Chapter 6 – Functions, relations and transformations

Solutions to Exercise 6A

1 $A = \{1, 2, 3, 5, 7, 11, 15\}$
$B = \{7, 11, 25, 30, 32\}$
$C = \{1, 7, 11, 25, 30\}$
$A \cap B$ means must be in both A and B
$A \cup B$ means must be in either or any
$A \backslash B$ means in A but not B

a $A \cap B = \{7, 11\}$

b $A \cap B \cap C = \{7, 11\}$

c $A \cup C = \{1, 2, 3, 5, 7, 11, 15, 25, 30\}$

d $A \cup B =$
$\{1, 2, 3, 5, 7, 11, 15, 25, 30, 32\}$

e $A \cup B \cup C =$
$\{1, 2, 3, 5, 7, 11, 15, 25, 30, 32\}$

f $(A \cap B) \cup C = \{1, 7, 11, 25, 30\}$

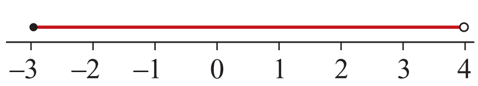
2 **a** $A \backslash B = \{1, 2, 3, 5, 15\}$

b $B \backslash A = \{25, 30, 32\}$

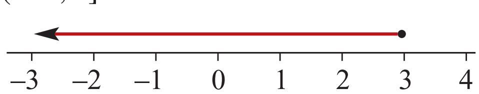
c $A \backslash C = \{2, 3, 5, 15\}$

d $C \backslash A = \{25, 30\}$

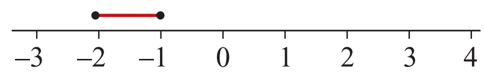
3 **a** $[-3, 4)$



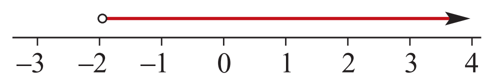
b $(-\infty, 3]$



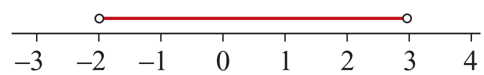
c $[-2, -1]$



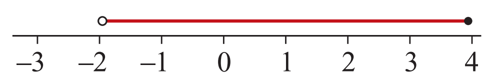
d $(-2, \infty)$



e $(-2, 3)$



f $(-2, 4]$



4 **a** $(-2, 1]$

b $[-3, 3]$

c $[-3, 2)$

d $(-1, 2)$

5 **a** $\{x: -1 \le x \le 2\} = [-1, 2]$

b $\{x: -4 < x \le 2\} = (-4, 2]$

c $\{y: 0 < y < \sqrt{2}\} = (0, \sqrt{2})$

d $\{y: -\frac{\sqrt{3}}{2} < y \le \frac{1}{\sqrt{2}}\} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right]$

e $\{x: x > -1\} = (-1, \infty)$

f $\{x: x \le -2\} = (-\infty, -2]$

g $\mathbb{R} = (-\infty, \infty)$

h $\mathbb{R}^+ \cup \{0\} = [0, \infty)$

i $\mathbb{R}^- \cup \{0\} = (-\infty, 0]$

6 $B = \{7, 11, 25, 30, 32\}$

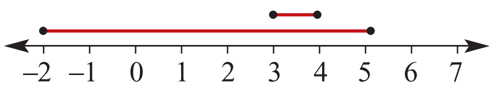
a $(-2, 10] \cap B = \{7\}$

b $(3, \infty) \cap B = \{7, 11, 25, 30, 32\} = B$

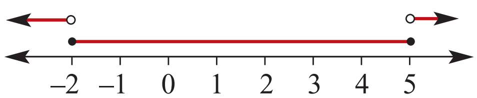
c $(2, \infty) \cup B = (2, \infty)$

d $(25, \infty) \cap B = \{30, 32\}$

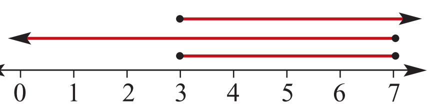
7 **a** $[-2, 5], [3, 4], [-2, 5] \cap [3, 4]$



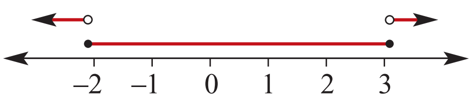
b $[-2, 5], \mathbb{R}\backslash[-2, 5]$



c $[3, \infty), (-\infty, 7], [3, \infty) \cap (-\infty, 7]$



d $[-2, 3], \mathbb{R}\backslash[-2, 3]$

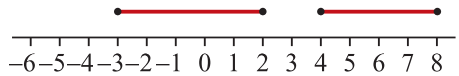


8 **a** $(-\infty, -2) \cup (-2, \infty)$

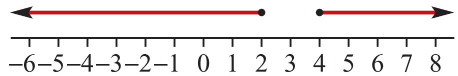
b $(-\infty, 3) \cup (3, \infty)$

c $(-\infty, 4) \cup (4, \infty)$

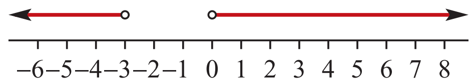
9 **a**



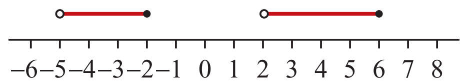
b



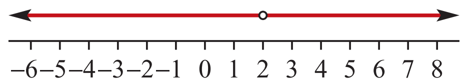
c



d



e



f

10 **a** $(-6, -3)$

b $\varnothing$

c $[-6, 0]$

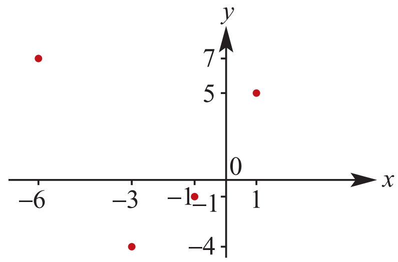
d $[-1, 2]$

e $\{1\}$

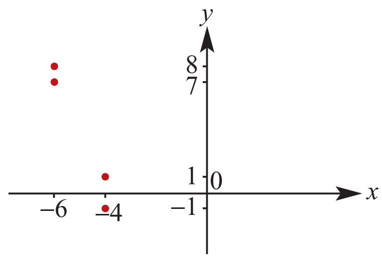
f $(-10, -1]$

Solutions to Exercise 6B

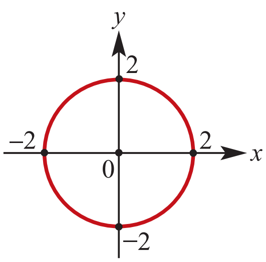
1 a Domain = $\{-3, -1, -6, 1\}$;
Range = $\{-4, -1, 7, 5\}$



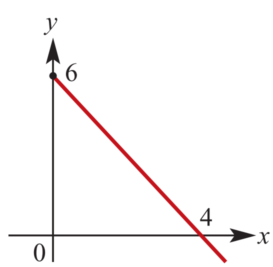
b Domain = $\{-4, -6\}$; Range = $\{-1, 1, 7, 8\}$



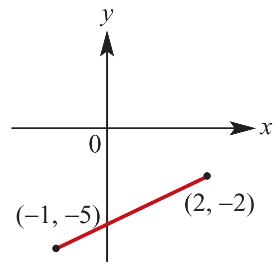
c Domain = $[-2, 2]$
Range = $[-2, 2]$



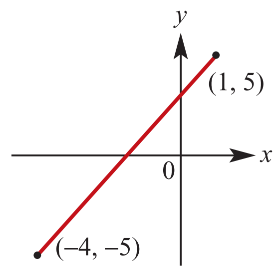
d Domain = $[0, \infty)$
Range = $(-\infty, 6]$



e Domain = $[-1, 2]$
Range = $[-5, -2]$



f Domain = $[-4, 1]$
Range = $[-5, 5]$



2 a Domain = $[-2, 2]$; Range = $[-1, 2]$

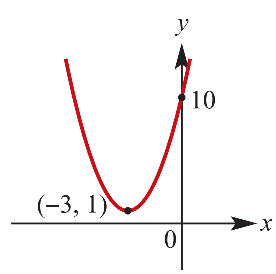
b Domain = $[-2, 2]$; Range = $[-2, 2]$

c Domain = $\mathbb{R}$; Range = $[-1, \infty)$

d Domain = $\mathbb{R}$; Range = $(-\infty, 4]$

3 a $x^2 + 6x + 10 = x^2 + 6x + 9 - 9 + 10$
$= (x + 3)^2 + 1$

$y = (x + 3)^2 + 1$
Range = $[1, \infty)$

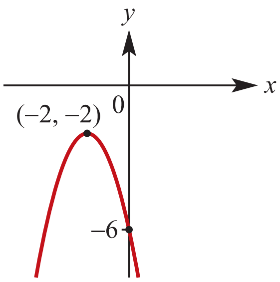


b

$$\begin{aligned} -x^2 - 4x - 6 &= -(x^2 + 4x + 6) \\ &= -[x^2 + 4x + 4 - 4 + 6] \\ &= -(x+2)^2 - 2 \end{aligned}$$

$y = -(x+2)^2 - 2$

Range $= (-\infty, -2]$

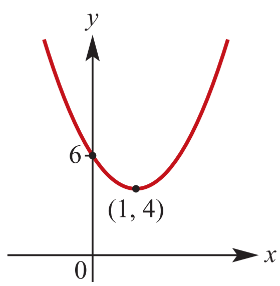


c

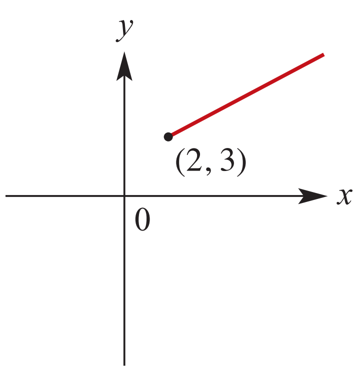
$$\begin{aligned} 2x^2 - 4x + 6 &= 2(x^2 - 2x + 3) \\ &= 2[x^2 - 2x + 1 - 1 + 3] \\ &= 2(x-1)^2 + 4 \end{aligned}$$

$y = 2(x-1)^2 + 4$

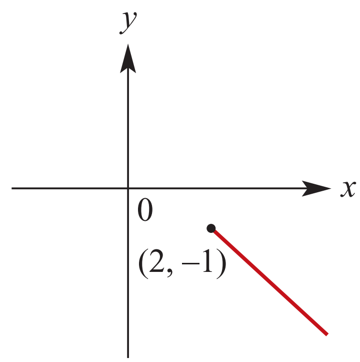
Range $= [4, \infty)$



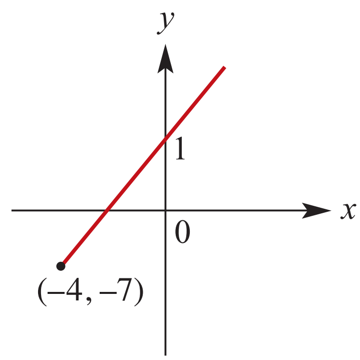
4 **a** $y = x + 1; x \in [2, \infty)$; Range $= [3, \infty)$



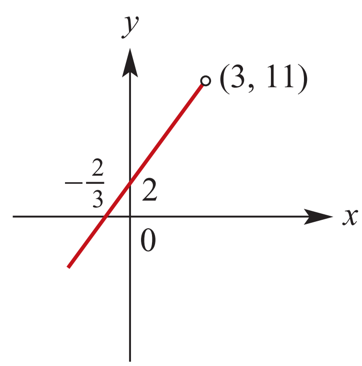
b $y = -x + 1; x \in [2, \infty)$;
Range $= (-\infty, -1]$



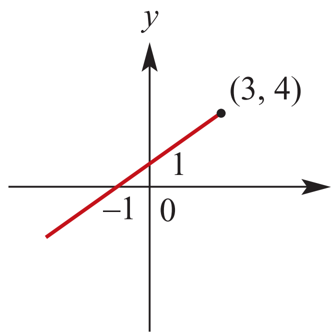
c $y = 2x + 1; x \in [-4, \infty)$;
Range $= [-7, \infty)$



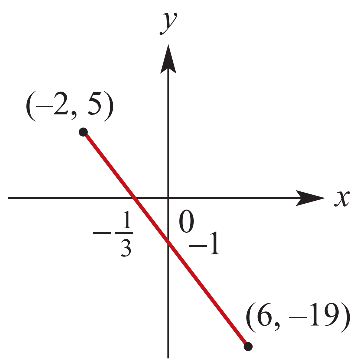
d $y = 3x + 2; x \in (-\infty, 3)$;
Range $= (-\infty, 11)$



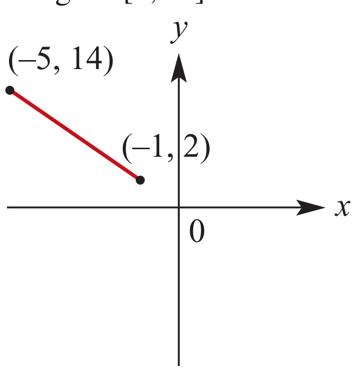
e $y = x + 1; x \in (-\infty, 3]$;
Range $= (-\infty, 4]$



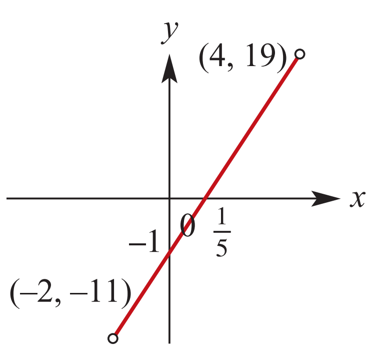
f $y = -3x - 1; x \in [-2, 6]$;
Range $= [-19, 5]$



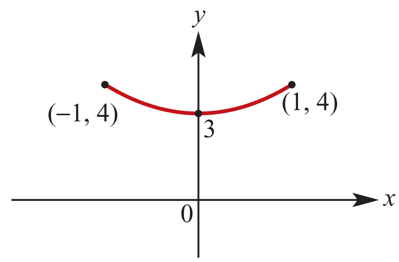
g $y = -3x - 1; x \in [-5, -1]$;
Range $= [2, 14]$



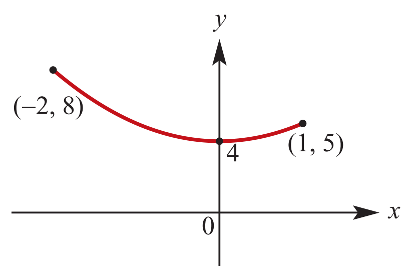
h $y = 5x - 1; x \in (-2, 4)$;
Range $= (-11, 19)$



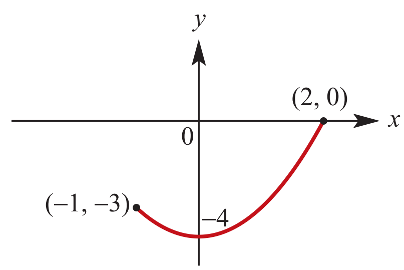
5 **a** $y = x^2 + 3,\ x \in [-1, 1]$
Range $= [3, 4]$



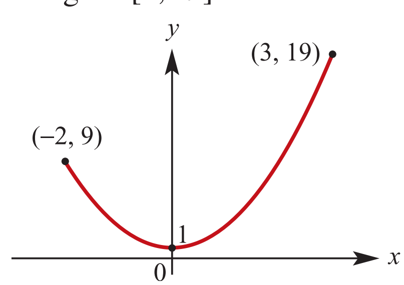
b $y = x^2 + 4,\ x \in [-2, 1]$
Range $= [4, 8]$



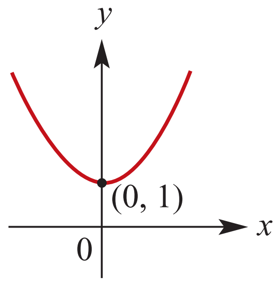
c $y = x^2 - 4,\ x \in [-1, 2]$
Range $= [-4, 0]$



