Year 12 Mathematics Methods Week 1 Practice

Name: ____________________

- **Differentiation revision to find local max and min**
- **Demonstrate understanding of domain.**

Question 1

Find the minimum and maximum values of $f(x) = 2x^3 - 3x^2 - 12x + 27$ over the interval $-3 \le x \le 3$.

Question 2

The function $ax^2 + bx - 3$ has the value 11 when $x = 2$ and has its minimum value $(a > 0)$ when $x = -\frac{3}{4}$. Find a and b.

Question 3

The curve $y = ax^2 + bx + c$ intersects the y-axis at (0, −1) and has gradient 3 there. If the point (2,13) also lies on this curve find the values of a, b and c.

Question 4

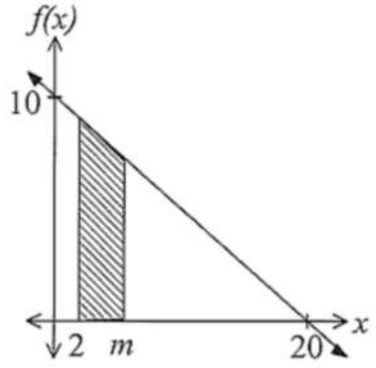
At any time t a particle moving in a line is distance x from a fixed point where x is a function of the time defined by $x(t) = 5 - 16t^2 + 8t^3 - t^4$. How far does the particle travel over the time interval $0 \leq t \leq 6$?

Question 5

Find the equation of the curve which passes through the point (1, 2) and has gradient $3x + 2$ at any point on the curve.

Question 6

A sketch of the function $y = 10 - \frac{x}{2}$ is shown below.



Determine the value of m so that the shaded area is 8.75 square units.

Question 7

Find the turning point of the quadratic function $y = a - (x+b)^2$ given that it passes through the points (0,2) and (−1,0).

Review SOLUTIONS

Year 12 Mathematics Methods Week 1 Practice

Name: ____________________

- **Differentiation revision to find local max and min**
- **Demonstrate understanding of domain.**

Question 1

Find the minimum and maximum values of $f(x) = 2x^3 - 3x^2 - 12x + 27$ over the interval $-3 \le x \le 3$.

$f(-3) = -18$

$f(3) = 18$

$f(-1) = 34$

$f(2) = 7$

$f'(x) = 6x^2 - 6x - 12$

$0 = 6(x^2 - x - 2)$

$0 = 6(x+1)(x-2)$

$x = -1$ or $x = 2$

t.p

$\therefore$ max $= 34$ at $(-1, 34)$ ✓

min $= -18$ at $(-3, -18)$ ✓

Question 2

The function $ax^2 + bx - 3$ has the value 11 when $x = 2$ and has its minimum value $(a > 0)$ when $x = -\frac{3}{4}$. Find a and b.

t.p at $x = -\frac{3}{4}$

$(2, 11)$

$11 = a(2)^2 + b(2) - 3$

$14 = 4a + 2b$ (1)

$f'(x) = 2ax + b$

$0 = 2a(-\frac{3}{4}) + b$ (2)

Solve (1) and (2) Simultaneously

$\therefore a = 2$ ✓

$b = 3$ ✓

Question 3

The curve $y = ax^2 + bx + c$ intersects the y-axis at $(0, -1)$ and has gradient 3 there. If the point $(2,13)$ also lies on this curve find the values of a, b and c.

$c = -1$ ✓

$y' = 2ax + b$

$y' = 3$ at point $(0,-1)$

$3 = 2a(0) + b$

$\therefore b = 3$ ✓

$y = ax^2 + 3x - 1$

sub $(2,13)$ $\rightarrow$ $13 = 2a + 6 - 1$

$\therefore a = 2$ ✓

Question 4

At any time t a particle moving in a line is distance x from a fixed point where x is a function of the time defined by $x(t) = 5 - 16t^2 + 8t^3 - t^4$. How far does the particle travel over the time interval $0 \le t \le 6$?

Particle turns around when $x'(t) = 0$

$x'(t) = -32t + 24t^2 - 4t^3$

$0 = -4t(8 - 6t + t^2)$

$0 = -4t(t-4)(t-2)$

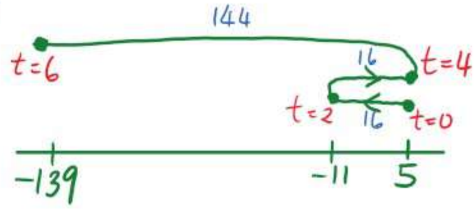
$t = 0$, $t = 4$, $t = 2$

$x(0) = 5$

$x(6) = -139$

$x(2) = -11$

$x(4) = 5$



$\therefore$ Total distance

$= 16 + 16 + 144$

$= 176$ units ✓

Question 5

Find the equation of the curve which passes through the point $(1, 2)$ and has gradient $3x + 2$ at any point on the curve.

Given $f'(x) = 3x + 2$

$f(x) = \int 3x + 2 \, dx$

$f(x) = \dfrac{3x^2}{2} + 2x + C$

sub $(1,2)$ to solve "C"

$2 = \dfrac{3(1)^2}{2} + 2(1) + C$

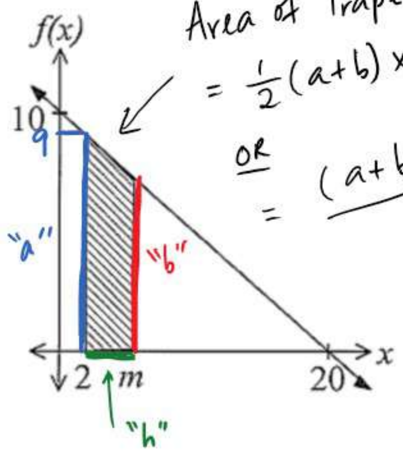
$2 = \dfrac{3}{2} + 2 + C$

$\therefore C = -\dfrac{3}{2}$

$\therefore f(x) = \dfrac{3x^2}{2} + 2x - \dfrac{3}{2}$

Question 6

A sketch of the function $y = 10 - \frac{x}{2}$ is shown below.



Area of trapezium $= \frac{1}{2}(a+b) \times h$

OR $= \frac{(a+b) \times h}{2}$

$f(2) = 9 \leftarrow$ "a"

$f(m) = 10 - \frac{m}{2} \leftarrow$ "b"

$m - 2 \leftarrow$ "h"

$$8.75 = \frac{\left(9 + 10 - \frac{m}{2}\right)(m-2)}{2}$$

$$17.5 = 19m - 38 - \frac{m^2}{2} + m$$

$$35 = 38m - 76 - m^2 + 2m$$

$$m^2 - 40m + 11 = 0$$

$$(m-3)(m-37) = 0$$

$\therefore m = 3$ ✓ or $\cancel{m = 37}$

Determine the value of m so that the shaded area is 8.75 square units.

other method:

Area $\boxed{\text{Big}}$ $= \frac{1}{2}(9)(18)$

$= 81$

Area $\boxed{\text{Small}}$ $= \frac{1}{2}\left(10 - \frac{m}{2}\right)(20 - m)$

$$8.75 = 81 - \left[\frac{1}{2}\left(10 - \frac{m}{2}\right)(20 - m)\right]$$

$\therefore m = 3$ ✓ or $\cancel{m = 37}$

Question 7

Find the turning point of the quadratic function $y = a - (x+b)^2$ given that it passes through the points (0,2) and (−1,0).

sub $(0,2) \rightarrow$ $2 = a - (0+b)^2$

$2 = a - b^2$

$2 + b^2 = a$ ①

sub $(-1,0) \rightarrow$ $0 = a - (-1+b)^2$

$0 = a - 1 + 2b - b^2$ ②

Sub ① → ②

$0 = 2 + \cancel{b^2} - 1 + 2b - \cancel{b^2}$

$0 = 1 + 2b$

$\therefore b = -\frac{1}{2}$ ✓

sub $b = -\frac{1}{2} \rightarrow$ ①

$2 + \left(-\frac{1}{2}\right)^2 = a$

$\therefore a = 2\frac{1}{4}$ ✓

$\therefore y = 2\frac{1}{4} - \left(x - \frac{1}{2}\right)^2$

$y = -\left(x - \frac{1}{2}\right)^2 + 2\frac{1}{4}$

t.p form!

$\therefore$ t.p $= \left(\frac{1}{2}, 2\frac{1}{4}\right)$ ✓

Product Rule

$$\boxed{\begin{array}{l} \text{If } y = f(x)\,g(x) \text{ then } y' = g(x)f'(x) + f(x)g'(x) \\ \text{If } y = uv \text{ then } \dfrac{dy}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx} \end{array}}$$

Examples:

1) Differentiate with respect to x:

a) $y = (\overset{u}{x+2})(\overset{v}{x-3})$

Use Product Rule

$$\frac{dy}{dx} = (x-3)(1) + (x+2)(1)$$

$$= 2x - 1 \checkmark$$

Check : Expand then differentiate ☺

b) $(2x^2+1)(5x^3+16)$

Let $u = 2x^2 + 1 \longrightarrow u' = 4x$

$v = 5x^3 + 16 \longrightarrow v' = 15x^2$

$(x+2)^5$

$$\frac{dy}{dx} = vu' + uv'$$

$$\frac{dy}{dx} = (5x^3+16)(4x) + (2x^2+1)(15x^2)$$

$$\frac{dy}{dx} = (5x^3+16)(4x) + (2x^2+1)(15x^2)$$

$$= 20x^4 + 64x + 30x^4 + 15x^2$$

$$= 50x^4 + 15x^2 + 64x$$

Quotient Rule

If $y = \frac{f(x)}{g(x)}$ then $y' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Example : Differentiate

1) $y = \frac{x^2 + 3x}{x}$ $\leftarrow u$, $\leftarrow v$

Method 1 :

$y = \frac{x^{\cancel{2}}}{\cancel{x}} + \frac{3\cancel{x}}{\cancel{x}}$

$y = x + 3$

$\frac{dy}{dx} = 1$ ✓

Method 2 : Quotient Rule

$\frac{dy}{dx} = \frac{x(2x+3) - (x^2+3x)(1)}{x^2}$

$= \frac{2x^2 + 3x - x^2 - 3x}{x^2}$

$= \frac{x^2}{x^2}$

$= 1$ ✓

2) $y = \frac{x-2}{x^2 + 4x + 1}$

Let $u = x - 2 \longrightarrow \frac{du}{dx} = 1$

Let $V = x^2 + 4x + 1 \longrightarrow \frac{dv}{dx} = 2x + 4$

$$\frac{dy}{dx} = \frac{(x^2 + 4x + 1)(1) - (x-2)(2x+4)}{(x^2 + 4x + 1)^2}$$

$$\frac{dy}{dx} = \frac{x^2 + 4x + 1 - (2x^2 - 8)}{(x^2 + 4x + 1)^2}$$

$$= \frac{x^2 + 4x + 1 - 2x^2 + 8}{(x^2 + 4x + 1)^2}$$

$$= \frac{-x^2 + 4x + 9}{(x^2 + 4x + 1)^2}$$ ✓

Chain Rule

- Used for Composite Function

$$\boxed{y = [f(x)]^n \rightarrow \frac{dy}{dx} = n[f(x)]^{n-1} f'(x) \quad \text{OR} \quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

← Function Notation

← Leibniz Notation

Example: differentiate $(3x^2+4)^5$

Method 1 : Leibniz Notation

Let $y = (3x^2+4)^5$,

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

Let $y = u^5 \longrightarrow \frac{dy}{du} = 5u^4$

$u = 3x^2+4 \longrightarrow \frac{du}{dx} = 6x$

$$\frac{dy}{dx} = 5u^4 \times 6x$$

$$\frac{dy}{dx} = 30x\,u^4$$

$$\frac{dy}{dx} = 30x(3x^2+4)^4 \checkmark$$

Method 2 : Function Notation

let $y = (3x^2+4)^5$

$$\frac{dy}{dx} = 5(3x^2+4)^4 \times 6x$$

$$\frac{dy}{dx} = 30x(3x^2+4)^4 \checkmark$$

Example 2: Differentiate

a) $y = \frac{1}{(4x^3-5x)^2}$

$y = (4x^3-5x)^{-2}$

$$\frac{dy}{dx} = -2(4x^3-5x)^{-3} \times (12x^2-5)$$

$$= \frac{-2(12x^2-5)}{(4x^3-5x)^3} \checkmark$$

b) $f(x) = \sqrt{3x^2+4x}$

$f(x) = (3x^2+4x)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2}(3x^2+4x)^{-\frac{1}{2}} \times (6x+4)$$

$$= (3x^2+4x)^{-\frac{1}{2}} \times (3x+2)$$

$$= \frac{3x+2}{\sqrt{3x^2+4x}} \checkmark$$

c) $y = x^2 + (x-1)^5$

$y' = 2x + 5(x-1)^4(1)$

$= 2x + 5(x-1)^4$ ✓

d) Find the gradient of the tangent to the curve $y = \dfrac{16}{3x^2+1}$ at $(1,4)$

$y = 16(3x^2+1)^{-1}$

$y' = -16(3x^2+1)^{-2} \times (6x)$

$= -96x(3x^2+1)^{-2}$ ✓

$\underline{(3x^2+1)^2}$

at $(1,4)$, $x = 1$

$\therefore$ gradient $= \dfrac{-96(1)}{(3(1)^2+1)^2}$

$= -6$ ✓

Ex1D page 17
Q1, 4, 9, 10 (Leibniz Method)
Q16, 20, 22, 26, 28 (Function Method)
Q31 (Classpad)

Examples: differentiate with respect to x

1) $x^3(3x-5)^4$ ← product + Chain Rule

Let $y = \overset{u}{x^3}\,\overset{v}{(3x-5)^4}$

$u = x^3 \longrightarrow u' = 3x^2$

$v = (3x-5)^4 \longrightarrow v' = 4(3x-5)^3 \times 3$

$v' = 12(3x-5)^3$

$$\frac{dy}{dx} = vu' + uv'$$

$$= (3x-5)^4(3x^2) + x^3\left(12(3x-5)^3\right)$$

$$= 3x^2(3x-5)^4 + 12x^3(3x-5)^3$$

$$= 3x^2(3x-5)^3\left[(3x-5) + 4x\right]$$

$$= 3x^2(3x-5)^3(7x-5) \checkmark$$

2) $y = \dfrac{3\sqrt{2x+1}}{x^3-5}$ ← Quotient + Chain

Let $u = 3(2x+1)^{\frac{1}{2}} \longrightarrow u' = \frac{1}{2}(3)(2x+1)^{-\frac{1}{2}}(2)$

$u' = 3(2x+1)^{-\frac{1}{2}}$

$v = x^3 - 5 \longrightarrow v' = 3x^2$

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

$$= \frac{(x^3-5)\left(3(2x+1)^{-\frac{1}{2}}\right) - 3(2x+1)^{\frac{1}{2}}(3x^2)}{(x^3-5)^2}$$

$$= \frac{\dfrac{3(x^3-5)}{\sqrt{2x+1}} - 9x^2\sqrt{2x+1}}{(x^3-5)^2}$$ ✓

3-1c

Curve Sketching

Review: Summary of graphs of gradient function $f'(x)$

Function $f(x)$	Gradient function $f'(x)$
Increasing function (positive gradient)	$f'(x) > 0$ (above x-axis)
Decreasing function (negative gradient)	$f'(x) < 0$ (below x-axis)
Turning point (zero gradient)	$f'(x) = 0$ (root or x-ints)
Horizontal point of inflection (zero gradient)	Turning point (also a root)
oblique point of inflection (positive or negative gradient)	Turning point (Not a root)

Second derivatives $f''(x)$

If $f''(x) < 0 \rightarrow f(x)$ is concave down

If $f''(x) > 0 \rightarrow f(x)$ is concave up

If $f''(x) = 0 \rightarrow f(x)$ is a point of inflection

oblique p.o.i if only $f''(x) = 0$

horizontal p.o.i if $f'(x) = 0$ and $f''(x) = 0$

* The 2nd derivative is useful to determine the nature of the t.p on a curve (instead of using the "sign test")

Example : Curve Sketching

Without a Classpad, sketch $y = x^3 - 3x^2$, showing all the features

① y-int

set $x = 0 \longrightarrow y = 0$

$\therefore$ y-int $= (0, 0)$

② x-ints

set $y=0 \longrightarrow$ $0 = x^3 - 3x^2$

$0 = x^2(x-3)$

$x^2 = 0$ and $x - 3 = 0$

$\therefore$ x-ints : $x = 0$ ✓ and $x = 3$ ✓

③ Turning Points

$y = x^3 - 3x^2$

$\frac{dy}{dx} = 3x^2 - 6x \rightarrow$ set $\frac{dy}{dx} = 0$ ↙ stationary point gradient = 0

$0 = 3x^2 - 6x$

$0 = 3x(x-2)$

$x = 0$ $\quad x = 2 \rightarrow$ sub into $y = x^3 - 3x^2$

$\therefore$ t.p = $(0,0)$ and $(2,-4)$ ✓

④ Testing the t.p with $f''(x)$

$\frac{dy}{dx} = 3x^2 - 6x$

$\frac{d^2y}{dx^2} = 6x - 6$

for t.p $(0,0) \rightarrow$ $y'' = 6(0) - 6$

$= -6$ negative

$\therefore$ $(0,0)$ is a max t.p ✓

for t.p $(2,-4) \rightarrow$ $y'' = 6(2) - 6$

$= 6$ positive

$\therefore$ $(2,-4)$ is a min t.p ✓

⑤ Check for p.o.i using $f''(x)$

⑤ Check for p.o.i using $f''(x)$

Set $\frac{d^2y}{dx^2} = 0$

$0 = 6x - 6$

$x = 1$ $\rightarrow$ sub into $y=$

$\therefore (1, -2)$ is an oblique p.o.i

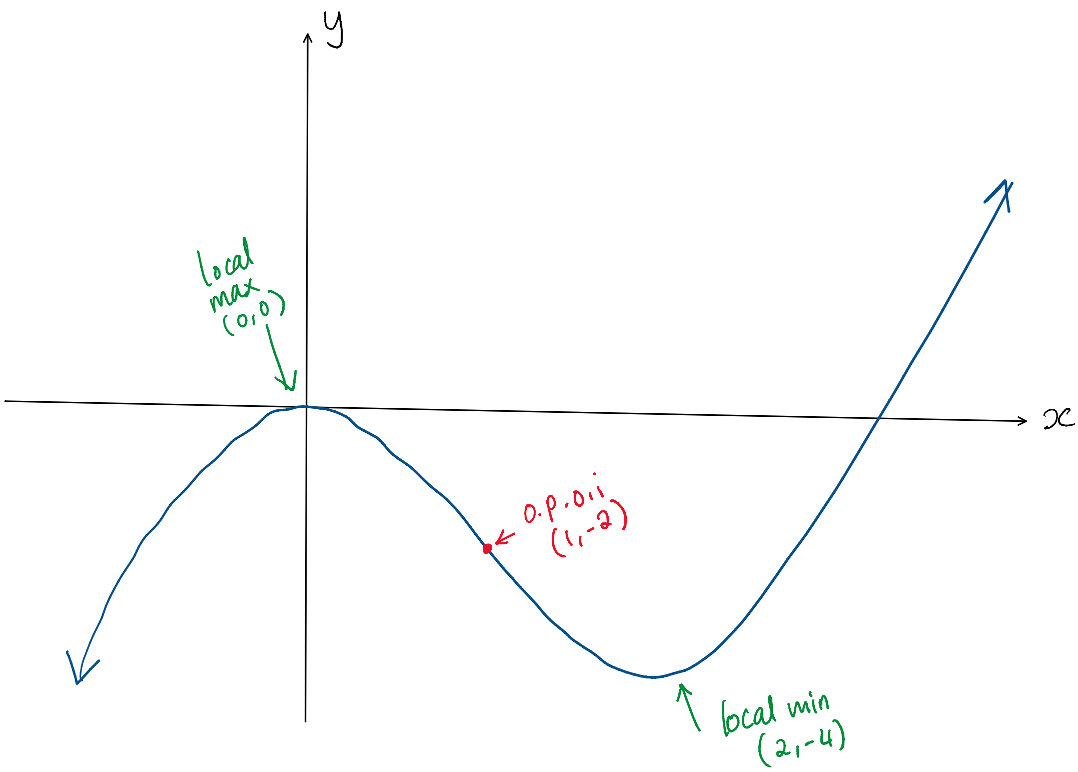
⑥ Behaviour of End points

As $x \rightarrow \infty$, $y = \infty$

As $x \rightarrow -\infty$, $y = -\infty$

shape $\swarrow$ $\nearrow$

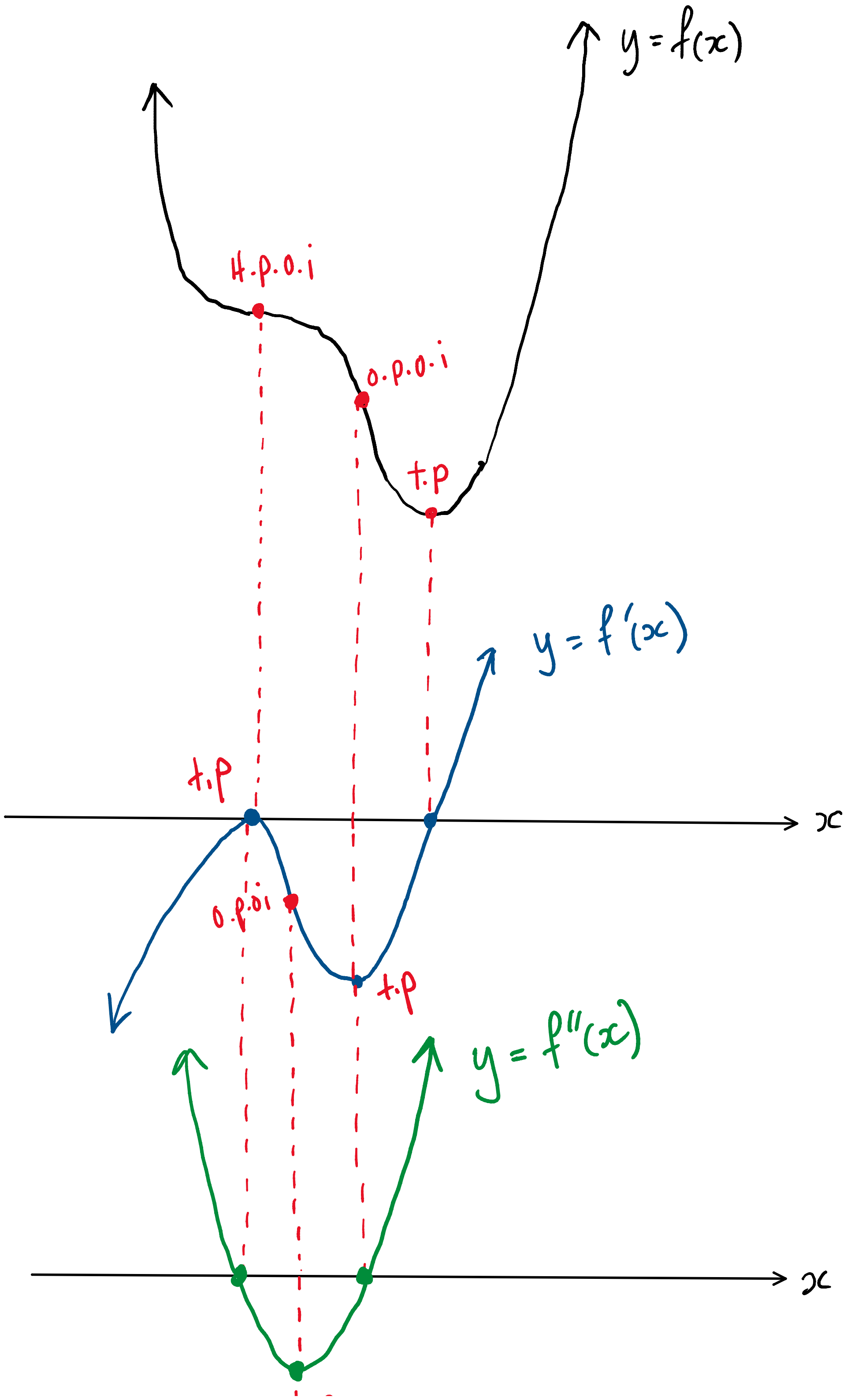
⑦ Sketch

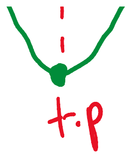


Ex2A
Q1b
Q5, 7, 9, 12, 13, 16

Example:

Given the sketch of $y=f(x)$, sketch a possible $y=f'(x)$ and $y=f''(x)$.




t.p

Rate of Change

Average rate of change $= \dfrac{f(b)-f(a)}{b-a}$ OR $\dfrac{\Delta y}{\Delta x}$

Instantaneous rate of change $= f'(a)$ OR $\dfrac{dy}{dx}$

Examples:

1) The population of sheep at a farm is given by
$P(t) = 50 + 45t - 7t^2 + \dfrac{t^3}{3}$ where t is time in years.

a) Find the average rate of population growth for $3 \le t \le 6$

b) Find the rate of population growth when $t = 3$.

c) Find when the population of sheep is stable. ← population growth = 0

a) $P(3) = 131$

$P(6) = 140$

∴ Ave rate of change

$= \dfrac{P(6) - P(3)}{\text{time period}}$

$= \dfrac{140 - 131}{6 - 3}$

$= 3$ sheep per year ✓

b) $P(t) = 50 + 45t - 7t^2 + \dfrac{t^3}{2}$

$P'(t) = 45 - 14t + t^2$

$P'(3) = 12$

∴ rate of change = 12 sheep per year ✓

c) $P'(t) = 0$

$0 = 45 - 14t + t^2$

$$0 = 45 - 14t + t^2$$

$$0 = (t-5)(t-9)$$

$$t = 5 \qquad t = 9$$

2) A balloon develops a leak and gradually decreases in volume. Its volume $V\,cm^3$ at time t seconds is

$$V = 600 - 10t - \frac{t^2}{100}, \quad t \geq 0.$$

a) Find the rate of change of volume after 10 seconds

b) For how long could this model be valid?

a) $V'(t) = -10 - \frac{t}{50}$

at $t = 10 \rightarrow V'(10) = -10\frac{1}{5}$ or $-10.2\ cm^3/s$

b) Not valid for $V < 0$

Consider $V = 0$

$$0 = 600 - 10t - \frac{t^2}{100}$$

$t = 56.8$ or $\cancel{t = -1056}$

$\therefore$ the model may be suitable for $0 \leq t \leq 56.8$ seconds

$\therefore$ the model may be suitable for $0 \leq t \leq 56.8$ seconds

Ex2A
Q1b
Q5, 7, 9, 12, 13, 16

Ex2B page 33
Q9, 10

$$V = \frac{dx}{dt} \quad , \quad a = \frac{dV}{dt}$$

$$\underset{\substack{x \\ (\text{or } s)}}{\text{Displacement}} \xrightarrow{\text{differentiate}} \underset{V}{\text{Velocity}} \xrightarrow{\text{differentiate}} \underset{a}{\text{Acceleration}}$$

Example:

A particle moves along a straight line such that its position x m at time t seconds is given by

$x = t^3 - 6t^2 + 9t$. Find:

a) at what position the particle will have zero velocity

b) its acceleration at those instances

c) its velocity when its acceleration is zero.

a) $V = 0$, $x = ?$

$$x' = V = 3t^2 - 12t + 9$$

$$0 = 3(t^2 - 4t + 3)$$

$$0 = 3(t-3)(t-1)$$

$$t = 3 \quad \text{or} \quad t = 1$$

at $t = 1 \rightarrow x(1) = 4\text{m}$ ✓

at $t=1 \rightarrow x(1) = 4m$ ✓

at $t=3 \rightarrow x(3) = 0m$ ✓

b) $V' = a = 6t-12$

at $t=1 \rightarrow a(1) = -6\ m/s^2$ ✓

at $t=3 \rightarrow a(3) = 6\ m/s^2$ ✓

c) $a=0$, $V=?$

$$0 = 6t-12$$

$$0 = 6(t-2)$$

$$0 = t-2$$

$$t = 2$$ ✓

$t=2 \rightarrow V(2) = -3m/s$ ✓

Optimisation is the process of obtaining the optimum or best solution to a practical problem by finding the maximum or minimum values.

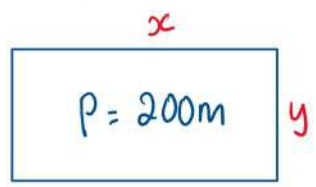
* Remember: Profit = Revenue − Cost

$$P(x) = R(x) - C(x)$$

Optimisation Examples:

1. A farmer has sufficient fencing to make a rectangular pen of perimeter 200 metres. What dimensions will give an enclosure of maximum area?

1) Diagram



Let x = length
y = width

2) Write the function to be optimised

$A = xy$ ← we want to max this!

Given $P = 200$

$$2x + 2y = 200$$
$$2y = 200 - 2x$$
$$y = 100 - x$$

sub → $A = xy$

$$A = x(100 - x)$$
$$\therefore A = 100x - x^2$$

3) Differentiate

set $A'(x) = 0$ → $A'(x) = 100 - 2x$

or $\frac{dA}{dx} = 0$ $\qquad 0 = 100 - 2x$

set $A'(x) = 0$

OR $\frac{dA}{dx} = 0$

$0 = 100 - 2x$

$x = 50$ ✓

4) Use 2nd derivative $\frac{d^2A}{dx^2}$ to determine max or min

$A''(x) = -2$

$A''(x) < 0 \quad \therefore$ maximum

5) Give values to all variables (answer the question!)

$x = 50 \xrightarrow{\text{sub}} y = 100 - x$

$y = 50$

Max Area $= 50 \times 50$

$= 2500\,m^2$ ✓

Dimensions:

length = 50m ✓

width = 50m ✓

1) Diagram / Table

Fee ($)	No. of subcribers	Revenue ($)
5	1000	5×1000
~~4.90~~	~~1100~~	~~4.90 × 1100~~
$5 - 0.1(1)$	$1000 + 100(1)$	$(5-0.1)(1000+100)$
$5 - 0.1(2)$	$1000 + 100(2)$	$(5-0.2)(1000+200)$
⋮	⋮	⋮
$5 - 0.1x$	$1000 + 100x$	

2) Function

2) Function

$$R(x) = (5-0.1x)(1000+100x)$$
$$R(x) = 5000 + 400x - 10x^2 \checkmark$$

3) Maximise $R(x)$

$$R'(x) = 0$$
$$R'(x) = 400 - 20x$$
$$0 = 400 - 20x$$
$$x = 20 \checkmark$$

4) 2nd derivative

$$R''(x) = -20$$
$$R''(x) < 0 \qquad \therefore \text{maximum}$$

5) Find all the values and answer the question

when $x = 20$

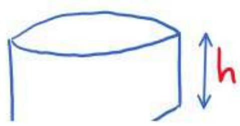
$$\text{Fee} = \$5 - \$0.1(20)$$
$$= \$3$$

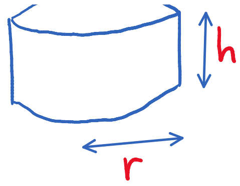
$$\text{Revenue (Max)} = 5000 + 400(20) - 10(20)^2$$
$$= \$9000 \checkmark$$

1) Diagram

Given $SA = 100\ cm^2$

$$SA = 2\pi r^2 + 2\pi rh$$





$SA = 2\pi r^2 + 2\pi rh$

Rearrange : $100 = 2\pi r^2 + 2\pi rh$

$$100 - 2\pi r^2 = 2\pi rh$$

$$h = \frac{100 - 2\pi r^2}{2\pi r}$$ ✓

Sub

2) Function

$$V = \pi r^2 h$$

$$V = \cancel{\pi} r^{\cancel{2}} \left(\frac{100 - 2\pi r^2}{2 \cancel{\pi} \cancel{r}} \right)$$

$$V = r(50 - \pi r^2)$$

$$V = 50r - \pi r^3$$ ✓

3) Maximise Volume

$V'(x) = 0 \rightarrow$ $V'(x) = 50 - 3\pi r^2$

$$0 = 50 - 3\pi r^2$$

$r \approx \cancel{-2.3}$ or $r \approx 2.3$ ✓

invalid

4) 2nd derivative

$$V''(x) = -6\pi r$$

$$= -6\pi(2.3)$$

$V''(x) < 0$ $\therefore$ maximum ☺

5) Answer the question!

$V_{max} = 76.78\,cm^3$ when $r = 2.3\,cm$

But!!! Given that radius can be at most 2cm,

$V(2) = 74.87\,cm^3$

Small Change and % Change

δx is the small increment in x

δy is the small increment in y

If δx is very small, $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

$$\boxed{\delta y \approx \frac{dy}{dx} \times \delta x}$$

* % change for y is $\frac{\delta y}{y}$
* % change for x is $\frac{\delta x}{x}$

<u>Small Change Example:</u>

1. If $y = 2x^3 - 4x$, use differentiation to find the approximate change in y when x changes from 5 to 5.01

Given $x = 5$, $\delta x = 0.01$, $\delta y = ?$

$y = 2x^3 - 4x$

$\frac{dy}{dx} = 6x^2 - 4$

$\delta y \approx \frac{dy}{dx} \times \delta x$

$\delta y \approx (6x^2 - 4) \times 0.01$

Given $x = 5 \rightarrow \delta y \approx (6(5)^2 - 4) \times 0.01$

≈ 1.46 ✓

* Check by substitution (without using calculus)

when $x = 5 \rightarrow y = 230$

when $x = 5.01 \rightarrow y = 231.463$

$\therefore$ change in $y = 1.463$ (very close to the approximation)

2. Find the radius of a sphere of volume 1000cm³. Use calculus to determine the approximate change in radius of the sphere to cause the volume to change from 1000cm³ to 1010cm³.

