are in GP $\Rightarrow b = \sqrt{ac}$, b is the geometric mean

(ii) Sum of infinite terms of a GP series $S_\infty = \dfrac{a}{1-r}$ where $0 < r < 1$

(iii) If a is first term, r is common ratio of a GP consisting m terms, then the n^{th} term from the end will be $= ar^{m-n}$

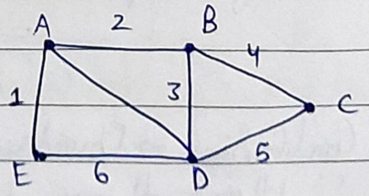
(iv) The n^{th} term from the end of GP with last term l and common ratio is $\dfrac{l}{[r^{(n-1)}]}$

- **Radius, Diameter, Central Point, Center, Circumference, Girth of a Graph**

Eccentricity (of a vertex) is the distance of the vertex from all other vertices. Take the max value

1) Radius $\rightarrow$ The minimum eccentricity from all the vertices is considered as the radius of a graph G.

2) Diameter $\rightarrow$ The maximum eccentricity from all the vertices is considered as the diameter of a graph G.



$e(A) = 2$, $e(B) = 2$, $e(C) = 2$

$e(D) = 1$, $e(E) = 2$

Min value $\rightarrow e(D) = 1$

$\therefore$ Radius is 1

Max value $\rightarrow e(A) = e(B) = e(C) = e(E) = 2$

$\therefore$ Diameter $= 2$

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3) Central Point

If the eccentricity of a graph vertex is equal to its radius,

if $e(V) = r(V)$, then V is a central point of the graph.

In the previous diagram, $r(G) = 1$

$\Rightarrow$ D becomes a central point (Since $e(D) = r(G)$)

Central points can be more than one.

4) Centre – Set of all central points of a graph is called centre of graph.

Centre = {D}

5) Circumference – Longest cycle of a graph is called circumference

Circumference of given graph = 6

6) Girth – Shortest cycle of a graph is called girth.

Girth of given graph = 3

- Combinatorics

(i) Factorial $\Rightarrow n! = n \times (n-1) \times (n-2) \times \ldots \quad 2 \times 1$

$0! \Rightarrow$ unity

(ii) Permutations $\rightarrow$ Ordered arrangement of objects

$${}^nP_r = \frac{n!}{(n-r)!}, \quad r \leq n$$

r objects chosen from n and these r objects have to be arranged in some fashion.

(iii) Permutations with Repetitions

'n' objects of which n_1 are alike, n_2 are alike and so on.

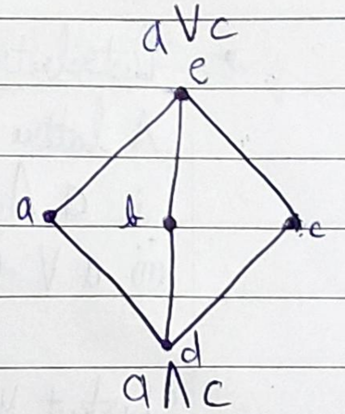
$$= \frac{n!}{n_1!\, n_2! \ldots n_r!}$$

(iv) Combinations $\rightarrow$ Unordered arrangement of objects

'n' different things, taken 'r' at a time is given by

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

- **Properties of Lattices**



1) **Idempotency law**

$$a \wedge a = a \qquad a \vee a = a$$

2) **Commutative law**

$$a \wedge b = b \wedge a \quad , \quad a \vee b = b \vee a$$

3) **Associative law**

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$
$$(a \vee b) \vee c = a \vee (b \vee c)$$

4) **Absorption law**

$$a \wedge (a \vee b) = a$$
$$a \vee (a \wedge b) = a$$

5) **Consistency law**

$$a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$$

- **Duality in Lattice**

$$\leq \qquad\qquad \geq$$
$$a \vee b,\ l.u.b \longrightarrow g.l.b,\ a \wedge b$$
$$a \wedge b,\ g.l.b. \longrightarrow l.u.b,\ a \vee b$$

- **Sub-lattice**

A non-empty subset S of a lattice P is called sublattice if $\forall\, a, b \in S \Rightarrow a \vee b,\ a \wedge b \in S$. Basically, when a subset of a lattice becomes a lattice itself, it is called the sub lattice of lattice P.

- **Complement of an element in lattice**

An element 'x' in a bounded lattice (closed loop) 'P' is said to be a complement of an element $a \in P$ if $a \vee x = 1$ & $a \wedge x = 0$.

- **Distributive lattice**

A lattice P is called distributive if for all $a, b, c \in P$

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ holds

Shortcut Method

A lattice 'P' is said to be distributive if every element in 'P' has "atmost one complement".

