# ÉLÉMENTS DE GÉOMÉTRIE ALGÉBRIQUE

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CHAPTER IV

# LOCAL STUDY OF SCHEMES AND MORPHISMS OF SCHEMES

Trasnlated (by *Syü Gau*) from the **I.H.É.S.** version

## § 16 Differential invariants; Differentially smooth morphisms

In this section, we present, in a global form, some notions of differential calculus particularly useful in algebraic geometry. We ignore many developments, classic in differential geometry (connections, infinitesimal transformations associated with a vector field, jets, etc.), although these concepts are written in a particularly natural way in the framework of schemes. We also overlook here the special phenomena with the characteristic $p > 0$ (some of which are studied, in the affine context, in (**0**, 21)). For some complements on the differential formalism in the schemes, the reader will be able to consult the presentations II and VII of [42], as well as the later chapters of thiTreaty.

### 16.1 Normal invariants of an immersion

**16.1.1** Let $(\mathrm{X},\mathscr{O}_\mathrm{X}),(\mathrm{Y},\mathscr{O}_\mathrm{Y})$ be two ringed spaces, $f = (\psi,\theta)\colon \mathrm{Y} \to \mathrm{X}$ a morphism of ringed spaces such that the homomorphism

$$\theta^\sharp\colon \psi^*(\mathscr{O}_\mathrm{X}) \longrightarrow \mathscr{O}_\mathrm{Y}$$

is surjective, so that $\mathscr{O}_\mathrm{Y}$ is identified with a quotient $\psi^*(\mathscr{O}_\mathrm{X})/\mathscr{J}_f$. We can then provide $\psi^*(\mathscr{O}_\mathrm{X})$ with $\mathscr{J}_f$-adic filtration.

**16.1.2** The $\mathscr{O}_\mathrm{Y}$-augmented sheaf of rings $\psi^*(\mathscr{O}_\mathrm{X})/\mathscr{J}_f^{n+1}$ is called the $n$-th ***normal invariant*** of $f$; the ringed space $(\mathrm{Y},\psi^*(\mathscr{O}_\mathrm{X})/\mathscr{J}_f^{n+1})$ is called the $n$-th ***infinitesimal neighborhood*** of Y with respect to the morphism $f$, denoted by $Y_f^{(n)}$ or simply $Y^{(n)}$. The sheaf of graded rings associated to the sheaf of filtered rings $\psi^*(\mathscr{O}_X)$

$$\mathscr{G}r_\bullet(f) = \bigoplus_{n\geqslant 0}(\mathscr{J}_f^n/\mathscr{J}_f^{n+1}) \tag{16.1.2.1}$$

is called the ***sheaf of graded rings associated to*** $f$. The sheaf of rings $\mathscr{G}r_1(f) = \mathscr{J}_f/\mathscr{J}_f^2$ is called the ***conormal sheaf*** of $f$ (also denoted by $\mathscr{N}_{Y/X}$ if no confusion).

It is clear that $\mathscr{O}_{Y^{(n)}} = \psi^*(\mathscr{O}_X)/\mathscr{J}_f^{n+1}$ (also denoted by $\mathscr{O}_{Y_f^{(n)}}$) form a projective system of sheaves of rings on Y, that the transition homomorphism $\varphi_{nm}\colon \mathscr{O}_{Y^{(m)}} \to \mathscr{O}_{Y^{(n)}}$ for $n \leqslant m$ identifies $\mathscr{O}_{Y^{(n)}}$ to a quotient of $\mathscr{O}_{Y^{(m)}}$ by the power $(\mathscr{J}_f/\mathscr{J}_f^{n+1})^m$ of the ideal of augmentation of $\mathscr{O}_{Y^{(n)}}$, a.e. kernel of $\varphi_{0n}\colon \mathscr{O}_{Y^{(n)}} \to \mathscr{O}_Y$. The $Y^{(n)}$ therefore form an inductive system of ringed spaces, all of which have the underlying space Y and we have canonical morphisms of ringed spaces $h_n\colon Y^{(n)} \to X$ equal to $(\psi, \theta_n)$, where $\theta_n$ is the canonical morphism $\psi^*(\mathscr{O}_X) \to \psi^*(\mathscr{O}_X)/\mathscr{J}_f^{n+1}$. It is clear that the sheaf $\mathscr{G}r_\bullet(f)$ is a sheaf of graded algebra over the sheaf of rings $\mathscr{O}_Y = \mathscr{G}r_0(f)$, and $\mathscr{G}r_k(f)$ are $\mathscr{O}_Y$-modules.

Like any sheaf of filtered rings, we have a canonical surjective homomorphism of graded $\mathscr{O}_Y$-algebras

$$\mathbb{S}^\bullet_{\mathscr{O}_Y}(\mathscr{G}r_1(f)) \longrightarrow \mathscr{G}r_\bullet(f) \tag{16.1.2.2}$$

coinciding in degree 0 and 1 with identical homomorphisms.

**Example 16.1.3** (i) Assume that X is a locally ringed space, that Y is reduced to a single point $y$ (provided with a ring $\mathscr{O}_y$) and that, if $x = \psi(y)$, $\theta^\sharp\colon \mathscr{O}_x \to \mathscr{O}_y$ is a surjective homomorphism of rings having kernel the maximal ideal $\mathfrak{m}_x$ of $\mathscr{O}_x$. Then $\mathscr{O}_{Y^{(n)}}$ are identified with the rings $\mathscr{O}_x/\mathfrak{m}_x^{n+1}$ and $\mathscr{G}r_\bullet(f)$ with the graded ring associated to the local ring $\mathscr{O}_x$ provided with its $\mathfrak{m}_x$-adic filtration.

(ii) Suppose that Y is a closed subset of an open subspace U of X and that $\mathscr{O}_Y$ is induced on Y by a quotient sheaf $\mathscr{O}_U/\mathscr{J}$, where $\mathscr{J}$ is an Ideal of $\mathscr{O}_U$ such that $\mathscr{J}_x = \mathscr{O}_x$, for all $x \notin Y$; if X is a locally ringed space, we will suppose further that $\mathscr{J}_x \neq \mathscr{O}_x$, for $x \in Y$, so that $(Y, \mathscr{O}_Y)$ is still a locally ringed space.

Let $\psi_0\colon Y \to U$ be the canonical injection, and let $\theta_0\colon \mathscr{O}_U \to (\psi_0)_*(\mathscr{O}_Y)$ be a homomorphism such that $\theta_0^\sharp$ is the canonical homomorphism $\psi_0^*(\mathscr{O}_U) = \mathscr{O}_U|_Y \to (\mathscr{O}_U/\mathscr{J})|_Y$ so that $j_0 = (\psi_0, \theta_0)\colon Y \to U$ is a morphism of ringed spaces (and locally ringed space if X is a locally ringed space); if $i\colon U \to X$ is the canonical injection (morphism of ringed spaces), $j = i \circ j_0$ is the morphism $(\psi, \theta)$ from Y to X, where $\psi\colon Y \to X$ is the canonical injection, and $\theta\colon \mathscr{O}_X \to \psi_*(\mathscr{O}_Y)$ is a homomorphism such that $\theta^\sharp = \theta_0^\sharp$. Since $\theta^\sharp$ is surjective, we can apply the previous definitions; $\mathscr{O}_{Y^{(n)}}$ is equal to $\psi_0^*(\mathscr{O}_U/\mathscr{J}^{n+1})$, and we have $(\psi_0)_*(\mathscr{O}_{Y^{(n)}}) = \mathscr{O}_U/\mathscr{J}^{n+1}$, and $\mathscr{G}r_n(j) = \mathscr{G}r_n(j_0) = \psi_0^*(\mathscr{J}^n/\mathscr{J}^{n+1}) = j_0^*(\mathscr{J}^n/\mathscr{J}^{n+1})$.

**Remark 16.1.4** Example (16.1.3, (ii)) shows that, in general, $\mathscr{O}_{Y^{(n)}}$ are not canonically equipped with a structure of $\mathscr{O}_Y$-module, nor a fortiori with a structure of $\mathscr{O}_Y$-algebra. The data of such a structure is equivalent to that of a homomorphism of sheaves of rings $\lambda_n\colon \mathscr{O}_Y \to \mathscr{O}_{Y^{(n)}}$, right inverse to the homomorphism of augmentation $\varphi_{0n}$; it also comes down to the data of a morphism of ringed spaces $(1_Y, \lambda_n)\colon Y^{(n)} \to Y$, left inverse to the canonical morphism $(1_Y, \varphi_{0n})\colon Y \to Y^{(n)}$.

**Proposition 16.1.5** *Let $f = (\psi, \theta)\colon Y \to X$ be a immersion of schemes. Then*

*(i) $\mathscr{G}r_\bullet(f)$ is a quasi-coherent graded $\mathscr{O}_Y$-algebra.*

*(ii) The $Y^{(n)}$ are schemes, canonically isomorphic to subschemes of X.*

*(iii) Any homomorphism of sheaves of rings $\lambda_n\colon \mathscr{O}_Y \to \mathscr{O}_{Y^{(n)}}$, right inverse to the homomorphism of augmentation $\varphi_{0n}$, makes $\mathscr{O}_{Y^{(n)}}$ and $\mathscr{O}_{Y^{(k)}}$ for $k \leqslant n$, quasi-coherent $\mathscr{O}_Y$-algebras; the $\mathscr{O}_Y$-module structures deduced from the previous structures on the $\mathscr{G}r_k(f)$ for $k \leqslant n$ coincide with those defined in (16.1.2).*

Proof: Since the question is local on X and on Y, we can confine ourselves to the case where Y is a closed subscheme of X defined by a quasi-coherent Ideal $\mathscr{J}$ of $\mathscr{O}_X$; since $\mathscr{O}_Y$ is the restriction to Y of $\mathscr{O}_X/\mathscr{J}$, the assertion (i) is obvious, and $Y^{(n)}$ is the closed subscheme of X defined by the quasi-coherent Ideal $\mathscr{J}^{n+1}$ of $\mathscr{O}_X$. Finally, to prove (ii), let us note that the data of $\lambda_n$ makes the ideal $\mathscr{J}/\mathscr{J}^{n+1}$ of the augmentation $\varphi_{0n}$ and of its quotients $\mathscr{J}/\mathscr{J}^{k+1}$ being $\mathscr{O}_Y$-modules, and it suffices to prove by induction on $k$ that $\mathscr{J}/\mathscr{J}^{k+1}$ are quasi-coherent $\mathscr{O}_Y$-modules and that the structure of quotient $\mathscr{O}_Y$-module that one deduces on $\mathscr{J}^k/\mathscr{J}^{k+1}$ is the same as that defined in (16.1.2). The second assertion is immediate, $\mathscr{J}^k/\mathscr{J}^{k+1}$ being annulled by $\mathscr{J}/\mathscr{J}^{n+1}$; the first results, by induction on $k$, that it is trivial for $k = 1$ and that $\mathscr{J}/\mathscr{J}^{k+1}$ is an extension of $\mathscr{J}/\mathscr{J}^k$ by $\mathscr{J}^k/\mathscr{J}^{k+1}$ (**III**,1.4.17). □

**Corollary 16.1.6** *Under the general hypotheses of (16.1.5), if the immersion $f$ is locally of finite presentation, the $\mathscr{G}r_n(f)$ are quasi-coherent $\mathscr{O}_Y$-modules of finite type.*

Proof: Indeed, with the notation of the proof of (16.1.5), $\mathscr{J}$ is a finite-type ideal of $\mathscr{O}_X$ (1.4.7), so the $\mathscr{J}^n/\mathscr{J}^{n+1}$ are $\mathscr{O}_Y$-modules of finite type hence the conclusion. □

**Corollary 16.1.7** *Under the general assumptions of (16.1.5), let* $g\colon \mathrm{X} \to \mathrm{Y}$ *be a scheme morphism left inverse to* $f$*. Then, for all* $n$*, the composite morphism* $(1, \lambda_n)\colon \mathrm{Y}^{(n)} \xrightarrow{h_n} \mathrm{X} \xrightarrow{g} \mathrm{Y}$ *defines a homomorphism of sheaves of rings* $\lambda_n\colon \mathscr{O}_{\mathrm{Y}} \to \mathscr{O}_{\mathrm{Y}^{(n)}}$ *right inverse to the augmentation* $\varphi_{0n}$*, making* $\mathscr{O}_{\mathrm{Y}^{(n)}}$ *a quasi-coherent* $\mathscr{O}_{\mathrm{Y}}$*-algebra; for these homomorphisms, the transition homomorphisms* $\varphi_{nm}\colon \mathscr{O}_{\mathrm{Y}^{(m)}} \to \mathscr{O}_{\mathrm{Y}^{(n)}} (n \leqslant m)$ *are homomorphisms of* $\mathscr{O}_{\mathrm{Y}}$*-algebras. Moreover, if* $g$ *is locally of finite type, then* $\mathscr{O}_{\mathrm{Y}^{(n)}}$ *are quasi-coherent* $\mathscr{O}_{\mathrm{Y}}$*-modules of finite type.*

Proof: The first assertion follows immediately from the definitions and from (16.1.5). On the other hand, if $g$ is locally of finite type, then $f$ is locally of finite presentation (1.4.3, (v)); then $\mathscr{G}r_n(f)$ being quasi-coherent $\mathscr{O}_{\mathrm{Y}}$-modules of finite type by (16.1.6), it is the same as the $\mathscr{O}_{\mathrm{Y}}$-modules $\mathscr{J}/\mathscr{J}^{n+1}$, which are extensions of a finite number of $\mathscr{G}r_k(f)$ (**III**,1.4.17). □

**Proposition 16.1.8** *Let* X *be a locally noetherian scheme,* $j\colon \mathrm{Y} \to \mathrm{X}$ *an immersion. Then* $\mathrm{Y}^{(n)}$ *are locally noetherian schemes, the* $\mathscr{G}r_n(j)$ *are coherent* $\mathscr{O}_{\mathrm{Y}}$*-modules, and* $\mathscr{G}r_\bullet(j)$ *is a coherent sheaf of rings on the space* Y*.*

Proof: Since everything is local on X and Y, we are reduced to the case where X is affine and $j$ to a closed immersion, and then all the assertions are obvious except the last one, which results from that if A is a noetherian ring and $\mathfrak{J}$ an ideal of A, $\mathrm{gr}^\bullet_{\mathfrak{J}}(\mathrm{A})$ is a noetherian ring, considering the exactness of the functor $\psi^*$ and $(\mathbf{0_I},5.3.7)$. □

**Proposition 16.1.9** *Let* X *be a scheme,* $j\colon \mathrm{Y} \to \mathrm{X}$ *an immersion locally of finite presentation,* $y$ *a point of* Y*. The following conditions are equivalent:*

*(a) There exists an open neighborhood* U *of* $y$ *in* Y *such that* $j|_{\mathrm{U}}$ *is a homeomorphism from* U *onto an open set of* X*.*

*(b) There exists an integer* $n > 0$ *such that the canonical homomorphism*

$$(\varphi_{n-1,n})_y\colon \mathscr{O}_{\mathrm{Y}^{(n)},y} \longrightarrow \mathscr{O}_{\mathrm{Y}^{(n-1)},y}$$

*is bijective.*

*(c) There exists an integer* $n > 0$ *such that* $(\mathscr{G}r_n(j))_y = 0$*.*

*Moreover, if the integer* $n$ *satisfies b) or c), there exists a neighborhood* V *of* $y$ *in* Y *such that* $\mathscr{G}r_m(j)|_{\mathrm{V}} = 0$ *for* $m \geqslant n$ *and that* $\varphi_{nm}\big|_{\mathrm{V}}\colon \mathscr{O}_{\mathrm{Y}^{(m)}}|_{\mathrm{V}} \to \mathscr{O}_{\mathrm{Y}^{(n)}}|_{\mathrm{V}}$ *is bijective for* $m \geqslant n$*.*

Proof: Since the question is local on Y, we can confine ourselves to the case where $j$ is a closed immersion, Y being defined by a quasi-coherent ideal of finite type $\mathscr{J}$ of $\mathscr{O}_{\mathrm{X}}$. The equivalence of b) and c), for a given $n$, is then immediate; moreover, since $\mathscr{J}^n/\mathscr{J}^{n+1}$ is a $\mathscr{O}_X$-module of finite type, there exists an open neighborhood U of $y$ in X such that $\mathscr{J}^n|_U = \mathscr{J}^{n+1}\big|_U$ ($\mathbf{0_I}$,5.2.2), so also $\mathscr{J}^n|_U = \mathscr{J}^m|_U$ for $m \geqslant n$, which proves the last assertions. To prove that a) implies b), we can confine ourselves to the case where the space underlying Y is equal to the space underlying X, and where $\mathscr{J}$ is generated by a finite number of its sections above X: as $\mathscr{J}$ is then contained in the nilradical $\mathscr{N}$ of $\mathscr{O}_X$ (I,5.1.2), it is nilpotent, which proves b). Finally, to prove that b) implies a), we can also confine ourselves to the case where $\mathscr{J}^n = \mathscr{J}^{n+1}$; then, for all $y \in Y$, as $\mathscr{J}_y \subset \mathfrak{m}_y$, maximal ideal of $\mathscr{O}_{X,y}$, we necessarily have $\mathscr{J}_y^n = 0$ by virtue of Nakayama's lemma, since $\mathscr{J}_y$ is an ideal of finite type. The set of $x \in X$ such that $\mathscr{J}_x = 0$ is therefore an open set U of X containing Y ($\mathbf{0_I}$,5.2.2); as on the other hand $\mathscr{J}_x \neq 0$ for $x \notin Y$, we necessarily have $U = Y$. □

**Corollary 16.1.10** *For the restriction of the immersion $j$ to a neighborhood of $y$ in* Y *to be an open immersion (in other words, to have a* local isomorphism *at the point $y$), it is necessary and sufficient that* $(\mathscr{G}r_1(j))_y = (\mathscr{N}_{Y/X})_y = 0$.

PROOF: The condition is obviously necessary, and the previous reasoning, applied to $n = 1$, proves that it is sufficient. □

**Remark 16.1.11** (i) Under the conditions of the definition (16.1.1), the projective limit of the projective system $(\mathscr{O}_{Y^{(n)}}, \varphi_{nm})$ of sheaves of rings on Y is called the ***normal invariant of infinite order of*** $f$, and sometimes denoted by $\mathscr{O}_{Y^{(\infty)}}$. When X is a locally noetherian scheme, $j\colon Y \to X$ is a closed immersion, Y is then a closed subscheme of X defined by a coherent ideal $\mathscr{J}$, $\mathscr{O}_{Y^{(\infty)}}$ is none other than the formal completion of $\mathscr{O}_X$ along Y (**I**,10.8.4), and Y the formal scheme completion of X along Y (**I**,10.8.5). In any case, we can say that $Y^{(\infty)}$ is the formal neighborhood of Y in X (with respect to the morphism $f$). In the particular case we have just considered, it is thus the inductive limit formal scheme of the infinitesimal neighborhoods of order $n$.

(ii) Note that for a morphism of schemes $f = (\psi, \theta)\colon Y \to X$, it may be that the homomorphism $\theta^\sharp\colon \psi^*(\mathscr{O}_X) \to \mathscr{O}_Y$ is surjective without $f$ being a local immersion, and without $f$ being injective. We have an example taking for Y a sum of schemes $Y_\lambda$ all isomorphic to a $\mathrm{Spec}(\mathscr{O}_x)$, where $x \in X$, and for $f$ the morphism equal to the canonical morphism on each $Y_\lambda$.

## 16.2 Functorial properties of normal invariants

**16.2.1** Let $f = (\psi, \theta)\colon Y \to X, f' = (\psi', \theta')\colon Y' \to X'$ be two morphisms of ringed spaces such that the homomorphisms $\theta^\sharp$ and $\theta'^\sharp$ are surjective; consider a commutative diagram of ringed space morphisms

$$\begin{tikzcd} \mathrm{Y} \arrow[r, "f"] & \mathrm{X} \\ \mathrm{Y}' \arrow[u, "u"] \arrow[r, "f'"'] & \mathrm{X}' \arrow[u, "v"'] \end{tikzcd} \tag{16.2.1.1}$$

Let $u = (\rho, \lambda), v = (\sigma, \mu)$. We have $\rho^*(\psi^*(\mathscr{O}_{\mathrm{X}})) = \psi'^*(\sigma^*(\mathscr{O}_{\mathrm{X}}))$ and we have consequently a commutative diagram of homomorphisms of sheaves of rings on $\mathrm{Y}'$

$$\begin{tikzcd} \rho^*(\psi^*(\mathscr{O}_{\mathrm{X}})) = \psi'^*(\sigma^*(\mathscr{O}_{\mathrm{X}})) \arrow[r, "\psi'^*(\mu^\sharp)"] \arrow[d, "\rho^*(\theta^\sharp)"'] & \psi'^*(\mathscr{O}_{\mathrm{X}'}) \arrow[d, "\theta'^\sharp"] \\ \rho^*(\mathrm{Y}) \arrow[r, "\lambda^\sharp"'] & \mathscr{O}_{\mathrm{Y}'} \end{tikzcd}$$

from which we conclude, if $\mathscr{J}$ and $\mathscr{J}'$ are the kernels of $\theta^\sharp$ and $\theta'^\sharp$, then we have $\psi'^*(\mu^\sharp)(\rho^*(\mathscr{J})) \subset \mathscr{J}'$, considering the exactness of the functor $\rho^*$. We deduce from this that for any integer $n$, $\psi'^*(\mu^\sharp)(\rho^*(\mathscr{J}^n)) \subset \mathscr{J}'^n$, which shows that $\psi'^*(\mu^\sharp)$ defines, by passage to the quotients, a homomorphism of sheaves of rings

$$\nu_n \colon \rho^*(\psi^*(\mathscr{O}_{\mathrm{X}})/\mathscr{J}^n) \longrightarrow \psi'^*(\mathscr{O}_{\mathrm{X}'})/\mathscr{J}'^n \tag{16.2.1.2}$$

and consequently a morphism of ringed spaces $w_n = (\rho, \nu_n) \colon \mathrm{Y}'^{(n)} \to \mathrm{Y}^{(n)}$ (which, for $n = 0$, is none other than $u$). It follows immediately from this definition that the diagrams

$$\begin{tikzcd} \mathrm{Y}^{(n)} \arrow[r, "h_{mn}"] & \mathrm{Y}^{(m)} \arrow[r, "h_m"] & \mathrm{X} \\ \mathrm{Y}'^{(n)} \arrow[u, "w_n"] \arrow[r, "h'_{mn}"'] & \mathrm{Y}'^{(m)} \arrow[u, "w_m"] \arrow[r, "h'_m"'] & \mathrm{X}' \arrow[u, "v"'] \end{tikzcd} \qquad (n \leqslant m)$$

(where the horizontal arrows are the canonical morphisms in (16.1.2)) are commutative.

By passing to quotients from homomorphisms (16.2.1.2), and taking into account the exactness of the functor $\rho^*$, one obtains a di-homomorphism of graded algebras (with respect to the homomorphism $\lambda^\sharp \colon \rho^*(\mathscr{O}_{\mathrm{Y}}) \to \mathscr{O}_{\mathrm{Y}'}$)

$$\mathrm{gr}(u) \colon \rho^*(\mathscr{G}r_\bullet(f)) \longrightarrow \mathscr{G}r_\bullet(f') \tag{16.2.1.3}$$

(or, if you want, a $\rho$-morphism $(\mathbf{0_I}, 3.5.1)$ $\mathscr{G}r_\bullet(f) \to \mathscr{G}r_\bullet(f')$), and in particular a di-homomorphism of conormal sheaves

$$\mathrm{gr}_1(u) \colon \rho^*(\mathscr{G}r_1(f)) \longrightarrow \mathscr{G}r_1(f').$$

It is also immediate that these homomorphisms give rise to a commutative diagram

$$
\begin{tikzcd}
\rho^*(\mathbb{S}^{\bullet}_{\mathscr{O}_Y}(\mathscr{G}r_1(f))) \arrow[r] \arrow[d, "\mathbb{S}(\mathrm{gr}_1(u))"'] & \rho^*(\mathscr{G}r_{\bullet}(f)) \arrow[d, "\mathrm{gr}(u)"] \\
\mathbb{S}^{\bullet}_{\mathscr{O}_{Y'}}(\mathscr{G}r_1(f')) \arrow[r] & \mathscr{G}r_{\bullet}(f')
\end{tikzcd}
\tag{16.2.1.4}
$$

where the horizontal arrows are canonical homomorphisms (16.1.2.2).

Finally, if we have a commutative diagram of morphisms of ringed spaces

$$
\begin{tikzcd}
Y \arrow[r, "f"] & X \\
Y' \arrow[r, "f'"] \arrow[u, "u"] & X' \arrow[u, "v"'] \\
Y'' \arrow[r, "f''"] \arrow[u, "u'"] & X'' \arrow[u, "v'"']
\end{tikzcd}
$$

where $f'' = (\psi'', \theta'')$ such that $\theta''^{\sharp}$ is surjective, and if $w'_n$ and $w''_n$ are defined from $u', v'$ on the one hand, and from $u'' = u \circ u', v'' = v \circ v'$ on the other hand, then we have $w''_n = w_n \circ w'_n$, as follows immediately from the definitions and ($\mathbf{0_I}$,3.5.5); we have similarly $\mathrm{gr}(u'') = \mathrm{gr}(u') \circ \rho'^*(\mathrm{gr}(u))$ if $u' = (\rho', \lambda')$. Therefore we can say that $Y^{(n)}$ *and* $\mathscr{G}r_{\bullet}(f)$ *functorially depend on* $f$.

**Proposition 16.2.2** *With the notations and hypotheses of (16.2.1), suppose further that* $f$, $f'$, $u$ *and* $v$ *are morphisms of schemes. Then:*

*(i) The morphisms* $w_n \colon Y'^{(n)} \to Y^{(n)}$ *are morphisms of schemes.*

*(ii) If* $Y' = Y \times_X X'$, $u$ *and* $f$ *being the canonical projections, and if* $f$ *is an immersion or if* $v$ *is flat, we have* $Y'^{(n)} = Y^{(n)} \times_X X'$.

*(iii) If* $Y' = Y \times_X X'$ *and if* $v$ *is flat (resp. if* $f$ *is an immersion), then the homomorphism*

$$\mathrm{Gr}(u) = \mathrm{gr}(u) \otimes 1 \colon \mathscr{G}r_{\bullet}(f) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'} \longrightarrow \mathscr{G}r_{\bullet}(f')$$

*is bijective (resp. surjective).*

Proof: (i): The hypotheses lead immediately to the fact that for all $y' \in Y'$, $\rho^*_{y'}(\theta^{\sharp}_{\rho(y')})$ is a local homomorphism ($\mathbf{0_I}$,6.6.2), so $w_n$ is a morphism of schemes (**I**,2.2.1).

(ii) and (iii): If $f$ is an immersion, we can confine ourselves to the case where $f$ is a closed immersion, $Y'$ being therefore defined by the quasi-coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$, and $Y'^{(n)}$ by the ideal $\mathscr{J}^{n+1}$; the assertions then result from (**I**,4.4.5).