Volume of a Sphere : $V = \frac{4}{3}\pi r^3$, $\delta V = 10$, $\delta r = ?$, $V = 1000$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\frac{\delta V}{\delta r} \approx \frac{dV}{dr}$$

$$\frac{\delta r}{\delta V} \approx \frac{dr}{dV}$$

$$\delta r \approx \frac{dr}{dV} \times \delta V$$

$$1000 = \frac{4}{3}\pi r^3$$

$$r^3 = \frac{3000}{4\pi}$$

$$r = \sqrt[3]{\frac{3000}{4\pi}}$$

$$r = 6.2035$$

$$\delta r \approx \frac{1}{4\pi r^2} \times 10$$

$$\delta r \approx \frac{10}{4\pi(6.2035)^2} \approx 0.02\,\text{cm} ✓$$

Small Percentage Change Example:

If V = 2x³, use differentiation to find the approximate percentage change in V when x changes by 2%.

$V = 2x^3$, $\frac{\delta x}{x} = 2\%$, $\frac{\delta V}{V} = ?$

$$\frac{dV}{dx} = 6x^2$$

$$\frac{\delta V}{\delta x} \approx 6x^2$$

$$\delta V \approx 6x^2 \times \delta x$$

$$\frac{\delta V}{V} \approx \frac{6x^2 \times \delta x}{V}$$

$$\frac{\delta V}{V} \approx \frac{6\cancel{x^2}}{2x^{\cancel{3}}} \times \delta x$$ given $V = 2x^3$

$$\frac{\delta V}{V} \approx \frac{3}{x} \times \delta x$$

$$\frac{\delta V}{V} \approx 3 \times \frac{\delta x}{x}$$ 2% given $\frac{\delta x}{x} = 2\%$

$$\frac{\delta V}{V} \approx 3 \times 2\% \quad \text{OR} \quad 3 \times 0.02$$

$$\frac{\delta V}{V} \approx 6\% \checkmark$$

$\therefore$ % change in V is 6% ✓

Marginal Rate of Change

Profit Function:

$$P(x) = R(x) - C(x)$$

$P(x)$: Profit

$R(x)$: The income from selling x units

$C(x)$: The cost of producing x units

$\frac{dP}{dx}$ = Marginal Profit (Approx. extra profit produced by the sales of one more unit. ie: xth item sold)

$\frac{dR}{dx}$ = Marginal Revenue (Approx. extra revenue by the sale of one more unit. ie: xth item sold)

$\frac{dC}{dx}$ = Marginal Cost (Approx. cost of producing one more unit ie: xth unit produced)

Marginal rates of Change Example:

A factory produces and sells x items of a product. The total cost of producing these x items is $C:

C(x) = 6x + 10√x + 500

Use differentiation to determine the approximate cost of producing one more item at the stage in the production when x = 100.

$$C(x) = 6x + 10x^{\frac{1}{2}} + 500$$

$$\frac{dC}{dx} = C'(x) = 6 + 5x^{-\frac{1}{2}}$$

$$= 6 + \frac{5}{\sqrt{x}}$$ ← marginal cost!!!

When $x = 100$

$$C'(x) = 6 + \frac{5}{\sqrt{100}}$$

$$= 6.5$$

$\therefore$ It costs approx. \$6.50 to produce one more item at the stage of production when $x = 100$

Using small change:

$$\frac{\delta C}{\delta x} \approx \frac{dC}{dx}$$

Given $x = 100$, $\delta x = 1$

$$\frac{\delta C}{\delta x} \approx 6 + \frac{5}{\sqrt{x}}$$

$$\delta C \approx 6 + \frac{5}{\sqrt{x}} \times \delta x$$

$$\approx 6 + \frac{5}{\sqrt{100}}$$

$$\approx 6.5$$ (same as above ☺)

→ Euler's number, $e \approx 2.71828$ (natural base for exp function)

$$\boxed{e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n}$$

$n \to \infty$ then $(1+\frac{1}{n})^n \to 2.718\ldots$

* e is an irrational number like π.

→ e can be used to describe growth and decay:

$$\boxed{A = A_0 e^{kt}}$$

where A is the amount / quantity at time t

A_0 is the initial amount (at $t=0$)

k is the constant rate

$$\boxed{\begin{array}{c} k>0 \to \text{Growth} \\ A = A_0 e^{kt} \end{array}} \qquad \boxed{\begin{array}{c} k<0 \to \text{Decay} \\ A = A_0 e^{-kt} \end{array}}$$

Example:

The population of a town was 8000 at the beginning of 2008 and 15000 at the end of 2015.

Assume that the growth is exponential ← "e"

a) Find the population at the end of 2017.

b) In what year will the population be double of 2015?

$$A = A_0 e^{kt}$$

A: 15000 A_0: 8000 t: 8 years

at the end of 2015

$t = 8$, $A = 15000$

$$P = P_0 e^{8k}$$

$$15000 = 8000 e^{8k}$$

$$e^{8k} = \frac{15000}{8000}$$

$$e^{8k} = \frac{15}{8}$$ ← Solve on classpad

$$k \approx 0.079$$

$\therefore$ The rate of increase is 7.9% per annum ✓

a) $$P = 8000 e^{0.079t}$$

At the end of 2017 → $t = 10$

$$P = 8000 e^{0.079(10)}$$

$$P = 8000 e^{0.79}$$

$$P \approx 17627$$

$\therefore$ The population is approximately 17600 ✓

b) when $P = 30000$, $t = ?$

$$P = 8000e^{0.079t}$$

$$30000 = 8000e^{0.079t}$$

$$\frac{30}{8} = e^{0.079t}$$ ← solve on Classpad

$$t \approx 16.82$$

∴ The population reaches 30000 in year 2023

Derivative of e^x

$$\boxed{\begin{array}{l} y = e^x \rightarrow \frac{dy}{dx} = e^x \\ f(x) = e^x \rightarrow f'(x) = e^x \end{array}}$$

* The e^x function differentiates to itself!

$$\boxed{\begin{array}{l} y = e^{f(x)} \rightarrow \frac{dy}{dx} = f'(x)\, e^{f(x)} \\ f(x) = e^{kx} \rightarrow f'(x) = k e^{kx} \end{array}}$$

← chain rule

Examples:

1) Differentiate:

a) e^{3x}

$$\frac{d}{dx} e^{3x} = 3e^{3x}$$

b) $2e^x + x^2$

$$\frac{d}{dx} = 2e^x + 2x$$

c) $\frac{1}{e^{2x}} + e^x$

$$\frac{d}{dx}\ e^{-2x} + e^x$$

$$= -2e^{-2x} + e^x$$

$$= e^x - \frac{2}{e^{2x}}$$

d) $\overset{u}{x^2}\overset{v}{e^x}$ Product Rule!

Let $u = x^2 \rightarrow u' = 2x$

$v = e^x \rightarrow v' = e^x$

$$\frac{d}{dx} = e^x(2x) + x^2(e^x)$$

$$= e^x(2x + x^2)$$

$$= xe^x(2 + x)$$

e) e^{4x+1}

$$\frac{d}{dx} = 4e^{4x+1}$$

f) e^{x^2-2x+3}

$$\frac{d}{dx} = (2x-2)e^{x^2-2x+3}$$

$$\frac{d}{dx} = 4e^{4x+1}$$

$$\frac{d}{dx} = (2x-2)e^{x^2-2x+3}$$
$$= 2(x-1)e^{x^2-2x+3}$$

2) Find the gradient of the curve $y = e^{2x} + 4$ at point $(1, e^2+4)$.

$$\frac{dy}{dx} = 2e^{2x}$$

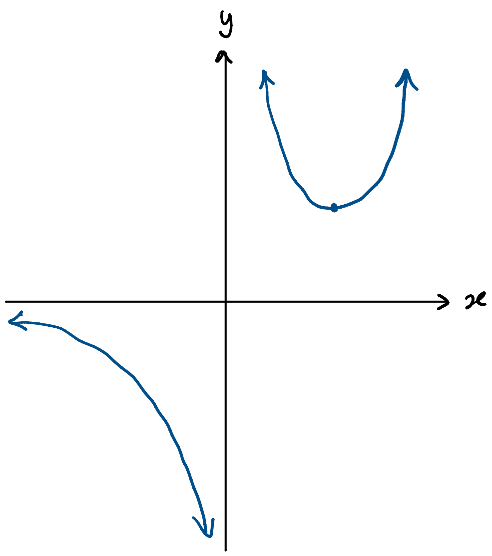
at $(1, e^2+4) \rightarrow x = 1$

$$\frac{dy}{dx} = 2e^{2(1)}$$
$$= 2e^2$$

$\therefore$ gradient at $(1, e^2+4)$ is $2e^2$

3) Show that there is only one stationary point on the curve $y = \frac{e^x}{x}$ shown below.

Determine the point of this local minimum.



$$y = \frac{e^x}{x}$$ ← Product Rule or Quotient Rule

$$\frac{dy}{dx} = \frac{xe^x - e^x}{x^2}$$

$$0 = \frac{xe^x - e^x}{x^2}$$

set $\frac{dy}{dx} = 0$ to find t.p.

set $\frac{dy}{dx} = 0$ to find t.p.

$$0 = e^x(x-1)$$

$$x = 1$$ (only one solution)

when $x = 1 \longrightarrow y = \frac{e^x}{x}$

$$y = \frac{e^1}{1}$$

$$y = e \longrightarrow \therefore (1, e)$$

min t.p at $(1, e)$

If $y = Ae^{kt} \rightarrow \frac{dy}{dt} = kAe^{kt}$

ie: $\frac{dy}{dt} = ky$

the rate of change of y with respect to t is proportional to y itself.
$\therefore$ Suitable to describe many growth and decay.

$$\boxed{\text{If } \frac{dP}{dt} = kP \rightarrow P = P_0 e^{kt}}$$

where P_0 is the value of P at time $t = 0$

Growth and Decay Example:

1. A country's population was monitored over 30 years from 1985 when the population was 2 000 000. The population was continuously growing with the instantaneous rate of increase in the population per year $\frac{dP}{dt} \approx \frac{P}{20}$

a) Estimate the population of this country at the end of 30 years.

b) If this pattern of growth continues, estimate the population in the year 2025.

Rate of increase : $\frac{dP}{dt} \approx \frac{P}{20}$

$$\frac{dP}{dt} \approx \overset{k}{0.05}\, P$$

If $\frac{dP}{dt} = kP \rightarrow P = P_0 e^{kt}$

$\frac{dP}{dt} = 0.05P \rightarrow P = P_0 e^{0.05t}$

When $t = 0$ (1985), $P_0 = 2000000$

$\therefore P = 2000000\, e^{0.05t}$ ✓

a) $t = 30 \rightarrow P \approx 2000000\, e^{0.05(30)}$

a) $t = 30 \rightarrow P \approx 2000000\, e^{0.05(30)}$

≈ 8960000 ✓

$\therefore$ The population is approx 9 million after 30 years

b) $t = 40$ (year 2025) $\rightarrow P \approx 2000000\, e^{0.05t}$

$P \approx 15000000$ ✓

$\therefore$ The population is approx. 15 million after 40 years.

2. A particular radioactive isotope decays continuously at a rate of 9% per year. 1kg of isotope is produced in a particular industrial process. How much remains undecayed after 20 years?

Let A = Amount (kg) of isotope remain undecayed after t years.

$$\frac{dA}{dt} = -0.09A \rightarrow A = A_0 e^{-0.09t}$$

when $t = 0$, $A_0 = 1$ kg $\rightarrow$ $A = 1e^{-0.09t}$

when $t = 20 \longrightarrow A = e^{-0.09(20)}$

≈ 0.165 kg

(0.165kg)

$\therefore$ Approx 165 grams remains undecayed after 20 years

3. A savings account is opened with a deposit of \$400 and attracts interest rate of 8% p.a. compounded continuously.

a) If the interest rate continues for 5 years, find the balance of the account at the end of this time

b) How many years will it take for the balance in the account to triple the initial deposit?

$$\frac{dP}{dt} = 0.08P \rightarrow P = P_0 e^{0.08t}$$

$$t = 0, \quad P_0 = 400 \rightarrow P = 400e^{0.08t}$$

a) when $t = 5 \rightarrow P = 400e^{0.08(5)}$

$= \$596.73$ ✓

b) $P = 400 \times 3$

$= \$1200 \rightarrow 1200 = 400e^{0.08t}$

$3 = e^{0.08t}$ (solve on classpad)

$t \approx 13.7$ years ✓

$y = \sin x \rightarrow \frac{dy}{dx} = \cos x$	
$y = \cos x \rightarrow \frac{dy}{dx} = -\sin x$	

* See next page for "first principle" derivative of $\sin x$ and $\cos x$.

Examples: Differentiate:

① Sum and Difference Rule

a) $y = 3\sin x + \cos x$

$$\frac{dy}{dx} = 3\cos x - \sin x$$

② Product Rule

a) $y = \overset{u}{2x^2}\ \overset{v}{\sin x}$

$u = 2x^2 \qquad V = \sin x$

$u' = 4x \qquad V' = \cos x$

$$\frac{dy}{dx} = 4x\sin x + 2x^2\cos x$$

b) $y = \overset{u}{(2 - \cos x)}\overset{v}{(1 + \sin x)}$

$$\frac{dy}{dx} = \sin x\,(1 + \sin x) + \cos x\,(2 - \cos x) \checkmark$$

OR

$$\sin x + \sin^2 x + 2\cos x - \cos^2 x \checkmark$$

c) $y = \overset{u}{\sin x}\,\overset{v}{\cos x}$

$$\frac{dy}{dx} = (\cos x)(\cos x) + (\sin x)(-\sin x)$$

$$= \cos^2 x - \sin^2 x \checkmark$$

③ Quotient Rule

a) $y = \dfrac{\overset{u}{\sin x}}{\underset{v}{x+1}} \quad (x \neq -1)$

$u = \sin x \qquad v = x + 1$

$u' = \cos x \qquad v' = 1$

$$\frac{dy}{dx} = \frac{(x+1)\cos x - \sin x}{(x+1)^2} \checkmark$$

b) $y = \tan x \quad \left(\text{use } \tan x = \dfrac{\overset{u}{\sin x}}{\underset{v}{\cos x}}\right)$

$$\frac{dy}{dx} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} \checkmark \quad = \sec^2 x \text{ (specialist only)}$$

Trigonometry Proof

Derivative by first principle:

If $y = f(x)$ then $\dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}.$

Remember the trigonometric identities?

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

If $y = \sin x$ then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\cos x \sin h}{h} - \lim_{h \to 0} \frac{\sin x - \sin x \cos h}{h} \\
&= \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{=1} - \sin x \underbrace{\lim_{h \to 0} \frac{1 - \cos h}{h}}_{=0}
\end{aligned}$$

* see proof below

$$\begin{aligned}
&= \cos x\,(1) - \sin x\,(0) \\
&= \cos x
\end{aligned}$$

Further, if $y = \cos x$ then

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\cos x \cos h - \cos x}{h} - \lim_{h \to 0} \frac{\sin x \sin h}{h} \\
&= -\cos x \lim_{h \to 0} \frac{1 - \cos h}{h} - \sin x \lim_{h \to 0} \frac{\sin h}{h} \\
&= -\cos x(0) - \sin x(1) \\
&= -\sin x
\end{aligned}$$

If $y = \sin x$ then $\dfrac{dy}{dx} = \cos x$

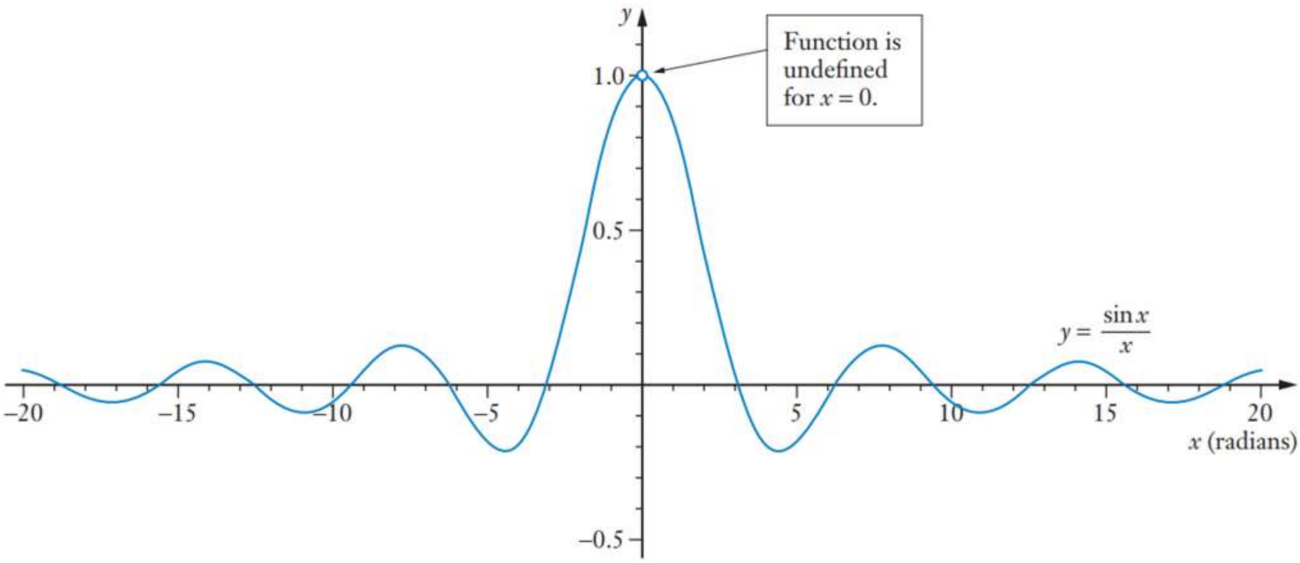
and

If $y = \cos x$ then $\dfrac{dy}{dx} = -\sin x$

✳ Graphical proof for :

$\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h}{h} = 0$

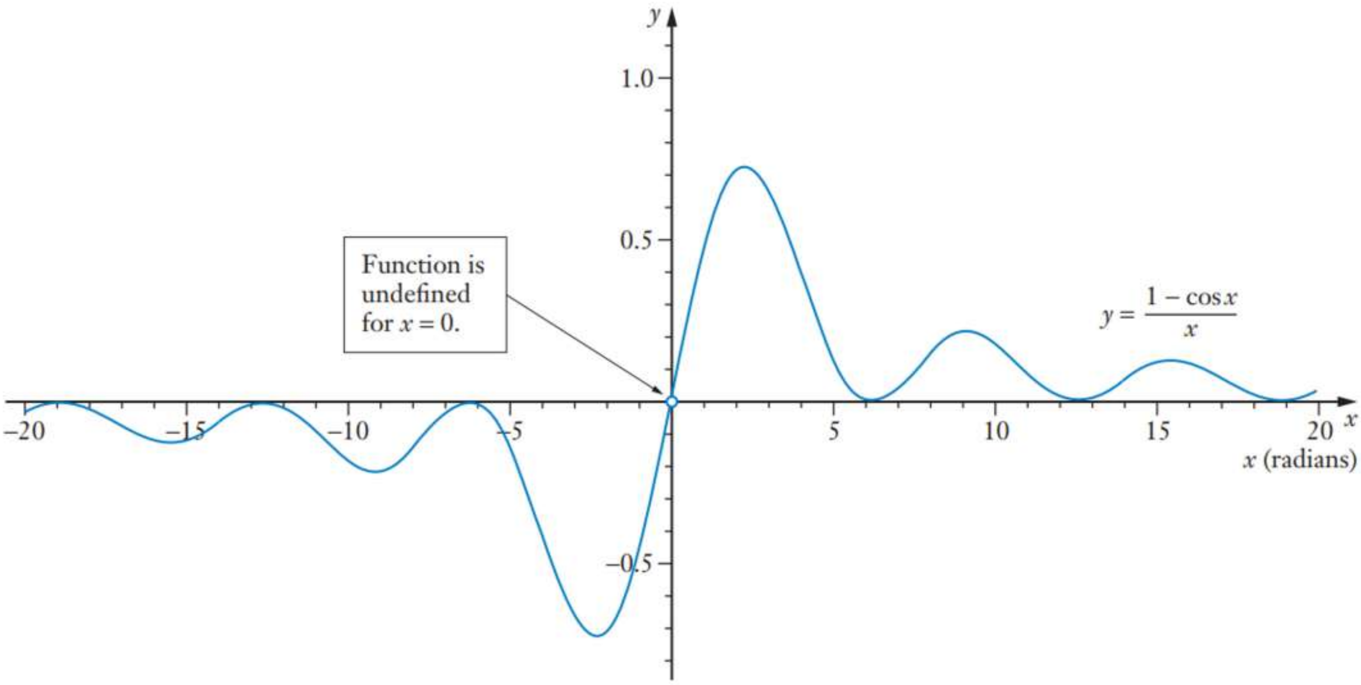
The graph of $y = \frac{\sin x}{x}$, for x in radians, is shown below. Notice that as we move along the graph, getting closer and closer to $x = 0$, from either the left side or the right side, the functional value seems to get closer and closer to 1.



Hence the graph of $y = \frac{\sin x}{x}$ supports the calculator statement, i.e. that $\lim_{x\to 0} \frac{\sin x}{x} = 1$,

(and hence that $\lim_{h\to 0} \frac{\sin h}{h} = 1$).

As we move along the graph of $y = \dfrac{1 - \cos x}{x}$, shown below, and get closer and closer to $x = 0$, from either the left side or the right side, the functional value seems to get closer and closer to 0.



Hence the graph of $y = \dfrac{1 - \cos x}{x}$ supports the calculator statement, i.e. that

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

(and hence that $\lim\limits_{h \to 0} \dfrac{1 - \cos h}{h} = 0$).

Diff Trig: Chain Rule

$$y = f(g(x)) \rightarrow y' = f'(g(x))\, g'(x)$$

$$y = f(u),\ u = g(x) \rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Examples:

a) $y = \sin(3x)$

$$\frac{dy}{dx} = \cos(3x) \times 3$$

Chain

$$= 3\cos(3x)$$

b) $y = \cos(5-2x)$

$$\frac{dy}{dx} = -\sin(5-2x) \times -2$$

$$= 2\sin(5-2x)$$

c) $y = \sin(4x^2+3)$

$$\frac{dy}{dx} = 8x\cos(4x^2+3)$$

d) $y = \sin^4 x$

$$y = (\sin x)^4$$

$$\frac{dy}{dx} = 4(\sin x)^3 \cos x$$

$$= 4\cos x \sin^3 x$$

e) $y = \sin^2(2x)$

e) $y = \sin^2(2x)$

$y = [\sin(2x)]^2$

$\frac{dy}{dx} = 2(\sin(2x)) \times \cos(2x) \times 2$

Chain Chain

$= 4\sin(2x)\cos(2x)$ ✓

Leibniz Method:

$y = [\sin(2x)]^2 \rightarrow u = \sin(2x)\ ,\ y = u^2$

$\frac{du}{dx} = 2\cos(2x)\ ,\ \frac{dy}{du} = 2u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$\frac{dy}{dx} = 2u \times 2\cos(2x)$

$= 2\sin(2x) \times 2\cos(2x)$

$= 4\sin(2x)\cos(2x)$ ✓

f) $y = \cos^3(4x+1)$

$y = [\cos(4x+1)]^3$

$\frac{dy}{dx} = 3[\cos(4x+1)]^2 \times -\sin(4x+1) \times 4$

$= -12\cos^2(4x+1)\sin(4x+1)$ ✓

Leibniz Method:

$$y = [\underbrace{\cos(4x+1)}_{u}]^3$$

$u = \cos(4x+1)$, $y = u^3$

$\frac{du}{dx} = -4\sin(4x+1)$, $y = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 3u^2 \times -4\sin(4x+1)$$

$$= 3[\cos(4x+1)]^2 \times -4\sin(4x+1)$$

$$= -12\cos^2(4x+1)\sin(4x+1) \checkmark$$

g) $y = \overset{u}{\cos(4x)}\ \overset{v}{\sin(2x)}$ ← (product + chain)

$u = \cos(4x)$ $\qquad$ $v = \sin(2x)$

$\frac{du}{dx} = \underline{-4\sin(4x)}$ $\qquad$ $\frac{dv}{dx} = \underline{2\cos(2x)}$

$$\frac{dy}{dx} = \sin(2x) \times -4\sin(4x) + \cos(4x) \times 2\cos(2x)$$

$$= -4\sin(2x)\sin(4x) + 2\cos(2x)\cos(4x) \checkmark$$

h) $y = e^{2x}\sin(2x+1)$ ← (e^x + chain + product)

Antidifferentiation is the opposite process of differentiation

$\int$ is the integral symbol

$\frac{dy}{dx}$	y
x^n	$\frac{x^{n+1}}{n+1} + c$
ax^n	$\frac{ax^{n+1}}{n+1} + c$
$(ax+b)^n$	$\frac{(ax+b)^{n+1}}{a(n+1)} + c$

* Remember to $+c$!!!

← Reverse chain Rule

Examples: Find the antiderivative:

1) $6x^5 + 4x^{-2} + 3$

$$\int 6x^5 + 4x^{-2} + 3 \; dx$$

$$= \frac{6x^6}{6} + \frac{4x^{-1}}{-1} + 3x + c$$

$$= x^6 - \frac{4}{x} + 3x + c \checkmark$$

2) $\frac{1}{x^3}$

$$\int x^{-3} \, dx$$

3) $3\sqrt{x}$

$$\int 3x^{\frac{1}{2}} \, dx$$

$$= \frac{x^{-2}}{-2} + c$$

$$= \frac{1}{-2x^2} + c \checkmark$$

$$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= 2x^{\frac{3}{2}} + c \checkmark$$

$$= 2\sqrt{x^3} + c \checkmark$$

4) $\frac{1}{(2x-1)^2}$

$$\int (2x-1)^{-2}\,dx$$

$$= \frac{(2x-1)^{-1}}{-1 \times 2} + c$$

$$= \frac{1}{-2(2x-1)} + c \checkmark$$

Ex3A page 53
Q15, 21, 24, 27, 28, 33, 35

$$\int f'(x)\, f(x)^n \, dx = \frac{f(x)^{n+1}}{n+1} + C$$

Scalar multiple of the derivative of $f(x)$

Example: Antidifferentiate

1) $x(6x^2 - 1)^2$

Scalar multiple of $12x$

derivative $= 12x$

2) $x(6x - 1)^2$

NOT SM of 6!

derivative $= 6$

leave the SM alone!

$$\int x(6x^2-1)^2 \, dx$$

$$= \frac{\cancel{x}(6x^2-1)^3}{3 \times 12\cancel{x}} + C$$

$$= \frac{(6x^2-1)^3}{36} + C$$

Expand first then integrate!

Check by differentiate

$$\frac{d}{dx} \frac{(6x^2-1)^3}{36}$$

$$= \frac{\cancel{3}(6x^2-1)^2}{\cancel{36}\,\cancel{3}} \times \cancel{12}x$$

$$= x(6x^2-1)^2$$

✓ :)

Sadler Ex 3B

Q30) $2x(1+x^2)^6$ ← Try this yourself

3) $\underbrace{(4x-x^2)}_{\text{SM} ✓}\underbrace{(6x^2-x^3)^2}_{\text{derivative } = 12x-3x^2}$

$$\int \underline{\quad\quad} \, dx = \frac{(4x-x^2)(6x^2-x^3)^3}{3(12x-3x^2)} + c$$

$$= \frac{(\cancel{4x-x^2})(6x^2-x^3)^3}{9(\cancel{4x-x^2})} + c$$

$$= \frac{(6x^2-x^3)^3}{9} + c$$

Rectilinear Motion

Displacement $x = \int v \, dt$

Velocity $V = \int a \, dt$

$$\begin{tikzcd} \text{Displacement } x \arrow[r, bend left, "\text{differentiate}"] & \text{Velocity } V \arrow[l, bend left] \arrow[r, bend left] & \text{Acceleration } a \arrow[l, bend left, "\text{Antidifferentiate}"] \end{tikzcd}$$

Rectilinear Motion Examples:

$t=0$ $V=0$ $x=3$

1. A particle starts from rest 3 metres from a fixed point and moves in a straight line with an acceleration of a = 6t +8. Find its position and velocity at t seconds.

Given $t=0$, $V=0$, $x=3$

$$a = 6t + 8$$

$$V = \int 6t + 8 \, dt$$

$$V = \frac{6t^2}{2} + 8t + C$$

Sub $t=0$, $V=0$ to find C $\rightarrow$ $\therefore C = 0$

$$\therefore V = 3t^2 + 8t \cancel{+0}$$

$$x = \int 3t^2 + 8t \, dt$$

$$x = \frac{3t^3}{3} + \frac{8t^2}{2} + C$$

Sub $t=0$, $x=3$ to find C $\rightarrow$ $\therefore C = 3$

$x = t^3 + 4t^2 + 3$ ✓

2. A body starts from O and moves in a straight line. After t seconds ($t \geq 0$) its velocity v m/s is given by $v = 2t - 4$.

a) Find its position after 3 seconds

b) Find the distance travelled in the first 3 seconds

Given $t = 0$, $x = 0$

$V = 2t - 4$

a) $x = \int 2t - 4 \, dt$

$x = \frac{2t^2}{2} - 4t + C$

at $t = 0$, $x = 0 \rightarrow \therefore C = 0$

$\therefore x = t^2 - 4t$ ✓

$x(3) = 3^2 - 4(3)$

$= -3$ ✓

The body is 3m left of O.

OR -3m displacement.

b) Set $V = 0 \leftarrow$ when it turns/stops/at rest

$V = 2t - 4$

$V = 2t - 4$

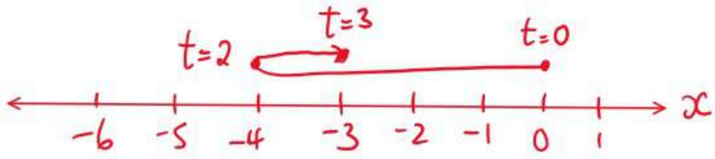
$0 = 2t - 4$

$\therefore t = 2$ ✓

when $t = 2 \longrightarrow$ $x = t^2 - 4t$

$x(2) = 2^2 - 4(2)$

$= -4$



$x(0) = 0$

$x(3) = -3$

$x(2) = -4$

$\therefore$ Total distance $= 4 + 1$

$= 5m$ ✓

3. A stone is projected vertically upward from the top of a building 20m high with an initial velocity of 15m/s. Assume that the acceleration due to gravity is approximately $-10m/s^2$. (hint: downwards -> negative direction).

   Find:

   a) The time taken for the stone to reach maximum height
   b) The maximum height reached by the stone
   c) The time taken for the stone to reach the ground
   d) The velocity of the stone as it hits the ground

Given : $t = 0$, $x = 20$, $V = 15$

a) max height is when $V = 0$

$V = \int a \, dt$

$$V = \int a \, dt$$

$$V = -10t + C$$

at $t = 0$, $V = 15 \rightarrow C = 15$

$$V = -10t + 15$$

$$0 = -10t + 15 \quad \leftarrow \text{set } V=0 \text{ for max height}$$

$$t = \frac{3}{2} \text{ or } 1.5 \text{ seconds} \checkmark$$

b) $$x = \int v \, dt$$

$$x = \int -10t + 15 \, dt$$

$$x = \frac{-10t^2}{2} + 15t + C$$

at $t=0$, $x = 20 \rightarrow \therefore C = 20$

$$\therefore x = -5t^2 + 15t + 20 \checkmark$$

at $t = 1.5 \rightarrow x(1.5) = -5(1.5)^2 + 15(1.5) + 20$

$$= 31.25\text{m}$$

$\therefore$ max height $= 31.25$ m $\checkmark$

c) when $x = 0$ the stone reaches the ground

$$0 = -5t^2 + 15t + 20$$

$$0 = -5(t^2 - 3t - 4)$$

$$0 = (t-4)(t+1)$$

$$t = 4 \qquad \cancel{t = -1} \qquad \checkmark$$

$t = 4$ $\quad$ $\cancel{t = -1}$

invalid

d) $V = -10t + 15$

at $t = 4 \rightarrow V(4) = -10(4) + 15$

$= -25 \text{ m/s}$

Integrating e

We know that:	Therefore:
$y = e^{x} \rightarrow \frac{dy}{dx} = e^{x}$	$\int e^{x}\,dx = e^{x} + c$
$y = e^{f(x)} \rightarrow \frac{dy}{dx} = f'(x)e^{f(x)}$	$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$
$y = e^{kx} \rightarrow \frac{dy}{dx} = ke^{kx}$	$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$

Examples:

1) Find:

a) $$\int e^{3x} + 4x\,dx = \frac{e^{3x}}{3} + \frac{4x^2}{2} + c$$

$$= \frac{e^{3x}}{3} + 2x^2 + c \checkmark$$

b) $$\int e^{-x} + e^{x}\,dx = \frac{e^{-x}}{-1} + e^{x} + c \checkmark$$

$$= -\frac{1}{e^{x}} + e^{x} + c \checkmark$$

$\frac{d}{dx} = 2x$

c) $$\int \underline{6x}\, e^{x^2}\,dx = \frac{{}^{3}\cancel{6}\cancel{x}\, e^{x^2}}{\cancel{2}\cancel{x}} + c$$

Scalar multiple of $2x$ ✓

$$= 3e^{x^2} + c \checkmark$$

Scalar multiple of $2x$ ✓

$$= 3e^{x^2} + c$$ ✓

d) $$\int 8e^{2x}\,dx = \frac{8e^{2x}}{2} + c$$

$$= 4e^{2x} + c$$ ✓

2) If the gradient of the tangent at a point (x, y) on a curve is $5e^{2x}$ and the curve passes through the point $(0, 7.5)$. Find the equation of the curve.

