Simulation of Nonstationary Stochastic Processes by Spectral Representation

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Abstract: This paper presents a rigorous derivation of a previously known formula for simulation of one-dimensional, univariate, nonstationary stochastic processes integrating Priestly's evolutionary spectral representation theory. Applying this formula, sample functions can be generated with great computational efficiency. The simulated stochastic process is asymptotically Gaussian as the number of terms tends to infinity. This paper shows that (1) these sample functions accurately reflect the prescribed probabilistic characteristics of the stochastic process when the number of terms in the cosine series is large, i.e., the ensemble averaged evolutionary power spectral density function (PSDF) or autocorrelation function approaches the corresponding target function as the sample size increases, and (2) the simulation formula, under certain conditions, can be reduced to that for nonstationary white noise process or Shinozuka's spectral representation of stationary process. In addition to derivation of simulation formula, three methods are developed in this paper to estimate the evolutionary PSDF of a given time-history data by means of the short-time Fourier transform (STFT), the wavelet transform (WT), and the Hilbert-Huang transform (HHT). A comparison of the PSDF of the well-known El Centro earthquake record estimated by these methods shows that the STFT and the WT give similar results, whereas the HHT gives more concentrated energy at certain frequencies. Effectiveness of the proposed simulation formula for nonstationary sample functions is demonstrated by simulating time histories from the estimated evolutionary PSDFs. Mean acceleration spectrum obtained by averaging the spectra of generated time histories are then presented and compared with the target spectrum to demonstrate the usefulness of this method.

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Introduction

Monte Carlo simulation methods play an important role in the response analysis of civil engineering structures and systems with stochastic properties and/or subjected to stochastic loadings. For the purpose of the Monte Carlo analysis, generation of sample functions of the stochastic processes involved in the problem must accurately provide the probabilistic characteristics of these processes. Among the various methods that have been developed to generate such sample functions, the spectral representation method pioneered by Shinozuka (Shinozuka and Jan 1972; Shinozuka 1972) appears to be most versatile and widely used today. Shinozuka (1972) developed an analytical expression for generating sample functions of the envelope of a stochastic process observing the usefulness of the Hilbert transformation for this purpose. Yang (1972, 1973) showed that the fast Fourier transform (FFT) technique can be used to dramatically improve the computational efficiency of the spectral representation algorithm, and also proposed a formula to simulate random envelop processes. Shinozuka (1974) extended the application of the FFT technique to multidimensional cases. In recent years, the spectral representation method has been extended in various ways, e.g., to simulate stochastic waves (Deodatis and Shinozuka 1989), non-Gaussian stochastic fields (Yamazaki and Shinozuka 1988), spatially incoherent multidimensional and multivariate random processes and fields (Ramadan and Novak 1993), multivariate ergodic stochastic processes (Deodatis 1996b), multivariate nonstationary stochastic processes (Deodatis 1996a), and non-Gaussian multidimensional multivariate fields (Popescu et al. 1998), etc.

From the rich bibliography related to the various methods for the simulation of nonstationary stochastic processes, representative studies are the following: Shinozuka and Sato (1967) by filtered Gaussian white noise process models, Shinozuka and Jan (1972) by spectral representation method with amplitude modulation, Ohsaki (1979) by phase difference method, Deodatis and Shinozuka (1988) by autoregressive method, Li and Kareem (1991) by fast Fourier transform technique, Ramadan and Novak (1993) by spectral representation method with amplitude modulation, and Deodatis (1996a) by spectral representation method with amplitude and frequency modulation. It should be noted that the modulated envelope functions used for making stationary stochastic process nonstationary in the previous references are usually based on the analytically expedient postulation derived from general observations of nonstationary time history data. However,

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the estimation of evolutionary power spectral density function (PSDF) from a particular nonstationary record (e.g., an earthquake ground motion acceleration) is necessary to represent spectral properties of the nonstationary process.

This paper presents a rigorous derivation of the computationally efficient cosine series formula presented by Shinozuka and Jan (Shinozuka and Jan 1972) for generation of sample functions of one-dimensional and univariate nonstationary stochastic processes. The spectral representation method, which is used to derive this simulation formula, integrates the theory of evolutionary process introduced and promoted by Priestly (1965, 1967). The simulated stochastic process is asymptotically Gaussian as the number of terms tends to infinity. This paper shows that (1) the sample functions generated by the present method accurately reflect the prescribed probabilistic characteristics of the stochastic process when the number of the terms in the cosine series is large, i.e., the ensemble averaged evolutionary PSDF and autocorrelation function approach the corresponding target function as the sample size increases, and (2) the simulation formula, under certain conditions, can be reduced to that for nonstationary white noise process or Shinozuka's stationary process (Shinozuka and Jan 1972; Shinozuka 1972).

In the later part, three methods are developed to estimate the evolutionary PSDF of a given time-history data by using the short-time Fourier transform (STFT), the wavelet transform (WT), and the Hilbert-Huang transform (HHT). Three evolutionary PSDFs thus developed maintain the same total energy possessed by the time history data. In fact, N-S component of the well-known 1940 El Centro earthquake record is used as an example of such a time history and the evolutionary PSDFs derived from the record by using these three methods are compared. Effectiveness of the simulation formula proposed here is demonstrated by generating time histories from the three evolutionary PSDFs, while using the same random number sequences for them. Further, 20 sample functions are generated for each of the three evolutionary PSDFs and the mean of 5% damped acceleration response spectra of these simulated motions are computed and compared with the target spectrum.

Spectral Representation of Nonstationary Stochastic Processes

Based on evolutionary spectral representation (Priestley 1965, 1967), a zero mean nonstationary stochastic process $f_0(t)$ (complex valued in general and defined in $-\infty < t < +\infty$) admits the representation

$$f_0(t) = \int_{-\infty}^{+\infty} A(t,\omega) e^{i\omega t} dZ(\omega) \tag{1}$$

where $A(t,\omega)$=deterministic modulating function of both t and ω; $Z(\omega)$=spectral process with orthogonal increments and has a distribution function in the interval $(-\infty, +\infty)$. The properties of $Z(\omega)$ can be expressed as

$$\varepsilon[dZ(\omega)] = 0 \tag{2}$$

$$\varepsilon[|dZ(\omega)|^2] = dH(\omega) \tag{3}$$

where ε represents the ensemble average and $H(\omega)$=integrated spectrum.

Now, if $H(\omega)$ is differentiable, one can write

$$dH(\omega) = S(\omega) d\omega \tag{4}$$

where $S(\omega)$=PSDF of the associated stationary stochastic process, and

$$\varepsilon[dZ^*(\omega) dZ(\omega')] = 0, \quad \omega \neq \omega' \tag{5}$$

with an asterisk indicating the complex conjugate. Then, one can also show that

$$\varepsilon[f_0(t)] = \int_{-\infty}^{\infty} A(t,\omega) e^{i\omega t} \varepsilon[dZ(\omega)] = 0 \tag{6}$$

and

$$\begin{aligned} R_{f_0 f_0}(t, t+\tau) &= \varepsilon[f_0^*(t) f_0(t+\tau)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^*(t,\omega) A(t+\tau,\omega') e^{-i\omega t} e^{i\omega'(t+\tau)} \varepsilon[dZ^*(\omega) dZ(\omega')] \\ &= \int_{-\infty}^{\infty} A^*(t,\omega) A(t+\tau,\omega) e^{i\omega\tau} \varepsilon[|dZ(\omega)|^2] \end{aligned} \tag{7}$$

For $\tau=0$, Eq. (7) can be written as

$$\varepsilon[f_0^2(t)] = \int_{-\infty}^{\infty} |A(t,\omega)|^2 S(\omega) d\omega \tag{8}$$

where

$$S_{f_0 f_0}(t,\omega) = |A(t,\omega)|^2 S(\omega) \tag{9}$$

is the (two-sided) evolutionary PSDF (Priestley 1965, 1967). It may be mentioned here that if $f_0(t)$ is stationary, it admits the following well-known spectral representation (e.g., Cramer and Leadbetter 1967):

$$f_0(t) = \int_{-\infty}^{+\infty} e^{i\omega t} dZ(\omega) \tag{10}$$

As $A(t,\omega)$ is complex, Eq. (1) can be written as

$$f_0^*(t) = \int_{-\infty}^{+\infty} A^*(t,\omega) e^{-i\omega t} dZ^*(\omega) = \int_{-\infty}^{+\infty} A^*(t,-\omega) e^{i\omega t} dZ^*(-\omega) \tag{11}$$

