Deligne's

Equations Différentielles à Points Singuliers Réguliers

Part I: Dictionnaire

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Abstract

This is a reading note of Part I (*Dictionnaire*) of Deligne's "*Equations Différentielles à Points Singuliers Réguliers*". Although closely follows the original French text, it is not a faithful English-translation: some supplementary materials are inserted and the numbering is thus different.

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§ 1 Local systems and fundamenal group

1.1 Definition Let X be a topological space. A *(complex) local system* on X is a sheaf of complex vector spaces on X which, locally on X, is isomorphic to the constant sheaf $\underline{\mathbb{C}^n}$ $(n \in \mathbb{N})$.

1.2 Let X be a *locally path-connected* and *locally simply connected* topological space, equipped with a base point $x_0 \in X$.

Let $\mathcal{F}$ be a locally constant sheaf on X. For each path $\alpha\colon [0,1] \to X$, the pullback $\alpha^*\mathcal{F}$ of $\mathcal{F}$ on $[0,1]$ is a locally constant sheaf, hence constant and there exists a qunive isomorphism between $\alpha^*\mathcal{F}$ and the constant sheaf defined by the set $(\alpha^*\mathcal{F})_0 = \mathcal{F}_{\alpha(0)}$. This isomorphism defines an isomorphism $\alpha(\mathcal{F})$ between $(\alpha^*\mathcal{F})_0$ and $(\alpha^*\mathcal{F})_1$, i.e. an isomorphism

$$\alpha(\mathcal{F})\colon \mathcal{F}_{\alpha(0)} \longrightarrow \mathcal{F}_{\alpha(1)}.$$

This isomorphism depends only on the homotopy class of α and satisfies $\alpha\beta(\mathcal{F}) = \alpha(\mathcal{F}) \circ \beta(\mathcal{F})$. In particular, $\pi_1(X, x_0)$ acts (on the left) on the stalk $\mathcal{F}_{x_0}$ of $\mathcal{F}$ at x_0.

1.3 Proposition *Under the hypothesis 1.2, with X being connected, the functor $\mathcal{F} \mapsto \mathcal{F}_{x_0}$ is an equivalence between the category of locally constant sheaves on X and the category of left $\pi_1(X, x_0)$-sets.*

1.3.1 Let $p\colon Y \to X$ be a covering space. For each path $\alpha\colon [0,1] \to X$ and $y \in Y_{\alpha(0)} := p^{-1}(\alpha(0))$, there exists a unique path $\widetilde{\alpha}\colon [0,1] \to Y$ such that $p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}(0) = y$. Thus $\alpha.y := \widetilde{\alpha}(1) \in Y_{\alpha(1)}$. This construction defines a bijective map from $Y_{\alpha(0)}$ to $Y_{\alpha(1)}$ and depends only on the homotopy class of α and satisfies $\alpha\beta.y = \alpha.(\beta.y)$. In particular, it defines an left action (the *monodromy action*) of $\pi_1(X, x_0)$ on the fiber Y_{x_0}.

Remark Let $p\colon Y \to X$ be a covering space and α be a path in X. Then the pullback α^*Y of Y along α is a covering space of $[0,1]$, hence trivial.

$$\begin{tikzcd}
\alpha^*Y \arrow[r, "\alpha'"] \arrow[d, "p'"] & Y \arrow[d, "p"] \\
{[0,1]} \arrow[r] \arrow[u, dashed, bend left] & X
\end{tikzcd}$$

For each point y in the fiber $Y_{\alpha(0)}$, there is a unique preimage y' in the fiber $(\alpha^*Y)_0$. As p' is a trivial covering, such a point uniquely determines a section of p'. It is clear that the composition of this section with α' gives a lifting of α into Y starting from y and any such lifting is given in this way.

1.3.2 Proposition *Under the hypothesis 1.2, with X being connected, the monodromy actions define an equivalence bewteen the category of covering spaces of X and the category of left $\pi_1(X, x_0)$-sets.*

1.3.3 Let $\mathcal{F}$ be a sheaf on X. We construct a space $X_{\mathcal{F}}$ over X as follows: the underlying set is the disjoint union of all stalks of $\mathcal{F}$ and the projection is induced by the maps $\mathcal{F}_x \to \{x\}$; the topology is the coarsest one in which every section $s \in \mathcal{F}(U)$ gives an open set $s(U) := \{(s_x, x) : x \in U\}$. When $\mathcal{F}$ is locally constant, this construction gives a covering space of X whose sheaf of sections is isomorphic to $\mathcal{F}$.

1.3.4 Proposition *The functor $\mathcal{F} \to X_{\mathcal{F}}$ is an equivalence between the category of locally constant sheaves and the category of covering spaces of* X*.*

Combining the above two propositions, after verfying the composition of those equivalences cocides with the functor $\mathcal{F} \mapsto \mathcal{F}_{x_0}$, the statement follows.

1.4 Corollary *Under the hypothesis 1.2, with* X *being connected, the functor $\mathcal{F} \mapsto \mathcal{F}_{x_0}$ is an equivalence between the category of local systems on* X *and the category of finite-dimensional complex representations of $\pi_1(X, x_0)$.*

1.5 Under the hypothesis 1.2, if α is a path and β a loop from $\alpha(0)$, so $\alpha(\beta) := \alpha\beta\alpha^{-1}$ is a loop from $\alpha(1)$ and its homotopy class depends only on those of α and β. This construction defines an isomorphism between $\pi_1(X, \alpha(0))$ and $\pi_1(X, \alpha(1))$.

1.6 Proposition *Under the hypothesis 1.5, there exists, uniquely up to a unique isomorphism, a locally constant sheaf of groups $\Pi_1(X)$ on* X *(the fundamental groupoid), equipped with, for every $x_0 \in X$, an isomorphism*

$$\Pi_1(X)_{x_0} \xrightarrow{\sim} \pi_1(X, x_0) \tag{1.6.1}$$

such that, for any path α, the isomorphism in 1.5 between $\pi_1(X, \alpha(0))$ and $\pi_0(X, \alpha(1))$ is identified via (1.6.1) *with the isomorphism 1.2 between $\Pi_1(X)_{\alpha(0)}$ and $\Pi_1(X)_{\alpha(1)}$.*

Moreover, if X *is connected with base point x_0, the sheaf $\Pi_1(X)$ corresponds, via equivalence 1.3, to the group $\pi_1(X, x_0)$ with its action on itself by internal automorphisms.*

Instead of proving the proposition directly, I prefer to relate this definition with the more common one.

1.6.2 For X a topological space, its *fundamental groupoid* is the category $\Pi_1(X)$ whose objects are points of X and whose morphisms are homotopy classes of paths.

It turns out that $\Pi_1(X)$ is a *topological groupoid*. Let $\widetilde{X}$ be the set of all morphisms in $\Pi_1(X)$. Then there is a topology on $\widetilde{X}$ such that all the operations (source, target, identity and composition) are continuous maps.

In this way, we get a covering space

$$\widetilde{X} \xrightarrow{(s,t)} X \times X,$$

where s and t are the source and target operations.

Remark In detail, the topology on $\widetilde{X}$ is given as follows. First, given any point $x \in X$, there is a basis of neighborhoods $\mathcal{B}_x$ whose members are path-connected open neighborhoods $U \subset X$ of x such that the induced homomorphism $\pi_1(U, x) \to \pi_1(X, x)$ is trivial. Then, each of such open neighborhood U can be lifted into a subset $\widetilde{U}_x$ of $\widetilde{X}$:

$$\widetilde{U}_x := \Big\{[\alpha] : \alpha \text{ is a path in } U \text{ starting from } x\Big\}.$$

For each point y in $\widetilde{X}$ with a path γ presenting it, the family

$$\Big\{\widetilde{V}_{\gamma(1)}\gamma\widetilde{U}^{-1}_{\gamma(0)}\Big\}_{U\in\mathcal{B}_{\gamma(0)}, V\in\mathcal{B}_{\gamma(1)}},$$

where

$$\widetilde{V}_{\gamma(1)}\gamma\widetilde{U}^{-1}_{\gamma(0)} = \Big\{[\beta\gamma\alpha^{-1}] : \alpha \in \widetilde{U}_{\gamma(0)}, \beta \in \widetilde{V}_{\gamma(1)}\Big\},$$

form a basis of neighborhoods of y in $\widetilde{X}$.

1.6.3 Let $\Delta\colon X \to X \times X$ be the diagonal map. Then the pullback of the covering $\widetilde{X} \to X \times X$ along Δ gives a covering space $\widetilde{X}_\Delta \to X$.

For each point x of X, the fiber of $\widetilde{X}_\Delta$ at x is $\pi_1(X, x)$, and the monodromy actions are given by the inner automorphisms. Hence, $\widetilde{X}_\Delta$ is a group bundle over X.

Let α be a path in X and β a loop presenting a point y in the fiber $\widetilde{X}_{\Delta,\alpha(0)}$. Then, the unique lifting of α starting from y is given by $\widetilde{\alpha}(t) = \alpha_t\beta\alpha_t^{-1}$, where α_t is the path $\alpha_t(s) := \alpha(ts)$. Therefore, the bijective map $\widetilde{X}_{\Delta,\alpha(0)} \to \widetilde{X}_{\Delta,\alpha(1)}$ given by such liftings is nothing but the conjugate operation $[\beta] \mapsto [\alpha][\beta][\alpha]^{-1}$.

Now, consider the sheaf of sections of $\widetilde{X}_\Delta \to X$, denoted also by $\Pi_1(X)$. The observations in previous paragraphs implies proposition 1.6.

1.7 Proposition *If $\mathcal{F}$ is a locally constant sheaf on X, there exists a canonical action of $\Pi_1(X)$ on $\mathcal{F}$ which, at each $x_0 \in X$, induces the action 1.2 of $\pi_1(X, x_0)$ on $\mathcal{F}_{x_0}$.*

It suffices to give the action of the group bundle $\widetilde{X}_\Delta$ on an arbitrary covering space $Y \to X$. This action is precisely the monodromy action, which is already defined in 1.3.1.

§ 2 Integrable connections and local systems

From now on, an *analytic space* means a complex analytic space locally of finite dimension and supposed to be σ-compact, while not necessarily separated; a *complex analytic manifold* means a non-singular (or smooth) analytic space.

2.1 Let X be an analytic space. A *(holomorphic) vector bundle* on X is a locally free $\mathcal{O}_X$-module of finite type. If $\mathcal{V}$ is a vector bundle on X and x is a point of X, we denote by $\mathcal{V}_{(x)}$ the finite free $\mathcal{O}_{(x)}$-module of germs of sections of $\mathcal{V}$. If $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{(x)}$, the *fiber* at x of the vector bundle $\mathcal{V}$ is the following vector space of finite rank:

$$\mathcal{V}_x := \mathcal{V}_{(x)} \otimes_{\mathcal{O}_{(x)}} \mathcal{O}_{(x)}/\mathfrak{m}_x.$$

If $f\colon \mathrm{X} \to \mathrm{Y}$ is a morphism of analytic spaces, the *pullback* $f^*\mathcal{V}$ on X of the vector bundle $\mathcal{V}$ on Y is

$$f^*\mathcal{V} := \mathcal{O}_\mathrm{X} \otimes_{f^{-1}\mathcal{O}_\mathrm{Y}} f^{-1}\mathcal{V}.$$

In particular, if $x\colon \mathsf{pt} \to \mathrm{X}$ is morphism from the punctual sapce pt to X defined by the point x of X, we have

$$\mathcal{V}_x \cong x^*\mathcal{V}.$$

2.2 Let X be a complex analytic manifold and $\mathcal{V}$ be a vector bundle on X. The ancients would have defined *(holomorphic) connection* on $\mathcal{V}$ as the data: for any pair (x, y) of infinitesimally near points of first order of X, an isomorphism $\gamma_{x,y}\colon \mathcal{V}_x \to \mathcal{V}_y$, this isomorphism depends holomorphically on (x, y) and satisfies $\gamma_{x,x} = \mathrm{Id}$.