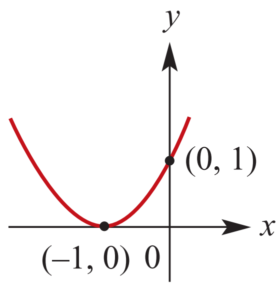
d $y = 2x^2 + 1,\ x \in [-2, 3]$
Range $= [1, 19]$



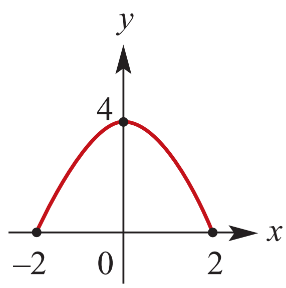
6 **a** $\{(x, y) : y = x^2 + 1\}$;
Range $= [1, \infty)$



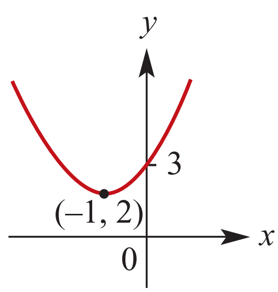
b $\{(x, y): y = x^2 + 2x + 1\}$;
Range = $[0, \infty)$



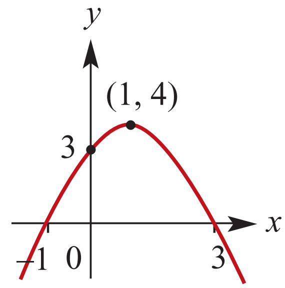
c $\{(x, y): y = 4 - x^2; x \in [-2, 2]\}$;
Range = $[0, 4]$



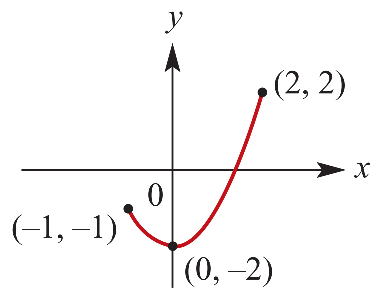
d $\{(x, y): y = x^2 + 2x + 3\}$;
Range = $[2, \infty)$



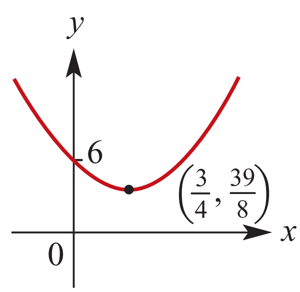
e $\{(x, y): y = -x^2 + 2x + 3\}$;
Range = $(-\infty, 4]$



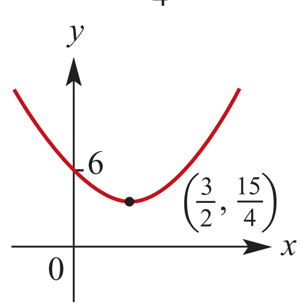
f $\{(x, y): y = x^2 - 2; x \in [-1, 2]\}$;
Range = $[-2, 2]$



g $\{(x, y): y = 2x^2 - 3x + 6\}$;
Range = $[\frac{39}{8}, \infty)$

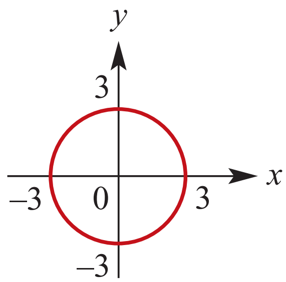


h $\{(x, y): y = 6 - 3x + x^2\}$;
Range = $[\frac{15}{4}, \infty)$

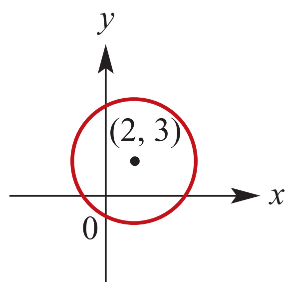


7 a $\{(x, y): x^2 + y^2 = 9\}$
Max. Domain = $[-3, 3]$

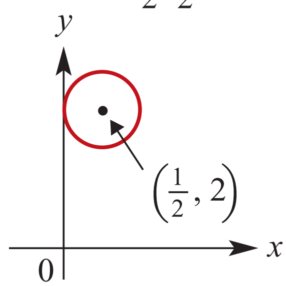
Range $= [-3, 3]$



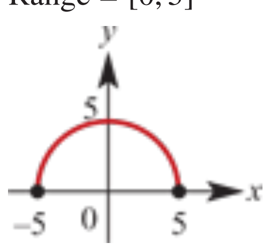
b $\{(x, y) : (x-2)^2 + (y-3)^2 = 16\}$
Max. Domain $= [-2, 6]$
Range $= [-1, 7]$



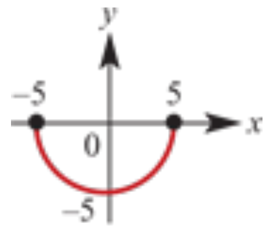
c $\{(x, y) : (2x-1)^2 + (2y-4)^2 = 1\}$
Max. Domain $= [0, 1]$
Range $= [\frac{3}{2}, \frac{5}{2}]$



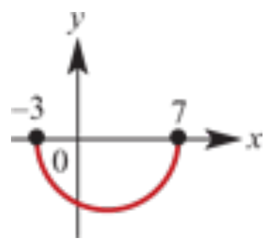
d $\{(x, y) : y = \sqrt{25 - x^2}\}$
Max. Domain $= [-5, 5]$,
Range $= [0, 5]$



e $\{(x, y) : y = -\sqrt{25 - x^2}\}$
Max. Domain $= [-5, 5]$,
Range $= [-5, 0]$



f $(x, y) : y = -\sqrt{25 - (x-2)^2}\}$
Max. Domain $= [-3, 7]$
Range $= [-5, 0]$



8 **a** Domain $= \mathbb{R} \setminus \left\{\frac{5}{2}\right\}$; Range $= \mathbb{R} \setminus \{3\}$

When $x = 0$, $y = -\frac{2}{5} + 3 = \frac{13}{5}$

When $y = 0$

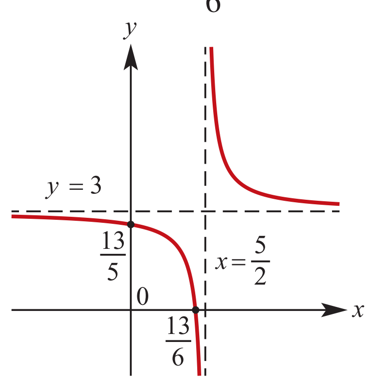
$$\frac{2}{2x-5} + 3 = 0$$

$$\frac{2}{2x-5} = -3$$

$$2 = -3(2x-5)$$

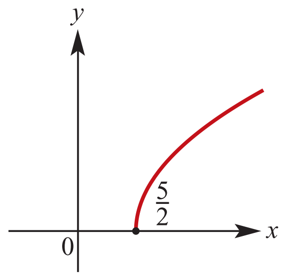
$$2 = -6x + 15$$

$$x = \frac{13}{6}$$



b Domain $= \left[\frac{5}{2}, \infty\right)$

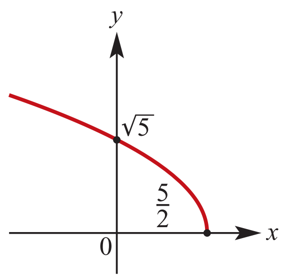
Range = $\mathbb{R}^+ \cup \{0\}$



c $y = \sqrt{5 - 2x} = \sqrt{-(2x - 5)}$

Domain = $\left(-\infty, \frac{5}{2}\right]$

Range = $\mathbb{R}^+ \cup \{0\}$



d $y = \sqrt{4 - (x - 5)^2}$

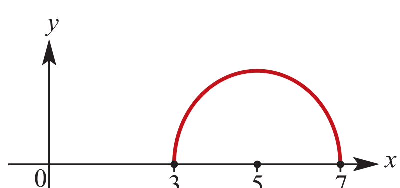
Squaring gives:

$y^2 = 4 - (x - 5)^2$

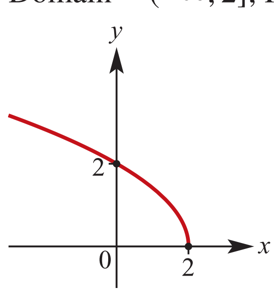
$(x - 5)^2 + y^2 = 4$ This last equation is that of a circle of radius 2 and centre $(5, 0)$.

It is the 'top half' of the circle.

Domain = $[3, 7]$; Range = $[0, 2]$

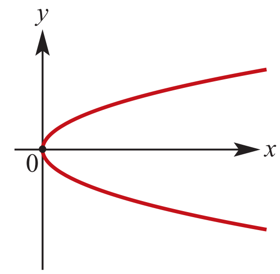


e Domain = $(-\infty, 2]$; Range= $[0, \infty)$



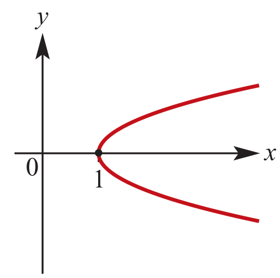
9 **a** Domain = $[0, \infty)$

Range = $\mathbb{R}$



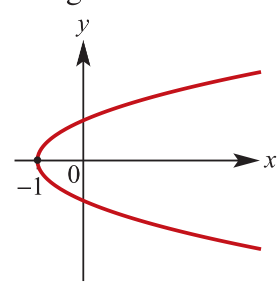
b Domain = $[1, \infty)$

Range = $\mathbb{R}$



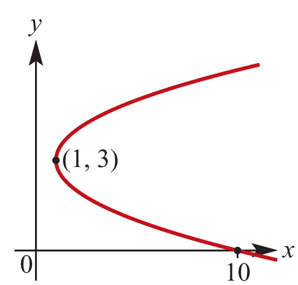
c Domain = $[-1, \infty)$

Range = $\mathbb{R}$



d Domain = $[1, \infty)$

Range = $\mathbb{R}$



Solutions to Exercise 6C

1 a $\{(0,1),(0,2),(1,2),(2,3),(3,4)\}$ is not a function because it is 1– many; (0, 1) and (0,2) Domain = $\{0, 1, 2, 3\}$; Range = $\{1, 2, 3, 4\}$

b $\{(-2,-1),(-1,-2),(0,2),(1,4),(2,-5)\}$ is a function because it is $1 - 1$; Domain = $\{-2,-1,0,1,2\}$; Range = $\{-5,-2,-1,2,4\}$

c Not a function; Domain = $\{-1,0,3,5\}$; Range = $\{1,2,4,6\}$

d $\{(1,3),(2,3),(4,3),(5,3),(6,3)\}$ is a function because it is many -1; Domain = $\{1,2,4,5,6\}$; Range = $\{3\}$

2 a Function

b Function

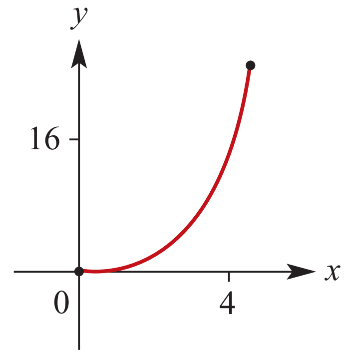
c Function

d Not a function

e Function

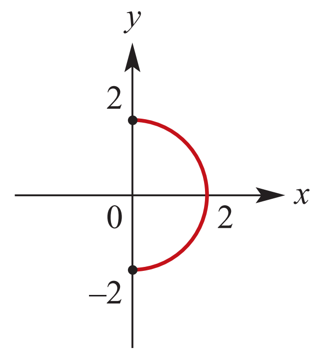
f Not a function

3 a $y = x^2; x \in [0,4]$; Range = $[0,16]$; function because $1 - 1$ relation



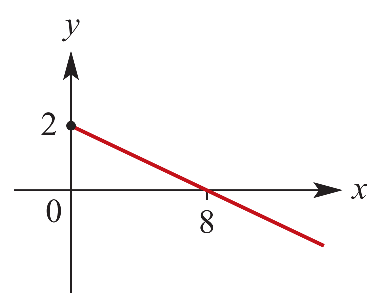
a function
Domain = [0, 4]
Range = [0, 16]

b $\{(x,y): x^2 + y^2 = 4\}; x \in [0,2]$; Range = $[-2,2]$; not a function because 1– many relation



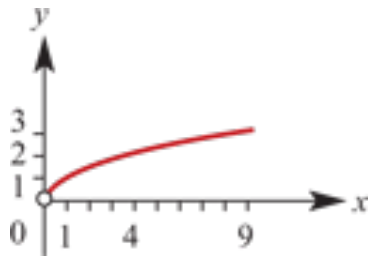
not a function
Domain = [0, 2]
Range = [–2, 2]

c $\{(x,y): 2x + 8y = 16; x \in [0,\infty)\}$; Range = $(-\infty, 2]$; function because $1 \to 1$ relation

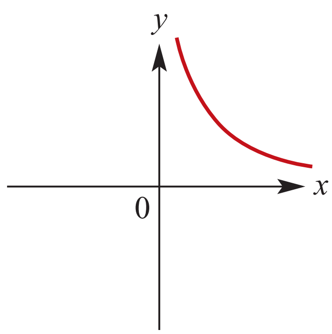


A function
Domain = $[0, \infty)$
Range = $(-\infty, 2]$

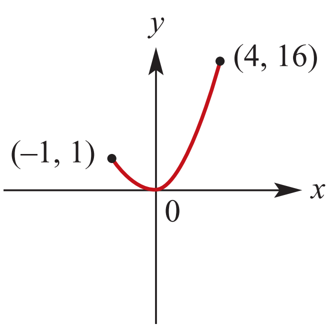
d $y = \sqrt{x}; x \in \mathbb{R}^+$, function because $1 \to 1$
relation; Range = R^+ or $(0, \infty)$



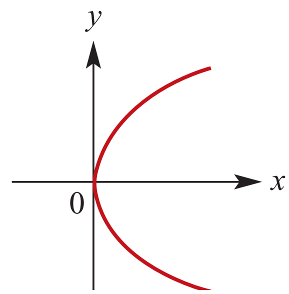
e $\{(x, y): y = \dfrac{1}{x}; x \in \mathbb{R}^+\}$; function because $1 \to 1$ relation; Range = $\mathbb{R}^+$ or $(0, \infty)$
Domain = $\mathbb{R}^+$



f $y = x^2; x \in [-1, 4]$; Range = $[0, 16]$; function because many $\to$ 1 relation



g $\{(x, y): x = y^2; x \in \mathbb{R}^+\}$;
Range = $\mathbb{R}\backslash\{0\}$; not a function because $1 \to$ many relation



4 a $\{(x, y): y = 3x + 2\}$ can be expressed as $f: \mathbb{R} \to \mathbb{R}, f(x) = 3x + 2$

b $\{(x, y): 2y + 3x = 12\}$

$$2y + 3x = 12$$

$$\therefore \quad 2y = 12 - 3x$$

$$\therefore \quad y = 6 - \frac{3x}{2}$$

$f: \mathbb{R} \to \mathbb{R}, f(x) = 6 - \dfrac{3x}{2}$

c $\{(x, y): y = 2x + 3, x \geq 0\}$
can be expressed as
$f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, f(x) = 2x + 3$

d $y = 5x + 6, -1 \leq x \leq 2$ can be expressed as
$f: [-1, 2] \to \mathbb{R}, f(x) = 5x + 6$

e $y + x^2 = 25, -5 \leq x \leq 5$
$\therefore\ y = 25 - x^2$ can be expressed as
$f: [-5, 5] \to \mathbb{R}, f(x) = 25 - x^2$

f $y = 5x - 7, 0 \leq x \leq 1$ can be expressed as
$f: [0, 1] \to \mathbb{R}, f(x) = 5x - 7$