- **Complemented Lattice**

A bounded lattice 'P' is said to be complemented if every element in 'P' has "atleast one complement".

- **Theorems on Lattices**

(i) For any $a, b \in L$

$$a \leq a \vee b$$

and $a \wedge b \leq a$

(ii) If $a \leq b$ and $c \leq d$ for any $a, b, c, d \in L$, then

$$a \vee c \leq b \vee d$$

and $a \wedge c \leq b \wedge d$

(iii) Let $(L, \leq)$ be a lattice. Then for any $a, b, c \in L$, the following results hold:

1) $a \vee a = a$ 2) $a \wedge a = a$

3) $a \vee b = b \vee a$ 4) $a \wedge b = b \wedge a$

5) $a \vee (b \vee c) = (a \vee b) \vee c$ 6) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

7) $a \vee (a \wedge b) = a$ 8) $a \wedge (a \vee b) = a$

(iv) Let $(L, \leq)$ be a lattice with universal lower bound 0 and universal upper bound as 1. Then for any a in L

$a \vee 1 = 1$ $a \wedge 1 = a$

$a \vee 0 = a$ $a \wedge 0 = 0$

(v) Let $(L, \leq)$ be a lattice with universal lower and ub 0 & 1 respectively, then for a in L, $\exists\, b \in L$, st

$$a \vee b = 1, \quad a \wedge b = 0$$

then, b is said to be complement of a.

We denote complement of an element a by $\bar{a}$. So

$$\bar{0} = 1, \quad \bar{1} = 0$$

$$a \vee \bar{a} = 1 \quad \& \quad a \wedge \bar{b} = 0$$

(vi) A lattice is said to be bounded if it has both 0 & 1.

(vii) In a distributive lattice $(L, \leq)$, if an element a has a complement, then it is unique.

(viii) <u>De-Morgan's laws</u>

Let $(L, \leq)$ be a complemented distributive lattice. Then for any $a, b \in L$

$$\overline{a \vee b} = \bar{a} \wedge \bar{b}$$

and $$\overline{a \wedge b} = \bar{a} \vee \bar{b}$$

- <u>Propositional or Sentence</u>

An expression consisting of some symbols, letters and words is called a sentence if it is true or false. Ex→

1) Jaipur is capital of RJ.
2) $3+5=8$
4) Mumbai is in America
5) "Wish you happy life" is not a proposition because true or false is not certain.

T → True F → False

- <u>Simple Proposition</u> : The proposition having one subject and one predicate is called simple proposition

Ex→ This flower is pink, Every even no. is divisible by 2.

- Compound Proposition : Two or more simple proposition when combined by various connectives into a single composite sentence is called compound proposition. Ex The earth is round and revolves around the sun.
- Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

p	$\sim p$
T	F
F	T

- Tautologies & Contradictions
A proposition is said to be a tautology if it contains only T in the last column of the truth table and contradiction if it contains only F in the last column of truth table.
- Conditional Statements
Many statements are of the form "If p then q", they're called conditional statements denoted by $p \Rightarrow q$ or $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
F	F	T
T	F	F
F	T	T

- Biconditional Statements
A statement "p if and only if q", they're called biconditional statements denoted by $p \Leftrightarrow q$ or $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- Converse of conditional statement

Let $p \to q$ be a conditional statement, then $q \to p$ is called its converse.

- Inverse of conditional statement

Let $p \to q$ be a conditional statement, then $\sim p \to \sim q$ is called its inverse.

- Contrapositive of conditional statement

Let $p \to q$ be a conditional statement, then $\sim q \to \sim p$ is called contrapositive.

- Some Imp laws

1) Idempotent law

$$p \wedge p \Leftrightarrow p \quad \text{and} \quad p \vee p \Leftrightarrow p$$

2) Commutative law

$$p \wedge q \Leftrightarrow q \wedge p \quad \text{and} \quad p \vee q \Leftrightarrow q \vee p$$

3) Associative law

$$(p \vee q) \vee r = p \vee (q \vee r)$$
$$\text{and } (p \wedge q) \wedge r = p \wedge (q \wedge r)$$

4) De-Morgan law

$$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$$
$$\text{and } \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$$

- Group Theory

Semi Group: An algebraic structure $(G, *)$ is called a semi group if the binary operations $*$ satisfies associative property i.e. $(a*b)*c = a*(b*c)$

Monoids

A semi group is called monoid if there exists an identity element 'e' in G such that: $e*a = a*e = a \ \forall a \in G$

- Group: Definition

An algebraic structure of set G and a binary operation $*$ defined in G i.e. $(G, *)$ is called a group if $*$ satisfies the following postulate:

1) Closure: $a \in G, b \in G \Rightarrow a*b \in G, \forall a, b \in G$

2) <u>Associatively</u> : The composition is associative in G i.e. $(a^*b)^*c = a^*(b^*c)$ $\forall a, b, c \in G$

3) <u>Existence of Identity</u> : There exist an identity element e in G such that $e^*a = a^*e = a, \forall a \in G$

4) <u>Existence of Inverse</u> : Each element of G is invertible i.e. for every $a \in G$, there exist a^{-1} in G such that
$$a^* a^{-1} = a^{-1} * a = e \text{ (Identity)}$$

- <u>Predicates</u>

A predicate P(x) is a sentence that contains a finite numbers of variables and becomes a proposition when specific values are substituted for the variables. P(x) is the propositional function and x is the predicate variable.