But $f_0(t)$ is real valued if and only if $f_0(t)=f_0^*(t)$, i.e., if and only if both $A(t,\omega)$ and $dZ(\omega)$ are Hermitian for all ω

$$A(t,\omega) = A^*(t,-\omega) \tag{12}$$

$$dZ(\omega) = dZ^*(-\omega) \tag{13}$$

Hence, introducing

$$dU_0(\omega) = \mathrm{Re}[dZ(\omega)], \quad dV_0(\omega) = -\mathrm{Im}[dZ(\omega)] \tag{14}$$

it can be shown that

$$dU_0(\omega) = dU_0(-\omega), \quad dV_0(\omega) = -dV_0(-\omega) \quad \text{for all } \omega \tag{15}$$

In Eq. (14), Re is used to denote the real part and Im is used to denote the imaginary part. Modulating function $A(t,\omega)$ can be written in the polar form as follows:

$$A(t,\omega) = \alpha(t,\omega) + i\beta(t,\omega) = |A(t,\omega)| e^{i\theta(t,\omega)} \tag{16}$$

with

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$$|A(t,\omega)| = (\alpha^2(t,\omega) + \beta^2(t,\omega))^{1/2} \tag{17}$$

and

$$\theta(t,\omega) = \tan^{-1}(\mathrm{Im}(A(t,\omega))/\mathrm{Re}(A(t,\omega))) = \tan^{-1}(\beta(t,\omega)/\alpha(t,\omega)) \tag{18}$$

Then, with the aid of Eqs. (13)–(15), the nonstationary process $f_0(t)$ defined in Eq. (1) can be written as

$$\begin{aligned} f_0(t) &= \int_{-\infty}^{\infty} [\alpha(t,\omega) + i\beta(t,\omega)][\cos \omega t + i \sin \omega t] \\ &\quad \times [dU_0(\omega) - idV_0(\omega)] \\ &= \int_{-\infty}^{\infty} [\cos \omega t + i \sin \omega t]\{[\alpha(t,\omega)dU_0(\omega) + \beta(t,\omega)dV_0(\omega)] \\ &\quad + i[\beta(t,\omega)dU_0(\omega) - \alpha(t,\omega)dV_0(\omega)]\} \\ &= \int_{-\infty}^{\infty} \cos \omega t[\alpha(t,\omega)dU_0(\omega) + \beta(t,\omega)dV_0(\omega)] \\ &\quad - \sin \omega t[\beta(t,\omega)dU_0(\omega) - \alpha(t,\omega)dV_0(\omega)] \end{aligned} \tag{19}$$

Eq. (19) can further be written as

$$\begin{aligned} f_0(t) &= \int_{0}^{\infty} \cos \omega t[\alpha(t,\omega)dU(\omega) + \beta(t,\omega)dV(\omega)] \\ &\quad - \sin \omega t[\beta(t,\omega)dU(\omega) - \alpha(t,\omega)dV(\omega)] \end{aligned} \tag{20}$$

where

$$dU(\omega) = dU_0(\omega) + dU_0(-\omega) = 2dU_0(\omega) \quad \text{for } \omega > 0 \tag{21}$$

$$dU(0) = dU_0(0) \tag{22}$$

and

$$dV(\omega) = dV_0(\omega) - dV_0(-\omega) = 2dV_0(\omega) \quad \text{for } \omega \geq 0 \tag{23}$$

On simplification, Eq. (20) can further be expressed as

$$f_0(t) = \int_0^{\infty} \cos \omega t \, dU_t(\omega) - \sin \omega t \, dV_t(\omega) \tag{24}$$

where

$$dU_t(\omega) = \alpha(t,\omega)dU(\omega) - \beta(t,\omega)dV(\omega) \tag{25}$$

$$dV_t(\omega) = \beta(t,\omega)dU(\omega) - \alpha(t,\omega)dV(\omega) \tag{26}$$

As $U(\omega)$ and $V(\omega)$=real and imaginary part, respectively, of the spectral process $Z(\omega)$ defined for nonnegative ω, they admit the following conditions (Cramer and Leadbetter 1967):

$$\varepsilon[dU(\omega)] = \varepsilon[dV(\omega)] = 0, \quad \omega \geqslant 0 \tag{27}$$

$$\varepsilon[dU^2(\omega)] = \varepsilon[dV^2(\omega)] = 2S(\omega)d\omega, \quad \omega \geqslant 0 \tag{28}$$

$$\varepsilon[dU(\omega)dU(\omega')] = \varepsilon[dV(\omega)dV(\omega')] = 0, \quad \omega \neq \omega', \quad \omega,\omega' \geqslant 0 \tag{29}$$

and

$$\varepsilon[dU(\omega)dV(\omega')] = 0, \quad \omega,\omega' \geqslant 0 \tag{30}$$

Then, it can be easily verified from their definitions that for ω, $\omega' \geq 0$, $U_t(\omega)$ and $V_t(\omega)$ satisfy following equations:

$$\varepsilon[dU_t(\omega)] = \alpha(t,\omega)\varepsilon[dU(\omega)] + \beta(t,\omega)\varepsilon[dV(\omega)] = 0$$

$$\varepsilon[dV_t(\omega)] = 0, \quad \text{(following the previous steps)} \tag{31}$$

$$\begin{aligned} \varepsilon[dU_t^2(\omega)] &= \alpha^2(t,\omega)\varepsilon[dU^2(\omega)] + \beta^2(t,\omega)\varepsilon[dV^2(\omega)] \\ &\quad + 2\alpha(t,\omega)\beta(t,\omega)\varepsilon[dU(\omega)dV(\omega)] \\ &= (\alpha^2(t,\omega) + \beta^2(t,\omega))2S(\omega)d\omega \\ &= 2|A(t,\omega)|^2 S(\omega)d\omega \\ &= 2S_{f_0f_0}(t,\omega)d\omega \end{aligned}$$

$$\begin{aligned} \varepsilon[dV_t^2(\omega)] &= 2|A(t,\omega)|^2 S(\omega)d\omega \\ &= 2S_{f_0f_0}(t,\omega)d\omega, \quad \text{(following the previous steps)} \end{aligned} \tag{32}$$

$$\begin{aligned} \varepsilon[dU_t(\omega)dU_t(\omega')] &= \alpha(t,\omega)\alpha(t,\omega')\varepsilon[dU(\omega)dU(\omega')] \\ &\quad + \beta(t,\omega)\beta(t,\omega')\varepsilon[dV(\omega)dV(\omega')] \\ &\quad + \alpha(t,\omega)\beta(t,\omega')\varepsilon[dU(\omega)dV(\omega')] \\ &\quad + \beta(t,\omega)\alpha(t,\omega')\varepsilon[dV(\omega)dU(\omega')] = 0 \end{aligned}$$

$$\varepsilon[dV_t(\omega)dV_t(\omega')] = 0, \quad \text{(following the previous steps)} \tag{33}$$

and

$$\begin{aligned} \varepsilon[dU_t(\omega)dV_t(\omega')] &= \alpha(t,\omega)\beta(t,\omega')\varepsilon[dU(\omega)dU(\omega')] \\ &\quad - \beta(t,\omega)\alpha(t,\omega')\varepsilon[dV(\omega)dV(\omega')] \\ &\quad - \alpha(t,\omega)\alpha(t,\omega')\varepsilon[dU(\omega)dV(\omega')] \\ &\quad + \beta(t,\omega)\beta(t,\omega')\varepsilon[dV(\omega)dU(\omega')] = 0 \end{aligned} \tag{34}$$

Therefore, $U_t(\omega)$ and $V_t(\omega)$ are each orthogonal, and in addition, they are mutually orthogonal, constituting the spectral processes of nonstationary stochastic process $f_0(t)$. It may be noted that when $U_t(\omega)$ and $V_t(\omega)$ are time independent, i.e., $dU_t(\omega) = dU(\omega)$ and $dV_t(\omega) = dV(\omega)$, Eq. (24) is then reduced to the spectral representation of a stationary stochastic process (Cramer and Leadbetter 1967). Now, Eq. (24) can be rewritten in the following discrete form:

$$f_0(t) = \sum_{k=0}^{\infty} [\cos(\omega_k t)dU_t(\omega_k) - \sin(\omega_k t)dV_t(\omega_k)] \tag{35}$$

where

$$\omega_k = k\Delta\omega \tag{36}$$

with sufficiently small but finite $\Delta\omega$ such that Eq. (35) can be used for Eq. (24). If $dU_t(\omega_k)$ and $dV_t(\omega_k)$ are defined as

$$dU_t(\omega_k) = \sqrt{2}B_{t,k} \cos \Phi_k \tag{37}$$

$$dV_t(\omega_k) = \sqrt{2}B_{t,k} \sin \Phi_k \tag{38}$$

where

$$B_{t,k} = [2S_{f_0f_0}(t,\omega_k)\Delta\omega]^{1/2} \tag{39}$$

and Φ_k=independent random phase angles uniformly distributed in the range $[0, 2\pi]$, then it can be shown that Eqs. (31)–(34)

