TP 10 : Revisions

Exercice 1 (Bézier curves). Write the function $f(x) = 4x^3 - 6x^2 + 3x + 1$ for $x \in [0, 1]$ as a cubic Bézier curve.

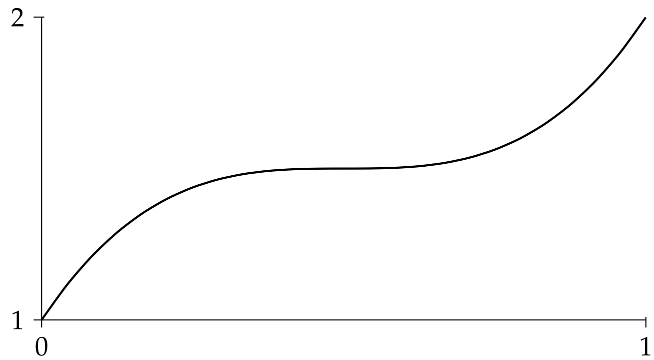


Figure 1: Graph of $f(x) = 4x^3 - 6x^2 + 3x + 1$ for $x \in [0, 1]$.

Exercice 2 (B-spline curves). Given a degree k, control polygon $\mathbf{d}_0, \dots, \mathbf{d}_n$, and a knot vector (a non-decreasing sequence) $t_0 \le t_1 \le \cdots \le t_m$ with $m = n + k + 1$, the *B-spline curve* $\mathbf{S}(t)$ is defined as

$$\mathbf{S}(t) = \sum_{j=0}^{n} \mathbf{d}_j \mathrm{N}_j^k(t), \quad t \in [t_k, t_{m-k}).$$

The N_j^k are the recursively-defined *B-spline basis functions*:

$$\mathrm{N}_j^0(t) = \begin{cases} 1 & t \in [t_j, t_{j+1}) \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mathrm{N}_j^k(t) = \underbrace{\frac{t - t_j}{t_{j+k} - t_j}}_{w_{j,k}(t)} \mathrm{N}_j^{k-1}(t) + \underbrace{\frac{t_{j+k+1} - t}{t_{j+k+1} - t_{j+1}}}_{1 - w_{j+1,k}(t)} \mathrm{N}_{j+1}^{k-1}(t).$$

(De Boor algorithm). Fix a parameter $t \in [t_i, t_{i+1}) \subset [t_k, t_{m-k})$. For $j = i - k, \dots, i$, set $\mathbf{d}_j^0 = \mathbf{d}_j$. Compute

$$\mathbf{d}_j^{r+1} = \left(1 - w_{j,k-r}\right) \mathbf{d}_{j-1}^r + w_{j,k-r} \mathbf{d}_j^r$$

for $r = 1, \dots, k$ and $j = i - k + r, \dots, i$. The weights are computed as follows:

$$w_{j,k-r} = \frac{t - t_j}{t_{j+k-r} - t_j}.$$

The curve point is $\mathbf{S}(t) = \mathbf{d}_i^k$.
Consider these five control points in $\mathbb{R}^2$ (see Figure 2 on the next page):

$$\mathbf{d}_0 = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \quad \mathbf{d}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \quad \mathbf{d}_2 = \begin{bmatrix} 3 \\ -3 \end{bmatrix} \quad \mathbf{d}_3 = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \mathbf{d}_4 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

We are given four uniform knot vectors:

$$\begin{aligned} \mathtt{T}_1 &= (t_0, \dots, t_6) = (0, 1, 2, 3, 4, 5, 6)\,, \\ \mathtt{T}_2 &= (t_0, \dots, t_7) = (0, 1, 2, 3, 4, 5, 6, 7)\,, \\ \mathtt{T}_3 &= (t_0, \dots, t_8) = (0, 1, 2, 3, 4, 5, 6, 7, 8)\,, \\ \mathtt{T}_4 &= (t_0, \dots, t_9) = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)\,. \end{aligned}$$

Together with the control points $\mathbf{d}_i$, each knot vector $\mathtt{T}_j$ defines a unique B-spline curve $\mathbf{S}_j(t)$.

(i) For each knot vector $\mathtt{T}_j$: what is the *degree* k of the corresponding B-spline curve $\mathbf{S}_j$?

