MvMM derivation

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Introduction

This is a simple but detailed derivation for MvMM [Link]. The prerequisite is GMM-EM which you can refer to [GMM] The PDF version is in my Github Res, where you can give me a precious star if you think it's useful for you and I'll appreciate it :)

Contents

Notation

- Common space: Ω
    - $x \in \Omega$: pixel
    - $I(x)/I_i(x)$: intensity
- Image / Sequence / Modality:
    - $i \in N_I$
    - $N_I = 3$ in this task: LGE, T2, bSSFP
- Type: defined on common space
    - $k \in K$
    - Latent variable: $s(x) = k$
    - Prior: π_{kx} or π_k
- Subtype: defined on a specific image and its type
    - $c \in C_{ik}$ (NOTE: related to image and type)
    - Latent variable: $z_i(x) = c$ or c_{ik}
    - Prior: τ_{ikc}
    - Gaussian parameters: $\mu_{ikc}, \sigma^2_{ikc}$

Likelihood function

First we derive the incomplete likelihood function for $\theta = \{\pi, \tau, \mu, \sigma\}$

$$\begin{aligned} L(\theta \mid I) = & \prod_{x \in \Omega} \sum_{k \in K} \pi_{kx} \prod_{i \in N_I} \sum_{c \in C_{ik}} \tau_{ikc} \Phi(I_i(x); \mu_{ikc}, \sigma^2_{ikc}) \\ := & P(I \mid \theta) \\ = & \prod_{x \in \Omega} P(I(x) \mid \theta) \\ = & \prod_{x \in \Omega} \sum_{k \in K} P(I(x) \mid s(x) = k, \theta) P(s(x) = k \mid \theta) \\ = & \prod_{x \in \Omega} \sum_{k \in K} \pi_{kx} \prod_{i \in N_I} P(I_i(x) \mid s(x) = k, \theta) \\ = & \prod_{x \in \Omega} \sum_{k \in K} \pi_{kx} \prod_{i \in N_I} \sum_{c \in C_{ik}} P(z_i(x) = c \mid s(x) = k, \theta) P(I_i(x) \mid z_i(x) = c, \theta) \\ = & \prod_{x \in \Omega} \sum_{k \in K} \pi_{kx} \prod_{i \in N_I} \sum_{c \in C_{ik}} \tau_{ikc} \Phi(I_i(x); \mu_{ikc}, \sigma^2_{ikc}) \end{aligned}$$

EM solution

Recall the EM solution, we need the complete log-likelihood function in E-step, so we now derive the complete likelihood function.

Complete likelihood

Here we use the trick, introduce the indicator function $1\{(s(x) = k)\}$ and $1\{(z_i(x) = c)\}$.

$$\begin{aligned} L(\theta \mid I, S, Z) := & P(I, S, Z \mid \theta) \\ = & \prod_{x \in \Omega} \prod_{k \in K} \{\pi_{kx} \prod_{i \in N_I} \prod_{c \in C_{ik}} \{\tau_{ikc} \Phi(I_i(x); \mu_{ikc}, \sigma^2_{ikc})\}^{1\{z_i(x)=c\}}\}^{1\{s(x)=k\}} \end{aligned}$$

Complete log-likelihood

Take log on both sides.

$$\begin{aligned} l(\theta \mid I, S, Z) = & \log L(\theta \mid I, S, Z) \\ = & \sum_{x \in \Omega} \sum_{k \in K} 1_{(s(x)=k)} \log \pi_{kx} \sum_{i \in N_I} \sum_{c \in C_{ik}} 1_{(z_i(x)=c)} \{\log \tau_{ikc} + \log \Phi(I_i(x); \mu_{ikc}, \sigma^2_{ikc})\} \\ = & \sum_{x \in \Omega} \sum_{k \in K} 1_{(s(x)=k)} \log \pi_{kx} + \sum_{x \in \Omega} \sum_{k \in K} \sum_{i \in N_I} \sum_{c \in C_{ik}} 1_{(s(x)=k)} 1_{(z_i(x)=c)} \{\log \tau_{ikc} + \log \Phi(I_i(x); \mu_{ikc}, \sigma^2_{ikc})\} \end{aligned}$$

E-step: Q function

In E-step, we are going to take conditioanl expectation for complete log-likelihood function we have derived above, with respect to latent varaible given obervation $I(x)$ and current pamameter $\theta^{[m]}$, i.e. $S, Z \mid I(x), \theta^{[m]}$.