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are satisfied. Indeed, the following expressions can be written for Eq. (31)

$$\varepsilon[dU_t(\omega_k)] = \varepsilon[\sqrt{2}B_{t,k}\cos\Phi_k] = \sqrt{2}B_{t,k}\varepsilon[\cos\Phi_k]$$
$$= \sqrt{2}B_{t,k}\int_{-\infty}^{+\infty}\cos\phi_k p_{\Phi_k}(\phi_k)d\phi_k \qquad (40)$$

where $p_{\Phi_k}(\phi_k)$=probability density function of random phase angle Φ_k given by

$$p_{\Phi_k}(\phi_k) = \begin{cases} 1/2\pi, & 0 \le \phi_k \le 2\pi \\ 0 & \text{otherwise} \end{cases} \qquad (41)$$

Now, Eq. (40) can be written as

$$\varepsilon[dU_t(\omega_k)] = \sqrt{2}B_{t,k}\int_0^{2\pi}\cos\phi_k\frac{1}{2\pi}d\phi_k = 0 \qquad (42)$$

and in exactly the same way it can be shown that $\varepsilon[dV_t(\omega_k)]=0$. The requirement described in Eq. (32) can be expressed as

$$\varepsilon[dU_t^2(\omega_k)] = \varepsilon[2B_{t,k}^2\cos^2\Phi_k]$$
$$= 2B_{t,k}^2\varepsilon[\cos^2\Phi_k]$$
$$= 2B_{t,k}^2\int_{-\infty}^{+\infty}\cos^2\phi_k p_{\Phi_k}(\phi_k)d\phi_k$$
$$= 2B_{t,k}^2\int_0^{2\pi}\frac{1}{2}(1+\cos 2\phi_k)\frac{1}{2\pi}d\phi_k$$
$$= B_{t,k}^2$$
$$= 2S_{f_0f_0}(t,\omega_k)\Delta\omega \qquad (43)$$

and in exactly the same way it can be shown that $\varepsilon[dV_t^2(\omega_k)] = 2S_{f_0f_0}(t,\omega_k)\Delta\omega$.

The condition described in Eq. (33) can be written as

$$\varepsilon[dU_t(\omega_k)dU_t(\omega_k')] = \varepsilon[2B_{t,k}B_{t,k}'\cos\Phi_k\cos\Phi_k']$$
$$= 2B_{t,k}B_{t,k}'\varepsilon[\cos\Phi_k\cos\Phi_k']$$
$$= B_{t,k}B_{t,k}'\varepsilon[\cos\Phi_k]\varepsilon[\cos\Phi_k'] \qquad (44)$$

and the last equality in Eq. (44) is valid as Φ_k and Φ_k' are independent random phase angles for $k \neq k'$. Eventually, Eq. (44) is written as

$$\varepsilon[dU_t(\omega_k)dU_t(\omega_k')] = B_{t,k}B_{t,k}'\int_0^{2\pi}\cos\phi_k\frac{1}{2\pi}d\phi_k \times \int_0^{2\pi}\cos\phi_k'\frac{1}{2\pi}d\phi_k' = 0 \quad \text{for } k \neq k' \qquad (45)$$

and in exactly the same way it can be shown that $\varepsilon[dV_t(\omega_k)dV_t(\omega_k')]=0$, for $k \neq k'$.

Finally, the requirement described in Eq. (34) can be expressed as

$$\varepsilon[dU_t(\omega_k)dV_t(\omega_k')] = \varepsilon[2B_{t,k}B_{t,k}'\cos\Phi_k\sin\Phi_k']$$
$$= 2B_{t,k}B_{t,k}'\varepsilon[\cos\Phi_k\sin\Phi_k'] \qquad (46)$$

When $k \neq k'$, the expected value appearing in the last term of Eq. (46) can be written as follows: as Φ_k and Φ_k' are independent random phase angles

$$\varepsilon[\cos\Phi_k\sin\Phi_k'] = \varepsilon[\cos\Phi_k]\varepsilon[\sin\Phi_k'] = 0 \quad \text{for } k \neq k' \qquad (47)$$

When $k=k'$, the term $\varepsilon[\cos\Phi_k\sin\Phi_k']$ can be written as

$$\varepsilon[\cos\Phi_k\sin\Phi_k'] = \varepsilon[\cos\Phi_k\sin\Phi_k] = \varepsilon\left[\frac{1}{2}\sin 2\Phi_k\right]$$
$$= \int_0^{2\pi}\sin 2\phi_k\frac{1}{2\pi}d\phi_k = 0 \quad \text{for } k = k' \qquad (48)$$

Combining Eqs. (47) and (48), the following result is established:

$$\varepsilon[dU_t(\omega_k)dV_t(\omega_k')] = 0 \quad \text{for } k \neq k' \text{ and } k = k' \qquad (49)$$

Therefore, all the requirements [Eqs. (31)–(34)] imposed on $dU_t(\omega_k)$ and $dV_t(\omega_k)$ are satisfied by the expressions given in Eqs. (37) and (38). Then, substituting Eqs. (37) and (38) into Eq. (35), the following series representation is obtained:

$$f_0(t) = \sum_{k=0}^{\infty}\{\cos(\omega_k t)\sqrt{2}[2S_{f_0f_0}(t,\omega_k)\Delta\omega]^{1/2}\cos\Phi_k - \sin(\omega_k t)\sqrt{2}[2S_{f_0f_0}(t,\omega_k)\Delta\omega]^{1/2}\sin\Phi_k\}$$
$$= \sqrt{2}\sum_{k=0}^{\infty}[2S_{f_0f_0}(t,\omega_k)\Delta\omega]^{1/2}\cos(\omega_k t + \Phi_k) \qquad (50)$$

Eq. (50) can also be written as

$$f_0(t) = \sqrt{2}\sum_{k=0}^{\infty}[2|A(t,\omega_k)|^2 S(\omega_k)\Delta\omega]^{1/2}\cos(\omega_k t + \Phi_k) \qquad (51)$$

where $S_{f_0f_0}(t,\omega_k)\Delta\omega = |A(t,\omega_k)|^2 S(\omega_k)\Delta\omega$, which represents the discretization of evolutionary PSDF introduced by Priestley (1965, 1967) as shown in Eq. (9).

It is important to note that the derivations of Eqs. (50) and (51) initially retained the terms involving $\theta(t,\omega)$, but they did not appear in the final result. In the present study, each of $dU_0(\omega)$ and $dV_0(\omega)$ [or, $dU(\omega)$ and $dV(\omega)$] is considered to represent a white noise with $S(\omega)=1$ with appropriate dimension. Therefore, if it is estimated for example, by the STFT (Flanagan 1972), $A(t,\omega)$ is nothing but the evolutionary PSDF.

Simulation of Nonstationary Stochastic Processes

Simulation Formula

Consider a stochastic process $f_0(t)$ with mean value equal to zero, autocorrelation function $R_{f_0f_0}(t,t+\tau)$ and two-sided PSDF $S_{f_0f_0}(t,\omega)$. In the following, distinction will be made between the stochastic process $f_0(t)$ and its simulation $f(t)$.

From the infinite series representation shown in Eq. (50), it follows that the stochastic process $f_0(t)$ can be simulated by the following series as $N \to \infty$;

$$f(t) = \sqrt{2}\sum_{n=0}^{N-1}[2S_{f_0f_0}(t,\omega_n)\Delta\omega]^{1/2}\cos(\omega_n t + \Phi_n) \qquad (52)$$

where

$$\Delta\omega = \omega_u/N \qquad (53)$$

$$\omega_n = n\Delta\omega, \quad n = 0,1,2,\ldots,N-1 \qquad (54)$$

and it is assumed that

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$$S_{f_0f_0}(t,\omega_0=0)=0 \tag{55}$$

In Eq. (53), ω_u represents an upper cutoff frequency beyond which the evolutionary PSDF $S_{f_0f_0}(t,\omega)$ may be assumed to be zero for either mathematical or physical reasons. As such, ω_u=fixed value and hence $\Delta\omega\rightarrow 0$ as $N\rightarrow\infty$ so that $N\Delta\omega=\omega_u$. The $\Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_{N-1}$ in Eq. (52) are independent random phase angles distributed uniformly over the interval $[0,2\pi]$. The simulated stochastic process $f(t)$ is asymptotically Gaussian as $N\rightarrow\infty$ because of the central limit theorem. It will be shown next that the ensemble expected value $\varepsilon[f(t)]$ and the ensemble autocorrelation function $R_{ff}(t,t+\tau)$ of the simulated stochastic process $f(t)$ are identical to the corresponding targets $\varepsilon[f_0(t)]$ and $R_{f_0f_0}(t,t+\tau)$, respectively.