Let the curve be $y = f(x)$

Then $f'(x) = 5e^{2x}$

↑ gradient function!

$$\int 5e^{2x}\,dx = \frac{5}{2}e^{2x} + c$$

$$\therefore f(x) = \frac{5}{2}e^{2x} + c$$

at point $(0, 7.5)$:

$$7.5 = \frac{5}{2}e^{0} + c$$

$$7.5 = 2.5e^{0} + c$$

$$c = 5$$

$$\therefore f(x) = \frac{5}{2}e^{2x} + 5 \checkmark$$

Ex6D page 125

Q11, 12, 14, 15, 22, 23

Integration of Trigonometry

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos(kx+a) \, dx = \frac{1}{k}\sin(kx+a) + c$$

$$\int \sin(kx+a) \, dx = -\frac{1}{k}\cos(kx+a) + c$$

$$\int f'(x)\cos f(x) \, dx = \sin f(x) + C$$

$$\int f'(x)\sin f(x) \, dx = -\cos f(x) + C$$

Examples: Find the antiderivative of:

a) $\int 3\sin x + \cos x \, dx$

$$= \int 3\sin x \, dx + \int \cos x \, dx$$

$$= -3\cos x + \sin x + C \checkmark$$

b) $\int \sin\left(3x+\frac{\pi}{4}\right) dx$

$$= \frac{-\cos\left(3x+\frac{\pi}{4}\right)}{3} + c \quad \checkmark$$

c) $\int \frac{1}{4}\sin 4x + 8\cos(2x+1)\, dx$

$$= \int \frac{\sin 4x}{4}\, dx + \int 8\cos(2x+1)\, dx$$

$$= \frac{-\cos(4x)}{4 \times 4} + \frac{8\sin(2x+1)}{2} + c$$

$$= \frac{-\cos 4x}{16} + 4\sin(2x+1) + c \quad \checkmark$$

d) $\int \cos^4 x \sin x \, dx$

Scalar multiple of the derivative of $\cos x$ (leave it! ignore it!)

$$= \int (\cos x)^4 \sin x \, dx$$

$$= \frac{(\cos x)^5 \ \cancel{\sin x}}{\quad} + C$$

← ignore!

$$= \frac{(\cos x)\ \cancel{\sin x}}{5 \times (-)\cancel{\sin x}} + C$$

$$= \frac{\cos^5 x}{-5} + c$$

e) $\int 3 \sin^3 4x \underbrace{\cos 4x}_{} \, dx$

S.M of the derivative of Sin4x

IGNORE IT !!!

$$\downarrow$$

$$3(\sin 4x)^3$$

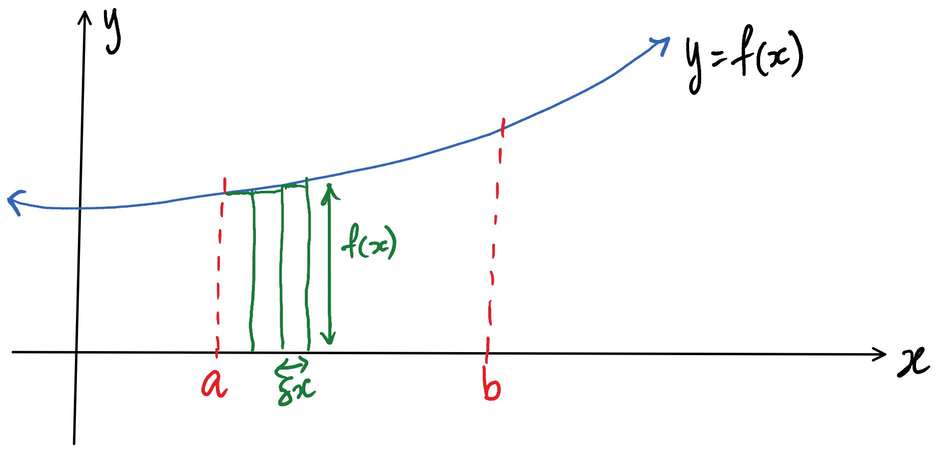
$$= \frac{3(\sin 4x)^4 \cos \cancel{4x}}{4 \times 4 \cos \cancel{4x}} + c$$

$$= \frac{3 \sin^4 4x}{16} + c$$

Ex7B page148

Q7, 9, 11, 15, 19, 21, 22, 23

- To determine the area under $y=f(x)$ from $x=a$ to $x=b$, we divide the area into rectangular strips of width δx and height $f(x)$



- The sum of the areas of each rectangle is the approximate area under $y=f(x)$
- As the number of rectangles increase and δx tends to 0, the sum will tend to the limiting value (Area)
- Area under the curve $\approx \sum f(x)\,\delta x$ sum
- Area between $y=f(x)$ and the x-axis from $x=a$ to $x=b$ is:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x$$

- We can write the limit of a sum as integration.

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x \longrightarrow \int_a^b f(x)\,dx$$

$$\boxed{\begin{aligned} \text{Area} = \int_a^b f(x)\,dx &= F(b) - F(a) \\ &= \big[\, F(x) \,\big]_a^b \end{aligned}}$$

where $F(x)$ is the antiderivative of $f(x)$

* we can omit the "$+c$"

(See example in Sadler page 71)

Ex 4A page 72

Q2

Past Exam Questions:

Definite Integral

Indefinite integral : $\int f(x)\,dx$ (with "+c")

Definite integral : $\int_a^b f(x)\,dx$ (can omit "+c")

$$\boxed{\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a) \\ &= \big[F(x)\big]_a^b\end{aligned}}$$

Examples:

1) $\int_1^2 \frac{1}{x^2}\,dx$

$= \int_1^2 x^{-2}\,dx$

$= \left[\frac{x^{-1}}{-1}\right]_1^2$

$= \left[-\frac{1}{x}\right]_1^2$

$= -\frac{1}{2} - \left(-\frac{1}{1}\right)$

$= -\frac{1}{2} + 1$

$= \frac{1}{2}$ ✓

2) $\int_0^1 6(2x+1)^3\,dx$

$= \left[\frac{\overset{3}{\cancel{6}}\,(2x+1)^4}{4 \times \cancel{2}}\right]_0^1$

$= \left[\frac{3(2x+1)^4}{4}\right]_0^1$

$= \frac{3(2(1)+1)^4}{4} - \frac{3(2(0)+1)^4}{4}$

$= \frac{243}{4} - \frac{3}{4}$

$= \frac{240}{4} = 60$ ✓

Ex4B page 79
Q16, 17, 25

Extra practise:
Q4, 11, 24

Linearity properties of definite integrals:

$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

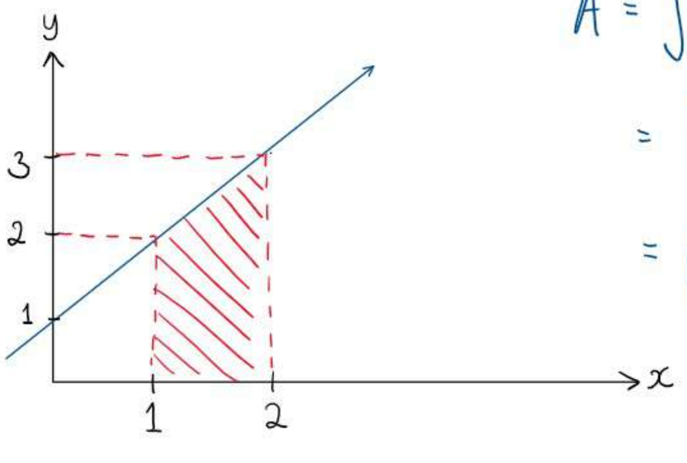
$$\int_a^b k\, f(x)\, dx = k \int_a^b f(x)\, dx$$

$$\int_a^a f(x)\, dx = 0$$

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$$

Examples:

1) Find the area under the line $y = x + 1$ between $x = 1$ and $x = 2$.



$$A = \int_1^2 x + 1\, dx$$

$$= \left[\frac{x^2}{2} + x\right]_1^2$$

$$= \left(\frac{2^2}{2} + 2\right) - \left(\frac{1^2}{2} + 1\right)$$

$$= 4 - \frac{3}{2}$$

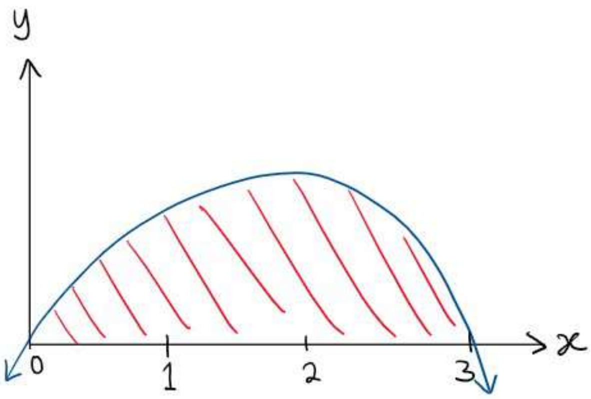
$$= \frac{5}{2} \text{ units}^2$$

* Trapezium Area $= \frac{1}{2}(a+b)h$

$$= \frac{1}{2}(2+3) \times 1$$

$$= \frac{5}{2} \text{ units}^2$$

2) Find the area under the curve $y = 3x - x^2$ between $x = 0$ and $x = 3$.



$$A = \int_0^3 (3x - x^2)\, dx$$

$$= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3$$

$$= \left[\frac{3(3)^2}{2} - \frac{3^3}{3} \right] - 0$$

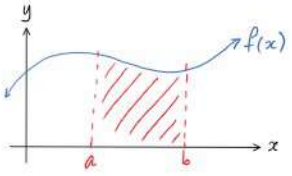
$$= \frac{27}{2} - \frac{27}{3}$$

$$= \frac{9}{2} \text{ units}^2$$

Area of a region under the curve

- If $f(x) \geq 0$ (above x-axis)

$$\text{Area} = \int_a^b f(x)\,dx$$

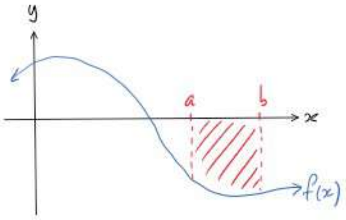


- If $f(x) \leq 0$ (below x-axis)

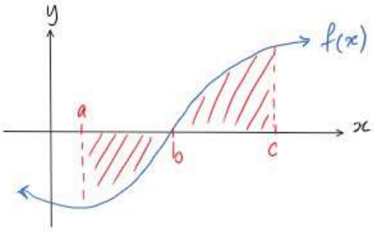
$$\text{Area} = -\int_a^b f(x)\,dx$$

OR $$\text{Area} = \int_b^a f(x)\,dx$$

OR $$\text{Area} = \int_a^b |f(x)|\,dx$$



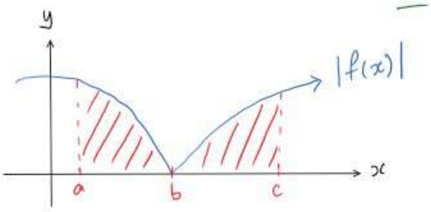
- The shaded area shown below is:



$$\text{Area} = \int_b^c f(x)\,dx + \left(-\int_a^b f(x)\,dx\right)$$

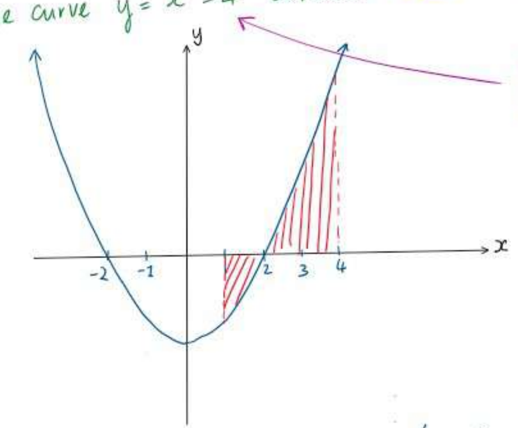
OR $$\text{Area} = \int_b^c f(x)\,dx + \int_a^b |f(x)|\,dx$$

OR $$\text{Area} = \int_a^c |f(x)|\,dx$$



* classpad only

Example:
Find the area of the region between the x-axis and the curve $y = x^2 - 4$ between $x = 1$ and $x = 4$



$y = x^2 - 2^2$

$y = (x-2)(x+2)$

$0 = (x-2)(x+2)$

$x = \pm 2$

$$\text{Area} = \int_2^4 (x^2 - 4)\,dx + \left(-\int_1^2 (x^2-4)\,dx\right)$$

$$= \left[\frac{x^3}{3} - 4x\right]_2^4 - \left[\frac{x^3}{3} - 4x\right]_1^2$$

$$= \left[\frac{x^3}{3} - 4x\right]_2 - \left[\frac{x^3}{3} - 4x\right]_1$$

$$= \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) - \left[\left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)\right]$$

$$= \frac{56}{3} - 8 - \left(\frac{7}{3} - 4\right)$$

$$= \frac{37}{3} \text{ units}^2 \checkmark$$

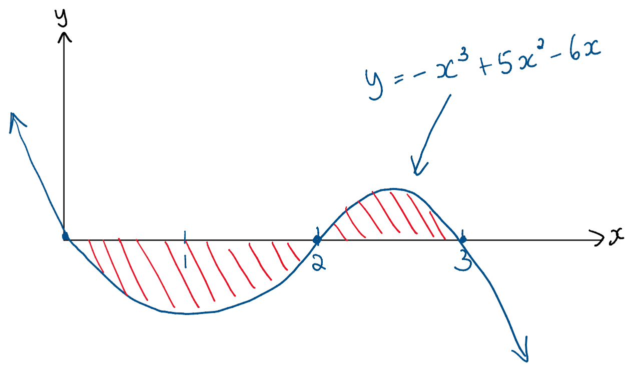
Example 2:

Find the area of the regions enclosed by the graph $y = x(2-x)(x-3)$ and the x-axis

Sketch the graph first!

x-ints: $x = 0$, $x = 2$, $x = 3$

Shape: ↖↗↘ (∿)



$$\text{Area} = \int_2^3 (-x^3 + 5x^2 - 6x)\,dx + \left[-\int_0^2 (-x^3 + 5x^2 - 6x)\,dx\right]$$

$$= \left[-\frac{x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_2^3 - \left[-\frac{x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_0^2$$

$$= \left(-\frac{81}{4} + 54 - 27\right) - \left(-4 + \frac{40}{3} - 12\right) - \left[\left(-4 + \frac{40}{3} - 12\right) - 0\right]$$

$$= -\frac{81}{4} + 18 + 32 - \frac{80}{3}$$

$$= \frac{37}{12} \text{ units}^2 \checkmark$$

Ex4C page 86
Q1, 2, 12, 13

* Classpad:

① Main → keyboard: Math1
|□| absolute value

eg: $\int_1^4 |x^2 - 4|\,dx$

② Graph & Table → Analysis
→ G-Solve
→ Integral → $\int dx$
→ Type in boundaries

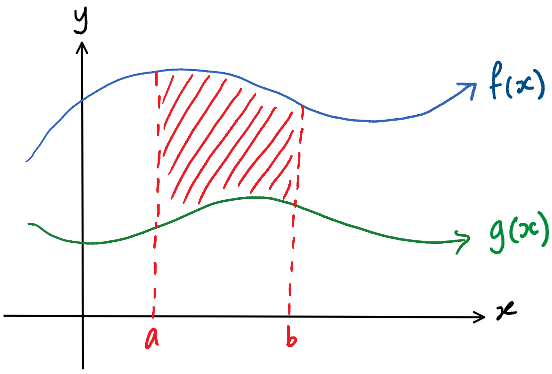
http://www.classpad.com.au/cp2/index.php?me=BA4&mn=349

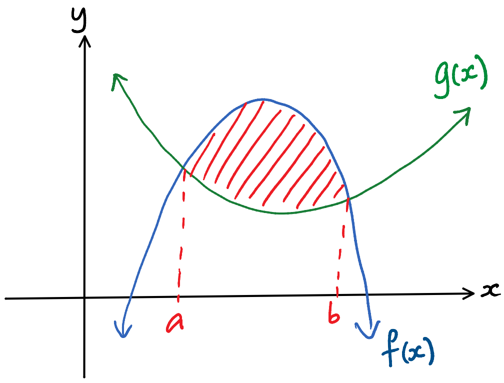
Given $f(x)$ and $g(x)$ are functions on the interval $[a, b]$ such that :

$$f(x) \geq g(x)$$

$$\text{Area} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

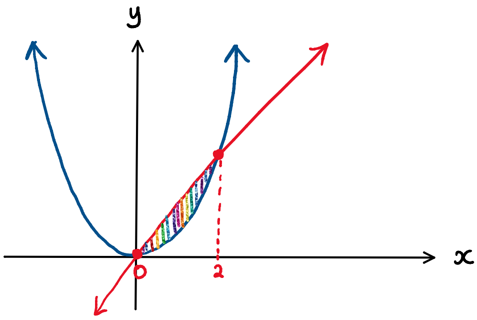
$$\boxed{\text{Area} = \int_a^b f(x) - g(x)\,dx}$$





Examples :

1) Find the area bounded by $y = x^2$ and $y = 2x$



* intersection points :

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$x = 0 \qquad x = 2$$

linear

$$\text{Area} = \int_0^2 2x - x^2\,dx$$

$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$

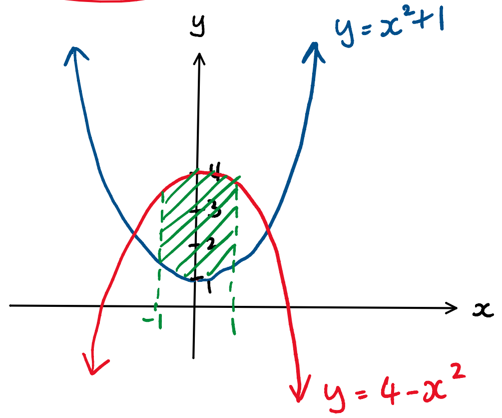
$$= 4 - \frac{8}{3}$$

$$= \frac{4}{3} \text{ units}^2 \checkmark$$

$4 - \frac{}{3}$

$= \frac{4}{3}$ units^2 ✓

2) Calculate the area enclosed by $y = x^2 + 1$ and $y = 4 - x^2$ between $x = -1$ and $x = 1$



$$\text{Area} = \int_{-1}^{1} 4 - x^2 - (x^2 + 1)\, dx$$

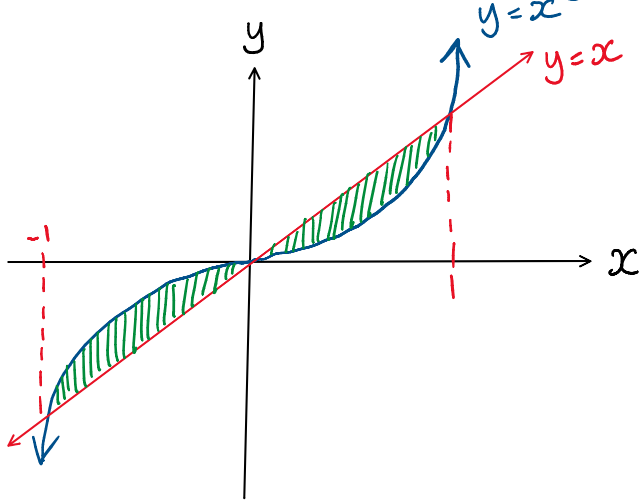
$$= \int_{-1}^{1} 3 - 2x^2\, dx$$

$$= \left[3x - \frac{2x^3}{3}\right]_{-1}^{1}$$

$$= 3 - \frac{2}{3} - \left(-3 + \frac{2}{3}\right)$$

$$= \frac{14}{3} \text{ units}^2$$ ✓

3) Find the area of the region enclosed by $f(x) = x^3$ and $g(x) = x$



$f(x) \geq g(x)$ for $-1 \leq x \leq 0$

$g(x) \geq f(x)$ for $0 \leq x \leq 1$

Intersection points:

$$x^3 = x$$

$$x^3 - x = 0$$

$x(x^2-1)=0$

$x=0$ ✓ $\quad x^2-1=0$

$x=\pm 1$ ✓

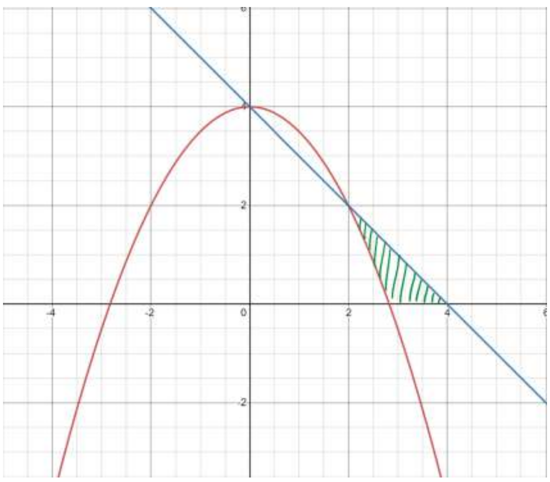
$$\text{Area} = \int_{-1}^{0} x^3 - x \, dx + \int_{0}^{1} x - x^3 \, dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2} \text{ units}^2$$ ✓

4) Find the shaded area bounded by $y = 4 - \frac{x^2}{2}$, $y = -x + 4$ and the x-axis



Set $y=0$ to find x-ints for $y = 4 - \frac{x^2}{2}$

$0 = 4 - \frac{x^2}{2}$

$x^2 = 8$

$x = \pm\sqrt{8}$ or $\pm 2\sqrt{2}$

Shaded Area :

$$\int_2^4 (-x+4)\,dx - \int_2^{2\sqrt{2}} \left(4 - \frac{x^2}{2}\right) dx$$

$$= \left[\frac{-x^2}{2} + 4x\right]_2^4 - \left[4x - \frac{x^3}{6}\right]_2^{2\sqrt{2}}$$

$$= 2 - \left(\frac{16\sqrt{2}}{3} - \frac{20}{3}\right)$$

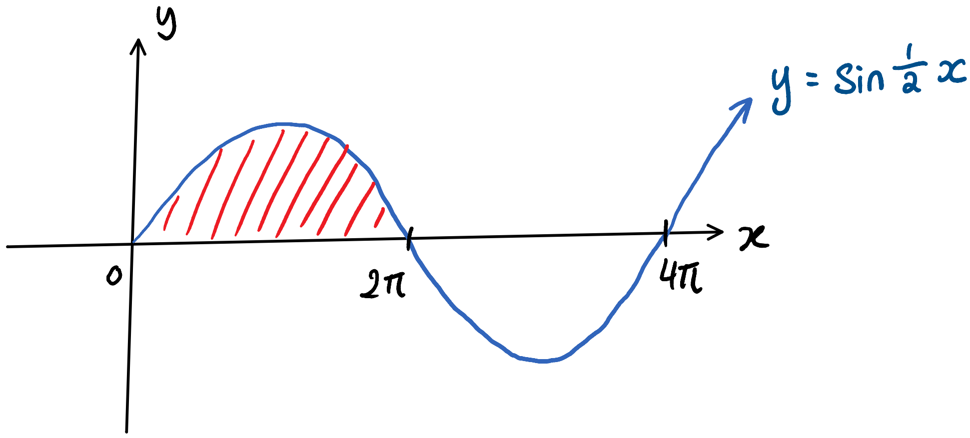
$$= \frac{26 - 16\sqrt{2}}{3} \text{ units}^2 \checkmark$$

Ex4C page 87
Q15, 16, 17, 18

Examples:

Find the exact area of the shaded region:

1)



$$\text{Area} = \int_0^{2\pi} \sin\tfrac{1}{2}x \; dx$$

$$= \left[\frac{-\cos\frac{1}{2}x}{\frac{1}{2}} \right]_0^{2\pi}$$

$$= \left[-2\cos\tfrac{1}{2}x \right]_0^{2\pi}$$

$$= -2\cos\tfrac{1}{2}(2\pi) - \left(-2\cos\tfrac{1}{2}(0)\right)$$

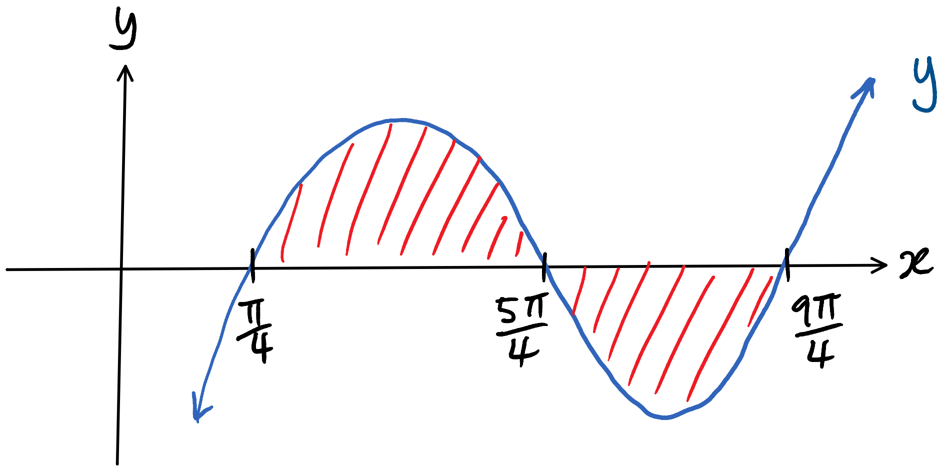
$$= -2\cos\pi + 2\cos 0$$

$$= -2(-1) + 2(1)$$

$= -2(-1) + 2(1)$

$= 4 \text{ units}^2$ ✓

2)



$$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta$$

$$= \left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -\cos\left(\frac{5\pi}{4} - \frac{\pi}{4}\right) - \left[-\cos\left(\frac{\pi}{4} - \frac{\pi}{4}\right)\right]$$

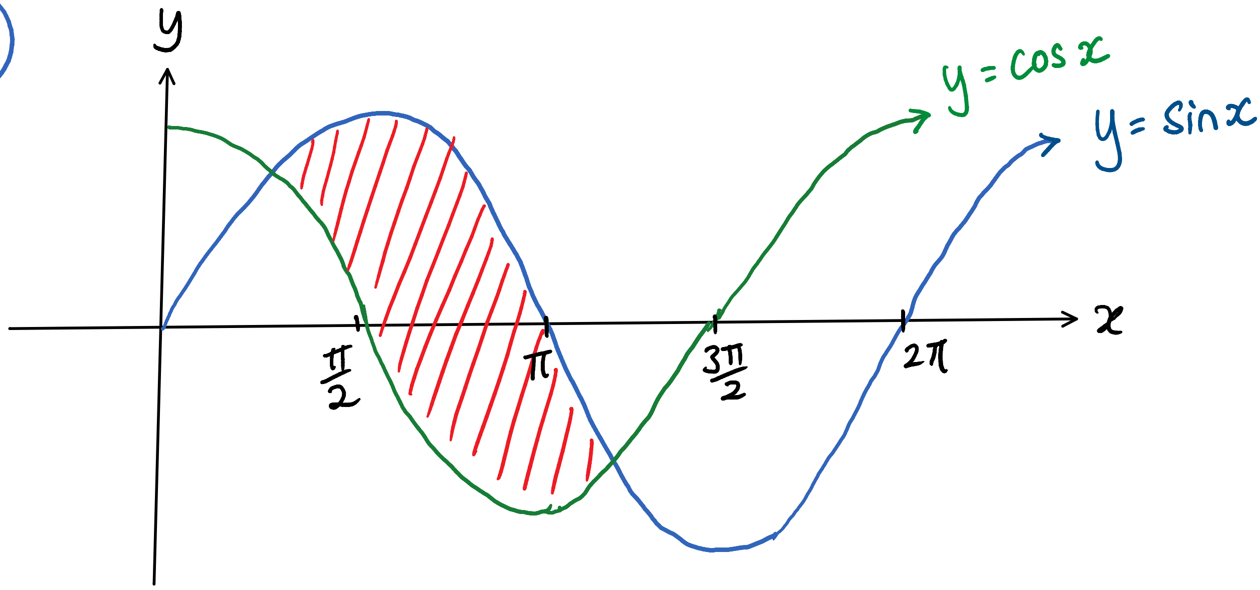
$$= -\cos\pi + \cos 0$$

$$= -(-1) + 1$$

$$= 2 \text{ units}^2$$ ✓ $\times 2$ ← symmetrical on other side :)

Total Area = 4 units^2 ✓

3)



intersection: $\sin x = \cos x$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$\therefore x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$

$$\text{Area} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx$$

$$= \left[-\cos x - \sin x \right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4} - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4} \right)$$

$$= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2} \text{ units}^2 \checkmark$$

Ex7B page 148 (Trigonometry)
Q30 -> Q36
Ex6D page 126 (e^x)
Q16 -> Q24

Total Change from rate of change

- A quantity Q changes with respect to a variable r such that $\frac{dQ}{dr}$ = rate of change.

Then $\int_a^b \frac{dQ}{dr}\, dr$ = Total change in Q when r changes from a to b.

Example:

An oil tank is drained of oil such that if V kL of oil are in the tank t minutes after draining commences, then $\frac{dV}{dt} = 2t - 20$.

a) write an expression that would give the amount of oil drained from the tank in the first minute.

b) Determine the number of kL of oil drained in the fourth minute.

a) $\int_0^1 2t - 20\, dt$

$-\int_0^1 2t - 20\, dt$ → decreasing volume

b) $$\int_3^4 2t - 20\, dt = \left[t^2 - 20t \right]_3^4$$

$$= 4^2 - 20(4) - (3^2 - 20(3))$$

$$= -64 - (-51)$$

$$= -13$$ → 13 kL drained

∴ 13 kL are drained from the tank in the fourth minute

Note:

* O.T. Lee uses "Net change" instead of "Total change".

* Remember:

Total change in the first 4 minutes → $\int_0^4 \cdots$

Total change in the fourth minute → $\int_3^4 \cdots$

* Total change can be used in Rectilinear Motion

eg: If given $V = \ldots\ldots$, find the change in distance between 2 to 5 minutes

$$= \int_2^5 |V| \, dt$$

Ex4D page 90
Q3 -> Q9

Worksheet

$$\int_3^a 2x - 6 \, dx = 16$$

$$\left[x^2 - 6x \right]_3^a = 16$$

$$a^2 - 6a - (3^2 - 18) = 16$$

$$a^2 - 6a - 7 = 0$$

$$(a - 7)(a + 1) = 0$$

$$a = 7 \quad \underline{\text{OR}} \quad a = -1$$

a) $V = \int a \, dt$

$V = \int \frac{t}{2} - 5 \, dt$

$V = \frac{t^2}{4} - 5t + C$ $\quad$ $V = 25$ at $t = 0$

$V = \frac{t^2}{4} - 5t + 25$ ✓

$V(4) = \frac{4^2}{4} - 5(4) + 25$

$= 9 m/s$ ✓

b) $V = 0$ when stationary

$0 = \frac{t^2}{4} - 5t + 25$

$0 = t^2 - 20t + 100$

$0 = (t - 10)(t - 10)$

$\therefore t = 10$ seconds $\leftarrow$ stationary

Distance change between 0 to 10 seconds

Distance change between 0 to 10 seconds

$$x = \int_0^{10} V \, dt$$

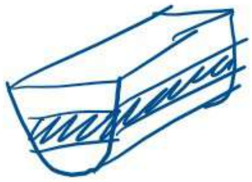
$$= \int_0^{10} \frac{t^2}{4} - 5t + 25 \, dt$$

$$= \left[\frac{t^3}{12} - \frac{5t^2}{2} + 25t \right]_0^{10}$$

$$= \frac{10^3}{12} - \frac{5(10)^2}{2} + 25(10) - 0$$

$$= 83\tfrac{1}{3} \text{ m}$$

EXTRA QUESTION

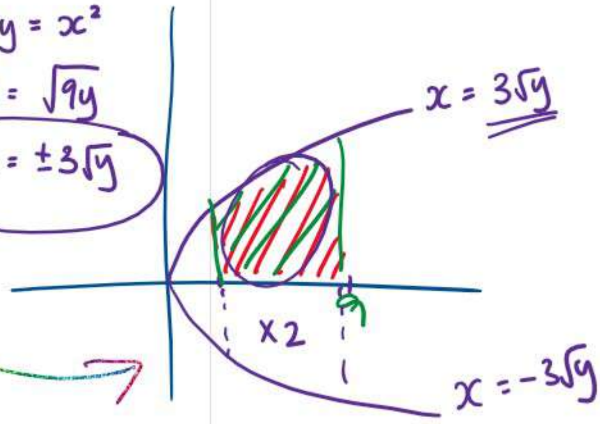


$y = \frac{1}{9}x^2 \leftarrow$ inverse

$9y = x^2$

$x = \sqrt{9y}$

$x = \pm 3\sqrt{y}$



Area $= 2\int_1^9 3y^{\frac{1}{2}}\,dy$

$= 2\left[\frac{3y^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^9$

$= 2\left[2y^{\frac{3}{2}}\right]_1^9$

$= 2\left(2(9^{\frac{3}{2}}) - 2(1^{\frac{3}{2}})\right)$

$= 2(54 - 2)$

$= 2(52)$

$= 104\ \text{cm}^2$

$V = 104 \times 18$

$= \_\_\_\_\ \text{cm}^3$

Area under curve -...