So the Q-function is,

$$
\begin{aligned}
Q(\theta \mid \theta^{[m]}) &= \mathbb{E}_{S,Z|\theta^{[m]}} l(\theta \mid I, S, Z) \\
&= \sum_{x \in \Omega} \sum_{k \in K} \mathbb{E}_{S,Z|\theta^{[m]}} \{1_{(s(x)=k)}\} \log \pi_{kx} \\
&+ \sum_{x \in \Omega} \sum_{k \in K} \sum_{i \in N_I} \sum_{c \in C_{ik}} \mathbb{E}_{S,Z|\theta^{[m]}} \{1_{(s(x)=k)} 1_{(z_i(x)=c)}\} \{\log \tau_{ikc} + \log \Phi(I_i(x); \mu_{ikc}, \sigma^2_{ikc})\}
\end{aligned}
$$

E-step: Compute posterior function

We can find that it is the indicator functions that are related to the expected variable, which are the posterior,

$$
\begin{aligned}
\mathbb{E}_{S,Z|I,\theta^{[m]}} \{1_{(s(x)=k)}\} &= P(s(x) = k \mid I, \theta^{[m]}) \\
&= \frac{P(I(x) \mid s(x) = k, \theta^{[m]}) P(s(x) = k \mid \theta^{[m]})}{\sum_{l \in K} P(I(x) \mid s(x) = l, \theta^{[m]}) P(s(x) = l \mid \theta^{[m]})} \\
&= \frac{P(I(x) \mid s(x) = k, \theta^{[m]}) \pi^{[m]}_{kx}}{\sum_{l \in K} P(I(x) \mid s(x) = l, \theta^{[m]}) \pi^{[m]}_{lx}} \\
&:= P^{[m+1]}_{kx}
\end{aligned}
$$

And then

$$
\begin{aligned}
&\mathbb{E}_{S,Z|I,\theta^{[m]}} \{1_{(s(x)=k)} 1_{(z_i(x)=c)}\} \\
&= P(s(x) = k, z_i(x) = c_{ik} \mid I, \theta^{[m]}) \\
&= P(z_i(x) = c_{ik} \mid s(x) = k, I, \theta^{[m]}) P(s(x) = k \mid I, \theta^{[m]}) \\
&= P(z_i(x) = c_{ik} \mid s(x) = k, I, \theta^{[m]}) P^{[m+1]}_{kx} \\
&= \frac{P(z_i(x) = c_{ik} \mid s(x) = k, \theta^{[m]}) P(I(x) \mid z_i(x) = c_{ik}, s(x) = k, \theta^{[m]})}{P(I(x) \mid s(x) = k, \theta^{[m]})} P^{[m+1]}_{kx} \\
&= \frac{\tau_{ikc} \Phi(I_i(x) \mid \mu^{[m]}_{ikc}, \sigma^{2[m]}_{ikc})}{P(I_i(x) \mid s(x) = k, \theta^{[m]})} P^{[m+1]}_{kx} \\
&:= P^{[m+1]}_{ikcx}
\end{aligned}
$$