1. **Show that:** $\varepsilon[f(t)]=\varepsilon[f_0(t)]=0$.

Proof: Utilizing the property that the operations of mathematical expectation and summation are commutative, the ensemble expected value of the simulated stochastic process $f(t)$ can be written as

$$\varepsilon[f(t)]=\varepsilon\left[\sqrt{2}\sum_{n=0}^{N-1}[2S_{f_0f_0}(t,\omega_n)\Delta\omega]^{1/2}\cos(\omega_n t+\Phi_n)\right]$$
$$=\sqrt{2}\sum_{n=0}^{N-1}[2S_{f_0f_0}(t,\omega_n)\Delta\omega]^{1/2}\varepsilon[\cos(\omega_n t+\Phi_n)] \tag{56}$$

where the expected value $\varepsilon[\cos(\omega_n t+\Phi_n)]$ can be shown to be equal to zero in the following way:

$$\varepsilon[\cos(\omega_n t+\Phi_n)]=\int_{-\infty}^{+\infty}p_{\Phi_n}(\phi_n)\cos(\omega_n t+\phi_n)d\phi_n$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\cos(\omega_n t+\phi_n)d\phi_n=0 \tag{57}$$

Combining Eqs. (56), (57), and (6), it is easy to show that

$$\varepsilon[f(t)]=0=\varepsilon[f_0(t)] \tag{58}$$

2. **Show that:** $R_{ff}(t,t+\tau)=R_{f_0f_0}(t,t+\tau)$.

Proof:

$$R_{ff}(t,t+\tau)=\varepsilon[f(t)f(t+\tau)]=\varepsilon\left[\sqrt{2}\sum_{n=0}^{N-1}(2S_{f_0f_0}(t,\omega_n)\Delta\omega)^{1/2}\cos(\omega_n(t)+\Phi_n)\times\sqrt{2}\sum_{m=0}^{N-1}(2S_{f_0f_0}(t+\tau,\omega_m)\Delta\omega)^{1/2}\cos(\omega_m(t+\tau)+\Phi_m)\right]$$
$$=2\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}(2S_{f_0f_0}(t,\omega_n)\Delta\omega)^{1/2}(2S_{f_0f_0}(t+\tau,\omega_m)\Delta\omega)^{1/2}\times\varepsilon[\cos(\omega_n t+\Phi_n)\cos(\omega_m(t+\tau)+\Phi_m)] \tag{59}$$

As random phase angles Φ_n $(n=0,1,2,\ldots,N-1)$ are independent, for $n\neq m$, the expected value shown in Eq. (59) can be written as

$$\varepsilon[\cos(\omega_n t+\Phi_n)\cos(\omega_m(t+\tau)+\Phi_m)]$$
$$=\varepsilon[\cos(\omega_n t+\Phi_n)]\varepsilon[\cos(\omega_m(t+\tau)+\Phi_m)]=0 \tag{60}$$

It may be noted that only the terms with $n=m$ remain in Eq. (59). Hence, the ensemble autocorrelation function of the simulated process $f(t)$ can be written as

$$R_{ff}(t,t+\tau)=2\sum_{n=0}^{N-1}(2S_{f_0f_0}(t,\omega_n)\Delta\omega)^{1/2}(2S_{f_0f_0}(t+\tau,\omega_n))^{1/2}$$
$$\cdot\,\varepsilon[\cos(\omega_n t+\Phi_n)\cos(\omega_n(t+\tau)+\Phi_n)] \tag{61}$$

where

$$\varepsilon[\cos(\omega_n t+\Phi_n)\cos(\omega_n(t+\tau)+\Phi_n)]$$
$$=\tfrac{1}{2}\varepsilon[\cos(\omega_n t+\omega_n(t+\tau)+2\Phi_n)+\cos(\omega_n(t+\tau)-\omega_n t)]$$
$$=\tfrac{1}{2}\varepsilon[\cos(\omega_n\tau)]=\tfrac{1}{2}\cos\omega_n\tau \tag{62}$$

Then, Eq. (61) takes a more compact expression

$$R_{ff}(t,t+\tau)=2\sum_{n=0}^{N-1}(2S_{f_0f_0}(t,\omega_n)\Delta\omega)^{1/2}$$
$$\times(2S_{f_0f_0}(t+\tau,\omega_n)\Delta\omega)^{1/2}\frac{1}{2}\cos(\omega_n\tau) \tag{63}$$

Taking the limit as $\Delta\omega\rightarrow 0$ and $N\rightarrow\infty$ and keeping in mind that $\omega_u=N\Delta\omega$ is constant and that $S_{f_0f_0}(t,\omega)=0$ for $|\omega|\geq\omega_u$, leads to

$$R_{ff}(t,t+\tau)=2\int_0^{\infty}[S_{f_0f_0}(t,\omega)]^{1/2}[S_{f_0f_0}(t+\tau,\omega)]^{1/2}\cos(\omega\tau)d\omega \tag{64}$$

From Eq. (7), we have

$$R_{f_0f_0}(t,t+\tau)=\int_{-\infty}^{\infty}A^*(t,\omega)A(t+\tau,\omega)e^{i\omega\tau}S(\omega)d\omega$$
$$=\int_{-\infty}^{\infty}[S_{f_0f_0}(t,\omega)]^{1/2}[S_{f_0f_0}(t+\tau,\omega)]^{1/2}e^{i\omega\tau}d\omega \tag{65}$$

Observing Eqs. (64) and (65), it is easy to show

$$R_{ff}(t,t+\tau)=R_{f_0f_0}(t,t+\tau) \tag{66}$$

then

$$S_{ff}(t,\omega)=S_{f_0f_0}(t,\omega) \tag{67}$$

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The fact that the two-sided PSDF $S_{f_0f_0}(t,\omega)$ is a real and even function of the frequency ω was used in deriving Eq. (66).

Two Special Cases of the Simulation Formula

It was shown earlier that sample functions generated by Eq. (52) possess the properties represented by Eqs. (58) and (66). Keeping the total energy a constant, Eq. (66) may be relaxed to some extent as given by following equations:

$$R_{ff}(t,t) = R_{f_0f_0}(t,t) \tag{68}$$

or further

$$R_{ff}(t,t+\tau) = R_{f_0f_0}(t,t+\tau) = \delta(\tau)\int_{-\infty}^{\infty} S_{f_0f_0}(t,\omega)d\omega \tag{69}$$

and

$$R_{ff}(\tau) = R_{f_0f_0}(\tau) \tag{70}$$

In the following two special cases, it will be shown that by relaxing Eq. (66) to Eq. (68), the energy is randomly distributed with frequency at any time instant of a sample function; and further by Eq. (69) the sample functions are nonstationary white noise processes. Also considering Eq. (70), the energy is averaged with time, and thus, Eq. (52) is reduced to the well-known Shinozuka's simulation formula (Shinozuka and Jan 1972; Shinozuka 1972) for stationary stochastic processes.

Special Case (I)

If $dU_t(\omega_k)$ and $dV_t(\omega_k)$ in Eq. (35) are defined as

$$dU_t(\omega_k) = \sqrt{2}B_{t,k}\cos\Phi_k(t) \tag{71}$$

$$dV_t(\omega_k) = \sqrt{2}B_{t,k}\sin\Phi_k(t) \tag{72}$$

where

$$B_{t,k} = [2S_{f_0f_0}(t,\omega)\Delta\omega]^{1/2} \tag{73}$$

and $\Phi_k(t)$=independent random phase angles uniformly distributed in the range $[0, 2\pi]$, and $\Phi_k(t_1)\neq\Phi_k(t_2)$ for $t_1\neq t_2$. It can be shown that Eqs. (71) and (72) satisfy Eqs. (31)–(34).