If interpreted correctly, this "definition" coincides with the definition in 2.2.4 below (which will not be used in the rest of the section).

To obtain it, it suffices to interpret "point" as "point valued in any analytic space":

2.2.1 A *point of the analytic space* X *with values in the analytic space* S is a morphism from S to X.

2.2.2 If Y is a subspace of X, the *n-th infinitesimal neighborhood* of Y in X is the subspace of X locally defined by the $(n+1)$-th power of the ideal of $\mathcal{O}_\mathrm{X}$ defining Y.

2.2.3 Two points x, y of X with values in S is said to be *infinitesimally near of first order* if the map $(x, y)\colon \mathrm{S} \to \mathrm{X} \times \mathrm{X}$ they defined factors through the first infinitesimal neighborhood of the diagonal of $\mathrm{X} \times \mathrm{X}$.

2.2.4 If X is a complex analytic manifold and $\mathcal{V}$ is a vector bundle on X, a *(holomorphic) connection* γ on $\mathcal{V}$ consists of the following data:

- for every pair (x,y) of points of X with values in any analytic space S, with x and y being infinitesimally near of first order, we give $\gamma_{x,y}\colon x^*\mathcal{V} \to y^*\mathcal{V}$;

this data is subject to the conditions:

1. For any $f\colon \mathrm{T} \to \mathrm{S}$ and two points $x,y\colon \mathrm{S} \rightrightarrows \mathrm{X}$ infinitesimally near of first order, we have $f^*(\gamma_{x,y}) = \gamma_{xf,yf}$.
2. We have $\gamma_{x,x} = \mathrm{Id}$.

2.3 Let X_1 be the first infinitesimal neighborhood of the diagonal X_0 of $\mathrm{X}\times\mathrm{X}$, and p_1, p_2 the two projections of X_1 to X. By definition, the vector bundle $\mathrm{P}^1(\mathcal{V})$ of the *jets of sections of first order* of $\mathcal{V}$ is the bundle $p_{1*}p_2^*\mathcal{V}$. We denote by j^1 the differential operator of first order which associates each section of $\mathcal{V}$ its jet of first order:

$$j^1\colon \mathcal{V} \longrightarrow \mathrm{P}^1(\mathcal{V}) \cong \mathcal{O}_{\mathrm{X}_1} \otimes_{\mathcal{O}_\mathrm{X}} \mathcal{V}.$$

A connection in the sense of 2.2.4 can be interpreted as a homomorphism (automatically isomorphism)

$$\gamma\colon p_1^*\mathcal{V} \longrightarrow p_2^*\mathcal{V}$$

which induces the identity above X_0. Since

$$\mathrm{Hom}_{\mathrm{X}_1}(p_1^*\mathcal{V}, p_2^*\mathcal{V}) \cong \mathrm{Hom}(\mathcal{V}, p_{1*}p_2^*\mathcal{V}),$$

a connection is also interpreted as a ($\mathcal{O}$-linear) homomorphism

$$\mathrm{D}\colon \mathcal{V} \longrightarrow \mathrm{P}^1(\mathcal{V})$$

such that the following composition

$$\mathcal{V} \xrightarrow{\mathrm{D}} \mathrm{P}^1(\mathcal{V}) \longrightarrow \mathcal{V}$$

is the identity. The sections $\mathrm{D}(s)$ and $j^1(s)$ of $\mathrm{P}^1(\mathcal{V})$ have the same image in $\mathcal{V}$, and $j^1(s) - \mathrm{D}(s)$ identifies with a section ∇s of $\Omega^1_\mathrm{X} \otimes \mathcal{V} \cong \mathrm{Ker}(\mathrm{P}^1(\mathcal{V}) \to \mathcal{V})$:

$$\nabla\colon \mathcal{V} \longrightarrow \Omega^1_\mathrm{X}(\mathcal{V}).$$

In other words, a connection (2.2.4), permiting to compare two neighboring fibers of $\mathcal{V}$, also permit to define the *differential* ∇s of a section s of $\mathcal{V}$.

Conversely, the formula

$$j^1(s) = \mathrm{D}(s) + \nabla s \tag{2.3.1}$$

permit to define D hence γ from the covariant derivative ∇. For D to be linear, it is necessary and sufficient that ∇ satisfies the identity

$$\nabla(fs) = \mathrm{d}f \otimes s + f.\nabla s. \tag{2.3.2}$$

The definition 2.2.4 is therefore equivalent to the following definition, due to J.L. Koszul.

2.4 Definition Let $\mathcal{V}$ be a (holomorphic) vector bundle on a complex analytic manifold X. A *holomorphic connection* (or simply, *connection*) on $\mathcal{V}$ is $\mathbb{C}$-linear homomorphism

$$\nabla\colon \mathcal{V} \longrightarrow \Omega^1_X(\mathcal{V}) := \Omega^1_X \otimes \mathcal{V}$$

satisfying the Leibniz identity (2.3.2) for f and s any local sections of $\mathcal{O}$ and $\mathcal{V}$. We call ∇ the *covariant derivative* defined by the connection.

2.5 If the vector bundle $\mathcal{V}$ is provided with a connection Γ of covariant derivative ∇, and if w is a holomorphic vector field on X, we put, for every local section v of $\mathcal{V}$ on open U of X,

$$\nabla_w(v) := \langle \nabla v, w\rangle \in \mathcal{V}(\mathrm{U}).$$

We call $\nabla_w\colon \mathcal{V} \to \mathcal{V}$ the *covariant derivative along the vector field* w.

2.6 If ${}_1\Gamma$ and ${}_2\Gamma$ are two connections, of covariant derivatives ${}_1\nabla$ and ${}_2\nabla$, then ${}_2\nabla - {}_1\nabla$ is an $\mathcal{O}$-linear homomorphism from $\mathcal{V}$ to $\Omega^1_X(\mathcal{V})$. Conversely, the sum of ${}_1\nabla$ and such a homomorphism defines a connection on $\mathcal{V}$. Thus, the connections on $\mathcal{V}$ form a *homogeneous principal space* (or *torsor*) under $\mathcal{H}om(\mathcal{V}, \Omega^1_X(\mathcal{V})) \cong \Omega^1_X(\mathcal{E}nd(\mathcal{V}))$.

2.7 If vector bundles are provided with connections, any vector bundle which is deduced by a tensor operation is still provided with a connection. This is evident on 2.2.4. Specifically, let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two vector bundles with connections of covariant derivatives ${}_1\nabla$ and ${}_2\nabla$.

2.7.1 We define a connection on $\mathcal{V}_1 \oplus \mathcal{V}_2$ by the formula

$$\nabla_w(v_1 + v_2) = {}_1\nabla_w(v_1) + {}_2\nabla_w(v_2).$$

2.7.2 We define a connection on $\mathcal{V}_1 \otimes \mathcal{V}_2$ by the formula

$$\nabla_w(v_1 \otimes v_2) = {}_1\nabla_w(v_1) \otimes v_2 + v_1 \otimes {}_2\nabla_w(v_2).$$

2.7.3 We define a connection on $\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)$ by the formula

$$(\nabla_w f)(v_1) = {}_2\nabla_w(f(v_1)) - f({}_1\nabla_w(v_1)).$$

The canonical connection on $\mathcal{O}$ is the connection for which $\partial f = \mathrm{d}f$. Let $\mathcal{V}$ be a vector bundle with a connection.

2.7.4 We define a connection on the dual $\mathcal{V}^\vee$ of $\mathcal{V}$ via 2.7.3 and the isomorphism of the definition $\mathcal{V}^\vee = \mathcal{H}om(\mathcal{V}, \mathcal{O})$. We have

$$\langle \nabla_w v', v\rangle = \partial_w\langle v', v\rangle - \langle v', \nabla_w v\rangle.$$

2.8 An $\mathcal{O}$-homomorphism f between vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ equipped with connections is said *compatible with connections* if

$$ {}_2\nabla.f = f.{}_1\nabla. $$

According to 2.7.3, this is to say that $\nabla f = 0$, if f is regarded as a section of $\mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2)$. For example, according to 2.7.3, the canonical homomorphism

$$ \mathcal{H}om(\mathcal{V}_1, \mathcal{V}_2) \otimes \mathcal{V}_1 \longrightarrow \mathcal{V}_2 $$

is compatible with connections.

2.9 A local section v of $\mathcal{V}$ is said to be *horizontal* if $\nabla v = 0$. If f is a homomorphism between vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ equipped with connections, it is then the same to say that f is horizontal and that f is compatible with connections.

2.10 Let $\mathcal{V}$ be a holomorphic vector bundle on X. We put $\Omega^p_{\mathrm{X}} = \bigwedge^p \Omega^1_{\mathrm{X}}$ and $\Omega^p_{\mathrm{X}}(\mathcal{V}) = \Omega^p_{\mathrm{X}} \otimes_{\mathcal{O}} \mathcal{V}$ (sheaf of the *external differential p-forms with values in* $\mathcal{V}$). Suppose that $\mathcal{V}$ is provided with a holomorphic connection. We thus define a $\mathbb{C}$-linear homomorphism

$$ \nabla \colon \Omega^p_{\mathrm{X}}(\mathcal{V}) \longrightarrow \Omega^{p+1}_{\mathrm{X}}(\mathcal{V}) \tag{2.10.1} $$

characterized by the following formula

$$ \nabla(\alpha \otimes v) = \mathrm{d}\alpha \otimes v + (-1)^p \alpha \wedge \nabla v, \tag{2.10.2} $$

where α is a local section of Ω^p_{X}, v is a local section of $\mathcal{V}$ and d is the external differential. To verify that the right hand side $\mathrm{II}(\alpha, v)$ of (2.10.2) defines a homomorphism (2.10.1), it is sufficient to verify that $\mathrm{II}(\alpha, v)$ is $\mathbb{C}$-bilinear and that

$$ \mathrm{II}(f\alpha, v) = \mathrm{II}(\alpha, fv). $$

In fact, we have

$$ \begin{aligned} \mathrm{II}(f\alpha, v) &= \mathrm{d}(f\alpha) \otimes v + (-1)^p f\alpha \wedge \nabla v \\ &= \mathrm{d}\alpha \otimes fv + \mathrm{d}f \wedge \alpha \otimes v + (-1)^p f\alpha \wedge \nabla v \\ &= \mathrm{d}\alpha \otimes fv + (-1)^p \alpha \wedge (f\nabla v + \mathrm{d}f \otimes v) \\ &= \mathrm{II}(\alpha, fv). \end{aligned} $$

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be two vector bundles with connections and let $\mathcal{V}$ be their tensor product. We denote by $\wedge$ the evident morphism

$$ \wedge \colon \Omega^p_{\mathrm{X}}(\mathcal{V}_1) \otimes \Omega^q_{\mathrm{X}}(\mathcal{V}_2) \longrightarrow \Omega^{p+q}_{\mathrm{X}}(\mathcal{V}) $$

such that, for α, β, v_1, v_2 local sections of Ω_X^p, Ω_X^q, $\mathcal{V}_1$, $\mathcal{V}_2$, we have

$$(\alpha \otimes v_1) \wedge (\beta \otimes v_2) = (\alpha \wedge \beta) \otimes (v_1 \otimes v_2).$$

If ν_1 (resp. ν_2) is a local section of $\Omega_X^p(\mathcal{V}_1)$ (resp. $\Omega_X^q(\mathcal{V}_2)$), we have

$$\nabla(\nu_1 \wedge \nu_2) = \nabla \nu_1 \wedge \nu_2 + (-1)^p \nu_1 \wedge \nabla \nu_2. \tag{2.10.3}$$