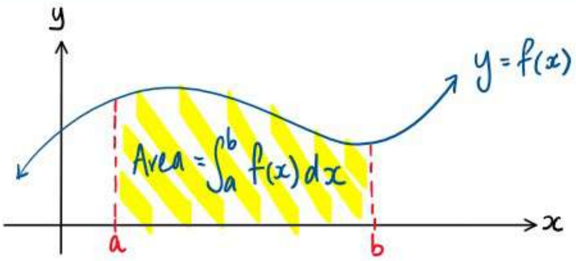
Area under curve -...

① The F.T.O.C states that the limit of a sum can be determined by definite integral.

$$\text{Area} = \lim_{\delta x \to 0} \sum f(x)\,\delta x = \int_a^b f(x)\,dx$$

$$\boxed{\therefore \int_a^b f(x)\,dx = F(b) - F(a)}$$

where $F(x)$ is the antiderivative of $f(x)$



② The F.T.O.C also states that : Integrating the derivative of a function "gives the function back", and differentiating the integral of a function "gives the function back".

$$\boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$

where "a" is a constant

Examples:

1) Find :

a) $\frac{d}{dx}\left(\int x^2 + 6\,dx\right)$

$= \frac{d}{dx}\left(\frac{x^3}{3} + 6x + c\right)$

b) $\int \left(\frac{d}{dx}\, x^2 + 6\right) dx$

$= \int 2x\,dx$

$= x^2 + 6$ ✓ Same as the original function

$= x^2 + C$ ✓

Don't forget to $+C$

$$\boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$

where "a" is a constant

2) $\frac{d}{dx}\left(\int_2^x t^2 + 6 \; dt\right)$

$= x^2 + 6$ ✓

3) $\frac{d}{dx}\int_1^x \frac{1+t^3}{\sqrt{t^3}} \; dt$

$= \frac{1+x^3}{\sqrt{x^3}}$ ✓

4) Find the derivative of

$\int_x^3 \sin 2t \; dt$

$= \frac{d}{dx}\int^3 \sin 2t \; dt$

$$= \frac{d}{dx} \int_x^3 \sin 2t \, dt$$

$$\boxed{\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx}$$

$$= \frac{d}{dx}\left(- \int_3^x \sin 2t \, dt \right)$$

$$= - \frac{d}{dx} \int_3^{x} \sin 2t \, dt$$

$$= - \sin 2x \; \checkmark$$

Examples : (Continue from previous lesson)

② F.T.O.C

$$\boxed{\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$$

where "a" is a constant

not x ∴ use chain rule!

5a) $\frac{d}{dx}\int_0^{2x}(3t^2 - 4t + 1)\,dt$

$= \left(3(2x)^2 - 4(2x) + 1\right) \times 2$

$= 24x^2 - 16x + 2$ ✓

use chain rule!

b) Find the derivative of $F(x) = \int_1^{x^2} \frac{\cos t}{t}\,dt$

$= \frac{\cos x^2}{x^2} \times 2x$

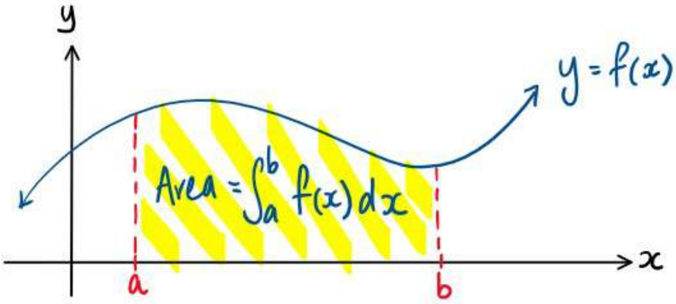
$= \frac{2\cos x^2}{x}$ ✓

① F.T.O.C

$$\text{Area} = \lim_{\delta x \to 0} \sum f(x)\,\delta x = \int_a^b f(x)\,dx$$

$$\boxed{\therefore \int_a^b f(x)\,dx = F(b) - F(a)}$$

where $F(x)$ is the antiderivative of $f(x)$

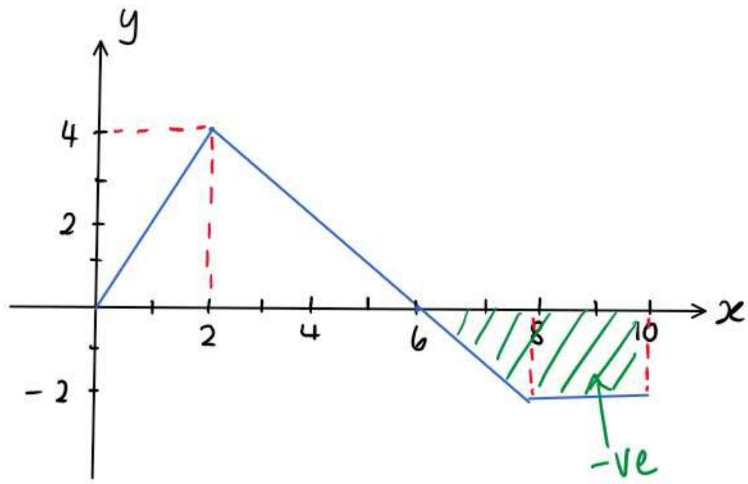


6) For $0 \leq x \leq 10$ the function $y = f(x)$.

Determine:

a) $\int_0^4 f(x)\,dx$

b) $\int_2^{10} f(x)\,dx$



a) $\text{Area} = \frac{1}{2}(6)(4) - \frac{1}{2}(2)(2)$

$= 12 - 2$

$= 10$

$\therefore \int_0^4 f(x)\,dx = 10$ ✓

$$\therefore \int_0 f(x)\,dx = 10 \checkmark$$

b) Area of $\triangle = \frac{1}{2}(4)(4) = 8$

Area of ⏢ $= \frac{1}{2}(4+2)(2) = 6$

$$\therefore \int_2^{10} f(x)\,dx = 8 + (-6)$$

$$= 2 \checkmark$$

-ve because under x-axis

Discrete Random Variable (Notes)

- A random variable is a function that assigns a number to each outcome in the sample space.

- A discrete random variable is one that can only take a countable number of values.

- The probability function of a DRV is a function formed by the values of the variable and their probabilities. For random variable X, the probability function is:

$$\boxed{P(X = x) \text{ or } p(x)}$$

- A discrete probability function / discrete probability distribution shows the probability of a DRV as a list, table or graph.

- A discrete probability function P(X = x) or p(x) has the following properties:

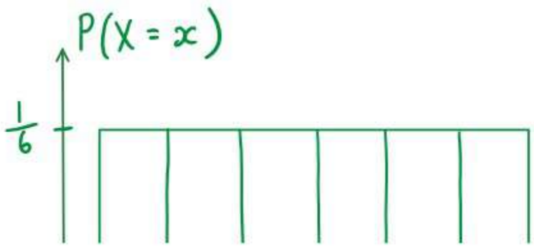
$$\boxed{\begin{array}{l} \text{①} \quad 0 \leq P(X = x) \leq 1 \text{ for all values of } x \\ \text{②} \quad \sum P(X = x) = 1 \end{array}}$$

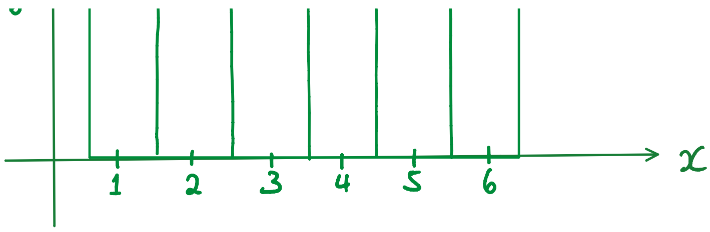
- A uniform discrete probability distribution has "n" outcomes with equal probabilities, where

$$\boxed{P(X = x) = \frac{1}{n}}$$

Eg:

X is the DRV of "when a normal six-sided die is rolled."




1
2
3
4
5
6
x

Discrete Random Variable (Examples)

Example 1:

Let X be the discrete random variable "number of heads obtained" when a fair coin is tossed three times.
Show the probability distribution of X in a:

a) List
b) Table
c) Graph
d) Cumulative probability distribution table and graph

a) List

No heads in 3 tosses

$$P(0) = P(X=0) = Pr(\{\overset{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}{TTT}\}) = \frac{1}{8}$$

$$P(1) = P(X=1) = Pr(\{\overset{\frac{1}{8}}{HTT} + \overset{\frac{1}{8}}{THT} + \overset{\frac{1}{8}}{TTH}\}) = \frac{3}{8}$$

$$P(2) = P(X=2) = Pr(\{\overset{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}{HHT} + HTH + THH\}) = \frac{3}{8}$$

$$P(3) = P(X=3) = Pr(\{\overset{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}{HHH}\}) = \frac{1}{8}$$

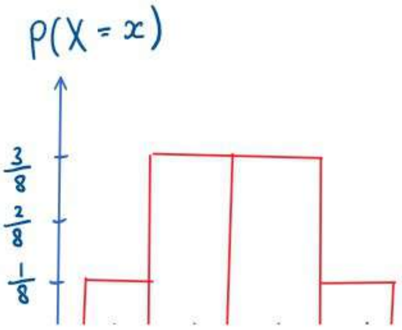
b) Table

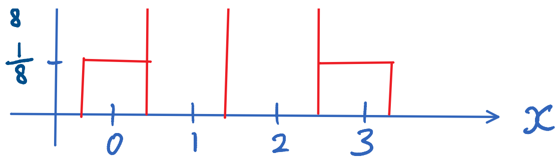
X = No. of heads in 3 tosses

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Draw a tree diagram to find the probability if you need to :)

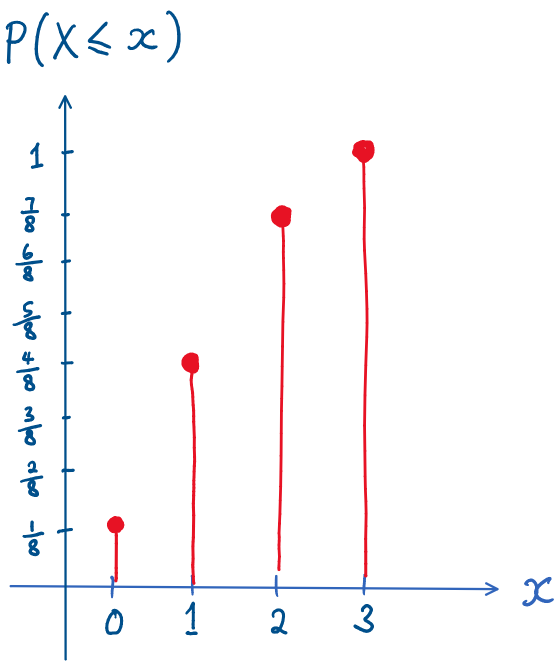
c) Graph





d) Cumulative probabilities

x	0	1	2	3
$P(X \leq x)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	1

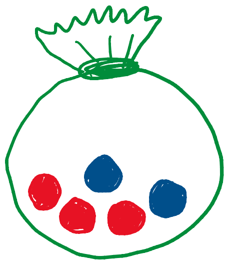


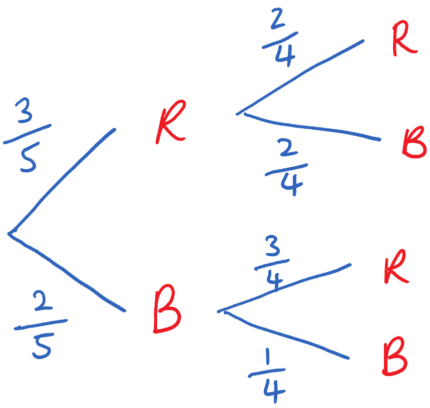
Example 2:

Two marbles are randomly selected from the bag (without replacement).
Draw a table of the probability distribution if:

Two marbles are randomly selected from the bag (without replacement).
Draw a table of the probability distribution if:

a) X = number of red marbles selected
b) X = number of blue marble selected





$P(RR) = \frac{6}{20} = 0.3$

$P(RB) = \frac{6}{20} = 0.3$

$P(BR) = \frac{6}{20} = 0.3$

$P(BB) = \frac{2}{20} = 0.1$

a)

x	0	1	2
$P(X=x)$	0.1	0.6	0.3

red marbles

b)

x	0	1	2
$P(X=x)$	0.3	0.6	0.1

blue marbles

* Note: $\sum P(X=x) = 1$

DRV Examples

1. The random variable X represents the number of chocolate chips in a certain brand of biscuit and have the following probability distribution:

x	2	3	4	5	6	7
$P(X = x)$	~~k~~ 0.01	0.25	0.4	0.3	~~2K~~ 0.02	~~2K~~ 0.02

a) Find the value of k
b) $P(X \geq 4)$
c) $P(X \geq 4 \mid X > 2)$
d) $P(X < 5 \mid X > 2)$

a) $\sum P(X = x) = 1$ (must add to 1)

$k + 0.25 + 0.4 + 0.3 + 2k + 2k$

$5k + 0.95 = 1$

$\therefore k = 0.01$

b) $P(X \geq 4) = 0.4 + 0.3 + 0.02 + 0.02$

$= 0.74$ ✓

c) $P(X \geq 4 \mid X > 2)$

$= \dfrac{P(X \geq 4)}{P(X > 2)}$

$= \dfrac{0.74}{0.99} \leftarrow 1 - 0.01$

$= \dfrac{74}{99}$ ✓

d) $P(X < 5 \mid X > 2) = \dfrac{P(2 < X < 5)}{P(X > 2)}$

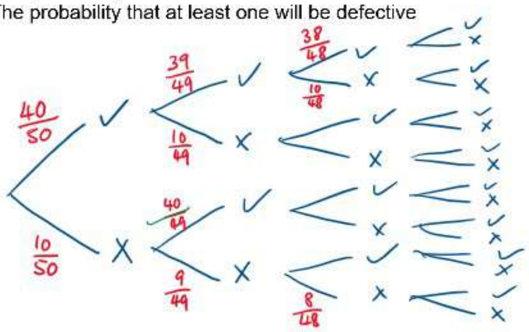
$= \dfrac{0.25 + 0.4}{0.99}$

$= \dfrac{0.65}{0.99} = \dfrac{65}{99}$ ✓

10 out of 50

2. A box contains 50 electrical safety switches, of which 20% are known to be faulty. A sample of 4 switches is selected at random and tested. If X is the number of faulty switches, determine:

a) The probability distribution of X
b) The probability that two will be defective
c) The probability that at least one will be defective



(✓ ✓ ✓ ✓) No faulty!

a) $P(X = 0) = \dfrac{40}{50} \times \dfrac{39}{49} \times \dfrac{38}{48} \times \dfrac{37}{47}$

$= 0.396830$

$(\checkmark \; \checkmark \; \checkmark \; \times) + (\checkmark\checkmark\times\checkmark) + (\checkmark\times\checkmark\checkmark) + (\times\checkmark\checkmark\checkmark)$

$P(X=1) = \frac{40}{50} \times \frac{39}{49} \times \frac{38}{48} \times \frac{10}{47} \times 4$

$= 0.429006$

$P(X=2) = \times 6 \quad (\checkmark\checkmark\times\times) + (\checkmark\times\checkmark\times) + (\checkmark\times\times\checkmark) + (\times\times\checkmark\checkmark) + (\times\checkmark\times\checkmark) + (\times\checkmark\checkmark\times)$

$P(X=3) = \times 4 \quad (\checkmark\times\times\times) + (\times\checkmark\times\times) + (\times\times\checkmark\times) + (\times\times\times\checkmark)$

$P(X=4) = \times 1 \quad (\times\times\times\times)$

OR

No faulty

$P(\underline{X}=0) = \dfrac{{}^{10}C_0 \times {}^{40}C_4}{{}^{50}C_4} = 0.396830$ ✓

$P(X=1) = \dfrac{{}^{10}C_1 \times {}^{40}C_3}{{}^{50}C_4} = 0.429006$ ✓

$P(X=2) = \dfrac{{}^{10}C_2 \times {}^{40}C_2}{{}^{50}C_4} = 0.152410$ ✓

$P(X=3) = \dfrac{{}^{10}C_3 \times {}^{40}C_1}{{}^{50}C_4} = 0.020842$ ✓

$P(X=4) = \dfrac{{}^{10}C_4 \times {}^{40}C_0}{{}^{50}C_4} = 0.000912$ ✓

a) Probability distribution :

x	0	1	2	3	4

a) Probability distribution :

x	0	1	2	3	4
$P(X=x)$	0.396830	0.429006	0.152410	0.020842	0.000912

b) $P(X=2) = 0.152410$.

c) $P(X \geq 1) = 1 - P(X=0)$

$= 1 - 0.396830$

$= 0.60317$ ✓

3. A computer manufacturer orders 500 circuit boards from a supplier. 15 circuit boards are randomly selected to be tested. The manufacturer will only accept the order if there are no defective circuit boards in those that have been randomly selected for testing.
Find the probability of no faulty boards if 5% of the circuit boards are defective.

X = selecting a defective circuit board

$$P(X=0) = \frac{{}^{25}C_0 \times {}^{475}C_{15}}{{}^{500}C_{15}}$$

$= 0.458096$ ✓

More Examples (Sadler)
Page 160 - Example 2
Page 161 - Example 3
Page 152 - Example 4

Ex8A page 165
Q11, 12, 17, 18, 19, 21

More Examples (Textbook)

EXAMPLE 1

Suppose that a fair coin is flipped three times. If X is the discrete random variable 'Number of heads obtained', copy and complete the following table.

x	0	1	2	3
$P(X = x)$				

Hence determine **a** $P(X < 1)$, **b** $P(X \geq 2)$.

Solution

Considering the eight equiprobable events:

$$\underbrace{\text{TTT}}_{X=0} \quad \underbrace{\text{HTT} \quad \text{THT} \quad \text{TTH}}_{X=1} \quad \underbrace{\text{THH} \quad \text{HTH} \quad \text{HHT}}_{X=2} \quad \underbrace{\text{HHH}}_{X=3}$$

$P(X = 0) = \frac{1}{8}$ $\quad$ $P(X = 1) = \frac{3}{8}$ $\quad$ $P(X = 2) = \frac{3}{8}$ $\quad$ $P(X = 3) = \frac{1}{8}$

The table can then be completed:

x	0	1	2	3
$P(X = x)$	0.125	0.375	0.375	0.125

a $P(X < 1) = 0.125$

b $P(X \geq 2) = 0.375 + 0.125$
$= 0.5$

EXAMPLE 2

The probability function for a discrete random variable X is given by:

$$P(X=x)=\begin{cases} k(6-x) & \text{for} \quad x=1,2,3,4,5. \\ 0 & \text{for} \quad \text{all other values of } x. \end{cases}$$

Copy and complete the following probability distribution for X, giving the probabilities as numbers (i.e. k should be evaluated).

x	1	2	3	4	5
$P(X=x)$					

Determine **a** $P(X=\text{even})$ **b** $P(X>3)$ **c** $P(X=4 \mid X>3)$

Solution

$$\begin{aligned} P(X=1) &= k(6-1) \\ &= 5k \end{aligned} \qquad \begin{aligned} P(X=2) &= k(6-2) \\ &= 4k \end{aligned} \qquad \begin{aligned} P(X=3) &= k(6-3) \\ &= 3k \end{aligned}$$

$$\begin{aligned} P(X=4) &= k(6-4) \\ &= 2k \end{aligned} \qquad \begin{aligned} P(X=5) &= k(6-5) \\ &= k \end{aligned}$$

But these probabilities must add up to 1. Thus

$$\begin{aligned} 15k &= 1 \\ \therefore \quad k &= \frac{1}{15} \end{aligned}$$

The table can then be completed:

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{15}$

a
$$\begin{aligned} P(X=\text{even}) &= \frac{4}{15}+\frac{2}{15} \\ &= \frac{2}{5} \end{aligned}$$

b
$$\begin{aligned} P(X>3) &= \frac{2}{15}+\frac{1}{15} \\ &= \frac{1}{5} \end{aligned}$$

c
$$\begin{aligned} P(X=4 \mid X>3) &= \frac{2}{15}\div\frac{1}{5} \\ &= \frac{2}{3} \end{aligned}$$

EXAMPLE 3

A spinner shows the numbers 1, 2, 3.

For each spin of this spinner the probability associated with each outcome, 1, 2 or 3, is as shown in the table.

x	1	2	3
$P(X = x)$	0.5	0.3	k

a Determine the value of k.

The spinner is spun twice. Determine the probability of getting

b a 2 and then a 3,

c a 2 and a 3 in any order,

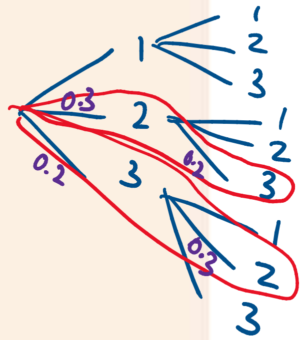
d the same number twice,

e a total of 4 when the two numbers obtained are added together,

f a 2 on the second spin given that the two spins give a total of 4.

4)

Solution



0.3×0.2 ✓

a The probabilities must sum to 1.

$$\therefore \quad 0.5 + 0.3 + k = 1$$

$$\text{Thus} \quad k = 0.2$$ ✓

b

$$\begin{aligned} P(2 \text{ then } 3) &= P(2 \text{ on 1st spin}) \times P(3 \text{ on 2nd spin} \mid 2 \text{ on 1st}) \\ &= 0.3 \times 0.2 \\ &= 0.06 \end{aligned}$$ ✓

c

$$\begin{aligned} P(2 \text{ and } 3 \text{ in any order}) &= P(2 \text{ then } 3) + P(3 \text{ then } 2) \\ &= 0.3 \times 0.2 + 0.2 \times 0.3 \\ &= 0.12 \end{aligned}$$ ✓

d

$$\begin{aligned} P(\text{same number twice}) &= P(1 \text{ then } 1) + P(2 \text{ then } 2) + P(3 \text{ then } 3) \\ &= 0.5 \times 0.5 + 0.3 \times 0.3 + 0.2 \times 0.2 \\ &= 0.38 \end{aligned}$$ ✓

e

$$\begin{aligned} P(\text{total of } 4) &= P(1 \text{ then } 3) + P(3 \text{ then } 1) + P(2 \text{ then } 2) \\ &= 0.5 \times 0.2 + 0.2 \times 0.5 + 0.3 \times 0.3 \\ &= 0.29 \end{aligned}$$ ✓

f

$$\begin{aligned} P(2 \text{ second} \mid \text{total of } 4) &= \frac{P(2 \text{ then } 2)}{P(1 \text{ then } 3) + P(3 \text{ then } 1) + P(2 \text{ then } 2)} \\ &= \frac{0.09}{0.29} \\ &= \frac{9}{29} \end{aligned}$$ ✓

EXAMPLE 4

A batch of 100 components include 5 that are faulty. Four components are randomly selected from the batch without replacement. If X is the number of faulty components in the selection determine the probability distribution for X.

Solution

Using ✓ for not faulty and ✗ for faulty:

$$\begin{aligned}
P(X=0) &= P(✓✓✓✓) \\
&= \frac{95}{100}\times\frac{94}{99}\times\frac{93}{98}\times\frac{92}{97} \\
&\approx 0.811\,875
\end{aligned}$$

$$\begin{aligned}
P(X=1) &= P(✓✓✓✗)+P(✓✓✗✓)+P(✓✗✓✓)+P(✗✓✓✓) \\
&= \frac{95}{100}\times\frac{94}{99}\times\frac{93}{98}\times\frac{5}{97}\times 4 \\
&\approx 0.176\,495
\end{aligned}$$

$$\begin{aligned}
P(X=2) &= P(✓✓✗✗)+P(✓✗✓✗)+P(✓✗✗✓)+P(✗✓✓✗)+P(✗✓✗✓)+P(✗✗✓✓) \\
&= \frac{95}{100}\times\frac{94}{99}\times\frac{5}{98}\times\frac{4}{97}\times 6 \\
&\approx 0.011\,387
\end{aligned}$$

$$\begin{aligned}
P(X=3) &= P(✓✗✗✗)+P(✗✓✗✗)+P(✗✗✓✗)+P(✗✗✗✓) \\
&= \frac{95}{100}\times\frac{5}{99}\times\frac{4}{98}\times\frac{3}{97}\times 4 \\
&\approx 0.000\,242
\end{aligned}$$

$$\begin{aligned}
P(X=4) &= P(✗✗✗✗) \\
&= \frac{5}{100}\times\frac{4}{99}\times\frac{3}{98}\times\frac{2}{97} \\
&\approx 0.000\,001
\end{aligned}$$

These probabilities are shown tabulated below:

x	0	1	2	3	4
P($X=x$)	0.811 875	0.176 495	0.011 387	0.000 242	0.000 001

Expected Value (mean), Variance, Standard Deviation

Review: $\sum P(X = x) = 1$

$$0 \leq P(X = x) \leq 1$$

- The expected value (mean) of a Discrete Random Variable X is found by summing the products of X and the probability of X:

$$\mu = \underbrace{E(X)}_{\text{Expected value of } X} = \sum x \cdot P(X = x)$$

$$\underline{\text{OR}} \quad \sum x \cdot p(x)$$

- The variance of a Discrete Random Variable X is the measure of spread of the probability distribution about its mean μ :

$$\sigma^2 = \underbrace{Var(X)}_{\text{Variance of } X} = \sum p(x) \cdot (x - \mu)^2$$

$$\underline{\underline{\text{OR}}} = E[(X - \mu)^2]$$

$$\underline{\underline{\text{OR}}} = E(X^2) - \mu^2$$

$$\underline{\underline{\text{OR}}} = E(X^2) - [E(X)]^2$$

- The standard deviation of a Discrete Random Variable X is:

$$\sigma = \underbrace{SD(X)}_{\text{Standard deviation of } X} = \sqrt{Var(X)}$$

$$\boxed{\sigma = SD(X) = \sqrt{Var(X)}}$$

Example:

The discrete random variable X can take the values 1, 2, 3 and 4 with the probability distribution as given in the table at right.

x	1	2	3	4
P($X = x$)	0.1	0.2	0.4	0.3

a Find E(X), the expected value of X, and Var(X), the variance of X.

$$E(X) = 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.4 + 4 \times 0.3$$
$$= 2.9 \checkmark$$

$$Var(X) = 0.1 \times (1-2.9)^2 + 0.2 \times (2-2.9)^2 + 0.4(3-2.9)^2 + 0.3(4-2.9)^2$$
$$= 0.89$$

$$SD(X) = \sqrt{0.89} = 0.9434 \text{ (to 4 d.p)} \checkmark$$

Ex8B page 171
Q2, 4, 5, 6

- Change of origin is when we add a constant to each data

$$E(X + b) = E(X) + b$$

$$Var(X + b) = Var(X)$$

- Change of scale is when we multiply a constant to each data

$$E(aX) = aE(X)$$

$$Var(aX) = a^2 Var(X)$$

- **SUMMARY**: Expected value and Variance of

$$aX + b$$

(where a and b are constants)

$$\boxed{E(aX + b) = aE(X) + b}$$

$$Var(aX + b) = a^2 Var(X)$$

Example:

The discrete random variable X can take the values 1, 2, 3 and 4 with the probability distribution as given in the table at right.

x	1	2	3	4
$P(X = x)$	0.1	0.2	0.4	0.3

a Find $E(X)$, the expected value of X, and $Var(X)$, the variance of X.

b If $Y = 3X$ find $E(Y)$, the expected value of Y, and $Var(Y)$, the variance of Y.

c If $Z = 3X + 4$ find $E(Z)$, the expected value of Z, and $Var(Z)$, the variance of Z.

a) $E(X) = 2.9$

$Var(X) = 0.89$

b) $E(3X) = 3E(X) = 3 \times 2.9$

$= 8.7$ ✓

$Var(3X) = 3^2 Var(X) = 9 \times 0.89$

$= 8.01$ ✓

c) $E(3X+4) = 3E(X) + 4 = 3 \times 2.9 + 4$

$= 12.7$ ✓

$Var(3X+4) = 3^2 Var(X) = 9 \times 0.89$

$= 8.01$ ✓

Ex8B page 172
Q7, 8, 9

Examples of DRV

1. A coin is biased in favour of heads such that the probability of obtaining a head on a toss is 0.6. The coin is tossed three times, if X is the number of heads obtained on three tosses, find:

a) E(X)
b) Var(X)
c) SD(X)
d) E(3X + 1)
e) Var (3X + 1)

1
1 1
1 2 1
1 3 3 1

Probability Distribution

x	0	1	2	3
$P(X=x)$	0.064	0.288	0.432	0.216

TTT
$0.4 \times 0.4 \times 0.4$

HTT, THT, TTH
$0.6 \times 0.4 \times 0.4$ (x3)

$0.6 \times 0.6 \times 0.4$ (x3)

HHH
$0.6 \times 0.6 \times 0.6$

* From Classpad:

$\bar{x}$	$=1.8$ ← E(X) or μ
Σx	$=1.8$
Σx^2	$=3.96$
σ_x	$=0.8485281$ ← SD(X)

a) $E(X) = (0 \times 0.064) + (1 \times 0.288) + (2 \times 0.432) + (3 \times 0.216)$

$= 1.8$ ✓

b) $Var(X) = E(X^2) - \mu^2$

$1.8^2 = 3.24$

$(0^2 \times 0.064) + (1^2 \times 0.288) + (2^2 \times 0.432) + (3^2 \times 0.216)$

$= 3.96 - 3.24$

$= 0.72$ ✓

c) $SD(X) = \sqrt{0.72}$

$= 0.848$ ✓

d) $E(3X+1) = 3(1.8) + 1 = 6.4$ ✓

e) $Var(3X+1) = 3^2(0.72) = 6.48$ ✓

2. There are 10 computers: four were made in Malaysia and the rest were made in Singapore. 5 computers are randomly chosen without replacement. We can define the random variable X as "number of Malaysian made computers chosen". Show the probability distribution in the following table and find the mean number of Malaysian made computers chosen.

x	$P(X = x)$
0	$\frac{{}^4C_0 \times {}^6C_5}{{}^{10}C_5} = \frac{1}{42}$ ← Choosing 0 malaysian made
1	$\frac{{}^4C_1 \times {}^6C_4}{{}^{10}C_5} = \frac{5}{21}$
2	$\frac{{}^4C_2 \times {}^6C_3}{{}^{10}C_5} = \frac{10}{21}$
3	$\frac{{}^4C_3 \times {}^6C_2}{{}^{10}C_5} = \frac{5}{21}$
4	$\frac{{}^4C_4 \times {}^6C_1}{{}^{10}C_5} = \frac{1}{42}$

Expected Mean = 2 ✓

3. A lucky ticket is sold for $1. You have a 20% chance of winning $2, a 5% chance of winning $11, otherwise you lose. Is this a good game to play?

Let X be the amount of profit from each game

x	-1	1	10
$P(X=x)$	0.75	0.2	0.05

Expected outcome

$E(X) = (-1 \times 0.75) + (1 \times 0.2) + (10 \times 0.05)$

$= -0.05$

$\therefore$ The expectation is to lose 5 cents per game.

4. A game consists of a wheel divided into 12 equal sectors numbered 1 through 12. You place a $2 bet on any number, if your number comes up, you get $15 (ie you win $13).

a) What is the house percentage?
b) How much should you win for the game to be fair?

(ie you win $13).

a) What is the house percentage?

b) How much should you win for the game to be fair?

Let X be the amount won

x	-2	13
$P(X=x)$	$\frac{11}{12}$	$\frac{1}{12}$

a) $E(X) = -2\left(\frac{11}{12}\right) + 13\left(\frac{1}{12}\right)$

$= -0.75$

$\therefore$ You would expect to lose 75 cents on each bet of \$2

$\therefore$ House Percentage $= \frac{0.75}{2} \times 100$

$= 37.5\%$ ✓

b)

x	-2	a ??
$P(X=x)$	$\frac{11}{12}$	$\frac{1}{12}$

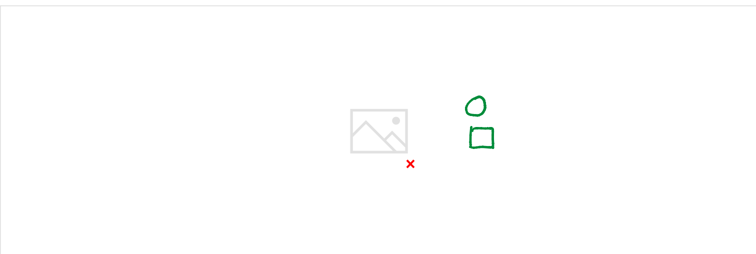
* $E(X) = 0$ for a fair game

$0 = -2\left(\frac{11}{12}\right) + a\left(\frac{1}{12}\right)$

$a = 22$

* you get back \$a + \$2

$\therefore$ You should win \$22, so you should get back \$24 ✓



Die 1

Die 2	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	$\boxed{10}$
5	6	7	8	9	$\boxed{10}$	$\boxed{11}$
6	7	8	9	$\boxed{10}$	$\boxed{11}$	$\boxed{12}$

← all possible outcomes!

Let X be the amount recieved.

x (\$)	0	5	8
$P(X=x)$	$\frac{20}{36}$	$\frac{10}{36}$	$\frac{6}{36}$

$$E(X) = 0\times\left(\frac{20}{36}\right) + 5\left(\frac{10}{36}\right) + 8\left(\frac{6}{36}\right)$$

$$= 2.72$$

Each game costs \$3 to play

$\therefore$ \$2.72 − \$3 = −\$0.28

In the long run, a loss of 28 cents is to be expected.