- <u>Domain</u>

The domain of a predicate variable is the set of all possible values that may be substituted in place of variables. Ex→ "x is a great tennis player".

- <u>Quantifiers</u>

Quantifiers are words that refer to quantities such as "some" or "all" and indicate how frequently a certain statement is true. Two types:
1) Universal Quantifier 2) Existential Quantifier

- 1) <u>Universal Quantifier</u> : The phrase "for all" denoted by $\forall$ is called the universal quantifier. Ex→ "All students are smart". Let P(x) denote "x is smart" ⇒ $\forall_x P(x)$

2) <u>Existential Quantifier</u> : The phrase "there exists" denoted by $\exists$ is called existential quantifier. Ex→ Let "There exists x such that $x^2 = 9$".
Let P(x) "$x^2 = 9$"

then the above statement can be written as $\exists_x P(x)$

$\exists x$ → There exists an x
→ There is an x
→ For some x
→ There is atleast one x

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- <u>Abelian group or Commutative group</u>

A group $(G, *)$ is said to be an abelian if $*$ is commutative. Group $(G, *)$ is an abelian group, if $a * b = b * a$, $\forall a, b \in G$

- G_1 Associativity — Semi Group
  G_2 Existence of identity — Monoid
  G_3 Existence of Inverse — Group
  G_4 Commutativity — Abelian Group

- <u>Finite and Infinite Group</u>

A group $(G, *)$ is said to be finite if its underlying set G is a finite set and a group which is not finite is called an infinite group.

- <u>Order of a group</u>

The no. of elements in a finite group is called the order of the group. Denoted by $O(G)$. Infinite groups have infinite order.

- <u>Properties of Groups</u>

Theorem 1: The identity element in a group is unique

Th 2: The inverse of an element in a group is unique

Th 3: If G is a group, then for $a, b \in G$

a) $(a^{-1})^{-1} = a$

b) $(ab)^{-1} = b^{-1} a^{-1}$ (Reversal law)

→ the inverse of the product of two elements is the product of their inverses in the reverse order.

Th 4: If a, b are elements of a group G, then the equations $ax = b$ and $ya = b$ have unique solutions in G.

Th 5: Alternative definition of a group: If for all elements a, b of a semigroup G, equations $ax = b$ and $ya = b$ have unique solutions in G, then G is a group

- <u>Subgroup</u>

A non-empty subset H of a group G is called subgroup of G if

(i) H is stable (closed) for the composition defined in G i.e

$a \in H, b \in H \Rightarrow ab \in H$

(iii) H itself is a group for the composition induced by that of G.

- Cyclic Group

A group G is a cyclic group if there exists an element $a \in G$ such that $G = [a]$ i.e. every element of G can be expressed as some integral power of a.
a is called the generator of G.

$G = \{ \dots a^{-3}, a^{-2}, a^{-1}, a^{0}, a^{1}, a^{2}, a^{3}, \dots \}$

If the operation of the group G is addition (+), then

$$G = [a] = \{ \dots -3a, -2a, -a, 0, a, 2a, 3a, \dots \}$$

- Normal Subgroup

A subgroup H of a group G is said to be a normal subgroup of G if $x \in G$ and $h \in H \Rightarrow x h x^{-1} \in H$. If H is a normal subgroup of G, then symbolically, we write it as $H \triangleleft G$. From this definition, we may observe that

$$H \triangleleft G \Leftrightarrow xHx^{-1} \subseteq H, \ \forall x \in G \text{ (another definition)}$$

- Coset

Let H be a subgroup of a group G and $a \in G$, then the set $aH = \{ah \mid h \in H\}$ is called a left coset of H in G and $Ha = \{ha \mid h \in H\}$ is called right coset of H in G.

Th 1: If H is a subgroup of a group G and $a \in G$. Then

$$a \in aH \quad \text{and} \quad a \in Ha$$

Th 2: Any two left (or right) cosets of a subgroup are either identical or disjoint.

Lagrange's Theorem
Th 3: The order of every subgroup of a finite group is a divisor of the order of the group.