In fact, if $\nu_1 = \alpha \otimes v_1$ and $\nu_2 = \beta \otimes v_2$, we have

$$\begin{aligned}
\nabla(\nu_1 \wedge \nu_2) &= \nabla(\alpha \wedge \beta \otimes v_1 \otimes v_2) \\
&= d(\alpha \wedge \beta) \otimes v_1 \otimes v_2 + (-1)^{p+q} \alpha \wedge \beta \wedge \nabla(v_1 \otimes v_2) \\
&= d\alpha \wedge \beta \otimes v_1 \otimes v_2 + (-1)^p \alpha \wedge d\beta \otimes v_1 \otimes v_2 \\
&\quad + (-1)^{p+q} \alpha \wedge \beta \wedge \nabla v_1 \otimes v_2 + (-1)^{p+q} \alpha \wedge \beta \otimes v_1 \wedge \nabla v_2 \\
&= d\alpha \otimes v_1 \wedge \nu_2 + (-1)^p \nu_1 \wedge d\beta \otimes v_2 \\
&\quad + (-1)^p \alpha \wedge \nabla v_1 \wedge \nu_2 + (-1)^{p+q} \nu_1 \wedge \beta \wedge \nabla v_2 \\
&= \nabla \nu_1 \wedge \nu_2 + (-1)^p \nu_1 \wedge \nabla \nu_2.
\end{aligned}$$

Let $\mathcal{V}$ be a vector bundle with a connection. If we apply the previous formula to $\mathcal{O}$ and $\mathcal{V}$, we find that for every local section α (resp, ν) of Ω_X^p (resp. $\Omega_X^q(\mathcal{V})$), we have

$$\nabla(\alpha \wedge \nu) = d\alpha \wedge \nu + (-1)^p \alpha \wedge \nabla \nu. \tag{2.10.4}$$

Repeating this formula provides

$$\begin{aligned}
\nabla\nabla(\alpha \wedge \nu) &= \nabla(d\alpha \wedge \nu + (-1)^p \alpha \wedge \nabla \nu) \\
&= dd\alpha \wedge \nu + (-1)^{p+1} d\alpha \wedge \nabla \nu + (-1)^p d\alpha \wedge \nabla \nu + \alpha \wedge \nabla\nabla \nu \\
&= \alpha \wedge \nabla\nabla \nu.
\end{aligned}$$

2.11 Definition Under the hypothesis 2.10, the *curvature* $\mathcal{R}$ of the connection on $\mathcal{V}$ is composed homomorphism:

$$\nabla \circ \nabla \colon \mathcal{V} \longrightarrow \Omega_X^2(\mathcal{V})$$

viewed as a section of $\mathcal{H}om(\mathcal{V}, \Omega_X^2(\mathcal{V})) \cong \Omega_X^2(\mathcal{E}nd(\mathcal{V}))$.

2.12 The formula (2.10.4) for $q = 0$ provides

$$\nabla\nabla(\alpha \otimes v) = \alpha \wedge \mathcal{R}(v), \tag{2.12.1}$$

which is also written as

$$\nabla\nabla(\nu) = \mathcal{R} \wedge \nu \qquad (\textit{Ricci identity}). \tag{2.12.2}$$

Providing $\mathcal{E}nd(\mathcal{V})$ with the connection 2.7.3, the formula $\nabla(\nabla\nabla) = (\nabla\nabla)\nabla$ can be written as $\nabla(\mathcal{R} \wedge \nu) = \mathcal{R} \wedge \nabla \nu$. According to 2.7.3, we have $\nabla\mathcal{R} \wedge \nu = \nabla(\mathcal{R} \wedge \nu) - \mathcal{R} \wedge \nabla \nu$ so that

$$\nabla\mathcal{R} = 0 \qquad (\textit{Bianchi identity}). \tag{2.12.3}$$

2.13 If α is a p-form, we know that

$$\begin{aligned}\langle d\alpha, X_0\wedge\cdots\wedge X_p\rangle &= \sum(-1)^i\partial_{X_1}\langle\alpha, X_0\wedge\cdots\widehat{X_i}\cdots\wedge X_p\rangle\\ &+\sum_{i<j}(-1)^{i+j}\langle\alpha,[X_i,X_j]\wedge X_0\wedge\cdots\widehat{X_i}\cdots\widehat{X_j}\cdots\wedge X_p\rangle.\end{aligned}$$

From this formula and (2.10.2), we find that for every local section ν of $\Omega^p_X(\mathcal{V})$ and holomorphic vector fields $X_0,\cdots,X_p$, we have

$$\begin{aligned}\langle \nabla\nu, X_0\wedge\cdots\wedge X_p\rangle &= \sum(-1)^i\nabla_{X_1}\langle\nu, X_0\wedge\cdots\widehat{X_i}\cdots\wedge X_p\rangle\\ &+\sum_{i<j}(-1)^{i+j}\langle\nu,[X_i,X_j]\wedge X_0\wedge\cdots\widehat{X_i}\cdots\widehat{X_j}\cdots\wedge X_p\rangle.\end{aligned}$$

In particular, for v a loacl section of $\mathcal{V}$, we have

$$\langle\nabla\nabla v, X_1\wedge X_2\rangle = \nabla_{X_1}\langle\nabla v, X_2\rangle - \nabla_{X_2}\langle\nabla v, X_1\rangle + \langle\nabla,[X_1,X_2]\rangle.$$

Let

$$\mathcal{R}(X_1,X_2)(v) := \nabla_{X_1}\nabla_{X_2}v - \nabla_{X_2}\nabla_{X_1}v - \nabla_{[X_1,X_2]}v.$$

2.14 Definition A connection is said to be *integrable* if its curvature is zero, i.e. if we have tha identity

$$\nabla_{[X,Y]} = [\nabla_X,\nabla_Y].$$

If $\dim(X)\leqslant 1$, every connection is integrable.

If Γ is a integrable connection on $\mathcal{V}$, the morphism ∇ in (2.10.1) satisfies $\nabla\nabla = 0$, so that $\Omega^p_X(\mathcal{V})$ forms a differential complex $\Omega^\bullet_X(\mathcal{V})$.

2.15 Definition Under the previous hypothesis, the complex $\Omega^\bullet_X(\mathcal{V})$ is called the *holomorphic De Rham complex* with values in $\mathcal{V}$.

The following results 2.16–2.19 will be presented more generally in 2.23.

2.16 Proposition *Let* V *be a complex local system on a complex analytic manifold* X *and* $\mathcal{V} = \mathcal{O}\otimes_{\mathbb{C}} V$.

(i) There exists a canonical connection on $\mathcal{V}$*, for which the horizontal sections of* $\mathcal{V}$ *are local sections of the subsheaf* V *of* $\mathcal{V}$*.*

(ii) The canonical connection on $\mathcal{V}$ *is integrable.*

(iii) For f *(resp.* v*) a local section of* $\mathcal{O}$ *(resp.* V*), we have*

(2.16.1) $$\nabla(fv) = df\otimes v.$$

If ∇ satisfies (i), then (2.16.1) is just a special case of (2.3.2). Conversely, the right hand side of (2.16.1) is $\mathbb{C}$-bilinear and extends uniquely to a $\mathbb{C}$-linear homomorphism $\nabla\colon\mathcal{V}\to\Omega^1_X(\mathcal{V})$, which satisfies the definition of a connection. The assumption (ii) is local on X, which permits to reduce to the case $V = \underline{\mathbb{C}}$. At the moment, $\mathcal{V} = \mathcal{O}$, $\nabla = d$ and $\nabla_{[X,Y]} = [\nabla_X,\nabla_Y]$ by definition of $[X,Y]$.

It is well known that

2.17 Theorem *Let* X *be a complex analytic manifold, the following functors*

a) sending a complex local system V *to the vector bundle* $\mathcal{V} = \mathcal{O} \otimes_{\mathbb{C}} \mathrm{V}$ *equipped with the canonical connection,*

b) sending a holomorphic vector bundle $\mathcal{V}$ *on* X*, equipped with an integrable connection, to the subsheaf* V *of horizontal sections of* $\mathcal{V}$*,*

form a pair of equivalences between the category of complex local systems on X *and the category of holomorphic vector bundles with integrable connections on* X *(with morphisms the horizontal morphisms of vector bundles).*

These equivalences are compatible with the formation of the tensor product, the internal Hom and the dual. The unit complex local system $\underline{\mathbb{C}}$ corresponds to the bundle $\mathcal{O}$, equipped with the connection such that $\nabla f = \mathrm{d}f$.

One deduces from (2.10.2) that

2.18 Proposition *If* V *is a complex local system on* X*, and if* $\mathcal{V} = \mathcal{O} \otimes_{\mathbb{C}} \mathrm{V}$*, then the system of isomorphisms* $\Omega^p_{\mathrm{X}} \otimes_{\mathbb{C}} \mathrm{V} \cong \Omega^p_{\mathrm{X}} \otimes_{\mathcal{O}} \mathcal{O} \otimes_{\mathbb{C}} \mathrm{V} = \Omega_{\mathrm{X}} \otimes_{\mathcal{O}} \mathcal{V}$ *is an isomorphism of complexes*

$$\Omega^{\bullet}_{\mathrm{X}} \otimes_{\mathbb{C}} \mathrm{V} \xrightarrow{\sim} \Omega^{\bullet}_{\mathrm{X}}(\mathcal{V}).$$

From this, the *holomorphic Poincaré lemma* results that

2.19 Proposition *Under the hypothesis of 2.16, the complex* $\Omega^{\bullet}_{\mathrm{X}}(\mathcal{V})$ *is a resolution of the sheaf* V.

2.20 Variants.

2.20.1 If X is a differential manifold, for the C^{∞}-connections on the C^{∞}-vector bundles, all the above results remains valid, *mutatis mutandis*.

2.20.2 Theorem 2.17 essentially requires the non-singularity of X; it is therefore of no interest to note that this hypothesis was not used in an essential way before 2.17.

2.20.3 The definition 2.4 of a connection and the definition 2.14 of integrablity are sufficiently formal to be transposed in the category of schemes, or in related situations.

2.21 Definition *(i)* Let $f\colon \mathrm{X} \to \mathrm{S}$ be a smooth morphism of schemes and $\mathcal{V}$ a quasi-coherent sheaf on X. A *relative connection* on $\mathcal{V}$ is an $f^{-1}\mathcal{O}_{\mathrm{S}}$-linear morphism of sheaves (called the *covariant derivative* defined by the connection)

$$\nabla\colon \mathcal{V} \longrightarrow \Omega^1_{\mathrm{X/S}}(\mathcal{V})$$

satisfying following identity, for f (resp. v) a local section of $\mathcal{O}_{\mathrm{X}}$ (resp. $\mathcal{V}$),

$$\nabla(fv) = \mathrm{d}f \otimes v + f\nabla v.$$

(ii) For $\mathcal{V}$ equipped with a relative connection, there exists a unique system of $f^{-1}\mathcal{O}_S$-homomorphisms of sheaves

$$\nabla^{(p)} \text{ or } \nabla\colon \Omega^p_{X/S}(\mathcal{V}) \longrightarrow \Omega^{p+1}_{X/S}(\mathcal{V})$$

satisfying the identity (2.10.4) and such that $\nabla^{(0)} = \nabla$.

(iii) The *curvature* of a connection is defined by

$$\mathcal{R} = \nabla^{(1)} \circ \nabla^{(0)} \in \mathcal{H}om(\mathcal{V}, \Omega^2_{X/S}(\mathcal{V})) \cong \Omega^2_{X/S}(\mathcal{E}nd\,\mathcal{V}).$$

The curvature satisfies the Ricci identity (2.12.2) and the Bianchi identity (2.12.3).

(iv) An *integrable connection* is a connection with zero curvature.

(v) The *De Rham complex* defined by an integrable connection is the complex $(\Omega^\bullet_{X/S}(\mathcal{V}), \nabla)$.