5 a $\{(x, -2): x \in \mathbb{R}\}$ is a function because it is many -1; Domain = R,
Range = $\{-2\}$

b $\{(3, y): y \in \mathbb{Z}\}$ is not a function because it is $1 \to$ many; Domain = $\{3\}$,
Range = Z

c $y = -x + 3$ is a function because it is $1 \to 1$; Domain $= \mathbb{R}$, Range $= \mathbb{R}$

d $y = x^2 + 5$ is a function because it is many $\to 1$; Domain $= \mathbb{R}$, Range $= [5, \infty)$

e $\{(x, y) : x^2 + y^2 = 9\}$ is not a function because it is many $\to$ many; Domain $= [-3, 3]$, Range $= [-3, 3]$

6 a $f(x) = 2x - 3$

i $f(0) = 2(0) - 3 = -3$

ii $f(4) = 2(4) - 3 = 5$

iii $f(-1) = 2(-1) - 3 = -5$

iv $f(6) = 2(6) - 3 = 9$

v $f(x - 1) = 2(x - 1) - 3 = 2x - 5$

vi $f\left(\dfrac{1}{a}\right) = \dfrac{2}{a} - 3$

b $g(x) = \dfrac{4}{x}$

i $g(1) = \dfrac{4}{1} = 4$

ii $g(-1) = \dfrac{4}{-1} = -4$

iii $g(3) = \dfrac{4}{3}$

iv $g(2) = \dfrac{4}{2} = 2$

c $g(x) = (x - 2)^2$

i $g(4) = (4 - 2)^2 = 4$

ii $g(-4) = (-4 - 2)^2 = 36$

iii $g(8) = (8 - 2)^2 = 36$

iv $g(a) = (a - 2)^2$

d $f(x) = 1 - \dfrac{1}{x}$

i $f(1) = 1 - \dfrac{1}{1} = 0$

ii $$f(1 + a) = 1 - \frac{1}{1 + a} = \frac{1 + a - 1}{1 + a} = \frac{a}{a + 1}$$

iii $$f(1 - a) = 1 - \frac{1}{1 - a} = \frac{1 - a - 1}{1 - a} = -\frac{a}{1 - a} = \frac{a}{a - 1}$$

iv $f(\dfrac{1}{a}) = 1 - \dfrac{1}{1/a} = 1 - a$

7 $f(x) = 2x + 1$

a $f(2) = 2 \times 2 + 1 = 5$ and $f(t) = 2t + 1$

b
$$\begin{aligned} f(x) &= 6 \\ 2x + 1 &= 6 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

c
$$\begin{aligned} f(x) &= 0 \\ 2x + 1 &= 0 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

d
$$\begin{aligned} f(t) &= t \\ 2t + 1 &= t \\ t &= -1 \end{aligned}$$

e $f(x) \geq x$

$2x + 1 \geq x$

$x \geq -1$

f $f(x) \leq 3x$

$2x + 1 \leq 3x$

$-x \leq -1$

$x \geq 1$

8 **a** $f(x) = 5x - 2 = 3$
$\therefore\ 5x = 5, \therefore\ x = 1$

b $f(x) = \frac{1}{x} = 6$
$\therefore 1 = 6x, x = \frac{1}{6}$

c $f(x) = x^2 = 9$
$\therefore x = \pm\sqrt{9} = \pm 3$

d $f(x) = (x + 1)(x - 4) = 0$
$\therefore x = -1, 4$

e $f(x) = x^2 - 2x = 3$
$\therefore x^2 - 2x - 3 = 0$
$\therefore (x - 3)(x + 1) = 0$
$\therefore x = -1, 3$

f $f(x) = x^2 - x - 6 = 0$
$\therefore (x - 3)(x + 2) = 0$
$\therefore x = -2, 3$

9 $g(x) = x^2 + 2x$ and
$h(x) = 2x^3 - x^2 + 6$

a $g(-1) = (-1)^2 + 2(-1) = -1$
$g(2) = (2)^2 + 2(2) = 8$
$g(-2) = (-2)^2 + 2(-2) = 0$

b $h(-1) = 2(-1)^3 - (-1)^2 + 6 = 3$
$h(2) = 2(2)^3 - (2)^2 + 6 = 18$
$h(-2) = 2(-2)^3 - (-2)^2 + 6 = -14$

c **i** $g(-3x) = (-3x)^2 + 2(-3x) =$
$9x^2 - 6x$

ii $g(x - 5) = (x - 5)^2 + 2(x - 5)$
$= x^2 - 8x + 15$

iii $h(-2x) = 2(-2x)^3 - (-2x)^2 + 6$
$= -16x^3 - 4x^2 + 6$

iv $g(x + 2) = (x + 2)^2 + 2(x + 2)$
$= x^2 + 6x + 8$

v $h(x^2) = 2(x^2)^3 - (x^2)^2 + 6$
$= 2x^6 - x^4 + 6$

10 $f(x) = 2x^2 - 3$

a $f(2) = 2(2)^2 - 3 = 5$
$f(-4) = 2(-4)^2 - 3 = 29$

b The Range of f is $[-3, \infty)$

11 $f(x) = 3x + 1$

a The image of $2 = 3(2) + 1 = 7$

b The pre-image of 7: $3x + 1 = 7$
so $3x = 6$ and $x = 2$

c $\{x\colon f(x) = 2x\}$:
$3x + 1 = 2x,\ \therefore\ x = -1$

12 $f(x) = 3x^2 + 2$

a The image of $0 = 3(0)^2 + 2 = 2$

b The pre-image(s) of 5:
$3x^2 + 2 = 5$
$\therefore\ 3x^2 = 3, \therefore\ x = \pm 1$

c $\{x\colon f(x) = 11\}$

$\therefore\ 3x^2 + 2 = 11$

$\therefore\ 3x^2 = 9$

$\therefore\ x^2 = 3, \therefore\ x = \pm\sqrt{3}$

13 $f(x) = 7x + 6$ and $g(x) = 2x + 1$

a $\{x\colon f(x) = g(x)\}$

$\therefore\ 7x + 6 = 2x + 1$

$\therefore\ 5x = -5, \therefore\ x = -1$

b $\{x\colon f(x) > g(x)\}$

$\therefore\ 7x + 6 > 2x + 1$

$\therefore\ 5x > -5, \therefore x > -1$

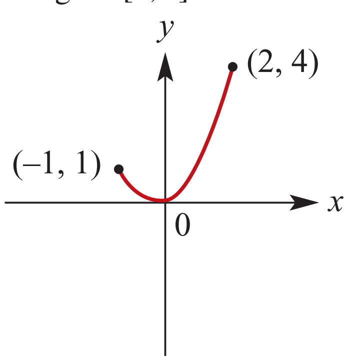
c $\{x\colon f(x) = 0\}$

$\therefore\ 7x + 6 = 0$

$\therefore\ 7x = -6, \therefore x = -\dfrac{6}{7}$

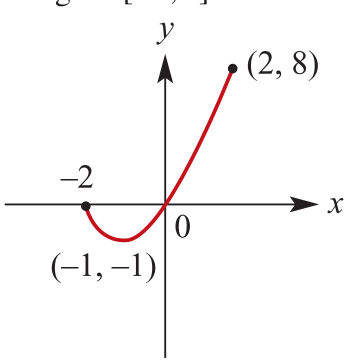
14 **a** $f\colon [-1, 2] \to \mathbb{R}, f(x) = x^2$

Range $= [0, 4]$



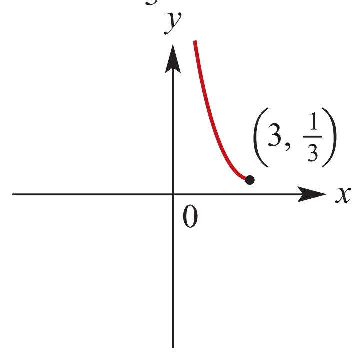
b $f\colon [-2, 2] \to \mathbb{R}, f(x) = x^2 + 2x$

Range $= [-1, 8]$



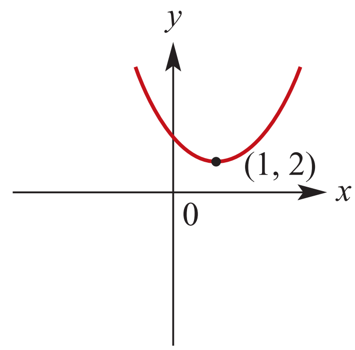
c $f\colon (0, 3] \to \mathbb{R}, f(x) = \dfrac{1}{x}$

Range $= [\frac{1}{3}, \infty)$



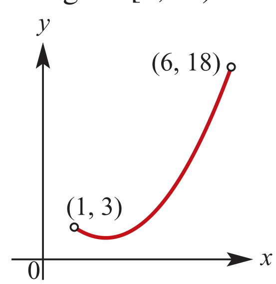
d $f\colon \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 2x + 3$

Range $= [2, \infty)$



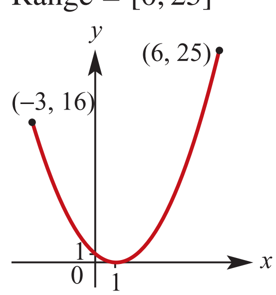
e $f\colon (1, 6) \to \mathbb{R}, f(x) = x^2 - 4x + 6$

Range $= [2, 18)$



f $f\colon [-3, 6] \to \mathbb{R}, f(x) = x^2 - 2x + 1$

Range $= [0, 25]$



15 a $y = 7 - x$,
Max. Domain $\mathbb{R}$, Range $\mathbb{R}$

b $y = 2\sqrt{x}$
Max. Domain $[0, \infty)$, Range $[0, \infty)$

c $y = x^2 + 1$,
Max. Domain $\mathbb{R}$, Range $[1, \infty)$

d $y = -\sqrt{9 - x^2}$,
Max. Domain $[-3, 3]$ because
$9 - x^2 \geq 0$,
Range $[-3, 0]$

e $y = \dfrac{1}{\sqrt{x}}$,
Max. Domain $\mathbb{R}^+$, Range $\mathbb{R}^+$
(Different from **b** because you can't have $\frac{1}{0}$.)

f $y = 3 - 2x^2$,
Max. Domain $\mathbb{R}$, Range $(-\infty, 3]$

g $y = \sqrt{x - 2}$,
Max. Domain $[2, \infty)$ because
$x - 2 \geq 0$,
Range $[0, \infty)$

h $y = \sqrt{2x - 1}$,
Max. Domain $[\frac{1}{2}, \infty)$ because
$2x - 1 \geq 0$,
Range $[0, \infty)$

i $y = \sqrt{3 - 2x}$,
Max. Domain $(-\infty, \frac{3}{2}]$ because
$3 - 2x \geq 0$,
Range $[0, \infty)$

j $y = \dfrac{1}{2x - 1}$,
Max. Domain $\mathbb{R}\backslash\{\frac{1}{2}\}$ because
$2x - 1 \neq 0$,
Range $\mathbb{R}\backslash\{0\}$ because $\dfrac{1}{2x - 1} \neq 0$

k $y = \dfrac{1}{(2x - 1)^2} - 3$,
Max. Domain $\mathbb{R}\backslash\{\frac{1}{2}\}$ because
$2x - 1 \neq 0$,
Range $(-3, \infty)$ because $\dfrac{1}{(2x - 1)^2} > 0$

l $y = \dfrac{1}{2x - 1} + 2$,
Max. Domain $\mathbb{R}\backslash\{\frac{1}{2}\}$ because
$2x - 1 \neq 0$,
Range $1/\{2\}$ because $\dfrac{1}{2x - 1} \neq 0$

16 a Domain $= [4, \infty)$; Range $= [0, \infty)$

b Domain $= (-\infty, 4]$; Range $= [0, \infty)$

c Domain $= [2, \infty)$; Range $= [3, \infty)$

d Domain $= \mathbb{R} \setminus \{4\}$; Range $= \mathbb{R} \setminus \{0\}$

e Domain $= \mathbb{R} \setminus \{4\}$; Range $= \mathbb{R} \setminus \{3\}$

f Domain $= \mathbb{R} \setminus \{-2\}$; Range $= \mathbb{R} \setminus \{-3\}$

17 a $f(x) = 3x + 4$;
Max. Domain $\mathbb{R}$, Range $\mathbb{R}$

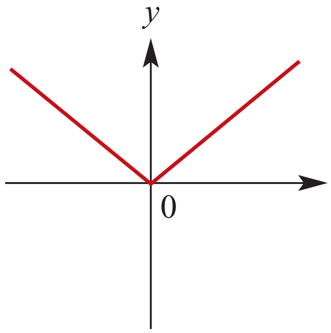
b $g(x) = x^2 + 2$,
Max. Domain $\mathbb{R}$, Range $[2, \infty)$

c $y = -\sqrt{16 - x^2}$,
Max. Domain $[-4, 4]$ because
$16 - x^2 \geq 0$,
Range $[-4, 0]$

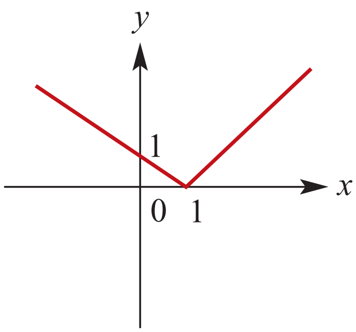
d $y = \dfrac{1}{x + 2}$,
Max. Domain $\mathbb{R}\backslash\{-2\}$ because
$x + 2 \neq 0$,
Range $\mathbb{R}\backslash\{0\}$ because $\dfrac{1}{x + 2} \neq 0$

Solutions to Exercise 6D

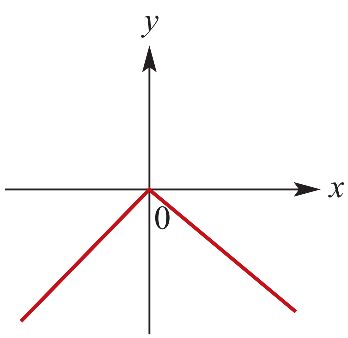
1 a $h(x) = x, x \geq 0$ and $h(x) = -x, x < 0$;
Range $= [0, \infty)$



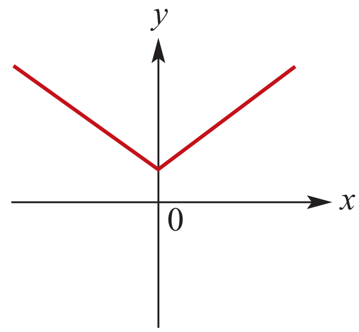
b $h(x) = x - 1, x \geq 1$ and $h(x) = 1 - x$, $x < 1$; Range $= [0, \infty)$



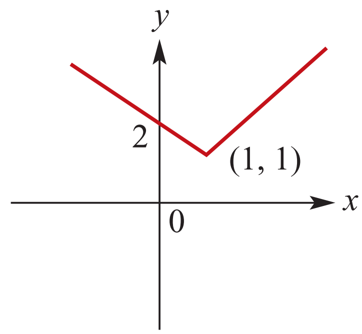
c $h(x) = -x, x \geq 0$ and $h(x) = x, x < 0$;
Range $= (-\infty, 0]$