So the Q function is,

$$
\begin{aligned}
Q(\theta \mid \theta^{[m]}) &= \sum_{x \in \Omega} \sum_{k \in K} \mathbb{E}_{S,Z|I,\theta^{[m]}} \{1_{(s(x)=k)}\} \log \pi_{kx} \\
&+ \sum_{x \in \Omega} \sum_{k \in K} \sum_{i \in N_I} \sum_{c \in C_{ik}} \mathbb{E}_{S,Z|I,\theta^{[m]}} \{1_{(s(x)=k)} 1_{(z_i(x)=c)}\} \{\log \tau_{ikc} + \log \Phi(I_i(x); \mu_{ikc}, \sigma^2_{ikc})\} \\
&= \sum_{x \in \Omega} \sum_{k \in K} P^{[m+1]}_{kx} \log \pi_{kx} \\
&+ \sum_{x \in \Omega} \sum_{k \in K} \sum_{i \in N_I} \sum_{c \in C_{ik}} P^{[m+1]}_{ikcx} \{\log \tau_{ikc} + \log \Phi(I_i(x); \mu_{ikc}, \sigma^2_{ikc})\}
\end{aligned}
$$

M-Step: Maximize the Q function

In the M-step, we want to maximize $Q(\theta \mid \theta^{[m]})$, with respect to $\theta = (\pi_{kx}, \tau_{ikc}, \mu_{ikc}, \sigma^2_{ikc})$. We can obtain the closed-form solution by letting the derivative equal to zero.

for τ

$$\tau_{ikc}^{[m+1]} = \arg\max_{\tau_{ikc}} Q(\theta \mid \theta^{[m]})$$
$$s.t. \sum_{d \in C_{ik}} \tau_{ikd} = 1$$

Construct Lagrange multiplier

$$L(\theta, \lambda) = Q(\theta \mid \theta^{[m]}) + \lambda(1 - \sum_{d \in C_{ik}} \tau_{ikd})$$
$$\frac{\partial L}{\partial \tau_{ikc}} = \sum_{x \in \Omega} \frac{P_{ikcx}^{[m+1]}}{\tau_{ikc}} + \lambda = 0$$
$$\sum_{d \in C_{ik}} \tau_{ikd} = 1$$

Solution:

$$\tau_{ikc} = \frac{\sum_{x \in \Omega} P_{ikcx}^{[m+1]}}{\sum_{d \in C_{ik}} \sum_{x \in \Omega} P_{ikdx}^{[m+1]}}$$

For π_{kx}

when no spatial constraint is applied, $\pi_{kx} = \pi_k$.

$$\pi_k^{[m+1]} = \arg\max_{\pi_{kx}} Q(\theta \mid \theta^{[m]})$$
$$s.t. \sum_{k \in K} \pi_k = 1$$

Solution

$$\pi_k^{[m+1]} = \frac{\sum_{x \in \Omega} P_{kx}^{[m+1]}}{\sum_{l \in K} \sum_{x \in \Omega} P_{kx}^{[m+1]}}$$

For μ_{ikc}

$$\mu_{ikc}^{[m+1]} = \arg\max_{\mu_{ikc}} Q(\theta \mid \theta^{[m]})$$

Solution

$$\frac{\partial Q(\theta \mid \theta^{[m]})}{\partial \mu_{ikc}} = \sum_{x \in \Omega} P_{ikcx}^{[m+1]} (I_i(x) - \mu_{ikc}) = 0$$
$$\mu_{ikc}^{[m+1]} = \frac{\sum_{x \in \Omega} P_{ikcx}^{[m+1]} I_i(x)}{\sum_{x \in \Omega} P_{ikcx}^{[m+1]}}$$