Substituting Eqs. (71) and (72) into Eq. (35), the following series representation is obtained

$$\begin{aligned} f_0(t) &= \sum_{k=0}^{\infty}\{\cos(\omega_k t)\sqrt{2}[2S_{f_0f_0}(t,\omega_k)\Delta\omega]^{1/2}\cos\Phi_k(t) \\ &\quad - \sin(\omega_k t)\sqrt{2}[2S_{f_0f_0}(t,\omega_k)\Delta\omega]^{1/2}\sin\Phi_k(t)\} \\ &= \sqrt{2}\sum_{k=0}^{\infty}[2S_{f_0f_0}(t,\omega_k)\Delta\omega]^{1/2}\cos(\omega_k t + \Phi_k(t)) \end{aligned} \tag{74}$$

Now, stochastic process $f_0(t)$ can be simulated by finite series representation

$$f(t) = \sqrt{2}\sum_{n=0}^{N-1}[2S_{f_0f_0}(t,\omega_n)\Delta\omega]^{1/2}\cos(\omega_n t + \Phi_n(t)) \tag{75}$$

where all the variables have the same meaning as those in Eq. (52) except for $\Phi_k(t)$. Next, it will be shown that the sample functions by Eq. (75) satisfy Eqs. (58) and (68). Now

$$\begin{aligned} \varepsilon[f(t)] &= \varepsilon[\sqrt{2}\sum_{n=0}^{N-1}[2S_{f_0f_0}(t,\omega_n)\Delta\omega]^{1/2}\cos(\omega_n t + \Phi_n(t)) \\ &= \sqrt{2}\sum_{n=0}^{N-1}[2S_{f_0f_0}(t,\omega_n)\Delta\omega]^{1/2}\varepsilon[\cos(\omega_n t + \Phi_n(t))] \end{aligned} \tag{76}$$

where

$$\begin{aligned} \varepsilon[\cos(\omega_n t + \Phi_n(t))] &= \int_{-\infty}^{\infty} p_{\Phi_n(t)}(\phi_n^{(t)})\cos(\omega_n t + \phi_n^{(t)})d\phi_n^{(t)} \\ &= \frac{1}{2\pi}\int_0^{2\pi}\cos(\omega_n t + \phi_n^{(t)})d\phi_n^{(t)} = 0 \end{aligned} \tag{77}$$

Combining Eqs. (77) and (6), it is easy to show that Eq. (58) is satisfied. Now, for Eq. (68), one can write,

$$\begin{aligned} R_{ff}(t,t) &= \varepsilon[f^2(t)] = \varepsilon\Bigg[\sqrt{2}\sum_{n=0}^{N-1}(2S_{f_0f_0}(t,\omega_n)\Delta\omega)^{1/2}\cos(\omega_n t + \Phi_n(t)) \\ &\quad \times \sqrt{2}\sum_{m=0}^{N-1}(2S_{f_0f_0}(t,\omega_m)\Delta\omega)^{1/2}\cos(\omega_m t + \Phi_m(t))\Bigg] \\ &= 2\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}(2S_{f_0f_0}(t,\omega_n)\Delta\omega)^{1/2}(2S_{f_0f_0}(t,\omega_m)\Delta\omega)^{1/2} \\ &\quad \times \varepsilon[\cos(\omega_n t + \Phi_n(t))\cos(\omega_m t + \Phi_m(t))] \end{aligned} \tag{78}$$

As random phase angles $\Phi_0(t)$, $\Phi_1(t)$, $\Phi_2(t)$,…, $\Phi_{N-1}(t)$ are independent at any time instant, the expected value in Eq. (78), for $n\neq m$, can be written as

$$\begin{aligned} &\varepsilon[\cos(\omega_n t + \Phi_n(t))\cos(\omega_m t + \Phi_m(t))] \\ &\quad = \varepsilon[\cos(\omega_n t + \Phi_n(t))]\varepsilon[\cos(\omega_m t + \Phi_m(t))] = 0 \end{aligned} \tag{79}$$

Therefore, considering $n=m$, Eq. (79) can be expressed as

$$R_{ff}(t,t) = 2\sum_{n=0}^{N-1}(2S_{f_0f_0}(t,\omega_n)\Delta\omega)\varepsilon[\cos^2(\omega_n t + \Phi_n(t))] \tag{80}$$

where

$$\varepsilon[\cos^2(\omega_n t + \Phi_n(t))] = \tfrac{1}{2}\varepsilon[\cos(2\omega_n t + 2\Phi_n(t)) + 1] = \tfrac{1}{2} \tag{81}$$

Further

$$R_{ff}(t,t) = \varepsilon[f^2(t)] = \sum_{n=0}^{N-1} 2S_{f_0f_0}(t,\omega_n)\Delta\omega \tag{82}$$

Taking the limit as $\Delta\omega\to 0$ and $N\to\infty$ and keeping in mind that $\omega_u=N\Delta\omega$ is constant and that $S_{f_0f_0}(t,\omega)=0$ for $|\omega|\geq\omega_u$, lead to

$$R_{ff}(t,t) = 2\int_0^{\infty} S_{f_0f_0}(t,\omega)d\omega = \int_{-\infty}^{\infty} S_{f_0f_0}(t,\omega)d\omega \tag{83}$$

Note that

$$R_{f_0f_0}(t,t) = \int_{-\infty}^{\infty} S_{f_0f_0}(t,\omega)d\omega \tag{84}$$

and it is easy to show that

$$R_{ff}(t,t) = R_{f_0f_0}(t,t) \tag{85}$$

Further, if independent random phase angles $\Phi_n(t)$ are distributed uniformly over $[0,2\pi]$ for any n and any t, i.e., $\Phi_n(t)$ is a

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random field, it can be proven that Eqs. (71) and (72) satisfy Eqs. (31)–(34), and Eq. (75) satisfies Eqs. (58) and (69). Thus, the sample functions generated by Eq. (75) are nothing but nonstationary white noise processes. The detail will not be given here for saving space.

Special Case (II)

If one relaxes the energy distribution along with time, e.g., if one takes its average, the evolutionary PSDF $S_{f_0 f_0}(t,\omega)$ of a nonstationary process is then reduced to PSDF $S_{f_0 f_0}(\omega)$ of a stationary process as follows:

$$S_{f_0 f_0}(\omega) = \frac{1}{T}\int_{-\infty}^{\infty} S_{f_0 f_0}(t,\omega)dt \tag{86}$$

where T=effective duration of evolutionary PSDF $S_{f_0 f_0}(t,\omega)$. Then, Eq. (52) is simplified into

$$f(t) = \sqrt{2}\sum_{n=0}^{N-1}[2S_{f_0 f_0}(\omega_n)\Delta\omega]^{1/2}\cos(\omega_n t + \Phi_n) \tag{87}$$

which is nothing but the well-known Shinozuka's simulation formula (Shinozuka and Jan 1972; Shinozuka 1972) for stationary stochastic processes.

Generation of Evolutionary Power Spectral Density Function

There are potentially a number of ways in which evolutionary PSDF is estimated from a ground motion time-history record. This study develops and compares simple methods of estimating evolutionary PSDF by means of the STFT (Flanagan 1972), the WT (Daubechies 1992), and the HHT (Huang et al. 1998). It may be noted here that the STFT and the WT represent energy-time-frequency distribution of nonstationary processes with a chosen windowing function or wavelet, while the HHT represents energy-time-frequency distribution by using a simple and adaptive technique of obtaining instantaneous frequencies. Following subsections present the methods to estimate evolutionary PSDFs from a ground motion record using the three abovementioned transforms. It may also be mentioned that the evolutionary PSDFs estimated from these three methods obey Parseval's identity of total energy.

Short-Time Fourier Transform

The STFT $F(t,\omega)$ of a function $f(t)$ is defined by the following convolution integral:

$$F(t,\omega) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)e^{-i\omega\tau}d\tau \tag{88}$$

where $h(t)$ is an appropriate time window. The evolutionary PSDF $S_{f_0 f_0}(t,\omega)$ can be written as

$$|F(t,\omega)|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau_1)f_0(\tau_2)h(t-\tau_1)h(t-\tau_2)e^{-i\omega\tau_1}e^{i\omega\tau_2}d\tau_1 d\tau_2 \tag{89}$$

The total energy of $f(t)$ can be estimated by

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(t,\omega)|^2 dt d\omega &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau_1)f(\tau_2)h(t-\tau_1)h(t-\tau_2) \\ &\quad \times e^{-i\omega(\tau_1-\tau_2)}d\tau_1 d\tau_2 dt d\omega \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f^2(\tau)h^2(t-\tau)d\tau dt\end{aligned} \tag{90}$$

In deriving Eq. (90), the following equation is used:

$$\int_{-\infty}^{\infty} e^{-i\omega(\tau_1-\tau_2)}d\omega = \delta(\tau_1-\tau_2) \tag{91}$$

If $h^2(t)=\delta(t)$, the total energy in Eq. (90) is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(t,\omega)|^2 dt d\omega = \int_{-\infty}^{\infty} f^2(t)dt \tag{92}$$

This means that if the time window is chosen such that

$$\int_{-\infty}^{\infty} h^2(t)dt = 1 \tag{93}$$

the total energy can be kept identical (Parseval's identity) in estimating the evolutionary PSDFs. For example, the selected time window-squared functions may be

$$h^2(t) = \frac{1}{\sqrt{2\pi}\sigma}e^{-t^2/2\sigma^2} \quad (\sigma = 0.25) \tag{94}$$

or

$$h^2(t) = 1 \quad (0 \le t \le 1) \tag{95}$$

which are nothing but the Gaussian time window squared with standard deviation σ=0.25 s, and the rectangular time window squared with height and duration both equal to 1, respectively. One might observe that both time windows satisfy the condition in Eq. (93).