Suppose secondly that $v$ is flat; we can confine ourselves to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, $\mathrm{Y} = \mathrm{Spec}(\mathrm{B})$, $\mathrm{X}' = \mathrm{Spec}(\mathrm{A}')$ are affine, $\mathrm{A}'$ being a flat A-module; then $\mathrm{Y}' = \mathrm{Spec}(\mathrm{B}')$ with $\mathrm{B}' = \mathrm{B} \otimes_{\mathrm{A}} \mathrm{A}'$; furthermore, if $\mathfrak{J}$ is the kernel of the homomorphism $\mathrm{A} \to \mathrm{B}$, the kernel $\mathfrak{J}'$ of $\mathrm{A}' \to \mathrm{B}'$ identifies with $\mathfrak{J} \otimes_{\mathrm{A}} \mathrm{A}'$ by flatness, and $\mathfrak{J}'^n/\mathfrak{J}'^{n+1} = (\mathfrak{J}^n/\mathfrak{J}^{n+1}) \otimes_{\mathrm{A}} \mathrm{A}'$. We deduce from this, considering ($\mathbf{0_I}$,4.3.3), that the $\mathscr{O}_{\mathrm{Y}'}$-module $\mathscr{J}'^n/\mathscr{J}'^{n+1}$ is equal to

$$\begin{aligned}\psi'^*(\sigma^*((\mathfrak{J}^n/\mathfrak{J}^{n+1})^\sim) &\otimes_{\sigma^*(\mathscr{O}_\mathrm{X})} \mathscr{O}_{\mathrm{X}'}) \\ &= \psi'^*(\sigma^*((\mathfrak{J}^n/\mathfrak{J}^{n+1})^\sim) \otimes_{\psi'^*(\sigma^*(\mathscr{O}_\mathrm{X}))} \psi'^*(\mathscr{O}_{\mathrm{X}'}) \\ &= \rho^*(\mathscr{J}^n/\mathscr{J}^{n+1}) \otimes_{\rho^*(\psi^*(\mathscr{O}_\mathrm{X}))} \psi'^*(\mathscr{O}_{\mathrm{X}'})\end{aligned}$$

and as in particular for $n = 0$, we have

$$\mathscr{O}_{\mathrm{Y}'} = \rho^*(\mathscr{O}_\mathrm{Y}) \otimes_{\rho^*(\psi^*(\mathscr{O}_\mathrm{X}))} \psi'^*(\mathscr{O}_{\mathrm{X}'})$$

we obtain an canonical isomorphism from $\mathscr{J}'^n/\mathscr{J}'^{n+1}$ to

$$\rho^*(\mathscr{J}^n/\mathscr{J}^{n+1}) \otimes_{\rho^*(\mathscr{O}_\mathrm{Y})} \mathscr{O}_{\mathrm{Y}'} = \mathscr{J}^n/\mathscr{J}^{n+1} \otimes_{\mathscr{O}_\mathrm{Y}} \mathscr{O}_{\mathrm{Y}'}$$

which proves (iii). Now let's put $\mathrm{C}_n = \Gamma(\mathrm{Y}, \mathscr{O}_{\mathrm{Y}^{(n)}})$, $\mathrm{C}'_n = \Gamma(\mathrm{Y}', \mathscr{O}_{\mathrm{Y}'^{(n)}})$. Since Y and Y′ are affine schemes (16.1.5), the kernel $\mathfrak{K}_n$ (resp. $\mathfrak{K}'_n$) of $\mathrm{C}_n \to \mathrm{C}_{n-1}$ (resp. $\mathrm{C}'_n \to \mathrm{C}'_{n-1}$) is $\Gamma(\mathrm{Y}, \mathscr{J}^n/\mathscr{J}^{n+1})$ (resp. $\Gamma(\mathrm{Y}', \mathscr{J}'^n/\mathscr{J}'^{n+1})$); it follows that $\mathfrak{K}'_n = \mathfrak{K}_n \otimes_{\mathrm{A}} \mathrm{A}'$. Now, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}_n \otimes_\mathrm{A} \mathrm{A}' \arrow[r] \arrow[d] & \mathrm{C}_n \otimes_\mathrm{A} \mathrm{A}' \arrow[r] \arrow[d, "s_n"'] & \mathrm{C}_{n-1} \otimes_\mathrm{A} \mathrm{A}' \arrow[r] \arrow[d, "s_{n-1}"'] & 0 \\
0 \arrow[r] & \mathfrak{K}'_n \arrow[r] & \mathrm{C}'_n \arrow[r] & \mathrm{C}'_{n-1} \arrow[r] & 0
\end{tikzcd}$$

where the left vertical arrow is bijective and the two sequences are exact (A′ being a flat A-module). From this it is deduced that $s_n$ is bijective for all $n$, since it is hypothetically true for $n = 0$, and is deduced by applying the 5-lemma for any $n$. This proves the second assertion of (ii). □

**Corollary 16.2.3** *Let $g\colon \mathrm{X} \to \mathrm{Y}$, $u\colon \mathrm{Y}' \to \mathrm{Y}$ be two morphisms of schemes, $\mathrm{X}' = \mathrm{X} \times_\mathrm{Y} \mathrm{Y}'$, $g'\colon \mathrm{X}' \to \mathrm{Y}'$ and $v\colon \mathrm{X}' \to \mathrm{X}$ the canonical projections. Let $f\colon \mathrm{Y} \to \mathrm{X}$ be a* Y*-section of* X *(hence an immersion), $f' = f_{(\mathrm{Y}')}\colon \mathrm{Y}' \to \mathrm{X}'$ the* Y′*-section of* X′ *deduced from $f$ by the base change $u$. Then:*

*(i) The morphism $w_n\colon \mathrm{Y}'^{(n)}_{f'} \to \mathrm{Y}^{(n)}_f$ corresponding to $f, f', u, v$ (16.2.1) and the canonical morphism $h'_n\colon \mathrm{Y}'^{(n)}_{f'} \to \mathrm{X}'$ identify $\mathrm{Y}'^{(n)}_{f'}$ to the product $\mathrm{Y}^{(n)}_f \times_\mathrm{X} \mathrm{X}'$.*

*(ii) If we equip $\mathscr{O}_{\mathrm{Y}_f^{(n)}}$ (resp. $\mathscr{O}_{\mathrm{Y}'^{(n)}_{f'}}$) with the structure of $\mathscr{O}_{\mathrm{Y}}$-algebra defined by g (resp. the structure of $\mathscr{O}_{\mathrm{Y}'}$-algebra defined by $g'$) (16.1.6), the homomorphism of $\mathscr{O}_{\mathrm{Y}'}$-algebras*

$$\rho^*(\mathscr{O}_{\mathrm{Y}_f^{(n)}}) \otimes_{\mathscr{O}_{\mathrm{Y}}} \mathscr{O}_{\mathrm{Y}'} \longrightarrow \mathscr{O}_{\mathrm{Y}'^{(n)}_{f'}} \tag{16.2.3.1}$$

*deduced from the homomorphism $\nu_n$ (16.2.1.2) is bijective. In addition, the homomorphism of $\mathscr{O}_{\mathrm{Y}'}$-modules*

$$\mathrm{Gr}_1(u)\colon \mathscr{G}r_1(f) \otimes_{\mathscr{O}_{\mathrm{Y}}} \mathscr{O}_{\mathrm{Y}'} \longrightarrow \mathscr{G}r_1(f')$$

*is bijective.*

Proof: (i): Let us first note that the morphisms $f'\colon \mathrm{Y}' \to \mathrm{X}'$ and $u\colon \mathrm{Y}' \to \mathrm{Y}$ identify $\mathrm{Y}'$ with the product $\mathrm{Y} \times_{\mathrm{X}} \mathrm{X}'$ (for the structural morphisms $f\colon \mathrm{Y} \to \mathrm{X}$ and $v\colon \mathrm{X}' \to \mathrm{X}$) (14.5.11.1). The conclusion of (i) then results from (16.2.2,(ii)), the morphism $g$ being an immersion.

(ii): The commutative diagram

$$\begin{tikzcd}
\mathrm{Y}_f^{(n)} \arrow[d, "h_n"'] & \mathrm{Y}'^{(n)}_{f'} \arrow[l, "w_n"'] \arrow[d, "h'_n"] \\
\mathrm{X} \arrow[d, "g"'] & \mathrm{X}' \arrow[l, "v"] \arrow[d, "g'"] \\
\mathrm{Y} & \mathrm{Y}' \arrow[l, "u"]
\end{tikzcd}$$

identifies $\mathrm{Y}'^{(n)}_{f'}$ with the product $\mathrm{Y}_f^{(n)} \times_{\mathrm{X}} \mathrm{X}'$ and $\mathrm{X}'$ with the product $\mathrm{X} \times_{\mathrm{Y}} \mathrm{Y}'$, therefore (**I**,3.3.9) it identifies (for the morphisms $g' \circ h'_n$ and $w_n$) $\mathrm{Y}'^{(n)}_{f'}$ with the product $\mathrm{Y}_f^{(n)} \times_{\mathrm{Y}} \mathrm{Y}'$. Since $\mathrm{Y}_f^{(n)}$ (resp. $\mathrm{Y}'^{(n)}_{f'}$) is the affine scheme above Y (or Y′) associated with $\mathscr{O}_{\mathrm{Y}}$-algebra $\mathscr{O}_{\mathrm{Y}_f^{(n)}}$ (or $\mathscr{O}_{\mathrm{Y}'}$-algebra $\mathscr{O}_{\mathrm{Y}'^{(n)}_{f'}}$), the fact that canonical homomorphism (16.2.3.1) is bijective results from (**II**,1.5.2). Finally, the canonical homomorphism (16.2.3.1) is compatible with the augmentation $\mathscr{O}_{\mathrm{Y}_f^{(n)}} \to \mathscr{O}_{\mathrm{Y}}$ and $\mathscr{O}_{\mathrm{Y}'^{(n)}_f} \to \mathscr{O}_{\mathrm{Y}'}$; Since $\mathscr{O}_{\mathrm{Y}_f^{(n)}}$ is a direct sum (as $\mathscr{O}_{\mathrm{Y}}$-module) of $\mathscr{O}_{\mathrm{Y}}$ and of the augmentation ideal $\mathscr{J}/\mathscr{J}^{n+1}$, we see that canonical homomorphism (16.2.3.1), restricted to $(\mathscr{J}/\mathscr{J}^{n+1}) \otimes_{\mathscr{O}_{\mathrm{Y}'}} \mathscr{O}_{\mathrm{Y}}$, is a bijection from the latter to $\mathscr{J}'/\mathscr{J}'^{n+1}$. For $n=1$, this shows that $\mathrm{Gr}_1(u)$ is bijective. □

Note that, under the hypotheses of (16.2.3), the homomorphisms $\mathrm{Gr}_n(u)$ are surjective by virtue of the above, but are not bijective in general for $n \geqslant 2$. However:

**Corollary 16.2.4** *Under the assumptions of (16.2.3), suppose that $u\colon \mathrm{Y}' \to \mathrm{Y}$ is a flat morphism (resp. that the $\mathscr{G}r_n(f)$ are flat $\mathscr{O}_{\mathrm{Y}}$-modules for $n \leqslant m$). Then the homomorphism*

$$\mathrm{Gr}_n(u)\colon \mathscr{G}r_n(f) \otimes_{\mathscr{O}_{\mathrm{Y}}} \mathscr{O}_{\mathrm{Y}'} \longrightarrow \mathscr{G}r_n(f')$$

*is bijective for all $n$ (resp. for $n \leqslant m$).*

Proof: If $u$ is flat, so is $v\colon \mathrm{X}' \to \mathrm{X}$ which is deduced by base change, and we already know in this case that $\mathrm{Gr}(u)$ is bijective (16.2.2,(iii)). If $\mathscr{G}r_n(f)$ are flat for $n \leqslant m$, we first see by induction on $n$ that so is $\mathscr{J}/\mathscr{J}^{n+1}$ for $n \leqslant m$, by virtue of the exact sequences

$$0 \longrightarrow \mathscr{J}^n/\mathscr{J}^{n+1} \longrightarrow \mathscr{J}/\mathscr{J}^{n+1} \longrightarrow \mathscr{J}/\mathscr{J}^n \longrightarrow 0$$

($\mathbf{0_I}$,6.1.2); in addition, we then have commutative diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathscr{J}^n/\mathscr{J}^{n+1}) \otimes_{\mathscr{O}_{\mathrm{Y}}} \mathscr{O}_{\mathrm{Y}'} & \longrightarrow & (\mathscr{J}/\mathscr{J}^{n+1}) \otimes_{\mathscr{O}_{\mathrm{Y}}} \mathscr{O}_{\mathrm{Y}'} & \longrightarrow & (\mathscr{J}/\mathscr{J}^n) \otimes_{\mathscr{O}_{\mathrm{Y}}} \mathscr{O}_{\mathrm{Y}'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{J}'^n/\mathscr{J}'^{n+1} & \longrightarrow & \mathscr{J}'/\mathscr{J}'^{n+1} & \longrightarrow & \mathscr{J}'/\mathscr{J}'^n & \longrightarrow & 0
\end{array}$$

in which the lines are exact (the first by flatness ($\mathbf{0_I}$,6.1.2)) and the last two vertical arrows are bijective under (16.2.3,(ii)); hence the conclusion. □

**Remark 16.2.5** (i) The reasoning in (16.2.2,(i)) still applies when in (16.2.1.1) all arrows refer to morphisms of locally ringed spaces ($\mathbf{Err_{II}}$,(1.8.2)).

(ii) In (16.2.2,(ii)), the conclusion is no longer necessarily valid when one supposes only that $v$ and $f$ are morphisms of schemes ($f$ satisfying the condition of (16.1.1)). For example (with the notation in the proof of (16.2.2,(ii)), it can happen that $\mathfrak{J} = 0$ but that the kernel $\mathfrak{J}'$ of $\mathrm{A}' \to \mathrm{B}' = \mathrm{B} \otimes_{\mathrm{A}} \mathrm{A}'$ is not zero and that $\mathrm{B}' \neq 0$, in which case $\mathrm{Y}^{(n)} = \mathrm{Y}$ for all $n$, but $\mathrm{Y}'^{(n)} \neq \mathrm{Y}'$. We have an example of this by taking $\mathrm{A} = \mathbb{Z}$, $\mathrm{B} = \mathbb{Q}$, $\mathrm{A}' = \prod_{h=1}^{\infty} (\mathbb{Z}/m^h\mathbb{Z})$ where $m > 1$.

**16.2.6** Consider the particular case of the diagram (16.2.1.1) where $\mathrm{X}' = \mathrm{X}$, $v$ being the identity, X is an scheme, Y is a subscheme of X, Y′ is a subscheme of Y, $f$, $u$ and $f' = f \circ u$ the canonical injections; the di-homomorphism (16.2.1.3) gives, by tensor with $\mathscr{O}_{\mathrm{Y}'}$ above $\rho^*(\mathscr{O}_{\mathrm{Y}})$, a homomorphism of graded $\mathscr{O}_{\mathrm{Y}'}$-algebras

$$u^*(\mathscr{G}r_\bullet(f)) \longrightarrow \mathscr{G}r_\bullet(f'). \tag{16.2.6.1}$$

On the other hand, we identify $\mathscr{O}_{\mathrm{Y}}$ with $\psi^*(\mathscr{O}_{\mathrm{X}})/\mathscr{J}_f$, $\mathscr{O}_{\mathrm{Y}'}$ with $\rho^*(\mathscr{O}_{\mathrm{Y}})/\mathscr{J}_u$; since $\rho^*$ is an exact functor, we have $\rho^*(\mathscr{O}_{\mathrm{Y}}) = \rho^*(\psi^*(\mathscr{O}_{\mathrm{X}}))/\rho^*(\mathscr{J}_f) = \psi'^*(\mathscr{O}_{\mathrm{X}})/\rho^*(\mathscr{J}_f)$, and since $\mathscr{O}_{Y'}$ is also identified with $\psi'^*(\mathscr{O}_X)/\mathscr{J}_{f'}$, we see that we have $\mathscr{J}_u = \mathscr{J}_{f'}/\rho^*(\mathscr{J}_f)$. We deduce, for all integers $n$, a canonical homomorphism $\mathscr{J}_{f'}^n/\mathscr{J}_{f'}^{n+1} \to \mathscr{J}_u^n/\mathscr{J}_u^{n+1}$, hence a canonical homomorphism of graded $\mathscr{O}_{Y'}$-algebras

$$\mathscr{G}r_\bullet(f') \longrightarrow \mathscr{G}r_\bullet(u). \tag{16.2.6.2}$$

**Proposition 16.2.7** *Let* X *be a scheme,* Y *a subscheme of* X*,* Y′ *a subscheme of* Y*,* $j\colon Y' \to Y$ *the canonical injection. We then have an exact sequence of conormal sheaves (as* $\mathscr{O}_{Y'}$*-modules)*

$$j^*(\mathscr{N}_{Y/X}) \longrightarrow \mathscr{N}_{Y'/X} \longrightarrow \mathscr{N}_{Y'/Y} \longrightarrow 0 \tag{16.2.7.1}$$

*where the arrows are the components of degree* 1 *of the canonical homomorphisms (16.2.6.1) and (16.2.6.2).*

Proof: Since the question is local, we can confine ourselves to the case where $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(A/\mathfrak{J})$, $Y' = \mathrm{Spec}(A/\mathfrak{K})$, $\mathfrak{J}$ and $\mathfrak{K}$ being two ideals of A such that $\mathfrak{J} \subset \mathfrak{K}$; everything comes back to see whereas the sequence of canonical homomorphisms

$$\mathfrak{J}/\mathfrak{K}\mathfrak{J} \longrightarrow \mathfrak{K}/\mathfrak{K}^2 \longrightarrow (\mathfrak{K}/\mathfrak{J})/(\mathfrak{K}/\mathfrak{J})^2 \longrightarrow 0$$

is exact, which is immediate since the image of $\mathfrak{J}/\mathfrak{K}\mathfrak{J}$ in $\mathfrak{K}/\mathfrak{K}^2$ is $(\mathfrak{J}+\mathfrak{K}^2)/\mathfrak{K}^2$ and that $(\mathfrak{K}/\mathfrak{J})/(\mathfrak{K}/\mathfrak{J})^2$ is identified with $\mathfrak{K}/(\mathfrak{J}+\mathfrak{K}^2)$.

It is easy to give examples where the sequence (16.2.7.1) prolonged on the left by an 0 is no longer exact; with the previous notations, it is enough to take $A = k[T]$, $\mathfrak{J} = AT^2$, $\mathfrak{K} = AT$, because one then has $(\mathfrak{J}+\mathfrak{K}^2)/\mathfrak{K}^2 = 0$ and $\mathfrak{J}/\mathfrak{K}\mathfrak{J} \neq 0$. See, however, (16.9.13) and (19.1.5) for useful cases where the prolonged sequence is still exact. □

## 16.3 Fundamental differential invariants of a morphism of schemes

**Definition 16.3.1** *Let* $f\colon X \to S$ *be a morphism of schemes,* $\Delta_f\colon X \to X \times_S X$ *the corresponding diagonal morphism, which is an immersion (**I**,5.3.9 and* $\mathbf{Err}_{\mathbf{III}}$*,10). We denote by* $\mathscr{P}_f^n$ *or* $\mathscr{P}_{X/S}^n$*, and we call* **sheaf of principal parts of order** $n$ **of the** S**-scheme** X*, the sheaf of* $\mathscr{O}_X$*-augmented rings,* $n$*-th normal invariant of* $\Delta_f$ *(16.1.2). We put* $\mathscr{P}_f^\infty = \mathscr{P}_{X/S}^\infty = \varprojlim_n \mathscr{P}_{X/S}^n$*,* $\mathscr{G}r_n(\mathscr{P}_f) = \mathscr{G}r_n(\mathscr{P}_{X/S}) = \mathscr{G}r_n(\Delta_f)$ *(16.1.2); the* $\mathscr{O}_X$*-Module* $\mathscr{G}r_1(\Delta_f)$*, ideal of augmentation of* $\mathscr{P}_{X/S}^1$*, is also noted by* $\Omega_f^1$ *or* $\Omega_{X/S}^1$ *and called the* $\mathscr{O}_X$*-Module of* 1**-differentials** *of* $f$*, or of* X *with respect to* S*, or of the* S*-scheme* X*.*

It follows from this definition that $\mathscr{P}_{X/S}^0$ identifies canonically with $\mathscr{O}_X$ (16.1.2).

We have (16.1.2.2) a surjective canonical homomorphism of graded $\mathscr{O}_X$-algebras

$$\mathbf{S}^{\bullet}_{\mathscr{O}_X}(\Omega^1_{X/S}) \longrightarrow \mathscr{G}r_{\bullet}(\mathscr{P}_{X/S}) \tag{16.3.1.1}$$

It also follows from the definition (16.3.1) that for every open set U of X, we have $\mathscr{P}^n_{f|_U} = \mathscr{P}^n_f\big|_U$, $\mathscr{P}^\infty_{f|_U} = \mathscr{P}^\infty_f\big|_U$, $\mathscr{G}r_n(\mathscr{P}_{f|_U}) = \mathscr{G}r_n(\mathscr{P}_f)\big|_U$, $\Omega^1_{f|_U} = \Omega^1_f\big|_U$ (that is, the concepts introduced are *local* on X).

**16.3.2** Let $p_1$, $p_2$ be the two canonical projections of the product $X \times_S X$; as $\Delta_f$ is an X-section of $X \times_S X$ for both the morphisms $p_1$ and $p_2$, each of these morphisms defines, for all $n$, a homomorphism of sheaves of rings $\mathscr{O}_X \to \mathscr{P}^n_{X/S}$, right inverse to $\mathscr{P}^n_{X/S} \to \mathscr{O}_X$ (16.1.7); we can still say that one thus defines on $\mathscr{P}_{X/S}$ *two* structures of *quasi-coherent augmented* $\mathscr{O}_X$*-algebras*; the corresponding structures of $\mathscr{O}_X$-module on $\mathscr{G}r_n(\mathscr{P}_{X/S})$ are the same. We have similarly, by passing to the limit, two structures of $\mathscr{O}_X$-algebra on $\mathscr{P}^\infty_{X/S}$.

**16.3.3** The morphism $s = (p_2, p_1)_S \colon X \times_S X \to X \times_S X$ is an involution of $X \times_S X$, called ***canonical symmetry***, such that

$$p_1 \circ s = p_2, \quad p_2 \circ s = p_2, \quad s \circ \Delta_f = \Delta_f. \tag{16.3.3.1}$$

If we put $s = (\rho, \lambda)$, $p_i = (\pi_i, \mu_i) (i = 1, 2)$, $\Delta_f = (\delta, \nu)$, then $\lambda^\sharp$ is an isomorphism from $\rho^*(\pi_1^*(\mathscr{O}_X))$ to $\pi_2^*(\mathscr{O}_X)$, and $\delta^*(\lambda^\sharp)$ fixes $\delta^*(\mathscr{O}_{X\times_X S})$ and the kernel $\mathscr{J}$ of the homomorphism $\nu^\sharp \colon \delta^*(\mathscr{O}_{X\times_X S}) \to \mathscr{O}_X$. As a result:

**Proposition 16.3.4** *The homomorphism $\sigma = \delta^*(\lambda^\sharp)$ deduced from $s$ (and still called* canonical symmetry*) is an involution of the projective system $(\mathscr{P}^n_{X/S})$ of $\mathscr{O}_X$-augmented sheaves of rings, and hence also of their projective limit $\mathscr{P}^\infty_{X/S}$. This automorphism permutes the two structures of $\mathscr{O}_X$-algebra on $\mathscr{P}^n_{X/S}$ and on $\mathscr{P}^\infty_{X/S}$.*

**16.3.5** In the following, the two structures of $\mathscr{O}_X$-algebra defined on $\mathscr{P}^n_{X/S}$ and on $\mathscr{P}^\infty_{X/S}$ will play very different roles: *we will now agree, unless expressly stated otherwise, that when $\mathscr{P}^n_{X/S}$ or $\mathscr{P}^\infty_{X/S}$ is considered to be an $\mathscr{O}_X$-algebra, it is of the structure of algebra defined by $p_1$ that it will be.*

For any open set U of X and every section $t \in \Gamma(U, \mathscr{O}_X)$, we will simply denote $t.1$ or even $t$ the image of $t$ by the structural homomorphism $\Gamma(U, \mathscr{O}_X) \to \Gamma(U, \mathscr{P}^n_{X/S})$ (resp. $\Gamma(U, \mathscr{O}_X) \to \Gamma(U, \mathscr{P}^\infty_{X/S})$) (that is to say the homomorphism corresponding to $p_1$).

**Definition 16.3.6** *We denote by $d^n_f$, or $d^n_{X/S}$ (resp. $d^\infty_f$, or $d^\infty_{X/S}$), or simply $d^n$ (resp. $d^\infty$), the homomorphism of sheaves of rings $\mathscr{O}_X \to \mathscr{P}^n_f = \mathscr{P}^n_{X/S}$ (resp. $\mathscr{O}_X \to \mathscr{P}^\infty_f = \mathscr{P}^\infty_{X/S}$) deduced from $p_2$. For every open set* U *of* X*, and* *every* $t \in \Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$, $d^n t$ *(resp.* $d^\infty t$*) is called* **the principal part of order** $n$ *(resp.* **the principal part of infinite order***) of* $t$. *We put* $dt = d^1 t - t$, *and we say that* $dt$ *is the* **differential** *of* $t$ *(element of* $\Gamma(\mathrm{U}, \Omega^1_{\mathrm{X/S}})$*), also denoted by* $d_{\mathrm{X/S}}(t)$.

It follows immediately from this definition that we have

$$d(t_1 t_2) = t_1 dt_2 + t_2 dt_1 \tag{16.3.6.1}$$

whenever $t_1, t_2 \in \Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$, in other words, $d$ is a *derivation* of the ring $\Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$ on the $\Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$-module $\Gamma(\mathrm{U}, \Omega^1_{\mathrm{X/S}})$.

In all the notations introduced in (16.3.1) and (16.3.6), we sometimes replace S with A when $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$.

**16.3.7** Suppose in particular that $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ and $\mathrm{X} = \mathrm{Spec}(\mathrm{B})$ are affine schemes, B being an A-algebra. Then $\Delta_f$ corresponds to the canonical surjective homomorphism $\pi\colon \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{B}$ such that $\pi(b \otimes b') = bb'$, with kernel $\mathfrak{J} = \mathfrak{J}_{\mathrm{B/A}}$ (**0**.20.4.1); $\mathscr{P}^n_f$ is the structural sheaf of the scheme $\mathrm{Spec}(\mathrm{P}^n_{\mathrm{B/A}})$, where

$$\mathrm{P}^n_{\mathrm{B/A}} = (\mathrm{B} \otimes_{\mathrm{A}} \mathrm{B})/\mathfrak{J}^{n+1};$$

$\mathscr{G}r_\bullet(f)$ is the quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module corresponding to the graded B-module

$$\mathrm{gr}^\bullet_{\mathfrak{J}}(\mathrm{B} \otimes_{\mathrm{A}} \mathrm{B}) = \bigoplus_{n \geqslant 0} (\mathfrak{J}^n/\mathfrak{J}^{n+1});$$

in particular $\Omega^1_f = \Omega^1_{\mathrm{X/S}}$ is the quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module corresponding to the B-module of the 1-differentials of B with respect to A, $\Omega^1_{\mathrm{B/A}}$ (**0**,20.4.3). The projection morphisms $p_1$, $p_2$ correspond to the two ring homomorphisms $j_1$, $j_2$ such that $j_1(b) = b \otimes 1$, $j_2(b) = 1 \otimes b$, so that (by the convention of (16.3.5)), $\mathrm{P}^n_{\mathrm{B/A}}$ is always considered as a B-algebra for the composited homomorphism $\mathrm{B} \xrightarrow{j_1} \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{P}^n_{\mathrm{B/A}}$; the ring homomorphism $\mathrm{B} \xrightarrow{j_2} \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{P}^n_{\mathrm{B/A}}$ is denoted by $d^n_{\mathrm{B/A}}$ and corresponds to $d^n_{\mathrm{X/S}}$ operating on $\Gamma(\mathrm{X}, \mathscr{O}_{\mathrm{X}})$; for all $t \in \mathrm{B}$, $dt$ is equal to $d_{\mathrm{B/A}} t$, defined in (**0**,20.4.6) (cf. $\mathbf{Err_{IV}}$, 11).

If $\pi_n\colon \mathrm{B} \otimes_{\mathrm{A}} \mathrm{B} \to \mathrm{P}^n_{\mathrm{B/A}}$ is the canonical homomorphism, then we have, by virtue of the previous definitions,

$$\pi_n(b \otimes b') = b.\pi_n(1 \otimes b') = b.d^n_{\mathrm{B/A}}(b'), \quad \text{for } b, b' \in \mathrm{B}. \tag{16.3.7.1}$$

**Proposition 16.3.8** *The image of the homomorphism* $d^n_{\mathrm{X/S}}\colon \mathscr{O}_{\mathrm{X}} \to \mathscr{P}^n_{\mathrm{X/S}}$ *generates the* $\mathscr{O}_{\mathrm{X}}$*-module* $\mathscr{P}^n_{\mathrm{X/S}}$.

Proof: We immediately return to the case where $\mathrm{X} = \mathrm{Spec}(\mathrm{B})$ and $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ are affine and the proposition results from (16.3.7.1) since $\pi_n$ is surjective. Note that in general $d^n_{\mathrm{X/S}}$ is not surjective (even already for $n = 1$). □

**Proposition 16.3.9** *Suppose that $f\colon \mathrm{X} \to \mathrm{S}$ is a morphism locally of finite type. Then $\mathscr{P}^n_f$ and $\mathscr{G}r_n(\mathscr{P}_f)$ are quasi-coherent $\mathscr{O}_\mathrm{X}$-modules of finite type.*

PROOF: This results from (16.1.6) and from the fact that $\Delta_f$ is locally of finite presentation (1.4.3.1). □

## 16.4 Functorial properties of differential invariants

**16.4.1** Consider a commutative diagram of morphisms of schemes.

(16.4.1.1)
$$\begin{tikzcd} \mathrm{X} \arrow[d, "f"'] & \mathrm{X}' \arrow[l, "u"'] \arrow[d, "f'"] \\ \mathrm{S} & \mathrm{S}' \arrow[l, "w"] \end{tikzcd}$$

We deduce a commutative diagram

(16.4.1.2)
$$\begin{tikzcd} \mathrm{X} \arrow[d, "\Delta_f"'] & \mathrm{X}' \arrow[l, "u"'] \arrow[d, "\Delta_{f'}"] \\ \mathrm{X}\times_\mathrm{S}\mathrm{X} & \mathrm{X}'\times_{\mathrm{S}'}\mathrm{X}' \arrow[l, "v"] \end{tikzcd}$$

where $v$ is the composited homomorphism (**I**,5.3.5 and 5.3.15)

(16.4.1.3)
$$\mathrm{X}'\times_{\mathrm{S}'}\mathrm{X}' \xrightarrow{(p'_1,p'_2)_\mathrm{S}} \mathrm{X}'\times_\mathrm{S}\mathrm{X}' \xrightarrow{u\times_\mathrm{S} u} \mathrm{X}\times_\mathrm{S}\mathrm{X}$$

From $u$ and $v$, as explained in (16.2.1), we deduce homomorphisms of sheaves of augmented rings

(16.4.1.4)
$$v_n\colon \rho^*(\mathscr{P}^n_{\mathrm{X}/\mathrm{S}}) \longrightarrow \mathscr{P}^n_{\mathrm{X}'/\mathrm{S}'}$$

(where we put $u = (\rho,\lambda)$); these homomorphisms form a projective system, and thus give the limit a homomorphism of sheaves of augmented rings

(16.4.1.5)
$$v_\infty\colon \rho^*(\mathscr{P}^\infty_{\mathrm{X}/\mathrm{S}}) \longrightarrow \mathscr{P}^\infty_{\mathrm{X}'/\mathrm{S}'};$$

on the other hand, by passing to the quotients, the homomorphisms $v_n$ give a di-homomorphism of graded algebras (with respect to $\lambda^\sharp$):

(16.4.1.6)
$$\mathrm{gr}(u)\colon \rho^*(\mathscr{G}r_\bullet(\mathscr{P}_{\mathrm{X}/\mathrm{S}})) \longrightarrow \mathscr{G}r_\bullet(\mathscr{P}_{\mathrm{X}'/\mathrm{S}'}).$$

**16.4.2** If we have a commutative diagram

$$\begin{tikzcd} \mathrm{X} \arrow[d, "f"'] & \mathrm{X}' \arrow[l, "u"'] \arrow[d, "f'"'] & \mathrm{X}'' \arrow[l, "u'"'] \arrow[d, "f''"] \\ \mathrm{S} & \mathrm{S}' \arrow[l, "w"] & \mathrm{S}'' \arrow[l, "w'"] \end{tikzcd}$$

we deduce a commutative diagram

$$\begin{array}{ccccc}
X & \xleftarrow{\ u\ } & X' & \xleftarrow{\ u'\ } & X'' \\
{\scriptstyle\Delta_f}\downarrow & & {\scriptstyle\Delta_{f'}}\downarrow & & \downarrow{\scriptstyle\Delta_{f''}} \\
X\times_S X & \xleftarrow[v]{} & X'\times_{S'} X' & \xleftarrow[v']{} & X''\times_{S''} X''
\end{array}$$

where $v'$ is defined from $u', w', f', f''$ as $v$ from $u, w, f, f'$. We check immediately that if $u'' = u\circ u'$, $w'' = w\circ w'$, then the composited morphism $v\circ v'$ is equal to the morphism $v''$ defined from $u'', w'', f, f''$ as $v$ from $u, w, f, f'$. If we put $u' = (\rho', \lambda')$, $u'' = (\rho'', \lambda'')$, it follows then from (16.2.1) that homomorphism $v''_n\colon \rho''^*(\mathscr{P}^n_{X/S}) \to \mathscr{P}^n_{X''/S''}$ is equal to the composition

$$\rho'^*(\rho^*(\mathscr{P}^n_{X/S})) \xrightarrow{\rho'^*(v_n)} \rho'^*(\mathscr{P}^n_{X'/S'}) \xrightarrow{v'_n} \mathscr{P}^n_{X''/S''}$$

and we have analogous transitive properties for the homomorphisms (16.4.1.4) and (16.4.1.5), which allows us to say that $\mathscr{P}^n_{X/S}$, $\mathscr{P}^\infty_{X/S}$ and $\mathscr{G}r_\bullet(\mathscr{P}_{X/S})$ *functorially dependent on* $f$.