Ex8B page 171
Q10, 11, 13, 14

Extra Example

A discrete probability distribution is defined by

$P(X=0) = p$ and $P(X=n+1) = (n+1)[P(X=n)]$

where $n \in \{0, 1, 2, 3\}$

a) find p

b) determine: i) $P(X=2)$

ii) $P(X \leq 1 \mid X \leq 2)$

c) Create a cumulative probability: i) table

ii) graph

a) Given $P(X=0) = p$

$n=0 \rightarrow P(X=1) = 1[P(X=0)]$

$= 1p$

$n=1 \rightarrow P(X=2) = 2[P(X=1)]$

$= 2p$

$n=2 \rightarrow P(X=3) = 3[2p]$

$n=2 \rightarrow P(X=3) = 3[2p]$

$= 6p$

$n=3 \rightarrow P(X=4) = 4[6p]$

$= 24p$

$p + p + 2p + 6p + 24p = 1$

$34p = 1$

$\therefore p = \frac{1}{34}$ ✓

x	0	1	2	3	4
$P(X=x)$	p	p	$2p$	$6p$	$24p$
	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
	$\frac{1}{34}$	$\frac{1}{34}$	$\frac{2}{34}$	$\frac{6}{34}$	$\frac{24}{34}$

b) i) $P(X=2)$

$= \frac{2}{34}$ ✓

ii) $P(X \leq 1 \mid X \leq 2)$

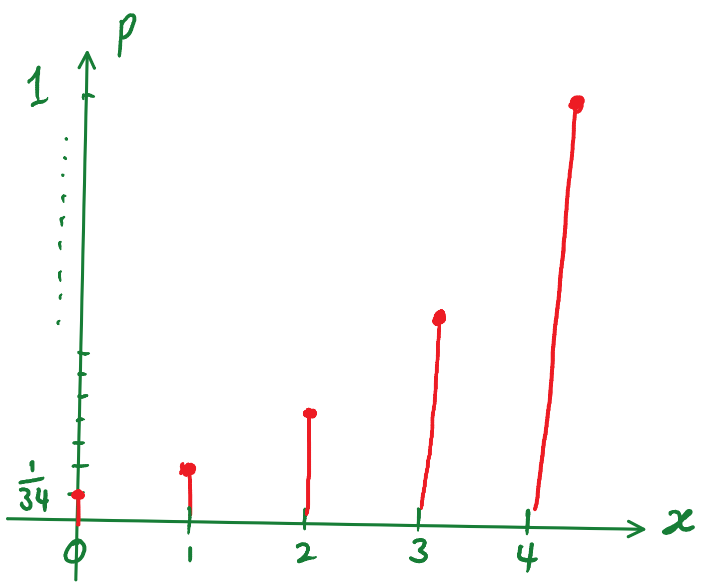
$$= \frac{2p}{4p}$$

$$= \frac{1}{2} \checkmark$$

c) i) Table ← Cumulative probability!

x	0	1	2	3	4
$P(X=x)$	$\frac{1}{34}$	$\frac{1}{17}$	$\frac{2}{17}$	$\frac{5}{17}$	1

ii) Graph



Bernoulli Distribution

- A Bernoulli distribution has two possible outcomes, called "success" and "failure", with a fixed probability of success.
- The probability of "success" is p and "failure" is 1 - p or q such that p + q = 1
- The Bernoulli random variable X is given the values 1 for "success" and 0 for "failure".
- The Bernoulli distribution with parameter p:

x	0	1
$P(X=x)$	$1-p$	p

or q

$$E(X) = \mu = p$$

$$Var(X) = p(1-p) \quad \text{OR} \quad Var(X) = pq$$

$$SD(X) = \sigma = \sqrt{p(1-p)} \quad \text{OR} \quad SD(X) = \sqrt{pq}$$

Example:

What is the standard deviation of the random variable X from the selection of a card from a normal deck to be hearts?

X : 1 for hearts and 0 for other suits

Success (under 1) · failure (under 0)

$$p = \frac{1}{4}$$

$$\sigma = \sqrt{pq}$$

$$= \sqrt{\frac{1}{4}\left(\frac{3}{4}\right)}$$

$$= \frac{\sqrt{3}}{4} \checkmark$$

Binomial Distribution

- The Binomial distribution is the probability distribution of the number of successes from a fixed number of Bernoulli trials. The properties are:
    - A fixed number n of repeated trials
    - Each trial has only two outcomes: Success and failure with probabilities p and q
    - The trials are independent (with replacement)
    - The probability of success does not change for each trial

- If the binomial random variable X is the number of successes from n independent trials, each with probability of success p:

$$\boxed{\begin{aligned} P(X=x) &= {}^{n}C_{x} \cdot p^{x}(1-p)^{n-x} \\ P(X=x) &= \binom{n}{x} p^{x}(1-p)^{n-x} \end{aligned}}$$

- This can be written as:

$$X \sim Bin(n, p)$$

OR

$$X \sim b(n, p)$$

Example:

Find the probability of obtaining three heads when a coin is tossed seven times

X = number of heads obtained.

$$X \sim Bin(\underset{\underset{n}{\uparrow}}{7}, \underset{\underset{p}{\uparrow}}{0.5}) \quad ✓$$

$$P(X = x) = {}^{n}C_{x} \cdot p^{x} (1-p)^{n-x}$$

$$P(X = \underset{\underset{\text{3 heads}}{\uparrow}}{3}) = {}^{7}C_{3} \; (0.5)^{3} \; (1-0.5)^{4}$$

$$= 0.2734$$

* Classpad :

Main → Interactive → distribution → discrete

binomialPDF (x , n , p)

binomialCDF (Lx , Ux , n , p)

↑

$P(X < 3)$

OR $P(X \geq 4)$ etc...

OR $P(X \geqslant 4)$ etc...

Binomial Examples

1. When driving to work, Josh encounters 8 sets of traffic lights. If the probability that Josh has to stop at each light is a constant 0.4, find the probability that he has to stop at:
   a) Exactly six set of lights
   b) At least six set of lights

$$X \sim Bin(8, 0.4)$$

X = Number of times Josh stops at a set of lights

a) $P(X=6) = {}^{8}C_{6}\,(0.4)^{6}(0.6)^{2}$

$= 0.0413$

* Classpad : BinomialPDf (6, 8, 0.4)

b) $P(X \geqslant 6) = P(X=6) + P(X=7) + P(X=8)$

$= 0.0498$

* Classpad : BinomialCDf (6, 8, 8, 0.4)

2. The probability of a netballer scoring a goal is 0.3. Find the probability that out of six attempts the netballer scores a goal:
   a) Four times
   b) Four times, given that she scores at least one goal

$$X \sim Bin(6, 0.3)$$

Let X = the number of goals scored

a) $P(X=4) = {}^{6}C_{4}\,(0.3)^{4}(0.7)^{2}$

a) $P(X=4) = {}^6C_4\,(0.3)^4(0.7)^2$

≈ 0.059535

* Classpad : BinomialPDf (4, 6, 0.3)

b) $P(X=4 \mid X \geqslant 1)$

0.059535 from (a)

$= \dfrac{P(X=4)}{P(X\geqslant 1)}$

$P(X\geqslant 1) = 1 - P(X=0)$

$= 1 - {}^6C_0\,(0.3)^0(0.7)^6$

$= 1 - 0.7^6$

OR BinomialCDf (1, 6, 6, 0.3)

$= \dfrac{0.059535}{0.882351}$

$= 0.0675$ ✓

Ex9A page 188
Q1, 3, 4, 5, 6, 8, 11, 12, 13

Properties of the Binomial Distribution

- Expected value, Variance and standard deviation of a Binomial distribution are given by:

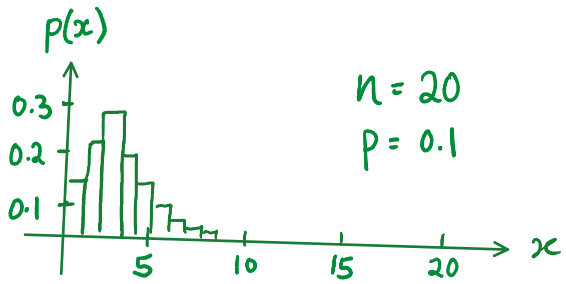
$$E(X) = \mu = np$$

$$Var(X) = np(1-p) \quad \underline{\underline{OR}} \quad Var(X) = npq$$

$$SD(X) = \sigma = \sqrt{np(1-p)} \quad \underline{\underline{OR}} \quad SD(X) = \sqrt{npq}$$

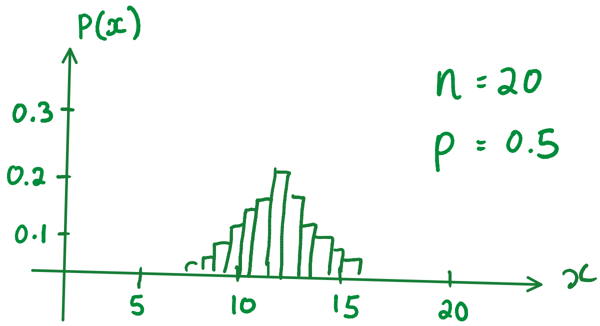
- For $p < 0.5$ the distribution is skewed to the right (positive skew)

eg:



- For $p = 0.5$ the distribution is symmetrical

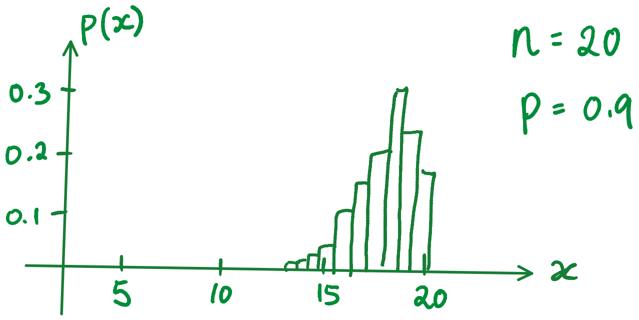
eg:



- For $p > 0.5$ the distribution is skewed to the left (negative skew)

(negative skew)

eg:



*Read more on Sadler page 187

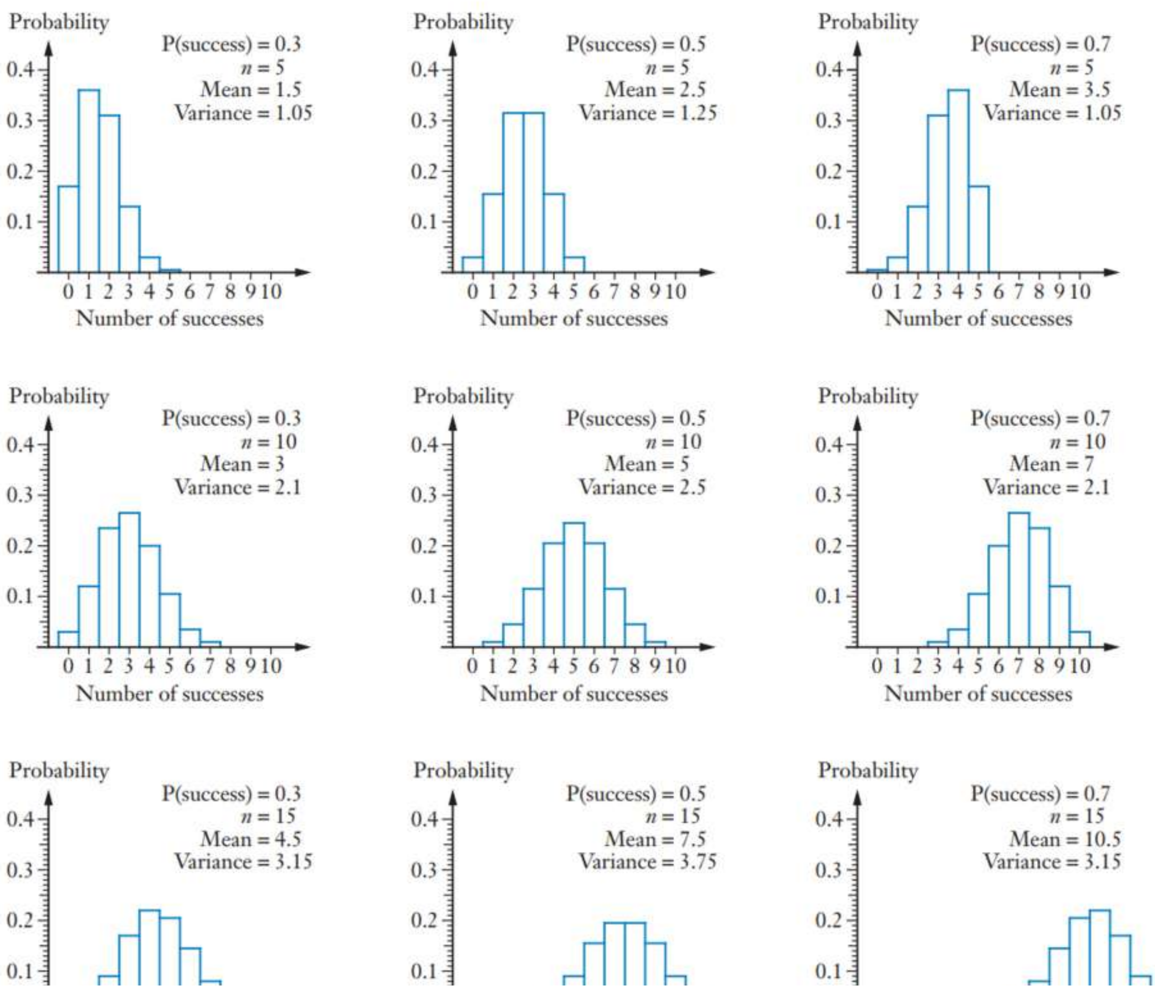
Graphs of binomial distributions

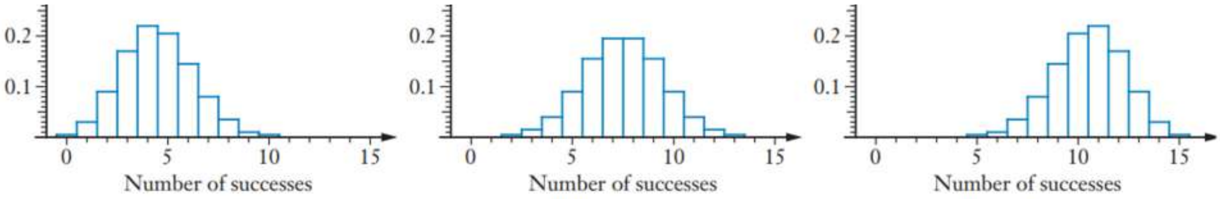
The graphs for the binomial distributions with

$$p = 0.3, \quad p = 0.5 \quad \text{and} \quad p = 0.7$$

are shown below for

$$n = 5, \quad n = 10 \quad \text{and} \quad n = 15$$





Note
- the symmetrical nature of the graphs for which P(success) = 0.5.
- the skewed nature of the graphs for which P(success) ≠ 0.5.
- the graphs of P(success) = k and P(success) = 1 − k are mirror images.
- the graphs for P(success) ≠ 0.5 appear to move towards a more symmetrical distribution as n increases.

More Examples

1. In the general population, 15% of people are left-handed. A class has 24 students:
   a) How many in the class would you expect to be left-handed?
   b) What is the probability that the number of left-handed students in the class is within one standard deviation of the mean?

$X \sim Bin(\underset{n}{\underset{\uparrow}{24}}, \underset{p}{\underset{\uparrow}{0.15}})$

a) $E(X) = np$

$= 24 \times 0.15$

$= 3.6$

$\therefore$ you would expect 3 or 4 left-handers. ✓

b) $SD(X) = \sqrt{np(1-p)}$

$= \sqrt{24 \times 0.15 \times 0.85}$

≈ 1.75

$\mu - \sigma \leq X \leq \mu + \sigma$ ← Values within 1 sd from the mean

$3.6 - 1.75 \leq X \leq 3.6 + 1.75$

$1.85 \leq X \leq 5.35$

$\therefore \quad 2 \leq X \leq 5$

$P(2 \leq X \leq 5) \approx 0.754637$

* classpad: Binomial CDf $(\underset{Lx}{\underset{\uparrow}{2}}, \underset{Ux}{\underset{\uparrow}{5}}, \underset{n}{\underset{\uparrow}{24}}, \underset{p}{\underset{\uparrow}{0.15}})$

$\therefore$ Approximately 75% probability that the number of left-handed students in the class is within one s.d. of the mean. ✓

left-handed students in the class is within one s.d. of the mean. ✓

2. A study found that 38% of fish sold in a particular seafood market contained levels of mercury that were above FSA recommended maximum levels. Food inspectors took a random sample of twelve pieces of fish from the market for testing. Find the probability that:
   a) Five of the pieces have mercury levels above FSA maximum
   b) At most four of the pieces have mercury levels above FSA maximum
   c) At least three of the pieces have mercury levels above FSA maximum

$X \sim Bin(\overset{n}{12}, \overset{p}{0.38})$

a) $P(X=5) = {}^{12}C_5 (0.38)^5 (0.62)^7$

$= 0.220996$

No need this if you use classpad

BinomialPDf (5, 12, 0.3

8)

$\therefore$ The probability that five of the pieces are bad is approximately 22%. ✓

b) $P(X \leq 4) \approx 0.495719$

* classpad : BinomialCDf (0, 4, 12, 0.38)

$\therefore$ The probability of at most 4 bad fish is approx. 49.6% ✓

c) $P(X \geq 3) \approx 0.893056$

* classpad : Binomial CDf (3, 12, 12, 0.38)

$\therefore$ The probability of at least 3 bad fish is approx. 89.3% ✓

3. At a particular telemarketing company, it is known that the probability that a call to a potential client results in a sale is 0.05. What is the least number of calls that must be made to ensure that the probability of marking at least 2 sales is 90%?

$X \sim Bin(n, 0.05)$

$P(X \geq 2) = 0.9$ $\quad n = ?$

X = successful calls

$P(X \geq 2) = 0.9$

$P(X=0) + P(X=1) = 0.1$ ← Complement of $P(X \geq 2)$

$$0.1 = \underbrace{{}^nC_0}_{1} \underbrace{(0.05)^0}_{1} (0.95)^{n-0} + \underbrace{{}^nC_1}_{n} (0.05)^1 (0.95)^{n-1}$$

$$0.1 = (0.95)^n + n(0.05)(0.95)^{n-1}$$

Solve on Classpad!

$n = \cancel{-18.2551039}$, $n = 76.3$

↑ invalid

$\therefore n \approx 76.34$

At least 77

3670305

Calls must be made!

Ex9B page 192
Q4, 6, 10, 11, 12, 15, 18

Sunday, 31 October 2021 11:31 AM

$$X \sim Bin(\underset{\substack{\uparrow \\ n}}{7}, \underset{\substack{\uparrow \\ p}}{0.5}) \quad \checkmark$$

$$P(X = x) = {}^{n}C_{x} \cdot p^{x} (1-p)^{n-x}$$

$$P(X = \underset{\substack{\uparrow \\ \text{3 heads}}}{3}) = {}^{7}C_{3} (0.5)^{3} (1-0.5)^{4}$$

$$= 0.2734$$

Assuming the question is $X \sim B(5, \frac{4}{5})$

$$P(X = x) = {}^{n}C_{x} \cdot p^{x} (1-p)^{n-x}$$

$$P(X = 2) = {}^{5}C_{2} \left(\frac{4}{5}\right)^{2} \left(\frac{1}{5}\right)^{3}$$

$$= 10 \times \frac{4}{5} \times \frac{4}{5} \times \frac{1}{5} \times \frac{1}{5} \times \frac{1}{5}$$

$$= \frac{{}^{2}\cancel{10} \times 16}{\ }$$

$$1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$^5C_0 \quad ^5C_1 \quad ^5C_2 \quad ^5C_3 \quad ^5C_4 \quad ^5C_5$$

$$= \frac{{}^{2}\cancel{10} \times 16}{\cancel{5} \times 5^4}$$

$$= \frac{32}{5^4} = \frac{32}{625}$$

mean!

$$e^{\ln 4} - 4e^{\ln 2}$$

$$= 4 - 4(2)$$

because $\boxed{e^{\ln x} = x}$

Explanation:

Let $y = e^{\ln x}$

$\ln y = \ln e^{\ln x}$ (ln both sides)

$\ln y = \ln x \cdot \cancel{\ln e}$

$\ln e = 1$ ☺

$y = x$

$\therefore e^{\ln x} = x$

So just have to remember $e^{\ln x} = x$

think of it as $e^{\ln}$ cancels out like $\ln e$

Logarithms

- The logarithm of b to the base a is the exponent x such that $a^x = b$ (where $a, b > 0$ and $a \neq 1$)

$$a^x = b \iff \log_a b = x$$

- A common logarithm is in base 10 and is normally written without the base.

Examples: Find

a) $\log_2 16$

let $\log_2 16 = x$

$2^x = 16$

$x = 4$

$\therefore \log_2 16 = 4$ ✓

b) $\log_4 8 = x$

$4^x = 8$

$2^{\boxed{2x}} = 2^{\boxed{3}}$

$2x = 3$

$x = \frac{3}{2}$

$\log_4 8 = \frac{3}{2}$ ✓

c) $\log_5 0.2$

d) $\log_{10}\left(\frac{1}{10}\right) = x$

c) $\log_5 0.2$

let $\log_5\left(\frac{1}{5}\right) = x$

$$5^x = \frac{1}{5}$$

$$5^x = 5^{-1}$$

$$x = -1$$

$\therefore \log_5 0.2 = -1$ ✓

d) $\log_{10} \ldots$

$$10^x = \frac{1}{10}$$

$$10^x = 10^{-1}$$

$$x = -1$$

$\therefore \log\left(\frac{1}{10}\right) = -1$ ✓

e) $\log_{\frac{1}{5}} 25 = x$

$$\left(\frac{1}{5}\right)^x = 25$$

$$5^{-x} = 5^{2}$$

$$-x = 2$$

$$x = -2$$

$\therefore \log_{\frac{1}{5}} 25 = -2$

f) $\log_{\frac{1}{2}}\left(\frac{1}{32}\right) = x$

$$\left(\frac{1}{2}\right)^x = \frac{1}{32}$$

$$2^{-x} = 2^{-5}$$

$$x = 5$$

$\therefore \log_{\frac{1}{2}}\left(\frac{1}{32}\right) = 5$ ✓

g) $\log_{96} 1$

$$\log_{96} 1 = x$$

$96^x = 1 \leftarrow$ anything to power of 0 $= 1$:)

$$x = 0$$

$x = 0$

$\therefore \log_{96} 1 = 0$ ✓

Ex1A page 8
Right side every other question

Laws of Logarithms

Law 1: $\log_a(bc) = \log_a b + \log_a c$

Law 2: $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$

Law 3: $\log_a(b^n) = n\log_a b$

Law 4: $\log_a\left(\frac{1}{b}\right) = -\log_a b$

Law 5: $\log_a a = 1 \leftarrow (a^1 = a)$

$\log_a 1 = 0 \leftarrow (a^0 = 1)$

- Logarithms to the base e are Natural logs

$\log_e x$ can be written as $\ln x$

ie: $\log x \rightarrow \log_{10} x$

$\ln x \rightarrow \log_e x$

- Change of base formula:

$$\log_a b = \frac{\log_c b}{\log_c a}$$

eg: $\log_2 7 = \dfrac{\ln 7}{\ln 2}$

Examples

1) *Simplify / Evaluate*

a) $\log_2 12 + \log_2 36 - 3\log_2 3$

$= \log_2 12 + \log_2 36 - \log_2 3^3$

$= \log_2 \left(\frac{12 \times 36}{27} \right)$

$= \log_2 16$

$= \log_2 2^4$

$= 4 \cancel{\log_2 2}$

$= 4$ ✓

b) $\log_6 18 + \log_6 12 - 2$

$= \log_6 (18 \times 12) - 2$

$= \log_6 (6 \times \underline{3 \times 2} \times 6) - 2$

$= \log_6 6^3 - 2$

$= 3 \cancel{\log_6 6} - 2$

$= 3 - 2$

$= 1$ ✓

2) *Express as a single logarithm.*

a) $3\log_2 x - 4\log_2 (x+3) + \log_2 y$

$= \log_2 x^3 - \log_2 (x+3)^4 + \log_2 y$

$= \log_2 \left(\frac{x^3 y}{(x+3)^4} \right)$ ✓

b) $2\log_5 \sqrt{x} - \log_5 5x$

$= \log_5 (\sqrt{x})^2 - (\log_5 5 + \log_5 x)$

$= \cancel{\log_5 x} - \log_5 5 \cancel{- \log_5 x}$

$= -1$ ✓

3) *If* $\log_a 4 = p$ *and* $\log_a 5 = q$, *express in terms of* p *and* q:

a) $\log_a 20$

$= \log_a (4 \times 5)$

$= \log_a 4 + \log_a 5$

$= p + q$ ✓

b) $\log_a 0.8$

$= \log_a \left(\frac{4}{5} \right)$

$= \log_a 4 - \log_a 5$

$= p - q$

c) $\log_a 100a^2$

$= \log_a 100 + \log_a a^2$

$= \log_a (4 \times 5^2) + 2 \cancel{\log_a a}$

$= \log_a 4 + 2\log_a 5 + 2$

$= p + 2q + 2$ ✓

4) *Solve for* x:

4) Solve for x:

a) $\log_2(2x-1) = 4$

$$2^4 = 2x - 1$$
$$16 = 2x - 1$$
$$17 = 2x$$
$$x = \frac{17}{2} \checkmark$$

b) $\log_x 27 = \frac{3}{2}$

$$x^{\frac{3}{2}} = 27$$
$$(\sqrt{x})^{\cancel{3}} = 3^{\cancel{3}}$$
$$\sqrt{x} = 3$$
$$x = 9$$

c) $\ln(2x+1) - \ln(x-1) = 4$

$$\log_e\left(\frac{2x+1}{x-1}\right) = 4$$
$$e^4 = \frac{2x+1}{x-1}$$
$$e^4(x-1) = 2x+1$$
$$xe^4 - e^4 = 2x + 1$$
$$xe^4 - 2x = e^4 + 1$$
$$x(e^4 - 2) = e^4 + 1$$
$$x = \frac{e^4+1}{e^4-2}$$

d) $\ln(x-1) + \ln(x+1) = 1$

$$\log_e(x-1)(x+1) = 1$$
$$\log_e(x^2-1) = 1$$
$$e^1 = x^2 - 1$$
$$x^2 = e + 1$$
$$x = \pm\sqrt{e+1} \checkmark$$

5) Solve for x:

a) $\log(x-1) + \log(x+2) = \log(6x-8)$

$$\cancel{\log}(x-1)(x+2) = \cancel{\log}(6x-8)$$
$$x^2 + x - 2 = 6x - 8$$

$$x^2 - 5x + 6 = 0$$

$$(x-3)(x-2) = 0$$

$$x = 3 \qquad x = 2 \quad \checkmark$$

b) $\log_2 x - \log_2 (7-2x) = \log_2 6$

$$\cancel{\log_2} \frac{x}{7-2x} = \cancel{\log_2} 6$$

$$x = 6(7-2x)$$

$$x = 42 - 12x$$

$$13x = 42$$

$$x = \frac{42}{13} \quad \checkmark$$

c) $\log_2 6 = x \log_2 3 + 1$

$$\log_2 6 = \log_2 3^x + \log_2 2$$

$$\cancel{\log_2} 6 = \cancel{\log_2} (3^x \times 2)$$

$$6 = 3^x \times 2$$

$$3 = 3^x$$

$$x = 1 \quad \checkmark$$

Ex1B page 11
Q2 to Q30 (right side, every other question)

Solving with Logs

Examples: Solve for x

a) $3^{2x-1} = 28$

$2x - 1 = \log_3 28$

$2x = \log_3 28 + 1$

$x = \dfrac{\log_3 28 + 1}{2}$ ✓

b) $3^{x+1} = 5^{x-4}$

$\log 3^{x+1} = \log 5^{x-4}$

$(x+1)\log 3 = (x-4)\log 5$

$x\log 3 + \log 3 = x\log 5 - 4\log 5$

$\log 3 + 4\log 5 = x\log 5 - x\log 3$

$x(\log 5 - \log 3) = \log 3 + 4\log 5$

$x = \dfrac{\log 3 + 4\log 5}{\log 5 - \log 3}$ ✓

c) $4^x = 2^{x+1} + 3$

$2^{2x} = 2^{x+1} + 3$

$(2^x)^2 = 2^1 \times 2^x + 3$

Let $a = 2^x$

$a^2 = 2a + 3$

$a^2 - 2a - 3 = 0$

$(a-3)(a+1) = 0$

$a = 3$ or $a = -1$

$2^x = 3$ or $\cancel{2^x = -1}$ invalid

$2^x = 3$

$x = \log_2 3$ ✓

d) $e^{x+1} = 5$

$\log_e 5 = x + 1$

$\ln 5 = x + 1$

$x = \ln 5 - 1$ ✓

e) Find the inverse of the function $f(x) = e^{x} + 2$

Consider $x = e^{y} + 2$

$x - 2 = e^{y}$

$\therefore \log_e(x-2) = y$

$\therefore$ The inverse function : $f^{-1}(x) = \ln(x-2)$ ✓

f) Write the equation $y = 2\log_e x + 3$ with x as the subject.

$$y = 2\log_e x + 3$$

$$y - 3 = 2\log_e x$$

$$\frac{y-3}{2} = \log_e x$$

$$x = e^{\frac{y-3}{2}}$$ ✓

Ex1C

Q12, 15, 17, 19, 20

More Examples (Change of Base)

1) Simplify (Answer in log base 10)

$\log_5 0.6$

$= \log_5 \frac{3}{5}$

$= \log_5 3 - \cancel{\log_5 5}^{\,1}$

$= \frac{\log 3}{\log 5} - 1$ ✓

2) Express in terms of natural logs and prime numbers

$\downarrow$

$\log_e$ or $\ln$

$\log_6 9$

$= \frac{\ln 9}{\ln 6}$

$= \frac{\ln 3^2}{\ln(2\times 3)}$

$= \frac{2\ln 3}{\ln 2 + \ln 3}$ ✓

More Examples 2

Solve for x the following equations, giving answers in exact form.

a) $\log(1-x) = -1 + \log(2x+1)$

b) $e^{2x} - e^{x-2} = 0$

c) $3e^{2x} - 1 = 5$

d) $2\ln x = 4 - \ln 9$

a) $\log(1-x) - \log(2x+1) = -1$

$\log_{10}\left(\frac{1-x}{2x+1}\right) = -1$

$10^{-1} = \frac{1-x}{2x+1}$

$\frac{1}{10} = \frac{1-x}{2x+1}$

$2x+1 = 10(1-x)$

$2x+1 = 10 - 10x$

$12x = 9$

$x = \frac{9}{12} = \frac{3}{4}$ ✓

d) $2\ln x = 4 - \ln 9$

$\ln x^2 + \ln 9 = 4$

$\ln(9x^2) = 4$

$\ln(3x)^2 = 4$

$2\ln(3x) = 4$

$\log_e(3x) = 2$

b) $e^{2x} - e^{x-2} = 0$

$\cancel{e}^{2x} = \cancel{e}^{x-2}$

$2x = x - 2$

$x = -2$ ✓

c) $3e^{2x} - 1 = 5$

$3e^{2x} = 6$

$e^{2x} = 2$

$\ln e^{2x} = \ln 2$

$2x \underbrace{\cancel{\ln e}}_{}^{1} = \ln 2$

$2x = \ln 2$

$x = \frac{\ln 2}{2}$ ✓

$$\log_e(3x) = 2$$

$$e^2 = 3x$$

$$x = \frac{e^2}{3}$$ ✓

Logarithms Application Examples

1. A type of bacteria cell divides into two new cells every T_D minutes (where T_D is the "generation time").
   Let N_o be the initial number of cells, after t minutes, the number of cells N is given by $N = N_o\ 2^{\frac{t}{TD}}$

   T_D ?

   What is the generation time of a bacteria population that increases from 5000 cells to 100 000 cells in four hours of growth?

   N $\quad t = 240 \quad N_0$

$$100\,000 = 5000 \times 2^{\frac{240}{T_D}}$$

$$20 = 2^{\frac{240}{T_D}}$$

$$\log_2 20 = \frac{240}{T_D}$$

$$T_D = \frac{240}{\log_2 20}$$ ← Non Calc section leave it here

$$T_D \approx 55.53 \text{ min}$$ ✓

2. After 1000 years, a sample of radium has decayed to 64.7% of its original mass. Find the half-life of the radium.

Formula: $A = A_0 e^{-kt}$

When $t = 1000 \longrightarrow A = 0.647 A_0$

$$0.647 \cancel{A_0} = \cancel{A_0} e^{-1000k}$$

$$0.647 = e^{-1000k}$$

$$\log_e 0.647 = -1000k$$

$$k = \frac{\ln 0.647}{-1000}$$

$$\therefore k = 0.000435$$ ✓

Find k first

$$A = A_0 e^{-0.000435t}$$ ← Complete equation :)

Half life is when $A = 0.5 A_0$

$e^{-0.000435t}$

Half life is when $A = 0.5A_0$

$$0.5\cancel{A_0} = \cancel{A_0}e^{-0.000435t}$$

$$0.5 = e^{-0.000435t}$$

$$\log_e 0.5 = -0.000435t$$

$$t = -\frac{\ln 0.5}{0.000435}$$

$$\therefore t \approx 1591.95$$

find t

$\therefore$ The half life of the radium is approx. 1592 years.