Wavelet Transform

The WT of $f \in \mathbf{L}^2(R)$ (finite energy function $\int|f(t)|^2 dt < +\infty$) at time u and scale s, and the corresponding inverse relationship are given by Daubechies (1992)

$$W_\psi f(u,s) = \frac{1}{\sqrt{s}}\int_{-\infty}^{\infty} f(t)\psi^*\left(\frac{(t-u)}{s}\right)dt, \quad s>0 \tag{96}$$

and

$$f(t) = \frac{1}{2\pi C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{s^2}W_\psi f(u,s)f(t)\psi\left(\frac{(t-u)}{s}\right)duds, \quad s>0 \tag{97}$$

where

$$C_\psi = \int_{-\infty}^{\infty}\frac{|\hat{\psi}(\omega)|^2}{|\omega|}d\omega < +\infty \quad \text{(admissibility condition)} \tag{98}$$

In Eqs. (96)–(98), the wavelet function $\psi \in \mathbf{L}^2(R)$ known as "basic" or "mother" wavelet with zero average

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$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \tag{99}$$

is centered in the neighborhood of $t=0$, and as normalized $\|\psi\|=1$. $\hat{\psi}(\omega)$ denotes the Fourier transform of $\psi(t)$ and is given by

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(t)e^{-i\omega t}dt \tag{100}$$

It may be noted that the WT decomposes signal $f(t)$ over dilated and translated wavelets. As $W_\psi f(u,s)$ is the convolution of $f(t)$ with $(1/\sqrt{s})\psi^*(-t/s)$, $W_\psi f(u,s)$ represents the contribution of the function $f(t)$ in the neighborhood of $t=u$ and in the frequency band corresponding to scale s. It can also be shown that (see Daubechies 1992)

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{s^2}|W_\psi f(u,s)|^2 du ds \tag{101}$$

Now, if any wavelet function satisfies the condition

$$\int_{-\infty}^{\infty} |\hat{\psi}_{u,s}(\omega)|^2 d\omega = 1 \tag{102}$$

one can write Eq. (101) as

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{s^2}|W_\psi f(u,s)|^2 du ds\right] \times |\hat{\psi}_{u,s}(\omega)|^2 d\omega \tag{103}$$

In Eqs. (102) and (103), $\hat{\psi}_{u,s}(\omega)$ represents the Fourier transform of $\psi[(t-u)/s]$ and can be expressed as $\hat{\psi}_{u,s}(\omega)=\sqrt{s}\hat{\psi}(s\omega)e^{i\omega u}$. Then, using Parseval's identity, one can write

$$|F(\omega)|^2 = \frac{1}{2\pi C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{s^2}|W_\psi f(u,s)|^2 |\hat{\psi}_{u,s}(\omega)|^2 du ds \tag{104}$$

where $F(\omega)$=Fourier transform of $f(t)$. As the wavelet coefficients $W_\psi f(u,s)$ provides the localized information of signal $f(t)$ at $t=u$, from Eq. (104) the evolutionary PSDF $S_{f_0 f_0}(t,\omega)$ can be expressed as

$$|F(\omega,t)|^2 = \frac{1}{2\pi C_\psi}\int_{-\infty}^{\infty} \frac{1}{s^2}|W_\psi f(t,s)|^2 |\hat{\psi}_{t,s}(\omega)|^2 ds \tag{105}$$

It may be noted that the previous expression of evolutionary PSDF obeys total energy equilibrium. Therefore, one can use any wavelet basis, which satisfies Eq. (102), to generate evolutionary PSDF [e.g., modified Littlewood-Paley basis as proposed by Basu and Gupta (1998)] that maintains total energy.

Hilbert-Huang Transform

The Hilbert transform $Y(t)$ of any function $X(t)$ of class L^p is expressed by the following convolution integral (see Bendat and Piersol 1986)

$$Y(t) = \frac{1}{\pi}P\int_{-\infty}^{\infty} \frac{X(t')}{t-t'}dt' \tag{106}$$

where P indicates the Cauchy principle value. Here, $X(t)$ and $Y(t)$ form the complex conjugate pair and thus an analytic function $Z(t)$ can be constructed as

$$Z(t) = X(t) + iY(t) = a(t)e^{i\theta(t)} \tag{107}$$

in which

$$a(t) = \sqrt{X^2(t) + Y^2(t)} \tag{108}$$

$$\theta(t) = \tan^{-1} Y(t)/X(t) \tag{109}$$

One can observe in Eq. (107) that $X(t)$ is the real part of $Z(t)$ $[=(a(t)e^{i\theta(t)})]$ expressed in the polar coordinate to clarify the local nature of $X(t)$ by representing it in an amplitude- and phase-varying trigonometric function.

Based on the definition of Hilbert transform, the instantaneous frequency of $\omega(t)$ can be expressed as (Huang et al. 1998)

$$\omega(t) = \frac{d\theta(t)}{dt} \tag{110}$$

However, as discussed by Huang et al. (1998), for this definition to be meaningful, the data must be locally symmetric with respect to the zero mean. Unfortunately, the majority of data encountered do not satisfy this condition. Empirical mode decomposition, which is the first part of the HHT, decomposes the data into a number of intrinsic mode functions (IMF) to satisfy local symmetry condition by using an adaptive numerical technique (see Huang et al. 1998). Empirical modes are obtained through a shifting process, the result of which decomposes data into n empirical modes and a residue r_n, which can either be the mean trend of the data or a constant and often negligible compared to the other IMF components. After that, the instantaneous frequencies are obtained through Hilbert transform.

Let $C_j(t)$ be the jth empirical mode of $X(t)$, then $X(t)$ can be expressed as

$$X(t) = \sum_{j=1}^{n} C_j(t) + r_n \tag{111}$$

Now, for a negligible residue, the original data can be approximated as the real part of the sum of the transform of each empirical modes

$$X(t) = RP\sum_{j=1}^{n} a_j(t)e^{i\int\omega_j(t)dt} \tag{112}$$

where $a_j(t)=(C_j^2(t)+Y_j^2(t))^{1/2}$, $\int\omega_j(t)dt=\theta_j(t)=\arctan(Y_j(t)/C_j(t))$, and $Y_j(t)$ being the Hilbert transform of $C_j(t)$. Both the frequency and the amplitude of each component are functions of time, thus, the final representation is a curve in the three-dimensional space of frequency-amplitude-time. This frequency-time distribution of the amplitude is termed as the Hilbert amplitude spectrum, or simply, the Hilbert spectrum. By expressing $X^2(t)$ in terms of $C_j(t)$ from Eq. (111) and integrating with respect to t, one can write

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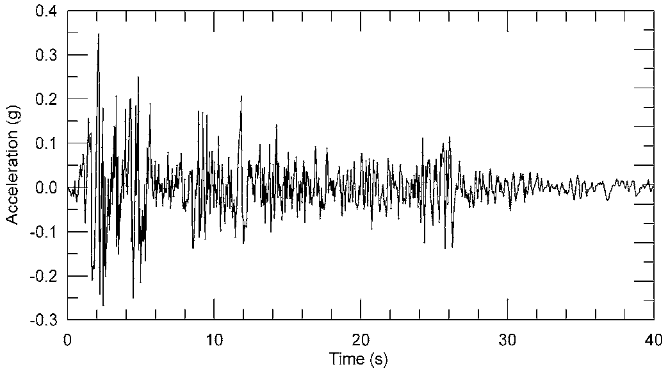

Fig. 1. Acceleration time history: N-S component of 1940 El Centro earthquake record

$$\int_{\infty}^{\infty} X^2(t)dt = \int_{\infty}^{\infty} \sum_{j=1}^{n} \sum_{k=1}^{n} C_j(t)C_k(t)dt$$
$$= \int_{\infty}^{\infty} \sum_{j=1}^{n} C_j^2(t)dt + 2\int_{\infty}^{\infty} \sum_{j=1}^{n} \sum_{k=1;j\neq k}^{n} C_j(t)C_k(t)dt \quad (113)$$

Now, if the IMF components are orthogonal with respect to each other, then the cross terms given in the right-hand side of Eq. (113) are zero. Although, in most of the cases, these cross terms are not zero, by virtue of the decomposition, each IMF component represents narrow banded data with minimal frequency content overlaps. Therefore, the cross terms are negligible compared to the first term of Eq. (113). Further, using this argument along with Parseval identity, and keeping in mind that the instantaneous frequency is a function of time t, the evolutionary PSDF $S_{f_0 f_0}(t,\omega)$ of $X(t)$ can be expressed as

$$S_{f_0 f_0}(t,\omega) = |F(\omega,t)|^2 = \sum_{j=1}^{n} \frac{1}{2}\delta[\omega - \omega_j(t)]a_j^2(t) \quad (114)$$

It may be noted that the previous expression of evolutionary PSDF obey energy equilibrium when IMF components are orthogonal to each other and the residue r_n is zero. In all other cases, we need to introduce a correction term c_r for energy equilibrium in the right-hand side of Eq. (114), which can be obtained by

$$c_r = \frac{\int_{\infty}^{\infty} X^2(t)dt}{\int_{\infty}^{\infty} \sum_{j=1}^{n} C_j^2(t)dt} \quad (115)$$

Now, if filtering and smoothing functions are applied to give a better look of the estimated evolutionary PSDF, the energy spreads over time and frequency. Thus, further correction terms must be introduced to preserve total energy of the time history in generated evolutionary PSDFs.