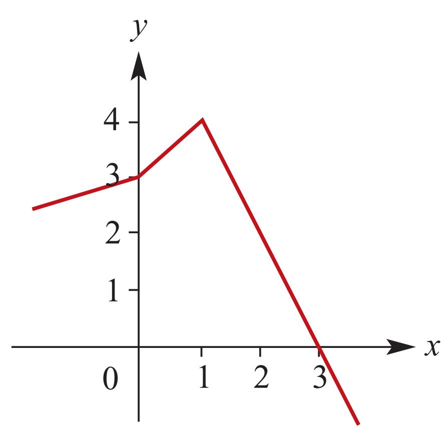
d $h(x) = 1 + x, x \geq 0$ and $h(x) = 1 - x$, $x < 0$; Range $= [1, \infty)$



e $h(x) = x, x \geq 1$ and
$h(x) = 2 - x, x < 1$;
Range $= [1, \infty)$

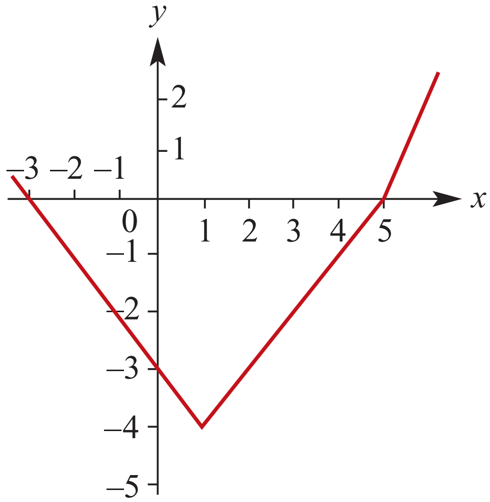


2 a $f(x) = \dfrac{2}{3}x + 3, x < 0$
$f(x) = x + 3, 0 \leq x \leq 1$
$f(x) = -2x + 6, x > 1$
Axis intercepts at $(-\dfrac{9}{2}, 0)$, (0,3) and (3,0)

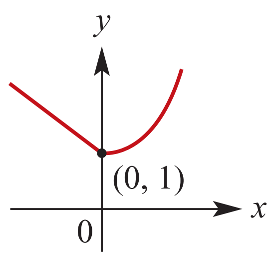


b Range $= (-\infty, 4]$

3 $g(x) = -x - 3, x < 1$
$g(x) = x - 5, 1 \leq x \leq 5$
$g(x) = 3x - 15, x > 5$
Axis intercepts at $(-3, 0)$, $(0, -3)$ and $(5, 0)$

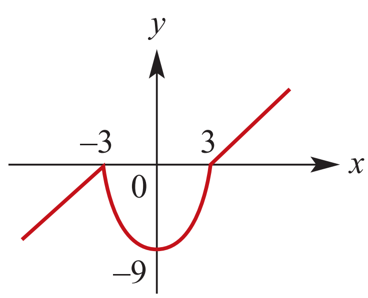


4 **a** $h(x) = x^2 + 1, x \geq 0$
$h(x) = 1 - x, x < 0$



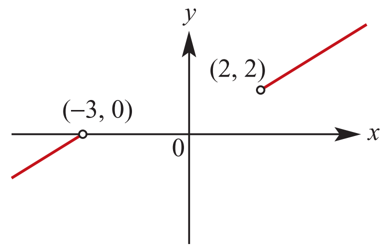
b Range = $[1, \infty)$

5 **a** $f(x) = -x + 3, x < -3$
$f(x) = x^2 - 9, -3 \leq x \leq 3$
$f(x) = x - 3, x > 3$

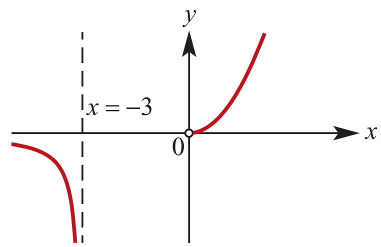


b Range = $\mathbb{R}$

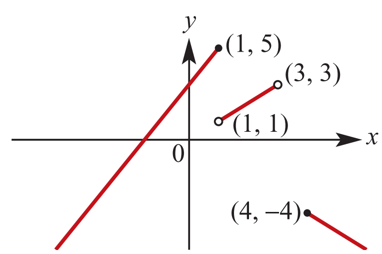
6 **a** Range = $(-\infty, 0) \cup (2, \infty)$



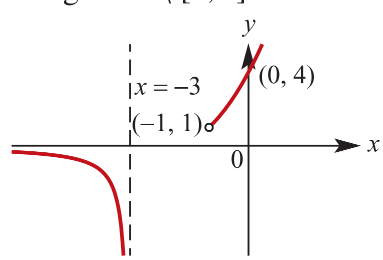
b Range = $\mathbb{R} \setminus \{0\}$



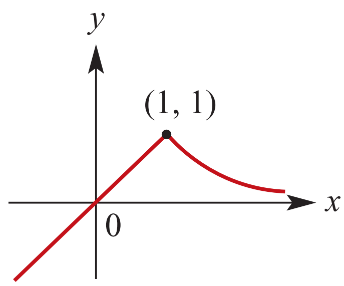
c Range = $(-\infty, 5]$



d Range = $\mathbb{R} \setminus [0, 1]$



7 **a** $f(x) = \dfrac{1}{x}, x > 1$
$f(x) = x, x \leq 1$



b Range $= (-\infty, 1]$

8 Line connecting $(-3, 0)$ and $(-1, 2)$ has gradient $= \dfrac{2-0}{-1-(-3)} = 1$

Using $(-3, 0)$: $y - 0 = 1(x - -3)$

$\therefore\ y = x + 3$ for $[-3, -1]$

Line connecting $(-1, 2)$ and $(2, -1)$ has gradient $= \dfrac{-1-2}{2-(-1)} = -1$

Using $(-1, 2)$: $y - 2 = -1(x - -1)$

$\therefore\ y = 1 - x$ for $[-1, 2]$

Line connecting $(2, -1)$ and $(4, -2)$ has gradient $= \dfrac{-2 - -1}{4-2} = -\dfrac{1}{2}$

Using $(2, -1)$: $y + 1 = -\dfrac{1}{2}(x - 2)$

$\therefore\ y = -\dfrac{x}{2}$ for $[2, 4]$

$$f(x) = \begin{cases} x + 3; -3 \le x < -1 \\ 1 - x; -1 \le x < 2 \\ -\dfrac{x}{2}; 2 \le x \le 4 \end{cases}$$

Solutions to Exercise 6E

1 **a** $L(C) = 0.002C + 25; -273 \le C \le 1000$
Most metals will melt at over 1000 degrees and $C = -273$ is absolute zero.

b **i** $L(30) = (0.002)30 + 25 = 25.06$ cm

ii $L(16) = (0.002)16 + 25 = 25.032$ cm

iii $L(100) = (0.002)100 + 25 = 25.20$ cm

iv $L(500) = (0.002)500 + 25 = 26.00$ cm

2 **a** $f(x) = a + bx$

$$\begin{array}{ll} f(4) = -1 & \therefore\ a + 4b = -1 \\ f(8) = 1 & \therefore\ a + 8b = 1 \\ \hline & \therefore\ \quad 4b = 2 \end{array}$$

$\therefore\ b = \dfrac{1}{2}; a = -3$

b $f(x) = 0, \therefore \dfrac{x}{2} - 3 = 0$

$\therefore \quad x = 6$

3 If (fx) is parallel to $g(x) = 2 - 5x$ then the gradient of $f(x) = -5$ and
$f(x) = -5x + c$
$f(0) = 7, \therefore c = 7$
$f(x) = -5x + 7$

4 $f(x) = ax + b$

$$\begin{array}{ll} f(-5) = -12 & \therefore\ \ -5a + b = -12 \\ f(7) = 6 & \therefore\ \ \ \ 7a + b = \ \ 6 \\ \hline & \therefore\ 12a \qquad = 18 \end{array}$$

a **i** $f(0) = b = -\dfrac{9}{2}$

ii $f(1) = \dfrac{3}{2} - \dfrac{9}{2} = -3$

b $f(x) = \dfrac{1}{2}(3x - 9) = 0$
$\therefore 3x - 9 = 0, \therefore x = 3$

5 $f(x) = a(x - b)(x - c)$
$f(2) = f(4) = 0$ so $b = 2, c = 4$
If 7 is maximum then $a < 0$; Max. occurs halfway between 2 and 4, i.e. at $x = 3$:
$f(x) = a(3 - 2)(3 - 4) = 7$
$\therefore a = -7$
$\therefore f(x) = -7(x - 2)(x - 4)$
OR $f(x) = -7x^2 + 42x - 56$

6 $f(x) = x^2 - 6x + 16$

$= x^2 - 6x + 9 + 7$

$= (x - 3)^2 + 7$

Range of $f = [7, \infty)$

7 $f: \mathbb{R} \to \mathbb{R}, f(x) = m(x - p)(x - q)$
$f(4) = f(5) = 0$, so $p = 4, q = 5$
$f(0) = 2$, so $mpq = 2$ and $m = 0.1$
$f(x) = 0.1(x - 4)(x - 5)$

$= 0.1(x^2 - 9x + 20)$

$= 0.1x^2 - 0.9x + 2$

$a = 0.1, b = -0.9, c = 2$
OR Use $f(0) = 2$ so $c = 2$:
$f(4) = 0$, so $16a + 4b + 2 = 0$
$f(5) = 0$, so $25a + 9b + 2 = 0$
and use simultaneous equations or matrices.

8 $f(x) = ax^2 + bx + c$

$f(0) = 10$ so $c = 10$

Max. value $= 18$ at $x = -\dfrac{b}{2a}$:

$$f\left(-\frac{b}{2a}\right) = 18$$

$$= a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + 10$$

$$= \frac{b^2}{4a} - \frac{b^2}{2a} + 10$$

$$= -\frac{b^2}{4a} + 10$$

$\therefore\ -\dfrac{b^2}{4a} = 8$

$\therefore\ b^2 = -32a\ \ldots$**(1)**

$f(1) = 0$

$\therefore\ a + b + 10 = 0$

$\therefore\ b = -10 - a$

$\therefore\ b^2 = (10 + a)^2 \ldots$**(2)**

Equate **(1)** and **(2)**:

$\therefore\ (10 + a)^2 = -32a$

$\therefore\ a^2 + 20a + 100 = -32a$

$\therefore\ a^2 + 52a + 100 = 0$

$\therefore\ (a + 50)(a + 2) = 0$

$\therefore\ a = -2, -50$

If $a = -2, b = -8$; if $a = -50, b = 40$

$\therefore\ f(x) = -2x^2 - 8x + 10$

$g(x) = -50x^2 + 40x + 10$

OR $f(x) = -2(x - 1)(x + 5)$

$g(x) = -10(5x + 1)(x - 1)$

9 **a** $f(x) = 3x^2 - 5x - k$

$f(x) > 1$ for all real x

So $f(x) - 1 >$ for all real x

$13x^2 - 5x - (k + 1) > 0$ for all real x.

Then there are two real solutions to the equation $3x^2 - 5x - (k + 1) = 0$,

so $\Delta < 0$.

$\therefore\ 12k < -37$

$\therefore\ k < -\dfrac{37}{12}$

< 0 if $k < -\dfrac{37}{12}$

b $a > 0$ so the curve is an upright parabola, so the vertex is the minimum value which occurs at $x = -\dfrac{b}{2a}$

For $a = 3$ and $b = -5$, $x = \dfrac{5}{6}$

$f\left(\dfrac{5}{6}\right) = 3\left(\dfrac{5}{6}\right)^2 - 5\left(\dfrac{5}{6}\right) - k = 0$

$\therefore\ \dfrac{25}{12} - \dfrac{25}{6} - k = 0$

$\therefore\ k = -\dfrac{25}{12}$

Solutions to Exercise 6F

1 **a** $(-3, 4) \to (-3 + 2, 4 - 3) = (-1, 1)$

b $(-3, 4) \to (-3 - 2, 4 + 4) = (-5, 8)$

c $(-3, 4) \to (-3 - 3, 4 - 2) = (-6, 2)$

d $(-3, 4) \to (-3 - 4, 4 + 5) = (-7, 9)$

e $(-3, 4) \to (-3 - 2, 4 - 1) = (-5, 3)$

2 **a** $g(x) = \dfrac{1}{x - 2} - 1$

b $g(x) = \dfrac{1}{x - 4} + 3$

c $g(x) = (x + 2)^2 - 3$

d $g(x) = (x - 4)^2 - 2$

e $g(x) = \sqrt{x - 2} - 1$

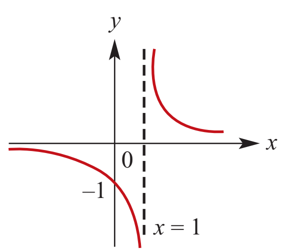
3 $y = f(x) = \dfrac{1}{x}$

a $y = f(x - 1) = \dfrac{1}{x - 1}$

Asymptotes at $x = 1$ and $y = 0$

y-intercept: $y = \dfrac{1}{0 - 1} = -1$

No x-intercept because $y = 0$ is an asymptote.



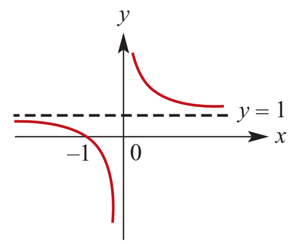
b $y = f(x) + 1 = \dfrac{1}{x} + 1$

Asymptotes at $x = 0$ and $y = 1$

x intercept: $y = \dfrac{1}{x} + 1 = 0$

$\therefore \dfrac{1}{x} = -1, \therefore x = -1$

No y intercept because $y = 0$ is an asymptote

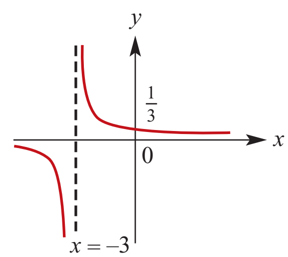


c $y = f(x + 3) = \dfrac{1}{x + 3}$

Asymptotes at $x = -3$ and $y = 0$

y-intercept: $y = \dfrac{1}{0 + 3} = \dfrac{1}{3}$

No x-intercept because $y = 0$ is an asymptote.



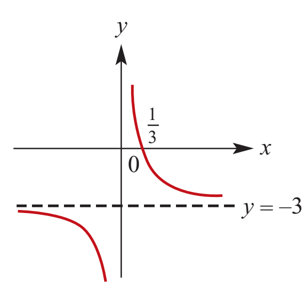
d $y = f(x) - 3 = \dfrac{1}{x} - 3$

Asymptotes at $x = 0$ and $y = -3$

x-intercept: $y = \dfrac{1}{x} - 3 = 0$

$\therefore \dfrac{1}{x} = 3, \therefore x = \dfrac{1}{3}$

No y-intercept because $x = 0$ is an asymptote.

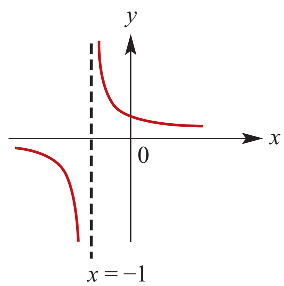


e $y = f(x+1) = \dfrac{1}{x+1}$

Asymptotes at $x = -1$ and $y = 0$

y-intercept: $y = \dfrac{1}{0+1} = 1$

No x-intercept because $y = 0$ is an asymptote.



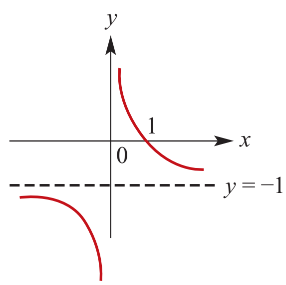
f $y = f(x) - 1 = \dfrac{1}{x} - 1$

Asymptotes at $x = 0$ and $y = -1$

x-intercept: $y = \dfrac{1}{x} - 1 = 0$

$\therefore \dfrac{1}{x} = 1, \therefore x = 1$

No y-intercept because $x = 0$ is an asymptote.



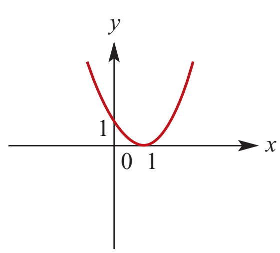
4 $y = f(x) = x^2$

a $y = f(x-1) = (x-1)^2$

x-intercept: $(x-1)^2 = 0, \therefore x = 1$

y-intercept: $f(0-1) = 1$

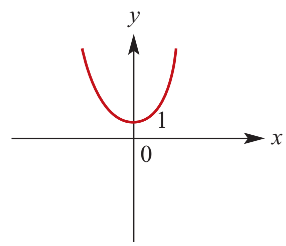
$y = (x-1)^2 = 0, x = 1$



b $y = f(x) + 1 = x^2 + 1$

No x-intercept because $f(x+1) > 0$ for all real x.