(ii) For each knot vector $\mathtt{T}_j$: (approximately) plot the corresponding B-spline basis functions N_i^k for $i = 0, \ldots, 4$. What is the support for each basis function N_i^k? (Support refers to the parameter interval where the function is non-zero.)

(iii) For each curve $\mathbf{S}_j$: what is the domain – the interval of definition?

(iv) For each curve $\mathbf{S}_j$: (approximately) plot the curve; to that end, you can use the grid provided in Figure 2. What is the degree of smoothness between curve's segments?

(v) For the quadratic curve: evaluate the curve point $t = 4.5$ using the De Boor algorithm.

(vi) For the quadratic curve: what is the curve's tangent vector at the point computed in the previous question? ($t = 4.5$)

Consider now the knot vector

$$\mathtt{T}_5 = (t_0, \ldots, t_9) = (0, 0, 0, 0, 0, 1, 1, 1, 1, 1)\,.$$

(vii) What are the basis functions N_i^0?

(viii) What are the basis functions $\mathrm{N}_0^4, \ldots, \mathrm{N}_4^4$ for a quartic curve defined by $\mathbf{d}_0, \ldots, \mathbf{d}_4$ and $\mathtt{T}_5$?

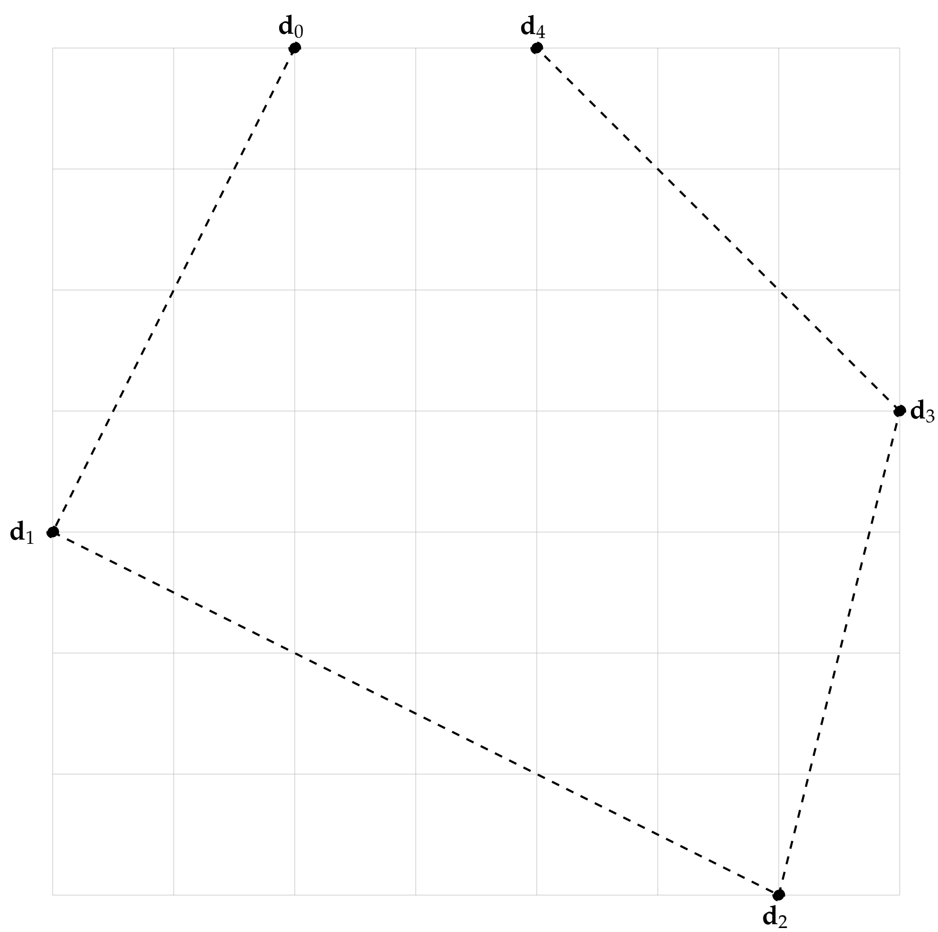
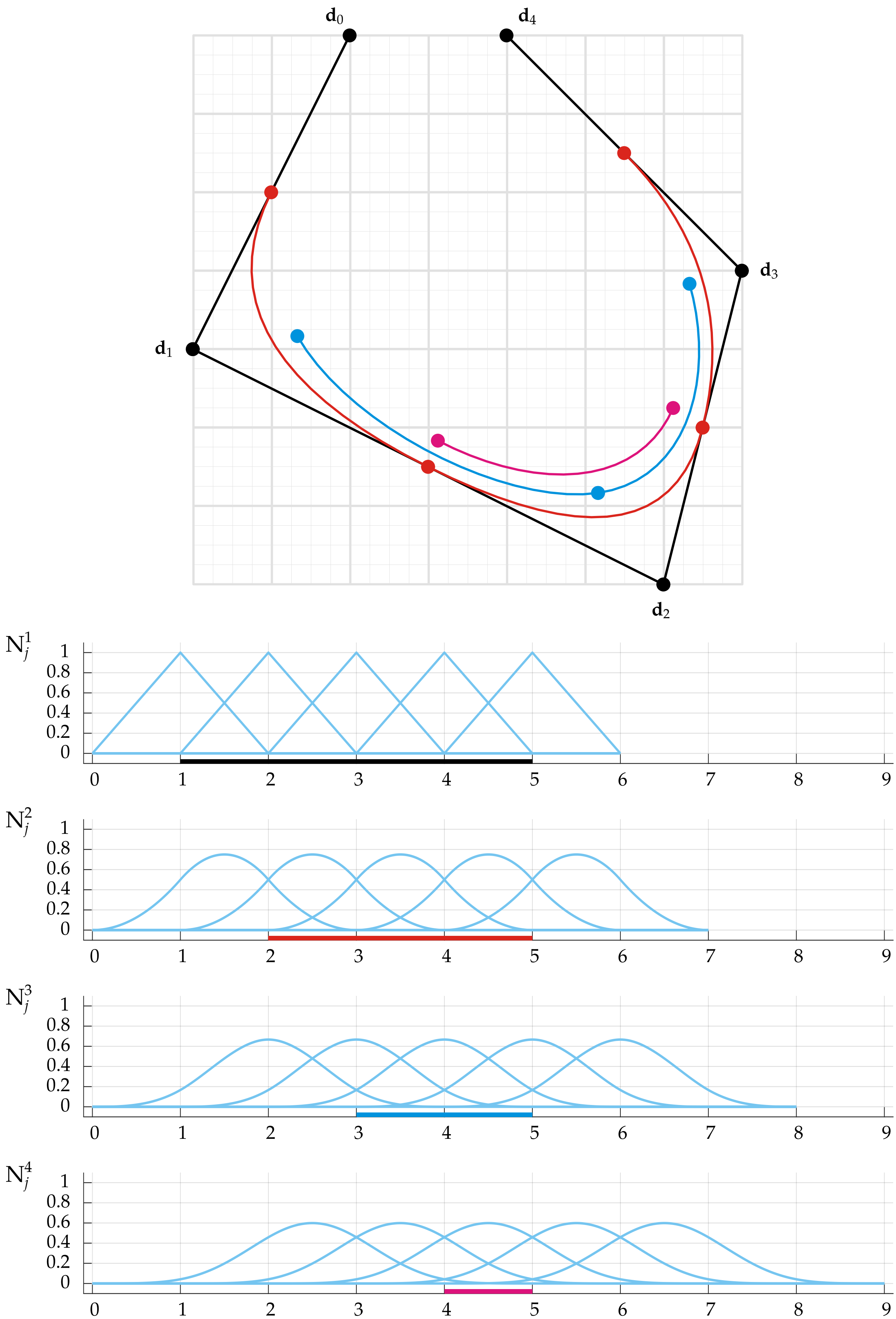


Figure 2: The control points $\mathbf{d}_i$. You can use use this figure to plot the B-spline curves $\mathbf{S}_j(t)$.



$\mathbf{d}_0$
$\mathbf{d}_4$
$\mathbf{d}_3$
$\mathbf{d}_1$
$\mathbf{d}_2$
N_j^1
N_j^2
N_j^3
N_j^4
1
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8 9