For $\sigma_{ikc}^{2[m+1]}$

$$\sigma_{ikc}^{2[m+1]} = \arg\max_{\sigma_{ikc}^2} Q(\theta \mid \theta^{[m]})$$

$$\frac{\partial Q(\theta \mid \theta^{[m]})}{\partial \sigma_{ikc}^2} = \sum_{x\in\Omega} P_{ikcx}^{[m+1]}\left(-\frac{1}{2}\frac{1}{\sigma_{ikc}^2} + \frac{(I_i(x)-\mu_{ikc}^{[m+1]})^2}{2(\sigma_{ikc}^2)^2}\right) = 0$$
$$\sigma_{ikc}^{2[m+1]} = \frac{\sum_{x\in\Omega} P_{ikcx}^{[m+1]}(I_i(x)-\mu_{ikc}^{[m+1]})^2}{\sum_{x\in\Omega} P_{ikcx}^{[m+1]}}$$

We can note that, here we use the newest $\mu^{[m+1]}$ for the update of $\sigma^{2[m+1]}$

Spatial regularization / constraint

The motivation of spatial constrain is, pixels with the same intensity distribution in medical images can come from different structures.

Probabilistic Atlas

Probabilistic atlas is a common used method for spatial constraint [Link].

After the introducing of atlas, the prior now is,

$$\pi_{kx} = P(s(x)=k \mid \theta) = \frac{\pi_k P_A(s(x)=k)}{NF} = \frac{\pi_k P_A(s(x)=k)}{\sum_{l\in K}\pi_l P_A(s(x)=l)}$$

where NF is normalization factor. $NF = \sum_{l\in K}\pi_l P_A(s(x)=l)$.

Similarly, we want to maximize the Q-function, we denote Q_π referred to the π part.

$$\begin{aligned}
Q_\pi &= \sum_{x\in\Omega}\sum_{k\in K} P_{kx}^{[m+1]} \log \pi_{kx} = \sum_{x\in\Omega}\sum_{k\in K} P_{kx}^{[m+1]} \log \frac{\pi_k P_A(s(x)=k)}{\sum_{l\in K}\pi_l P_A(s(x)=l)} \\
&= \sum_{x\in\Omega}\sum_{k\in K} P_{kx}^{[m+1]} \log \pi_k - \sum_{x\in\Omega}\sum_{k\in K} P_{kx}^{[m+1]} \log \frac{\sum_{l\in K}\pi_l P_A(s(x)=l)}{P_A(s(x)=k)} \\
&= \sum_{x\in\Omega}\sum_{k\in K} P_{kx}^{[m+1]} \log \pi_k - \sum_{x\in\Omega}\sum_{k\in K} P_{kx}^{[m+1]} \{\log \sum_{l\in K}\pi_l P_A(s(x)=l) - \log \sum_{k\in K} P_A(s(x)=k)\}
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial Q_\pi}{\partial \pi_k} &= \sum_{x\in\Omega} \frac{P_{kx}^{[m+1]}}{\pi_k} - \sum_{x\in\Omega}\sum_{j\in K} \frac{P_{jx}^{[m+1]} P_A(s(x)=k)}{\sum_{l\in K}\pi_l P_A(s(x)=l)} \\
&= \sum_{x\in\Omega} \frac{P_{kx}^{[m+1]}}{\pi_k} - \sum_{x\in\Omega} \frac{P_A(s(x)=k)\sum_{j\in K} P_{jx}^{[m+1]}}{\sum_{l\in K}\pi_l P_A(s(x)=l)} \\
&= \sum_{x\in\Omega} \frac{P_{kx}^{[m+1]}}{\pi_k} - \sum_{x\in\Omega} \frac{P_A(s(x)=k)}{\sum_{l\in K}\pi_l P_A(s(x)=l)} = 0
\end{aligned}$$

Since there is no closed-form solution. So we assume on the left hand side, $\pi_k = \pi_k^{[m+1]}$, the right hand side, $\pi_l = \pi_l^{[m]}$. And we denote the constant $C_x^{[m]} = \sum_{l \in K} \pi_l^{[m]} P_A(s(x) = l)$. So then we have a iterative $\pi_k^{[m+1]}$,

$$\pi_k^{[m+1]} = \frac{\sum_{x \in \Omega} P_{kx}^{[m+1]}}{\sum_{x \in \Omega} (P_A(s(x) = k) / C_x^{[m]})}$$