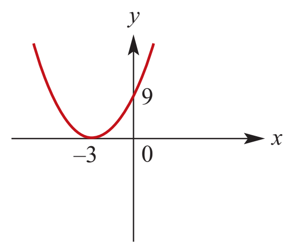
y-intercept: $f(0) + 1 = 1$



c $y = f(x+3) = (x+3)^2$

x-intercept: $(x+3)^2 = 0, \therefore x = -3$

y-intercept: $f(0+3) = 3^2 = 9$



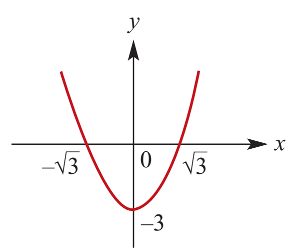
d $y = f(x) - 3 = x^2 - 3$

x-intercepts:

$y = f(x) - 3 = 0, \therefore x^2 - 3 = 0$

$\therefore x^2 = 3, \therefore x = \pm\sqrt{3}$

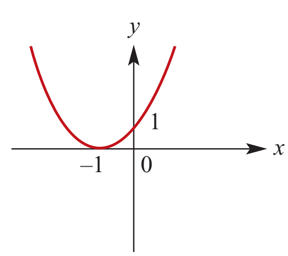
y-intercept: $f(0) - 3 = -3$



e $y = f(x+1) = (x+1)^2$

x-intercept: $(x+1)^2 = 0, \therefore\ x = -1$

y-intercept: $f(0+1) = 1^2 = 1$



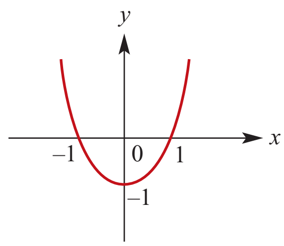
f $y = f(x) - 1 = x^2 - 1$

x-intercepts:

$y = f(x) - 3 = 0, \therefore x^2 - 3 = 0$

$\therefore x^2 = 1, \therefore x = \pm 1$

y-intercept: $f(0) - 1 = -1$



5 $y = f(x) = x^2$

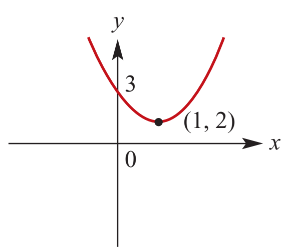
a $y = f(x-1) + 2 = (x-1)^2 + 2$

No x-intercepts because

$f(x-1) + 2 > 0$ for all real x.

y-intercept:

$f(0-1) + 1 = (-1)^2 + 2 = 3$



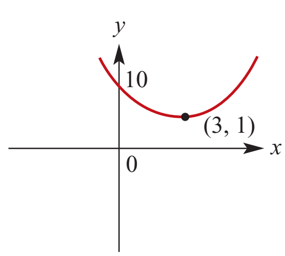
b $y = f(x-3) + 1 = (x-3)^2 + 1$

No x-intercepts because

$f(x-3) + 1 > 0$ for all real x.

y-intercept:

$f(0-3) + 1 = (-3)^2 + 1 = 10$



c $y = f(x+3) - 5 = (x+3)^2 - 5$

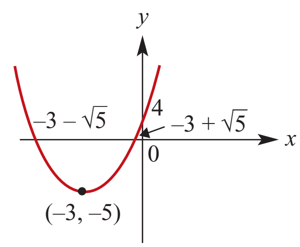
x-intercepts: $y = f(x+3) - 5 = 0$

$\therefore\ (x+3)^2 - 5 = 0$

$\therefore\ (x+3)^2 = 5$

$\therefore\ x + 3 = \pm\sqrt{5},\ \therefore\ x = -3 \pm \sqrt{5}$

y-intercept: $f(0+3) - 5 = 9 - 5 = 4$



d $y = f(x+1) - 3 = (x+1)^2 - 3$

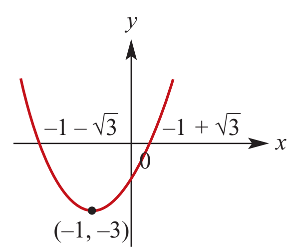
x-intercepts: $y = f(x+1) - 3 = 0$

$\therefore\ (x+1)^2 - 3 = 0$

$\therefore\ (x+1)^2 = 3$

$\therefore\ x + 1 = \pm\sqrt{3},\ \therefore\ x = -1 \pm \sqrt{3}$

y-intercept: $f(0+1) - 3 = 1 - 3 = -2$



e $y + 2 = f(x+1), \therefore\ y = f(x+1) - 2$

$y = f(x+1) - 2 = (x+1)^2 - 2$

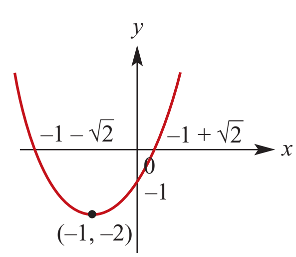
x-intercepts: $y = f(x+1) - 2 = 0$

$\therefore\ (x+1)^2 - 2 = 0$

$\therefore\ (x+1)^2 = 2$

$\therefore\ x + 1 = \pm\sqrt{2}, x = -1 \pm \sqrt{2}$

y-intercept: $f(0+1) - 2 = 1 - 2 = -1$



f $y = f(x-5) - 1 = (x-5)^2 - 1$

x-intercepts: $y = f(x-5) - 1 = 0$

$\therefore\ (x-5)^2 - 1 = 0$

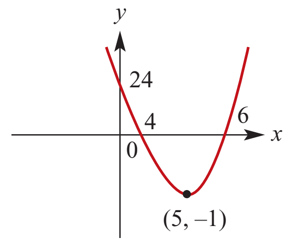
$\therefore\ (x-5)^2 = 1$

$\therefore\ x - 5 = \pm 1$

$\therefore\ x = 5 \pm 1 = 4; 6$

y-intercept:

$f(0-5) - 1 = (-5)^2 - 1 = 24$



Solutions to Exercise 6G

1 **a** $(-2,-3) \to (-2,3)$

b $(-2,-3) \to (2,-3)$

c $(-2,-3) \to (-2,-12)$

d $(-2,-3) \to (-8,-3)$

2 **a** $y = x^2$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \left(\frac{x}{0.5}\right)^2 = 4x^2$

ii A dilation of factor 5 from the y-axis

$\therefore y = \left(\frac{x}{5}\right)^2 = \frac{x^2}{25}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}(x)^2 = \frac{2x^2}{3}$

iv A dilation of factor 4 from the x-axis

$\therefore y = 4(x)^2 = 4x^2$

v A reflection in the x-axis

$\therefore y = -(x)^2 = -x^2$

vi A reflection in the y-axis

$\therefore y = (-x)^2 = x^2$

b $y = \frac{1}{x}$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \frac{0.5}{x} = \frac{1}{2x}$

ii A dilation of factor 5 from the y-axis

$\therefore y = \frac{5}{x}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}\left(\frac{1}{x}\right) = \frac{2}{3x}$

iv A dilation of factor 4 from the x-axis

$\therefore y = \frac{4}{x}$

v A reflection in the x-axis

$\therefore y = -\frac{1}{x}$

vi A reflection in the y-axis

$\therefore y = \frac{1}{-x} = -\frac{1}{x}$

c $y = \sqrt{x}$

i A dilation of factor $\frac{1}{2}$ from the y-axis

$\therefore y = \sqrt{\frac{x}{0.5}} = \sqrt{2x}$

ii A dilation of factor 5 from the y-axis

$\therefore y = \sqrt{\frac{x}{5}}$

iii A dilation of factor $\frac{2}{3}$ from the x-axis

$\therefore y = \frac{2}{3}\sqrt{x}$

iv A dilation of factor 4 from the x-axis

$\therefore y = 4\sqrt{x}$

v A reflection in the x-axis

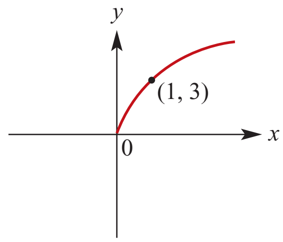
$\therefore y = -\sqrt{x}$

vi A reflection in the y-axis

$\therefore y = \sqrt{-x}; x \leq 0$

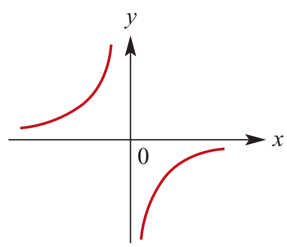
3 **a** $y = 3\sqrt{x}$

Starting point at (0,0)



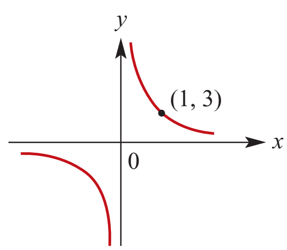
b $y = -\dfrac{1}{x}$

Asymptotes at $x = 0$ and $y = 0$



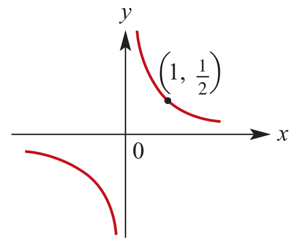
c $y = \dfrac{3}{x}$

Asymptotes at $x = 0$ and $y = 0$



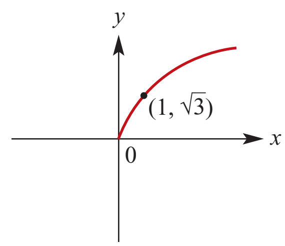
d $y = \dfrac{1}{2x}$

Asymptotes at $x = 0$ and $y = 0$



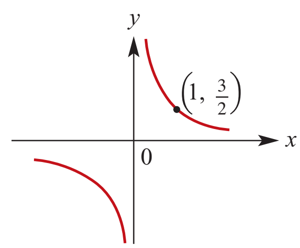
e $y = \sqrt{3x}$

Starting point at (0, 0)



f $y = \dfrac{3}{2x}$

Asymptotes at $x = 0$ and $y = 0$



Solutions to Exercise 6H

1 a Translation of 2 units in the positive direction of the x-axis:
$y = \sqrt{x}$ becomes $y = \sqrt{x-2}$
followed by a dilation of factor 3 from the x-axis: $y = 3\sqrt{x-2}$

b Translation of 3 units in the negative direction of the x-axis:
$y = \sqrt{x}$ becomes $y = \sqrt{x+3}$
followed by a reflection in the x-axis:
$y = -\sqrt{x+3}$

c Reflection in the x-axis:
$y = \sqrt{x}$ becomes $y = -\sqrt{x}$
followed by a dilation of factor 3 from the x-axis: $y = -3\sqrt{x}$

d Reflection in the x-axis: $y = -\sqrt{x}$
followed by a dilation of factor 2 from the y-axis:
$y = -\sqrt{\frac{x}{2}}$

e Dilation of factor 2 from the x-axis:
$y = 2\sqrt{x}$
followed by a translation of 2 units in the positive direction of the x-axis:
$y = 2\sqrt{x-2}$
and 3 units in the negative direction of the y-axis: $y = 2\sqrt{x-2} - 3$

f Dilation of factor 2 from the y-axis:
$y = \sqrt{\frac{x}{2}}$
followed by a translation of 2 units in the negative direction of the x-axis:
$y = \sqrt{\frac{x+2}{2}}$
and 3 units in the negative direction of the y-axis:
$y = \sqrt{\frac{x+2}{2}} - 3$

2 $y = \frac{1}{x}$

a Translation of 2 units in the positive direction of the x-axis:
$y = \frac{1}{x}$ becomes $y = \frac{1}{x-2}$
followed by a dilation of factor 3 from the x-axis: $y = \frac{3}{x-2}$

b Translation of 3 units in the negative direction of the x-axis: $y = \frac{1}{x+3}$
followed by a reflection in the x-axis:
$y = -\frac{1}{x+3}$

c Reflection in the x-axis: $y = -\frac{1}{x}$
followed by a dilation of factor 3 from the x-axis: $y = -\frac{3}{x}$

d Reflection in the x-axis: $y = -\frac{1}{x}$
followed by a dilation of factor 2 from the y-axis: $y = -\frac{2}{x}$

e Dilation of factor 2 from the x-axis:
$y = \frac{2}{x}$
followed by a translation of 2 units in the positive direction of the x-axis:
$y = \frac{2}{x-2}$
and 3 units in the negative direction of the y-axis: $y = \frac{2}{x-2} - 3$

f Dilation of factor 2 from the y-axis:
$y = \frac{2}{x}$
followed by a translation of 2 units in the negative direction of the x-axis:
$y = \frac{2}{x+2}$
and 3 units in the negative direction of the y-axis: $y = \frac{2}{x+2} - 3$

Solutions to Exercise 6I

1 a i Write $y' = 2(x' - 1)^2 + 3$

$\therefore \dfrac{y' - 3}{2} = (x' - 1)^2$

Choose $x = x' - 1$ and $y = \dfrac{y' - 3}{2}$

$\therefore x' = x + 1$ and $y' = 2y + 3$

In summary,

A dilation of factor 2 from the x-axis, then a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

ii Write $y' = -(x' + 1)^2 + 2$

$\therefore -y' + 2 = (x' + 1)^2$

Choose $x = x' + 1$ and $y = -y' + 2$

$\therefore x' = x - 1$ and $y' = -y + 2$

In summary,

A reflection in the x-axis, then a translation of 1 unit in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

iii Write $y' = (2x' + 1)^2 - 2$

$\therefore y' + 2 = (2x' + 1)^2$

Choose $x = 2x' + 1$ and $y = y' + 2$

$\therefore x' = \dfrac{x - 1}{2} = \dfrac{x}{2} - \dfrac{1}{2}$ and $y' = y - 2$

In summary,

A dilation of factor $\frac{1}{2}$ from the y-axis, then a translation of $\frac{1}{2}$ unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

b i Write $y' = \dfrac{2}{x' + 3}$

$\therefore \dfrac{y'}{2} = \dfrac{1}{x' + 3}$

Choose $x = x' + 3$ and $y = \dfrac{y'}{2}$

$\therefore x' = x - 3$ and $y' = 2y$

In summary,

A dilation of factor 2 from the x-axis, then a translation of 3 units in the negative direction of the x-axis

ii Write $y' = \dfrac{1}{x' + 3} + 2$

$\therefore \dfrac{y' - 2}{=} \dfrac{1}{x' + 3}$

Choose $x = x' + 3$ and $y = y' - 2$

$\therefore x' = x - 3$ and $y' = y + 2$

In summary,

A translation of 3 units in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

iii Write $y' = \dfrac{1}{x' - 3} - 2$

$\therefore y' + 2 = \dfrac{1}{x' - 3}$

Choose $x = x' - 3$ and $y = y' + 2$

$\therefore x' = x + 3$ and $y' = y - 2$

In summary,

A translation of 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

c i Write $y' = \sqrt{x' + 3} + 2$

$\therefore y' - 2 = \sqrt{x' + 3}$

Choose $x = x' + 3$ and $y = y' - 2$

$\therefore x' = x + 3$ and $y' = y - 2$

In summary,

A translation of 3 units in the negative direction of the x-axis and 2 units in the positive direction of the y-axis

ii Write $y' = 2\sqrt{3x'}$

$\therefore \dfrac{y'}{2} = \sqrt{3x'}$

Choose $x = 3x'$ and $y = \dfrac{y'}{2}$

$\therefore x' = \frac{1}{3}x$ and $y' = 2y$

In summary,

A dilation of factor $\frac{1}{3}$ from the y-axis, then a dilation of factor 2 from the x-axis

iii Write $y' = -\sqrt{x'} + 2$

$\therefore -y' + 2 = \sqrt{x'}$

Choose $x = x'$ and $y = -y' + 2$

$\therefore x' = x$ and $y' = -y + 2y$

In summary,

A reflection in the x-axis, then a translation of 2 units in the positive direction of the y-axis