2.22 Let $f\colon X \to S$ be a *smooth* morphism of complex analytic spaces, that means, locally on X, it is isomorphic to the projection from $D^n \times S$ to S, where D^n is an open polydisc. A *relative local system* on X is an $f^{-1}\mathcal{O}_S$-module, locally isomorphic to a pullback of a coherent analytic sheaf on S. If $\mathcal{V}$ is a coherent analytic sheaf on X, a *relative connection* on $\mathcal{V}$ is an $f^{-1}\mathcal{O}_S$-linear homomorphism

$$\nabla\colon \mathcal{V} \longrightarrow \Omega^1_{X/S}(\mathcal{V})$$

satisfying following identity, for f (resp. v) a local section of $\mathcal{O}_X$ (resp. $\mathcal{V}$),

$$\nabla(fv) = \mathrm{d}f \otimes v + f\nabla v.$$

A *(horizontal) morphism* between vector bundles with relative connections is a morphism bewteen vector bundles commuting with ∇. We define as in 2.13 and 2.21 the *curvature* $\mathcal{R} \in \Omega^2_{X/S}(\mathcal{E}nd(\mathcal{V}))$ of a relative connection. A connection is said to be *integrable* if $\mathcal{R} = 0$, in which case we have the *relative De Rham complex* $\Omega^\bullet_{X/S}(\mathcal{V})$ with values in $\mathcal{V}$, defined as in 2.15 and 2.21.

The "absolute" statements 2.17, 2.18 and 2.19 have "relative" (i.e. "with parameters") analogies.

2.23 Theorem *Under the hypothesis of 2.22, we have*

(i) *For every relative local system* V *on* X, *there exists a coherent analytic sheaf* $\mathcal{V} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} V$ *with a canonical relative connection, such that a local section v of $\mathcal{V}$ is horizontal ($\nabla v = 0$) if and only if v is a section of* V. *Moreover, this connection is integrable.*

(ii) *Given a relative local system* V *on* X*, the De Rham complex defined by* $\mathcal{V} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} V$*, equipped with its canonical connection, is a resolution of the sheaf* V*.*

(iii) *The following functors*

a) sending a relative local system V *to the coherent analytic sheaf* $\mathcal{V} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} V$ *equipped with the canonical connection,*

b) sending a coherent analytic sheaf $\mathcal{V}$ *on* X*, equipped with an integrable relative connection, to the subsheaf of horizontal sections of* $\mathcal{V}$*,*

form a pair of equivalences between the category of relative local systems on X *and the category of coherent analytic sheaves on* X*, equipped with integrable relative connections.*

Proof of (i). To verify that $\mathcal{V}$ is coherent, it suffices to do it locally, for $V = f^{-1}V_0$, in which case $\mathcal{V}$ is the pullback, in the sense of coherent analytical sheaves, of V_0. The canonical relative connection necessarily verifies, for f (resp. v) a loal section of $\mathcal{O}$ (resp. $\mathcal{V}$),

$$\nabla(fv) = \mathrm{d}f \otimes v. \tag{2.23.1}$$

The right hand side $\mathrm{II}(f,v)$ is biadditive in f and v, and satisfies, for g a local section of $f^{-1}\mathcal{O}_S$, the identity

$$\mathrm{II}(fg,v) = \mathrm{II}(f,gv),$$

(using that $\mathrm{d}g = 0$ in $\Omega^1_{X/S}$). We can deduce the existence and the uniqueness of a relative connection ∇ satisfying (2.23.1). We finally have

$$\nabla\nabla(fv) = \nabla(\mathrm{d}f \otimes v) = \mathrm{d}\mathrm{d}f \otimes v = 0;$$

the canonical connection is thus integrable. That the sections of V are the only horizontal ones is a special case of (ii) proving below.

2.23.2 Let's first look at the particular case of (ii) where $S = D^n$, $X = D^n \times D^m$, $f = \mathrm{pr}_2$ and where the relative local system V is the pullback of $\mathcal{O}_S$. The complex of global sections

$$0 \longrightarrow \Gamma(f^{-1}\mathcal{O}_S) \longrightarrow \Gamma(\mathcal{O}_X) \xrightarrow{\mathrm{d}} \Gamma(\Omega^1_{X/S}) \longrightarrow \cdots$$

is acyclic, because it admits the following homotopy operator.

a) $\mathrm{H}\colon \Gamma(\mathcal{O}_X) \to \Gamma(f^{-1}\mathcal{O}_S) = \Gamma(S,\mathcal{O}_S)$ is the pullback by zero section of f;

b) $\mathrm{H}\colon \Gamma(\Omega^p_{\mathrm{X/S}}) \to \Gamma(\Omega^{p-1}_{\mathrm{X/S}})$ is given as follows: since H must be $\Gamma(f^{-1}\mathcal{O}_{\mathrm{S}})$-linear and $\Omega^p_{\mathrm{X/S}}$ has a basis $\{x^{\underline{n}}\mathrm{d}x_{\mathrm{I}} : \underline{n} \in \mathbb{N}^m, \mathrm{I} \subset [1,m], |\mathrm{I}| = p\}$, it suffices to define

$$\mathrm{H}(x^{\underline{n}}\mathrm{d}x_{\mathrm{I}}) = \frac{1}{m}\sum_{i\in\mathrm{I}} \frac{\mathrm{sgn}_{\mathrm{I}}(i)}{n_i+1} x^{\underline{n}+\epsilon_i}\mathrm{d}x_{\mathrm{I}\setminus\{i\}},$$

where $\mathrm{sgn}_{\mathrm{I}}(i)$ is the signature of i in the seuqence I, and ϵ_i is the i-th member of the standard basis of $\mathbb{N}^m$.

This remains true if we replace D^{m+n} by a smaller polycylinder. Therefore the complex of sheaves

$$0 \longrightarrow f^{-1}\mathcal{O}_{\mathrm{S}} \longrightarrow \mathcal{O}_{\mathrm{X}} \xrightarrow{\mathrm{d}} \Omega^1_{\mathrm{X/S}} \longrightarrow \cdots$$

is acyclic and thus $\Omega^\bullet_{\mathrm{X/S}}$ is a resolution of $f^{-1}\mathcal{O}_{\mathrm{S}}$.

2.23.3 Proof of (ii). The assertion (ii) is naturally local on X and S. Denoted by D the unit open disk, so we can go back to the case where S is a closed analytical subset of the polycylinder D^n, where $\mathrm{X} = \mathrm{D}^m \times \mathrm{S}$, with $f = \mathrm{pr}_2$, and where V is the pullback of a coherent analytic sheaf V_0 on S. Applying the syzygy theorem, and shrinking X and S, we can further assume that the pushforward of V_0 on D^n, which also denoted by V_0, admits a finite resolution $\mathcal{L}^\bullet$ by free coherent $\mathcal{O}_{\mathrm{D}^n}$-modules. To prove (ii), it is permissible to replace V_0 by its pushforward on D^n, which will be done henceforth.

If Σ_0 is a short exact sequence of coherent $\mathcal{O}_{\mathrm{S}}$-moduels

$$\Sigma_0\colon 0 \longrightarrow \mathrm{V}_0' \longrightarrow \mathrm{V}_0 \longrightarrow \mathrm{V}_0'' \longrightarrow 0,$$

let $\mathrm{V} = f^{-1}\mathrm{V}_0$ be the short exact sequence of relative local systems which is the pullback of Σ_0 (the sequence Σ is exact because f^{-1} is an exact functor) and let $\Omega^\bullet_{\mathrm{X/S}}(\Sigma)$ be the corresponding exact sequence of relative De Rham complexes

$$0 \longrightarrow \Omega^\bullet_{\mathrm{X/S}} \otimes_{f^{-1}\mathcal{O}_{\mathrm{S}}} f^{-1}\mathrm{V}_0' \longrightarrow \Omega^\bullet_{\mathrm{X/S}} \otimes_{f^{-1}\mathcal{O}_{\mathrm{S}}} f^{-1}\mathrm{V}_0 \longrightarrow \Omega^\bullet_{\mathrm{X/S}} \otimes_{f^{-1}\mathcal{O}_{\mathrm{S}}} f^{-1}\mathrm{V}_0'' \longrightarrow 0.$$

This sequence is exact because $\Omega^\bullet_{\mathrm{X/S}}$ is flat over $f^{-1}\mathcal{O}_{\mathrm{S}}$, being locally free on $\mathcal{O}_{\mathrm{X}}$ which is flat on $f^{-1}\mathcal{O}_{\mathrm{S}}$.

The snake lemma applied to $\Omega^\bullet_{\mathrm{X/S}}(\Sigma)$ shows that if the assertion (ii) is true for two of $f^{-1}\mathrm{V}_0'$, $f^{-1}\mathrm{V}_0$ and $f^{-1}\mathrm{V}_0''$, then it is also true for the third. We can deduce by induction that if V_0 admits a finite resolution by modules satisfying (ii), then so does V_0. This, applied to V_0 and $\mathcal{L}^\bullet$, complete the proof of (ii).

2.23.4 It follows from (ii) that the composition of functors in (iii) (in the order $b \circ a$) is canonically isomorphic to the identity; in addition, if V_1 and V_2 are two relative local systems, and $u\colon \mathcal{V}_1 \to \mathcal{V}_2$ is a homomorphism inducing 0 on V_1, then $u = 0$ since V_1 generates $\mathcal{V}_1$; it follows that the functor a is fully faithful. It remains to show that any vector bundle $\mathcal{V}$ equipped with a integrable relative connection ∇ locally comes from a relative local system.

Case 1. $S = D^n$, $X = D^{n+1} = D^n \times D$, $f = \mathrm{pr}_1$ and $\mathcal{V}$ is free.

Under these assumptions, if v is any section of the pullback of $\mathcal{V}$ along the zero section s_0 of f, there exists a unique horizontal section $\widetilde{v}$ of $\mathcal{V}$ which coincides with v on $s_0(S)$ (existence and uniqueness from a Cauchy problem with parameters). If (e_i) is a basis of $s_0^*\mathcal{V}$, then $\widetilde{e_i}$ form a horizontal basis of $\mathcal{V}$, and $(\mathcal{V}, \nabla)$ is defined by the relative local system $f^{-1}s_0^*\mathcal{V} = f^{-1}\mathcal{O}_S^k$.

Case 2. $S = D^n$, $X = D^{n+1} = D^n \times D$ and $f = \mathrm{pr}_1$.

Shrinking X and S, we may assume that $\mathcal{V}$ admits a free presentation

$$\mathcal{V}_1 \xrightarrow{d} \mathcal{V}_0 \xrightarrow{\epsilon} \mathcal{V} \longrightarrow 0$$

Shrinking further, we reduce to the case where $\mathcal{V}_0$ and $\mathcal{V}_1$ admit connections ${}_0\nabla$ and ${}_1\nabla$, such that ϵ and d are compatible with connections (if (e_i) is a basis of $\mathcal{V}_0$, ${}_0\nabla$ is determined by ${}_0\nabla e_i$, and it suffices to choose ${}_0\nabla e_i$ such that $\epsilon({}_0\nabla e_i) = \nabla(\epsilon(e_i))$; similarly for ${}_1\nabla$). The connections ${}_0\nabla$ and ${}_1\nabla$ are automatically integrable, since f is of relative dimension 1. Thus there exist (case 1) relative local systems V_0 and V_1 such that $(\mathcal{V}_i, {}_i\nabla) \cong \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} V_i$. We thus have

$$(\mathcal{V}, \nabla) \cong \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} (V_0/dV_1).$$

Case 3. f is of relative dimension 1.

We may assume that S is a closed analytical subset of the polycylinder D^n and that $X = \times D$, $f = \mathrm{pr}_1$. The relative local systems (resp. relative moduels with connection) on X then identify with relative local systems (resp. relative modules with connection) on $D^n \times D$ annihilated by the pullback of the ideal defining S, and we conclude by case 2.