**16.4.3** We check immediately (for example by referring to the affine case and using (16.3.7)) that with the notation of (16.4.1), the diagram

(16.4.3.1)
$$\begin{array}{ccc}
\rho^*(\mathscr{O}_X) & \xrightarrow{\ \lambda^\sharp\ } & \mathscr{O}_{X'} \\
\downarrow & & \downarrow \\
\rho^*(\mathscr{P}^n_{X/S}) & \xrightarrow[v_n]{} & \mathscr{P}^n_{X'/S'}
\end{array}$$

where the vertical arrows are those defining the structures of algebra chosen in (16.3.5) (that is, those from the first projections) is commutative; it is the same with the diagram

(16.4.3.2)
$$\begin{array}{ccc}
\rho^*(\mathscr{O}_X) & \xrightarrow{\ \lambda^\sharp\ } & \mathscr{O}_{X'} \\
{\scriptstyle\rho^*(d^n_{X/S})}\downarrow & & \downarrow{\scriptstyle d^n_{X'/S'}} \\
\rho^*(\mathscr{P}^n_{X/S}) & \xrightarrow[v_n]{} & \mathscr{P}^n_{X'/S'}
\end{array}$$

the vertical arrows defining the algebra structures here come from the second projections; moreover, if $\sigma$ and $\sigma'$ are the canonical symmetries corresponding to $f$ and $f'$ (16.3.4), we have

$$v_n\circ\rho^*(\sigma) = \sigma'\circ v_n$$

which passes from one of the previous diagrams to the other. We thus deduce from (16.4.3.1) a canonical homomorphism of *augmented* $\mathscr{O}_{X'}$*-algebras*

(16.4.3.3) $$\mathrm{P}^n(u)\colon u^*(\mathscr{P}^n_{X/S}) = \mathscr{P}^n_{X/S}\otimes_{\mathscr{O}_X}\mathscr{O}_{X'} \longrightarrow \mathscr{P}^n_{X'/S'}$$

and it follows from (16.4.3.2) that the diagram

$$\begin{array}{ccc} \mathscr{O}_{X'} & \xrightarrow{\mathrm{id}} & \mathscr{O}_{X'} \\ {\scriptstyle u^*(d^n_{X/S})}\downarrow & & \downarrow{\scriptstyle d^n_{X'/S'}} \\ u^*(\mathscr{P}^n_{X/S}) & \xrightarrow[\mathrm{P}^n(u)]{} & \mathscr{P}^n_{X'/S'} \end{array} \tag{16.4.3.4}$$

is commutative. From this we deduce a homomorphism of *graded* $\mathscr{O}_{X'}$-*algebras*

$$\mathrm{Gr}_\bullet(u)\colon u^*(\mathscr{G}r_\bullet(\mathscr{P}_{X/S})) \longrightarrow \mathscr{G}r_\bullet(\mathscr{P}_{X'/S'}) \tag{16.4.3.5}$$

and in particular a homomorphism of $\mathscr{O}_{X'}$-modules

$$\mathrm{Gr}_1(u)\colon \Omega^1_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \longrightarrow \Omega^1_{X'/S'} \tag{16.4.3.6}$$

giving rise to a commutative diagram

$$\begin{array}{ccc} \mathscr{O}_{X'} & \xrightarrow{\mathrm{id}} & \mathscr{O}_{X'} \\ {\scriptstyle d_{X/S}\otimes 1}\downarrow & & \downarrow{\scriptstyle d_{X'/S'}} \\ \Omega^1_{X/S} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} & \longrightarrow & \Omega^1_{X'/S'} \end{array} \tag{16.4.3.7}$$

**16.4.4** When $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, $X = \mathrm{Spec}(B)$, $X' = \mathrm{Spec}(B')$ are affine, so that we have a commutative diagram of homomorphisms of rings

$$\begin{array}{ccc} B & \longrightarrow & B' \\ \uparrow & & \uparrow \\ A & \longrightarrow & A' \end{array}$$

the image of $\mathfrak{J}_{B/A}$ in $B' \otimes_{A'} B'$ is contained in $\mathfrak{J}_{B'/A'}$, and the homomorphism $v_n$ corresponds to the homomorphism of rings $\mathrm{P}^n_{B/A} \to \mathrm{P}^n_{B'/A'}$ deduced from the homomorphism $B \otimes_A B \to B' \otimes_{A'} B'$ by passage to the quotients. The homomorphism (16.4.3.6) corresponds to the homomorphism defined in (**0**,20.5.4.1), and the commutative diagram (16.4.3.7) to the diagram (**0**,20.5.4.2).

**Proposition 16.4.5** *Suppose that* $X' = X \times_S S'$, $f'$ *and* $u$ *being the canonical projections. Then the canonical homomorphisms* $\mathrm{P}^n(u)$ *(16.4.3.3) and* $\mathrm{Gr}_1(u)$ *(16.4.3.6) are bijective.*

Proof: We have then $X' \times_{S'} X' = (X \times_S X) \times_S S'$, and we can therefore apply (16.2.3,(ii)) by replacing $g$ with the first projection $p_1$ and $f$ by the diagonal $\Delta_f$. $\square$

Note that under the assumptions of (16.4.5) the homomorphism $\mathrm{Gr}_\bullet(u)$ (16.4.3.5) is surjective, but not bijective in general. However (16.2.4):

**Corollary 16.4.6** *Hypotheses being those of (16.4.5), let us suppose further that* $w\colon \mathrm{S}' \to \mathrm{S}$ *is flat (resp. that* $\mathscr{G}r_n(\mathscr{P}_{\mathrm{X/S}})$ *are flat* $\mathscr{O}_\mathrm{X}$*-modules for* $n \leqslant m$*); then homomorphism*

$$\mathrm{Gr}_n(u)\colon u^*(\mathscr{G}r_n(\mathscr{P}_{\mathrm{X/S}})) \longrightarrow \mathscr{G}r_n(\mathscr{P}_{\mathrm{X'/S'}})$$

*is bijective for all* $n$ *(resp. for* $n \leqslant m$*).*

Proof: Indeed, if $w$ is flat, so is $v\colon \mathrm{X}' \times_{\mathrm{S}'} \mathrm{X}' \to \mathrm{X} \times_\mathrm{S} \mathrm{X}$, thus the conclusion results from (16.2.4). □

**16.4.7** Let S be a scheme, $\mathscr{E}$ a quasi-coherent $\mathscr{O}_\mathrm{S}$-module, and let $\mathrm{X} = \mathbb{V}(\mathscr{E})$ (**II**,1.7.8) be the vector bundle associated to $\mathscr{E}$, equal to $\mathrm{Spec}(\mathbb{S}_{\mathscr{O}_\mathrm{S}}(\mathscr{E}))$. Let $f\colon \mathrm{X} \to \mathrm{S}$ be the structural morphism. For any open set U of S, and any section $t \in \Gamma(\mathrm{U}, \mathscr{E})$, $t$ identifies with a section of $\mathbb{S}_{\mathscr{O}_\mathrm{S}}(\mathscr{E})$ above U; Let $t'$ be its image in $\Gamma(f^{-1}(\mathrm{U}), \mathscr{O}_\mathrm{X}) = \Gamma(\mathrm{U}, f_*(\mathscr{O}_\mathrm{X})) = \Gamma(\mathrm{U}, \mathbb{S}_{\mathscr{O}_\mathrm{S}}(\mathscr{E}))$, and put

$$\delta(t) = d^n_{\mathrm{X/S}}(t') - t' \in \Gamma(f^{-1}(\mathrm{U}), \mathscr{P}^n_{\mathrm{X/S}}) \tag{16.4.7.1}$$

it is clear that δ is a di-homomorphism of modules (corresponding to the ring homomorphism $\Gamma(\mathrm{U}, \mathscr{O}_\mathrm{S}) \to \Gamma(f^{-1}(\mathrm{U}), \mathscr{O}_\mathrm{X})$) from $\Gamma(\mathrm{U}, \mathscr{E})$ to $\Gamma(f^{-1}(\mathrm{U}), \mathscr{P}^n_{\mathrm{X/S}})$, whose image belongs moreover to the ideal of augmentation of $\Gamma(f^{-1}(\mathrm{U}), \mathscr{P}^n_{\mathrm{X/S}})$. We deduce (by varying U) a canonical homomorphism of $\mathscr{O}_\mathrm{X}$-algebras

$$f^*(\mathbb{S}_{\mathscr{O}_\mathrm{S}}(\mathscr{E})) \longrightarrow \mathscr{P}^n_{\mathrm{X/S}} \tag{16.4.7.2}$$

and according to the previous remark, if $\mathscr{K}$ is the kernel ideal of the augmentation $\mathbb{S}_{\mathscr{O}_\mathrm{S}}(\mathscr{E}) \to \mathscr{O}_\mathrm{S}$, the image of $\mathscr{K}^{n+1}$ by (16.4.7.2) is null, so that by factoring by $\mathscr{K}^{n+1}$, we finally have a canonical homomorphism

$$\delta_n\colon f^*(\mathbb{S}_{\mathscr{O}_\mathrm{S}}(\mathscr{E})/\mathscr{K}^{n+1}) \longrightarrow \mathscr{P}^n_{\mathrm{X/S}}. \tag{16.4.7.3}$$

**Proposition 16.4.8** *Under the conditions of (16.4.7), homomorphisms* $\delta_n$ *are bijective and form a projective system of isomorphisms; we deduce an isomorphism of* $\mathscr{O}_\mathrm{X}$*-algebras*

$$f^*(\mathbb{S}^\bullet_{\mathscr{O}_\mathrm{S}}(\mathscr{E})) \longrightarrow \mathscr{G}r_\bullet(\mathscr{P}_{\mathrm{X/S}}) \tag{16.4.8.1}$$

Proof: The fact that homomorphisms (16.4.7.3) form a projective system results immediately from their definition. To prove that they are isomorphisms, it will suffice to demonstrate that (16.4.8.1) is an isomorphism, the filtrations of the two terms of (16.4.7.3) being finite (Bourbaki, *Alg. Comm.*, Chap. III, §2, No. 8, cor. 3 of th. 1). For this, consider the exact sequence of $\mathscr{O}_{\mathrm{S}}$-modules

$$0 \longrightarrow \mathscr{E} \xrightarrow{u} \mathscr{E} \oplus \mathscr{E} \xrightarrow{v} \mathscr{E} \longrightarrow 0 \tag{16.4.8.2}$$

where, for any pair of sections $s, t$ of $\mathscr{E}$ above an open set U of S, we take $u(s) = (-s, s)$ and $v(s,t) = s + t$. We have

$$\mathrm{X} \times_{\mathrm{S}} \mathrm{X} = \mathrm{Spec}(\mathbb{S}_{\mathscr{O}_{\mathrm{S}}}(\mathscr{E}) \otimes_{\mathscr{O}_{\mathrm{S}}} \mathbb{S}_{\mathscr{O}_{\mathrm{S}}}(\mathscr{E})) = \mathrm{Spec}(\mathbb{S}_{\mathscr{O}_{\mathrm{S}}}(\mathscr{E} \oplus \mathscr{E}))$$

(**II**,1.4.6 and 1.7.11), and the diagonal morphism $\mathrm{X} \to \mathrm{X} \times_{\mathrm{S}} \mathrm{X}$ corresponds (**II**,1.2.7) to the homomorphism of $\mathscr{O}_{\mathrm{S}}$-algebras $\mathbb{S}(v) \colon \mathbb{S}_{\mathscr{O}_{\mathrm{S}}}(\mathscr{E} \oplus \mathscr{E}) \to \mathbb{S}_{\mathscr{O}_{\mathrm{S}}}(\mathscr{E})$ (**II**,1.7.4), so that if $\mathscr{J}$ is the kernel of this homomorphism, we have

$$\mathscr{P}^n_{\mathrm{X/S}} = f^*(\mathbb{S}_{\mathscr{O}_{\mathrm{S}}}(\mathscr{E} \oplus \mathscr{E})/\mathscr{J}^{n+1}).$$

The proposal will be consequence of the following lemma:

**Lemma 16.4.8.3** *Let* Y *be a ringed space,* $0 \longrightarrow \mathscr{F}' \xrightarrow{u} \mathscr{F} \xrightarrow{v} \mathscr{F}'' \longrightarrow 0$ *an exact sequence of* $\mathscr{O}_{\mathrm{Y}}$*-modules such that every point* $y \in \mathrm{Y}$ *has an open neighborhood* V *such that the sequence* $0 \to \mathscr{F}'|_{\mathrm{V}} \to \mathscr{F}|_{\mathrm{V}} \to \mathscr{F}''|_{\mathrm{V}} \to 0$ *is split. Let* $\mathscr{I}$ *the kernel of*

$$\mathbb{S}(v) \colon \mathbb{S}_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}) \longrightarrow \mathbb{S}_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}'')$$

*and let* $\mathrm{gr}^\bullet_{\mathscr{I}}(\mathbb{S}_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}))$ *be the graded* $\mathscr{O}_{\mathrm{Y}}$*-algebra associated to the* $\mathscr{O}_{\mathrm{Y}}$*-algebra* $\mathbb{S}_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F})$ equipped with the $\mathscr{I}$-adic filtration. *Then the homomorphism of graded* $\mathscr{O}_{\mathrm{Y}}$*-algebras*

$$\mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}') \otimes_{\mathscr{O}_{\mathrm{Y}}} \mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}'') \longrightarrow \mathrm{gr}^\bullet_{\mathscr{I}}(\mathbb{S}_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F})) \tag{16.4.8.4}$$

*(where the first term is the graded tensor product of symmetrical* $\mathscr{O}_{\mathrm{Y}}$*-algebras with their canonical grading (**II**,1.7.4 and 2.1.2)), from the canonical injection*

$$\mathscr{F}' \longrightarrow \mathscr{I}$$

*is bijective.*

Proof: The injection $\mathscr{F}' \to \mathscr{I}$ gives canonically a homomorphism of graded $\mathscr{O}_{\mathrm{Y}}$-algebras $\mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}') \to \mathrm{gr}^\bullet_{\mathscr{I}}(\mathbb{S}_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}))$, and since the second term is by definition an graded $\mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}'')$-algebra, we deduce the canonical homomorphism (16.4.8.4) by tensorization of the previous with $\mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}'')$. To prove the lemma, we can, since the question is local, confine ourself to the case where $\mathscr{F} = \mathscr{F}' \oplus \mathscr{F}''$, $u$ and $v$ being the canonical homomorphisms. Then the graded algebra $\mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F})$ identifies canonically with the graded tensor product $\mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}') \otimes_{\mathscr{O}_{\mathrm{Y}}} \mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}'')$ (**II**,1.7.4), and it is then immediate that $\mathscr{I}$ is the ideal $\mathscr{J}' \otimes_{\mathscr{O}_{\mathrm{Y}}} \mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}'')$, where $\mathscr{J}'$ is the ideal of the augmentation of $\mathbb{S}^\bullet_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{F}')$, that is to say the (direct) sum of $\mathbb{S}^m_{\mathscr{O}_Y}(\mathscr{F})$ for $m \geqslant 1$. It is concluded that $\mathscr{I}^n = \mathscr{J}'^n \otimes_{\mathscr{O}_Y} \mathbb{S}^{\bullet}_{\mathscr{O}_Y}(\mathscr{F}'')$, where this time $\mathscr{J}'^n$ is the (direct) sum of $\mathbb{S}^m_{\mathscr{O}_Y}(\mathscr{F})$ for $m \geqslant n$; we have consequently $\mathscr{I}^n/\mathscr{I}^{n+1} = \mathbb{S}^n_{\mathscr{O}_Y}(\mathscr{F}') \otimes_{\mathscr{O}_Y} \mathbb{S}^{\bullet}_{\mathscr{O}_Y}(\mathscr{F}'')$ which proves that (16.4.8.4) is bijective. □

This lemma being proved, it remains to be seen that homomorphism (16.4.8.1) is indeed the image by $f^*$ of the homomorphism (16.4.8.4) corresponding to the exact sequence (16.4.8.2); it is easy to see the results from the definition of $u$ (16.4.8.2) and $\delta$ (16.4.7.1), considering the definition of the $\mathscr{O}_X$-algebra structure on $\mathscr{P}^n_{X/S}$ and of $d^n_{X/S}$ (16.3.5 and 16.3.6). □

In particular:

**Corollary 16.4.9** *Under the conditions of (16.4.7), we have a canonical isomorphism*

$$\mathrm{gr}^1(\delta)\colon f^*(\mathscr{E}) \xrightarrow{\sim} \Omega^1_{X/S}. \tag{16.4.9.1}$$

**Corollary 16.4.10** *If* $S = \mathrm{Spec}(A), \mathscr{E} = \mathscr{O}_S^m$, *so that*

$$X = \mathrm{Spec}(A[T_1, \cdots, T_m]),$$

$\mathscr{P}^n_{X/S}$ *identifies canonically with the* $\mathscr{O}_X$*-algebra corresponding to the quotient* $A[T_1, \cdots, T_m]$*-algebra* $A[T_1, \cdots, T_m, U_1, \cdots, U_m]/\mathfrak{K}^{n+1}$, *where* $U_i (1 \leqslant i \leqslant m)$ *are* $m$ *new indeterminate and* $\mathfrak{K}$ *is the ideal generated by* $U_1, \cdots, U_m$.

We thus find in particular the structure of $\Omega^1_{X/S}$ in this case (**0**,20.5.13). Note further that $d^n_{X/S}$ then corresponds to a polynomial $F(T_1, \cdots .T_m)$ the $\bmod \mathfrak{K}^{n+1}$ class of $F(T_1 + U_1, \cdots, T_m + U_m)$, as it follows from the definition (16.4.7.1).

**Proposition 16.4.11** *Let* $f : X \to S$ *be a morphism,* $g : S \to X$ *an* $S$*-section of* $X$, $S^{(n)}_g$ *the* $n$*-th infinitesimal neighborhood of* $S$ *for the immersion* $g$ *(16.1.2). Then there exists a unique isomorphism of* $\mathscr{O}_S$*-algebras*

$$\varpi_n\colon g^*(\mathscr{P}^n_{X/S}) \longrightarrow \mathscr{O}_{S^{(n)}_g} \tag{16.4.11.1}$$

*(for the structure of* $\mathscr{O}_S$*-algebra on* $\mathscr{O}_{S^{(n)}_g}$ *defined by* $f$ *(16.1.7)), making the diagram commutative*

$$\begin{tikzcd} \mathscr{O}_S = g^*(\mathscr{O}_X) \arrow[rr, "\lambda_n"] \arrow[dr, "g^*(d^n_{X/S})"'] & & \mathscr{O}_{S^{(n)}_g} \\ & g^*(\mathscr{P}^n_{X/S}) \arrow[ur, "\varpi_n"'] & \end{tikzcd} \tag{16.4.11.2}$$

*(where* $\lambda_n$ *is the structural homomorphism).*

PROOF: In virtue of (**I**,5.3.7), where X, Y, S are replaced by S, X, S respectively and $f$ by $g$, the diagrams

$$(16.4.11.3)\qquad \begin{array}{ccc} \mathrm{S} & \xrightarrow{g} & \mathrm{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \Delta_f} \\ \mathrm{X} & \xrightarrow[(g\circ f,1_{\mathrm{X}})_{\mathrm{S}}]{} & \mathrm{X}\times_{\mathrm{S}}\mathrm{X} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{S} & \xrightarrow{g} & \mathrm{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \Delta_f} \\ \mathrm{X} & \xrightarrow[(1_{\mathrm{X}},g\circ f)_{\mathrm{S}}]{} & \mathrm{X}\times_{\mathrm{S}}\mathrm{X} \end{array}$$

identify S with the product of $(\mathrm{X}\times_{\mathrm{S}}\mathrm{X})$-schemes X and X for morphisms $\Delta_f$ and $(g\circ f,1_{\mathrm{X}})_{\mathrm{S}}$ (resp. $(1_{\mathrm{X}},g\circ f)_{\mathrm{S}}$). On the other hand, the diagrams

$$(16.4.11.4)\qquad \begin{array}{ccc} \mathrm{X} & \xrightarrow{(g\circ f,1_{\mathrm{X}})_{\mathrm{S}}} & \mathrm{X}\times_{\mathrm{S}}\mathrm{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_1} \\ \mathrm{S} & \xrightarrow[g]{} & \mathrm{X} \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{X} & \xrightarrow{(1_{\mathrm{X}},g\circ f)_{\mathrm{S}}} & \mathrm{X}\times_{\mathrm{S}}\mathrm{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathrm{S} & \xrightarrow[g]{} & \mathrm{X} \end{array}$$

identify X with the product of the X-schemes S and $\mathrm{X}\times_{\mathrm{S}}\mathrm{X}$ for the morphisms $g$ and $p_1$ (resp. $p_2$) (special case of the associativity formula (**I**,3.3.9.1)). We can say that $\Delta_f$, considered as X-section of $\mathrm{X}\times_{\mathrm{S}}\mathrm{X}$ (with respect to $p_1$ or $p_2$) plays the role of a "*universal section*" for the S-sections of X: each of these sections $g$ is in fact deduced by the *base change* $(g\circ f,1_{\mathrm{X}})_{\mathrm{S}}\colon \mathrm{X}\to \mathrm{X}\times_{\mathrm{S}}\mathrm{X}$. The definition of the homomorphism $\varpi_n$ and the fact that it is bijective thus result from these remarks and from (16.2.3,(ii)) applied to the first diagram of (16.4.11.4). The commutativity of the diagram (16.4.11.2) results similarly from (16.2.3,(ii)) applied this time to the second diagram of (16.4.11.4). To explain $\varpi_n$, we can confine ourselves to the case where $g$ is a closed immersion: indeed, for all $s\in \mathrm{S}$, there is an open neighborhood W of $s$ in S such that $g(\mathrm{W})$ is closed in an open U of X, and it is clear that $g|_{\mathrm{W}}$ is a W-section of the morphism $\mathrm{U}\cap f^{-1}(\mathrm{W})\to \mathrm{W}$, restriction of $f$, and $g(\mathrm{W})$ is *a fortiori* closed in $\mathrm{U}\cap f^{-1}(\mathrm{W})$. We can therefore assume that S is a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{K}$. Then the previous definitions show that if W is open in S, $t$ a section of $\mathscr{O}_{\mathrm{X}}$ above $f^{-1}(\mathrm{W})$, $\varpi_n(d^n t|_{\mathrm{W}})$ is equal to the canonical image of $t$ in $\Gamma(\mathrm{W},(\mathscr{O}_{\mathrm{X}}/\mathscr{K}^{n+1})\big|_{\mathrm{W}})$. The uniqueness of $\varpi_n$ therefore results from the fact that the image of $\mathscr{O}_{\mathrm{X}}$ by $d^n_{\mathrm{X}/\mathrm{S}}$ generates the $\mathscr{O}_{\mathrm{X}}$-module $\mathscr{P}^n_{\mathrm{X}/\mathrm{S}}$ (16.3.8). □

**Corollary 16.4.12** *Let $k$ be a field,* X *a $k$-scheme, $x$ a point of* X *rational over $k$. Then $(\mathscr{P}^n_{\mathrm{X}/k})_x\otimes_{\mathscr{O}_x}\kappa(x)$ is canonically isomorphic (as augmented $\kappa(x)$-algebra) to $\mathscr{O}_x/\mathfrak{m}_x^{n+1}$.*

PROOF: Consider the unique $k$-section $g$ of X such that $g(\mathrm{Spec}(k))=\{x\}$. □

**Corollary 16.4.13** *Let $f\colon \mathrm{X}\to\mathrm{S}$ be a morphism, $s\in\mathrm{S}$, $\mathrm{X}_s=\mathrm{X}\times_{\mathrm{S}}\mathrm{Spec}(\kappa(s))$ the fiber of $f$ at $s$. If $x\in\mathrm{X}_s$ is rational over $\kappa(s)$, $(\mathscr{P}^n_{\mathrm{X}/\mathrm{S}})_x\otimes_{\mathscr{O}_s}\kappa(s)$ is canonically* *isomorphic to* $\mathscr{O}_{X_s,x}/\mathfrak{m}'^{n+1}_x$, *where* $\mathfrak{m}'_x$ *is the maximal ideal of* $\mathscr{O}_{X_s,x}$*; precisely, this isomorphism corresponds to* $(d^n t)_x \otimes 1$ *(where* $t$ *is a section of* $\mathscr{O}_X$ *over an open neighborhood of* $x$ *in* X*) the class of* $t_x \otimes 1$ *modulo* $\mathfrak{m}'^{n+1}_x$.

Proof: This results from (16.4.5) and (16.4.12). □

The previous corollaries justify the terminology of "*sheaf of principal parts of order* $n$".

**Proposition 16.4.14** *Let* $\rho\colon A \to B$ *be a homomorphism of rings,* S *a multiplicative subset of* B*. Then the canonical homomorphisms*

(16.4.14.1) $$S^{-1}P^n_{B/A} \longrightarrow P^n_{S^{-1}B/A}$$

*deduced from the canonical homomorphisms* $P^n_{B/A} \to P^n_{S^{-1}B/A}$ *form a projective system and are bijective.*

Proof: It suffices to notice that $S^{-1}((B\otimes_A B)/\mathfrak{J}^{n+1}) = S^{-1}(B\otimes_A B)/(S^{-1}\mathfrak{J})^{n+1}$ by flatness, and that $S^{-1}(B\otimes_A B) = (S^{-1}B)\otimes_A (S^{-1}(B))$ (**I**,1.3.4). □

**Corollary 16.4.15** *With the notations of (16.4.14), let* R *be a multiplicative subset of* A *such that* $\rho(R) \subset S$*. Then we have canonical isomorphisms*

(16.4.15.1) $$S^{-1}P^n_{B/A} \xrightarrow{\sim} P^n_{S^{-1}B/R^{-1}A}$$

*forming a projective system.*

Proof: It suffices, of course, to define canonical isomorphisms

(16.4.15.2) $$P^n_{S^{-1}B/A} \xrightarrow{\sim} P^n_{S^{-1}B/R^{-1}A}$$

that is to say that we are reduced to the case where $\rho(R)$ is formed of invertible elements of B. But then the isomorphism (16.4.15.2) is simply deduced from the canonical isomorphism $B\otimes_A B \to B\otimes_{R^{-1}A} B$ by passing to the quotients ($\mathbf{0_I}$,1.5.3). □

**Corollary 16.4.16** *Let* $f\colon X \to S$ *be a scheme morphism,* $x$ *a point of* X*,* $s = f(x)$*. Then we have canonical isomorphisms*

(16.4.16.1) $$(\mathscr{P}^n_{X/S})_x \xrightarrow{\sim} P^n_{\mathscr{O}_x/\mathscr{O}_s}$$

*forming a projective system.*

We deduce therefrom isomorphisms for the associated gradients, and in particular a canonical isomorphism

(16.4.16.2) $$(\Omega^1_{X/S})_x \xrightarrow{\sim} \Omega^1_{\mathscr{O}_x/\mathscr{O}_s}.$$

**Corollary 16.4.17** *Let $k$ be a field,* K *the field of rational fractions* $k(\mathrm{T}_1,\cdots,\mathrm{T}_r)$*. Then, for all integers $n$, the homomorphism from* $\mathrm{K}[\mathrm{U}_1,\cdots,\mathrm{U}_r]$ *($\mathrm{U}_i$ are indeterminate) to $\mathrm{P}^n_{\mathrm{K}/k}$ which, for each $\mathrm{U}_i$, corresponds to $d^n\mathrm{T}_i-\mathrm{T}_i.1$, is surjective and defines an isomorphism from the quotient* $\mathrm{K}[\mathrm{U}_1,\cdots,\mathrm{U}_r]/\mathfrak{m}^{n+1}$ *(where $\mathfrak{m}$ is the ideal generated by the $\mathrm{U}_i$) to $\mathrm{P}^n_{\mathrm{K}/k}$.*

PROOF: This results from (16.4.8), (16.4.10) and (16.4.14), where we put $\mathrm{A}=k$, $\mathrm{B}=k[\mathrm{T}_1,\cdots,\mathrm{T}_r]$ and $\mathrm{S}=\mathrm{B}-\{0\}$. □

We thus find the fact that the $d\mathrm{T}_i$ form a base of the K-vector space $\Omega^1_{\mathrm{K}/k}$ (**0**,20.5.10).

**Proposition 16.4.18** *Let $f\colon \mathrm{X}\to\mathrm{Y}$, $g\colon \mathrm{Y}\to\mathrm{Z}$ be two morphisms of schemes, and consider the canonical homomorphisms of augmented $\mathscr{O}_\mathrm{X}$-algebras (16.4.3.3)*

(16.4.18.1) $$g_{\mathrm{X/Y/Z}}\colon \mathscr{P}^n_{\mathrm{X/Z}}\longrightarrow\mathscr{P}^n_{\mathrm{X/Y}},$$

(16.4.18.2) $$f_{\mathrm{X/Y/Z}}\colon f^*(\mathscr{P}^n_{\mathrm{Y/Z}})\longrightarrow\mathscr{P}^n_{\mathrm{X/Z}}.$$

*Then $g_{\mathrm{X/Y/Z}}$ is surjective, and its kernel is the ideal generated by the image of the ideal of the augmentation of $f^*(\mathscr{P}^n_{\mathrm{Y/Z}})$ by $f_{\mathrm{X/Y/Z}}$.*

PROOF: Let us note first that $g_{\mathrm{X/Y/Z}}$ corresponds to the case where in (16.4.3.3) we put $\mathrm{X}'=\mathrm{X}$, $\mathrm{S}'=\mathrm{Y}$, $\mathrm{S}=\mathrm{Z}$ and $u=1_\mathrm{X}$, and $f_{\mathrm{X/Y/Z}}$ to the case where we replace $\mathrm{X}',\mathrm{X},\mathrm{S},\mathrm{S}'$ with $\mathrm{X},\mathrm{Y},\mathrm{Z},\mathrm{Z}$ respectively and $u,f$ by $f,g$ respectively.