2 a Write $y' = (x')^2$ and $y - 5 = (3x + 2)^2$

Choose $y' = y - 5$ and $x' = 3x + 2$

$\therefore (x, y) \to (3x + 2, y - 5)$

b Write $y' = (x')^2$ and

$-y + 7 = 3(3x + 1)^2$

Choose $y' = \dfrac{-y + 7}{3}$ and $x' = 3x + 1$

$\therefore (x, y) \to \left(3x + 1, -\dfrac{y - 7}{3}\right)$

c Write $y' = \sqrt{x'}$ and $\dfrac{y}{2} = \sqrt{-(x - 4)}$

Choose $y' = \dfrac{y}{2}$ and $x' = \sqrt{-(x - 4)}$

$\therefore (x, y) \to \left(-(x - 4), \dfrac{y}{2}\right)$

d Write $y' = -\sqrt{x'} + 6$ and

$\dfrac{y - 3}{2} = \sqrt{-(x - 4)}$

Choose $-y' + 6 = \dfrac{y - 3}{2}$ and

$x' = \sqrt{-(x - 4)}$

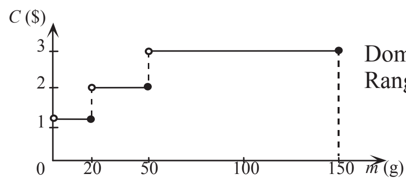
$\therefore (x, y) \to \left(-(x - 4), \dfrac{15 - y}{2}\right)$

Solutions to Exercise 6J

1 $C = 0.15n + 45$ where n is the number of calls

2 a $C = \begin{cases} 1.2 & 0 < m \le 20 \\ 2 & 20 < m \le 50 \\ 3 & 50 < m \le 150 \end{cases}$

b



Domain = (0, 150]
Range = {1.20, 2.00, 3.00}

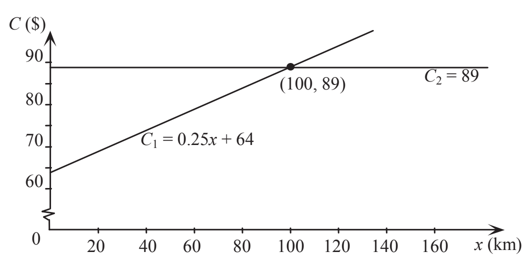
3 a $C_1 = 0.25x + 64$

$C_2 = 89$

b $0.25x + 64 = 89$

implies $0.25x = 25$

$\therefore \quad x = 100$



c Method 2 is cheaper than Method 1 if more than 100 km per day is travelled.

4 a Length = $(50 - x)$ cm

b $A(x) = x(50 - x)$

c $0 \le x \le 50$

d Maximum area = 625 cm^2 when $x = 25$

5 a i $A = (8 + x)y - x^2$

ii $P = y + (8 + x) + (y - x) + x + x + 8 = 2x + 2y + 16$

b i

If $P = 64$, $\quad 64 = 2x + 2y + 16$

$\therefore \quad 48 = 2(x + y)$

$\therefore \quad 24 = x + y$

$\therefore \quad y = 24 - x$

When $y = 24 - x$,

$A = (8 + x)(24 - x) - x^2$

$= 192 + 16x - 2x^2$

ii We know $y = 24 - x$

$, \therefore x < 24$

Also $y - x > 0$, i.e. $24 - 2x > 0$

$\therefore \; x < 12$

The allowable values for x are $\{x\colon 0 < x < 12\}$.

iii Turning point is at $\quad x = \dfrac{-b}{2a}$

and $a = -2, b = 16 \; \therefore \quad x = \dfrac{-16}{-4}$

$= 4$

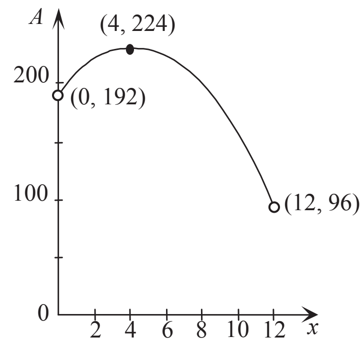
When $x = 4$, $\quad A = 192 + 16(4) - 2(4)^2$

$= 192 + 64 - 32 = 224$

When $x = 0$, $\quad A = 192$

When $x = 12$, $\quad A = 192 + 16(12) - 2(12)^2$

$= 192 + 192 - 288 = 96$



iv The maximum area occurs at the Turning point and is 224 cm^2.

Solutions to Review: Short-answer questions

1 a $[-2, 4)$

b $[-2, 4]$

c $[1, 8)$

d $(-1, 6]$

e $(-4, -2] \cup (1, 5]$

f $(-4, -2] \cup (2, \infty)$

g $(-\infty, -3] \cup (1, \infty)$

2 a $f(3) = 2 - 6(3) = -16$

b $f(-4) = 2 - 6(-4) = 26$

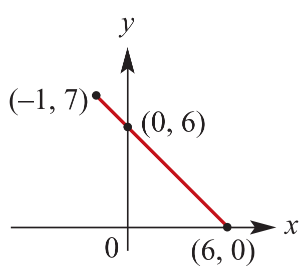
c $f(x) = 2 - 6x = 6$

$\therefore \quad -6x = 4$

$\therefore \quad x = -\dfrac{2}{3}$

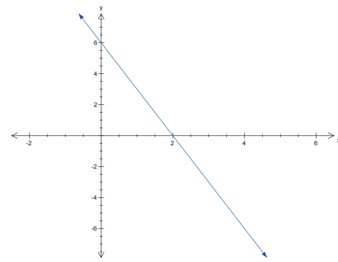
3 $f : [-1, 6] \to \mathbb{R}, f(x) = 6 - x$

a

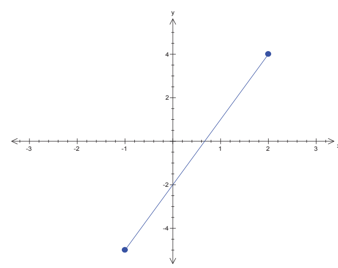


b Range of $f = [0, 7]$

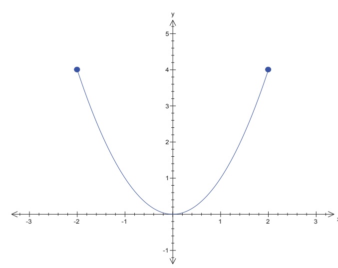
4 a $\{(x, y) : 3x + y = 6\}$; Range $= \mathbb{R}$



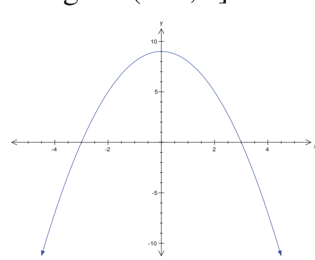
b $\{(x, y) : y = 3x - 2; x \in [-1, 2]\}$;
Range $= [-5, 4]$



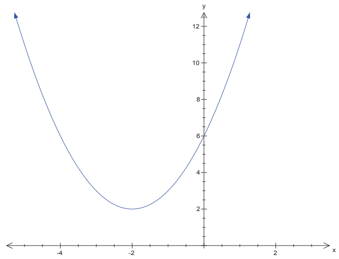
c $\{(x, y) : y = x^2; x \in [-2, 2]\}$;
Range $= [0, 4]$



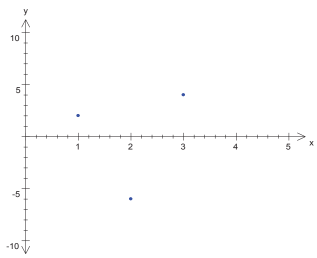
d $\{(x, y) : y = 9 - x^2\}$;
Range $= (-\infty, 9]$



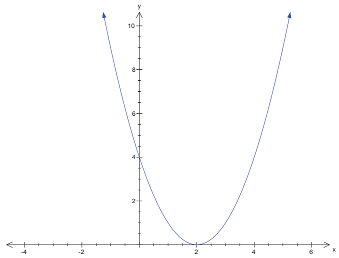
e $\{(x, y) : y = x^2 + 4x + 6\}$;
Range $= [2, \infty)$



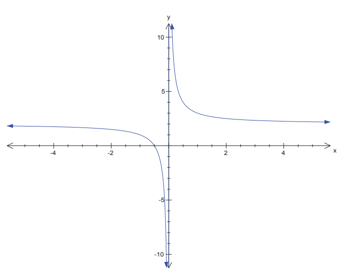
f $\{(1,2)(3,4)(2,-6)\}$;
Range $= \{-6, 2, 4\}$



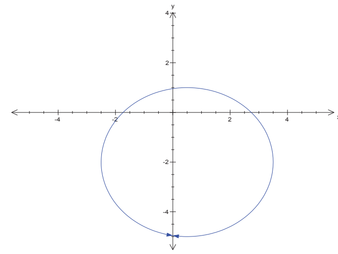
g $f: \mathbb{R} \to \mathbb{R}, f(x) = (x-2)^2$
Range $= [0, \infty)$



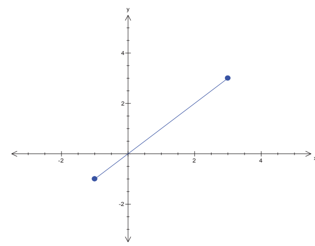
h $f: \mathbb{R}\backslash\{0\} \to \mathbb{R}, f(x) = \dfrac{1}{x} + 2$
Range $= \mathbb{R}\backslash\{2\}$



i $\left(x - \dfrac{1}{2}\right)^2 + (y+2)^2 = 9$
Range $= [-5, 1]$



j $f: [-1, 3] \to \mathbb{R}, f(x) = x$
Range $= [-1, 3]$



5 a $f(x) = \dfrac{a}{x} + b$

$f(1) = \dfrac{3}{2}, \therefore\ f(1) = a + b = \dfrac{3}{2}$

$\therefore\ b = \dfrac{3}{2} - a$

$f(2) = 9, \therefore\ f(2) = \dfrac{a}{2} + b = 9$

$\therefore\ \dfrac{a}{2} + \left(\dfrac{3}{2} - a\right) = 9$

$\therefore\ \dfrac{3}{2} - \dfrac{a}{2} = 9$

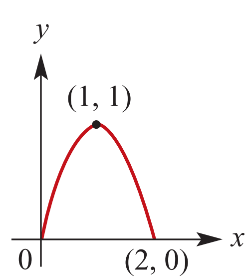
$\therefore\ 3 - a = 18$

$\therefore\ a = -15; b = \dfrac{33}{2}$

b Implied Domain of f is $\mathbb{R}\backslash\{0\}$.

6 $f: [0, 2] \to \mathbb{R}, f(x) = 2x - x^2$

a



b Range $= [0, 1]$

7 $f(x) = ax + b$

$$\begin{aligned} f(5) &= 10, & \therefore\ 5a + b &= 10 \\ f(1) &= -2 & \therefore\ a + b &= -2 \\ & & \therefore\ 4a \quad &= 12 \end{aligned}$$

$a = 3, b = -5$

8 $f(x) = ax^2 + bx + c$

$f(0) = 0, \therefore\ c = 0$

$$\begin{aligned} f(4) &= 0, & \therefore\ 16a + 4b &= 0 \\ & & \therefore\ 4a + b &= 0 \\ f(-2) &= -6 & \therefore\ 4a - 2b &= -6 \\ & & \therefore\ 3b &= 6 \end{aligned}$$

$\therefore\ b = 2; 4a = -2$

$a = -\dfrac{1}{2}, b = 2, c = 0$

9 **a** $y = \dfrac{1}{x-2}$;

implied Domain $= \mathbb{R}\backslash\{2\}$

b $f(x) = \sqrt{x-2}$;

implied Domain $= [2, \infty)$

c $y = \sqrt{25 - x^2}$;

implied Domain $= [-5, 5]$ since $25 - x^2 \geq 0$

d $f(x) = \dfrac{1}{2x-1}$;

implied Domain $= \mathbb{R}\backslash\{\dfrac{1}{2}\}$

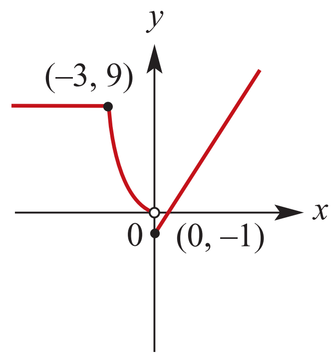
e $g(x) = \sqrt{100 - x^2}$;

implied Domain $= [-10, 10]$

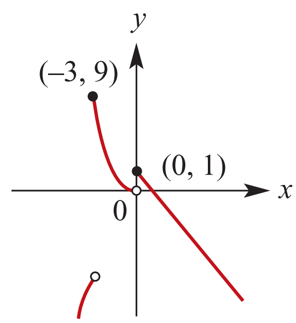
f $h(x) = \sqrt{4 - x}$;

implied Domain $= (-\infty, 4]$

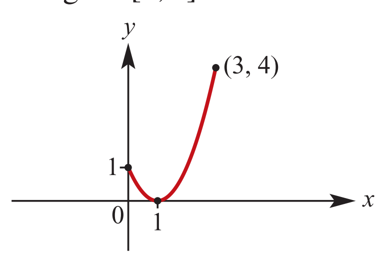
10 **a** $f(x) = \begin{cases} 3x - 1; x \in [0, \infty) \\ x^2; x \in [-3, 0) \\ 9; x \in (-\infty, -3) \end{cases}$



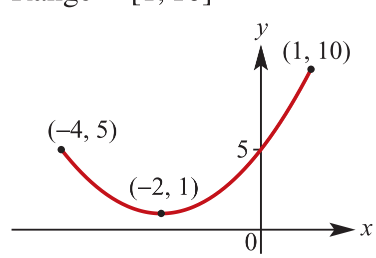
b $h(x) = \begin{cases} 1 - 2x; x \in [0, \infty) \\ x^2; x \in [-3, 0) \\ -x^2; x \in (-\infty, -3) \end{cases}$