Results and Discussion

In order to generate the evolutionary PSDF from a given earthquake record, N-S component of the well–known 1940 El Centro earthquake acceleration time history record is considered (see Fig. 1). The evolutionary PSDF $S_{f_0 f_0}(t,\omega)$ of the El Centro earthquake acceleration record is estimated on the basis of the methods developed by using the STFT, the WT, and the HHT. For the STFT, a Gaussian window squared defined by Eq. (93) with $\sigma = 0.25$ is used, and for the WT, a Morlet mother wavelet with

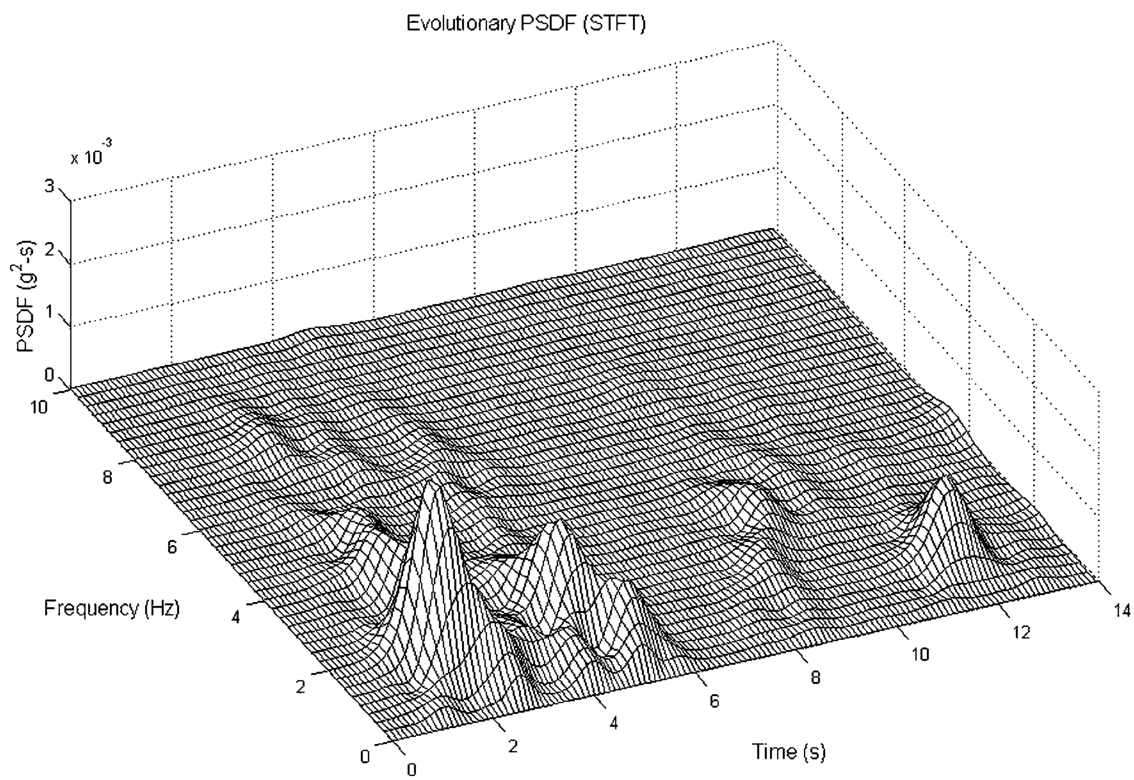

Fig. 2. Evolutionary PSDF of El Centro earthquake record in $g^2 - s$ using the method based on the STFT (Gaussian window)

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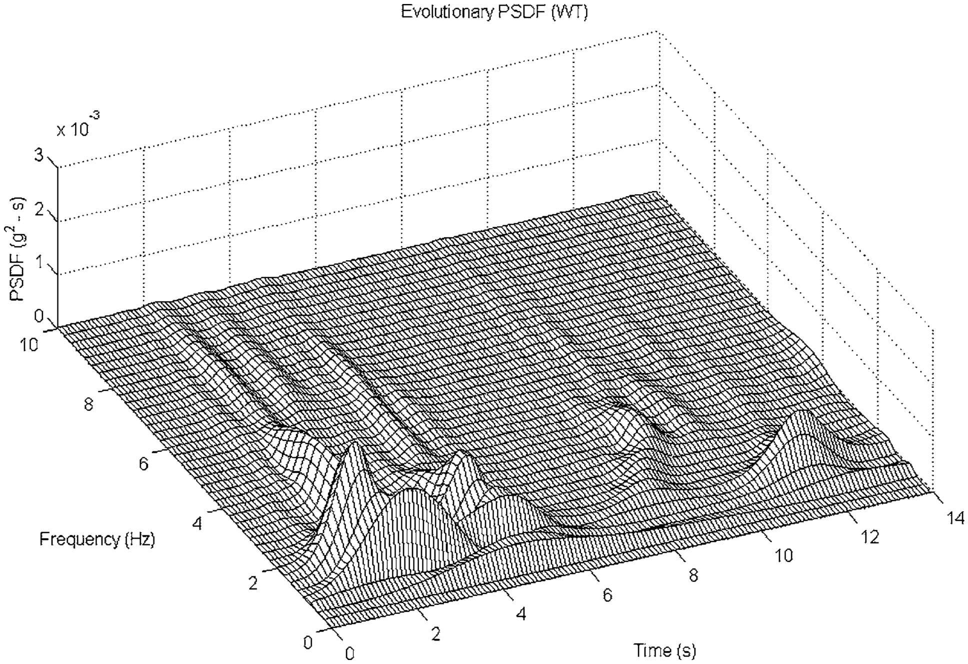

Fig. 3. Evolutionary PSDF of El Centro earthquake record in $g^2\!-\!s$ using the method based on the WT (Morlet)

$\omega_0=6$ is used. It was found that by using the proposed methods for generating evolutionary PSDFs, total energy of the time history is preserved in evolutionary PSDF with numerical errors of 0.01% for the method based on the STFT, 0.23% for the method based on the WT. The numerical error and the error introduced by nonorthogonality of the extracted empirical modes in case of the method based on the HHT is about 2.68%. Further, for the PSDFs estimated using this method, filtering and smoothing functions are applied and energy balance is ensured by performing double integral of evolutionary PSDF and using the Perseval's identity.

Figs. 2–4 show zoomed view of the evolutionary PSDFs estimated using the methods based on the STFT (Gaussian window), the WT (Morlet basis), and the HHT, respectively. By comparing Figs. 2–4, it may be observed that the methods based on the STFT and the WT give similar results in time-frequency representation. One can notice though that the method based on the WT gives better resolution in both high and low frequencies, whereas the method based on the STFT gives not-so-good resolution at high frequencies. On the other hand, the method based on the HHT shows that the energy is concentrated at some distinct frequencies only. Further, in this example, one can notice from the vertical axes of Figs. 2–4 that the energy concentration at certain frequency and time is approximately ten times higher in the case

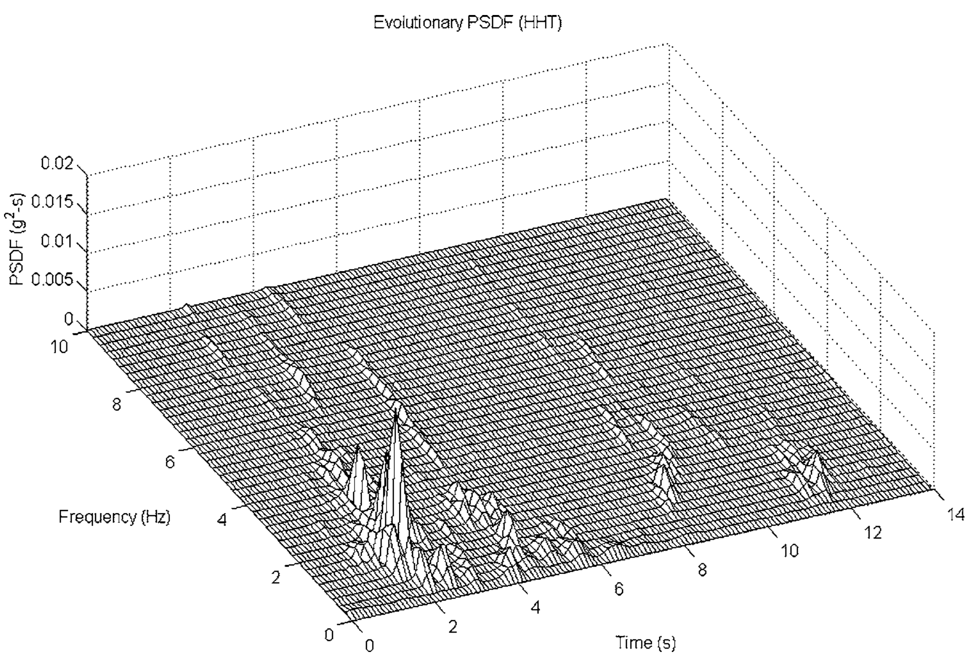

Fig. 4. Evolutionary PSDF of El Centro earthquake record in $g^2\!-\!s$ using the method based on the HHT