General case. Prove by induction on the relative dimension n of f. The case $n = 0$ is trivial. If $n \neq 0$, we can reduce to the case where $X = S \times D^{n-1} \times D$ and where $f = \mathrm{pr}_1$. The bundle with connection $(\mathcal{V}, \nabla)$ induces on $X_0 = S \times D^{n-1} \times \{0\}$ a bundle with connection $\mathcal{V}_0$ which, by inductive hypothesis, is of the type $(\mathcal{V}_0, {}_0\nabla) = \mathcal{O}_{X_0} \otimes_{\mathrm{pr}_1^{-1}\mathcal{O}_S} V$. The projection p from X to X_0 is of relative dimension 1, and the relative connection ∇ induces a relative connection for $\mathcal{V}$ on $X \to X_0$. According to case 3, there exists a vector bundle V_1 on X_0 and of an isomorphism of bundles with relative connection (for p) such that

$$\mathcal{V} \cong \mathcal{O}_X \otimes_{p^{-1}\mathcal{O}_{X_0}} p^{-1}V_1.$$

The vector bundle V_1 is identified with the restriction of $\mathcal{V}$ on X_0, hence an isomorphism of vector bundles

$$\alpha\colon \mathcal{V} \cong \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} V$$

satisfying

(i) The restriction of α to X_0 is horizontal.

(ii) α is "relatively horizontal" for p.

If v is a section of V, then condition (ii) means that

$$\nabla_{x_n} v = 0.$$

If $1 \leqslant i < n$, and since $\mathcal{R} = 0$, we have by the relative analogy of 2.11:

$$\nabla_{x_n} \nabla_{x_i} v = \nabla_{x_i} \nabla_{x_n} v = 0.$$

In other words, $\nabla_{x_i} v$ is a relative horizontal section, for p, of $\mathcal{V}$; according to (i), it vanishes on X_0, hence it is zero and we conclude that $\nabla v = 0$. The isomorphism α is thus horizontal and this finish the proof of 2.23.

Some general topology results (2.24–2.27) will be needed to deduce 2.28 from 2.23.

2.24 Recall Let, in a topological space X, Y be a closed subspace having a *paracompact* neighborhood. For each sheaf $\mathcal{F}$ on X, we have

$$\varinjlim \mathrm{H}^\bullet(\mathrm{U}, \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^\bullet(\mathrm{Y}, \mathcal{F}|_{\mathrm{Y}})$$

where U goes through neighborhoods of Y.

This is Godement [God73, II 4.11.1].

2.25 Corollary *Let $f : \mathrm{X} \to \mathrm{S}$ be a proper and separated map between topological spaces. Suppose that* S *is locally paracompact. Then, for any point s and any sheaf $\mathcal{F}$ on* X*, we have*

$$(\mathrm{R}^i f_* \mathcal{F})_s = \mathrm{H}^i(f^{-1}(s), \mathcal{F}|_{f^{-1}(s)}).$$

Since f is closed, the collection of $f^{-1}(\mathrm{U})$, where U goes through all neighborhoods of s, form a basis of neighborhoods of $f^{-1}(s)$. Moreover, if U is paracompact, so is $f^{-1}(\mathrm{U})$ because f is proper and separated. Note that

$$(\mathrm{R}^i f_* \mathcal{F})_s = \varinjlim_{s \in \mathrm{U}} \mathrm{H}^i(f^{-1}(\mathrm{U}), \mathcal{F}).$$

We conclude by 2.24.

2.25.1 Lemma *Let $f : \mathrm{X} \to \mathrm{S}$ be a surjective proper map between topological spaces. If* S *is paracompact, then so is* X*.*

For any subset Y of X, define

$$g(\mathrm{Y}) = \mathrm{S} \setminus f(\mathrm{X} \setminus \mathrm{Y}).$$

Then g maps opens of X to opens of S because f is closed. Let $\{\mathrm{U}_i\}_{i\in\mathrm{I}}$ be an open cover of X. Then since each fiber $f^{-1}(s)$ is quasi-compact, it admits a finite open cover $\{\mathrm{U}_i \cap f^{-1}(s)\}_{i\in\mathrm{I}_s}$, where I_s is a finite subset of I. Then the family

$$\Big\{g(\bigcup_{i\in\mathrm{I}_s}\mathrm{U}_i) : s\in\mathrm{S}\Big\}$$

is an open cover of S. Then it admits a locally finite cover $\{\mathrm{V}_j\}_{j\in\mathrm{J}}$. Now, for each $\mathrm{V}_j \in g(\bigcup_{i\in\mathrm{I}_s}\mathrm{U}_i)$, let $\mathrm{W}_{j,i} = f^{-1}(\mathrm{V}_j)\cap\mathrm{U}_i$ $(i\in\mathrm{I}_s)$. Then $\{\mathrm{W}_{j,i}\}_{j\in\mathrm{J},i\in\mathrm{I}}$ is a locally finite refinement of $\{\mathrm{U}_i\}_{i\in\mathrm{I}}$.

2.26 Recall Let X be a locally contractible paracompact space, i an integer and V a complex local system on X, satisfying $\dim_{\mathbb{C}}\mathrm{H}^i(\mathrm{X},\mathrm{V}) < \infty$. Then, for every vector space A over $\mathbb{C}$, can be of infinite dimension, qw have

$$\mathrm{A}\otimes_{\mathbb{C}}\mathrm{H}^i(\mathrm{X},\mathrm{V}) \xrightarrow{\sim} \mathrm{H}^i(\mathrm{X},\mathrm{A}\otimes_{\mathbb{C}}\mathrm{V}). \tag{2.26.1}$$

Denoted by $\mathrm{H}_\bullet(\mathrm{X},\mathrm{V}^*)$ the singular homology of X with coefficients in V^*. The universal coefficient theorem, valid here, give

$$\mathrm{H}^i(\mathrm{X},\mathrm{A}\otimes_{\mathbb{C}}\mathrm{V}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_i(\mathrm{X},\mathrm{V}^*),\mathrm{A}). \tag{2.26.2}$$

For $\mathrm{A}=\mathbb{C}$, we conclude that $\dim_{\mathbb{C}}\mathrm{H}_i(\mathrm{X},\mathrm{V}^*) < \infty$. Then formula (2.26.1) follows from (2.26.2).

2.27 Let $f\colon \mathrm{X}\to\mathrm{S}$ be a smooth and separated morphism bewteen complex analytic spaces and let V be a local system on X. The sheaf

$$\mathrm{V}_{rel} := f^{-1}\mathcal{O}_{\mathrm{S}}\otimes_{\mathbb{C}}\mathrm{V} \tag{2.27.1}$$

is then a relative local system. We denote by $\Omega^\bullet_{\mathrm{X/S}}(\mathrm{V})$ the corresponding De Rham complex. According to 2.23, $\Omega^\bullet_{\mathrm{X/S}}$ is a resolution of V_{rel}. We thus have

$$\mathrm{R}^i f_*(\mathrm{V}_{rel}) \xrightarrow{\sim} \mathrm{R}^i f_*(\Omega^\bullet_{\mathrm{X/S}}(\mathrm{V})) \tag{2.27.2}$$

where right hand side is a relative hypercohomology. From (2.27.1), we deduce a projection arrow

$$\mathcal{O}_{\mathrm{S}}\otimes_{\mathbb{C}}\mathrm{R}^i f_*\mathrm{V} \longrightarrow \mathrm{R}^i f_*(\mathrm{V}_{rel}), \tag{2.27.3}$$

which comes from the canonical arrow

$$\mathrm{L}f^{-1}(\mathcal{O}_{\mathrm{S}}\otimes^{\mathrm{L}}_{\mathbb{C}}\mathrm{R}f_*\mathrm{V}) = \mathrm{L}f^{-1}\mathcal{O}_{\mathrm{S}}\otimes^{\mathrm{L}}_{\mathbb{C}}\mathrm{L}f\mathrm{R}f_*\mathrm{V} \longrightarrow \mathrm{L}f^{-1}\mathcal{O}_{\mathrm{S}}\otimes^{\mathrm{L}}_{\mathbb{C}}\mathrm{V}.$$

From (2.27.3), by composition with (2.27.2), we get an arrow

$$\mathcal{O}_{\mathrm{S}}\otimes_{\mathbb{C}}\mathrm{R}^i f_*\mathrm{V} \longrightarrow \mathrm{R}^i f_*(\Omega^\bullet_{\mathrm{X/S}}(\mathrm{V})). \tag{2.27.4}$$

2.28 Proposition *Let $f : \mathrm{X} \to \mathrm{S}$ be a smooth and separated morphism bewteen complex analytic spaces, i an integer and V a complex local system on X. We suppose that*

a) locally on S, f is topologically trivial;

b) the fiber of f satisfies

$$\dim_{\mathbb{C}} \mathrm{H}^i(f^{-1}(s), \mathrm{V}) < \infty.$$

Then, the arrow (2.27.4) *is an isomorphism*

$$\mathcal{O}_\mathrm{S} \otimes_{\mathbb{C}} \mathrm{R}^i f_* \mathrm{V} \xrightarrow{\sim} \mathrm{R}^i f_*(\Omega^\bullet_{\mathrm{X}/\mathrm{S}}(\mathrm{V})).$$

Let $s \in \mathrm{S}$, $\mathrm{Y} = f^{-1}(s)$, and $\mathrm{V}_0 = \mathrm{V}|_\mathrm{Y}$. To verify (2.27.4) is an isomorphism, it suffices to construct a basis of neighborhoods T of s such that the arrows

$$\mathrm{H}^0(\mathrm{T}, \mathcal{O}_\mathrm{S}) \otimes \mathrm{H}^i(\mathrm{T} \times \mathrm{Y}, \mathrm{pr}_2^{-1} \mathrm{V}_0) \xrightarrow{\sim} \mathrm{H}^i(\mathrm{T} \times \mathrm{Y}, \mathrm{pr}_1^{-1} \mathcal{O}_\mathrm{S} \otimes \mathrm{pr}_2^{-1} \mathrm{V}_0) \tag{2.28.1}$$

are isomorphisms. If so, then the stalk of (2.27.3) at s, as an inductive limit of (2.28.1) because of locality of cohomology, will also be an isomorphism.

We'll prove this for T a contractible Stein compact neighborhood of s. Note that (2.28.1) can be also written as

$$\mathrm{H}^0(\mathrm{T}, \mathcal{O}_\mathrm{S}) \otimes \mathrm{H}^i(\mathrm{Y}, \mathrm{V}_0) \xrightarrow{\sim} \mathrm{H}^i(\mathrm{T} \times \mathrm{Y}, \mathrm{pr}_1^{-1} \mathcal{O}_\mathrm{S} \otimes \mathrm{pr}_2^{-1} \mathrm{V}_0). \tag{2.28.2}$$

Calculate the right hand side of (2.28.2) by Leray spectral sequence for $\mathrm{pr}_2 : \mathrm{T} \times \mathrm{Y} \to \mathrm{Y}$. First, we have

$$\mathrm{H}^i\big(\mathrm{Y}, \mathrm{R}^j(\mathrm{pr}_2)_*(\mathrm{pr}_1^{-1} \mathcal{O}_\mathrm{S} \otimes \mathrm{pr}_2^{-1} \mathrm{V}_0)\big) \Rightarrow \mathrm{H}^{i+j}(\mathrm{T} \times \mathrm{Y}, \mathrm{pr}_1^{-1} \mathcal{O}_\mathrm{S} \otimes \mathrm{pr}_2^{-1} \mathrm{V}_0).$$

By projection formula, we have

$$\mathrm{R}^j(\mathrm{pr}_2)_*(\mathrm{pr}_1^{-1} \mathcal{O}_\mathrm{S} \otimes \mathrm{pr}_2^{-1} \mathrm{V}_0) \cong \mathrm{R}^j(\mathrm{pr}_2)_* \mathrm{pr}_1^{-1} \mathcal{O}_\mathrm{S} \otimes \mathrm{V}_0.$$

According to 2.25, we have

$$(\mathrm{R}^j(\mathrm{pr}_2)_* \mathrm{pr}_1^{-1} \mathcal{O}_\mathrm{S})_y \xrightarrow{\sim} \mathrm{H}^j(\mathrm{T}, \mathcal{O}_\mathrm{S})$$

for all $y \in \mathrm{Y}$, hence it is a constant sheaf. Since T is Stein, $\mathrm{H}^j(\mathrm{T}, \mathcal{O}_\mathrm{S}) = 0$ for $j > 0$. Finally, we have

$$\mathrm{H}^i(\mathrm{T} \times \mathrm{Y}, \mathrm{pr}_1^{-1} \mathcal{O}_\mathrm{S} \otimes \mathrm{pr}_2^{-1} \mathrm{V}_0) = \mathrm{H}^i(\mathrm{Y}, \mathrm{H}^0(\mathrm{T}, \mathcal{O}_\mathrm{S}) \otimes \mathrm{V}_0).$$

We conclude by 2.26.