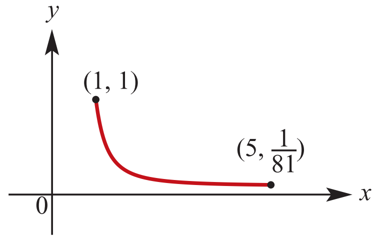
11 **a** Range $= [0, 4]$



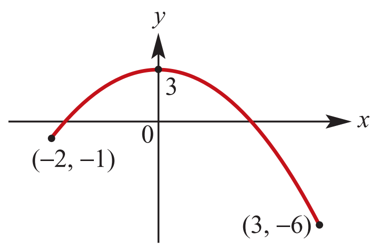
b Range $= [1, 10]$



c Range $= [\frac{1}{81}, 1]$



d Range $= [-6, 3]$



12 **a** Domain $= [1, \infty)$; Range $= [0, \infty)$

b Domain $= (-\infty, 1]$; Range $= [0, \infty)$

c Domain $= [0, \infty)$; Range $= (-\infty, 1]$

13 **a** Domain $= \mathbb{R} \setminus \{1\}$; Range $= \mathbb{R} \setminus \{0\}$

b Domain $= \mathbb{R} \setminus \{-1\}$; Range $= \mathbb{R} \setminus \{0\}$

c Domain $= \mathbb{R} \setminus \{1\}$; Range $= \mathbb{R} \setminus \{3\}$

14 **a** Domain $= [-1, 1]$; Range $= [0, 1]$

b Domain $= [-3, 3]$; Range $= [0, 3]$

c Domain $= [-1, 1]$; Range $= [3, 4]$

15 $f(x) = 2x + 5$

a $f(p) = 2p + 5$

b $f(p + h) = 2p + 2h + 5$

c $f(p + h) - f(p)$
$= (2p + 2h + 5) - (2p + 5) = 2h$

d $f(p + 1) - f(p)$
$= (2p + 2 + 5) - (2p + 5) = 2$

16 $f(x) = 3 - 2x$

$f(p + 1) - f(p)$

$= (3 - 2(p + 1)) - (3 - 2p)$

$= 3 - 2p - 2 - 3 + 2p$

$= -2$

17 **a** $f(x) = -2x^2 + x - 2$

$$= -2\left(x^2 - \frac{x}{2} + 1\right)$$

$$= -2\left(x^2 - \frac{x}{2} + \frac{1}{16} + \frac{15}{16}\right)$$

$$= -2\left(x - \frac{1}{4}\right)^2 - \frac{15}{8}$$

Range of $f = (-\infty, -\frac{15}{8}]$

b $f(x) = 2x^2 - x + 4$

$$= 2\left(x^2 - \frac{x}{2} + 2\right)$$

$$= 2\left(x^2 - \frac{x}{2} + \frac{1}{16} + \frac{31}{16}\right)$$

$$= 2\left(x - \frac{1}{4}\right)^2 + \frac{31}{8}$$

Range of $f = [\frac{31}{8}, \infty)$

c $f(x) = -x^2 + 6x + 11$

$= -(x^2 - 6x - 11)$

$= -(x^2 - 6x + 9 - 20)$

$= -(x - 3)^2 + 20$

Range of $f = (-\infty, 20]$

d $g(x) = -2x^2 + 8x - 5$

$$= -2\left(x^2 - 4x + \frac{5}{2}\right)$$

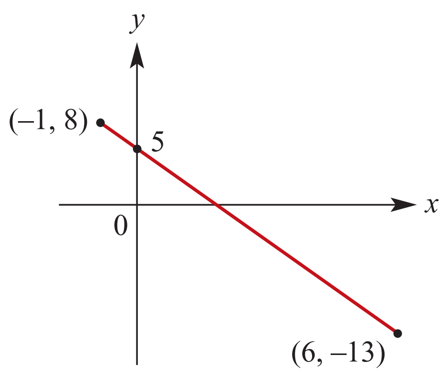
$$= -2\left(x^2 - 4x + 4 - \frac{3}{2}\right)$$

$$= -2(x-2)^2 + 3$$

Range of $g = (-\infty, 3]$

18 $f: [-1, 6] \to \mathbb{R}, f(x) = 5 - 3x$

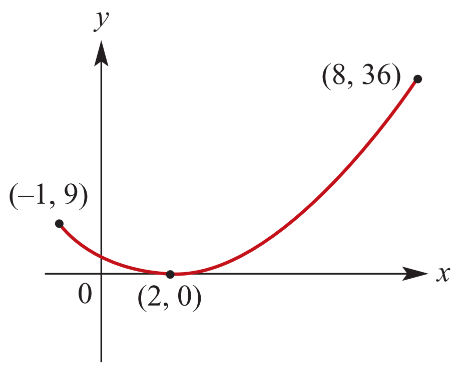
a



b Range of $f = [-13, 8]$

19 $f: [-1, 8] \to \mathbb{R}, f(x) = (x-2)^2$

a



b Range of $= [0, 36]$

20 **a** $x^2 + y^2 = 9$
Circle: radius 3, centre $(0, 0)$
Implied Domain $= [-3, 3]$,
Range $= [-3, 3]$

b $(x-2)^2 + y^2 = 1$
Circle: radius 1, centre (2,0)
Implied Domain $= [1, 3]$,
Range $= [-1, 1]$

c $(2x-1)^2 + (2y-1)^2 = 1$
Circle: radius $\frac{1}{2}$, centre $(\frac{1}{2}, \frac{1}{2})$
Implied Domain $= [0, 1]$,
Range $= [0, 1]$

d $(x-4)^2 + y^2 = 25$
Circle: radius 5, centre $(4, 0)$
Implied Domain $= [-1, 9]$,
Range $= [-5, 5]$

e $(y-2)^2 + x^2 = 16$
Circle: radius 4, centre (0,2)
Implied Domain $= [-4, 4]$,
Range $= [-2, 6]$

21 Domain of the function f is $\{1, 2, 3, 4\}$

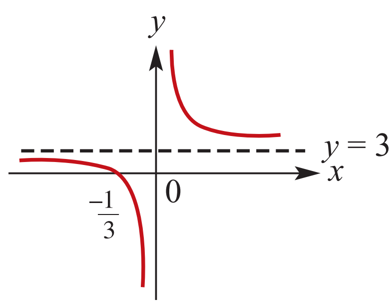
a $f(x) = 2x$ so Range $= \{2, 4, 6, 8\}$

b $f(x) = 5 - x$ so Range $= \{1, 2, 3, 4\}$

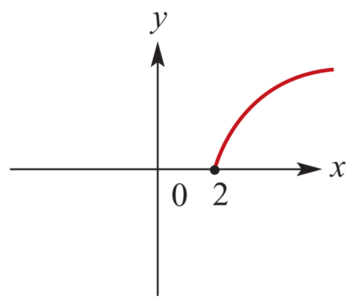
c $f(x) = x^2 - 4$ so Range $=$
$\{-3, 0, 5, 12\}$

d $f(x) = \sqrt{x}$ so Range $= \{1, \sqrt{2}, \sqrt{3}, 2\}$

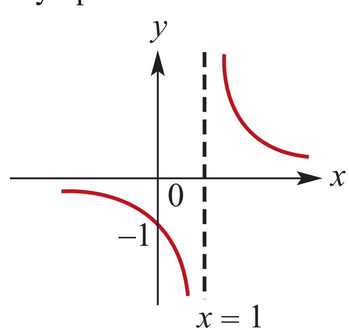
22 **a** $y = \dfrac{1}{x} + 3$
Asymptotes at $x = 0$ and $y = 3$
x-intercept: $y = \dfrac{1}{x} + 3 = 0$
$\therefore \dfrac{1}{x} = -3, \therefore x = -\dfrac{1}{3}$
No y-intercept because $x = 0$ is an asymptote.



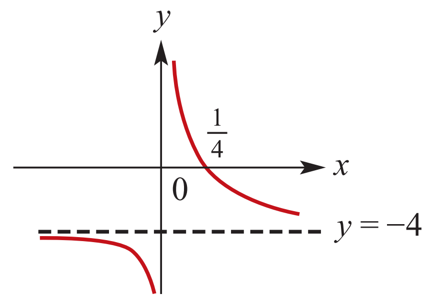
b $y = \sqrt{x-2}$
No asymptotes, starting point at $(2, 0)$.



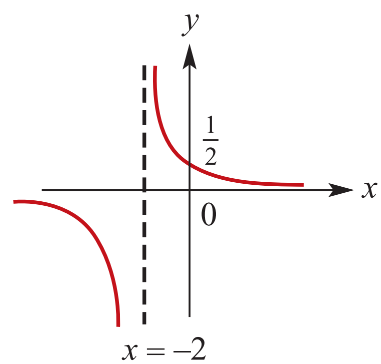
c $y = \dfrac{1}{x-1}$
Asymptotes at $x = 1$ and $y = 0$
y-intercept: $y = \dfrac{1}{0-1} = -1$
No x-intercept because $y = 0$ is an asymptote.



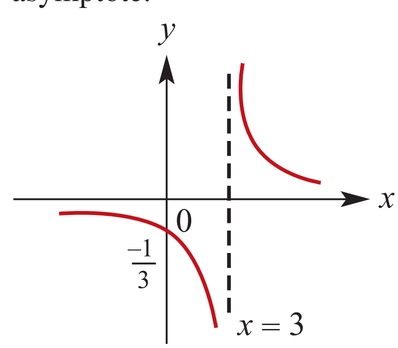
d $y = \dfrac{1}{x} - 4$
Asymptotes at $x = 0$ and $y = -4$
x-intercept: $y = \dfrac{1}{x} - 4 = 0$
$\therefore\ \dfrac{1}{x} = 4, \therefore\ x = \dfrac{1}{4}$
No y-intercept because $x = 0$ is an asymptote.



e $y = \dfrac{1}{x+2}$
Asymptotes at $x = -2$ and $y = 0$
y-intercept: $y = \dfrac{1}{0+2} = \dfrac{1}{2}$
No x-intercept because $y = 0$ is an asymptote.

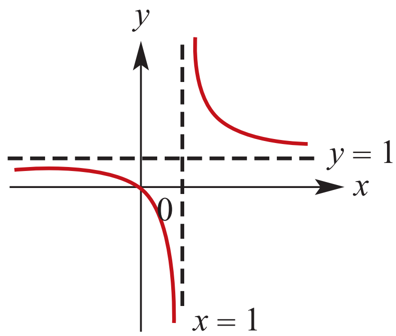


f $y = \dfrac{1}{x-3}$
Asymptotes at $x = 3$ and $y = 0$
y-intercept: $y = \dfrac{1}{0-3} = -\dfrac{1}{3}$
No x-intercept because $y = 0$ is an asymptote.



g $f(x) = \dfrac{1}{x-1} + 1$
Asymptotes at $x = 1$ and $y = f(x) = 1$
x-intercept: $y = \dfrac{1}{x-1} + 1 = 0$

$\therefore \dfrac{1}{x-1} = -1, \therefore x = 0$
y-intercept is also at (0,0).



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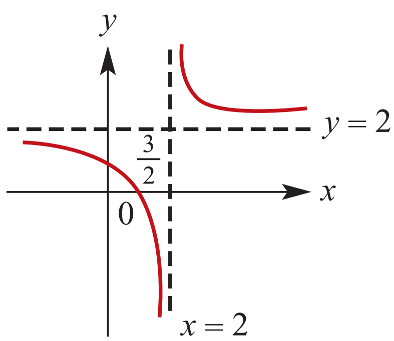


23 a i $(x, y) \to (x - 1, 3y + 2)$

ii $(x, y) \to (x - 2, -2y + 3)$

iii $(x, y) \to \left(\dfrac{x-1}{3}, y - 1\right)$

b i $(x, y) \to (x - 2, 4y)$

ii $(x, y) \to (x - 6, y - 12)$

iii $(x, y) \to (x + 3, 4y - 5)$

c i $(x, y) \to (x + 4, y + 2)$

ii $(x, y) \to \left(\dfrac{x}{2}, 2y\right)$

iii $(x, y) \to (x, -2y + 3)$

Solutions to Review: Multiple-choice questions

1 B $f(x) = 10x^2 + 2$

$\therefore f(2a) = 10(2a)^2 + 2$

$= 40a^2 + 2$

2 E Maximal Domain of $f(x) = \sqrt{3x+5}$ is $[-\frac{5}{3}, \infty)$

3 D Maximal Domain of $f(x) = \sqrt{6-2x}$ is $[-\infty, 3]$

4 B Range of $x^2 + y^2 > 9$ is *all* numbers outside the circle $x^2 + y^2 = 9$

Hence Range is $\mathbb{R}$.

5 E Range is [1,5]

6 E For $f : (a, b] \to \mathbb{R}, f(x) = 3 - x$

Max. value of Range $> 3 - a$

Min. value of Range $= 3 - b$

7 D $y = \frac{2}{x} + 3$ is reflected in the x-axis:

$y = -\frac{2}{x} - 3$

and then in the y-axis:

$y = -\frac{2}{-x} - 3 = \frac{2}{x} - 3$

8 C For $f : [-1, 5) \to \mathbb{R}, f(x) = x^2$

Min. value at $(0, 0)$;

$f(-1) = 1; f(5) = 25$:

the Range is $[0, 25)$.

9 D **D** is correct.

A $y = x^2 - x$ is a many $\to$ 1 function

B $y = \sqrt{4 - x^2}$ is a many $\to$ 1 function

C $y = 3, x > 0$ is a many $\to$ 1 function

D $x = 3$ is a 1 $\to$ many relation

E $y = 3x$ is a 1 $\to$ 1 function

10 C $(1, 7) \to (1, 10) \to (1, -10)$

11 D $(4, -3) \to (4, 1) \to (-4, 1)$

12 A $3a - 1 = 8$ and $b + 2 = 8$

$\therefore a = 3$ and $b = 6$

13 A $3a - 1 = a$ and $2b + 2 = b$

$\therefore a = \frac{1}{2}$ and $b = -2$

Solutions to Review: Extended-response questions

1 **a** For the first coach,

$$d = 80t \text{ for } 0 \le t \le 4$$

$$d = 320 \text{ for } 4 < t \le 4\frac{3}{4}$$

$$d = 320 + 80\left(t - 4\frac{3}{4}\right) \text{ for } 4\frac{3}{4} < t \le 7\frac{1}{4}$$

$$= 320 + 80t - 380 = 80t - 60$$

i.e. for the first coach,

$$d = \begin{cases} 80t & 0 \le t \le 4 \\ 320 & 4 < t \le 4\frac{3}{4} \\ 80t - 60 & 4\frac{3}{4} < t \le 7\frac{1}{4} \end{cases} \quad \text{Range: } [0, 520]$$

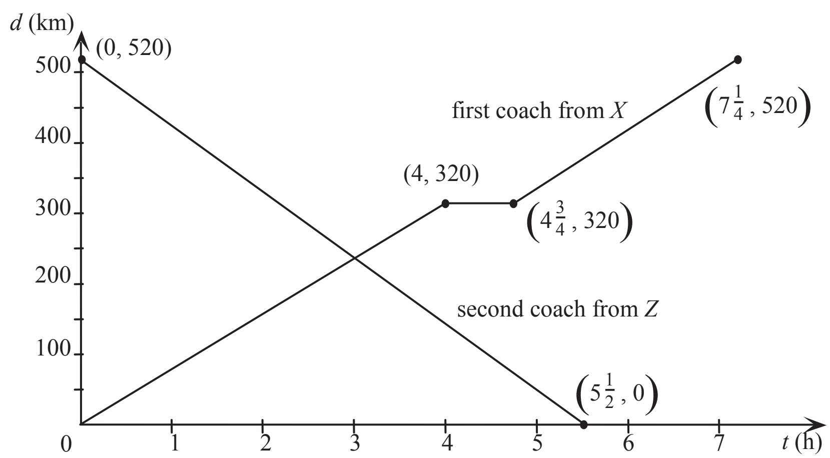
For the second coach,

$$v = \frac{d}{t}$$

$$= \frac{520}{5\frac{1}{2}} = \frac{1040}{11}$$

$$\therefore \; d = 520 - \frac{1040}{11}t, 0 \le t \le 5\frac{1}{2}, \text{ Range: } [0, 520]$$

b The point of intersection of the two graphs yields the time at which the two coaches pass and where this happens.



$$520 - \frac{1040}{11}t = 80t$$

$$\therefore \quad 520 = \frac{1920t}{11}$$

$$\therefore \qquad t = \frac{520 \times 11}{1920} = \frac{143}{48}$$

When $t = \dfrac{143}{48}$, $\quad d = \dfrac{143}{48}$

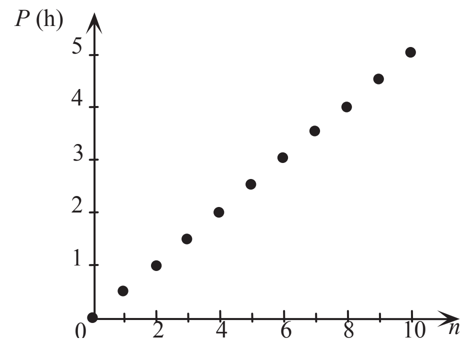
$$= \frac{80 \times 11440}{48} = 238\frac{1}{3}$$

i.e. the two coaches pass each other $238\frac{1}{3}$ km from X.