P_0

Let P = population at t years

$$P = P_0 e^{kt}$$ ($P_0 = 8000 \leftarrow$ given)

$$P = 8000e^{kt}$$

At the end of 2014, $t = 8$, $P = 15000$

$$15000 = 8000e^{8k}$$

$$\frac{15}{8} = e^{8k}$$

$$\log_e \frac{15}{8} = 8k$$

$$k = \frac{1}{8}\log_e \frac{15}{8}$$

$\therefore k \approx 0.079 \leftarrow$ rate of increase is 7.9% per year.

a) End of 2016 $\rightarrow t = 10$

$10(0.079)$

a) End of 2016 $\rightarrow$ $t = 10$

$$P = 8000e^{10(0.079)}$$

$$P \approx 17552.605$$

$\therefore$ population is approx. 17750 ✓

b) $P = 30000$

$$30000 = 8000e^{0.079t}$$

$$\frac{30}{8} = e^{0.079t}$$

$$\log_e \frac{15}{4} = 0.079t$$

$$t = \frac{\log_e \frac{15}{4}}{0.079}$$

$$\therefore t \approx 16.82$$ ✓

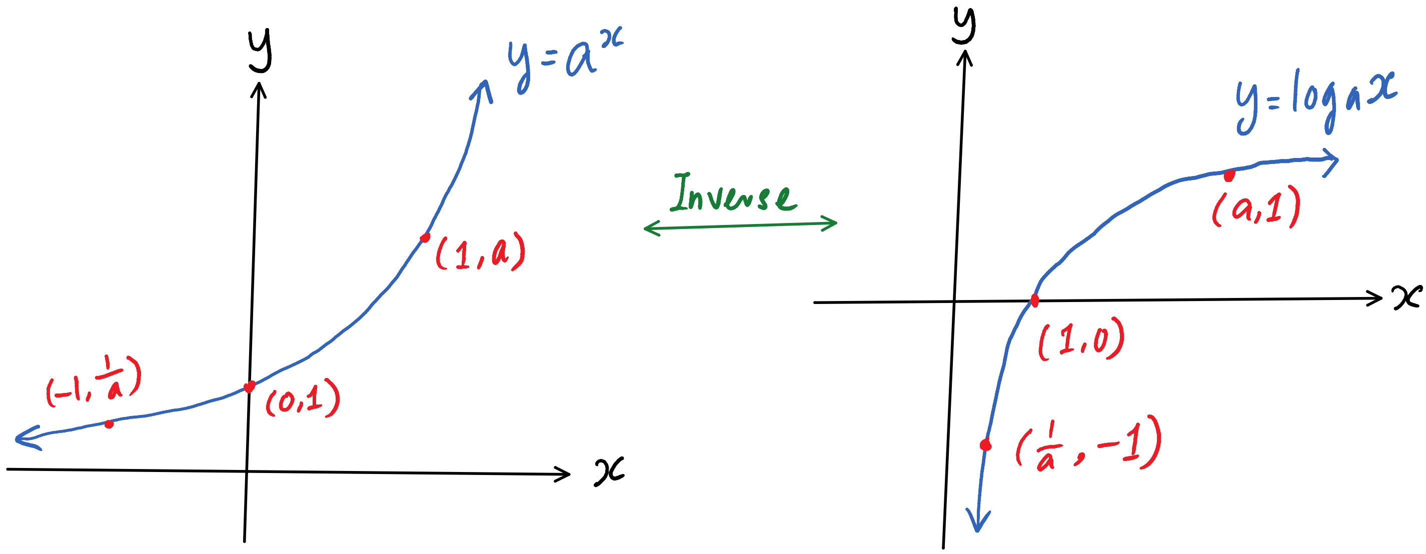
$\therefore$ year 2023 ✓

Ex1B
Q31, 32
Ex1C
Q23, 25, 27
Ex1D
Q6, 7, 12, 14, 23, 26

Logarithmic Functions

- The graph of logarithmic function $f(x) = \log_a x$ has the following features:
  - → $f(1) = 0$
  - → $f(a) = 1$ and $f\left(\frac{1}{a}\right) = -1$
  - → The y-axis is the vertical asymptote.
  - → A logarithmic function with $a > 1$ is an increasing function:

$$\left.\begin{array}{l} \text{as } x \to \infty, \quad y \to \infty \\ \text{as } x \to 0^+, \quad y \to -\infty \end{array}\right\} \text{right hand end behaviour.}$$



→ Transformation:

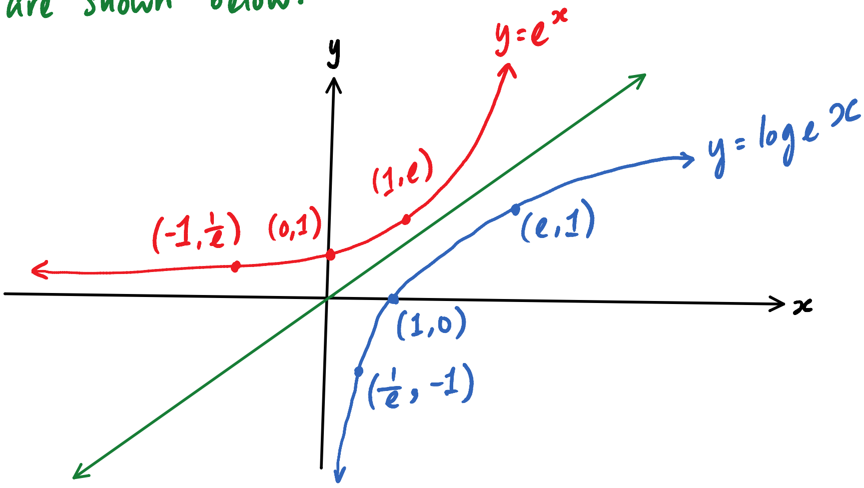
$$y = -\log_a(x + b) + c$$

Reflection in the x-axis

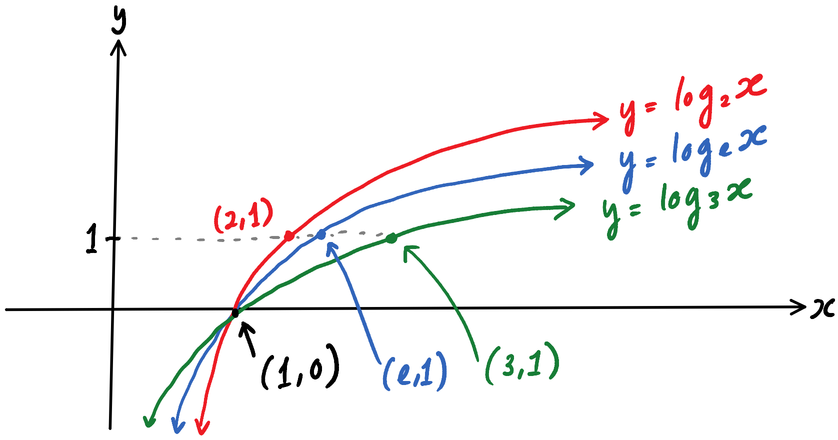
Horizontal Translation
$+b$ left
$-b$ right
Asymptote: $x = -b$
x-int: $(1-b, 0)$

Vertical Translation
$+c$ up
$-c$ down
Asymptote: y-axis
x-int: $(a^{-c}, 0)$

eg 1: The graph of $y = e^x$ and its inverse function $y = \log_e x$ are shown below:



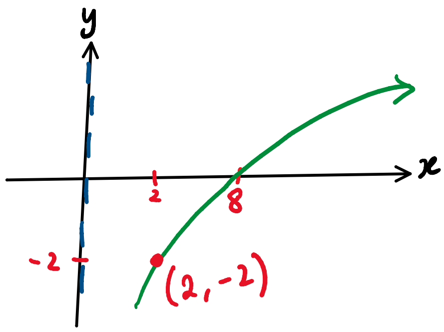
eg 2: The graphs of $y = \log_2 x$, $y = \log_e x$ and $y = \log_3 x$ are shown below:



Log Functions Examples

Examples: sketch the following graphs, labelling important features:

a) $y = \log_2 x - 3$



→ $\log_2 x$ is translated 3 units down

→ x-int: $(2^3, 0) = (8, 0)$

→ Vertical Asymptote: y-axis $(x = 0)$

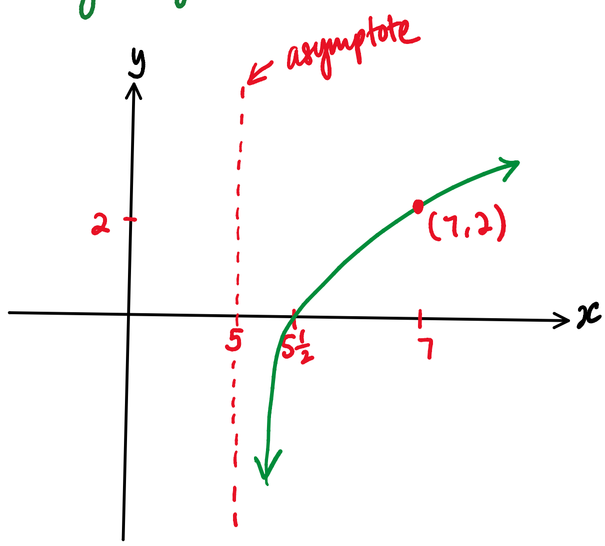
→ $(2, -2)$ is a point on the graph

↑

$y = \log_2 2 - 3$

$y = -2$ ✓

b) $y = \log_2(x - 5) + 1$



→ $\log_2 x$ is translated 5 right and 1 up

→ Vertical Asymptote at $x = 5$

→ x-int when $y = 0$

$\log_2(x-5) = -1$

$x - 5 = 2^{-1}$

$x = 5 + \frac{1}{2}$

$x = 5\frac{1}{2}$

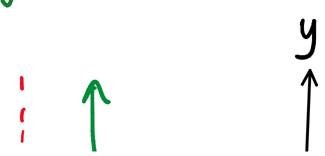
→ $(7, 2)$ lies on the graph

↑

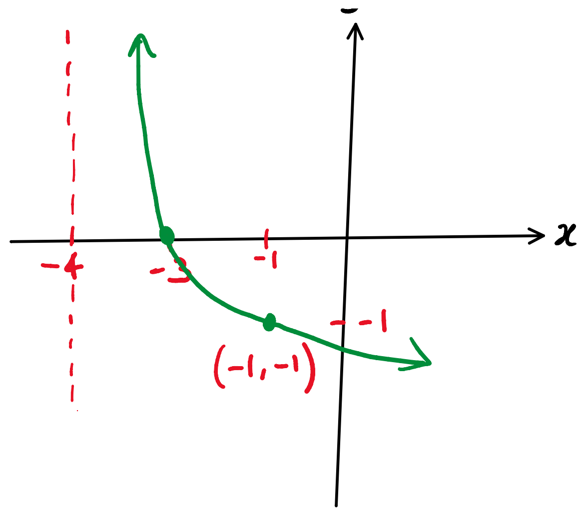
$y = \log_2(7-5) + 1$

$y = 1 + 1 = 2$

c) $y = -\log_3(x + 4)$



→ $\log_3 x$ is reflected on the x-axis and translated 4 units left



and translated 4 units left

→ Vertical Asymptote at $x = -4$

→ x-int $(1-4, 0) = (-3, 0)$

→ $(-1, -1)$ lies on the graph

$y = -\log_3(-1+4)$

$y = -1$ ✓

Ex1E page 20
Q1-6

Logarithmic Scale

The "multiplication effect" of a logarithmic scale can be used in a number of physical applications, such as:

1. The Richter Scale
   A seismograph measures vibrations from an earthquake graphically. (The base 10 logarithm means that each unit increases on the Richter scale means the amplitude of the vibrations is multiplied by ten).

Magnitude $\boxed{M(x) = \log\left(\frac{x}{x_0}\right)}$

where x_0 = magnitude of an earthquake with a seismograph reading of 0.001 mm at a distance of 100km from the epicentre.

2. The pH Scale
   The pH scale is a measure of the acidity and alkalinity of a solution. A pH of 7 is regarded neutral. (A solution with pH of 3 is ten times as acidic as a solution with pH of 4. A pH 10 is one hundred times a alkaline as pH 8).

Potential of Hydrogen, $\boxed{pH = -\log(H^+)}$

Where H^+ = the concentration of hydrogen ions moles/litre

3. Scale of Loudness/Intensity of Sound
   The decibel (dB) scale measures loudness and is based on multiples of ten. To compare relative sound intensities.

$$\boxed{S = 10\log\left(\frac{I}{I_0}\right)}$$

where I_0 is the threshold of human hearing (intensity of 1×10^{-12} watts/m^2) ← or 0dB

4. Music Scale
   In the scale of musical notes, each note is double the frequency of the note of the same name in the octave below. (There are 12 notes in an octave, this is called a chromatic scale).

$$\frac{f_2}{f_1} = 2^x \rightarrow \log \frac{f_2}{f_1} = x \log 2$$

No. of octaves, $\boxed{x = \frac{1}{\log 2} \times \log \frac{f_2}{f_1}}$

where x octaves between a note of f_1 Hertz (Hz) and one of f_2 Hz

Formula: $M(x) = \log\left(\frac{x}{x_0}\right)$

$x = 7943$ (converted to mm)

$x_0 = 0.001$ mm

$$M(x) = \log\left(\frac{7943}{0.001}\right)$$

$$= 6.9$$

a) $pH = -\log(H^+)$

$= -\log_{10}(10^{-7})$

$= 7 \cancel{\log_{10} 10}$

b) $pH = -\log(0.1)$

$= -\log(10^{-1})$

$= 1 \cancel{\log_{10} 10}$

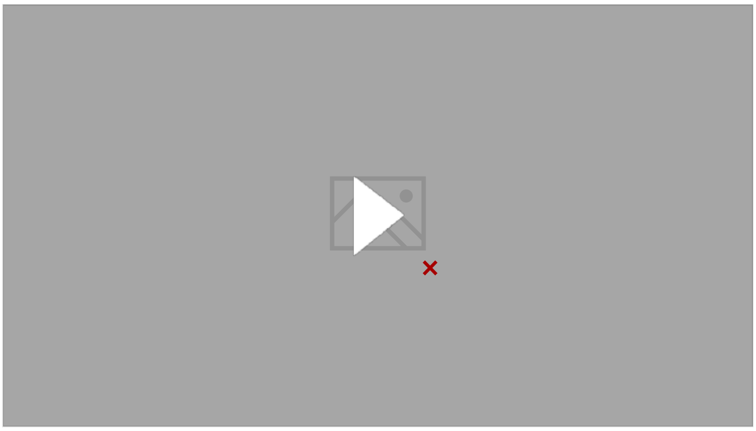
$= 7 \cancel{\log_{10} 10}$

$= 7$ ✓

$= 1 \cancel{\log_{10} 10}$

$= 1$ ✓

[Logarithmic scale | Logarithms | Algebra II | Khan Academy]



Ex1F
Q2, 3

Worksheet

MM4-1F Log Scales (wit...

Differentiating the Natural Logarithm

The natural log is given by $\ln x = \log_e x$.

Let $y = \ln x \rightarrow x = e^y$

$$\frac{dx}{dy} = e^y$$

$$\frac{dy}{dx} = \frac{1}{e^y}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x} \checkmark$$

$y = \ln x \rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$	← OR $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$
$y = \ln f(x) \rightarrow \dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$	← Chain Rule
$y = \log_a x \rightarrow \dfrac{dy}{dx} = \dfrac{1}{x \ln a}$	

Examples: Differentiate

1) $y = 5 \ln x$

$$\frac{dy}{dx} = \frac{5}{x} \checkmark$$

OR $y' = \dfrac{5}{x}$

2) $y = \log_e 5x$

$y = \ln (5x)$

$y' = \dfrac{1}{5x} \times 5$

Chain

$= \dfrac{5}{5x}$

$= \dfrac{1}{x} \checkmark$

3) $y = \ln (\underbrace{x^2+3}_{f(x)})$

$f(x)$ $\quad = \frac{\quad}{x}$ ✓

$$y' = \frac{f'(x)}{f(x)}$$

$$= \frac{2x}{x^2+3}$$ ✓

4) $y = (\log_e x)^2$

$$y' = 2(\ln x) \times \frac{1}{x}$$

$$= \frac{2\ln x}{x}$$ ✓

5) $y = \ln(2x-5)^2$

$$y' = \frac{2(2x-5)\times(2)}{(2x-5)^2}$$

$$= \frac{4\cancel{(2x-5)}}{(2x-5)^{\cancel{2}}}$$

$$= \frac{4}{2x-5}$$ ✓

6) $y = x^2 \ln x$

$u = x^2 \qquad v = \ln x$

$u' = 2x \qquad v' = \frac{1}{x}$

7) $y = \log_7 x$

Change of base:

$y = \frac{\log_e x}{\quad}$

$u' = 2x \quad v' = \frac{1}{x}$

$$\frac{dy}{dx} = 2x \ln x + \frac{x^2}{x}$$

$$= 2x \ln x + x$$

$$y = \frac{\log_e x}{\log_e 7}$$

$$y = \frac{1}{\ln 7} \times \ln x$$

$$y' = \frac{1}{\ln 7} \times \frac{1}{x}$$

$$y' = \frac{1}{x \ln 7}$$

OR use formula

Ex2A page 32
Q3-39 (right column)
Q40, 41, 42

Integration to give Logarithm Functions

$$y = \ln x \rightarrow \frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \boxed{\int \frac{1}{x}\,dx = \ln x + c} \leftarrow \text{for } x > 0$$

$$y = \ln f(x) \rightarrow \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

$$\therefore \boxed{\int \frac{f'(x)}{f(x)}\,dx = \ln f(x) + c} \leftarrow \text{for } f(x) > 0$$

* Any algebraic fraction for which the numerator is the derivative of the denominator will integrate to give a natural log function.

Examples:

1) $\int \frac{3}{x}\,dx$

$= 3\int \frac{1}{x}\,dx$

$= 3\ln x + c$

2) $\int \frac{2}{5x}\,dx$

$= \frac{2}{5}\int \frac{1}{x}\,dx$

$= \frac{2}{5}\ln x + c$

3) $\int \frac{2x-3}{x^2-3x+5}\,dx$ $\leftarrow f'(x)$

always check first!!!

3) $\displaystyle\int \frac{\quad}{\underbrace{x^2-3x+5}_{f(x)}}\,dx$ first !!!

$$= \ln(x^2-3x+5)+c$$

4) $\displaystyle\int \frac{15x^2}{x^3+1}\,dx$

$$= \int \frac{5 \times 3x^2}{x^3+1}\,dx$$

$$= 5\int \frac{\overbrace{3x^2}^{f'(x)}}{\underbrace{x^3+1}_{f(x)}}\,dx$$

$$= 5\ln(x^3+1)+c \checkmark$$

5) $\displaystyle\int \frac{\sin x}{\cos x}\,dx$

$$= -\int \frac{-\sin x}{\cos x}\,dx$$

$$= -\ln\cos x + c \checkmark$$

6) $\displaystyle\int \tan 5x\,dx$

$$= \int \frac{\sin 5x}{\cos 5x}\,dx$$

$* \frac{d}{dx}\cos 5x = -\sin 5x \times 5$

$= -5\sin 5x$

$$= -\frac{1}{5}\int \frac{-5\sin 5x}{\cos 5x}\,dx$$

$$= -\frac{1}{5}\ln\cos 5x + c \checkmark$$

7) $\displaystyle\int_4^5 \frac{1}{2x-5}\,dx$

$$= \frac{1}{2}\int_4^5 \frac{2}{2x-5}\,dx$$

$$= \frac{1}{2}\Big[\ln(2x-5)\Big]_4^5$$

$$= \frac{1}{2}(\ln 5 - \ln 3)$$

$$= \frac{1}{2}\ln\left(\frac{5}{3}\right) \checkmark$$

Ex2B page 37
Q1-23 (left column)
Q27, 29, 31

Continuous Random Variables

- Continuous Random Variables commonly occur when we are measuring Eg: height, weight, time etc...

- If X is a CRV that can take any value, x, in an interval, the probability of X taking an exact value is negligible ie: $P(X=x) = 0$

- We use "relative frequency" to determine the probability of a randomly selected range of data.

Example:

The length of 50 babies born in a hospital are recorded:

$48 < x \leq 49 \quad 49 < x \leq 50$
$48 \leq x < 49 \quad 49 \leq x < 50$ } does not matter!

Length (cm)	48-49	49-50	50-51	51-52	52-53	53-54
No. of babies	1	8	19	14	6	2

If X is the length of a randomly selected baby, determine:

a) $P(51 \leq X < 52) = \frac{14}{50} = 0.28$ ✓

b) $P(X < 50) = \frac{9}{50}$ ✓

c) $P(X \geq 49 \mid X < 51) = \frac{27}{28}$ ✓

d) $P(X = 53) = 0$ ✓

Probability Density Function

- A probability density function (pdf), $f(x)$ is a function for a continuous random variable X with domain on some interval $[a, b]$ ← or $a \leq x \leq b$

- Properties of a pdf $f(x)$:

1) $f(x) \geqslant 0$ for all x in $a < x < b$

2) The area under the graph $f(x)$ is equal to 1.

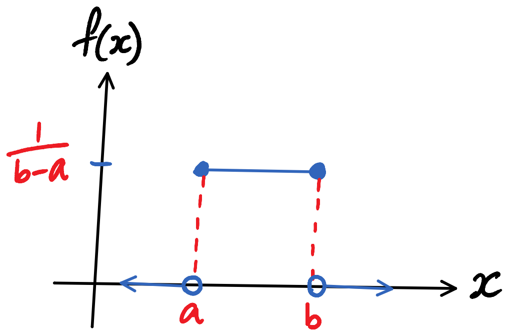
$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

3) $P(a < X < b)$ = area under the curve from $x = a$ to $x = b$

$$= \int_a^b f(x)\, dx$$

Uniform Distribution

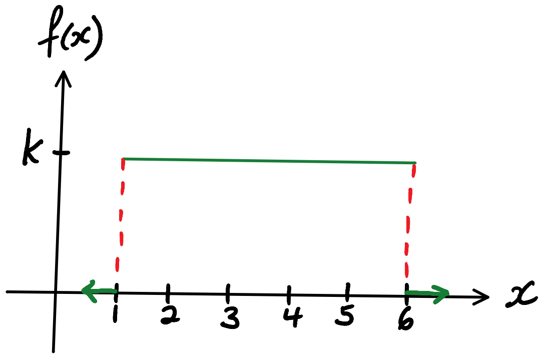
- The probability distribution for a uniformly distributed continuous random variable on the interval $a \le x \le b$.



$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for all other } x \text{ values} \end{cases}$$

Example:

The CRV X has the pdf shown below:



Determine:

a) k, hence the rule for the pdf $f(x)$

b) $P(X<4)$

c) $P(2<X<6)$

d) $P(X<4 \mid 2<X<6)$

a) Area $= 5k = 1$

$k = \frac{1}{5}$ ✓

$$f(x) = \begin{cases} \frac{1}{5} & \text{for } 1 \le x \le 6 \\ 0 & \text{for all other } x \text{ values} \end{cases}$$

b) $P(X<4) = 3k = \frac{3}{5}$ ✓

c) $P(2<X<6) = 4k = \frac{4}{5}$ ✓

d) $P(X<4 \mid 2<X<6) = \dfrac{\frac{2}{5}}{\frac{4}{5}}$ $\rightarrow$ $\frac{2}{5} \div \frac{4}{5}$

$= \frac{2}{\cancel{5}} \times \frac{\cancel{5}}{4}$

$= \frac{2}{4}$

$= \frac{1}{2}$

$= \frac{1}{2}$ ✓

Ex3B page 54
Q3, 4, 6, 7, 10, 13

Non-Uniform Distribution

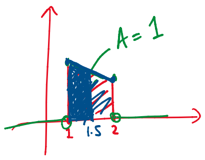
* Review: For pdf $f(x)$ on the interval $a<x<b$:
  1) $f(x) \geq 0$ for all x in the interval
  2) $\int_a^b f(x)\,dx = 1$

Examples:

1) The CRV X has probability density function given by: $f(x) = \begin{cases} 4-2x & \text{for } 1 \leq x \leq 2 \\ 0 & \text{for all other } x \text{ values} \end{cases}$

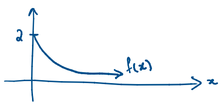
Determine $P(X<1.5)$

$$P(X<1.5) = \int_1^{1.5} (4-2x)\,dx$$
$$= \left[4x - \frac{\cancel{2}x^2}{\cancel{2}}\right]_1^{1.5}$$
$$= 4(1.5) - (1.5)^2 - (4-1)$$
$$= 0.75$$



2) Consider an exponential pdf:
$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x>0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Determine:
a) $P(X \leq 2)$
b) $P(3 \leq X \leq 5)$
c) the value of k for which $P(X \leq k) = 0.5$



a) $P(X \leq 2) = \int_0^2 2e^{-2x}\,dx$
$$= \left[\frac{2e^{-2x}}{-2}\right]_0^2$$
$$= \left[-e^{-2x}\right]_0^2$$
$$= -e^{-4} - (-e^0)$$
$$= -\frac{1}{e^4} + 1 \checkmark$$

b) $P(3 \leq X \leq 5) = \int_3^5 2e^{-2x}\,dx$
$$= \left[-e^{-2x}\right]_3^5$$
$$= -e^{-10} - (-e^{-6})$$
$$= -\frac{1}{e^{10}} + \frac{1}{e^6} \checkmark$$

c) If $P(X \leq k) = 0.5$

then $\int_0^k 2e^{-2x}\,dx = 0.5$
$$\left[-e^{-2x}\right]_0^k = 0.5$$
$$-e^{-2k} - (-e^0) = 0.5$$
$$-e^{-2k} + 1 = 0.5$$
$$e^{-2k} = 0.5$$
$$\ln e^{-2k} = \ln 0.5$$
$$-2k \cancel{\ln e} = \ln 0.5$$
$$k = -\frac{1}{2}\ln\frac{1}{2} \quad \underline{\text{OR}} \quad \frac{\ln 0.5}{-2} \quad \underline{\text{OR}} \quad \approx 0.3466$$

with classpad

3) A CRV X has probability density function:

$$f(x) = \begin{cases} kx & \text{for } 0 \le x \le 5 \\ 0 & \text{for all other values of } x \end{cases}$$

Determine:

a) k

b) $P(X \geq 2 \mid X \leq 4)$

a) $\int_0^5 kx \, dx = 1$

$\left[\frac{kx^2}{2} \right]_0^5 = 1$

$\frac{25k}{2} - 0 = 1$

$25k = 2$

$\therefore k = \frac{2}{25}$ OR $k = 0.08$

b) $\frac{P(2 \leq X \leq 4)}{P(X \leq 4)}$

$= \frac{\int_2^4 \frac{2}{25} x \, dx}{\int_0^4 \frac{2}{25} x \, dx}$

$= \frac{\left[\frac{x^2}{25} \right]_2^4}{\left[\frac{x^2}{25} \right]_0^4}$

$= \frac{\frac{12}{25}}{\frac{16}{25}} \rightarrow \frac{12}{25} \div \frac{16}{25}$

$= \frac{12}{\cancel{25}} \times \frac{\cancel{25}}{16}$

$= \frac{12}{16}$

$= \frac{12}{16}$

$= \frac{3}{4}$ OR 0.75

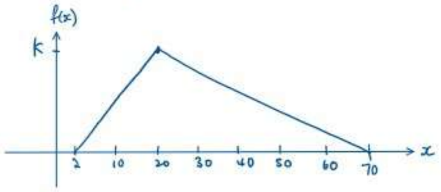
4) A merchant sells chicken feed by the kilogram. Their sales range from 2kg to 70kg with most sales being approximately 20 kg.

a) Draw a graph to show the probability density function.

b) Write an expression for the pdf.

c) Find the probability that the next sale is between 30 and 40 kg.

a)



Area $= 1$

$\frac{1}{2} \times b \times h = 1$

$\frac{1}{2} \times 68 \times k = 1$

$34k = 1$

$\therefore k = \frac{1}{34}$

b) * Equation of line for $2 \leq x < 20$

$m = \frac{\frac{1}{34}}{20-2} = \frac{1}{612}$

$y = \frac{1}{612} x + c$

sub $(2, 0) \rightarrow y = \frac{1}{612} x - \frac{2}{612}$

* Equation of line for $20 \leq x \leq 70$

$m = \frac{-\frac{1}{34}}{70-20} = -\frac{1}{1700}$

$y = -\frac{1}{1700} x + c$

sub $(70, 0) \rightarrow y = -\frac{1}{1700} x + \frac{70}{1700}$

$$\therefore f(x) = \begin{cases} \frac{1}{612}(x-2) & \text{for } 2 \leqslant x \leqslant 20 \\ -\frac{1}{1700}(x-70) & \text{for } 20 \leqslant x \leqslant 70 \\ 0 & \text{for all other } x \text{ values} \end{cases}$$

c) $P(30 \leqslant X \leqslant 40)$

$$= \int_{30}^{40} -\frac{1}{1700}(x-70)\,dx$$

$$= 0.20588$$

Expected Value, Variance, Standard Deviation

- For a CRV X, with pdf $f(x)$:

→ The expected value or long term mean value is:

OR $P(x)$

$$\boxed{\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx}$$

* $E(X) = \frac{a+b}{2}$ for uniform distribution OR symmetrical triangular distribution

→ The variance and standard deviation are:

$$\sigma^2 = Var(X) = \int_{-\infty}^{\infty} \left[f(x)\,(x-\mu)^2 \right] dx$$

$$\sigma = SD(X) = \sqrt{Var(X)}$$

* $Var(X) = \frac{(b-a)^2}{12}$ for uniform distribution

* $Var(X) = \frac{(b-a)^2}{24}$ for symmetrical triangular distribution

- Change of origin and change of scale:

$E(X)$ and $Var(X)$ of $aX + b$:

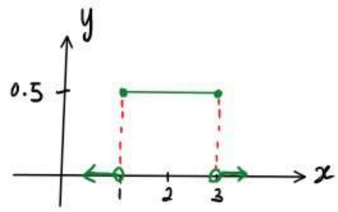
$$E(aX+b) = a\,E(X) + b$$

$$Var(aX+b) = a^2\,Var(X)$$

Examples:

1) Determine the $E(X)$, $Var(X)$ and $SD(X)$ for each pdf shown below:

a)



$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$E(X) = \frac{a+b}{2}$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$
$$= \int_1^3 (x \times 0.5)\, dx$$
$$= \int_1^3 \frac{x}{2}\, dx$$
$$= \left[\frac{x^2}{4}\right]_1^3$$
$$= \frac{9}{4} - \frac{1}{4}$$
$$= 2 \checkmark$$

OR

$$E(X) = \frac{a+b}{2}$$
$$= \frac{1+3}{2}$$
$$= \frac{4}{2}$$
$$= 2 \checkmark$$

$$Var(X) = \int_{-\infty}^{\infty} \left[f(x)(x-\mu)^2\right] dx$$
$$= \int_1^3 \left[0.5(x - \underset{E(X)}{2})^2\right] dx$$
$$= \frac{1}{3} \checkmark$$

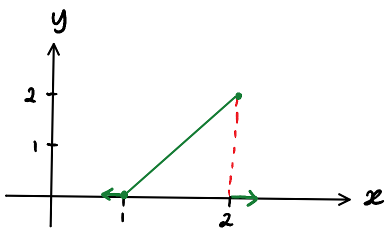
$$SD(X) = \sqrt{\frac{1}{3}}$$
$$= \frac{1}{\sqrt{3}} \checkmark$$

OR

$$Var(X) = \frac{(3-1)^2}{12}$$
$$= \frac{4}{12}$$
$$= \frac{1}{3} \checkmark$$

$$SD(X) = \sqrt{\frac{1}{3}}$$
$$= \frac{1}{\sqrt{3}} \checkmark$$

b)



$y = 2x + c$

sub $(2, 2)$

$2 = 2(2) + c$

$c = -2$

$\therefore y = 2x - 2$

$$E(X) = \int_1^2 x(2x-2)\, dx$$
$$= \int_1^2 2x^2 - 2x\, dx$$
$$= \left[\frac{2x^3}{3} - x^2\right]_1^2$$
$$= \left(\frac{2(8)}{3} - 4\right) - \left(\frac{2(1)}{3} - 1\right)$$
$$= \frac{16}{3} - \frac{12}{3} - \frac{2}{3} + \frac{3}{3}$$
$$= \frac{5}{3} \checkmark$$

$$Var(X) = \int_1^2 \left[(2x-2)\left(x - \frac{5}{3}\right)^2\right] dx$$
$$= \frac{1}{18} \checkmark$$

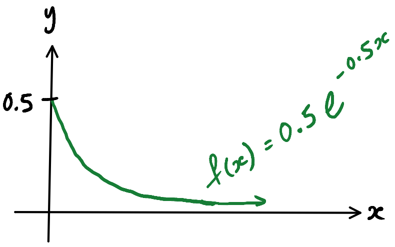
$$SD(X) = \frac{1}{\sqrt{18}} \leftarrow \frac{1}{\sqrt{9 \times 2}} = \frac{1}{3\sqrt{2}}$$ simplified :)

$$= \frac{1}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{2}}{6} \checkmark$$ rationalised :)

$= \frac{}{3}$ ✓

$= \frac{\sqrt{2}}{6}$ ✓ rationalised ☺

c)