We have a commutative diagram (**I**,5.3.5)

(16.4.18.3) $$\begin{tikzcd} \mathrm{X} \arrow[r, "\Delta_f"] \arrow[dr, "f"'] & \mathrm{X}\times_\mathrm{Y}\mathrm{X} \arrow[r, "j"] \arrow[d, "p"] & \mathrm{X}\times_\mathrm{Z}\mathrm{X} \arrow[d, "f\times_\mathrm{Z} f"] \\ & \mathrm{Y} \arrow[r, "\Delta_g"'] & \mathrm{Y}\times_\mathrm{Z}\mathrm{Y} \end{tikzcd}$$

where $j=(1_\mathrm{X},1_\mathrm{X})_\mathrm{Z}$ is an immersion, $j\circ\Delta_f=\Delta_{g\circ f}$ and $p$ is the structural morphism. As we can confine ourself to the case where $\mathrm{X},\mathrm{Y},\mathrm{Z}$ are affine, we can suppose the immersions $\Delta_f$, $\Delta_g$ and $j$ are closed, so that $\mathscr{O}_\mathrm{X}$ and $\mathscr{O}_{\mathrm{X}\times_\mathrm{Y}\mathrm{X}}$ identify with $\mathscr{O}_{\mathrm{X}\times_\mathrm{Z}\mathrm{X}}/\mathscr{J}$ and $\mathscr{O}_{\mathrm{X}\times_\mathrm{Z}\mathrm{X}}/\mathscr{L}$ respectively, where $\mathscr{J}\supset\mathscr{L}$ are the two quasi-coherent ideals corresponding to the immersions $\Delta_{g\circ f}$ and $j$ respectively. The $\mathscr{O}_\mathrm{X}$-algebra $\mathscr{P}^n_{\mathrm{X/Z}}$ thus identifies with $\mathscr{O}_{\mathrm{X}\times_\mathrm{Z}\mathrm{X}}/\mathscr{J}^{n+1}$, and $\mathscr{P}^n_{\mathrm{X/Y}}$ identifies with $\mathscr{O}_{\mathrm{X}\times_\mathrm{Y}\mathrm{X}}/(\mathscr{L}/\mathscr{J})^{n+1}$, that is, with $\mathscr{O}_{\mathrm{X}\times_\mathrm{Z}\mathrm{X}}/(\mathscr{J}^{n+1}+\mathscr{L})$, and hence with the quotient of $\mathscr{P}^n_{\mathrm{X/Z}}$ by $(\mathscr{J}^{n+1}+\mathscr{L})/\mathscr{J}^{n+1}$. But we know (*loc. cit.*) that $p$ and $j$ make $\mathrm{X}\times_\mathrm{Y}\mathrm{X}$ the product of $(\mathrm{Y}\times_\mathrm{Z}\mathrm{Y})$-schemes Y and $\mathrm{X}\times_\mathrm{Z}\mathrm{X}$, so if $\mathscr{O}_\mathrm{Y}$ is identified with $\mathscr{O}_{\mathrm{Y}\times_\mathrm{Z}\mathrm{Y}}/\mathscr{K}$, where $\mathscr{K}$ is the ideal corresponding to $\Delta_g$, $\mathscr{L}$ is equal to $(f\times_\mathrm{Z} f)^*(\mathscr{K}).\mathscr{O}_{\mathrm{X}\times_\mathrm{Z}\mathrm{X}}$ (**I**,4.4.5). Since $(\mathscr{J}^{n+1}+\mathscr{L})/\mathscr{J}^{n+1}$ is the ideal of $\mathscr{P}^n_{\mathrm{X/Z}}$ generated by the image of $\mathscr{L}$, we deduce the proposition. □

**Corollary 16.4.19** *With the notation of (16.4.18), we have an exact sequence of quasi-coherent $\mathscr{O}_X$-modules*

$$f^*(\Omega^1_{Y/Z}) \xrightarrow{f_{X/Y/Z}} \Omega^1_{X/Z} \xrightarrow{g_{X/Y/Z}} \Omega^1_{X/Y} \longrightarrow 0. \tag{16.4.19.1}$$

When $X, Y, Z$ are affine, we recover the exact sequence (**0**,20.5.7.1).

**Proposition 16.4.20** *Let $f\colon Y \to Z$ be a morphism, $j\colon X \to Y$ a closed immersion, $\mathscr{K}$ the quasi-coherent ideal of $\mathscr{O}_Y$ corresponding to $j$. Then we have $\mathscr{P}^n_{X/Y} = \mathscr{O}_X = \mathscr{O}_Y/\mathscr{K}$, the canonical homomorphism $j^*_{X/Y/Z}\colon j^*(\mathscr{P}^n_{Y/Z}) \to \mathscr{P}^n_{X/Z}$ is surjective, and its kernel is the ideal of $j^*(\mathscr{P}^n_{Y/Z})$ generated by $j^*(\mathscr{O}_Y.d^n_{Y/Z}(\mathscr{K}))$ (it should be noted that $d^n_{Y/Z}(\mathscr{K})$ is a subsheaf of commutative groups of $\mathscr{P}^n_{Y/Z}$, but not an $\mathscr{O}_Y$-module in general).*

Proof: We know (**I**,5.3.8) that the diagonal $\Delta_j\colon X \to X \times_Y X$ is an isomorphism, hence the first assertion follows. If $\varpi_1$ and $\varpi_2$ are the two homomorphisms of algebras $\mathscr{O}_Y \to \mathscr{P}^n_{Y/Z}$ corresponding to the two canonical projections $p_1$, $p_2$ of $Y \times_Z Y$ to $Y$ respectively, recall that by definition (16.3.5 and 16.3.6) $\varpi_1$ is the structural homomorphism of the $\mathscr{O}_Y$-algebra $\mathscr{P}^n_{Y/Z}$ and $\varpi_2 = d^n_{Y/Z}$. The $\mathscr{O}_X$-algebra $j^*(\mathscr{P}^n_{Y/Z})$ thus identifies with $\mathscr{P}^n_{Y/Z}/\varpi_1(\mathscr{K})\mathscr{P}^n_{Y/Z}$ and its quotient by the ideal generated by $j^*(d^n_{Y/Z}(\mathscr{K}))$ with $\mathscr{P}^n_{Y/Z}/(\varpi_1(\mathscr{K}) + \varpi_2(\mathscr{K}))\mathscr{P}^n_{Y/Z}$. Now note that we have a commutative diagram

$$\begin{tikzcd} Y \arrow[d, "\Delta_f"'] & X \arrow[l, "j"'] \arrow[d, "\Delta_{f\circ j}"] \\ Y \times_Z Y & X \times_Z X \arrow[l, "j \times_Z j"] \end{tikzcd}$$

identifying $X$ with the product of $(Y \times_Z Y)$-schemes $Y$ and $X \times_Z X$ (**I**,5.3.7). Since $j \times_Z j$ is an immersion, we deduce from this remark and from (16.2.2) that if $\Delta^n_{Y/Z}$ and $\Delta^n_{X/Z}$ denote the infinitesimal neighborhoods of order $n$ of $Y$ and $X$ for the canonical immersions $\Delta_f$ and $\Delta_{f\circ j}$ respectively, we have a diagram

$$\begin{tikzcd} \Delta^n_{Y/Z} \arrow[d] & \Delta^n_{X/Z} \arrow[l] \arrow[d] \\ Y \times_Z Y & X \times_Z X \arrow[l, "j \times_Z j"] \end{tikzcd}$$

making $\Delta^n_{X/Z}$ the product of $(Y \times_Z Y)$-schemes $\Delta_{Y/Z}$ and $X \times_Z X$. We can still say that $\mathscr{P}^n_{X/Z}$ identifies with the sheaf of rings $\mathscr{P}^n_{Y/Z} \otimes_{\mathscr{O}_{Y\times_Z Y}} \mathscr{O}_{X\times_Z X}$. But we see immediately (for example, by reducing ourselves to the affine case) that $\mathscr{O}_{X\times_Z X} = \mathscr{O}_{Y\times_Z Y}/(p_1^*(\mathscr{K}) + p_2^*(\mathscr{K}))\mathscr{O}_{Y\times_Z Y}$. Thus $\mathscr{P}^n_{X/Z}$ identifies with the quotient of $\mathscr{P}^n_{Y/Z}$ by the ideal generated by the image on $\mathscr{P}^n_{Y/Z}$ of $p_1^*(\mathscr{K}) + p_2^*(\mathscr{K})$. But by definition this ideal is also generated by $\varpi_1^*(\mathscr{K}) + \varpi_2^*(\mathscr{K})$. □

**Corollary 16.4.21** *Let $f: \mathrm{Y} \to \mathrm{Z}$ be a morphism, $j: \mathrm{X} \to \mathrm{Y}$ a closed immersion, we have an exact sequence of quasi-coherent $\mathscr{O}_{\mathrm{X}}$-modules*

$$\mathscr{N}_{\mathrm{X/Y}} \longrightarrow f^*(\Omega^1_{\mathrm{Y/Z}}) \longrightarrow \Omega^1_{\mathrm{X/Z}} \longrightarrow 0. \tag{16.4.21.1}$$

When X, Y*and*Z are affine, we recover the exact sequence (**0**,20.5.12.1).

**Corollary 16.4.22** *If $f: \mathrm{X} \to \mathrm{S}$ is a morphism locally of finite presentation, $\mathscr{P}^n_{\mathrm{X/S}}$ and $\Omega^1_{\mathrm{X/S}}$ are quasi-coherent $\mathscr{O}_{\mathrm{X}}$-modules of finite presentation.*

Proof: We are immediately reduced to the case where $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ is affine, $\mathrm{X} = \mathrm{Spec}(\mathrm{B})$, where $\mathrm{B} = \mathrm{A}[\mathrm{T}_1, \cdots, \mathrm{T}_r]/\mathfrak{K}$, $\mathfrak{K}$ being an ideal of finite type of $\mathrm{C} = \mathrm{A}[\mathrm{T}_1, \cdots, \mathrm{T}_r]$. We then apply (16.4.20) to $\mathrm{Z} = \mathrm{S}$, $\mathrm{Y} = \mathrm{Spec}(\mathrm{C})$ and $\mathscr{K} = \widetilde{\mathfrak{K}}$. Then $j^*(\mathscr{P}^n_{\mathrm{Y/Z}})$ is a free $\mathscr{O}_{\mathrm{X}}$-module of finite rank and the hypothesis on $\mathfrak{K}$ implies that $j^*(\mathscr{O}_{\mathrm{Y}}.d^n_{\mathrm{Y/Z}}(\mathscr{K}))$ generates a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-modules of finite type; hence the conclusion. ◻

**Proposition 16.4.23** *Let X, Y be two S-schemes, $\mathrm{Z} = \mathrm{X} \times_{\mathrm{S}} \mathrm{Y}$ their product, $p: \mathrm{X} \times_{\mathrm{S}} \mathrm{Y} \to \mathrm{X}$ and $q: \mathrm{X} \times_{\mathrm{S}} \mathrm{Y} \to \mathrm{Y}$ the canonical projections. Then the canonical homomorphism*

$$p_{\mathrm{Z/X/S}} \oplus q_{\mathrm{Z/X/S}}: p^*(\Omega^1_{\mathrm{X/S}}) \oplus q^*(\Omega^1_{\mathrm{Y/S}}) \longrightarrow \Omega^1_{\mathrm{X}\times_{\mathrm{S}}\mathrm{Y/S}} \tag{16.4.23.1}$$

*is bijective.*

Proof: The commutative diagram

$$\begin{tikzcd}
\mathrm{Y} \arrow[d, "g"'] & \mathrm{X} \times_{\mathrm{S}} \mathrm{Y} \arrow[l, "q"'] \arrow[d, "h"'] & \mathrm{X} \times_{\mathrm{S}} \mathrm{Y} \arrow[l, "\mathrm{id}"'] \arrow[d, "p"] \\
\mathrm{S} & \mathrm{S} \arrow[l, "\mathrm{id}"] & \mathrm{X} \arrow[l, "f"]
\end{tikzcd}$$

gives a factorization of canonical isomorphism $\mathrm{P}^n(p)$ (16.4.5)

$$p^*(\mathscr{P}^n_{\mathrm{X/S}}) \longrightarrow \mathscr{P}^n_{\mathrm{Z/S}} \longrightarrow \mathscr{P}^n_{\mathrm{Z/Y}}$$

and similarly, by interchange the roles of X and Y, we have a factorization of isomorphism $\mathrm{P}^n(q)$

$$q^*(\mathscr{P}^n_{\mathrm{Y/S}}) \longrightarrow \mathscr{P}^n_{\mathrm{Z/S}} \longrightarrow \mathscr{P}^n_{\mathrm{Z/X}}.$$

This proves that the canonical homomorphism (16.4.18.1)

$$p_{\mathrm{Z/X/S}}: p^*(\mathscr{P}^n_{\mathrm{X/S}}) \longrightarrow \mathscr{P}^n_{\mathrm{Z/S}} \quad (\text{resp. } q_{\mathrm{Z/X/S}}: q^*(\mathscr{P}^n_{\mathrm{Y/S}}) \longrightarrow \mathscr{P}^n_{\mathrm{Z/S}})$$

is injective, and that the kernel of the canonical surjective homomorphism (16.4.18.2)

$$\mathscr{P}^n_{\mathrm{Z/S}} \longrightarrow \mathscr{P}^n_{\mathrm{Z/Y}} \quad (\text{resp. } \mathscr{P}^n_{\mathrm{Z/S}} \longrightarrow \mathscr{P}^n_{\mathrm{Z/X}})$$

is the complement of the image of $p_{Z/X/S}$ (resp. $q_{Z/X/S}$). But on the other hand, this kernel is, in virtue of (16.4.18), generated by the image of the ideal of augmentation of $q^*(\mathscr{P}^n_{Y/S})$ (resp. $p^*(\mathscr{P}^n_{X/S})$) by $q_{Z/X/S}$ (resp. $p_{Z/X/S}$). We conclude the proposition by considering the case $n = 1$. □

We immediately generalize (16.4.23) to the case of a product of any finitely many S-schemes.

**Remark 16.4.24** (i) We will see (17.2.3) that when the morphism $f : X \to Y$ in (16.4.18) is smooth, the homomorphism $f_{X/Y/Z}$ in (16.4.19.1) is locally *invertible to the left* and in particular injective. Likewise, when the morphism $f \circ j : X \to Z$ in (16.4.20) is smooth, the left homomorphism in (16.4.21.1) is locally *invertible to the left* and *a fortiori* injective (17.2.5). In chapter V, we will also give a variant, in the case of modules on schemes, of the "modules of imperfection" studied in (**0**,20.6), and exact sequences where they appear.

(ii) Let X be topological space, $\mathscr{A}$ a sheaf of rings on X and $\mathscr{B}$ an $\mathscr{A}$-algebra on X. Then it is clear that

$$U \mapsto P^n_{\Gamma(U,\mathscr{B})/\Gamma(U,\mathscr{A})} \quad (U \text{ an open set on } X)$$

is a presheaf of augmented $\Gamma(U, \mathscr{B})$-algebras, so the associated sheaf $\mathscr{P}^n_{\mathscr{B}/\mathscr{A}}$ is an augmented $\mathscr{B}$-algebra. In the particular case where X is a scheme, $f = (\psi, \theta) : X \to S$ a morphism of schemes, it follows easily from (16.4.16) and from the exactness of the functor $\varinjlim$ that $\mathscr{P}^n_{X/S}$ is canonically isomorphic to $\mathscr{P}^n_{\mathscr{O}_X/\psi^*(\mathscr{O}_S)}$. It follows that the formalism developed in this section could be considered as a special case of a different formalism for ringed spaces provided with a sheaf of algebras over the structural sheaf. However, we did not want to start from this point of view, since it is less intuitive and less convenient for applications. It seems, moreover, that for the various species of "varieties", the "global" construction of $\mathscr{P}^n$ analogous to we use here is also better adapted to the applications.

## 16.5 Relative tangent sheaves and bundles; derivations

**16.5.1** Let $f = (\psi, \theta) : X \to S$ be a morphism of ringed spaces. For each $\mathscr{O}_X$-module $\mathscr{F}$, a S-***derivation*** (or (X/S)-***derivation***, or $f$-***derivation***) of $\mathscr{O}_X$ on $\mathscr{F}$ is a homomorphism of sheaves of additive groups $D : \mathscr{O}_X \to \mathscr{F}$ satisfying the following conditions:

a) for every open set V of X, and every pair of sections $(t_1, t_2)$ of $\mathscr{O}_X$ on V, we have

$$D(t_1 t_2) = t_1 D(t_2) + D(t_1) t_2; \tag{16.5.1.1}$$

b) for every open set V of X, every section $t$ of $\mathscr{O}_X$ on V and every section $s$ of $\mathscr{O}_S$ on an open set U of S such that $V \subset f^{-1}(U)$, we have

$$D((s|_V)t) = (s|_V)D(t). \tag{16.5.1.2}$$

It is clear that this is the same thing to say that for every $x \in X$, the homomorphism of additive groups $D_x \colon \mathscr{O}_x \to \mathscr{F}_x$ is an $\mathscr{O}_{f(x)}$*-derivation.*

Another interpretation is to consider the $\mathscr{O}_X$-algebra $\mathscr{D}_{\mathscr{O}_X}(\mathscr{F})$ equal to $\mathscr{O}_X \oplus \mathscr{F}$, the the algebra structure being defined by the condition that for any open set V of X, the product of two sections of $\mathscr{O}_X$ (resp. of a section of $\mathscr{O}_X$ and of a section of $\mathscr{F}$) on V is defined by the ring structure of $\Gamma(V, \mathscr{O}_X)$ (resp. the structure of $\Gamma(V, \mathscr{O}_X)$-module on $\Gamma(V, \mathscr{F})$), and the product of two sections of $\mathscr{F}$ on V is take to be 0; then, $\mathscr{F}$ is a n ideal of $\mathscr{D}_{\mathscr{O}_X}(\mathscr{F})$, kernel of the canonical augmentation $\mathscr{D}_{\mathscr{O}_X}(\mathscr{F}) \to \mathscr{O}_X$, and say that D is a S-derivation of $\mathscr{O}_X$ on $\mathscr{F}$ means that $1_{\mathscr{O}_X} + D$ is an $\mathscr{O}_S$-homomorphism of algebras from $\mathscr{O}_X$ to $\mathscr{D}_{\mathscr{O}_X}(\mathscr{F})$ which, composed with the augmentation, gives $1_{\mathscr{O}_X}$.

The S-derivations of $\mathscr{O}_X$ on $\mathscr{F}$ form a $\Gamma(X, \mathscr{O}_X)$-module $\mathrm{Der}_S(\mathscr{O}_X, \mathscr{F})$.

When $\mathscr{F} = \mathscr{O}_X$, a S-derivation of $\mathscr{O}_X$ on itself is simply called a ***S-derivation of*** $\mathscr{O}_X$.

**Proposition 16.5.2** *Let* A *be a ring,* B *an* A*-algebra,* L *a* B*-module; we put* $S = \mathrm{Spec}(A)$*,* $X = \mathrm{Spec}(B)$*,* $\mathscr{F} = \widetilde{L}$*. Then the map* $D \mapsto \Gamma(D)$ *which, for every* S*-derivation* $D$ *of* $\mathscr{O}_X$ *on* $\mathscr{F}$ *corresponds the map* $\Gamma(D) \colon t \mapsto D(t)$ *from* B *to* L*, is an isomorphism of* B*-modules from* $\mathrm{Der}_S(\mathscr{O}_X, \mathscr{F})$ *to* $\mathrm{Der}_A(B, L)$ *(cf. (**0**,20.1.2)).*

Proof: This results immediately from the interpretation given above of the S-derivations in terms of algebra homomorphisms, the analogous interpretation given in (**0**,20.1.6), and the canonical correspondence between homomorphisms of $\mathscr{O}_X$-algebras and homomorphisms of B-algebras (**I**,1.3.13 and 1.3.8). □

**Proposition 16.5.3** *Let* $f = (\psi, \theta) \colon X \to S$ *be a morphism of schemes.*

*(i) The differential* $d_{X/S} \colon \mathscr{O}_X \to \Omega^1_{X/S}$ *(16.3.6) is an* S*-derivation.*

*(ii) For each* $\mathscr{O}_X$*-module* $\mathscr{F}$*, the map* $u \mapsto u \circ d_{X/S}$ *is an isomorphism of* $\Gamma(X, \mathscr{O}_X)$*-modules*

$$\mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{X/S}) \xrightarrow{\sim} \mathrm{Der}_S(\mathscr{O}_X, \mathscr{F}). \tag{16.5.3.1}$$

Proof: Assertion (i) has already been shown in (16.3.6). On the other hand, it is immediate (in virtue of (**0**,20.4.8)) that $u \mapsto u \circ d_{X/S}$ is injective, considering the restriction of to a stalk $\mathscr{O}_x$ of both terms and using (16.4.16.2). To see that the homomorphism (16.5.3.1) is surjective, consider an S-derivation $D \colon \mathscr{O}_X \to \mathscr{F}$; for every open affine subset $V = \mathrm{Spec}(B)$

of X, such that $f(\mathrm{V})$ is contained in an open affine subset $\mathrm{U} = \mathrm{Spec}(\mathrm{A})$ of S, $D_{\mathrm{V}} : \mathrm{B} \to \Gamma(\mathrm{V}, \mathscr{F})$ is an A-derivation, hence there exists a *unique* B-homomorphism $u_{\mathrm{V}} : \Omega^1_{\mathrm{B/A}} \to \Gamma(\mathrm{V}, \mathscr{F})$ such that $D_{\mathrm{V}} = u_{\mathrm{V}} \circ d_{\mathrm{B/A}}$ (**0**,20.4.8); in additional to the uniqueness of $u_{\mathrm{V}}$, it shows immediately that for every open affine subset $\mathrm{W} \subset \mathrm{V}$, we have $u_{\mathrm{W}} = u_{\mathrm{V}}|_{\mathrm{W}}$, hence the $u_{\mathrm{V}}$ defines a homomorphism of $\mathscr{O}_{\mathrm{X}}$-modules $u : \mathscr{O}_{\mathrm{X}} \to \mathscr{F}$ responding to the question. □

**16.5.4** With the notations of (16.5.1), for every open U of X, $\mathrm{Der}_{\mathrm{S}}(\mathscr{O}_{\mathrm{U}}, \mathscr{F}|_{\mathrm{U}})$ is a $\Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$-module and it is clear that the map $\mathrm{U} \mapsto \mathrm{Der}_{\mathrm{S}}(\mathscr{O}_{\mathrm{U}}, \mathscr{F}|_{\mathrm{U}})$ is a presheaf; in fact, it is even a *sheaf* (hence an $\mathscr{O}_{\mathrm{X}}$-module), in virtue of the point characterization of S-derivations, seen in (16.5.1). This $\mathscr{O}_{\mathrm{X}}$-module is denoted by $\mathscr{D}er_{\mathrm{S}}(\mathscr{O}_{\mathrm{X}}, \mathscr{F})$ and is called the ***sheaf of* S*-derivations of* $\mathscr{O}_{\mathrm{X}}$ *on* $\mathscr{F}$**, and what we have just seen is still expressed by the following corollary.

**Corollary 16.5.5** *For every $\mathscr{O}_{\mathrm{X}}$-module $\mathscr{F}$, the homomorphism of $\mathscr{O}_{\mathrm{X}}$-modules deduced from $u \mapsto u \circ d_{\mathrm{X/S}}$*

$$\mathscr{H}om_{\mathscr{O}_{\mathrm{X}}}(\Omega^1_{\mathrm{X/S}}, \mathscr{F}) \longrightarrow \mathscr{D}er_{\mathrm{S}}(\mathscr{O}_{\mathrm{X}}, \mathscr{F}) \tag{16.5.5.1}$$

*is bijective.*

**Corollary 16.5.6** *(i) If the morphism $f : \mathrm{X} \to \mathrm{S}$ is locally of finite presentation and if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module, then $\mathscr{D}er_{\mathrm{S}}(\mathscr{O}_{\mathrm{X}}, \mathscr{F})$ is a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module.*

*(ii) If* S *is moreover locally noetherian and if $\mathscr{F}$ is coherent, then $\mathscr{D}er_{\mathrm{S}}(\mathscr{O}_{\mathrm{X}}, \mathscr{F})$ is a coherent $\mathscr{O}_{\mathrm{X}}$-module.*

Proof: Assertion (i) results from the isomorphism (16.5.5.1), from (16.4.22) and from (**I**,1.3.12); assertion (ii) results from ($\mathbf{0_I}$,5.3.5). □

**16.5.7** We put

$$\mathfrak{G}_{\mathrm{X/S}} = \mathscr{H}om_{\mathscr{O}_{\mathrm{X}}}(\Omega^1_{\mathrm{X/S}}, \mathscr{O}_{\mathrm{X}}) = \mathscr{D}er_{\mathrm{S}}(\mathscr{O}_{\mathrm{X}}, \mathscr{O}_{\mathrm{X}}) \tag{16.5.7.1}$$

and we call it the ***sheaf of* S*-derivations of* $\mathscr{O}_{\mathrm{X}}$**, or the ***sheaf of* S*-derivations of* X *with respect to* S**; it is therefore the dual of the $\mathscr{O}_{\mathrm{X}}$-module $\Omega^1_{\mathrm{X/S}}$. If $f$ is locally of finite presentation, $\mathscr{G}_{\mathrm{X/S}}$ is a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module; if S is moreover locally noetherian, $\mathscr{G}_{\mathrm{X/S}}$ is coherent (16.5.6).

**16.5.8** Suppose more specifically that $\Omega^1_{\mathrm{X/S}}$ is a locally free $\mathscr{O}_{\mathrm{X}}$-module (of finite rank) (it will be the case when $f$ is smooth (17.2.3)); then $\mathscr{G}_{\mathrm{X/S}}$ is a locally free $\mathscr{O}_{\mathrm{X}}$-module with the same rank of $\Omega^1_{\mathrm{X/S}}$ at each point. More precisely, suppose $\Omega^1_{\mathrm{X/S}}$ is of rank $n$ at a point $x$; there are then $n$ sections $s_i (1 \leqslant i \leqslant n)$ of $\mathscr{O}_{\mathrm{X}}$ on a open affine neighborhood U of $x$ such that the canonical images of $ds_i$ on $\Omega^1_{\mathrm{X/S}} \otimes_{\mathscr{O}_{\mathrm{X}}} \kappa(x)$ form a basis of this $\kappa(x)$-vector space; in virtue of the Nakayama's lemma, the germs $(ds_i)_x = d(s_i)_x$ of $ds_i$ at $x$ form a basis of the $\mathscr{O}_x$-module $(\Omega^1_{\mathrm{X/S}})_x$, hence, by shrinking U, we may assume that the $ds_i$ form a basis of the $\Gamma(\mathrm{U},\mathscr{O}_{\mathrm{X}})$-module $\Gamma(\mathrm{U},\Omega^1_{\mathrm{X/S}})$. Then the $\Gamma(\mathrm{U},\mathscr{O}_{\mathrm{X}})$-module $\Gamma(\mathrm{U},\mathscr{G}_{\mathrm{X/S}})$ is the dual of previous; we denote by $(\mathrm{D}_i)_{1\leqslant i\leqslant n}$ or $(\frac{\partial}{\partial s_i})_{1\leqslant i\leqslant n}$ the dual basis of $(ds_i)_{1\leqslant i\leqslant n}$, so that, by (16.5.3), we have

$$\text{(16.5.8.1)} \qquad \mathrm{D}_i s_j = \langle \mathrm{D}_i, ds_i\rangle = \langle \frac{\partial}{\partial s_i}, ds_i\rangle = \delta_{ij} \quad \text{(Kronecker index).}$$

Every $\Gamma(\mathrm{S},\mathscr{O}_{\mathrm{S}})$-derivation of the $\Gamma(\mathrm{S},\mathscr{O}_{\mathrm{S}})$-algebra $\Gamma(\mathrm{U},\mathscr{O}_{\mathrm{X}})$ therefore can be uniquely written as

$$\mathrm{D} = \sum_{i=1}^{n} a_i \mathrm{D}_i = \sum_{i=1}^{n} a_i \frac{\partial}{\partial s_i}$$

where the $a_i (1 \leqslant i \leqslant n)$ are sections of $\mathscr{O}_{\mathrm{X}}$ on U. For every section $g \in \Gamma(\mathrm{U},\mathscr{O}_{\mathrm{X}})$, if we put $dg = \sum_{i=1}^{n} c_i ds_i$, we have $c_i = \langle \mathrm{D}_i, dg\rangle = \mathrm{D}_i g$ in virtue of (16.5.8.1), in other words,

$$\text{(16.5.8.2)} \qquad dg = \sum_{i=1}^{n} (\mathrm{D}_i g) ds_i = \sum_{i=1}^{n} \frac{\partial g}{\partial s_i} ds_i.$$

**16.5.9** Let $D_1, D_2$ be two S-derivations of $\mathscr{O}_{\mathrm{X}}$. For every open U of X, if $\mathrm{D}_1^{\mathrm{U}}, \mathrm{D}_2^{\mathrm{U}}$ are the corresponding derivations of the ring $\Gamma(\mathrm{U},\mathscr{O}_{\mathrm{X}})$, the bracket

$$[\mathrm{D}_1^{\mathrm{U}}, \mathrm{D}_2^{\mathrm{U}}] = \mathrm{D}_1^{\mathrm{U}} \circ \mathrm{D}_2^{\mathrm{U}} - \mathrm{D}_2^{\mathrm{U}} \circ \mathrm{D}_1^{\mathrm{U}}$$

is also a derivation of this ring, hence the $\psi^*(\mathscr{O}_{\mathrm{S}})$-endomorphism of $\mathscr{O}_{\mathrm{X}}$

$$\text{(16.5.9.1)} \qquad [D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

is again an S-derivation; since we can verify immediately that this bracket satisfies the Jacobi identity, we see that we have thus defined on $\mathscr{D}er_{\mathrm{S}}(\mathscr{O}_{\mathrm{X}},\mathscr{O}_{\mathrm{X}})$ a structure of *Lie algebra* over $\Gamma(\mathrm{S}.\mathscr{O}_{\mathrm{S}})$. Since the definition of this structure commutes with the restriction to an open of X, we see that $\mathfrak{G}_{\mathrm{X/S}}$ is canonically provided with a structure of *Lie algebra* over $\psi^*(\mathscr{O}_{\mathrm{S}})$. Note that the map $(D_1, D_2) \mapsto [D_1, D_2]$ is not $\Gamma(\mathrm{X},\mathscr{O}_{\mathrm{X}})$-bilinear.