2.29 Under the hypothesis of 2.28, with S smooth, we define the *Gauss-Manin connection* on $\mathrm{R}^i f_*(\Omega_{\mathrm{X}/\mathrm{S}}(\mathrm{V}))$ as being the unique integrable connection admitting for horizontal local sections the local sections of $\mathrm{R}^i f_* \mathrm{V}$ (2.17).

§ 3 Translation in terms of partial differential equations of first order

3.1 Let X be a complex analytic manifold. If $\mathcal{V}$ is a holomorphic vector bundle defined by a $\mathbb{C}$-vector space V_0, we have seen that $\mathcal{V}$ admits a canonical connection of covariant derivative ${}_0\nabla$. If ∇ is the covariant derivative defined by another connection on $\mathcal{V}$, we have seen (2.6) that ∇ is written in the form

$$\nabla = {}_0\nabla + \Gamma, \qquad \text{where } \Gamma \in \Omega(\mathcal{E}nd(\mathcal{V})).$$

If we identify sections of $\mathcal{V}$ with holomorphic maps from X to V_0 (for example, the section $f \otimes v_0$ with $v_0 \in V_0$ can be viewed as such a map by $x \mapsto f(x) \otimes v_0$), then we have

$$\nabla v = \mathrm{d}v + \Gamma v. \tag{3.1.1}$$

If we choose a basis of V, i.e. an isomorphism $e \colon \mathbb{C}^n \to V_0$ of coordinates (identified with vectors of basis) $e_\alpha \colon \mathbb{C} \to V_0$, then Γ can be represented by a matrix of formal differential ω^α_β (the *matrix of forms of connection*), and (3.1.1) can be rewritten as

$$(\nabla v)^\alpha = \mathrm{d}v^\alpha + \sum_\beta \omega^\alpha_\beta v^\beta. \tag{3.1.2}$$

Let $\mathcal{V}$ be any holomorphic vector bundle on X. The choice of a basis $e \colon \mathbb{C}^n \to \mathcal{V}$ of $\mathcal{V}$ permit to consider $\mathcal{V}$ as defined by the constant vector space $\mathbb{C}^n$, and the previous considerations apply: the connection on $\mathcal{V}$ corresponds, via (3.1.2), to the $n \times n$ matrix of differential forms on X. If ω_e is the matrix of the connection ∇ under the basis e, and if $f \colon \mathbb{C}^n \to \mathcal{V}$ is a new basis of $\mathcal{V}$, of coordinate $A \in GL_n(\mathcal{O})$ ($A = ef^{-1}$), we have, by (3.1.2),

$$\begin{aligned}\nabla v &= e\mathrm{d}(e^{-1}v) + e\omega_e e^{-1} v \\ &= fA^{-1}\mathrm{d}(Af^{-1}v) + fA^{-1}\omega_e A f^{-1} v \\ &= f\mathrm{d}(f^{-1}v) + f(A^{-1}\mathrm{d}A + A^{-1}\omega_e A) f^{-1} v.\end{aligned}$$

Comparing with (3.1.2) under the basis f, we find that

$$\omega_f = A^{-1}\mathrm{d}A + A^{-1}\omega_e A. \tag{3.1.3}$$

Moreover, if (x^i) is a system of local coordinates on X, defining a bsis of Ω^1_X with basis vectors $\mathrm{d}x^i$, we put

$$\omega^\alpha_\beta = \sum_i \Gamma^\alpha_{\beta i} \mathrm{d}x^i$$

and we call the holomorphic functions $\Gamma^{\alpha}_{\beta i}$ the *coefficients of the connection*. The formula (3.1.2) can rewritten as

$$(\nabla_i v)^{\alpha} = \partial_i v^{\alpha} + \sum_{\beta} \Gamma^{\alpha}_{\beta,i} v^{\beta}. \tag{3.1.4}$$

The differential equation $\nabla v = 0$ of horizontal sections of $\mathcal{V}$ can be rewritten as the system of homogeneous linear partial differential equations of first order,

$$\partial_i v^{\alpha} = -\sum_{\beta} \Gamma^{\alpha}_{\beta,i} v^{\beta}. \tag{3.1.5}$$

3.2 With the notations of (3.1.2), using the summation convention of dummy indexes, we have

$$\begin{aligned}
\nabla\nabla v &= \nabla((\mathrm{d}v^{\alpha} + \omega^{\alpha}_{\beta} v^{\beta}).e_{\alpha}) \\
&= \mathrm{d}(\mathrm{d}v^{\alpha} + \omega^{\alpha}_{\beta} v^{\beta}).e_{\alpha} - (\mathrm{d}v^{\alpha} + \omega^{\alpha}_{\beta} v^{\beta}) \wedge \omega^{\gamma}_{\alpha}.e_{\gamma} \\
&= \mathrm{d}\omega^{\alpha}_{\beta}.v^{\beta}.e_{\alpha} - \omega^{\alpha}_{\beta} \wedge \mathrm{d}v^{\beta}.e_{\alpha} - \mathrm{d}v^{\alpha} \wedge \omega^{\gamma}_{\alpha}.e_{\gamma} - \omega^{\alpha}_{\beta} \wedge \omega^{\gamma}_{\alpha}.v^{\beta}.e_{\gamma} \\
&= (\mathrm{d}\omega^{\gamma}_{\beta} - \omega^{\alpha}_{\beta} \wedge \omega^{\gamma}_{\alpha}) v^{\beta}.e_{\gamma}
\end{aligned}$$

The matrix of the curvature tensor is then

$$\mathcal{R}^{\alpha}_{\beta} = \mathrm{d}\omega^{\alpha}_{\beta} + \sum_{\gamma} \omega^{\alpha}_{\gamma} \wedge \omega^{\gamma}_{\beta} \tag{3.2.1}$$

which is also written as

$$\mathcal{R} = \mathrm{d}\omega + \omega \wedge \omega. \tag{3.2.2}$$

The formula (3.2.1) provided, in a system of local coordinates (x^i),

$$\begin{cases} \mathcal{R}^{\alpha}_{\beta,i,j} &= (\partial_i \Gamma^{\alpha}_{\beta,j} - \partial_j \Gamma^{\alpha}_{\beta,i}) + (\Gamma^{\alpha}_{\gamma,i} \Gamma^{\gamma}_{\beta,j} - \Gamma^{\alpha}_{\gamma,j} \Gamma^{\gamma}_{\beta,i}) \\ \mathcal{R}^{\alpha}_{\beta} &= \sum_{i<j} \mathcal{R}^{\alpha}_{\beta,i,j} \mathrm{d}x^i \wedge \mathrm{d}x^j. \end{cases} \tag{3.2.3}$$

The condition $\mathcal{R}^{\alpha}_{\beta,i,j} = 0$ is the condition of integrability of the system (3.1.5) in the classical sense; it can be obtained by eliminating v^{α} from equations obtained by substituting (3.1.5) in the identity $\partial_i \partial_j v^{\alpha} = \partial_j \partial_i v^{\alpha}$.

§ 4 Differential equations of n-th order

4.1 The resolution of a homogeneous linear differential equation of n-th order

(4.1.1)
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}y = \sum_{i=1}^{n} a_i(x)\frac{\mathrm{d}^{n-i}}{\mathrm{d}x^{n-i}}y.$$

is equivalent to that of the system of equations of first order

(4.1.2)
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}y_i = y_{i+1} \quad (1 \leqslant i < n), \\ \frac{\mathrm{d}}{\mathrm{d}x}y_n = \sum_{i=1}^{n} a_i(x)y_{n+1-i}. \end{cases}$$

According to §3, this system can be described as the differential equation of the horizontal sections of a vector bundle of rank n equipped with a suitable connection, and that's what we propose to explain.

4.2 Let X be a nonsingular complex analytic manifold purely of dimension one, X_n the n-th infinitesimal neighborhood of the diagonal of $\mathrm{X} \times \mathrm{X}$ and p_1, p_2 the two projections from X_n to X. We denote by $\Pi_{l,k}$ the injection from X_l to X_k, for $l \leqslant k$.

Let $\Omega_{\mathrm{X}}^{\otimes n}$ be the n-th tensorial power of the invertible sheaf Ω_{X}^1 ($n \in \mathbb{Z}$), hence it is also the n-th symmetric power. If I is the ideal defining the diagonal of $\mathrm{X} \times \mathrm{X}$, we canonically have $\mathrm{I}/\mathrm{I}^2 \cong \Omega_{\mathrm{X}}^1$, and

(4.2.1)
$$\mathrm{I}^n/\mathrm{I}^{n+1} \cong \Omega_{\mathrm{X}}^{\otimes n}.$$

Let $\mathcal{L}$ be an invertible sheaf on X, we denote by $\mathrm{P}^n(\mathcal{L})$ the vector bundle of jets of sections of n-th order on $\mathcal{L}$.

(4.2.2)
$$\mathrm{P}^n(\mathcal{L}) = p_{1*}p_2^*\mathcal{L}.$$

The I-adic filtration of $\mathrm{P}_2^*\mathcal{L}$ defines a filtration of $\mathrm{P}^n(\mathcal{L})$

(4.2.3)
$$\mathrm{Gr}^i\,\mathrm{P}^n(\mathcal{L}) \cong \mathrm{Gr}^i\,\mathrm{P}^n(\mathcal{O}) \otimes \mathcal{L} \cong \Omega_{\mathrm{X}}^{\otimes i} \otimes \mathcal{L} \quad (0 \leqslant i \leqslant n).$$

Recall that we define by induction on n the *differential operator of order $\leqslant n$*: $\mathrm{A}\colon \mathcal{M} \to \mathcal{N}$ as being a moprhism of abelian sheaves satisfying

- if $n = 0$: A is $\mathcal{O}$-linear;
- if $n = m+1$: for any local section f of $\mathcal{O}$, $[\mathrm{A}, f]$ is of order $\leqslant m$.

For each local section s of $\mathcal{L}$, p_2^*s defines a local section $\mathrm{D}^n(s)$ of $\mathrm{P}^n(\mathcal{L})$ (4.2.2). The $\mathbb{C}$-linear morphism of sheaves $\mathrm{D}^n\colon \mathcal{L} \to \mathrm{P}^n(\mathcal{L})$ is the universal differential operator of order $\leqslant n$ from $\mathcal{L}$.

4.3 Definition *(i)* A *homogeneous linear differential equation of n-th order* on $\mathcal{L}$ is an $\mathcal{O}_{\mathrm{X}}$-linear homomorphism $\mathrm{E}\colon \mathrm{P}^n(\mathcal{L}) \to \Omega_{\mathrm{X}}^{\otimes n} \otimes \mathcal{L}$ which induces the identity on the submodule $\Omega_{\mathrm{X}}^{\otimes n} \otimes \mathcal{L}$ of $\mathrm{P}^n(\mathcal{L})$.

(ii) A local section s of $\mathcal{L}$ is a *solution* of the differential equation E if $\mathrm{E}(\mathrm{D}^n(s)) = 0$.

In fact, I cheated in this definition, in that I only consider the equations that are put in the form "resolute" (4.1.1).

