Supplementary exercises

Exercice 1. Compute the control points of a cubic Bézier curve $\mathbf{c}(t),\ t \in [0,1]$ such that:

$$\mathbf{c}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \mathbf{c}(0.5) = \begin{bmatrix} 4 \\ 7 \end{bmatrix} \qquad \mathbf{c}(1) = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad \mathbf{c}'(0.5) = \begin{bmatrix} 2 \\ -0.5 \end{bmatrix}$$

Recall that the derivative of a degree n Bézier curve

$$c(t) = \sum_{0}^{n} \mathbf{P}_i \mathrm{B}_i^n(t)$$

is a degree $(n-1)$ Bézier curve

$$c'(t) = n \sum_{0}^{n-1} (\mathbf{P}_{i+1} - \mathbf{P}_i)\ \mathrm{B}_i^{n-1}(t).$$

Exercice 2. We are given a cubic Bézier curve with the following control points:

$$\mathbf{P}_0 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad \mathbf{P}_1 = \begin{bmatrix} 9 \\ 6 \end{bmatrix} \qquad \mathbf{P}_2 = \begin{bmatrix} 13 \\ -1 \end{bmatrix} \qquad \mathbf{P}_3 = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

(i) Evaluate the curve for the parameter $t = \frac{1}{3}$ using de Casteljau's algorithm. Illustrate the algorithm graphically by plotting the intermediate control points and polygons.

(ii) Write the control points of the two cubic parts obtained by subdividing the curve at $t = \frac{1}{3}$. Hint: use the intermediate results from the previous question.

Exercice 3 (from TP10). Write the function $f(x) = 4x^3 - 6x^2 + 3x + 1$ for $x \in [0,1]$ as a cubic Bézier curve.

Exercice 4 (from TP10). Let $\mathbf{c}(t) = \sum_0^3 \mathbf{P}_i \mathrm{B}_i^3(t)$ be a cubic Bézier segment with

$$\mathbf{P}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \mathbf{P}_1 = \begin{bmatrix} 6 \\ 1 \end{bmatrix} \qquad \mathbf{P}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \qquad \mathbf{P}_3 = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

(i) Express $\mathbf{c}(t)$ in the monomial basis, i.e. find the scalar coefficients $a_i \in \mathbb{R}$ satisfying

$$\mathbf{c}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3.$$

(ii) Join $\mathbf{c}(t)$ with a quadratic Bézier segment $\mathbf{q}(t) = \sum_0^2 \mathbf{Q}_i \mathrm{B}_i^2(t)$ to obtain a $\mathcal{C}^1$ spline which satisfies the following conditions:

$$\mathbf{q}(1) = \mathbf{c}(0), \qquad \mathbf{q}'(1) = \mathbf{c}'(0), \qquad \mathbf{q}(0.5) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Exercice 5. We are given the following three points:

$$\mathbf{A} = \begin{bmatrix} -2 \\ 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

We want to interpolate these points using a cubic Bézier spline with two cubic segments $\mathbf{s}_0(t)$ and $\mathbf{s}_1(t)$ which are joined together $\mathcal{C}^2$-continuously. Compute the control points for both segments of the spline using

(i) natural boundary conditions $s_0''(0) = s_1''(1) = 0$;

(ii) clamped boundary conditions $s_0'(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $s_1'(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Each set of conditions should provide a different result.