**16.5.10** For every base change $f\colon \mathrm{S}' \to \mathrm{S}$, if we put $\mathrm{X}' = \mathrm{X} \times_{\mathrm{S}} \mathrm{S}'$, we saw (16.4.5) that we have a canonical isomorphism

$$\text{(16.5.10.1)} \qquad \Omega^1_{\mathrm{X/S}} \otimes_{\mathrm{S}} \mathrm{S}' \xrightarrow{\sim} \Omega^1_{\mathrm{X'/S'}}$$

from which we deduce, in virtue of (16.5.10.1), a canonical homomorphism (Bourbaki, *Alg.*, chap. II, 3 ed., §5, no 3)

$$\text{(16.5.10.2)} \qquad \mathfrak{G}_{\mathrm{X/S}} \otimes_{\mathscr{O}_{\mathrm{S}}} \mathscr{O}_{\mathrm{S}'} \longrightarrow \mathfrak{G}_{\mathrm{X'/S'}}$$

which in general is neither injective nor surjective. However:

**Proposition 16.5.11** *(i) If $g\colon \mathrm{S}' \to \mathrm{S}$ is a flat morphism and if $f$ is locally of finite type (resp. locally of finite presentation), the homomorphism (16.5.10.2) is injective (resp. bijective).*

*(ii) If $\Omega^1_{\mathrm{X}/\mathrm{S}}$ is a locally free $\mathscr{O}_\mathrm{X}$-module of finite type, the homomorphism (16.5.10.2) is bijective.*

Proof: In fact, assertion (ii) results from Bourbaki, *Alg.*, chap. II, 3 ed., §5, no 3, prop. 7. The assertion (i) results similarly from Bourbaki, *Alg. comm.*, chap. I, §2, no 10, prop. 11 and the fact that if $f$ is locally of finite type (resp. locally of finite presentation), $\Omega^1_{\mathrm{X}/\mathrm{S}}$ is an $\mathscr{O}_\mathrm{X}$-module of finite type (resp. of finite presentation) ((16.3.9) and (16.4.22)). $\square$

**16.5.12** Since $\Omega^1_{\mathrm{X}/\mathrm{S}}$ is a quai-coherent $\mathscr{O}_\mathrm{X}$-module, we can consider the fiber bundle on X defined by $\Omega^1_{\mathrm{X}/\mathrm{S}}$ (**II**,1.7.8)

$$\mathrm{T}_{\mathrm{X}/\mathrm{S}} = \mathbb{V}(\Omega^1_{\mathrm{X}/\mathrm{S}}) \tag{16.5.12.1}$$

which is called ***tangent bundle of*** X ***with respect to*** S. We thus have a canonical bijection (**II**,1.7.9)

$$\Gamma(\mathrm{T}_{\mathrm{X}/\mathrm{S}}/\mathrm{S}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_\mathrm{X}}(\Omega^1_{\mathrm{X}/\mathrm{S}}, \mathscr{O}_\mathrm{X}) = \Gamma(\mathrm{X}, \mathfrak{G}_{\mathrm{X}/\mathrm{S}})$$

by definition of $\mathfrak{G}_{\mathrm{X}/\mathrm{S}}$, and in this isomorphism, we can replace X with any open set U of X; therefore we can say that the *tangent sheaf* of X with respect to S is isomorphic to the *sheaf of germs of* S*-sections* of the tangent bundle of X with respect to S. If $f : \mathrm{X} \to \mathrm{Y}$ is an S-morphism, we saw (16.4.19) that we have a canonical homomorphism $f_{\mathrm{X}/\mathrm{Y}/\mathrm{S}}\colon f^*(\Omega^1_{\mathrm{Y}/\mathrm{S}}) \to \Omega^1_{\mathrm{X}/\mathrm{S}}$, which gives, considering that

$$\mathbb{V}(f^*(\Omega^1_{\mathrm{Y}/\mathrm{S}})) = \mathbb{V}(\Omega^1_{\mathrm{X}/\mathrm{S}}) \times_\mathrm{Y} \mathrm{X} \qquad (\mathbf{II},1.7.11),$$

an X-morphism $\mathrm{T}_{\mathrm{X}/\mathrm{S}}(f)\colon \mathrm{T}_{\mathrm{X}/\mathrm{S}} \to \mathrm{T}_{\mathrm{Y}/\mathrm{S}} \times_\mathrm{Y} \mathrm{X}$. If $g\colon \mathrm{Y} \to \mathrm{Z}$ is another S-morphism, we have $\mathrm{T}_{\mathrm{X}/\mathrm{S}}(g \circ f) = (\mathrm{T}_{\mathrm{Y}/\mathrm{S}}(g) \times 1_\mathrm{X}) \circ \mathrm{T}_{\mathrm{X}/\mathrm{S}}(f)$ (**0**,20.5.4.1).

It results from (16.5.10.1) and from (**II**,1.7.11) that for each base change $g\colon \mathrm{S}' \to \mathrm{S}$, we have a canonical isomorphism

$$\mathrm{T}_{\mathrm{X}'/\mathrm{S}'} \xrightarrow{\sim} \mathrm{T}_{\mathrm{X}/\mathrm{S}} \times_\mathrm{S} \mathrm{S}' = \mathrm{T}_{\mathrm{X}/\mathrm{S}} \times_\mathrm{X} \mathrm{X}'. \tag{16.5.12.2}$$

**16.5.13** For every point $x \in \mathrm{X}$, a ***tangent space of*** X ***at point*** $x$ (with respect to S) is the set of all points of the fiber $\mathrm{T}_{\mathrm{X}/\mathrm{S}} \times_\mathrm{X} \mathrm{Spec}(\kappa(x))$ *rational* over $\kappa(x)$, hence the set

$$\mathrm{T}_{\mathrm{X}/\mathrm{S}}(x) = \mathrm{Hom}_{\kappa(x)}(\Omega^1_{\mathrm{X}/\mathrm{S}} \otimes_{\mathscr{O}_\mathrm{X}} \kappa(x), \kappa(x)) \tag{16.5.13.1}$$

which is the *dual* of the $\kappa(x)$-vector space $\Omega^1_{\mathscr{O}_x/\mathscr{O}_s}/\mathfrak{m}_x.\Omega^1_{\mathscr{O}_x/\mathscr{O}_s}$. When $\Omega^1_{\mathrm{X}/\mathrm{S}}$ is an $\mathscr{O}_\mathrm{X}$-module of *finite type*, $\mathrm{T}_{\mathrm{X}/\mathrm{S}}(x)$ is then a vector space of finite rank over

$\kappa(x)$, and for every base change $g\colon S' \to S$, and every point $x' \in X' = X \times_S S'$ above $x$, we have a canonical isomorphism

$$T_{X'/S'}(x') \xrightarrow{\sim} T_{X/S}(x) \otimes_{\kappa(x)} \kappa(x'). \tag{16.5.13.2}$$

If $x$ is *rational* over $\kappa(s)$, where $s = f(x)$ (so that $\kappa(s) \to \kappa(x)$ is an isomorphism), it results from (16.4.13) that we have a canonical isomorphism

$$T_{X/S}(x) = T_{X_s/\kappa(s)}(x) = \operatorname{Hom}_{\kappa(s)}(\mathfrak{m}'_x/\mathfrak{m}'^2_x, \kappa(s)) \tag{16.5.13.3}$$

where $\mathfrak{m}'_x$ is the maximal ideal of $\mathscr{O}_{X_s,x} = \mathscr{O}_{X,x}/\mathfrak{m}_s\mathscr{O}_{X,x}$. In the case where S is the spectra of a field $k$, we recover Zariski's definition of tangent space of a point $x \in X$ *rational* on $k$, as the dual of $\mathfrak{m}_x/\mathfrak{m}_x^2$.

Let Y be another S-scheme and let $g\colon Y \to X$ be an S-morphism; we also have a canonical homomorphism of $\mathscr{O}_Y$-modules (16.4.19)

$$g_{Y/X/S}\colon g^*(\Omega^1_{X/S}) \longrightarrow \Omega^1_{Y/S}. \tag{16.5.13.4}$$

Now, note that if $y \in Y$ and $x = g(y)$, we have

$$g^*(\Omega^1_{X/S}) \otimes_{\mathscr{O}_Y} \kappa(y) = (\Omega^1_{X/S} \otimes_{\mathscr{O}_X} \kappa(x)) \otimes_{\mathscr{O}_Y} \kappa(y)$$

and consequently, if $\Omega^1_{X/S}$ is an $\mathscr{O}_X$-module of *finite type*, we can identify

$$\operatorname{Hom}_{\kappa(y)}(g^*(\Omega^1_{X/S}) \otimes_{\mathscr{O}_Y} \kappa(y), \kappa(y))$$

with $T_{X/S}(x) \otimes_{\kappa(x)} \kappa(y)$. We therefore also deduce from the homomorphism (16.5.13.4) a homomorphism of $\kappa(y)$-*vector spaces*

$$T_y(g)\colon T_{X/S}(y) \longrightarrow T_{X/S}(x) \otimes_{\kappa(x)} \kappa(y) \tag{16.5.13.5}$$

called ***the tangent linear map of*** $g$ ***at point*** $y$. When $y$ is *rational* over $\kappa(s)$, we can identify $\kappa(s)$, $\kappa(y)$ and $\kappa(x)$ and $T_y(g)$ is also a homomorphism of $\kappa(s)$-vector spaces $T_{Y/S}(y) \to T_{X/S}(x)$; it should be noted that in this case, $g^*(\Omega^1_{X/S}) \otimes_{\mathscr{O}_Y} \kappa(y)$ is identified to $\Omega^1_{X/S} \otimes_{\mathscr{O}_X} \kappa(x)$, and the previous homomorphism is then defined without any condition of finiteness on $\Omega^1_{X/S}$ and is none other than the homomorphism $T_{Y/S}(g)$ (16.5.12) restricted to the fiber at point $y$ of $T_{Y/S}$.

**16.5.14** The interpretation of derivations of an A-algebra B on a B-module L, given in (**0**,20.1.1), is translated in the language of schemes as follows.

Consider two morphisms of schemes $f\colon X \to S$, $g\colon Y \to S$, and a closed subscheme $Y_0$ of Y defined by a *square-zero* ideal $\mathscr{J}$ of $\mathscr{O}_Y$ (so that Y and $Y_0$

have the same underlying topological space). Suppose given an S-morphism $u_0\colon Y_0 \to X$, so that we have a commutative diagram

(16.5.14.1)
$$\begin{tikzcd} X \arrow[d, "f"'] & Y_0 \arrow[l, "u_0"'] \arrow[d, "j"] \\ S & Y \arrow[l, "g"] \arrow[ul, dotted, "u"'] \end{tikzcd}$$

and we are proposed to see if there exists an S-morphism $u\colon Y \to X$ such that $u_0 = u \circ j$ (in other words, if it is possible to complete the previous diagram by the dotted arrow $u$, so as to leave it commutative).

For this, consider an open affine $U = \mathrm{Spec}(C)$ of Y; its inverse image $j^{-1}(U)$ is the open affine $U_0 = \mathrm{Spec}(C/\mathfrak{L})$, where $\mathfrak{L} = \Gamma(U, \mathscr{J})$, a square-zero ideal of C; we'll assume U is sufficiently small so that $u_0(U_0)$ is contained in an open affine $V = \mathrm{Spec}(B)$ of X, and $g(U) = f(u_0(U_0))$ contained in an open affine $W = \mathrm{Spec}(A)$ of S, so that B and C are A-algebras and that $u_0|_{U_0}$ corresponds to an A-homomorphism $\psi$ from B to $C/\mathfrak{L}$; let $P(U_0)$ be the set of restrictions $u|_U$ of the required homomorphism, which canonically corresponds to the A-homomorphisms of algebras $\varphi\colon B \to C$ such that the composition $B \to C \to C/\mathfrak{L}$ is equal to $\psi$. So (**0**,20.1.1) we see that the set of these homomorphisms is empty or of the form $\varphi_1 + \mathrm{Der}_A(B, \mathfrak{L})$; when $P(U_0)$ is not empty, the additive group $\mathrm{Der}_A(B, \mathfrak{L})$ acts by addition on $P(U_0)$, which is then an *affine space* for the additive group $\mathrm{Der}_A(B, \mathfrak{L})$ (or a *principal homogeneous space* (or *torsor*) under $\mathrm{Der}_A(B, \mathfrak{L})$).

Now notice that, since $\mathfrak{L}$ is provided with a structure of B-module by means of $\psi$, we have an isomorphism $v \mapsto v \circ d_{B/A}$ from $\mathrm{Hom}_B(\Omega^1_{B/A}, \mathfrak{L})$ to $\mathrm{Der}_A(B, \mathfrak{L})$ (**0**,20.4.8). Moreover, since $\mathfrak{L}$ is square-zero, hence a $(C/\mathfrak{L})$-module, every B-homomorphism $v\colon \Omega^1_{B/A} \to \mathfrak{L}$ can be considered as a $(C/\mathfrak{L})$-homomorphisms $\Omega^1_{B/A} \otimes_A (C/\mathfrak{L}) \to \mathfrak{L}$. Since $\mathscr{J}$ is square-zero, it can be considered as a quasi-coherent $\mathscr{O}_{Y_0}$-module; introduce the $\mathscr{O}_{Y_0}$-module

(16.5.14.2)
$$\mathscr{G} = \mathscr{H}om(u_0^*(\Omega^1_{X/S}), \mathscr{J});$$

it also results from the fact $\Omega^1_{B/A} = \Gamma(V, \Omega^1_{X/S})$ (16.3.7) that we can write $\mathrm{Der}_A(B, \mathfrak{L}) = \Gamma(U_0, \mathscr{G})$.

Since $P(U_0)$ is defined as a set of S-morphisms $U \to X$, it is clear that $U_0 \mapsto P(U_0)$ is a sheaf of sets $\mathscr{P}$ on $Y_0$. Let's use this fact to prove that the map $h\colon \Gamma(U_0, \mathscr{G}) \times P(U_0) \to P(U_0)$ defining the structure of torsor on $P(U_0)$ is also independent of the choice of V and W, and further that, if $U' \subset U$ is another open affine of Y, $U'_0$ the inverse image of $Y_0$, the diagram

(16.5.14.3)
$$\begin{tikzcd} \Gamma(U_0, \mathscr{G}) \times P(U_0) \arrow[r, "h"] \arrow[d] & P(U_0) \arrow[d] \\ \Gamma(U'_0, \mathscr{G}) \times P(U'_0) \arrow[r, "h'"'] & P(U'_0) \end{tikzcd}$$

is commutative (the vertical arrows being the restriction maps). In virtue of the previous remark, it is necessary to prove the commutativity of the previous diagram when $h$ is defined as above from the open affines V, W and $h'$ from open affines $V' \subset V$ and $W' \subset W$. But in virtue of the above description of $h$, this results from the commutativity of the diagram (**0**,20.5.4.2).

The map $\Gamma(U_0,\mathscr{G}) \times P(U_0) \to P(U_0)$ hence defines a homomorphism of sheaves of sets $m\colon \mathscr{G} \times \mathscr{P} \to \mathscr{P}$ such that, for every open $U_0$ for which $\Gamma(U_0,\mathscr{P}) \neq \varnothing$, $m_{U_0}\colon \Gamma(U_0,\mathscr{G}) \times \Gamma(U_0,\mathscr{P}) \to \Gamma(U_0,\mathscr{P})$ is an external action defining on $\Gamma(U_0,\mathscr{P})$ a structure of torsor under the group $\Gamma(U_0,\mathscr{G})$.

**16.5.15** In general, when one gives on a topological space Z a sheaf of sets $\mathscr{P}$, a sheaf of groups $\mathscr{G}$ (not necessarily commutative) and a homomorphism of sheaves of sets $m\colon \mathscr{G} \times \mathscr{P} \to \mathscr{P}$ such that, for every open $U \subset Z$ for which $\Gamma(U,\mathscr{P}) \neq \varnothing$, $m_U\colon \Gamma(U,\mathscr{G}) \times \Gamma(U,\mathscr{P}) \to \Gamma(U,\mathscr{P})$ makes $\Gamma(U,\mathscr{P})$ a *torsor* under the group $\Gamma(U,\mathscr{G})$, we say that $\mathscr{P}$ is a ***pseudo-torsor*** (or ***formally principal homogeneous sheaf***) under the sheaf of groups $\mathscr{G}$. We say that $\mathscr{P}$ is a ***torsor*** (or ***principal homogeneous sheaf***) under $\mathscr{G}$ if furthermore $\Gamma(U,\mathscr{P}) \neq \varnothing$ for every open $U \neq \varnothing$ of a suitable basis of the topology of Z.

For the general theory of torsors, we refer to [42]; here we will confine ourselves to recalling the canonical correspondence between the isomorphic classes of these torsors (for a given $\mathscr{G}$) and the elements of the set of cohomology $H^1(Z,\mathscr{G})$. Consider in fact a torsor $\mathscr{P}$ under $\mathscr{G}$ and a open covering $(U_\lambda)$ of Z such that $\Gamma(U_\lambda,\mathscr{P}) \neq \varnothing$ for every $\lambda$; let us denote by $p_\lambda$ an element of $\Gamma(U_\lambda,\mathscr{P})$. For every pair of indexes $\lambda,\mu$ such that $U_\lambda \cap U_\mu \neq \varnothing$, there also exists a unique element $\gamma_{\lambda\mu}$ of $\Gamma(U_\lambda \cap U_\mu,\mathscr{G})$ such that $\gamma_{\lambda\mu}.(p_\mu|_{U_\lambda\cap U_\mu}) = p_\lambda|_{U_\lambda\cap U_\mu}$; moreover, if $\lambda$, $\mu$, $\nu$ are three indexes such that $U_\lambda \cap U_\mu \cap U_\nu \neq \varnothing$, the restrictions $\gamma'_{\lambda\mu}$, $\gamma'_{\mu\nu}$, $\gamma'_{\lambda\nu}$ of $\gamma_{\lambda\mu}$, $\gamma_{\mu\nu}$, $\gamma_{\lambda\nu}$ to $U_\lambda \cap U_\mu \cap U_\nu$ satisfy the condition $\gamma'_{\lambda\nu} = \gamma'_{\lambda\mu}\gamma'_{\mu\nu}$; in other words, $(\lambda,\mu) \mapsto \gamma_{\lambda\mu}$ is a 1-*cocycle* of the covering $(U_\lambda)$ with values in $\mathscr{G}$. If, for each $\lambda$, $p'_\lambda$ is another element of $\Gamma(U_\lambda,\mathscr{P})$, there exists a unique element $\beta_\lambda \in \Gamma(U_\lambda,\mathscr{G})$ such that $p'_\lambda = \beta_\lambda.p_\lambda$, and the 1-cocycle $(\gamma'_{\lambda\mu})$ corresponding to the family $(p'_\lambda)$ is given by $\gamma'_{\lambda\mu} = \beta_\lambda\gamma_{\lambda\mu}\beta_\mu^{-1}$, in other words, is *cohomologous* to $(\gamma_{\lambda\mu})$. Conversely, the data of a 1-cocycle $(\gamma_{\lambda\mu})$ defines, for each pair $(\lambda,\mu)$, an automorphism $\theta_{\lambda\mu}$ of sheaves of sets $\mathscr{G}|_{U_\lambda\cap U_\mu}$, namely the right translation by $\gamma_{\lambda\mu}$, and the fact that it is a cocycle shows that we can pick up the sheaves of sets $\mathscr{G}|_{U_\lambda}$ using automorphisms $\theta_{\lambda\mu}$ ($\mathbf{0_I}$,3.3.1); we thus clearly obtain a torsor under $\mathscr{G}$, namely $\mathscr{P}$, and if we take for $p_\lambda$ the unit section on $U_\lambda$, the corresponding 1-cocycle is none other than the given 1-cocycle $(\gamma_{\lambda\mu})$; moreover, if we replace $(\gamma_{\lambda\mu})$ with a 1-cocycle $\gamma'_{\lambda\mu} = \beta_\lambda\gamma_{\lambda\mu}\beta_\mu^{-1}$ which are cohomogeneous, we immediately see that the torsor obtained is isomorphic to $\mathscr{P}$.

In particular, if $(\gamma_{\lambda\mu})$ is a 1-coboundary, in other words, of the form $\gamma_{\lambda\mu} = \beta_\lambda\beta_\mu^{-1}$, the obtained torsor is isomorphic to $\mathscr{G}$ (considered as a torsor under itself for left translations); we say in this case that $\mathscr{P}$ is *trivial*, and the converse is obvious.

More particularly, it follows from (**III**,1.3.1) that we have:

**Proposition 16.5.16** *Let* Z *be an affine scheme,* $\mathscr{G}$ *a quasi-coherent* $\mathscr{O}_{\mathrm{Z}}$*-module; then every tosor under* $\mathscr{G}$ *is trivial.*

Returning to the problem considered in (16.5.13), we obtain:

**Proposition 16.5.17** *Let* X, Y *be two* S*-schemes,* $\mathrm{Y}_0$ *a closed subscheme of* Y *defined by a quasi-coherent ideal* $\mathscr{J}$ *of* $\mathscr{O}_{\mathrm{Y}}$ *such that* $\mathscr{J}^2 = 0$, $j\colon \mathrm{Y}_0 \to \mathrm{Y}$ *the canonical injection. Let* $u_0\colon \mathrm{Y}_0 \to \mathrm{X}$ *a* S*-morphism, and* $\mathscr{P}$ *the sheaf of sets on* Y *such that, for each open* U *of* Y, $\Gamma(\mathrm{U},\mathscr{P})$ *is the set of* S*-morphisms* $u\colon \mathrm{U} \to \mathrm{X}$ *such that* $u_0|_{\mathrm{U}_0} = u \circ (j|_{\mathrm{U}_0})$*, where* $\mathrm{U}_0 = j^{-1}(\mathrm{U})$*. Then on* $\mathscr{P}$*, there exists a structure of pseudo-torsor under the* $\mathscr{O}_{\mathrm{Y}_0}$*-module* $\mathscr{G} = \mathscr{H}om_{\mathscr{O}_{\mathrm{Y}_0}}(u_0^*(\Omega^1_{\mathrm{X/S}}),\mathscr{J})$*.*

In particular:

**Corollary 16.5.18** *With the notations of (16,5,16), suppose* Y *is affine and* $\Omega^1_{\mathrm{X/S}}$ *is of finite presentation; if there exists an open covering* $(\mathrm{U}_\alpha)$ *of* Y*, and, for each index* $\alpha$*, an* S*-morphism* $v_\alpha\colon \mathrm{U}_\alpha \to \mathrm{X}$ *such that, if* $\mathrm{U}^0_\alpha = j^{-1}(\mathrm{U}_\alpha)$*, we have* $v_\alpha \circ (j|_{\mathrm{U}^0_\alpha}) = u_0|_{\mathrm{U}^0_\alpha}$*, then there exists an* S*-morphism* $u\colon \mathrm{X} \to \mathrm{Y}$ *such that* $u \circ j = u_0$*.*

Proof: In fact, $\mathscr{G}$ is then a quasi-coherent $\mathscr{O}_{\mathrm{Y}_0}$-module (**I**,1.3.12); in virtue of (16.5.16) and the fact that $\mathrm{Y}_0$ is then affine, the sheaf $\mathscr{P}$, which is by hypothesis a torsor under $\mathscr{G}$, and not just a pseudo-torsor, is *trivial*; but if $w$ is an isomorphism from $\mathscr{G}$ to $\mathscr{P}$ (as torsors under $\mathscr{G}$), the image by $w$ of the section 0 of $\mathscr{G}$ is the desired S-morphism $u$. □

## 16.6 Sheaves of $p$-differential and exterior differential

**16.6.1** Let $f\colon \mathrm{X} \to \mathrm{S}$ be a morphism of schemes. By ***sheaf of $p$-differentials of*** X ***with respect to*** S ($p$ is an integer), we mean the *$p$-th external power* ($\mathbf{0_I}$,4.1.5) of the $\mathscr{O}_{\mathrm{X}}$-module $\Omega^1_{\mathrm{X/S}}$, denoted by

(16.6.1.1) $$\Omega^p_{\mathrm{X/S}} = \wedge^p(\Omega^1_{\mathrm{X/S}}).$$

We have then $\Omega^0_{\mathrm{X/S}} = \mathscr{O}_{\mathrm{X}}$, and $\Omega^p_{\mathrm{X/S}} = 0$ for $p < 0$; the $\Omega^p_{\mathrm{X/S}}$ are the homogeneous components of the external algebra on $\Omega^1_{\mathrm{X/S}}$

(16.6.1.2) $$\Omega^\bullet_{\mathrm{X/S}} = \wedge(\Omega^1_{\mathrm{X/S}}) = \bigoplus_{p\in\mathbb{Z}} \wedge^p(\Omega^1_{\mathrm{X/S}}),$$

which is then a quasi coherent anticommutative graded $\mathscr{O}_X$-algebra and whose elements of degree 1 are square-zero. For every open affine U of X, we have $\Gamma(U, \Omega^\bullet_{X/S}) = \wedge(\Gamma(U, \Omega^1_{X/S}))$, where $\Gamma(U, \Omega^1_{X/S})$ is considered as $\Gamma(U, (O)_X)$-module.

When $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine, B hence being an A-algebra, we have ($\mathbf{0_I}$,4.1.5) $\Omega^p_{X/S} = (\Omega^p_{B/A})^\sim$, by putting $\Omega^p_{B/A} = \wedge^p(\Omega^1_{B/A})$.

**Theorem 16.6.2** *There exists a unique endomorphism $d$ of the sheaf of additive groups $\Omega^\bullet_{X/S}$, having the following properties:*

*(i)* $d \circ d = 0$.

*(ii) For every open* U *of* X *and every section $f \in \Gamma(U, \mathscr{O}_X)$, we have $df = d_{X/S} f$.*

*(iii) For every open* U *of* X*, every pair of integers $p, q$ and every pair of sections $\omega'_p \in \Gamma(U, \Omega^p_{X/S})$, $\omega''_q \in \Gamma(U, \Omega^q_{X/S})$, we have*

(16.6.2.1) $$d(\omega'_p \wedge \omega''_q) = (d\omega'_p) \wedge \omega''_q + (-1)^p \omega'_p \wedge d\omega''_q.$$

*Moreover, $d$ is an endomorphism of graded $\psi^*(\mathscr{O}_X)$-modules, of degree $+1$.*

Proof: Suppose we have proved the existence of the endomorphism $d$. For every open affine U of X, every section of $\Omega^p_{X/S}$ on U is (in virtue of (i)) a linear combination of finitely many elements of the from $g(df_1 \wedge df_2 \wedge \cdots \wedge df_p)$, where $g$ and $f_i$ are sections of $\mathscr{O}_X$ on U ($\mathbf{0}$,20.4.7). The conditions (i) and (iii) then show, by induction on $p$, that we necessarily have

(16.6.2.2) $$d(g(df_1 \wedge df_2 \wedge \cdots \wedge df_p)) = dg \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_p.$$

This proves the uniqueness of $d$ and the last assertion of the theorem. In virtue of this property of uniqueness, to demonstrate the existence of $d$, we can confine oneself to the case where $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine. Now (Bourbaki, *Alg.*, chap.III, 3 ed., §10) to define an A-antiderivation D of degree +1 of the external algebra $\wedge(M)$ (where M is a B-module and B is an A-algebra), which takes values in an anticommutative graded A-algebra $C = \bigoplus_{n=0}^{\infty} C_n$, whose element of degree 1 are hence square-zero, it suffices to arbitrarily give an A-derivation $D_0$ of B on $C_1$ and an A-homomorphism $D_1$ from M to $C_2$; then there exists a unique A-antiderivation D of $\wedge(M)$ on C coinciding with $D_0$ on B and with $D_1$ on M.

In the present case, $D_0$ is necessarily equal to $d_{B/A}$ in virtue of (ii); everything reduces to see, considering (16.6.2.2), if there is an A-homomorphism $u$ from $\Omega^1_{B/A}$ to $\Omega^2_{B/A}$ such that we have

(16.6.2.3) $$u(g.df) = dg \wedge df$$

whatever $f, g \in A$; it suffices to show that there exists an A-homomorphism $v\colon B \otimes_A \Omega^1_{B/A} \to \Omega^2_{B/A}$ such that

$$v(g.\omega) = dg \wedge \omega \tag{16.6.2.4}$$

for $g \in B$ and $\omega \in \Omega^1_{B/A}$. Finally, since $\Omega^1_{B/A} = \mathfrak{J}/\mathfrak{J}^2$ (where $\mathfrak{J} = \mathfrak{J}_{B/A}$ is the kernel of the canonical homomorphism $B \otimes_A B \to B$) and that $\Omega^1_{B/A}$ is generated by the elements of the form $g.df$, it suffices to define an A-homomorphism $w\colon B \otimes_A (B \otimes_A B) \to \Omega^2_{B/A}$ such that

$$u(g' \otimes g \otimes f) = dg' \wedge (g.df) \tag{16.6.2.5}$$

and such that $w$ is 0 on the image of $B \otimes_A \mathfrak{J}^2$. Now, since the second term of (16.6.2.5) is A-trilinear in $g', g, f$, the existence of $w$ satisfying (16.6.2.5) is immediate. Since, on the other hand, $\mathfrak{J}$ is generated by the elements $1 \otimes x - x \otimes 1$ ($x \in B$), we are brought back to verify that when $z = (1 \otimes x - x \otimes 1)(1 \otimes y - y \otimes 1)$, we have $w(g' \otimes z) = 0$. Now, since $z = 1 \otimes (xy) + (xy) \otimes 1 - x \otimes y - y \otimes x$, the formula (16.6.2.4) shows that it suffices to see if $d(xy) - x.dy - y.dx = 0$, which expresses that $d$ is a derivation.

It remains to prove that $d$ satisfies condition (i). Now, the square of an antiderivation is a derivation (Bourbaki, *loc. cit.*), and since $\Omega^\bullet_{B/A}$ is generated by $\Omega^1_{B/A}$ as B-algebra, it suffices to verify that $d(dz) = 0$ for $z \in B$ and $z \in \Omega^1_{B/A}$; in the first case, this results from the formula (16.6.2.3) with $g = 1$; in the second, we can confine ourselves to the case where $z = g.df$ with $f, g \in B$, and then, in virtue of (16.6.2.1) and (16.6.2.3), we have

$$d(d(g.df)) = d(dg \wedge df) = (d(dg)) \wedge (df) - (dg) \wedge (d(df)) = 0. \qquad \square$$

**Definition 16.6.3** *The antiderivation* $d$ *defined in (16.6.2) (also noted by* $d_{X/S}$*) is called* **external differential** *on* X *(with respect to* S*).*

**Proposition 16.6.4** *For every base change* $g\colon S' \to S$*, if we put* $X' = X \times_S S'$*, the canonical homomorphism*

$$\Omega^\bullet_{X/S} \otimes_S S' \longrightarrow \Omega^\bullet_{X'/S'} \tag{16.6.4.1}$$

*deduced from the isomorphism (16.5.9.1) is bijective. In addition, if* $s$ *is a section of* $\Omega^\bullet_{X/S}$ *on an open* U *of* X*,* $s \otimes 1$ *is its inverse image, the section of* $\Omega^\bullet_{X'/S'}$ *on the preimage* U′ *of* U *on* X′*, we have* $d_{X'/S'}(s \otimes 1) = d_{X/S}(s) \otimes 1$.