4.4 Suppose that $\mathcal{L} = \mathcal{O}$ and that x is a local coordinate on X. The choice of x permits to identify $\mathrm{P}^k(\mathcal{O})$ with $\mathcal{O}^{[0,k]}$, the arrow D^k becoming

$$\mathrm{D}^k\colon \mathcal{O} \longrightarrow \mathrm{P}^k(\mathcal{O}) \cong \mathcal{O}^{[0,k]}\colon f \mapsto (\partial_x^i f)_{0\leqslant i\leqslant k}.$$

The choice of x also permits to identify Ω^1 and $\mathcal{O}$, so that a differential equation of order n is identified with a morphism $\mathrm{E} \in \mathrm{Hom}(\mathcal{O}^{[0,k]}, \mathcal{O})$, and such has coordinates $(b_i)_{0\leqslant i\leqslant n}$ with $b_n = 1$. The solution of E is then the (holomorphic) function f satisfying

$$\sum_{i=0}^{n} b_i(x)\partial_x^i f = 0 \quad (b_n = 1). \tag{4.4.1}$$

The theorem of existence and uniqueness of solutions to the Cauchy problem for (4.4.1) indicates that

4.5 Theorem (Cauchy) *Let* X *and* $\mathcal{L}$ *be as in 4.2, and* E *a differential equation of n-th order on* $\mathcal{L}$*. Then*

(i) The subsheaf of $\mathcal{L}$ *of solutions of* E *is a local system* $\mathcal{L}^{\mathrm{E}}$ *of rank n on* X*.*

(ii) The canonical arrow $\mathrm{D}^{n-1}\colon \mathcal{L}^{\mathrm{E}} \to \mathrm{P}^{n-1}(\mathcal{L})$ *induces an isomorphism*

$$\mathcal{O} \otimes_{\mathbb{C}} \mathcal{L}^{\mathrm{E}} \xrightarrow{\sim} \mathrm{P}^{n-1}(\mathcal{L}).$$

It follows in particular from (ii) and 2.17 that E defines a canonical connection on $\mathrm{P}^{n-1}(\mathcal{L})$, whose horizontal sections are images by D^{n-1} of solutions of E.

4.6 Having a differential equation E on $\mathcal{L}$, we thus associated

a) a holomorphic vector bundle $\mathcal{V}$ with connection (automatically integrable): the bundle $\mathrm{P}^{n-1}(\mathcal{L})$,

b) a surjective homomorphism (4.2.3) $(i = 0)$ $\lambda\colon \mathcal{V} \to \mathcal{L}$.

Moreover, the solutions of E are the images by λ of horizontal sections of $\mathcal{V}$. This is just another way of expressing the passage of 4.1.1 and 4.1.2.

4.7 Let $\mathcal{V}$ be a vector bundle of rank n on X equipped with a connection of covariant derivative ∇. Let v be a local section of $\mathcal{V}$ and w a vector field on X, which does not vanish at any point. We say that v is *cyclic* if the local sections $(\nabla_w)^i(v)$ of $\mathcal{V}$ $(0 \leqslant i < n)$ form a basis of $\mathcal{V}$. This condition doesn't depend on the choice of w, and if f is an invertible holomorphic function, then v is cyclic if and only if fv is cyclic. It is indeed verified by induction on i that $(\nabla_{gw})^i(fv)$ lies in the submodule of $\mathcal{V}$ generated by $(\nabla_w)^j(v)$ $(0 \leqslant j \leqslant i)$.

If $\mathcal{L}$ is an invertible sheaf, we say that a section v of $\mathcal{V} \otimes \mathcal{L}$ is *cyclic* if, for every local isomorphism between $\mathcal{L}$ and $\mathcal{O}$, the corresponding section of $\mathcal{V}$ is cyclic. This applies in particular to a section v of $\mathcal{H}om(\mathcal{V}, \mathcal{L}) = \mathcal{V}^\vee \otimes \mathcal{L}$.

4.8 Lemma *With hypothesis and notations in 4.6, λ is a cyclic section of $\mathcal{H}om(\mathcal{V}, \mathcal{L})$.*

The problem is local on X; we come back to the case where $\mathcal{L} = \mathcal{O}$ and where there exists a local coordinate x.

Using the notations from 4.4, a section (f^i) of $\mathrm{P}^{n-1}(\mathcal{O}) \cong \mathcal{O}^{[0,n-1]}$ is horizontal if and only if it satisfies

$$\begin{cases} \partial_x f^i = f^{i+1} & 0 \leqslant i \leqslant n-2 \\ \partial_x f^{n-1} = -\sum_{i=0}^{n-1} b_i f^i. \end{cases}$$

This gives us the coefficients of the connection: the matrix of connection is

$$\begin{pmatrix} 0 & -1 & & & \\ & 0 & -1 & & \\ & & \ddots & \ddots & \\ & & & 0 & -1 \\ b^0 & b^1 & \cdots & \cdots & b^{n-1} \end{pmatrix} \tag{4.8.1}$$

In the chosen coordinate system, $\lambda = e^0$ and we calculate that

$$\nabla_x^i \lambda = e^i \quad (0 \leqslant i \leqslant n-1)$$

what proves 4.8.

4.9 Proposition *The construction in 4.6 establishes an equivalence of the following categories, when we take for morphisms the isomorphisms:*

a) the category of invertible sheaves on X*, equipped with differential equations of order n;*

b) the category of triples consisting of a vector bundle $\mathcal{V}$ of rank n equipped with a connection, of an invertible sheaf $\mathcal{L}$ and of a cyclic homomorphism $\lambda\colon \mathcal{V} \to \mathcal{L}$.

Construct a quasi-inverse of the functor 4.6. Let $\mathcal{V}$ be a vector bundle with connection, and λ a homomorphism from $\mathcal{V}$ to an invertible sheaf $\mathcal{L}$. We denote by V the local system of horizontal sections of $\mathcal{V}$. For every $\mathcal{O}$-module $\mathcal{M}$, we have (2.17)

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{V},\mathcal{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\mathrm{V},\mathcal{M}).$$

In particular, we define a map γ^k from $\mathcal{V}$ to $\mathrm{P}^k(\mathcal{L})$ by putting, for every horizontal section v of $\mathcal{V}$,

$$\gamma^k(v) = \mathrm{D}^k(\lambda(v)). \tag{4.9.1}$$

4.9.2 Lemma *The homomorphism λ is cyclic if and only if*

$$\gamma^{n-1}\colon \mathcal{V} \longrightarrow \mathrm{P}^{n-1}(\mathcal{L})$$

is an isomorphism.

The problem is local on X. We come back to the case where $\mathcal{L} = \mathcal{O}$ and where we already have a local coordinate x. With the notations of 4.4, the morphism γ^k admits then for coordinates the morphisms $\partial_x^i\lambda = \nabla_x^i\lambda$ $(0 \leqslant i \leqslant k)$. For $k = n-1$, these form a basis of $\mathrm{Hom}(\mathcal{V},\mathcal{O})$ if and only if γ^{n-1} is an isomorphism.

For $k \geqslant l$, the diagram

$$\begin{tikzcd} & \mathcal{V} \arrow[dl, "\gamma^k"'] \arrow[dr, "\gamma^l"] & \\ \mathrm{P}^k(\mathcal{L}) \arrow[rr, "\Pi_{l,k}"] & & \mathrm{P}^l(\mathcal{L}) \end{tikzcd} \tag{4.9.3}$$

is commutative; if $lambda$ is cyclic, we deduce from this fact and 4.9.2 that $\gamma^n(\mathcal{V})$ is locally a direct factor of codimension one on $\mathrm{P}^n(\mathcal{L})$, and admits for supplement $\omega^{\otimes n} \otimes \mathcal{L} \cong \ker \Pi_{n-1,n}$. Then there exists one and only one differential equation of order n on $\mathrm{L}l$:

$$\mathrm{E}\colon \mathrm{P}^n(\mathcal{L}) \longrightarrow \Omega^{\otimes n} \otimes \mathcal{L}$$

such that $\mathrm{E} \circ \gamma^n = 0$.

According to (4.9.1), if v is a horizontal section of $\mathcal{V}$, then $\mathrm{E}\mathrm{D}^n\lambda v = \mathrm{E}\gamma^n v = 0$, so that λv is a solution of E. Endow $\mathrm{P}^{n-1}(\mathcal{L})$ the connection in 4.6 defined by E. If v is a horizontal section of $\mathcal{V}$, then $\gamma^{n-1}v = \mathrm{D}^{n-1}\lambda v$, with λv a solution of E, and thus $\gamma^{n-1}v$ is horizontal. We deduce that γ^{n-1} is compatible with connections. A special case of 4.9.3 shows that the diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\gamma^{n-1}"] \arrow[d, "\lambda"'] & \mathrm{P}^{n-1}(\mathcal{L}) \arrow[d, "\Pi_{0,n-1}"] \\ \mathcal{L} \arrow[r, equal] & \mathcal{L} \end{tikzcd}$$

is commutative, from where an isomorphism between $(\mathcal{V}, \mathcal{L}, \lambda)$ and the triple deduced by 4.6 from $(\mathcal{L}, \mathrm{E})$. The functor

$$(\mathcal{V}, \mathcal{L}, \lambda) \longmapsto (\mathcal{L}, \mathrm{E})$$

is then the quasi-inverse of the functor in 4.6.

4.10 Summarizations of the relation between the two systems $(\mathcal{V}, \mathcal{L}, \lambda)$ and $(\mathcal{L}, \mathrm{E})$ which corresponded by 4.6 and 4.9.

We have homomorphisms $\gamma^k \colon \mathcal{V} \to \mathrm{P}^k(\mathcal{L})$, such that

(4.10.1) For horizontal v, $\gamma^k(v) = \mathrm{D}^k \lambda(v)$.

(4.10.2) We have $\gamma^0 = \lambda$ and $\Pi_{l,k} \circ \gamma^k = \gamma^l$.

(4.10.3) γ^{n-1} is an isomorphism (λ is cyclic).

(4.10.4) $\mathrm{E}\gamma^n = 0$.

(4.10.5) λ induces an isomorphism between the local system V of horizontal sections of $\mathcal{V}$ and the local system $\mathcal{L}^{\mathrm{E}}$ of solutions of E.

§ 5 Differential equations of second order

In this section, we specialize the results of §4 to the case $n = 2$, and express, in a more geometric form, some of the results outlined in R.C.Gunning's [Gun66].

5.1 Let S be an analytic space, and let $q\colon \mathrm{X}_2 \to \mathrm{S}$ be an analytic space over S, locally isomorphic to the finite analytic space over S described by the $\mathcal{O}_\mathrm{S}$-algebra $\mathcal{O}_\mathrm{S}[\mathrm{T}]/(\mathrm{T}^3)$.

The fact that the group PGL_2 acts strict-transitively three times on $\mathbb{P}^1$ have the following infinitesimal analogy.

5.2 Lemma *Under the hypothesis of 5.1, let u and v be two* S*-immersions from* X_2 *to* $\mathbb{P}^1_\mathrm{S}$*:*

$$\mathrm{X}_2 \overset{u}{\underset{v}{\rightrightarrows}} \mathbb{P}^1_\mathrm{S}$$

There exists one and only one projective (= S*-automorphism) of* $\mathbb{P}^1_\mathrm{S}$ *which transforms u and v.*

The problem is local n S, which permits to suppose X_2 is defined by the $\mathcal{O}_\mathrm{S}$-algebra $\mathcal{O}_\mathrm{S}[\mathrm{T}]/(\mathrm{T}^3)$, and that $u(\mathrm{X}_2)$ and $v(\mathrm{X}_2)$ are contained in the same affine line, saying $\mathbb{A}^1_\mathrm{S}$. By translation, we may assume that $u(0) = v(0) = 0$. Giving u amounts to choosing $f(\mathrm{T}) \in \mathcal{O}_\mathrm{S}[\mathrm{T}]/(\mathrm{T}^3)$. The condition $u(0) = 0$ implies that f has zero constant term, so the choice really boils down to $f'(0)$ and $f''(0)$. Similarly for v, choosing $g(\mathrm{T})$. Now one such choice is clearly, $h(\mathrm{T}) = \mathrm{T}(\operatorname{mod}\mathrm{T}^3)$. Let $w\colon \mathrm{X}_2 \to \mathbb{A}^1_\mathrm{S}$ denote the corresponding map. Once we find the unique projective automorphisms mapping w to u and w to v, the unique projective automorphism mapping u to v follows by composition. Therefore, we reduce to the case u is given by $f(\mathrm{T}) = \mathrm{T}(\operatorname{mod}\mathrm{T}^3)$. Then such a projective $p(x)$ should satisfy that $v = p \circ u$, i.e. $f(p^*(\mathrm{T})) = g(\mathrm{T})$. Suppose $g(\mathrm{T}) = e\mathrm{T} + f\mathrm{T}^2(\operatorname{mod}\mathrm{T}^3)$, the condition is

$$p(x) \equiv ax + bx^2 \qquad (\operatorname{mod} x^3).$$

We must then verify the existence and uniqueness of such a projective. Since $p(0) = 0$, and p is written in a unique way in the form

$$\begin{aligned} p(x) &= \frac{cx}{1 - dx} \qquad (c \neq 0) \\ &= cx + cdx^2 \qquad (\operatorname{mod} x^3). \end{aligned}$$

The assertion follows immediately.