$E(X) = \int_0^\infty x\left(0.5e^{-0.5x}\right)\,dx$

$= 2$ ✓

$Var(X) = \int_0^\infty \left[0.5e^{-0.5x}(x - \overset{\mu}{2})^2\right]dx$

$= 4$

$SD(X) = 2$ ✓

2) X is a CRV with mean $\mu = 10$ and variance $\sigma^2 = 2$. Find:

a) $E(2X+1)$

b) $Var(1-3X)$

a) $E(2X+1) = 2(10) + 1$

$= 21$ ✓

b) $Var(1-3X) = (-3)^2\,Var(X)$

$= 9 \times 2$

$= 18$ ✓

Cumulative Distribution Function

The cumulative distribution function cdf for the probability density function pdf of $f(x)$ of a CRV X defined on the interval $[a, b]$ is given by:

$$F(x) = P_r(X \leq x)$$

$f(x)$ = pdf
$F(x)$ = cdf

$$\boxed{\therefore F(x) = \int_a^x f(x)\,dx}$$

$$\boxed{P(a \leq X \leq b) = F(b) - F(a)}$$

Examples:

1) For the following uniform pdf, determine the cdf and $P(3 \leq X \leq 5)$

$$f(x) = \begin{cases} 0.2 & \text{for } 1 \leq x \leq 6 \\ 0 & \text{for all other } x \text{ values} \end{cases}$$ pdf

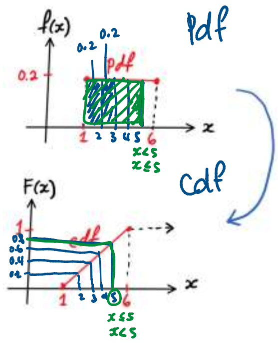
Cdf: $F(x) = P(X \leq x)$

$$= \int_1^x 0.2\,dx$$

$$= \left[0.2x\right]_1^x$$

$$= 0.2x - 0.2 \quad ✓$$

OR $0.2(x-1)$



$$P(X \leq x) = \begin{cases} 0 & \text{for } x < 1 \\ 0.2(x-1) & \text{for } 1 \leq x \leq 6 \\ 1 & \text{for } x > 6 \end{cases}$$ cdf

$$P(3 \leq X \leq 5) = P(X \leq 5) - P(X \leq 3)$$

$$= 0.2(5-1) - 0.2(3-1)$$

$$= 0.8 - 0.4$$

$$= 0.4 \quad ✓$$

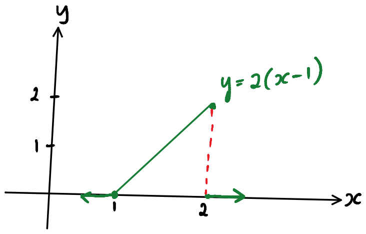
* Same as find area under the pdf for $3 \leq x \leq 5$

$$P(3 \leq x \leq 5) = \int_3^5 0.2\,dx$$

$$= \left[0.2x\right]_3^5$$

$$= 0.4 \quad ✓$$

2) Consider the triangular pdf below. Find the Cumulative distribution function.



cdf : $F(x) = P(X \leq x)$

$$= \int_1^x 2x - 2 \, dx$$

$$= \left[x^2 - 2x\right]_1^x$$

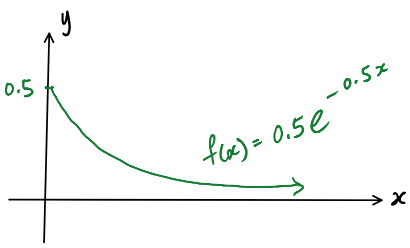
$$= x^2 - 2x - (1-2)$$

$$= x^2 - 2x + 1$$

$$= (x-1)^2$$

$$\therefore P(X \leq x) = \begin{cases} 0 & \text{for } x < 1 \\ (x-1)^2 & \text{for } 1 \leq x \leq 2 \\ 1 & \text{for } x > 2 \end{cases}$$ ✓

3) Consider the exponential pdf and determine $P(5 \leq X \leq 10)$



cdf : $F(x) = P(X \leq x)$

$$= \int_0^x 0.5e^{-0.5x} \, dx$$

$$= \left[-e^{-0.5x}\right]_0^x$$

$$= -e^{-0.5x} - (-e^{-0})$$

$$= -e^{-0.5x} + 1$$

$$= 1 - e^{-0.5x}$$

* Same solution can also be obtained by determining the area under the pdf :)

$$\therefore\ P(X \leq x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-0.5x} & \text{for } x \geq 0 \end{cases} \checkmark$$

$$\begin{aligned} P(5 \leq X \leq 10) &= P(X \leq 10) - P(X < 5) \\ &= \left(1 - e^{-0.5(10)}\right) - \left(1 - e^{-0.5(5)}\right) \\ &= \left(1 - e^{-5}\right) - \left(1 - e^{-2.5}\right) \\ &= 0.0753 \checkmark \end{aligned}$$

$$\begin{aligned} P(5 \leq X \leq 10) &= \int_5^{10} 0.5e^{-0.5x}\,dx \\ &= \left[-e^{-0.5x}\right]_5^{10} \\ &= \left(-e^{-0.5(10)}\right) - \left(-e^{-0.5(5)}\right) \\ &= -e^{-5} + e^{-2.5} \\ &= 0.0753 \checkmark \end{aligned}$$

Ex3C page 59
Q1-16 (even only)
Q19, 20, 21, 24, 25, 26, 29, 30 } Thursday

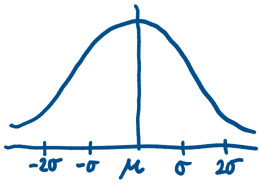
Ex3D page 82
Q1, 3, 5, 6, 9, 11, 12 } Friday

Ex3D page 83
Q14 - 21 } TODAY!

The Normal Distribution

The graph of a normal density function is symmetrical and bell-shaped with the following properties:

1. Its centre is determined by the mean μ of the distribution and its width is determined by the standard deviation σ
2. Probability corresponds to an area under the curve (Total area under the curve is 1)



Standard Scores

Standardising scores means converting normally distributed data to a new set of data which shows the number of standard deviation each data value lies from the mean of the distribution

<u>Consider this situation:</u>

- Lachlan achieves a mark of 27 in his Mathematics test. In the next test he scores 30. Has he improved?

  (Before answering this question we might first ask: What was each test out of?)

- If test 1 was out of 40 and test 2 was out of 50. Can we now decide whether he has improved?

  (Before answering we may want to know if the tests were of similar difficulty?)

- Suppose the mean in test 1 was 23 and in test 2 was 25. Now can we judge whether he has improved?

  (What is we also knew the standard deviation for each test?)

- If we know that the standard deviation in test 1 was 5 marks and in test 2 was 10 marks. Now can we suggest whether Lachlan's mark in test 2 was an improvement?

When comparing scores it is helpful to look at the <u>mean</u> and the <u>standard deviation</u>.

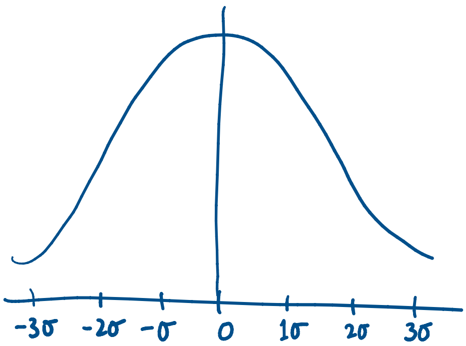
Expressing a score as a number of standard deviations above or below the mean is called **standardising the score**. The result is called the **standard score** or **z-score.**

$$\text{Standardised score} = \frac{\text{Raw score} - \text{mean}}{\text{standard deviation}}$$

$$z = \frac{x - \bar{x}}{\sigma}$$

OR

$$z = \frac{x - \mu}{\sigma}$$



Example:

$$z = \frac{x - \mu}{\sigma}$$

Jennifer scores 23, 35 and 17 in tests A, B and C respectively. If the mean and standard deviation in each of these tests are as given below express each of Jennifer's test scores as standardised scores.

Test A:	mean	30	standard deviation	5
Test B:	mean	32	standard deviation	6

Jennifer scores 23, 35 and 17 in tests A, B and C respectively. If the mean and standard deviation in each of these tests are as given below express each of Jennifer's test scores as standardised scores.

Test A:	mean	30	standard deviation	5
Test B:	mean	32	standard deviation	6
Test C:	mean	15	standard deviation	2·5

Test A $= -1.4$

Test B $= 0.5$

Test C $= 0.8$

Example 2:

Jason achieved 65% for English and 57% for Physics. The class mean and standard deviation are:

English	*Mean 62%*	*Standard deviation 8*
Physics	*Mean 52%*	*Standard deviation 10*

$$Z = \frac{x - \mu}{\sigma}$$

In which subject did he appear to do better?

$$Z(\text{English}) = \frac{65-62}{8} = 0.375 \checkmark$$

$$Z(\text{Physics}) = \frac{57-52}{10} = 0.5 \checkmark$$

$\therefore$ Jason is doing better in Physics

Example 3:

If a test has a mean of 65 and standard deviation on 12. Find the standardised score of a test result of 78

$$Z = \frac{78-65}{12}$$

$$= 1.083 \checkmark$$

Example 4:

If the Z score for a data set is -0.8 and the mean of a data is 24 and standard deviation is 5, determine the raw score

$$Z = \frac{x - \mu}{\sigma}$$

$$-0.8 = \frac{x - 24}{5}$$

$$-0.8 = \frac{x - 24}{5}$$

$$\therefore x = 20 \checkmark$$

Example 5:

For a set of data with mean of 52. Find the standard deviation if a raw score of 37 corresponds to a standardised score of -2.5

$$-2.5 = \frac{37 - 52}{\sigma}$$

$$-2.5 = \frac{-15}{\sigma}$$

$$\therefore \sigma = 6 \checkmark$$

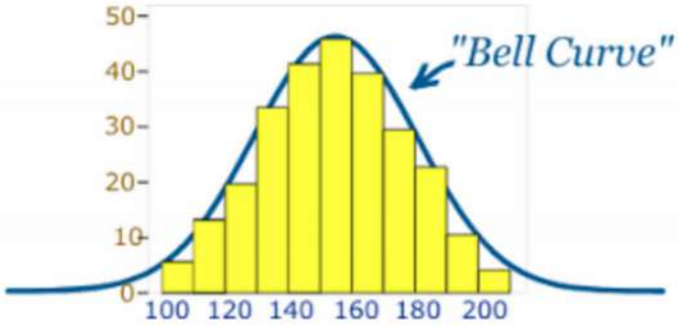
Ex4A Q3, 5

68-95-99.7% Rule

If we take many measurements of something occurring naturally …

eg adult height of females, weight, diastolic blood pressure, IQ tests across a very large group of people

… we often get a graph shaped as a "bell-shaped curve"

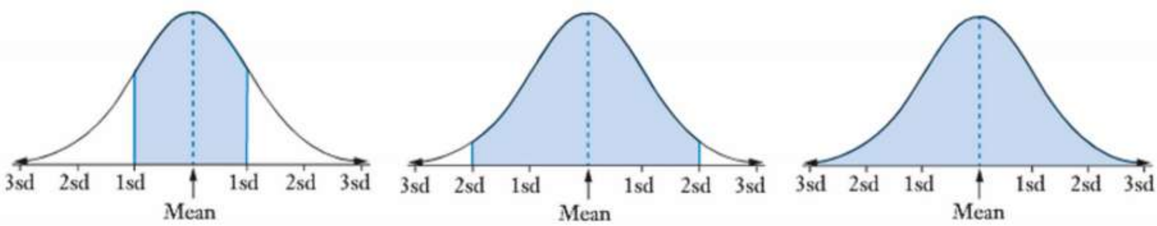


This is called a
NORMAL DISTRIBUTION

or Gaussian Distribution after Carl Gauss who discovered it

For a normally distributed random variable, approximately:
68% of data lies within *one standard deviation* of the mean (34% either side)
95% lies within *two standard deviations* of the mean (47.5% either side)
99.7% of data (if not all of it) lies within *three standard deviations* of the mean (49.85% either side)

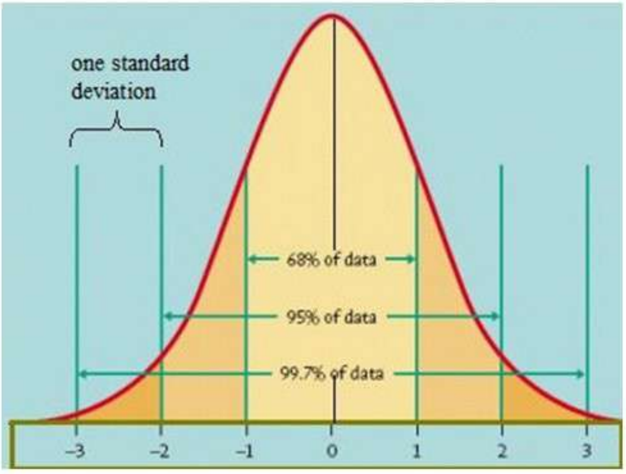
This is the 68%, 95% and 99.7% rule.



68.3% of area under curve shaded — Probability = 0.683

95.4% of area under curve shaded — probability = 0.954

99.7% of area under curve shaded — probability = 0.997



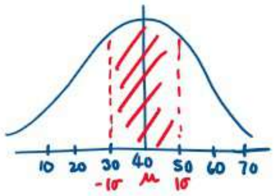
Notation: $X \sim N(\mu, \sigma^2)$

Example 1:

If $X \sim N(\underset{\mu}{40}, \underset{\sigma^2}{10^2})$, determine without a calculator:

a) $P(30 < X < 50)$ ← 1 sd within μ

$= 0.683$ or 0.68 ✓



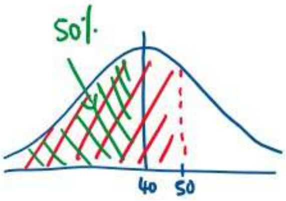
b) $P(20 < X < 60)$ ← 2sd within μ

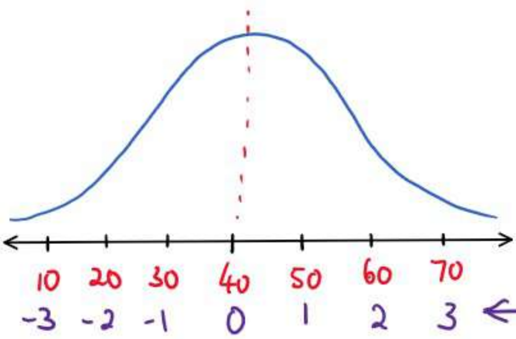
$= 0.954$ or 0.95 ✓

c) $P(40 < X < 60)$

$= \frac{0.95}{2} = 0.48$ ✓

d) $P(X \leqslant 50) = 0.5 + \frac{0.68}{2} = 0.84$ ✓

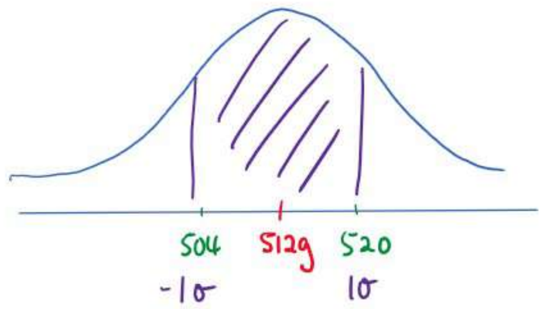




← Z scores!

Example 1:

A box of breakfast cereal has 'contains 500 grams of breakfast cereal' printed on it. Suppose that in fact the weight of breakfast cereal contained in these boxes is normally distributed with a mean of 512 grams and a standard deviation of 8 grams. Determine the probability that a randomly chosen box of this cereal contains between 504 grams and 520 grams.

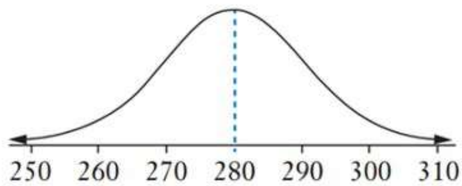


Probability = 0.68

68% of cereal will be between 504g and 520g

Example 2:

Let us suppose that the duration of pregnancy, for a naturally delivered human baby, is a normally distributed variable with a mean of 280 days and a standard deviation of 10 days.

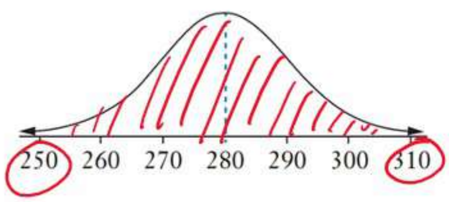


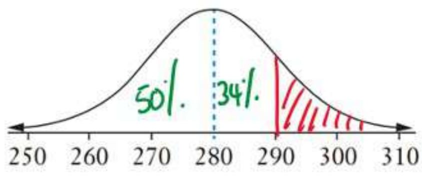
Using the 68%, 95%, 99.7% rule, and *not* the statistical capability of your calculator, determine estimates for the percentage of human pregnancies, for naturally delivered babies, that:

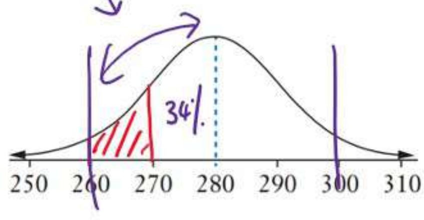
a are between 250 days and 310 days, 99.7%

b exceed 290 days, 16%

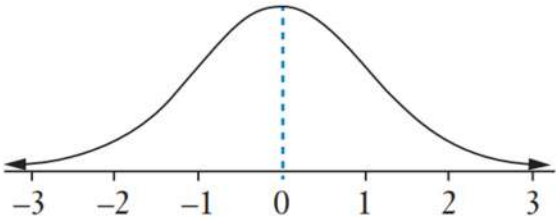
c are between 260 days and 270 days. → $\frac{95\%}{2} - 34\% = 13.5\%$







- Standardised scores are often called "Z" scores with a mean of zero



Example 3:

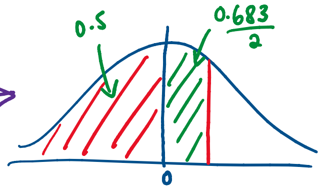
Using the 68-95-99.7 rule, determine the following probabilities for the standard normal variable Z.
(Note the capital Z – standardized score)

$X \sim N(0, 1)$

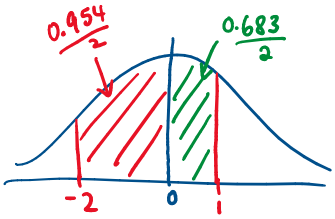
a) *P(-1<Z<1)* $= 0.683$ ✓

b) *P(-3<Z<3)* $= 0.997$ ✓

c) *P(Z<1)* $= 0.8415$ ✓



d) *P(-2<Z<1)* $= 0.815$ ✓



Ex4C Q14, 15, 16

Normal Distribution

NOTATION and CLASSPAD

- If X is normally distributed with mean μ and s.d σ

$$X \sim N(\mu, \sigma^2)$$

* Use proper Maths Notation in Exam! Not classpad "instructions"!

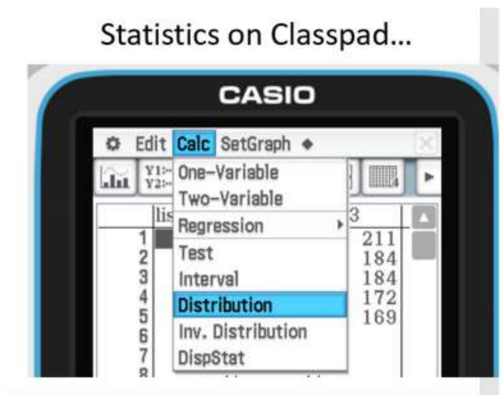
* Classpad : Statistics OR Main

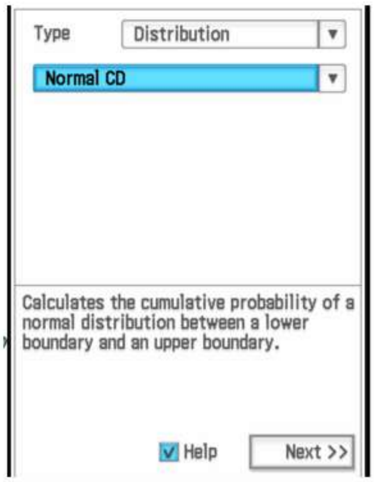
→ Calc

→ Distribution ↗ normCDf (L_x, U_x, σ, μ) ← find probability

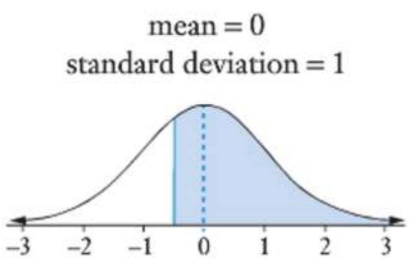
↘ invNormCDf (p, σ, μ) ← find k for $P(X \leq k)$

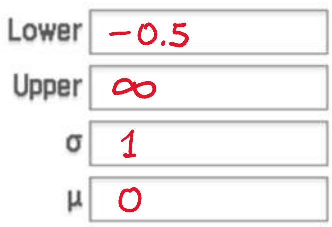
← graph





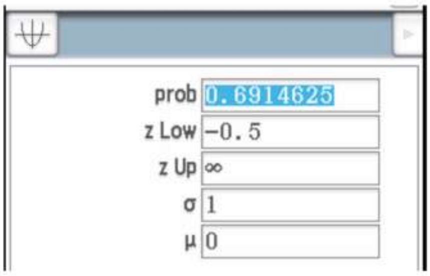
Example 1:

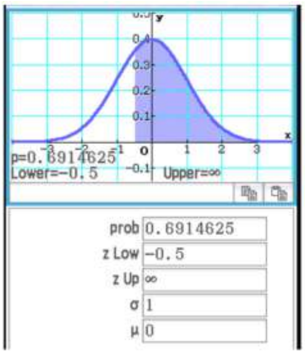




$P(X > -0.5) =$

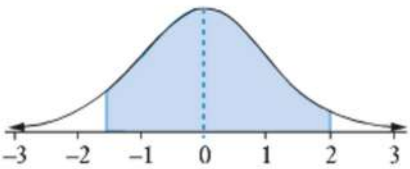
Use the graph function to make sure that what you have entered is the same as what you want!

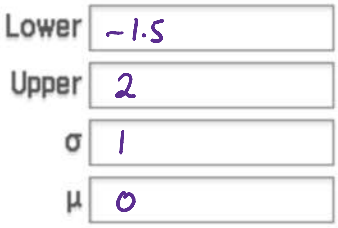




Example 2:

mean = 0
standard deviation = 1





$P(-1.5 < X < 2) =$ 0.9104

Example 3:
Find the following using your classpad:

a) $P(x < 4)$, $\mu = 5$, $\sigma = 1.4$ $X \sim N(5, 1.4^2)$

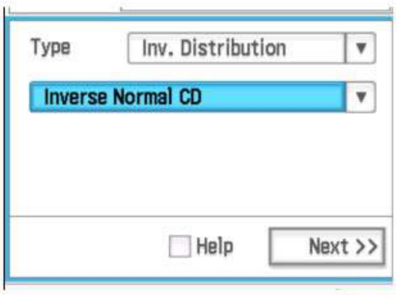
$L_x: -\infty$ $U_x: 4$

b) $P(x > 64)$, $\mu = 70$ $\sigma = 3$ $X \sim N(70, 9)$

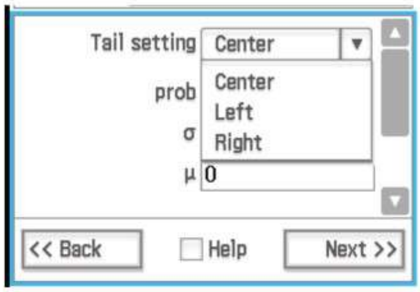
c) $P(52 < x < 60)$, $\mu = 56$, $\sigma = 4$

d) $P(0.1 < x < 0.6)$, $\mu = 0.3$, $\sigma = 0.1$

<u>INVERSE NORMAL DISTRIBUTION (CLASSPAD):</u>

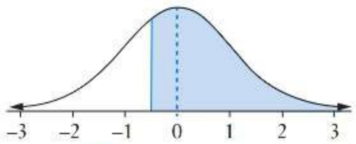


Select either Center, Left or Right



Example 4:
With μ = 0 and σ =1, determine the value of k for the following situations

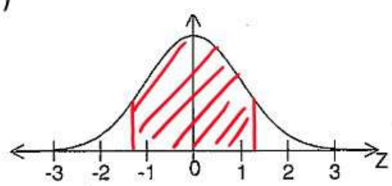
a)



P(X>k) = 0.6554217

k = −0.4 ✓

b)



P(−k<X<k) = 0.7698607

P(−1.2 < X < 1.2)

k = 1.2 ✓

Example 5:
Find the value of k in each of the following using your classpad:

$a)\ P(x < k) = 0.54, \quad \mu = 6, \quad \sigma = 2.5$

left tail

k = 6.25 ✓

$b)\ P(x > k), = 0.9, \quad \mu = 50 \quad \sigma = 8$

right tail

k = 39 ✓

Example 6:
A normal distribution is represented by X~N(40, 4) ← σ^2

a) What is the mean and standard deviation of the distribution?

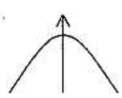
mean ↓ 40

standard deviation ↓ 2

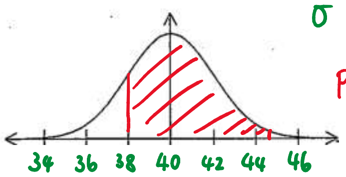
b) What is the probability, correct to 5 decimal places that a randomly selected score,
i) lies between 38 and 45?

$\mu = 40$
$\sigma = 2$

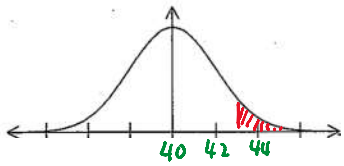
ii) is greater than 43?



$\mu = 40$
$\sigma = 2$



$P(38 < x < 45)$
$= 0.83514$ ✓



$P(x > 43) = 0.06681$ ✓

Example 7:

A box of breakfast cereal has 'contains 500 grams of breakfast cereal' printed on it. Suppose that in fact the weight of breakfast cereal contained in these boxes is normally distributed with a mean of 512 grams and a standard deviation of 8 grams.

a Determine the probability that a randomly chosen box of this cereal contains less than 500 grams.

b In a random sample of 100 boxes of this cereal approximately how many boxes should we expect to contain less than 500 g?

Example 8:

Let us suppose that the time from Simon getting out of bed until his arrival at school is normally distributed with a mean of 55 minutes and a standard deviation of 5 minutes. Simon's arrival at school is classified as being late if it occurs after 9.10 am.

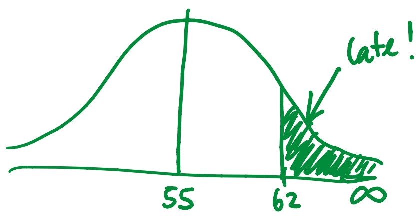
a One day Simon gets out of bed at 8.08 am. What is the probability of him arriving late?

b For a period of time Simon always gets out of bed at the same time but finds that he arrives late approximately 85% of the time! What time is he getting out of bed (to the nearest minute)?

Let X be the time from getting out of bed to arriving to school

$X \sim N(55, 5^2)$

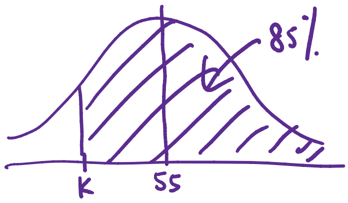
a) Simon has 62 mins to get to school



$P(X > 62) = 0.0808$ ✓

b) $P(X > k) = 0.85$ ← inverse!!!

tail = right!

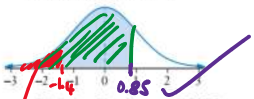


$k \approx 49.8$

Simon gets out of bed at 8:20 am ✓

Ex4B page 83

13 mean = 0
standard deviation = 1



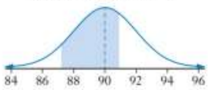
$P(\underline{-1.4} < X < k) = 0.7215$ ✓

0.0807567 +

$= 0.8022567$

$\therefore k = 0.85$ ✓

14 mean = 90
standard deviation = 2

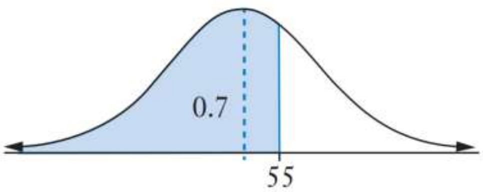


$P(87.2 < X < k) = 0.5964$

Quantiles

Quantiles are the values which a particular proportion of the distribution falls below.

For example, if 70% (0.7) of the distribution is below 55 then 55 is the 0.7 **quantile**.
(also refer to as being the 70th **percentile**).



Quartiles divide the distribution into 4 equal parts, **percentiles** into one hundred, **deciles** into ten, and **quintiles** into five.

Eg: **0.25 quantile** is also referred to as the **lower quartile** (or first quartile)
0.75 quantile as the **upper quartile** (or third quartile).

Example:

If $X \sim N(20, 3^2)$ determine

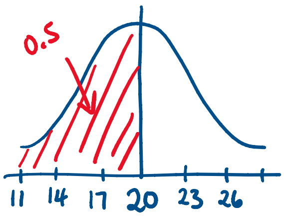
a the 0.5 quantile

b the 0.82 quantile

c the 24th percentile

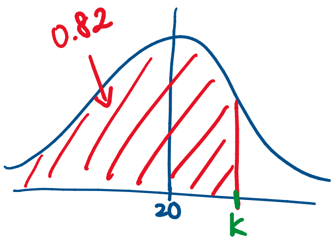
d the 62nd percentile

a)



0.5 quantile is 20 ✓

b)



Classpad → inverse Norm

k = 22.75 ✓

c) 17.9 ✓

d) 20.9 ✓

Ex4C Q12, 13

Normal Distribution to Model Data

- To check if the data set collected is appropriate to be model with a <u>Normal Distribution</u>:

① Histogram has the characteristic <u>bell shaped curve</u>.

② The data set gives proportions we expect from a normal distributed random variable <u>(68-95-99.7 rule)</u>

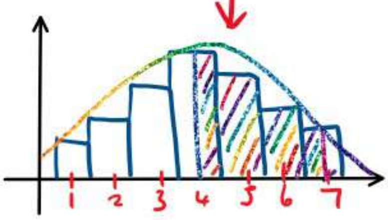
③ The distribution is <u>symmetrical</u> (mean = median = mode)

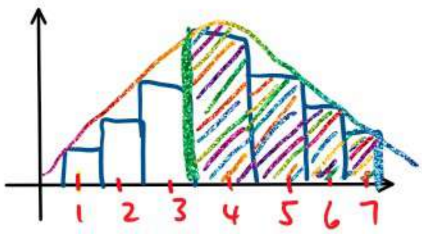
Example:

To determine $P(4 \leq X \leq 7)$ for a discrete distribution of integers : 1, 2, 3, 4, 5, 6, 7, 8

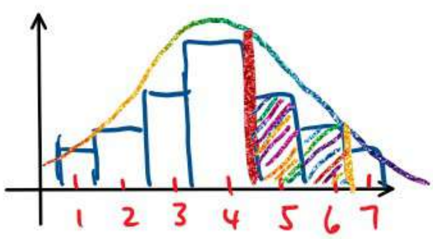
$$\text{Discrete} \xrightarrow{\text{adjust}} \text{Continuous}$$

$$P(4 \leq X \leq 7) \longrightarrow P(3.5 < X < 7.5)$$





$$P(4 < X < 7) \longrightarrow P(4.5 < X < 6.5)$$



other Examples :

$P(X < 50) \longrightarrow P(X < 49.5)$ ✓

$P(X \leqslant 50) \longrightarrow P(X < 50.5)$ ✓

Ex4D page 96
Q1 -> 5

Exercises

Z-scores	Ex4A Q3, 5
68-95-99.7 rule	Ex4C Q14, 15, 16
Quantiles	Ex4C Q12, 13
Adjusting data	Ex4D Q1 -> 5
Classpad	Ex4C Q2, 4, 6, 8, 10, 11
Word Problems	Ex4C Q18 - 32 (even)

Examples

Normal Distribution

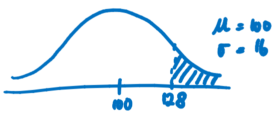
1. The scores obtained on an IQ test can be assumed to be normally distributed with mean μ = 100 and standard deviation σ = 16.

   a) Approximately what percentage of the distribution lies within one, two or three standard deviation of the mean?
   b) What percentage of scores are more than two standard deviation above or below the mean?
   c) 8240 students were given an IQ test,
      i) How many students (to the nearest ten), achieved a score in excess of 128?
      ii) What is the minimum and maximum scores of the middle 60% of students on this test?

a) 68% of scores lie between 84 and 116
95% " " " " 68 and 132
99.7% " " " " 52 and 148

b) 5%
2.5% of scores are below 68
2.5% of scores are above 132

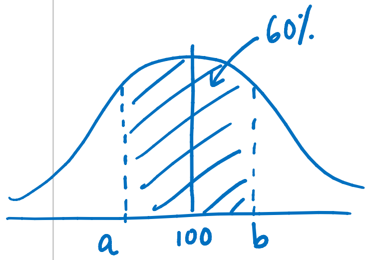
c) i) $P(X > 128)$ where $X \sim N(100, 16^2)$



$P(X > 128) = 0.0401$

students scoring more than 128
$= 0.0401 \times 8240$
≈ 330 students ✓

c) ii)



InvNormCD
Centre

$a = 86.53$ ✓, $b = 113.47$ ✓

2. If $X \sim N(40, 10^2)$, determine without a calculator:
   a) $P(30 < X < 50)$
   b) $P(20 < X < 60)$
   c) $P(40 < X < 60)$
   d) $P(X < 50)$

$X \sim N(40, 10^2)$

a) 0.68 ✓

b) 0.95 ✓

c) $0.95 \div 2 = 0.475$ ✓

d) $0.5 + \frac{0.68}{2} = 0.84$ ✓

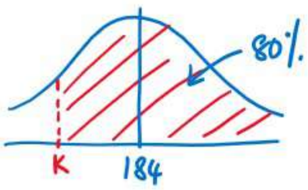
3. If Z is a standard normal random variable, find:
   a) $P(-1 < Z < 2.5)$
   b) $P(Z > 1)$

$\mu = 0$
$\sigma = 1$

a) 0.8351

b) 0.1587

4. The average height of Year 12 boys is 184cm with a standard deviation of 8.9cm. Estimate the height of a boy shorter than 80% of all Year 12 boys.