PROOF: The first assertion is immediate from the formation of the external algebra of a permutation module with any extension of the scalar ring. To prove the second, we can, in virtue of (16.6.2.2), confine ourselves to the case where $s \in \Gamma(U, \mathscr{O}_X)$, and in this case the assertion has already been proved in (16.4.3.7). $\square$

**16.6.5** Suppose that $\Omega^1_{X/S}$ is a locally free $\mathscr{O}_X$-module of rank $n$ at point $x$, so there are $n$ sections $s_i \in \Gamma(U,\mathscr{O}_X)$ such that $ds_i$ form a basis of the $\Gamma(U,\mathscr{O}_X)$-module $\Gamma(U,\Omega^1_{X/S})$ (16.5.8). Then, for every integer $p \geqslant 1$, the $p$-differentials $ds_{i_1} \wedge ds_{i_2} \wedge \cdots \wedge ds_{i_p}$ (for $i_1 < i_2 < \cdots < i_p$ being elements of [1,n]) form a basis of $\binom{n}{p}$ elements of $\Gamma(U,\Omega^p_{X/S})$ over $\Gamma(U,\mathscr{O}_X)$. Moreover, the formula (16.6.2.2) shows that for every section $g \in \Gamma(U,\mathscr{O}_X)$, we have

(16.6.5.1)

$$d(g.ds_{i_1} \wedge ds_{i_2} \wedge \cdots \wedge ds_{i_p}) = \sum_k (-1)^r \frac{\partial g}{\partial s_k} ds_{i_1} \wedge \cdots \wedge ds_{i_r} \wedge ds_k \wedge ds_{i_{r+1}} \wedge \cdots \wedge ds_{i_p}$$

where, in the second term, $k$ goes through all the $n-p$ indexes distinct from $i_k$, $i_r$ being the largest index $< k$.

Note that the relation $d(dg) = 0$ for every $g \in \Gamma(U,\mathscr{O}_X)$ can be expressed in the form

$$D_i(D_j g) = D_j(D_i g) \qquad \text{for } i \neq j$$

in other words, the derivations $D_i$ defined in (16.5.7) are two to two permutable.

## 16.7 The sheaves $\mathscr{P}^n_{X/S}(\mathscr{F})$

**16.7.1** Let $f: X \to S$ a morphism of schemes, $\mathscr{F}$ an $\mathscr{O}_X$-module. Denoted by $X^{(n)}_{\Delta_f}$ the *n-th infinitesimal neighborhood* of X for the diagonal morphism $\Delta_f : X \to X \times_S X$, by $h_n : X^{(n)}_{\Delta_f} \to X \times_S X$ the canonical morphism (16.1.2), abd consider the two compositions of morphisms

$$p_1^{(n)} : X^{(n)}_{\Delta_f} \xrightarrow{h_n} X \times_S X \xrightarrow{p_1} X, \qquad p_2^{(n)} : X^{(n)}_{\Delta_f} \xrightarrow{h_n} X \times_S X \xrightarrow{p_2} X,$$

so that, by definition, $p_1^{(n)}$ corresponds to the homomorphism of sheaves of rings $\mathscr{O}_X \to \mathscr{P}^n_{X/S}$ which we have chosen to define the $\mathscr{O}_X$-algebra structure on $\mathscr{P}^n_{X/S}$ (16.3.5), and $p_2^{(n)}$ to the homomorphism of sheaves of rings $d^n_{X/S} : \mathscr{O}_X \to \mathscr{P}^n_{X/S}$ (16.3.6). Since $X^{(n)}_{\Delta_f}$ and X have the same underlying space, we can write

$$\mathscr{P}^n_{X/S} = (p_1^{(n)})_*((p_2^{(n)})^*(\mathscr{O}_X)). \tag{16.7.1.1}$$

More generally, we put

$$\mathscr{P}^n_{X/S}(\mathscr{F}) = (p_1^{(n)})_*((p_2^{(n)})^*(\mathscr{F})) \tag{16.7.1.2}$$

so that $\mathscr{P}^n_{X/S} = \mathscr{P}^n_{X/S}(\mathscr{O}_X)$; by definition, $\mathscr{P}^n_{X/S}(\mathscr{F})$ is then an $\mathscr{O}_X$-module.

**16.7.2** If we go back to the definitions of inverse images of modules on the ringed spaces ($\mathbf{0_I}$,4.3.1) and take into account that $X_{\Delta_f}^{(n)}$ and X have the same underlying space, we see that we can also write the definition (16.7.1.2) in the form

$$\mathscr{P}^n_{X/S}(\mathscr{F}) = \mathscr{P}^n_{X/S} \otimes_{\mathscr{O}_X} \mathscr{F} \tag{16.7.2.1}$$

but where one has to be careful that, in the interpretation of the symbol $\otimes$, $\mathscr{P}^n_{X/S}$ is provided with its $\mathscr{O}_X$-module structure defined by the homomorphism of sheaves of rings $d^n_{X/S} \colon \mathscr{O}_X \to \mathscr{P}^n_{X/S}$. It follows immediately from this formula (or directly from (16.7.1.2)) that $\mathscr{P}^n_{X/S}(\mathscr{F})$ is canonically provided with a structure of $\mathscr{P}^n_{X/S}$-module.

**Proposition 16.7.3** *(i) The functor $\mathscr{F} \mapsto \mathscr{P}^n_{X/S}(\mathscr{F})$ from the category of $\mathscr{O}_X$-modules to that of $\mathscr{P}^n_{X/S}$-modules is right exact, and commutes with any inductive limits; it is exact when $\mathscr{P}^n_{X/S}$ is a flat $\mathscr{O}_X$-module.*

*(ii) If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module (resp. fo finite type, resp. of finite presentation), $\mathscr{P}^n_{X/S}(\mathscr{F})$ is a quasi-coherent $\mathscr{P}^n_{X/S}$-module (resp. fo finite type, resp. of finite presentation).*

Proof: The assertion of (i) follows immediately from the formula (16.7.2.1) and from the symmetry of $\mathscr{P}^n_{X/S}$ (16.3.4). The assertion of (ii) derive from the right exactness of the functor $\mathscr{F} \mapsto \mathscr{P}^n_{X/S}(\mathscr{F})$. □

**16.7.4** The two $\mathscr{O}_X$-module structures of $\mathscr{P}^n_{X/S}$ define on $\mathscr{P}^n_{X/S}(\mathscr{F})$ two $\mathscr{O}_X$-module structures, both permutable, hence a structure of $\mathscr{O}_X$-bimodule. It is convenient to write on the left that of these structures coming from the structural homomorphism $\mathscr{O}_X \to \mathscr{P}^n_{X/S}$ (chosen in (16.3.5)) and on the right from the homomorphism $d^n_{X/S} \colon \mathscr{O}_X \to \mathscr{P}^n_{X/S}$. In other words, for any open U of X, and any triplet of elements $a \in \Gamma(U, \mathscr{O}_X)$, $b \in \Gamma(U, \mathscr{P}^n_{X/S})$, $t \in \Gamma(U, \mathscr{F})$, we have by definition

$$a(b \otimes t) = (ab) \otimes t, \quad (b \otimes t)a = (b.d^n a) \otimes t = b \otimes (at) = (d^n a).(b \otimes t). \tag{16.7.4.1}$$

The $\mathscr{O}_X$-module structure from the definition (16.7.1.2) is then, with these conventions, the left $\mathscr{O}_X$-module structure.

If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, so is $\mathscr{P}^n_{X/S}(\mathscr{F})$ for both of its $\mathscr{O}_X$-module structures. If $\mathscr{F}$ is furthermore of finite type (resp. of finite presentation) and $f \colon X \to S$ locally of finite type (resp. locally of finite presentation), $\mathscr{P}^n_{X/S}(\mathscr{F})$ is (for both of its $\mathscr{O}_X$-module structures) of finite type (resp. of finite presentation), as follows from (16.3.9) and (16.4.22).

**16.7.5** The definition (16.7.2.1) implies the existence of a homomorphism of sheaves of commutative groups

$$d^n_{X/S,\mathscr{F}} \colon \mathscr{F} \longrightarrow \mathscr{P}^n_{X/S}(\mathscr{F}) \qquad \text{(also noted by } d^n_{X/S}) \tag{16.7.5.1}$$

such that, with the notations of (16.7.4), we have

(16.7.5.2) $$d^n_{X/S,\mathscr{F}}(t) = 1 \otimes t$$

and consequently, in virtue of (16.7.4.1)

(16.7.5.3) $$d^n_{X/S,\mathscr{F}}(at) = (1 \otimes t)a = (d^n_{X/S,\mathscr{F}}(t)).a$$

(16.7.5.4) $$d^n_{X/S,\mathscr{F}}(at) = (d^n_{X/S,\mathscr{F}}(a)).(1 \otimes t) = (d^n_{X/S,\mathscr{F}}(a))(d^n_{X/S,\mathscr{F}}(t)).$$

It is then $\mathscr{O}_X$-linear for the right $\mathscr{O}_X$-module structure on $\mathscr{P}^n_{X/S}(\mathscr{F})$, and *semi-linear* (with respect to the automorphism σ (16.3.4)) for the left $\mathscr{O}_X$-module structure.

**Proposition 16.7.6** *The left $\mathscr{O}_X$-module $\mathscr{P}^n_{X/S}(\mathscr{F})$ is generated by the image of $\mathscr{F}$ by the canonical homomorphism $d^n_{X/S,\mathscr{F}}$.*

Proof: This follows immediately from (16.7.5.3) and from the special case corresponding to $\mathscr{F} = \mathscr{O}_X$ (16.3.8). □

**16.7.7** The canonical homomorphism of sheaves of rings

$$\varphi_{nm} : \mathscr{P}^m_{X/S} \longrightarrow \mathscr{P}^n_{X/S}$$

for $n \leqslant m$ (16.1.2) define, in virtue of (16.7.2.1) of canonical homomorphism

$$\mathscr{P}^m_{X/S}(\mathscr{F}) \longrightarrow \mathscr{P}^n_{X/S}(\mathscr{F}) \quad (n \leqslant m)$$

which are homomorphisms of $\mathscr{O}_X$-bimodules in virtue of (16.1.6) and (16.7.4.1); in addition we have commutative diagrams

$$\begin{tikzcd} \mathscr{P}^m_{X/S}(\mathscr{F}) \arrow[rr] & & \mathscr{P}^n_{X/S}(\mathscr{F}) \\ & \mathscr{F} \arrow[ul, "d^m_{X/S,\mathscr{F}}"] \arrow[ur, "d^n_{X/S,\mathscr{F}}"'] & \end{tikzcd}$$

We thus have a projective system of $\mathscr{O}_X$-bimodules $(\mathscr{P}^n_{X/S}(\mathscr{F}))$, and we put

(16.7.7.1) $$\mathscr{P}^\infty_{X/S}(\mathscr{F}) = \varprojlim \mathscr{P}^n_{X/S}(\mathscr{F}).$$

Moreover, the previous shows that the homomorphisms (16.7.5.1) form a projective system of homomorphisms, and thus define a canonical homomorphism

(16.7.7.2) $$d^\infty_{X/S,\mathscr{F}} : \mathscr{F} \longrightarrow \mathscr{P}^\infty_{X/S}(\mathscr{F}).$$

**16.7.8** Let $\mathscr{F}$, $\mathscr{G}$ be two $\mathscr{O}_X$-modules; it follows immediately from the definition (16.7.2.1) that we have a canonical isomorphism of $\mathscr{P}^n_{X/S}$-modules

$$(16.7.8.1) \qquad \mathscr{P}^n_{X/S}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \xrightarrow{\sim} \mathscr{P}^n_{X/S}(\mathscr{F}) \otimes_{\mathscr{P}^n_{X/S}} \mathscr{P}^n_{X/S}(\mathscr{G})$$

(Bourbaki, *Alg.*, chap.II, 3 ed., §5, no. 1, prop. 3).

We conclude in particular (or we see directly from the definition (16.7.2.1)) that if $\mathscr{F}$ is provided with an $\mathscr{O}_X$-algebra structure (not necessarily associative), $\mathscr{P}^n_{X/S}(\mathscr{F})$ is canonically provided with a $\mathscr{P}^n_{X/S}$-algebra structure; the latter is associative (resp. commutative, resp. unitary, resp. a Lie algebra) when so is $\mathscr{F}$. In addition, the canonical homomorphisms $\mathscr{P}^m_{X/S}(\mathscr{F}) \to \mathscr{P}^n_{X/S}(\mathscr{F})$ for $n \leqslant m$ (16.7.7) are therefore di-homomorphisms of algebras; likewise, (16.7.5.1) is then a homomorphism of $\mathscr{O}_X$-algebras when $\mathscr{P}^n_{X/S}(\mathscr{F})$ is equipped with its $\mathscr{O}_X$-algebra structure from its right $\mathscr{O}_X$-module structure.

With the same notations, we also have a canonical homomorphism of $\mathscr{P}^n_{X/S}$-modules

$$(16.7.8.2) \qquad \mathscr{P}^n_{X/S}(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})) \longrightarrow \mathscr{H}om_{\mathscr{P}^n_{X/S}}(\mathscr{P}^n_{X/S}(\mathscr{F}),\mathscr{P}^n_{X/S}(\mathscr{G}))$$

(Bourbaki, *Alg.*, 3 ed., §5, no. 3), which is bijective when $\mathscr{P}^n_{X/S}$ is a locally free $\mathscr{O}_X$-module of finite type (*loc.cit.*, prop. 7).

**16.7.9** Suppose that we are in the situation described in (16.4.1); then, from the canonical homomorphism $\mathrm{P}^n(u)$ (16.4.3.3), we deduce immediately a canonical homomorphism of $\mathscr{O}_{X'}$-bimodules

$$(16.7.9.1) \qquad u^*(\mathscr{P}^n_{X/S}(\mathscr{F})) \longrightarrow \mathscr{P}^n_{X'/S'}(u^*(\mathscr{F})).$$

We leave to the reader the task of extending to this homomorphism the properties seen in (16.4) for the case $\mathscr{F} = \mathscr{O}_X$.

**Remark 16.7.10** The definition of $\mathscr{P}^n_{X/S}(\mathscr{F})$ in the form (16.7.1.2) keeps meaningful when $\mathscr{F}$ is any sheaf of sets (the inverse image of a sheaf of sets by $p_2^{(n)}$ being defined in ($\mathbf{0_I}$,3.7.1)); a variant of this definition makes it possible to define the "*jet scheme*" (with respect to S) of any X-scheme.

## 16.8 Differential operators

[1]

**Definition 16.8.1** *Let $f = (\psi,\theta)\colon X \to S$ be a morphism of schemes, $\mathscr{F}$, $\mathscr{G}$ two $\mathscr{O}_X$-modules, $n$ an integer $\geqslant 0$. We say that a homomorphism of sheaves of additive groups $D\colon \mathscr{F} \to \mathscr{G}$ is a* **differential operator of order** $\leqslant n$ *(with respect to* S*) if there exists a homomorphism of $\mathscr{O}_X$-modules $u\colon \mathscr{P}^n_{X/S}(\mathscr{F}) \to \mathscr{G}$ (where $\mathscr{P}^n_{X/S}(\mathscr{F})$ is equipped with its left $\mathscr{O}_X$-module structure (16.7.4)) such that we have $D = u \circ d^n_{X/S,\mathscr{F}}$.*

[1]For a more general formalism, see Exposé VII of [42] (due to P. Gabriel).

It is clear, in virtue of the existence of canonical homomorphisms

$$\mathscr{P}^m_{X/S}(\mathscr{F}) \longrightarrow \mathscr{P}^n_{X/S}(\mathscr{F})$$

for $n \leqslant m$ (16.7.7) that a differential operator of order $\leqslant n$ is also a differential operator of order $\leqslant m$ for all $m \geqslant n$. If $D\colon \mathscr{F} \to \mathscr{G}$ is a differential operator of order $\leqslant n$, then, for any open U of X, $D|_U : \mathscr{F}|_U \to \mathscr{G}|_U$ is also a differential operator of order $\leqslant n$.

We say that a homomorphism $D\colon \mathscr{F} \to \mathscr{G}$ of sheaves of additive groups underlying $\mathscr{F}$ and $\mathscr{G}$ is a ***differential operator*** (with respect to S) if, for every $x \in X$, there exists an open neighborhood U of $x$ and an integer $n \geqslant 0$ such that $D|_U : \mathscr{F}|_U \to \mathscr{G}|_U$ is a differential operator of order $\leqslant n$. The *order* of a differential operator $D\colon \mathscr{F} \to \mathscr{G}$ is the lower bound of integers $n$ such that $D$ is a differential order operator of order $\leqslant n$ (and hence $+\infty$ if there are no such integers); this order is always finite if X is quasi-compact. The differential order operators of order 0 are none other than the homomorphisms of $\mathscr{O}_X$-modules $\mathscr{F} \to \mathscr{G}$; we convenient that any differential operator of order $< 0$ is zero. For $n \geqslant 0$, a differential operator is usually not a homomorphism of $\mathscr{O}_X$-modules but is still a homomorphism of $\psi^*(\mathscr{O}_S)$-modules.

When $\mathscr{F} = \mathscr{O}_X$, a differential order operators of order $\leqslant 1$ of $\mathscr{O}_X$ on $\mathscr{G}$ can be uniquely written in the form $v + D$, where $v\colon \mathscr{O}_X \to \mathscr{G}$ is an $\mathscr{O}_X$-homomorphism, and $D$ is an S-derivation (16.5.1) of $\mathscr{O}_X$ on $\mathscr{G}$: this follows from the structure of $P^1_{B/A}$ (**0**,20.4.8).

**16.8.2** To describe more explicitly a differential operator of order $\leqslant n$, $D\colon \mathscr{F} \to \mathscr{G}$, it suffices, for any open affine U of X, whose image in S is contained in an open affine V, to characterize the homomorphism $\mathrm{D} = D_U\colon \Gamma(U,\mathscr{F}) \to \Gamma(U,\mathscr{G})$. If we put $\Gamma(V,\mathscr{O}_S) = A$, $\Gamma(U,\mathscr{O}_X) = B$, so that B is an A-algebra, we have $\Gamma(U,\mathscr{P}^n_{X/S}) = (B \otimes_A B)/\mathfrak{J}^{n+1}$, where we put $\mathfrak{J} = \mathfrak{J}_{B/A}$ for abbreviation. Let's put $M = \Gamma(U,\mathscr{F})$, $N = \Gamma(U,\mathscr{G})$; then the definition of $D$ implies that for each pair (U, V) satisfying the previous conditions, the A-homomorphism $\mathrm{D}\colon M \to N$ factorizes into

$$M \longrightarrow ((B \otimes_A B)/\mathfrak{J}^{n+1}) \otimes_B M \xrightarrow{v} N$$

where the first arrow is the canonical homomorphism $t \mapsto 1 \otimes t$, and $v$ is a B-homomorphism, the B-module structure of $((B \otimes_A B)/\mathfrak{J}^{n+1}) \otimes_B M$ is from the first factor B (whereas recall that in the formation of the tensor product over B, the B-module structure of $(B \otimes_A B)/\mathfrak{J}^{n+1}$ is from the second factor B). Now, note that the B-module $((B \otimes_A B)/\mathfrak{J}^{n+1}) \otimes_B M$ is isomorphic to $(B \otimes_A M)/\mathfrak{J}^{n+1}(B \otimes_A M)$, where $B \otimes_A M$ is considered as $(B \otimes_A B)$-module and its B-module structure is from the homomorphism $b \mapsto b \otimes 1$ from B to $B \otimes_A B$. Then let $\mathrm{D}'$ be the B-homomorphism from $B \otimes_A M$ to N such that $\mathrm{D}'(b \otimes t) = b\mathrm{D}(t)$; the condition of factorization on D is still expressed by saying that $\mathrm{D}'$ must be zero in the B-module $\mathfrak{J}^{n+1}(B \otimes_A M)$.

**16.8.3** It is clear that the set of differential operator of order $\leqslant n$ of $\mathscr{F}$ on $\mathscr{G}$ is an additive group, denoted by $\mathrm{Diff}^n_{\mathrm{X/S}}(\mathscr{F},\mathscr{G})$; when $\mathscr{F}=\mathscr{G}=\mathscr{O}_x$, we also write $\mathrm{Diff}^n_{\mathrm{X/S}}$ instead of $\mathrm{Diff}^n_{\mathrm{X/S}}(\mathscr{O}_{\mathrm{X}},\mathscr{O}_{\mathrm{X}})$.

We have seen (16.8.1) that we have for every two open $\mathrm{U}\supset\mathrm{V}$ of X a canonical homomorphism of restriction

$$\mathrm{Diff}^n_{\mathrm{U/S}}(\mathscr{F}|_{\mathrm{U}},\mathscr{G}|_{\mathrm{U}})\longrightarrow\mathrm{Diff}^n_{\mathrm{V/S}}(\mathscr{F}|_{\mathrm{V}},\mathscr{G}|_{\mathrm{V}})$$

then $\mathrm{U}\mapsto\mathrm{Diff}^n_{\mathrm{U/S}}(\mathscr{F}|_{\mathrm{U}},\mathscr{G}|_{\mathrm{U}})$ is a presheaf of additive groups; in fact it is also a *sheaf*, because for variable open U of X, the homomorphisms $u\mapsto u\circ d^n_{\mathrm{U/S},\mathscr{F}|_{\mathrm{U}}}$ are isomorphisms of additive groups

$$\mathrm{Hom}_{\mathscr{O}_{\mathrm{U}}}(\mathscr{P}^n_{\mathrm{U/S}}(\mathscr{F}|_{\mathrm{U}}),\mathscr{G}|_{\mathrm{U}})\xrightarrow{\sim}\mathrm{Diff}^n_{\mathrm{X/S}}(\mathscr{F}|_{\mathrm{U}},\mathscr{G}|_{\mathrm{U}}) \tag{16.8.3.1}$$

in virtue of the fact that the image of $\mathscr{F}$ by $d^n_{\mathrm{X/S},\mathscr{F}}$ generates $\mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F})$ (16.7.6). We denote this sheaf by $\mathscr{D}\mathit{iff}^n_{\mathrm{X/S}}(\mathscr{F},\mathscr{G})$, and then we have:

**Proposition 16.8.4** *The isomorphisms (16.8.3.1) define an isomorphism of sheaves of additive groups*

$$\mathscr{H}\mathit{om}_{\mathscr{O}_{\mathrm{X}}}(\mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F}),\mathscr{G})\xrightarrow{\sim}\mathscr{D}\mathit{iff}^n_{\mathrm{X/S}}(\mathscr{F},\mathscr{G}). \tag{16.8.4.1}$$

When $\mathscr{F}=\mathscr{G}=\mathscr{O}_{\mathrm{X}}$, we also write $\mathscr{D}\mathit{iff}^n_{\mathrm{X/S}}$ instead of $\mathscr{D}\mathit{iff}^n_{\mathrm{X/S}}(\mathscr{O}_{\mathrm{X}},\mathscr{O}_{\mathrm{X}})$; it follows from (16.8.4) that $\mathscr{D}\mathit{iff}^n_{\mathrm{X/S}}$ canonically identifies with the *dual* of the $\mathscr{O}_{\mathrm{X}}$-module $\mathscr{P}^n_{\mathrm{X/S}}$; we also write $\langle t,D\rangle$ instead of $u(t)$ if $t$ is a section of $\mathscr{P}^n_{\mathrm{X/S}}$ on an open and $u$ is the homomorphism from $\mathscr{P}^n_{\mathrm{X/S}}$ to $\mathscr{O}_{\mathrm{X}}$ corresponding to $D$.

**16.8.5** Since $\mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F})$ is equipped with an $\mathscr{O}_{\mathrm{X}}$-bimodule structure (16.7.4), we canonically deduce an $\mathscr{O}_{\mathrm{X}}$-bimodule structure on $\mathscr{H}\mathit{om}_{\mathscr{O}_{\mathrm{X}}}(\mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F}),\mathscr{G})$, hence also on $\mathscr{D}\mathit{iff}^n_{\mathrm{X/S}}(\mathscr{F},\mathscr{G})$ in virtue of (16.8.4.1). More precise, in virtue of the definition (16.8.1), the left $\mathscr{O}_{\mathrm{X}}$-module structure on $\mathscr{D}\mathit{iff}^n_{\mathrm{X/S}}(\mathscr{F},\mathscr{G})$ corresponding to the left $\mathscr{O}_{\mathrm{X}}$-module structure on $\mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F})$ can be explained as follow: for every open U of X, every section $a\in\Gamma(\mathrm{U},\mathscr{O}_{\mathrm{X}})$ and every differential operator $D\colon\mathscr{F}|_{\mathrm{U}}\to\mathscr{G}|_{\mathrm{U}}$, $aD$ is the differential operator which corresponds a section $t\in\Gamma(\mathrm{U},\mathscr{F})$ to the section

$$(aD)(t)=a(D(t)) \tag{16.8.5.1}$$

of $\Gamma(\mathrm{U},\mathscr{G})$. Similarly, the right $\mathscr{O}_{\mathrm{X}}$-module structure on $\mathscr{D}\mathit{iff}^n_{\mathrm{X/S}}(\mathscr{F},\mathscr{G})$ corresponding to the right $\mathscr{O}_{\mathrm{X}}$-module structure on $\mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F})$ can be explained as follow: with the same notations as above, $Da$ is the differential operator which corresponds a section $t\in\Gamma(\mathrm{U},\mathscr{F})$ to the section

$$(Da)(t)=D(at). \tag{16.8.5.2}$$

**Proposition 16.8.6** *If $f : \mathrm{X} \to \mathrm{S}$ is a morphism locally of finite presentation, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module of finite presentation and $\mathscr{G}$ a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module, then $\mathscr{D}iff^n_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$ is a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module for both structures defined in (16.8.5).*

Proof: The proposition follows from the fact that, under the assumptions, $\mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module of finite presentation (16.7.4), and of (**I**,1.3.12). □

**16.8.7** The set of differential operators of $\mathscr{F}$ on $\mathscr{G}$ (not of a specific order (16.8.1)) is denoted $\mathrm{Diff}_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$; we still see ad in (16.8.3) that $\mathrm{U} \mapsto \mathrm{Diff}_{\mathrm{U/S}}(\mathscr{F}|_{\mathrm{U}}, \mathscr{G}|_{\mathrm{U}})$ is a sheaf of additive groups, which we will denoted by $\mathscr{D}iff_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$. It is immediate that $\mathscr{D}iff_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$ is the union of the increasing filtration of subsheaves $\mathscr{D}iff^n_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$; if X is quasi-compact, $\mathrm{Diff}_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$ is similarly the union of subgroups $\mathrm{Diff}^n_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$ (16.8.1). The $\mathscr{O}_{\mathrm{X}}$-bimodule structures on $\mathscr{D}iff^n_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$ therefore define an $\mathscr{O}_{\mathrm{X}}$-bimodule structure on $\mathscr{D}iff_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$, still explained by (16.8.5.1) and (16.8.5.2).

Note that, for $n \leqslant m$, we have a commutative diagram

$$\begin{array}{ccc} \mathscr{H}om_{\mathscr{O}_{\mathrm{X}}}(\mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F}), \mathscr{G}) & \xrightarrow{\sim} & \mathscr{D}iff^n_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G}) \\ \downarrow & & \downarrow \\ \mathscr{H}om_{\mathscr{O}_{\mathrm{X}}}(\mathscr{P}^m_{\mathrm{X/S}}(\mathscr{F}), \mathscr{G}) & \xrightarrow{\sim} & \mathscr{D}iff^m_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G}) \end{array} \tag{16.8.7.1}$$

where the horizontal arrows are the isomorphisms (16.8.4.2) and the vertical arrow on the left is given by the canonical homomorphism $\mathscr{P}^m_{\mathrm{X/S}}(\mathscr{F}) \to \mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F})$ (16.7.7). For every open U of X, let us then provide $\Gamma(\mathrm{U}, \mathscr{P}^\infty_{\mathrm{X/S}}(\mathscr{F})) = \varprojlim \Gamma(\mathrm{U}, \mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F}))$ the projective limit topology of the discrete topologies on $\Gamma(\mathrm{U}, \mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F}))$, which makes $\Gamma(\mathrm{U}, \mathscr{P}^\infty_{\mathrm{X/S}}(\mathscr{F}))$ a topological $\Gamma(\mathrm{U}, \mathscr{O}_{\mathrm{X}})$-bimodule, so that $\mathscr{P}^\infty_{\mathrm{X/S}}(\mathscr{F})$ appears as a sheaf with values in the category of topological commutative groups ($\mathbf{0_I}$,3.2.6). Then (**G,II**,1.11) the limit of the inductive system of sheaves of commutative groups $\mathscr{H}om_{\mathscr{O}_{\mathrm{X}}}(\mathscr{P}^n_{\mathrm{X/S}}(\mathscr{F}), \mathscr{G})$ is none other than the sheaf of germs of continuous homomorphisms from $\mathscr{P}^\infty_{\mathrm{X/S}}(\mathscr{F})$ to $\mathscr{G}$ (provided with discrete topology): the continuous homomorphisms from $\Gamma(\mathrm{U}, \mathscr{P}^\infty_{\mathrm{X/S}}(\mathscr{F}))$ to $\Gamma(\mathrm{U}, \mathscr{G})$ corresponds in fact to inductive systems of group homomorphisms $\Gamma(\mathrm{U}, \mathscr{P}^\infty_{\mathrm{X/S}}(\mathscr{F})) \to \Gamma(\mathrm{U}, \mathscr{G})$. We can therefore still express (16.8.4) saying that we have a canonical isomorphism

$$\mathscr{H}om.cont_{\mathscr{O}_{\mathrm{X}}}(\mathscr{P}^\infty_{\mathrm{X/S}}(\mathscr{F}), \mathscr{G}) \xrightarrow{\sim} \mathscr{D}iff_{\mathrm{X/S}}(\mathscr{F}, \mathscr{G})$$

where the first term denotes the sheaf of germs of continuous homomorphisms from $\mathscr{P}^\infty_{\mathrm{X/S}}(\mathscr{F})$ to $\mathscr{G}$.