5.3 According to 5.2, there exists, up to a unique isomorphism, one and only one couple (u, P) consisting of a projective line P on S (with group structure $\mathrm{PGL}_2(\mathcal{O}_\mathrm{S})$) and an S-immersion u from X_2 to P. We call P the *osculating projective line* at X_2.

Let X be a smooth curve, X_2 the second infinitesimal neighborhood of the diagonal of $X \times X$ and q_1, q_2 the two projections from X_2 to X.

The morphism $q_1 : X_2 \to X$ is of the type considered in 5.1.

5.4 Definition We call the projective line bundle on X osculating to $q_1 : X_2 \to X$ the *projective line bundle osculating to* X and denote by P_{tg}.

By definition, we already have a canonical commutative diagram

$$
\begin{tikzcd}
X_2 \arrow[r, hook] \arrow[dr, "q_1"'] & P_{tg} \arrow[d] \\
& X.
\end{tikzcd}
\tag{5.4.1}
$$

In particular P_{tg} is equipped with a canonical section e, image of the diagonal section of X_2, and we have

$$
e^{-1}\Omega^1_{P_{tg}/X} \cong \Omega^1_X. \tag{5.4.2}
$$

5.5 If X is a projective line, then $\mathrm{pr}_1 : X \times X \to X$ is a projective bundle on X, so that P_{tg} identifies with the constant projective bundle of fiber X on X, provided with the inclusion from X_2 to $X \times X$

$$
\begin{tikzcd}
X_2 \arrow[r, hook] \arrow[dr, "q_1"'] & X \times X \arrow[d, "\mathrm{pr}_1"] \\
& X.
\end{tikzcd}
$$

In this particular case, we have a canonical commutative diagram

$$
\begin{tikzcd}
X_3 \arrow[dr, hook] \arrow[ddr] & \\
X_2 \arrow[u, hook] \arrow[r, hook] \arrow[dr, "q_1"'] & P_{tg} \arrow[d] \\
& X.
\end{tikzcd}
$$

Let X be a smooth curve again.

5.6 Definition (local form) A *projective connection* on X is a sheaf on X of germs of local isomorphisms from X to $\mathbb{P}^1$, that is a homogeneous principal sheaf (=torsor) under the constant sheaf of groups with value $\mathrm{PGL}_2(\mathbb{C})$.

If X is provided a projective connection, then every local construction on $\mathbb{P}^1$, invariant under the projective group, can be transported to X; in particular, the construction 5.5 provides us a morphism γ inserted in a commutative diagram

$$
\begin{tikzcd}
\mathrm{X}_3 \arrow[dr, hook, "\gamma"] \arrow[ddr] & \\
\mathrm{X}_2 \arrow[u, hook] \arrow[r, hook] \arrow[dr, "q_1"'] & \mathrm{P}_{tg} \arrow[d] \\
& \mathrm{X}.
\end{tikzcd}
\tag{5.6.1}
$$

It is not difficult to verify that such a morphism γ is defined by one and only one projective connection (a demonstration will be given in 5.10), so that the definition 5.6 is equivalent to the following.

5.6 Definition (infinitesimal form) A *projective connection* on X is a morphism $\gamma\colon \mathrm{X}_3 \to \mathrm{P}_{tg}$ making the diagram (5.6.1) commutative.

Intuitively, giving a projective connection (infinitesimal form) permits to define the birapport (= anharmonic ratio) of 4 infinitesimally near points one thus has to define the blrapport of 4 neighboring points (local form).

5.7 Put $\Omega^{\otimes n} = (\Omega^1_{\mathrm{X}})^{\otimes n}$. The ideal sheaf on X_3 defining X_2 is canonically isomorphic to Ω^3 and is killed by the ideal sheaf defining the diagonal. According to this, if Δ is the diagonal map, we have (5.4.2)

$$\Delta^{-1}\gamma^{-1}\Omega^1_{\mathrm{P}_{tg}/\mathrm{X}} \cong \Omega^1.$$

§ 6 Multivalued functions of finite determination

6.1 Let X be a non-empty connected, locally path-connected and locally simply connected topological space, and let x_0 be a point of X. We denoted by $\widetilde{X}_{x_0}$ the universal covering of (X, x_0) and by $\widetilde{x_0}$ the base point of $\widetilde{X}_{x_0}$.

If $\mathcal{F}$ is a sheaf on X, we put

6.2 Definition A *multivalued section* of $\mathcal{F}$ on X is a global section of the pullback $\pi^{-1}\mathcal{F}$ of $\mathcal{F}$ on $\widetilde{X}_{x_0}$.

If s is a multivalued section of $\mathcal{F}$ on X, a *determination* of s at a point x of X is an element of the stalk $\mathcal{F}_{(x)}$ of $\mathcal{F}$ at x defined by pullback of s by a local section of π at x. Each point of $\pi^{-1}(x)$ defines such a determination of s at x. The *determination of base* of s at x_0 is the determination defined by $\widetilde{x_0}$. The *determination* of s on an open U of X is a section of $\mathcal{F}$ on U whose germ at each point is the determination of s at that point.

6.3 Definition We say that $\mathcal{F}$ satisfies the *principle of analytic continuation* if the coincidence place of two local sections of $\mathcal{F}$ is always (open and) closed.

6.4 Example If $\mathcal{F}$ is a coherent analytic sheaf on a complex analytic space, $\mathcal{F}$ satisfies the principle of analytic continuation if and only if $\mathcal{F}$ is without immerged components.

Let X be the open unit disk $\{|z| < 1\}$ and $\mathcal{F} = \mathcal{O}_X/(z)$. Then the sections of $\mathcal{F}$, $u = 0$ and $v =$ anything not vanishing at 0 violate the principle of analytic continuation since the locus where they coincide is the punctured disc, not closed in X.

6.5 Proposition *Let* X *and* x_0 *as in 6.1 and* $\mathcal{F}$ *a sheaf of* $\mathbb{C}$*-vector spaces on* X *satisfying the principal of analytic continuation. For every multivalued section* s *of* $\mathcal{F}$*, the following conditions are equivalent.*

- *(i) The determinations of* s *at* x_0 *generate a finite dimensional subspace of* $\mathcal{F}_{x_0}$.
- *(ii) The subsheaf of* $\mathbb{C}$*-vector spaces of* $\mathcal{F}$ *generated by the determinations of* s *is a complex local system.*

(ii) $\Rightarrow$ (i) is trivial. Let's prove (i) $\Rightarrow$ (ii). Let x be a point of X at which the determinations of s generate a finite dimensional subspace of $\mathcal{F}_x$ and let U be a connected open neighborhood of x above which $\widetilde{X}_{x_0}$ is trivial: $(\pi^{-1}(U), \pi) \cong (U \times I, \mathrm{pr}_1)$ for some suitable set I. This implies on U that the determinations of s generate a complex local system: each $i \in I$ defines a determination s_i of s and on U the subsheaf of vector spaces of $\mathcal{F}$ generated by the determinations of s is generated by $(s_i)_{i \in I}$; if this sheaf is constant, the hypothesis on x implies that it is a complex local system. We have

6.6 Lemma *If a sheaf of* $\mathbb{C}$*-vector spaces on* $\mathcal{F}$ *on a connected space satisfies the principal of analytic continuation, then the subsheaf of vector spaces of* $\mathcal{F}$ *generated by a family of global sections* s_i *is a constant sheaf.*

The sections s_i define

$$a\colon \underline{\mathbb{C}}^{(\mathrm{I})} \longrightarrow \mathcal{F}$$

with image the subsheaf of vector spaces $\mathcal{G}$ of $\mathcal{F}$ generated by s_i. If a relation $\sum \lambda_i s_i = 0$ is satisfied at a point, it is true everywhere by the principal of analytic continuation.

The sheaf $\ker(a)$ is then a constant subsheaf of $\underline{\mathbb{C}}^{(\mathrm{I})}$ and the assertion follows.

We conclude the demonstration of 6.5 by note that, according to the above, the largest open of X on which the determinations of s generate a local system is closed and contains x_0.

6.7 Definition Under the hypothesis of 6.5, a multivalued section s of $\mathcal{F}$ is *of finite determination* if it satisfies the equivalent conditions in 6.5.

6.8 Under the hypothesis of 6.5, let s be a multivalued section of finite determination of $\mathcal{F}$. This section defines

a) the local system V generated by its determinations;

b) a germ of sections of V at x_0, saying v_0, corresponding to the determination of base of s;

c) an inclusion $\lambda\colon \mathrm{V} \to \mathcal{F}$.

The triple form of V_{x_0}, of v_0 and of the representation of $\pi_1(\mathrm{X}, x_0)$ on V_{x_0} defined by V (1.4) is called the *monodromy* of s. The triple $(\mathrm{V}, v_0, \lambda)$ satisfies the following two conditions.

6.8.1 v_0 is a cyclic vector of the $\pi_1(\mathrm{X}, x_0)$-module V_{x_0}, i.e. generates the $\pi_1(\mathrm{X}, x_0)$-module V_{x_0}.

This simply means that V is generated by the set of determinations of the unique multivalued section of V having determination of base v_0.

6.8.2

$$\lambda\colon \mathrm{V}_{x_0} \longrightarrow \mathcal{F}_{x_0}$$

is injective.

6.9 Let W_0 be a finite dimensional complex representation of $\pi_1(\mathrm{X}, x_0)$ equipped with a cyclic vector w_0. The multivalued section s of $\mathcal{F}$ is said of *subordinate monodromy* at (W_0, w_0) if it is of finite determination and if, with the notations in 6.8, there exists a homomorphism of $\pi_1(\mathrm{X}, x_0)$-representations from W_0 to V_{x_0} sending w_0 to v_0. Let W be the local system defined by W_0, and w the unique multivalued section of W of determination of base w_0. It is clear that, under the hypothesis of 6.5, we have

6.10 Proposition *The function* $\lambda \mapsto \lambda(w)$ *is a bijection between* $\mathrm{Hom}_{\mathbb{C}}(\mathrm{W}, \mathcal{F})$ *and the set of multivalued sections of* $\mathcal{F}$ *of subordinate monodromy at* (W_0, w_0).

6.11 Corollary *Let* X *be a connected reduced complex analytic space equipped with a base point* x_0, W_0 *a finite dimensional complex representation of* $\pi_1(\mathrm{X}, x_0)$ *equipped with a cyclic vector* w_0, W *the local system defined by* W_0, $\mathcal{W} = \mathcal{O} \otimes_{\mathbb{C}} \mathrm{W}$ *the associated vector bundle,* w *the unique multivalued section of* $\mathcal{W}$ *of determination of base* w_0, *and* $\mathcal{W}^\vee$ *the dual vector bundle of* $\mathcal{W}$. *The function*

$$\lambda \longmapsto \langle \lambda, w \rangle$$

from $\Gamma(\mathrm{X}, \mathcal{W}^\vee)$ *to the set of multivalued holomorphic functions on* X *of subordinate monodromy at* (W_0, w_0), *is a bijection.*

6.12 Corollary *If* X *is Stein, there exist multivalued holomorphic functions on* X *having any monodromy* (W_0, w_0) *given in advance.*

If X is Stein, then any coherent sheaf on it is generated by global sections. Then the bijection gives the desired data.

References

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