2 **a** $P = \frac{1}{2}n, n \leq 200, n \in Z$

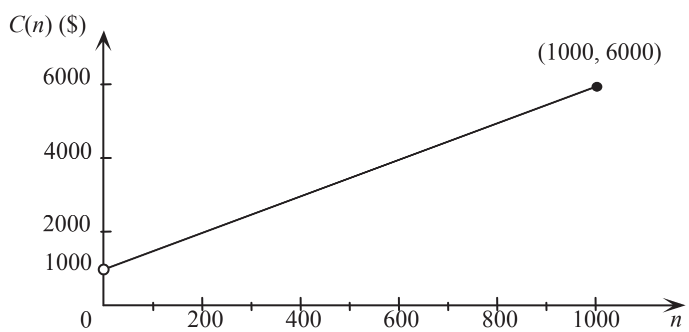
b Domain = $\{n : n \in \mathrm{Z}, 0 \leq n \leq 200\}$

Range = $\{\frac{n}{2} : n \in \mathrm{Z}, 0 \leq n \leq 200\}$



3 **a** **i** $C(n) = 5n + 1000, n > 0$

ii

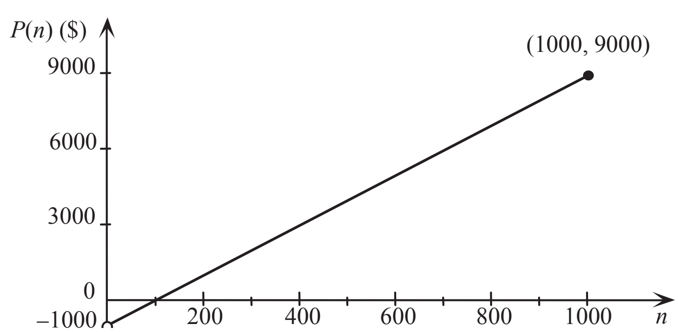


b **i** $P(n) = 15n - C(n)$

$= 15n - (5n + 1000)$

$= 10n - 1000, n > 0$

ii



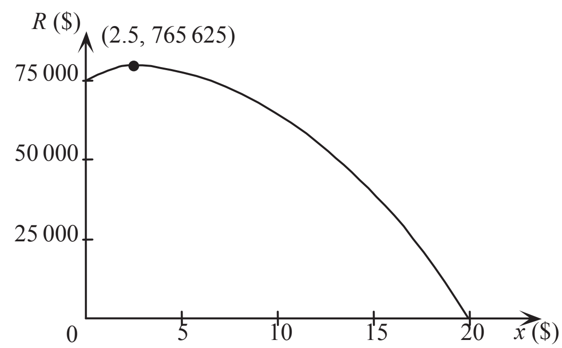
4 $V = 8000 - 0.05 \times 8000 \times n = 8000 - 400n, n \geq 0$

5 **a** R = price of ticket $\times$ number of tickets sold

$$= (15 + x)(50\,000 - 2500x)$$

$$= 2500(x + 15)(20 - x), 0 \leq x \leq 20$$

b



x-intercepts occur when $R = 0$ $\quad \therefore x = -15$ or 20(but $x \geq 0$, so $x = 20$)

R-intercept occurs when $x = 0$ $\quad \therefore R = 750000$

Turning point occurs when $\quad x = \dfrac{-15 + 20}{2} = 2.5$

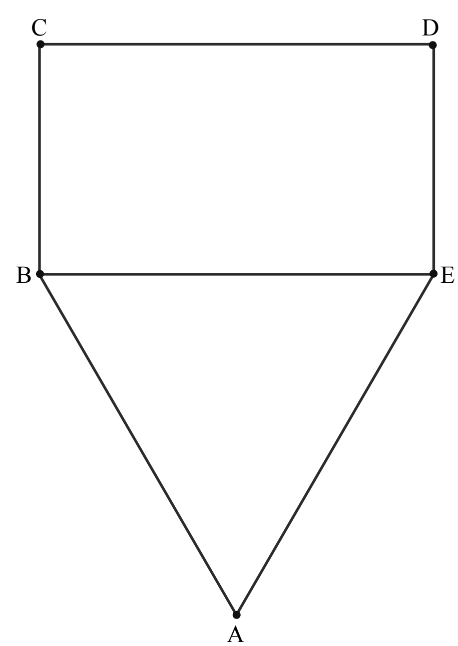
When $x = 2.5$, $\quad R = 2500(2.5 + 15)(20 - 2.5) = 765625$

c The price which will maximise the revenue is \$ 17.50 (i.e. when $x = 2.5$). This assumes that price can be increased by other than dollar amounts.

6 **a** $BE = CD = x$, as $BCDE$ is a rectangle.
$AB = AE = BE = x$,
as ABE is an equilateral triangle.
$DE = BC$, as $BCDE$ is a rectangle.
Perimeter of pentagon

$$= 3x + 2BC = a$$

$$\therefore \quad BC = \frac{1}{2}(a - 3x)$$



$$\begin{aligned}
A(x) &= \text{area of rectangle } + \text{ area of triangle}\\
&= \text{length} \times \text{width} + \frac{1}{2}\text{base} \times \text{height}\\
&= x \times \frac{1}{2}(a-3x) + \frac{1}{2}x \times \sqrt{x^2 - \left(\frac{1}{2}x\right)^2}\\
&= \frac{ax}{2} - \frac{3x^2}{2} + \frac{x}{2}\sqrt{x^2\left(1-\frac{1}{4}\right)} = \frac{ax}{2} - \frac{3x^2}{2} + \frac{x^2\sqrt{3}}{4}\\
&= \frac{2ax - 6x^2 + x^2\sqrt{3}}{4}\\
&= \frac{x}{4}(2a - (6-\sqrt{3})x)
\end{aligned}$$

b From geometry: $x > 0$

Also BC > 0

So $a - 3x > 0$

Giving $x < \dfrac{a}{3}$

Therefore allowable values for x are $\{x: 0 < x < \frac{a}{3}\}$.

c Maximum area occurs when $x = \dfrac{0 + \frac{2a}{6-\sqrt{3}}}{2} = \dfrac{a}{6-\sqrt{3}}$

When $x = \dfrac{a}{6-\sqrt{3}}$, $\quad A(x) = \dfrac{a}{4(6-\sqrt{3})}\left(2a - \dfrac{(6-\sqrt{3})a}{6-\sqrt{3}}\right)$

$$= \frac{a^2}{4(6-\sqrt{3})}$$

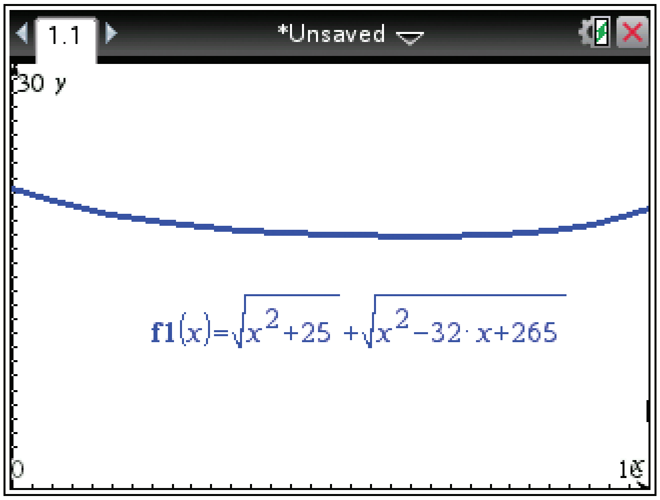
i.e. the maximum area is $\dfrac{a^2}{4(6-\sqrt{3})}$ cm^2.

7 a i $d(x) = AP + PD$

$$\begin{aligned}
&= \sqrt{AB^2 + BP^2} + \sqrt{PC^2 + CD^2}\\
&= \sqrt{5^2 + x^2} + \sqrt{(16-x)^2 + 3^2}\\
&= \sqrt{x^2 + 25} + \sqrt{x^2 - 32x + 265}
\end{aligned}$$

ii $0 \le BO \le BC \qquad \therefore\ 0 \le x \le 16$

b **i**



ii On a CAS calculator, sketch the graphs of $\boldsymbol{f1 = \sqrt{}(x\hat{}2 + 25) + \sqrt{}(x\hat{}2 - 32x + 265)}$ and $\boldsymbol{f2 = 20}$. The point of intersection is (1.5395797, 20). Therefore $d(x) = 20$ when $x \approx 1.54$.
Alternatively, enter **solve** $\boldsymbol{(\sqrt{}(x\hat{}2 + 25) + \sqrt{}(x\hat{}2 - 32x + 265) = 20, x)}$ to find

$$x = \frac{80 \pm 25\sqrt{7}}{9} \approx 1.54, 16.24$$

However, $0 \le x \le 16$, just one answer of 1.54.

iii Repeat **b ii** using $\boldsymbol{f2 = 19}$. The points of intersection are (3.3968503, 19) and (15.041245, 19). Therefore $d(x) = 19$ when $x \approx 3.40$ or $x \approx 15.04$.
Alternatively, enter **solve** $\boldsymbol{(\sqrt{}(x\hat{}2 + 25) + \sqrt{}(x\hat{}2 - 32x + 265) = 19, x)}$ to find

$$x = \frac{1936 \pm 19\sqrt{4141}}{210} \approx 15.04, 3.40$$

c **i** Use CAS calculator to yield
to yield (9.9999998, 17.88544). Therefore the minimum value of $d(x)$ is 17.89 when $x \approx 10.00$.

ii Range = [17.89, 21.28]. Exact answer is $[8\sqrt{5}, 5 + \sqrt{265}]$.

8 **a** On a CAS calculator, sketch the graphs of $\boldsymbol{f1 = (x + 1)(6 - x)}$ and $\boldsymbol{f2 = 2x}$.
Points of intersection are (−1.372 281, −2.744 563) and (4.372 281 3, 8.744 562 6).
The coordinates of A and B are (4.37, 8.74) and (−1.37, −2.74) respectively, correct to 2 decimal places.

Or consider $(x+1)(6-x) = 2x$

$\therefore \quad -x^2 + 5x + 6 = 2x$

$\therefore \quad x^2 - 3x - 6 = 0$

$\therefore \quad x^2 - 3x + \frac{9}{4} - \frac{33}{4} = 0$

$\therefore \quad \left(x - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{33}}{2}\right)^2 = 0$

$\therefore \quad \left(x - \frac{3}{2} + \frac{\sqrt{33}}{2}\right)\left(x - \frac{3}{2} - \frac{\sqrt{33}}{2}\right) = 0$

$\therefore \quad x = \frac{3 \pm \sqrt{33}}{2}$ and $y = 3 \pm \sqrt{33}$

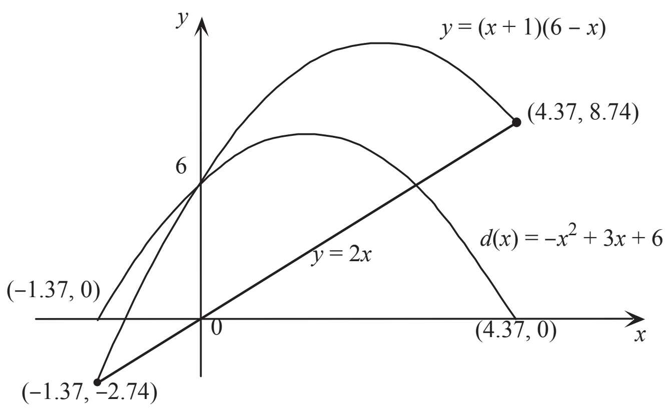
yielding $A\left(\frac{3+\sqrt{33}}{2}, 3+\sqrt{33}\right), B\left(\frac{3-\sqrt{33}}{2}, 3-\sqrt{33}\right)$

b **i** $d(x) = (x+1)(6-x) - 2x$

$= 6x + 6 - x^2 - x - 2x$

$= -x^2 + 3x + 6$

ii



c **i** with $\boldsymbol{f1 = -x\text{^}2 + 3x + 6}$,
Use CAS calculator
to yield (1.5000015, 8.25).
Therefore the maximum value of $d(x)$ is 8.25.

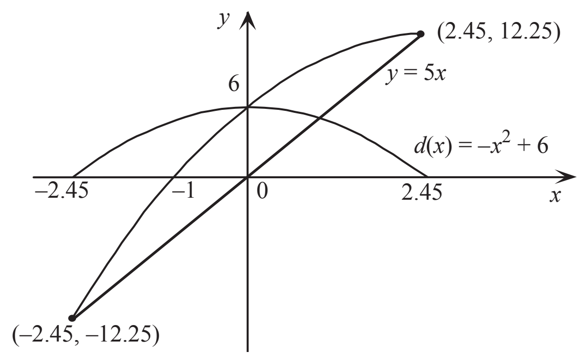
ii Range = $[0, 8.25]$

d $d(x) = (x+1)(6-x) - 5x$

$= 6x + 6 - x^2 - x - 5x$

$= -x^2 + 6$

The maximum value of $d(x)$ is 6 and the Range is $[0, 6]$.



9 **a** $f(x) + k = x^2 + k$

Consider the equations:

$y = x$

$y = x^2 + k$

Solving simultaneously

$$x = x^2 + k$$

$x^2 - x + k = 0 \ldots (1)$

For $y = x$ to be a tangent to $y = x^2 + k$ the discriminant of the quadratic in equation (1) is zero. That is $\Delta = 0$

$\therefore 1 - 4k = 0$

$\therefore k = \frac{1}{4}$

b $f(x-h) = (x-h)^2$

Consider the equations:

$y = x$

$y = (x-h)^2$

Solving simultaneously

$$x = (x-h)^2$$

$$x = x^2 - 2xh + h^2$$

$$x^2 - 2xh - x - h^2 = 0$$

$x^2 - (1+2h)x + h^2 = 0 \ldots (1)$

For $y = x$ to be a tangent to $y = (x-h)^2$ the discriminant of the quadratic in equation (1) is zero. That is $\Delta = 0$

$$(1+2h)^2 - 4h^2 = 0$$

$$1 + 4h + 4h^2 - 4h^2 = 0$$

$$1 + 4h = 0$$

$$h = -\frac{1}{4}$$

10 **a** $7(1+h)^2 = 8$

$$\therefore \quad 1 + h = \pm\sqrt{8}$$

$$\therefore \quad h = -1 \pm 2\sqrt{2}$$

b

$$\begin{aligned} \text{Let } g(x) &= f(ax) \\ &= (ax)^2 \\ &= a^2x^2 \end{aligned}$$

Now $g(1) = 8, \therefore \quad a^2 = 8$

$$\therefore \quad a = \pm 2\sqrt{2}$$

c

$$\begin{aligned} y &= ax^2 + bx \\ &= a\left(x^2 + \frac{b}{a}x\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} \end{aligned}$$

The vertex has coordinates $(1, 8)$, $\therefore \dfrac{-b}{2a} = 1$ and $\dfrac{-b^2}{4a} = 8$.

Substituting $\dfrac{-b}{2a} = 1$ into $\dfrac{-b^2}{4a} = 8$ gives

$$\frac{b}{2} = 8$$

$$\therefore b = 16$$

Substituting $b = 16$ into $\dfrac{-b}{2a} = 1$ gives

$$\frac{-16}{2a} = 1$$

$$a = -8$$

11 **a** $g(x) = x^2 + 4x - 6$

$$g(x) + k = 0$$

$$x^2 + 4x - 6 + k = 0$$

One solution when $\Delta = 0$

$$\begin{aligned} \Delta &= 0 \\ 16 - 4(k - 6) &= 0 \\ 16 - 4k + 24 &= 0 \\ 40 - 4k &= 0 \\ k &= 10 \end{aligned}$$

b

$$\begin{aligned} x^2 + 4x - 6 &= 0 \\ x^2 + 4x + 4 - 4 - 6 &= 0 \\ (x + 2)^2 - 10 &= 0 \\ (x + 2)^2 &= 10 \\ x &= -2 \pm \sqrt{10} \end{aligned}$$

i For two positive solutions $h > 2 + \sqrt{10}$

ii for two negative solutions $h < 2 - \sqrt{10}$

iii One positive and one negative $2 - \sqrt{10} < h < 2 + \sqrt{10}$