**Proposition 16.8.8** *Let $\mathscr{F}$, $\mathscr{G}$ be two $\mathscr{O}_{\mathrm{X}}$-modules, $D\colon \mathscr{F} \to \mathscr{G}$ a homomorphism of $\psi^*(\mathscr{O}_{\mathrm{S}})$-module, $n$ an integer $\geqslant 0$. The following conditions are equivalent:*

a) *$D$ is a differential operator of order $\leqslant n$.*

b) *For every section $a$ of $\mathscr{O}_{\mathrm{X}}$ on an open $\mathrm{U}$, the homomorphism $D_a\colon \mathscr{F}|_{\mathrm{U}} \to \mathscr{G}|_{\mathrm{U}}$ such that, for every section $t$ of $\mathscr{F}$ on an open $\mathrm{V} \subset \mathrm{U}$, we have*

$$D_a(t) = D(at) - aD(t) \tag{16.8.8.1}$$

*is a differential operator of order $\leqslant n-1$.*

c) *For every open $\mathrm{U}$ of $\mathrm{X}$, every family $(a_i)_{1\leqslant i\leqslant n+1}$ of $n+1$ sections of $\mathscr{O}_{\mathrm{X}}$ on $\mathrm{U}$ and every section $t$ of $\mathscr{F}$ on $\mathrm{U}$, we have identity*

$$\sum_{\mathrm{H}\subset \mathrm{I}_{n+1}} (-1)^{|\mathrm{H}|}\Big(\prod_{i\in \mathrm{H}} a_i\Big)D\Big(\Big(\prod_{j\notin \mathrm{H}} a_j\Big)t\Big) = 0 \tag{16.8.8.2}$$

*(where $\mathrm{I}_{n+1}$ is the interval $1 \leqslant i \leqslant n+1$ of $\mathbb{N}$).*

Proof: Let us first prove the equivalence of a) and c). By definition, to prove that $D$ is a differential operator of order $\leqslant n$, it suffices to show that this is true for the restriction $D|_{\mathrm{U}}\colon \mathscr{F}|_{\mathrm{U}} \to \mathscr{G}|_{\mathrm{U}}$ to any open affine open $\mathrm{U}$ of $\mathrm{X}$, and on the other hand property c) is valid for any open $\mathrm{U}$ of $\mathrm{X}$ if it is valid for any open affine. We can therefore confine ourselves to the case where $\mathrm{S} = \mathrm{Spec}(\mathrm{A})$ and $\mathrm{X} = \mathrm{Spec}(\mathrm{B})$ are affine. In virtue of (16.8.2) (we keep whose notations), the condition a) then implies that the A-homomorphism $\mathrm{D}'\colon \mathrm{B}\otimes_{\mathrm{A}} \mathrm{M} \to \mathrm{N}$ such that $\mathrm{D}'(b\otimes t) = b\mathrm{D}'(t)$ is annulled by $\mathfrak{J}^{n+1}(\mathrm{B}\otimes_{\mathrm{A}}\mathrm{M})$, which, in virtue of (**0**,20.4.4), is equivalent to say that $\mathrm{D}'$ is annulled by every elements of the form

$$\Big(\prod_{i=1}^{n+1}(a_i\otimes 1 - 1\otimes a_i)\Big).(1\otimes t)$$

where $a_i \in \mathrm{B}$ and $t \in \mathrm{M}$. Now this element is written as $\sum_{\mathrm{H}\subset \mathrm{I}_{n+1}} (-1)^{|\mathrm{H}|}(\prod_{i\in\mathrm{H}} a_i)\otimes ((\prod_{j\notin\mathrm{H}} a_j)t)$, and the value of $\mathrm{D}'$ for this element is the first term of (16.8.8.2), which proves the equivalence of a) and c).

Now prove the equivalence of b) and c). Reason by induction on $n$, the assertion being trivial for $n = 0$. Write $a_{n+1}$ instead of $a$ in the condition b), we see, in virtue of the inductive hypothesis, that the condition b) implies that for any family $(a_i)_{1\leqslant i\leqslant n}$ of $n$ sections of $\mathscr{O}_{\mathrm{X}}$ on $\mathrm{U}$ and any section $t$ of $\mathscr{F}$ on $\mathrm{U}$, we have

$$\sum_{\mathrm{H}'\subset \mathrm{I}_n} (-1)^{|\mathrm{H}'|}\Big(\prod_{i\in \mathrm{H}'} a_i\Big)D_{a_{n+1}}\Big(\Big(\prod_{j\notin \mathrm{H}'} a_j\Big)t\Big) = 0$$

But if we replace $D_{a_{n+1}}$ with its definition (16.8.8.1) in this relation, we immediately note that we obtain, with a sign near, the first term of (16.8.8.2); hence the conclusion. □

**Proposition 16.8.9** *If $D\colon \mathscr{F} \to \mathscr{G}$ is a differential operator of order $\leqslant n$, and $D'\colon \mathscr{G} \to \mathscr{H}$ a differential operator of order $\leqslant n'$, the $D' \circ D\colon \mathscr{F} \to \mathscr{H}$ is a differential operator of order $\leqslant n+n'$.*

Proof: By assumption, we can write $D = u \circ d^n_{X/S,\mathscr{F}}$ and $D' = v \circ d^n_{X/S,\mathscr{G}}$, where $u\colon \mathscr{P}^n_{X/S} \otimes_{\mathscr{O}_X} \mathscr{F} \to \mathscr{G}$ and $v\colon \mathscr{P}^{n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{H}$ are $\mathscr{O}_X$-homomorphisms. Everything comes back to show that the composed homomorphism of sheaves of additive groups

$$\mathscr{F} \xrightarrow{d^n_{X/S,\mathscr{F}}} \mathscr{P}^n_{X/S} \otimes_{\mathscr{O}_X} \mathscr{F} \xrightarrow{u} \mathscr{G} \xrightarrow{d^{n'}_{X/S,\mathscr{G}}} \mathscr{P}^{n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{G}$$

factorizes into

$$\mathscr{F} \xrightarrow{d^{n+n'}_{X/S,\mathscr{F}}} \mathscr{P}^{n+n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{F} \xrightarrow{w} \mathscr{P}^{n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{G}$$

where $w$ is an $\mathscr{O}_X$-homomorphism. It suffices to show that

**Lemma 16.8.9.1** *There exists a unique $\mathscr{O}_X$-homomorphism*

$$\delta\colon \mathscr{P}^{n+n'}_{X/S} \to \mathscr{P}^{n'}_{X/S}(\mathscr{P}^n_{X/S}) = \mathscr{P}^{n'}_{X/S} \otimes_{\mathscr{O}_X} \mathscr{P}^n_{X/S} \tag{16.8.9.2}$$

*making the diagram commutative*

$$\begin{tikzcd} \mathscr{O}_X \arrow[r, "d^{n+n'}_{X/S}"] \arrow[d, "d^n_{X/S}"'] & \mathscr{P}^{n+n'}_{X/S} \arrow[d, "\delta"] \\ \mathscr{P}^n_{X/S} \arrow[r, "d^{n'}_{X/S,\mathscr{P}^n_{X/S}}"'] & \mathscr{P}^{n'}_{X/S}(\mathscr{P}^n_{X/S}) \end{tikzcd} \tag{16.8.9.3}$$

We will then have in fact a commutative diagram deduced from (16.8.9.3) by tensor with $\mathscr{F}$

$$\begin{tikzcd} \mathscr{F} \arrow[r, "d^{n+n'}_{X/S,\mathscr{F}}"] \arrow[d, "d^n_{X/S,\mathscr{F}}"'] & \mathscr{P}^{n+n'}_{X/S}(\mathscr{F}) \arrow[d, "\delta"] \\ \mathscr{P}^n_{X/S}(\mathscr{F}) \arrow[r, "d^{n'}_{X/S,\mathscr{P}^n_{X/S}(\mathscr{F})}"'] & \mathscr{P}^{n'}_{X/S}(\mathscr{P}^n_{X/S}(\mathscr{F})) \end{tikzcd}$$

and on the other hand, we immediately see from the definition (16.7.5) that the diagram

$$\begin{tikzcd} \mathscr{P}^n_{X/S}(\mathscr{F}) \arrow[r, "u"] \arrow[d, "d^{n'}_{X/S,\mathscr{P}^n_{X/S}(\mathscr{F})}"'] & \mathscr{G} \arrow[d, "d^{n'}_{X/S,\mathscr{G}}"] \\ \mathscr{P}^{n'}_{X/S}(\mathscr{P}^n_{X/S}(\mathscr{F})) \arrow[r, "1\otimes u"'] & \mathscr{P}^{n'}_{X/S}(\mathscr{G}) \end{tikzcd}$$

is commutative. We will answer the question by taking $w$ to be the composed $\mathscr{O}_X$-homomorphism

$$\mathscr{P}^{n+n'}_{X/S}(\mathscr{F}) \xrightarrow{\delta\otimes 1} \mathscr{P}^{n'}_{X/S}(\mathscr{P}^{n}_{X/S}(\mathscr{F})) \xrightarrow{1\otimes u} \mathscr{P}^{n'}_{X/S}(\mathscr{G}).$$

It remains to prove the lemma (16.8.9.1). Given (16.7.6), which proves the uniqueness of $\delta$, we are brought back to the case where $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine; put $\mathfrak{J} = \mathfrak{J}_{B/A}$, it is about to define a canonical homomorphism of B-modules

$$\varphi\colon (B\otimes_A B)/\mathfrak{J}^{n+n'+1} \longrightarrow (B\otimes_A B)/\mathfrak{J}^{n'+1} \otimes_B (B\otimes_A B)/\mathfrak{J}^{n+1}$$

the B-module structures of both terms come from the first factor B; recall that in the tensor product of the second term, $(B\otimes_A B)/\mathfrak{J}^{n'+1}$ must be considered as right B-module by the second factor B and $(B\otimes_A B)/\mathfrak{J}^{n+1}$ as left B-module by the first factor B (16.7.2). It is the same thing to define a homomorphism of B-modules

$$\varphi_0\colon B\otimes_A B \longrightarrow (B\otimes_A B)/\mathfrak{J}^{n'+1} \otimes_B (B\otimes_A B)/\mathfrak{J}^{n+1}$$

and prove that it is annulled by $\mathfrak{J}^{n+n'+1}$. Now, such a homomorphism is immediately defined by the condition that

$$\varphi_0(b\otimes b') = \pi_{n'}(b\otimes 1)\otimes \pi_n(1\otimes b') \qquad \text{for } b,b'\in B$$

with the notations of (16.3.7). In addition, it is immediate that $\varphi_0$ is a homomorphism of rings. Now, we can write

$$\varphi_0(b\otimes 1-1\otimes b) = \pi_{n'}(b\otimes 1-1\otimes b)\otimes\pi_n(1\otimes 1)+\pi_{n'}(1\otimes b)\otimes\pi_n(1\otimes 1)-\pi_{n'}(1\otimes 1)\otimes\pi_n(1\otimes b)$$

and we have

$$\pi_{n'}(1\otimes b)\otimes\pi_n(1\otimes 1) = \pi_{n'}(1\otimes 1)b\otimes\pi_n(1\otimes 1) = \pi_{n'}(1\otimes 1)\otimes b\pi_n(1\otimes 1) = \pi_{n'}(1\otimes 1)\otimes\pi_n(b\otimes 1)$$

hence finally

$$\varphi_0(b\otimes 1-1\otimes b) = \pi_{n'}(b\otimes 1-1\otimes b)\otimes\pi_n(1\otimes 1)+\pi_{n'}(1\otimes 1)\otimes\pi_n(b\otimes 1-1\otimes b).$$

A product of $n+n'+1$ terms of the form (16.8.9.4) is necessarily nil, because it is then a product of $n+1$ terms of the form $\pi_n(b\otimes 1-1\otimes b)$ and $n'+1$ terms of the form $\pi_{n'}(b\otimes 1-1\otimes b)$. The conclusion therefore results from (**0**,20.4.4). ◻

**Corollary 16.8.10** *The sheaf $\mathscr{D}iff_{X/S}(\mathscr{O}_X,\mathscr{O}_X)$ (also denoted by $\mathscr{D}iff_{X/S}$) is canonically equipped with a structure of sheaf of rings, the $\mathscr{D}iff^n_{X/S}$ form an increasing filtration compatible with this structure.*

In particular, $\mathscr{D}iff^0_{X/S}$ is a sheaf of subrings of $\mathscr{D}iff_{X/S}$, which is canonically identified with $\mathscr{O}_X$ (16.8.1). The formulas (16.8.5.1) and (16.8.5.2) show that the $\mathscr{O}_X$-bimodule structure of $\mathscr{D}iff_{X/S}$ provided by multiplication from left and from right by sections of $\mathscr{O}_X$ considered as sheaf of subrings of $\mathscr{D}iff_{X/S}$.

**Remark 16.8.11** (i) Suppose $\mathscr{F} = \oplus_{\lambda\in L}\mathscr{F}_\lambda$; then it is clear (16.7.2.1) that $\mathscr{P}^n_{X/S}(\mathscr{F}) = \oplus_{\lambda\in L}\mathscr{P}^n_{X/S}(\mathscr{F}_\lambda)$; since the functor $\mathscr{F} \mapsto \Gamma(U,\mathscr{F})$ commutes with the formation of any direct sums, $d^n_{X/S,\mathscr{F}}$ is the homomorphism whose restriction to each $\mathscr{F}_\lambda$ is $d^n_{X/S,\mathscr{F}_\lambda}\colon \mathscr{F}_\lambda\mathscr{P}^n_{X/S}(\mathscr{F}_\lambda)$; we immediately conclude that we have

$$\mathrm{Diff}^n_{X/S}(\mathscr{F},\mathscr{G}) = \prod_{\lambda\in L}\mathrm{Diff}^n_{X/S}(\mathscr{F}_\lambda,\mathscr{G}),$$

and consequently ($\mathbf{0_I}$,3.2.6)

$$\mathscr{D}iff^n_{X/S}(\mathscr{F},\mathscr{G}) = \prod_{\lambda\in L}\mathscr{D}iff^n_{X/S}(\mathscr{F}_\lambda,\mathscr{G}).$$

On the other hand, if $\mathscr{G} = \prod_{\mu\in M}\mathscr{G}_\mu$, we have ($\mathbf{0_I}$,3.2.6)

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{P}^n_{X/S}(\mathscr{F}),\mathscr{G}) = \prod_{\mu\in M}\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{P}^n_{X/S}(\mathscr{F}),\mathscr{G}_\mu)$$

every homomorphism $u$ from $\mathscr{P}^n_{X/S}(\mathscr{F})$ to $\mathscr{G}$ corresponds uniquely to the family of its compounds $u_\mu\colon \mathscr{P}^n_{X/S}(\mathscr{F}) \to \mathscr{G} \to \mathscr{G}_\mu$. We thus have

$$\mathrm{Diff}^n_{X/S}(\mathscr{F},\mathscr{G}) = \prod_{\mu\in M}\mathrm{Diff}^n_{X/S}(\mathscr{F},\mathscr{G}_\mu),$$

and consequently

$$\mathscr{D}iff^n_{X/S}(\mathscr{F},\mathscr{G}) = \prod_{\mu\in M}\mathscr{D}iff^n_{X/S}(\mathscr{F},\mathscr{G}_\mu).$$

(ii) So far, we have encountered few differential operators $\mathscr{F} \to \mathscr{G}$ only when $\mathscr{F}$ and $\mathscr{G}$ are locally free $\mathscr{O}_X$-modules of finite rank, in which case their structure is locally reduced, in virtue of (i), to that of the sheaf $\mathscr{D}iff_{X/S}$; the last will be studied further (16.11) in a particular case.

## 16.9 Regular and quasi-regular immersions

**Definition 16.9.1** *Let* X *be a ringed space. We say that an ideal* $\mathscr{J}$ *of* $\mathscr{O}_X$ *is* **regular** *(resp.* **quasi-regular***) if, for every point* $x \in \mathrm{Supp}(\mathscr{O}_X/\mathscr{J})$*, there exists an open neighborhood* U *of* $x$ *on* X *and a regular (resp. quasi-regular) sequence (**0**,15.2.2) of elements of* $\Gamma(U,\mathscr{O}_X)$ *which generates* $\mathscr{J}|_U$*.*

We will say that a regular (resp. quasi-regular) sequence of sections of $\mathscr{O}_{\mathrm{X}}$ on U which generates $\mathscr{J}|_{\mathrm{U}}$ is a ***regular (resp. quasi-regular) system of generators*** of $\mathscr{J}|_{\mathrm{U}}$.

**Definition 16.9.2** *Let* $j\colon \mathrm{Y} \to \mathrm{X}$ *be an immersion of schemes and* U *an open of* X *such that* $j(\mathrm{Y}) \subset \mathrm{U}$ *and that* $j$ *is a closed immersion from* Y *to* U*. We say that* $j$ *is* **regular** *(resp.* **quasi-regular***) if, the closed subscheme* $j(\mathrm{Y})$ *of* U *associated to* $j$ *is defined by a regular (resp. quasi-regular) ideal of* $\mathscr{O}_{\mathrm{U}}$ *(condition independent of the chosen open* U*).*

We say that a subscheme Y of an scheme X is ***regularly immersed*** (resp. ***quasi-regularly immersed***) if the canonical injection $j\colon \mathrm{Y} \to \mathrm{X}$ is a regular (resp. quasi-regular) immersion. If Y is a closed subscheme of X and $\mathscr{J}$ the ideal of $\mathscr{O}_{\mathrm{X}}$ which defines Y, it is the same to say that $\mathscr{J}$ is regular (resp. quasi-regular).

For example, if A is an *integral* ring, $f$ an element $\neq 0$ of A, the subscheme $\mathrm{V}(f)$ of $\mathrm{Spec}(\mathrm{A})$ (isomorphic to $\mathrm{Spec}(\mathrm{A}/f\mathrm{A})$) is *regularly immersed* on $\mathrm{Spec}(\mathrm{A})$.

Every regular ideal is quasi-regular (**0**,15.2.2); every regular immersion is quasi-regular (cf. (16.9..11) for converse).

**Proposition 16.9.3** *Let* X *be a ringed space,* $\mathscr{J}$ *an ideal of* $\mathscr{O}_{\mathrm{X}}$*,* $(f_i)_{1\leqslant i\leqslant m}$ *a finite sequence of sections of* $\mathscr{O}_{\mathrm{X}}$ *on* X *generating* $\mathscr{J}$*. For* $(f_i)$ *to be a quasi-regular sequence (***0***,15.2.2), it is necessary and sufficient that the following conditions be satisfied:*

*(i) The canonical image of* $f_i$ *on* $\mathscr{J}/\mathscr{J}^2$ *form a basis of this* $(\mathscr{O}_{\mathrm{X}}/\mathscr{J})$*-module.*

*(ii) The canonical surjective homomorphism (16.1.2.2)*

$$\mathbb{S}^{\bullet}_{\mathscr{O}_{\mathrm{X}}/\mathscr{J}}(\mathscr{J}/\mathscr{J}^2) \longrightarrow \mathscr{G}r_{\bullet}(\mathscr{O}_{\mathrm{X}})$$

*is bijective.*

*In addition, if this is the case, any sequence* $(f'_i)_{1\leqslant i\leqslant n}$ *of* $n$ *sections of* $\mathscr{J}$ *on* X *which generates* $\mathscr{J}$ *is quasi-regular.*

Proof: The two conditions of the statement only translate the definition given in (**0**,15.2.2), considering the definition of canonical homomorphisms (**0**,15.2.1.1). The last assertion follows from the fact that, if a module M over a commutative ring A admits a base of $n$ elements, every system of generators of M having $n$ elements is a base of M (Bourbaki, *Alg. comm.*, Chapter II, §3, cor. 5 of section 1). □

**Corollary 16.9.4** *Let* X *be a locally ringed space,* $\mathscr{J}$ *an ideal of* $\mathscr{O}_{\mathrm{X}}$*. For* $\mathscr{J}$ *to be quasi-regular, it is necessary and sufficient that the following conditions be satisfied:*

*(i)* $\mathscr{J}$ *is of finite type.*

*(ii)* $\mathscr{J}/\mathscr{J}^2$ *is a locally free* $(\mathscr{O}_X/\mathscr{J})$*-module.*

*(iii) The canonical homomorphism*

$$\mathbb{S}^{\bullet}_{\mathscr{O}_X/\mathscr{J}}(\mathscr{J}/\mathscr{J}^2) \longrightarrow \mathscr{G}r_{\bullet}(\mathscr{O}_X)$$

*is bijective.*

Proof: The necessity follows immediately from (16.9.3). To see that these conditions are sufficient, it is necessary to show, in virtue of (16.9.3), that, at a point $x \in \mathrm{Supp}(\mathscr{O}_X/\mathscr{J})$, there exists an open neighborhood U of $x$ on X and $n$ sections $f_i(1 \leqslant i \leqslant n)$ of $\mathscr{J}$ on U whose canonical images in $\mathscr{J}/\mathscr{J}^2$ form a basis of $(\mathscr{J}/\mathscr{J}^2)\big|_U$ over $\mathscr{O}_X/\mathscr{J}|_U$, then there is an open neighborhood $V \subset U$ of $x$ such that $f_i|_V$ generate $\mathscr{J}|_V$.

Now, by the assumption, we have $\mathscr{J}_x \neq \mathscr{O}_x$, hence $\mathscr{J}_x$ is contained in the maximal ideal of $\mathscr{O}_x$; since $\mathscr{J}_x$ is an $\mathscr{O}_x$-module of finite type and that the classes of $(f_i)_x$ in $\mathscr{J}_x/\mathscr{J}_x^2$ generate this $(\mathscr{O}_x/\mathscr{J}_x)$-module, the Nakayama's lemma shows that $(f_i)_x$ generate $\mathscr{J}_x$. Since $\mathscr{J}$ is of finite type, we conclude by $(\mathbf{0_I}$,5.2.2). □

**Corollary 16.9.5** *Let* X *be a locally ringed space,* $\mathscr{J}$ *a quasi-regular ideal of* $\mathscr{O}_X$*,* $(f_i)_{1 \leqslant i \leqslant n}$ *a sequence of sections of* $\mathscr{J}$ *on* X*,* $x$ *a point of* $\mathrm{Supp}(\mathscr{O}_X/\mathscr{J})$*. The following conditions are equivalent:*

*a) There exists an open neighborhood* U *of* $x$ *on* X *such that* $f_i|_U$ *form a quasi-regular sequence of elements of* $\Gamma(U, \mathscr{O}_X)$ *generating* $\mathscr{J}|_U$*.*

*b)* $(f_i)_x$ *form a system of generators of* $\mathscr{J}_x$ *whose number of elements is as small as possible (in other words,* $(f_i)_x$ *form a minimal system of generators of* $\mathscr{J}_x$*).*

*c) If* $\overline{f_i}$ *is the canonical image of* $f_i$ *in* $\Gamma(X, \mathscr{J}/\mathscr{J}^2)$*, then* $(\overline{f_i})_x$ *form a basis of the* $(\mathscr{O}_x/\mathscr{J}_x)$*-module* $\mathscr{J}_x/\mathscr{J}_x^2$*.*

Proof: By the assumptions, $\mathscr{O}_x$ is a local ring, $\mathscr{J}_x$ a ideal of finite type of $\mathscr{O}_x$ contained in the maximal ideal of $\mathscr{O}_x$; the equivalence of b) and c) then follows from Nakayama's lemma (Bourbaki, *Alg. comm.*, chap. II, §3, no. 2, prop. 5). It is clear that a) causes c) in virtue of (16.9.3); on the other hand, it follows from (($\mathbf{0_I}$,5.2.2) that if the condition c) is satisfied (hence so is b)), there is a neighborhood U of $x$ on X such that $(\mathscr{J}/\mathscr{J}^2)\big|_U$ has a constant rank equal to $n$, and that $f_i|_U$ generate $\mathscr{J}|_U$; then, it suffices to apply to U the last assertion of (16.9.3). □

**Remark 16.9.6** (i) Under the general assumptions of (16.9.5), it is not enough that $(\overline{f_i})_y$ form a basis of the $(\mathcal{O}_y/\mathcal{J}_y)$-module $\mathcal{J}_y/\mathcal{J}_y^2$ for every $y \in X$ for the sequence $(f_i)$ to generate $\mathcal{J}$. We have an example taking $X = \operatorname{Spec}(A)$, where A is a Dedekind domain, and $\mathcal{J} = \widetilde{\mathfrak{J}}$, where $\mathfrak{J}$ is a non-principal prime ideal of A; we then have indeed $\mathcal{J}_y/\mathcal{J}_y^2 = 0$ at all points distinct from the point $x \in X$ corresponding to $\mathfrak{J}$, and $\mathcal{J}_x/\mathcal{J}_x^2$ is of rank 1 over the field $\mathcal{O}_x/\mathcal{J}_x$; in addition, $\mathcal{J}$ is obviously a regular ideal.

(ii) In (16.9.5), one can not replace "quasi-regular" with "regular", even when X is an scheme (cf (16.9.12)). Let us denote by B the ring of germs of functions infinitely differentiable at the point 0 in $\mathbb{R}$; it has a maximal ideal $\mathfrak{m}$ generated by the germ $t$ of the identical mapping of $\mathbb{R}$ at the point 0, and the intersection $\mathfrak{n}$ of $\mathfrak{m}^k$ for $k > 0$ is not reduced to 0. Now let A be the quotient ring $B[T]/\mathfrak{n}TB[T]$, and let $f_1, f_2$ be the images of $t$ and T from B[T] to A. The sequence $(f_1, f_2)$ is *regular* on A: indeed, $f_1$ is not a divisor of 0 on A, because the relation $tP \in \mathfrak{n}TB[T]$, for a polynomial $P \in B[T]$, implies that the products of $t$ by the coefficients of P belong to the ideal $\mathfrak{n}$, and it follows immediately that these coefficients are themselves in $\mathfrak{n}$, so $P \in \mathfrak{n}TB[T]$. Since $B/tB$ is isomorphic to $\mathbb{R}$, $A/f_1A$ is isomorphic to the polynomial ring $\mathbb{R}[T]$, hence integral, and the image of $f_2$ on $A/f_1A$, being equal to T, is therefore not a divisor of 0, which proves our assertion. However, $f_2$ is divisor of 0 in A, because for every nonzero element $x \in \mathfrak{n}$, the image of $x$ in A is $\neq 0$, but the image of $xT$ is zero. We conclude that the sequence $(f_2, f_1)$ is not regular on A; on the other hand, the ideal $\mathfrak{J} = f_1A + f_2A$ is distinct from A, hence the condition b) and c) of (16.9.5) do not lead to condition a) when replacing "quasi-regular" with "regular".

**16.9.7** If $X = \operatorname{Spec}(A)$ is an affine scheme, we will say that an ideal $\mathfrak{J}$ of A is ***regular*** (resp. ***quasi-regular***) if, the ideal $\mathcal{J} = \widetilde{\mathfrak{J}}$ of $\mathcal{O}_X$ is regular (resp. quasi-regular); note that this notion is *local* and does not imply the existence of a system of generators of $\mathfrak{J}$ forming on A a regular (or quasi-regular) sequence as shown in the example (16.9.6,(i)); however, this is the case when A is *local* (16.9.5).

Proposition (16.9.4) translates in terms of quasi-regular immersions as follows:

**Proposition 16.9.8** *Let $j : Y \to X$ be a morphism of schemes; for $j$ to be a quasi-regular immersion, it is necessary and sufficient that $j$ satisfies the following conditions:*

*(i) $j$ is an immersion locally of finite presentation.*

*(ii) The conormal sheaf* $\mathscr{G}r_1(j) = \mathscr{N}_{\mathrm{Y/X}}$ *(16.1.2) is a locally free* $\mathscr{O}_{\mathrm{X}}$*-module.*

*(iii) The canonical homomorphism*

$$\mathbf{S}^{\bullet}_{\mathscr{O}_{\mathrm{Y}}}(\mathscr{G}r_1(j)) \longrightarrow \mathscr{G}r_{\bullet}(j)$$

*(16.1.2.2) is bijective.*

Proof: Since the question is local on Y, we can confine ourselves to the case where $j$ is the canonical injection of a closed sub-scheme Y of X, and then the translation of (16.9.4) into (16.9.8) follows from the explanation of $\mathscr{G}r_1(j)$ and $\mathscr{G}r_{\bullet}(j)$ in terms of the ideal $\mathscr{J}$ of $\mathscr{O}_{\mathrm{X}}$ defining the subscheme Y (16.1.3,(ii)). □

**Corollary 16.9.9** *Let* Y *be a scheme,* X *a* Y*-scheme,* $j\colon \mathrm{Y} \to \mathrm{X}$ *a* Y*-section of* X*, so that the n-th normal invariant* $\mathscr{A}^{(n)}$ *of j (16.1.2) is an augmented* $\mathscr{O}_{\mathrm{Y}}$*-algebra (16.1.7); put* $\mathscr{A}^{(\infty)} = \varprojlim \mathscr{A}^{(n)}$*. For j to be a quasi-regular immersion, it is necessary and sufficient that j is locally of finite presentation, and that every* $y \in \mathrm{Y}$ *admits an affine open neighborhood* U *such that* $\mathscr{A}^{(\infty)}\big|_{\mathrm{U}}$ *is isomorphic, as augmented topological* $\mathscr{O}_{\mathrm{U}}$*-algebra, to* $\mathscr{O}_{\mathrm{U}}[[\mathrm{T}_1, \cdots, \mathrm{T}_n]]$*.*

Proof: We can confine ourselves to the case where $j$ is a closed immersion by restricting to a neighborhood which is small enough (see the reasoning of (16.4.11)), and then $\mathscr{O}_{\mathrm{Y}}$ identifies to a quotient algebra $\mathscr{O}_{\mathrm{X}}/\mathscr{J}$ and the canonical surjective homomorphism $\mathscr{O}_{\mathrm{X}} \to \mathscr{O}_{\mathrm{Y}}$ admits a right inverse (16.1.7). We can therefore assume $\mathrm{X} = \mathrm{Spec}(\mathrm{B})$ and $\mathrm{Y} = \mathrm{Spec}(\mathrm{A})$ being affine, B being an augmented A-algebra, and the ideal of augmentation $\mathfrak{J}$ being of finite type. Since then $\mathscr{A}^{(n)}$ identifies to $(\mathrm{B}/\mathfrak{J}^{n+1})^{\sim}$, the corollary follows from the equivalence of b) and c) in (**0**,19.5.4) since $\mathrm{B}/\mathfrak{J} = \mathrm{A}$. □

Note that, in the affine case considered above, the fact that $j$ is a quasi-regular immersion is still equivalent, in virtue of (**0**,19.5.4), to say that B is a *formally smooth* A-algebra for the $\mathfrak{J}$-adic topology.

Note also that the condition that $j$ is an immersion locally of finite presentation is always satisfied when the morphism $\mathrm{X} \to \mathrm{Y}$ is locally of finite type (1.4.3,(v)).

**Proposition 16.9.10** *Let* X *be a locally noetherian scheme,* Y *a subscheme of* X*,* $j\colon \mathrm{Y} \to \mathrm{X}$ *the canonical injection, y a point of* Y*.*

*(i) For there to be an open neighborhood* U *of y on* X *such that the restriction* $\mathrm{Y} \cap \mathrm{U} \to \mathrm{Y}$ *of j is a regular immersion, it is necessary and sufficient that the kernel* $\mathscr{J}_y$ *of the surjective homomorphism* $\mathscr{O}_{\mathrm{X},y} \to \mathscr{O}_{\mathrm{Y},y}$ *is generated by a regular sequence of elements of* $\mathscr{O}_{\mathrm{X},y}$*.*

*(ii) For the immersion j to be regular, it is necessary and sufficient that it is quasi-regular.*

Proof: (i) We can confine ourselves to the case where Y is a closed subscheme of X defined by a coherent ideal $\mathscr{J}$ of $\mathscr{O}_{\mathrm{X}}$. The condition is obviously necessary. Conversely, if $\mathscr{J}_y$ is generated by a regular sequence $(s_i)_y$, where $s_i$ are sections of $\mathscr{J}$ on an open neighborhood U of $y$ in X, we can assume that $s_i$ generate $\mathscr{J}|_{\mathrm{U}}$ ($\mathbf{0_I}$,5.2.2) and form a regular sequence ($\mathbf{0}$,15.2.4), hence the assertion follows.