$P(x > k) = 0.8$

$\therefore k = 176.5$ ✓

5. Pre-packed bananas are marked as "contains approximately 2kg". Let us suppose that the weight of the contents are normally distributed with mean 2.015kg and standard deviation 0.01kg.

a) What is the probability that the contents of a randomly chosen bag of bananas has a weight of less than 2kg?
b) If five bags of bananas are randomly selected, what is the probability that three or more will have contents weighing less than 2kg?

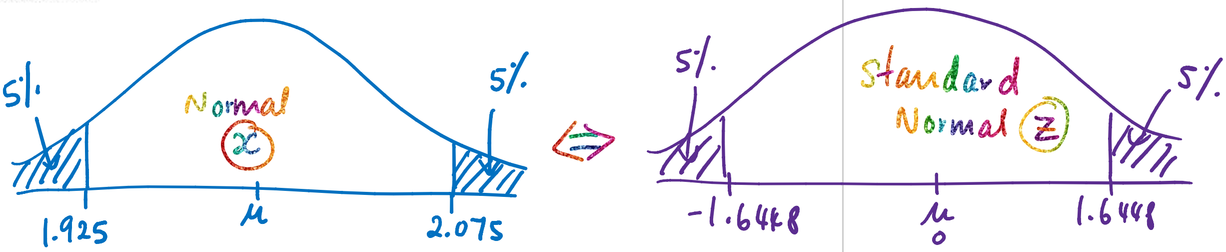
a) $X \sim N(2.015, 0.01^2)$

$P(X < 2) = 0.06681$ ✓

b) Binomial : $Y \sim B(5, 0.06681)$

$P(Y \geq 3) = 0.002691$ ✓

6. Limits of acceptability imposed on the lengths of metal rods are 1.925 cm and 2.075 cm. On average, 5% are rejected as undersized and 5% are rejected as oversized.
If the lengths are normally distributed, find the mean and standard deviation of the distribution.



$Z = -1.6448$ $\qquad Z = 1.6448$ $\qquad$ (use InvNorm $\mu = 0$, $\sigma = 1$)

$$Z = \frac{x - \mu}{\sigma}$$

$$-1.6448 = \frac{1.925 - \mu}{\sigma} \quad \text{and} \quad 1.6448 = \frac{2.075 - \mu}{\sigma}$$

Solve simultaneously

$\therefore \mu = 2, \quad \sigma = 0.045597$ ✓

Extra Questions

1. A new battery for an electric car has a charge that is normally distributed, with a mean of 150 km and standard deviation of 21km.
   To improve the consistency of the battery performance, engineers decide that at most 0.27% of batteries should last less than 100 km. Calculate the value of the new standard deviation if the mean remains the same.

2. The percentage marks of students sitting for a national Maths competition were found to be normally distributed.
   Certificates of Distinction were awarded to 15% of the candidates for achieving a mark in excess of 78% and the top 5% of the candidates in addition to the Certificate of Distinction received a prize for achieving a mark in excess of 88%.
   Find the mean and standard deviation of the Maths competition marks, giving answers to 1 decimal place.

Ex4C Q18 - 32 (even)

CRV, Normal, Uniform

Exam
Questions -...

Exam
Questions -...

Exam
Questions -...

Calculus

Question 6 **(5 marks)**

(a) Differentiate $y = \dfrac{2x+1}{e^x}$, simplifying your answer. (3 marks)

(b) Evaluate $\int_1^2 \left(\dfrac{1-2x}{e^x}\right) dx$. (2 marks)

Random Samples and Bias

- Population: (Census : survey the entire population)
  A population is the entire pool from which a statistical sample is drawn

- Sample:
  A sample is a group of members of a population selected to represent the population

- Types of Sampling:

  - Simple Random Sampling
  - Stratified Random Sampling
  - Systematic Sampling
  - Cluster Sampling

  } Random

  - Convenience Sampling
  - Quota Sampling
  - Self-Selection / Volunteer Sampling

  } biased

- A parameter is a characteristic value of a particular population (eg mean)
- A statistic is an estimate of a parameter obtained using a sample

- Bias:
  - The statistics from a fair sample are likely to be close to the parameters of the population
  - Non-random samples are most likely to be biased
  - Source of bias:
    - Selection bias - from choice of the sample, avoided by using random sample.
    - Design flaw bias - Use objective (instead of subjective) measures.
    - Data collection bias - from differences in ways that data is collected.

Stratified vs. Quota
- Stratified sampling requires random sampling for each strata
- Quota sampling does not require random sampling for each quota

**See Sheep example on page 104 Sadler Unit 4:*

Collecting Data,...

Collecting Data,...

12MM 0807 Sampling...

Variability of Random Samples

In Maths, we may run a simulation to collect data without carrying out the real event.

<u>Cereal example</u>

*See other Simulations example on page 112-

MATHS12T...

Random Sampling -...

<u>Variability of Random Samples</u>

- Unless we use the whole population, we would expect variations between random samples even when they are taken from the same distribution.

- The larger the samples, the more similar they are likely to be in term of mean, standard deviation and shape of the distribution.
  (eg: Uniform, Normal, Binomial or Bernoulli)

Simulating random samples on the Classpad:

1. <u>Uniform Distribution</u>

eg: $X = 1, 2, 3, 4, 5, 6$ $\quad E(X) = 4.5$

$SD(X) = 2.29$

Main → randList$(100, \overset{L}{1}, \overset{u}{6})$ ⇒ List 1

Statistics → Calc → 1-Variable

<u>OR</u>

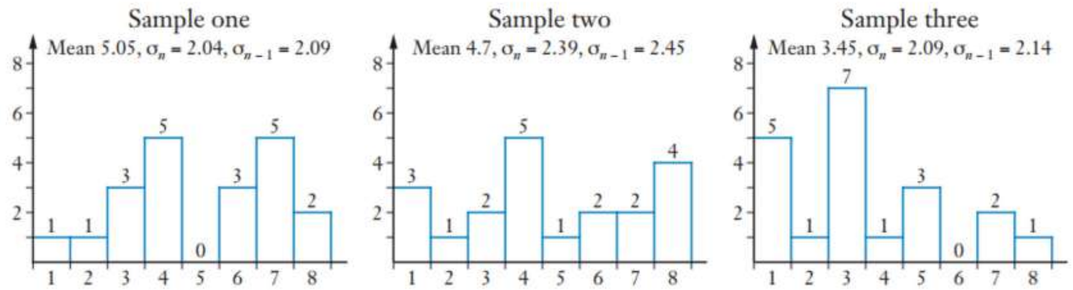
Spreadsheet → rand(1,6) → Edit → Fill range

→ Calc → 1-Variable, graph Histogram

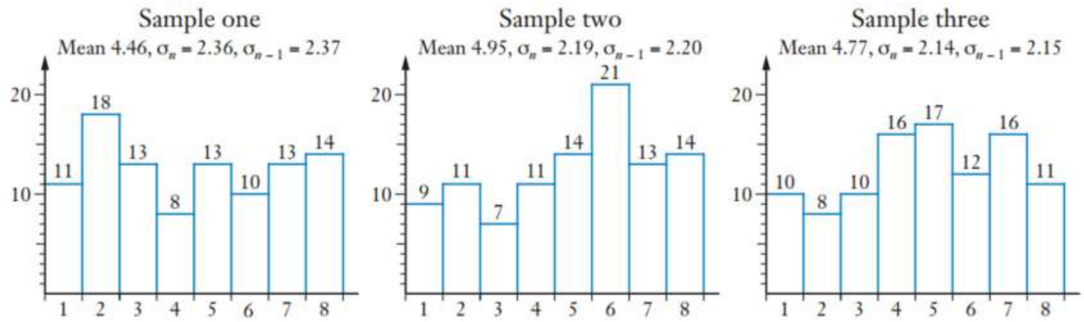
The following graphs each show the distribution of a sample of 20 numbers generated from a random variable, X, with X uniformly distributed across the integers

$$1, 2, 3, 4, 5, 6, 7, 8.$$

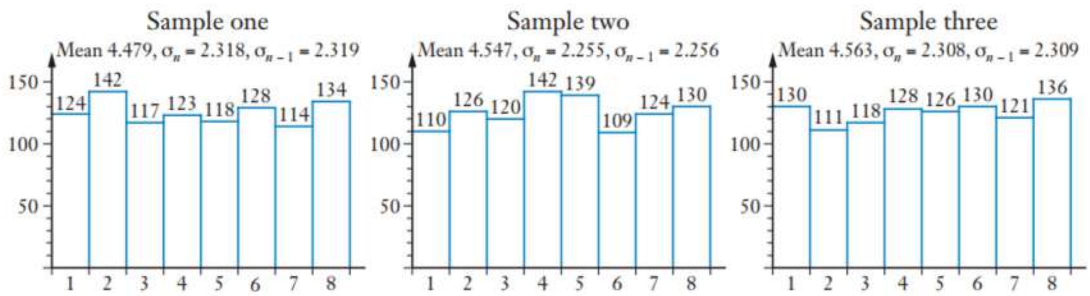
Note: $E(X) = 4.5$, $SD(X) = 2.29$ (2 decimal places).



The following graphs each show the distribution of a sample of 100 numbers generated from a random variable, X, with X uniformly distributed across the integers 1 to 8.



The following graphs each show the distribution of a sample of 1000 numbers generated from a random variable, X, with X uniformly distributed across the integers 1 to 8.



2) Normal Distribution

eg: $X \sim N(6, 1.2^2)$

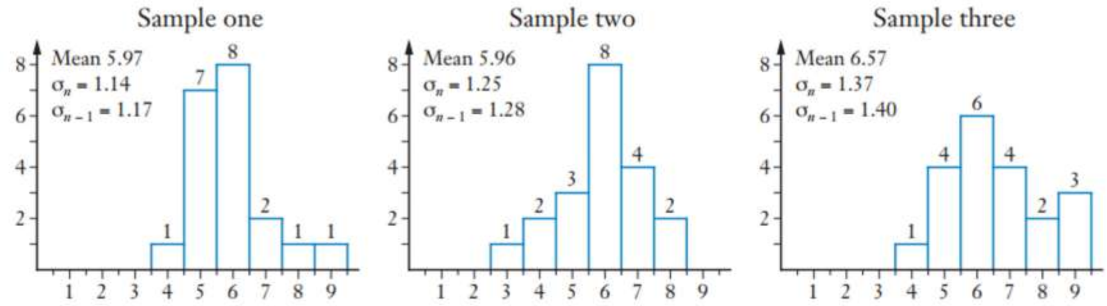
Main → randNorm$(\overset{\sigma}{1.2}, \overset{\mu}{6}, 100)$ ⇒ List 1

OR

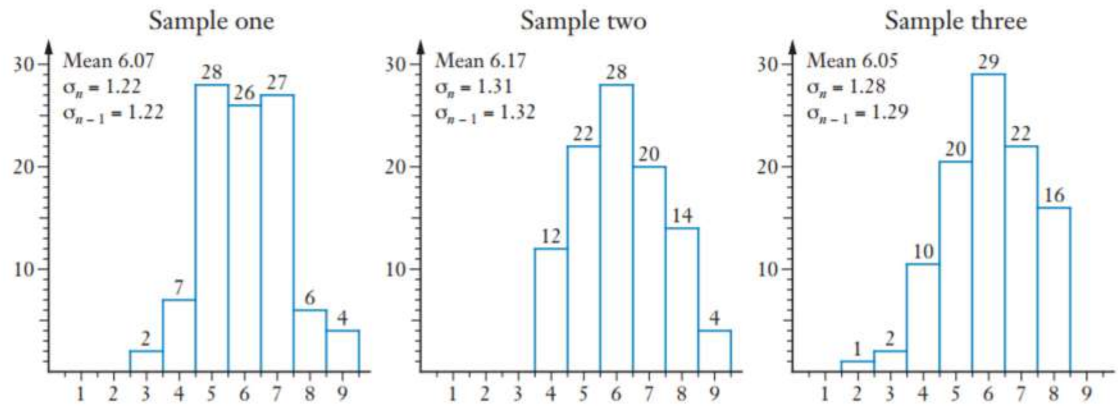
Spreadsheet → randNorm (1.2, 6) → Edit → Fill range

The following graphs each show the distribution of a sample of 20 numbers generated from a random variable, X, with $X \sim N(6, 1.2^2)$

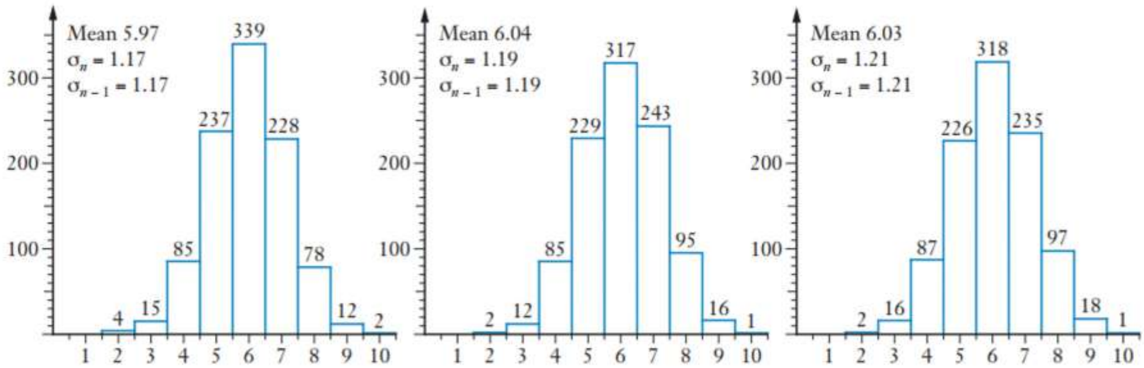
Note: On this page, whilst each graph shows the numbers grouped into columns, each mean and standard deviation have been calculated from the original numbers.



The following graphs each show the distribution of a sample of 100 numbers generated from a random variable, X, with $X \sim N(6, 1.2^2)$.



The following graphs each show the distribution of a sample of 1000 numbers generated from a random variable, X, with $X \sim N(6, 1.2^2)$.



3) Bernoulli / Binomial Distribution

eg: X with $P(0) = 0.4$, $P(1) = 0.6$, $E(X) = 0.6$

$SD(X) = \sqrt{0.6(1-0.6)}$
$= 0.49$

Main $\rightarrow$ randBin $(\overset{\text{success}}{1}, \overset{p}{0.6}, 100) \Rightarrow$ List 1

Main → randBin (1 , 0.6 , 100) ⇒ List 1

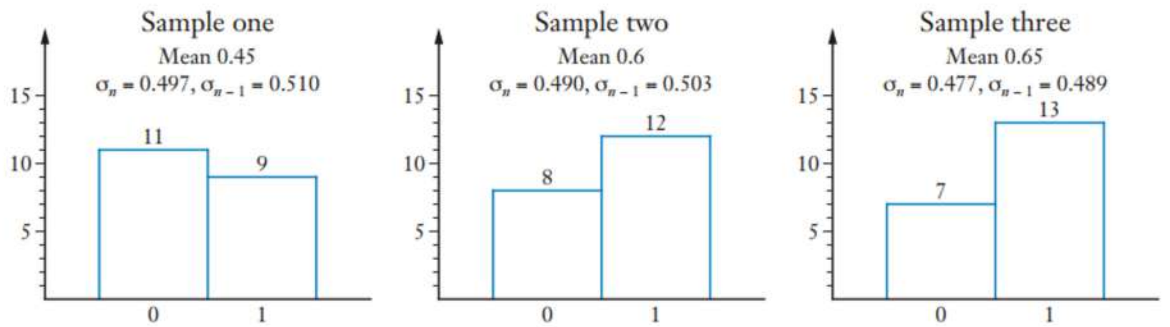
success p

OR

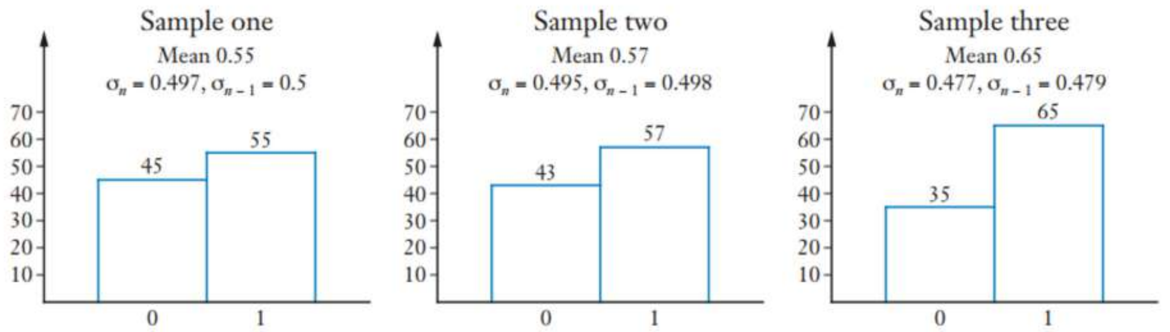
Spreadsheet → randBin (1, 0.6) → Edit → Fill range

The following graphs each show the distribution of a sample of 20 numbers generated from a Bernoulli distribution, X, with $P(0) = 0.4$ and $P(1) = 0.6$.

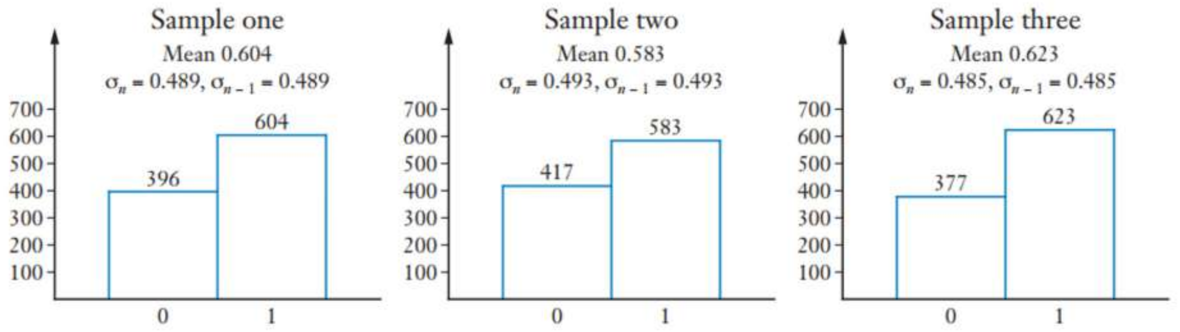
Note: $E(X) = 0.6$, $SD(X) = \sqrt{0.6(1-0.6)} \approx 0.49$.



The following graphs each show the distribution of a sample of 100 numbers generated from a Bernoulli distribution with $P(0) = 0.4$ and $P(1) = 0.6$.

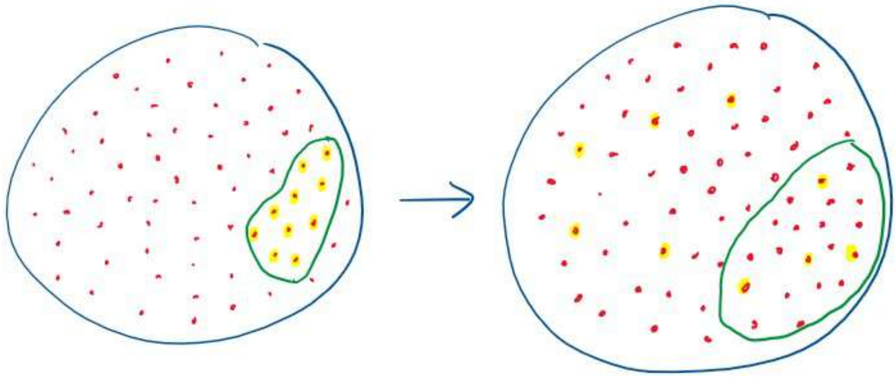


The following graphs each show the distribution of a sample of 1000 numbers generated from a Bernoulli distribution with $P(0) = 0.4$ and $P(1) = 0.6$.



Capture Recapture

Capture-Recapture is a method that uses sampling of a population to **estimate the size of the population**



Formula:

$$\frac{\text{No. of tagged}}{\text{Population}} = \frac{\text{No. of tagged (recapture)}}{\text{Total no. of recapture}}$$

Example:

$$\frac{10}{P} = \frac{4}{20} \quad \therefore \text{Estimated total population} \approx 50 \text{ fish}$$

- Repeat the process multiple times and find the mean total population for a better estimation!

Assumptions:

- No deaths or breeding occurred
- No tags fall off
- No migration (if applicable)
- The tagged will mingle with the rest.

Ex5A page 109
Q1, 4, 5, 7, 8

Sample Proportions

- The population proportion p is constant/fixed

$$p = \frac{\text{No. in population with attribute}}{\text{population size}}$$

- The sample proportion $\hat{p}$ varies from sample to sample

$$\hat{p} = \frac{\text{No. in sample with attribute}}{\text{Sample size}}$$

OR $$\hat{p} = \frac{X}{n}$$

where X is the no. of favourable outcome in a sample size n.

- The sample proportion $\hat{p}$ is an estimator of the probability p in the population. * point estimate

- The distribution of a statistic for many samples is called a sample distribution

- For samples that are small compared to a large population, the sample proportion $\hat{p}$ can be determined by a binomial random variable with parameters n and p

$$E(\hat{p}) = p$$

$$SD(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

$$SD(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

* See explanation below:

- Counting the number of "successes" in n trials, involves a Binomial Distribution Bin(n,p) where p is the probability of "success".
- For Bin(n,p) we have:

Mean $\mu = np$

S.D $\sigma = \sqrt{np(1-p)}$

- The proportion of successes
  The distribution of sample proportions will have:

Mean $\mu = \frac{np}{n} = p$

S.D $\sigma = \frac{\sqrt{np(1-p)}}{n} = \sqrt{\frac{p(1-p)}{n}}$

*Show Classpad video 571: http://www.classpad.com.au/cp2/index.php?me=BA3&mn=571

Examples

Example 1:

Police at a roadside checkpoint stopped 55 cars to check their roadworthiness. 7 of the drivers were issued with notices to have faults fixed within a week and 2 cars had such severe problems that they were immediately stopped from driving any further.

a Use the information to estimate the probability that a randomly selected car has a fault.

b State any problems with treating this as a reliable estimate.

a) sample proportion

$$\hat{p} \approx \frac{9}{55} \text{ or } 0.16$$

b) justify !!!

Example 2:

The probability of a normal car tyre lasting more than 60 000 km is about 0.34. What is the variance and standard deviation of the proportion of samples of 30 such tyres lasting more than 60 000 km?

$p = 0.34$, $q = 0.66$, $n = 30$ ← parameters

$$Var(\hat{p}) = \frac{pq}{n} = \frac{0.34 \times 0.66}{30} \sim 0.00748$$

$$SD(\hat{p}) = \sqrt{\frac{pq}{n}} = \sqrt{0.00748} \approx 0.0865$$

Example 3:

A class of Year 7 students investigated the results of dealing a card from a well-shuffled pack and checking its suit. Each student in the class dealt a card 50 times, replacing and shuffling the cards before dealing the next one. They each counted the number of times the card was a heart and recorded the proportion of times out of 50 as a decimal. What would be the mean and standard deviation of these results?

parameters: $p = 0.25$, $q = 0.75$, $n = 50$

$$E(\hat{p}) = p = 0.25$$

$$SD(\hat{p}) = \sqrt{\frac{pq}{n}} = \sqrt{\frac{0.25 \times 0.75}{50}} \approx 0.061$$

Central Limit Theorem

$n = 30$

- As the sample size n increases, the sample distribution of the means will approach a Normal distribution.

<u>Normal distribution</u>

mean $\mu = p$ and sd $\sigma = \sqrt{\dfrac{p(1-p)}{n}}$

* As sample size increases, the variation decreases and becomes closer to the population mean and sd.

- To be confident that the sample distribution will approximate the normal distribution:

$$np \geq 5 \text{ and } n(1-p) \geq 5$$

Central Limit Theorem: Bunnies and Dragons

https://www.youtube.com/watch?v=jvoxEYmQHNM

Example:
Simulate the sample proportion for blue eyes (p = 0.32) for 100 samples of 10 students

Classpad :

Main → randBin $(10, 0.32, 100) / 10 \Rightarrow$ List 1

(Histogram on statistics)

(Histogram on statistics)

* Repeat with samples of 50 students etc...

Examples

Sample Proportion Examples

1. When the spinner was spun 200 times, an "A" occurred 43 times. (Draw a spinner)
   a) What is the population proportion p for "A"s?

$p = 0.2$ ✓

   b) What is the sample proportion $\hat{p}$ for "A"s?

$\hat{p} = \frac{43}{200} \approx 0.215$ ✓

   c) Calculate the mean and s.d. of $\hat{p}$ for the sample of 200 spins.

$E(\hat{p}) = p$
$= 0.2$ ✓

$SD(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$
$= \sqrt{\frac{0.2(0.8)}{200}} \approx 0.0283$ ✓

2. Assuming that 60% of people have a driver's license. Using the normal approximation, find the probability that in a randomly selected sample of size 200, more than 65% of people have a driver's license.

$n = 200$, $p = 0.6$

(n is large , $np = 120$ and $n(1-p) = 80$
∴ distribution of $\hat{p}$ is approx. normal)

$\mu = p$
$= 0.6$

$\sigma = \sqrt{\frac{0.6(1-0.6)}{200}}$
$= 0.03464$

$\therefore Pr(\hat{p} > 0.65) = 0.7445$ ✓

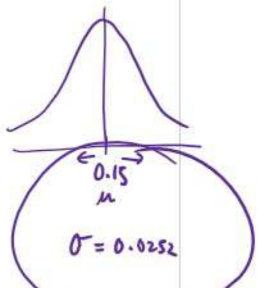
* Classpad : NormCDf $(\overset{L}{0.65}, \overset{U}{\infty}, \overset{\sigma}{0.03464}, \overset{\mu}{0.6})$

Do not write Classpad instructions in Exam!

3. Given that about 15% of Australians are left-handed, what is the probability that in a sample of 200 Australians, 20 to 30 of them are left-handed?

* Check $np = 0.15 \times 200 = 30$
$n(1-p) = 0.85 \times 200 = 170$

∴ Normal distribution can be used.



$\mu = p$
$= 0.15$

$\sigma = \sqrt{\frac{0.15(0.85)}{200}}$
$= 0.0252$

$= 0.0252$

(* Use 19.5 to 30.5 for integers 20 to 30)

$\hat{p} = \frac{19.5}{200}$

$= \underset{L}{0.0975}$

$\hat{p} = \frac{30.5}{200}$

$= \underset{U}{0.1525}$

$\text{Norm CDf}\,(\overset{L}{0.0975}, \overset{U}{0.1525}, \overset{\sigma}{0.0252}, \overset{\mu}{0.15})$

$= 0.52$ ✓

$X \sim N(0.15, 0.0252^2)$

$P(0.0975 \leq X \leq 0.1525)$

$= 0.52$

∴ The probability that 20 - 30 from a sample of 200 Australians are left handed is 0.52 ✓

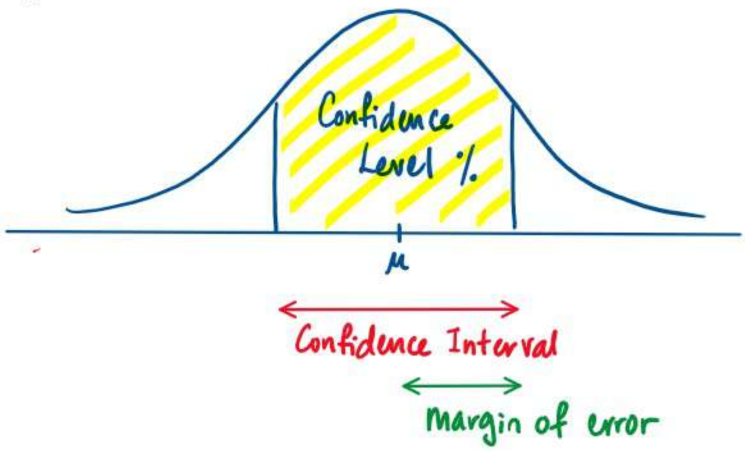
Confidence Interval

MM4-6B Notes for Confidence Interval

<u>Notes for Confidence Interval</u>

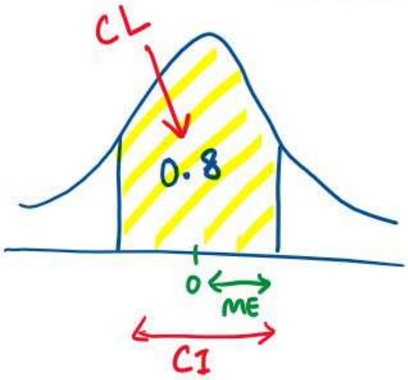
- A <u>point estimate</u> is a *single value* of the sample proportion $\hat{p}$ and can be used to estimate the population proportion p.
- An <u>interval estimate</u> for the population proportion p is called a *confidence interval*.

- For a <u>confidence interval</u> symmetrical around the mean in a statistical distribution:
    - The <u>confidence level</u> is the proportion of values that lie within the interval.
    - The <u>margin of error</u> is the distance of the ends of the interval from the mean.



Eg: Find the confidence interval and margin of error needed to give a confidence level of 80% for a standard normal variable.

$\mu = 0$, $\sigma = 1$



Classpad :

InvNorm → Tail : Centre

prob = 0.8

$\sigma = 1$

$\mu = 0$

$\therefore$ CI $= -1.28 < z < 1.28$

$\therefore$ ME ≈ 1.28

$\therefore ME \approx 1.28$

- A C% **confidence interval** is given by:

$$\hat{p} - k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Where:
p is the population proportion (unknown)
p̂ is a value of the sample proportion
n is the size of the sample from which p̂ was calculated
k is the number of standard deviation for C% confidence interval

* Commonly used levels of confidence for standard normal distribution:

Level of confidence, C%	90%	95%	98%	99%
z-score, "k"	1.645	1.960	2.326	2.576

To find "k" on classpad:
InvNormCDf: Tail -> center
Prob -> 0.95 (for C% = 95%)
σ -> 1
μ -> 0

- The **margin or error** (ME) is given by:

$$ME = k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \quad \underline{\underline{OR}} \quad ME = k\left(\frac{\sigma}{\sqrt{n}}\right)$$

Population S.D

Note:
*To be more confident that the confidence interval captures the true value of the population proportion p, a wider interval is required.

*To increase the precision of the estimate (lower margin of error), the sample size needs to be increased.

*There is 95% chance of the intervals containing P
*NOT that 95% chance that P is within the interval

eg: If you take large random samples over and over again from the same population, and make 95% confidence intervals for P, about 95% of the intervals should contain P.

Examples

CONFIDENCE INTERVAL EXAMPLES:

1. Twelve students from a sample of 30 students said that they liked the "Star Wars" movies. Estimate the 90% confidence interval for the probability of liking these movies. Interpret your result.

$$p \approx \hat{p} = \frac{12}{30} \text{ or } 0.4$$

$$CI: 0.4 - \underbrace{1.645}_{k \text{ for } 90\% \, CI = 1.645}\sqrt{\frac{0.4(1-0.4)}{30}} \le p \le 0.4 + 1.645\sqrt{\frac{0.4(1-0.4)}{30}}$$

$$CI: \quad 0.2529 \le p \le 0.5471 \checkmark$$

$$* \quad ME = 1.645\sqrt{\frac{0.4(1-0.4)}{30}} = 0.1471 \checkmark$$

Classpad Instructions:

Statistics → Calc → Interval

→ One-prop Z Int

→ C-Level : 0.9

x : 12

n : 30

Answer: Lower 0.25, Upper 0.55, $\hat{p} = 0.4$, $n = 30$

2. The probability of a Year 12 student driving to school is about 6%. How large a sample of students should be used to establish the probability to a 95% confidence level within 0.5% of the true probability?

↑margin of error !

$$ME = k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$0.005 = 1.96\sqrt{\frac{0.06(1-0.06)}{n}}$$

$$n \approx 8667$$

$\therefore$ About 9000 Yr 12 students would need to be surveyed!!!

NOT Practical! To decrease sample size, a larger margin of error must be accepted.

3. About 60% of Year 12 students obtain their driver's license before they complete Year 12. If 260 Year 12 students are asked, find the level of confidence if the reported rate must be within 5%.

↑ME !!

$$ME = k\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$0.05 = k\sqrt{\frac{0.6(1-0.6)}{260}}$$

$$k = 1.6457$$

$\therefore$ 90% confident!

Classpad:

NormCDf Lower: -1.6457
Upper: 1.6457
$\sigma = 1$
$\mu = 0$

Ex6A page 142
Q4, 6, 8, 14

Ex6B page 154
Q2, 4, 6, 10, 15