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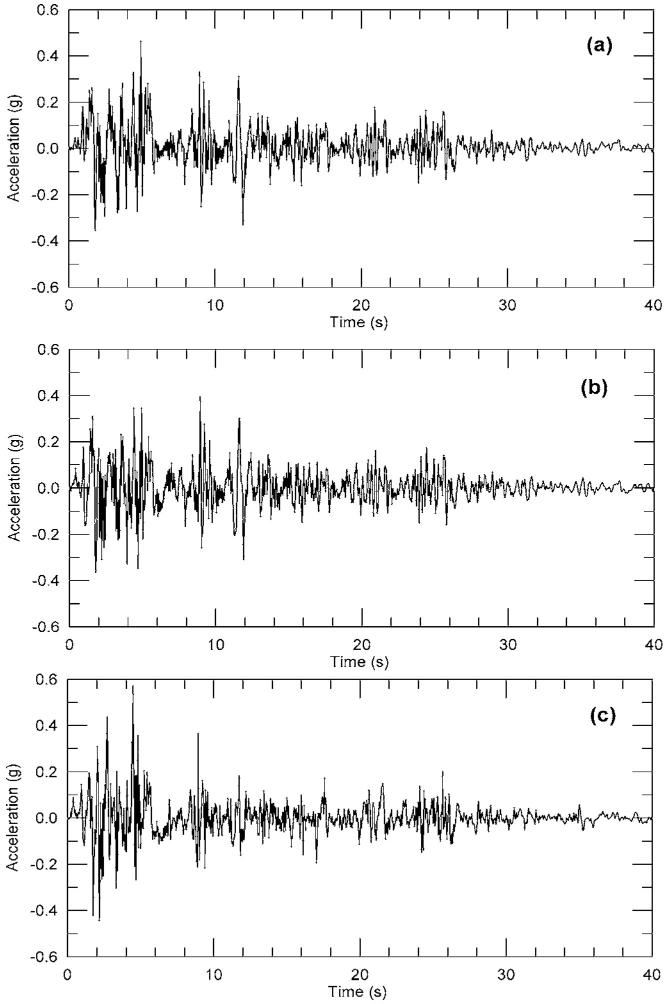

Fig. 5. Synthetic accelerogram using the same random number sequence and the methods based on (a) the STFT; (b) WT; and (c) HHT

of the HHT when compared with the one associated with the STFT or the WT.

In order to study the three estimated evolutionary PSDF in a different way, and to illustrate the methodology of simulating nonstationary ground motions, one sample function for each of the three estimated evolutionary PSDFs (see Figs. 2–4) is generated. Three corresponding acceleration time histories generated by the methods based on the STFT, WT, and HHT are shown in Fig. 5. Since the generated sample functions depend on selected random number sequence, the same sequence of the random numbers is used for all the three cases in order to compare the results. From Fig. 5, one can observe that the method based on the HHT gives higher estimate of the peak ground acceleration when compared with those based on the STFT and the WT. This may be due to a higher energy concentration at some high frequencies for HHT as observed in Fig. 4. Comparing Fig. 1 with Fig. 5, one can notice that all simulated time histories reproduce the arrivals of earthquake waves very well.

Figs. 6 and 7, respectively, show the plot of the mean and the coefficient of variation of response spectra (with 5% damping) of 20 simulated sample functions obtained from each of the three estimated evolutionary PSDF of El Centro record (Figs. 2–4). The same twenty sequences of random numbers are used for the three simulations. From Fig. 6, it is found that, in general, the response

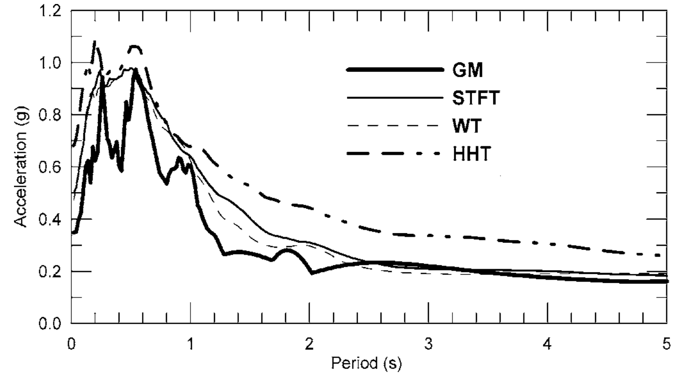

Fig. 6. 5% damped ground motion acceleration response spectrum (GM) and the mean of five percent damped acceleration spectra considering twenty sample functions generated from each of the three evolutionary PSDFs using the STFT, WT, and HHT

spectra estimated by all the three methods capture the trend of the target ground motion spectrum well with slight over estimation at higher frequencies. The discrepancies observed in spectral ordinates may be partly due to discretization scheme with some numerical error, and partly due to the variation of sample functions providing different levels of coefficient of variation in the spectral values as a function of natural period (see Fig. 7). Perhaps, a large number of sample functions may give a better estimate. Also, one may notice that out of the three methods, the method based on the HHT predicts higher spectral ordinates, though it captures the trend of the target spectrum well. This anomaly in spectral ordinates may be due to a larger energy concentration at some discrete frequencies.

Conclusions

This paper, integrating Priestly's evolutionary spectral representation theory, presents a rigorous derivation of the computationally efficient cosine series formula introduced by Shinozuka and Jan in 1972 for simulation of one-dimensional, univariate, nonstationary stochastic processes. The stochastic process thus simulated is asymptotically Gaussian as the number of terms tends to infinity. This paper shows that (1) the sample functions accurately reflect

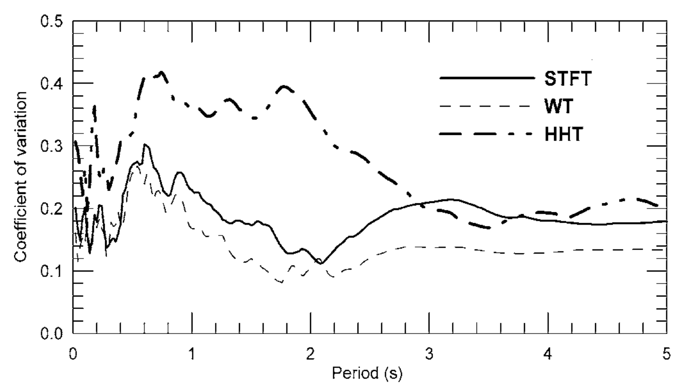

Fig. 7. Coefficient of variation of 5% damped acceleration spectra as a function of natural period considering twenty sample functions generated from each of the three evolutionary PSDFs using the STFT, WT, and HHT

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the prescribed probabilistic characteristics of the stochastic process when the number of terms in the cosine series is large, and (2) the simulation formula can be reduced to that for nonstationary white noise process or Shinozuka's stationary process under certain conditions.

Three methods are developed to estimate the evolutionary PSDF of a given time-history data by the STFT, WT, and HHT. A comparison of the generated PSDF using the well-known El Centro earthquake record shows that the method based the STFT and the WT give similar results, while the method based on the HHT gives concentrated energy at certain frequencies. Using the same random number sequence, it is found that the method based on the HHT gives higher estimate of the peak ground acceleration when compared with that of the methods based on the STFT and the WT. More importantly, it is found that the sample functions generated by these three methods all preserve the history of arrivals of earthquake waves very well. Further, using 20 sample functions, it is observed that, in general, the mean of the 5% damped acceleration response spectra obtained from each of the three estimated PSDFs captures the trend of the target spectrum well with some discrepancies at high frequencies. Perhaps, a better discretization scheme and a larger number of sample functions will provide better results. It is also concluded that the evolutionary PSDF estimated by the method based on the HHT overpredicts mean spectral ordinates.

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