(ii) The fact that a quasi-regular immersion is regular follows from (i) and the identity of quasi-regular sequences and regular sequences in $\mathscr{O}_{\mathrm{X},y}$, formed of elements of the maximal ideal ($\mathbf{0}$,15.1.11). □

When (without noetherian assumption on X) the kernel $\mathscr{J}_y$ of $\mathscr{O}_{\mathrm{X},y} \to \mathscr{O}_{\mathrm{Y},y}$ is generated by a regular sequence of elements of $\mathscr{O}_{\mathrm{X},y}$, we say that the immersion $j$ is ***regular at the point*** $y$.

**Corollary 16.9.11** *Let* X *be a locally noetherian scheme; then every quasi-regular ideal of* $\mathscr{O}_{\mathrm{X}}$ *is regular.*

**Remark 16.9.12** (i) Note that a regular immersion is in general not a flat morphism, nor *a fortiori* a regular morphism in the sense of (6.8.1).

(ii) Let A be a local noetherian ring; it follows immediately from (16.9.4) and from ($\mathbf{0}$,17.1.1) that for A to be *regular*, it is necessary and sufficient that its maximal ideal $\mathfrak{m}$ is *quasi-regular* (or *regular*, which amounts to the same since A is noetherian). For a noetherian affine scheme X to be *regular*, it is necessary and sufficient that for every closed point $x \in \mathrm{X}$, the canonical injection $\mathrm{Spec}(\kappa(x)) \to \mathrm{X}$ is a *regular immersion.*

**Proposition 16.9.13** *Let* X *be a locally noetherian scheme,* Y *a subscheme of* X*,* Y′ *a subscheme of* Y*, such that the canonical injection* $j\colon \mathrm{Y}' to \mathrm{Y}$ *is regular. Then the sequence of* $\mathscr{O}_{\mathrm{Y}'}$*-modules*

$$0 \longrightarrow j^*(\mathscr{N}_{\mathrm{Y}/\mathrm{X}}) \longrightarrow \mathscr{N}_{\mathrm{Y}'/\mathrm{X}} \longrightarrow \mathscr{N}_{\mathrm{Y}'/\mathrm{Y}} \longrightarrow 0 \tag{16.9.13.1}$$

*is exact; in addition, for every* $x \in \mathrm{X}$*, there exists an open neighborhood* U *of* $x$ *such that the restriction to* U *of homomorphisms in (16.9.13.1) form a split exact sequence.*

Proof: Let's prove first the following lemma:

**Lemma 16.9.13.2** *Let* A *be a ring,* $\mathfrak{J}$ *an ideal of* A*,* $\mathrm{A}' = \mathrm{A}/\mathfrak{J}$*,* $(f_i)_{1\leqslant i\leqslant r}$ *a sequence of elements of* A *which is* A′*-regular,* $\mathfrak{K} = \sum_i f_i\mathrm{A}$*,* $\mathfrak{L} = \mathfrak{J} + \mathfrak{K}$*,* $\mathfrak{K}' = \sum_i f_i\mathrm{A}'$*, so that* $\mathrm{C} = \mathrm{A}/\mathfrak{L}$ *is isomorphic to* $\mathrm{A}'/\mathfrak{K}'$*. Then for every integer* $n > 0$ *and every integer* $\mathrm{N} \geqslant n$*, we have*

$$\mathfrak{J} \cap \mathfrak{K}^n = \mathfrak{J}\mathfrak{K}^n + \mathfrak{J} \cap \mathfrak{K}^{\mathrm{N}}. \tag{16.9.13.3}$$

Proof: It suffices, of course, to prove that every element of the first term is contained in the second, and, by induction on $n$, we return to the case where $\mathrm{N} = n+1$. An element of the first term of (16.9.13.3), being in $\mathfrak{K}^n$, is written as $\mathrm{P}(f_1, \cdots, f_r)$, where $\mathrm{P} \in \mathrm{A}[\mathrm{T}_1, \cdots, \mathrm{T}_r]$ is homogeneous of degree $n$. If $f_i'$ is the canonical image of $f_i$ in $\mathrm{A}'$, the assumption $\mathrm{P}(f_1, \cdots, f_r) \in \mathfrak{J}$ implies that $\mathrm{P}(f_1', \cdots, f_r') = 0$. But $\mathrm{P}(f_1', \cdots, f_r') \in \mathfrak{K}'^n$, so the canonical image of $\mathrm{P}(f_1', \cdots, f_r')$ in $\mathfrak{K}'^n/\mathfrak{K}'^{n+1}$ is zero. Now, the assumption that the sequence $(f_i)$ is $\mathrm{A}'$-regular implies that the canonical homomorphism $\mathbb{S}^n_{\mathrm{C}}(\mathfrak{K}'/\mathfrak{K}'^2)$ is bijective (**0**,15.1.9); we conclude that the coefficients of P belong to $\mathfrak{L} = \mathfrak{J} + \mathfrak{K}$. It follows immediately that we have $\mathrm{P}(f_1, \cdots, f_r) \in \mathfrak{J}\mathfrak{K}^n + \mathfrak{J} \cap \mathfrak{K}^{n+1}$, which proves the lemma. □

Taking quotients of the two terms of (16.9.13.3) by $\mathfrak{J}\mathfrak{K}^n$ we see that the relations (16.9.13.3) for $\mathrm{N} \geqslant n$ lead to

$$(\mathfrak{J} \cap \mathfrak{K}^n)/\mathfrak{J}\mathfrak{K}^n \subset \bigcap_{\mathrm{N} \geqslant n} \mathfrak{K}^{\mathrm{N}}.(\mathrm{A}/(\mathfrak{J}\mathfrak{K}^n)). \tag{16.9.13.4}$$

We deduce the

**Corollary 16.9.13.5** *Assume that the hypotheses of (16.9.13.2) are satisfied, and suppose further that the ring* A *is noetherian and that* $\mathfrak{K}$ *is contained in the radical of* A*. Then for all integers* $n > 0$*,*

$$\mathfrak{J} \cap \mathfrak{K}^n = \mathfrak{J}\mathfrak{K}^n. \tag{16.9.13.6}$$

Proof: In fact, the second term of (16.9.13.4) is zero, since $\mathrm{A}/\mathfrak{J}\mathfrak{K}^n$ is an A-module of finite type (Bourbaki, *Alg. comm.*, chap. III, §3, no. 3, prop. 6). □

Take in particular $n = 2$ in (16.9.13.6), and notice that we have $\mathfrak{L}^2 = \mathfrak{J}^2 + \mathfrak{J}\mathfrak{K} + \mathfrak{K}^2 = \mathfrak{J}\mathfrak{L} + \mathfrak{K}^2$; since $\mathfrak{J}\mathfrak{L} \subset \mathfrak{L}^2$, we deduce that

$$\mathfrak{J} \cap \mathfrak{L}^2 = \mathfrak{J}\mathfrak{L} + (\mathfrak{J} \cap \mathfrak{K}^2) = \mathfrak{J}\mathfrak{L} + \mathfrak{J}\mathfrak{K}^2 = \mathfrak{J}\mathfrak{L},$$

in other words,

$$\mathfrak{J} \cap \mathfrak{L}^2 = \mathfrak{J}\mathfrak{L}, \tag{16.9.13.7}$$

which can also be expressed by saying that the canonical homomorphism

$$\mathfrak{J}/\mathfrak{J}\mathfrak{L} \longrightarrow (\mathfrak{J} + \mathfrak{L}^2)/\mathfrak{L}^2$$

is bijective.

These lemmas being proved, let us prove the first assertion of (16.9.13): it is sufficient, of course, to prove that the sequence of talks of the sheaves contained in (16.9.13.1) at a point $x \in \mathrm{Y}'$ is exact. Now, if we put A =

$\mathscr{O}_{X,x}$, we can write $\mathscr{O}_{Y,x} = A' = A/\mathfrak{J}$, where $\mathfrak{J}$ is an ideal contained in the maximal ideal of A, then $\mathscr{O}_{Y',x} = A'/\mathfrak{K}'$, where $\mathfrak{K}'$ is generated by an A'-regular sequence of elements of A', themselves being images of the elements of an A'-regular sequence of elements of A belonging to the maximal ideal of A. If $\mathfrak{K}$ is the ideal generated by these latter and $\mathfrak{L} = \mathfrak{J} + \mathfrak{K}$, we have $\mathscr{O}_{Y',x} = A/\mathfrak{L}$, and since we are in the situation of (16.9.13.5), the canonical homomorphism $\mathfrak{J}/\mathfrak{J}\mathfrak{L} \to (\mathfrak{J}+\mathfrak{L}^2)/\mathfrak{L}^2$ is bijective. But that shows the sequence

$$0 \longrightarrow \mathfrak{J}/\mathfrak{J}\mathfrak{L} \longrightarrow \mathfrak{L}/\mathfrak{L}^2 \longrightarrow (\mathfrak{L}/\mathfrak{L})/(\mathfrak{L}/\mathfrak{J})^2 \longrightarrow 0$$

is exact (see the proof of (16.2.7)), and the modules appearing in this sequence are precisely the stalks of the sheaves in (16.9.13.1) at $x$. The second assertion follows from that $\mathscr{N}_{Y'/Y}$ is a *locally free* $\mathscr{O}_{Y'}$-module (16.9.8) and (Bourbaki, *Alg.*, chap. II, 3 ed., §1, no. 11, prop. 21). $\square$

## 16.10 Differentially smooth morphisms

**Definition 16.10.1** *We say that a morphism of schemes* $f: X \to S$ *is* **differentially smooth** *(or that* X *is* **differentially smooth** *on* S*) if it satisfies the following conditions:*

*(i)* $\Omega^1_{X/S}$ *is a locally projective* $\mathscr{O}_X$*-module, that is to say, every point of* X *admits an open neighborhood* U *such that* $\Gamma(U, \Omega^1_{X/S})$ *is a* projective $\Gamma(U, \mathscr{O}_X)$*-module (not necessarily of finite type).*

*(ii) The canonical homomorphism (16.3.1.1)*

$$\mathbf{S}^{\bullet}_{\mathscr{O}_X}(\Omega^1_{X/S}) \longrightarrow \mathscr{G}r_{\bullet}(\mathscr{P}_{X/S})$$

*is bijective.*

*In particular, if* $\Omega^1_{X/S}$ *is locally free of finite rank,* $\mathscr{P}^n_{X/S}$ *are locally free* $\mathscr{O}_X$*-modules of finite rank (being extensions of such modules).*

We say that $f$ is ***differentially smooth at a point*** $x \in X$ (or that X is differentially smooth on S at point $x$) if there exists an open neighborhood U of $x$ on X such that $f|_U$ is differentially smooth.

We will see later (17.12.4) that a smooth morphism is differentially smooth, which justifies the terminology; but the converse is incorrect; indeed, a *monomorphism* $f: X \to S$ is difirentially smooth when $\Omega^1_{X/S} = 0$ in virtue of (**I**,5.3.8), and consequently the surjective homomorphism (16.3.1.1) is obviously bijective; but a monomorphism is not even necessarily flat, nor *a fortiori* smooth. Let us limit ourselves here to note the following proposition:

**Proposition 16.10.2** *Let* A *be a ring,* B *be an* A*-algebra formally smooth for the discrete topology (**0**,19.3.1). Then* Spec(B) *is differentially smooth on* Spec(A)*.*

PROOF: In fact, $B \otimes_A B$ is (for the discrete topology) a formally smooth B-algebra (for one or the other canonical homomorphisms $b \mapsto b \otimes 1$, $b \mapsto 1 \otimes b$ from B to $B \otimes_A B$) (**0**,19.3.5,(iii)); then $B \otimes_A B$ is also a formally smooth A-algebra for the discrete topology (**0**,19.3.5,(ii)). Put $\mathfrak{J} = \mathfrak{J}_{B/A}$, it follows that $B \otimes_A B$ is also a formally smooth A-algebra for the $\mathfrak{J}$-adic topology (**0**,19.3.8); since the assumption $B = (B \otimes_A B)/\mathfrak{J}$ is a formally smooth A-algebra for the discrete topology, the proposition follows from the equivalence of a) and b) in (**0**,19.5.4). □

**Proposition 16.10.3** *For a morphism $f : X \to S$ to be differentially smooth, it is necessary and sufficient that for every $x \in X$, there exists an open neighborhood* U *of $x$, and ring* A*, such that $\Gamma(U, \mathscr{P}^{\infty}_{X/S})$ is an augmented topological* A*-algebra isomorphic to the complete algebra $\widehat{B}$, where $B = \mathbb{S}_A(V)$,* V *being a projective* A*-module and* B *being equipped with the $B^+$-adic topology (where $B^+$ is the ideal of augmentation). If $\Omega^1_{X/S}$ is locally free of finite rank, we can replace $\widehat{B}$ with the algebra of formal series $A[[T_1, \cdots, T_n]]$.*

PROOF: The notion of differentially smooth morphism being obviously local on X, we can confine ourselves to the case where $S = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(C)$. Consider $C \otimes_B C$ as a C=algebra (for the first factor); put $\mathfrak{J} = \mathfrak{J}_{C/B}$ and provide $C \otimes_B C$ the $\mathfrak{J}$-adic topology; we can apply to the topological C-algebra $C \otimes_B C$ and the ideal $\mathfrak{J}$ of $C \otimes_B C$ the equivalence of b) and c) in (**0**,19.5.4), since $(C \otimes_B C)/\mathfrak{J} = C$ is obviously a formally smooth C-algebra for the discrete topology. The topology on $\Gamma(U, \mathscr{P}^{\infty}_{X/S})$ is obviously the projective limit topology of this ring (16.1.11).

Note that the integer $n$ in the statement of (16.10.3) is the *rank* of $\Omega^1_{X/S}$ at point $x$. We will see later (17.13.5) that when $f$ is differentially smoothly and locally of finite type, $n$ is also equal to the dimension of the fiber $f^{-1}(f(x))$ at point $x$. □

**Proposition 16.10.4** *Let $f : X \to S$, $g : S' \to S$ be two morphisms, and put $X' = X \times_S S'$, $f' = f_{(S')} : X' \to S'$.*

*(i) If $f$ is differentially smooth, so is $f'$.*

*(ii) Conversely, if $g$ is faithfully flat and quasi-compact, and if $f'$ is differentially smooth and $\Omega^1_{X'/S'}$ is an $\mathscr{O}_{X'}$-module of finite type, $f$ is differentially smooth and $\Omega^1_{X/S}$ is an $\mathscr{O}_X$-module of finite type.*

PROOF: In fact, if $f$ is differentially smooth, then $\mathscr{G}r_n(\mathscr{P}_{X/S})$ is a *flat* $\mathscr{O}_X$-module; consequently (16.4.6), the homomorphism $\mathscr{G}r_n(\mathscr{P}_{X/S}) \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} To \mathscr{G}r_n(\mathscr{P}_{X'/S'})$ is bijective for every $n$, and in virtue of the commutative diagram (16.2.1.3), if follows from the definition (16.10.1) that $f'$ is differentially smooth. On the other hand, if $g$ is faithfully flat and quasi-compact, if follows immediately from (16.4.6) that $\mathscr{G}r_n(\mathscr{P}_{X/S}) \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} To \mathscr{G}r_n(\mathscr{P}_{X'/S'})$ is bijective for every

$n$. Suppose then $f'$ is differentially smooth and $\Omega^1_{X'/S'}$ is of finite rank. Since the canonical projection $X' \to X$ is a faithfully flat and quasi-compact morphism, it follows first from (2.5.2) that $\Omega^1_{X/S}$ is a locally free $\mathscr{O}_X$-module of finite rank, then from (2.2.7) that the canonical homomorphism (16.3.1.1) is bijective, therefore $f$ is differentially smooth. □

**Proposition 16.10.5** *For a locally of finite type morphism $f: X \to S$ to be differentially smooth, it is necessary and sufficient that the diagonal immersion $\Delta_f: X \to X \times_S X$ is quasi-regular.*

Proof: The question being local, we can confine ourselves to the case where S and X are affine, and consequently the diagonal subscheme of $X \times_S X$ is closed. The assumption that $f$ is locally of finite type implies that $\Delta_f$ is locally of finite presentation (1.4.3.1), thus the diagonal subscheme of $X \times_S X$ is of finite type. The proposition then follows immediately from the comparison of the conditions in (16.10.1) and (16.9.4). □

**Remark 16.10.6** Let $f: X \to S$ be a morphism such that the $\mathscr{O}_X$-module $\Omega^1_{X/S}$ is locally free of finite rank. It follows then from (**0**,20.4.7) that every $x \in X$ admits an open neighborhood U such that there exits a finite family $(z_\lambda)_{\lambda \in L}$ of sections of $\mathscr{O}_X$ on U for which $(dz_\lambda)_{\lambda \in L}$ form a basis of the $\Gamma(U, \mathscr{O}_X)$-module $\Gamma(U, \Omega^1_{X/S})$.

## 16.11 Differential operators on a differentially smooth S-scheme

**16.11.1** Let $f: X \to S$ be a morphism, U an open of X, $(z_\lambda)_{\lambda \in L}$ a family of sections of $\mathscr{O}_X$ on U such that $dz_\lambda$ form a system of generators of $\Omega^1_{X/S}\big|_U = \Omega^1_{U/S}$. Let $m$ be an integer or the symbol $\infty$, and put, for every $\lambda$,

(16.11.1.1) $$\zeta_\lambda = \delta z_\lambda = d^m z_\lambda - z_\lambda \in \Gamma(U, \mathscr{P}^m_{X/S}).$$

We will also use the usual notation of analysis; for every $\mathbf{p} = (p_\lambda) \in \mathbb{N}^{(L)}$ (with $p_\lambda = 0$ except for a finite number of indexes), we put

(16.11.1.2) $$|\mathbf{p}| = \sum_\lambda p_\lambda, \qquad \mathbf{p}! = \prod_\lambda (p_\lambda!)$$

(16.11.1.3) $$\binom{\mathbf{p}}{\mathbf{q}} = \frac{\mathbf{p}!}{\mathbf{q}!(\mathbf{p}-\mathbf{q})!} \quad \text{for } \mathbf{p}, \mathbf{q} \text{ in } \mathbb{N}^{(L)}, \ \mathbf{q} \leqslant \mathbf{p}$$

and we convient $\binom{\mathbf{p}}{\mathbf{q}} = 0$ if $\mathbf{q} \nleqslant \mathbf{p}$,

(16.11.1.4) $$\mathbf{z}^{\mathbf{p}} = \prod_\lambda (z_\lambda)^{p_\lambda}, \qquad \zeta^{\mathbf{p}} = \prod_\lambda (\zeta_\lambda)^{p_\lambda}.$$

We will have, with these notations

$$d^m(\mathbf{z}^{\mathbf{p}}) = (d^m(\mathbf{z}))^{\mathbf{p}} = (\zeta + \mathbf{z})^{\mathbf{p}} = \sum_{\mathbf{q}\leqslant\mathbf{p}} \binom{\mathbf{p}}{\mathbf{q}} \mathbf{z}^{\mathbf{p}-\mathbf{q}}\zeta^{\mathbf{q}} \tag{16.11.1.5}$$

$$\zeta^{\mathbf{p}} = (d^m\mathbf{z} - \mathbf{z})^{\mathbf{p}} = \sum_{\mathbf{q}\leqslant\mathbf{p}} (-1)^{|\mathbf{p}-\mathbf{q}|}\binom{\mathbf{p}}{\mathbf{q}} \mathbf{z}^{\mathbf{p}-\mathbf{q}} f^m(\mathbf{z}^{\mathbf{q}}). \tag{16.11.1.6}$$

Since $dz_\lambda$ generate $\Omega^1_{\mathrm{U/S}}$ and are the images of $\delta z_\lambda$, and that the canonical homomorphism (16.3.1.1) is surjective, we conclude that for finite $m$, $\delta z_\lambda$ generate the $\mathscr{O}_{\mathrm{U}}$-algebra $\mathscr{P}^m_{\mathrm{U/S}}$ (Bourbaki, *Alg. comm.*, chap. III, §2, no. 8, cor. 2 of th. 1). Then $\zeta^{\mathbf{p}}$ (for $\mathbf{p} \leqslant m$) generate the $\mathscr{O}_{\mathrm{U}}$-module $\mathscr{P}^m_{\mathrm{U/S}}$. A differential operator $D \in \mathrm{Diff}^m_{\mathrm{U/S}}$ is thus entirely determined by the values $\langle \zeta^{\mathbf{p}}, D\rangle$ for $|\mathbf{p}| \leqslant m$, or, which amounts to the same by (16.11.1.5) and (16.11.1.6), by the values $\langle d^m(\mathbf{z}^{\mathbf{p}}), D\rangle = D(\mathbf{z}^{\mathbf{p}})$ for $|\mathbf{p}| \leqslant m$; precisely, it follows from (16.11.1.5) that we have

$$D(\mathbf{z}^{\mathbf{p}}) = \langle d^m(\mathbf{z}^{\mathbf{p}}), D\rangle = \sum_{\mathbf{q}\leqslant\mathbf{p}} \binom{\mathbf{p}}{\mathbf{q}} \langle \zeta^{\mathbf{p}}, D\rangle \mathbf{z}^{\mathbf{p}-\mathbf{q}}. \tag{16.11.1.7}$$

**Theorem 16.11.2** *Let $f : \mathrm{X} \to \mathrm{S}$ be a morphism, $\mathrm{U}$ an open of $\mathrm{X}$, $(z_\lambda)_{\lambda\in\mathrm{L}}$ a family of sections of $\mathscr{O}_{\mathrm{X}}$ on $\mathrm{U}$ such that the family $(dz_\lambda)_{\lambda\in\mathrm{L}}$ generates $\Omega^1_{\mathrm{X/S}}\big|_{\mathrm{U}} = \Omega^1_{\mathrm{U/S}}$. The following conditions are equivalent:*

*a) $f|_{\mathrm{U}}$ is differentially smooth and $(dz_\lambda)$ is a basis of the $\mathscr{O}_{\mathrm{U}}$-module $\Omega^1_{\mathrm{U/S}}$.*

*b) There exists a family $(D_{\mathbf{p}})_{\mathbf{p}\in\mathbb{N}^{(\mathrm{L})}}$ of differential operators of $\mathscr{O}_{\mathrm{U}}$ on itself, satisfying the condition*

$$D_{\mathbf{p}}(\mathbf{z}^{\mathbf{q}}) = \binom{\mathbf{q}}{\mathbf{p}} \mathbf{z}^{\mathbf{q}-\mathbf{p}} \quad (\mathbf{p}, \mathbf{q} \in \mathbb{N}^{(\mathrm{L})}). \tag{16.11.2.1}$$

*In addition, when these conditions are satisfied, the family $(D_{\mathbf{p}})$ is uniquely determined by the conditions (16.11.2.1) and the relations*

$$D_{\mathbf{p}} \circ D_{\mathbf{q}} = D_{\mathbf{q}} \circ D_{\mathbf{p}} = \frac{(\mathbf{p}+\mathbf{q})!}{\mathbf{p}!\mathbf{q}!} D_{\mathbf{p}+\mathbf{q}} \quad (\mathbf{p}, \mathbf{q} \in \mathbb{N}^{(\mathrm{L})}). \tag{16.11.2.2}$$

*Finally, if $\mathrm{L}$ is finite, for every integer $m$, these $D_{\mathbf{p}}$ such that $|\mathbf{p}| \leqslant m$ form a base of the $\mathscr{O}_{\mathrm{U}}$-module $\mathscr{D}iff^m_{\mathrm{U/S}}$, in other words, every differential operator of order $\leqslant m$ on $\mathrm{U}$ can be uniquely written in the form*

$$D = \sum_{|\mathbf{p}|\leqslant m} a_{\mathbf{p}} D_{\mathbf{p}}$$

*where $a_{\mathbf{p}}$ are sections of $\mathscr{O}_{\mathrm{U}}$ on $\mathrm{U}$.*

Proof: Let us first note that in virtue of (16.11.1.6) and (16.11.1.5), we immediately verify that the conditions (16.11.2.1) are equivalent to

$$(16.11.2.3) \qquad \langle \zeta^{\mathbf{p}}, D_{\mathbf{q}} \rangle = \delta_{\mathbf{pq}} \quad \text{(Kronecker index)}$$

The existence of the family $(D_{\mathbf{p}})$ satisfying these relations therefore implies first (by taking $|\mathbf{p}| = 1$) that $dz_\lambda$ are linearly independent, hence form a basis of the $\mathscr{O}_{\mathrm{U}}$-module $\Omega^1_{\mathrm{U/S}}$. Then, for every integer $m \geqslant 1$, we deduce from (16.11.2.3) that $\zeta^{\mathbf{p}}$ such that $|\mathbf{p}| \leqslant m$ are linearly independent; therefore the canonical homomorphism (16.3.1.1) is injective, hence bijective, and this proves that b) implies a). The converse follows immediately from the definition (16.10.1), the fact that $\zeta^{\mathbf{p}}$ form a basis of $\mathscr{P}^m_{\mathrm{U/S}}$ for $|\mathbf{p}| \leqslant m$ leading to the existence and uniqueness of a family of homomorphisms $u_{\mathbf{q},m} \colon \mathscr{P}^m_{\mathrm{U/S}} \to \mathscr{O}_{\mathrm{U}}$ such that $\langle \zeta^{\mathbf{p}}, u_{\mathbf{q},m} \rangle = \delta_{\mathbf{pq}}$ for $|\mathbf{p}| \leqslant m$, $|\mathbf{q}| \leqslant m$. For a given value of $\mathbf{q}$, the differential operators corresponding to $u_{\mathbf{q},m}$ for $|\mathbf{q}| \leqslant m$ are identified with the same operator $D_{\mathbf{p}}$. This proves that a) implies b), and further that the family $(D_{\mathbf{p}})$ is uniquely determined and that, if L is finite, for $|\mathbf{p}| \leqslant m$, $D_{\mathbf{p}}$ form a basis of the dual $\mathscr{D}\mathit{iff}^m_{\mathrm{U/S}}$ of $\mathscr{P}^m_{\mathrm{U/S}}$. Finally, the relations (16.11.2.2) follow immediately from the expression of the values of the three operators considered for $\mathbf{z}^{\mathbf{p}}$, and from the fact that $\zeta^{\mathbf{p}}$ generate $\mathscr{P}^m_{\mathrm{U/S}}$. □

**Remark 16.11.3** (i) The fact that $D_{\mathbf{p}}$ are two-to-two permutable in virtue of (16.11.2.2) does not naturally imply that the $\mathscr{O}_{\mathrm{U}}$-algebra $\mathscr{D}\mathit{iff}_{\mathrm{U/S}}$ is commutative, for $D_{\mathbf{p}}$ not permuting with products by sections of $\mathscr{O}_{\mathrm{U}}$ unless $n = 0$.

(ii) The index $\mathbf{p}$ such that $|\mathbf{p}| = 1$ are $\varepsilon_\lambda = (\varepsilon_{\lambda\mu})_{\mu\in\mathrm{L}}$ with $\varepsilon_{\lambda\mu} = 0$ if $\mu \neq \lambda$ and $\varepsilon_{\lambda\lambda} = 1$; when L is finite, the operators $D_{\varepsilon_\lambda}$ are none other than the S-derivations $\mathrm{D}_i$ introduced in (16.5.7). Note that in general (and contrary to what happens in classical analysis), it is not true that a differential operator of any order can be written as a linear combination of powers of $\mathrm{D}_i$ (cf. (16.12)).

(iii) For every integer $r \geqslant 1$, we can define the notion of ***differentially smooth up to the order*** $r$ ***morphism*** by replacing in (16.10.1) the condition (ii) with the condition that the homomorphisms

$$\mathbf{S}^m_{\mathscr{O}_{\mathrm{X}}}(\Omega^1_{\mathrm{X/S}}) \longrightarrow \mathscr{G}r_m(\mathscr{P}_{\mathrm{X/S}})$$

are bijective *for ever* $m \leqslant r$. The reasoning of (16.11.2) then proves that if, in condition a), we replace "differentially smooth" with "differentially smooth up to the order $r$", this condition is equivalent to condition b) in which we confines itself to $\mathbf{p}, \mathbf{q} \in \mathbb{N}^{(\mathrm{L})}$ such that $|\mathbf{p}| \leqslant r$, $|\mathbf{q}| \leqslant r$.

## 16.12 Characteristic 0 case: Jacobian criterion for differentially smooth morphisms

**16.12.1** We say that a scheme X is ***of characteristic*** $p$ ($p$ equal to 0 or a prime number) if, for every open affine U of X, the ring $\Gamma(U, \mathscr{O}_X)$ is of characteristic $p$ (**0**,21.1.1). It follows from (**0**,21.1.3) that for X to be of characteristic 0, it is necessary and sufficient that for every *closed* point $x$ of X, the residual field $\kappa(x)$ be of characteristic 0, or that X can be provided with a (necessarily unique) structure of $\mathbb{Q}$-scheme.

**Theorem 16.12.2** *Let* X *be a scheme of characteristic* 0*,* $f : X \to S$ *a morphism. If* $\Omega^1_{X/S}$ *is a locally free* $\mathscr{O}_X$*-module (not necessarily of finite type),* $f$ *is differentially smooth.*

PROOF: Since the question is local on X, we may assume that there exists a family $(z_\lambda)$ of sections of $\mathscr{O}_X$ on X such that $(dz_\lambda)$ is a basis of the $\mathscr{O}_X$-module $\Omega^1_{X/S}$. Applying the criterion (16.11.2), it is sufficient to show that the operators

$$D_{\mathbf{p}} = (\mathbf{p}!)^{-1} \prod_\lambda D_\lambda^{p_\lambda}$$

(where $D_\lambda$ are the coordinate forms corresponding to the basis $(dz_\lambda)$) satisfy the relations (16.11.2.1), which is a consequence of the fact that $D_\lambda$ are derivations □

**16.12.3** The previous theorem is no longer correct when we abandon the assumption that X is of characteristic 0. For example, if $S = \mathrm{Spec}(k)$, where $k$ is a field of characteristic $p > 0$, $X = \mathrm{Spec}(K)$ where $K = k(\alpha)$ with $\aleph \notin k, \alpha^p \in k$, we check immediately that $\Omega^1_{X/S}$ is of rank 1, and that the morphism $X \to S$ is differentially smooth up to the order $p-1$ (16.11.3,(iii)), but not up to the order $p$. However, the proof of (16.12.2) shows that if $\Omega^1_{X/S}$ is locally free, and if $n!1_{\mathscr{O}_X}$ is invertible in $\Gamma(X, \mathscr{O}_X)$, then X is differentially smooth on S up to order $n$.

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