We need to prove that $\pi_k^{[m+1]}$ can increase the likelihood,

$$\begin{aligned}
Q_\pi^{[m]} &= \sum_x \sum_k P_{kx}^{[m+1]} \log \frac{\pi_k^{[m]} P(A_{kx})}{C_x^{[m]}} \\
Q_\pi^{[m+1]} &= \sum_x \sum_k P_{kx}^{[m+1]} \log \frac{\pi_k^{[m+1]} P(A_{kx})}{C_x^{[m+1]}} \\
Q_\pi^{[m+1]} - Q_\pi^{[m]} &= \sum_x \sum_k P_{kx}^{[m+1]} \log \frac{\pi_k^{[m+1]}}{\pi_k^{[m]}} \frac{C_x^{[m]}}{C_x^{[m+1]}}
\end{aligned}$$

We see

$$\begin{aligned}
\frac{\pi_k^{[m+1]}}{\pi_k^{[m]}} \frac{C_x^{[m]}}{C_x^{[m+1]}} &= \frac{\sum_{y \in \Omega} P_{ky}^{[m+1]}}{\sum_{y \in \Omega} (P_A(s(y) = k) / C_y^{[m]})} \frac{1}{\pi_k^{[m]}} \frac{C_x^{[m]}}{C_x^{[m+1]}} \\
&= \frac{\sum_{y \in \Omega} P_{ky}^{[m+1]}}{\sum_y \pi_k^{[m]} P_A(s(y) = k) / C_y^{[m]}} \frac{C_x^{[m]}}{C_x^{[m+1]}} \\
&= \frac{\sum_{y \in \Omega} P_{ky}^{[m+1]}}{\sum_y \pi_{ky}^{[m]}} \frac{C_x^{[m]}}{C_x^{[m+1]}} = \frac{\sum_{y \in \Omega} P_{ky}^{[m+1]} / C_x^{[m+1]}}{\sum_y \pi_{ky}^{[m]} / C_x^{[m]}}
\end{aligned}$$

So we have,

$$\begin{aligned}
Q_\pi^{[m+1]} - Q_\pi^{[m]} &= \sum_x \sum_k P_{kx}^{[m+1]} \log \frac{\sum_{y \in \Omega} P_{ky}^{[m+1]} / C_x^{[m+1]}}{\sum_y \pi_{ky}^{[m]} / C_x^{[m]}} \\
&= C_x^{[m+1]} \sum_k \sum_x P_{kx}^{[m+1]} / C_x^{[m+1]} \log \frac{\sum_{y \in \Omega} P_{ky}^{[m+1]} / C_x^{[m+1]}}{\sum_y \pi_{ky}^{[m]} / C_x^{[m]}} \\
&= C_x^{[m+1]} |\Omega| \sum_k \sum_x P_{kx}^{[m+1]} / (C_x^{[m+1]} |\Omega|) \log \frac{\sum_{y \in \Omega} P_{ky}^{[m+1]} / (C_x^{[m+1]} |\Omega|)}{\sum_y \pi_{ky}^{[m]} / (C_x^{[m]} |\Omega|)} \\
&= C_x^{[m+1]} |\Omega| \cdot KL(P \| \pi) \geq 0
\end{aligned}$$

So, we proved that Q_π increases with the update of π_k. It is the GEM (General EM) algorithm.

Initialization

We start from $\theta^{[0]}$, and then compute the posterior $P_{ik}^{[1]}$ and $P_{ickx}^{[1]}$, and then iteratively.

For $\pi_k^{[0]}$:

$$\pi_k^{[0]} = \frac{\sum_x P(A_{kx})}{\sum_l \sum_x P(A_{lx})}$$

For $\tau_{ikc}^{[0]}$:

$$\tau_{ikc}^{[0]} = \frac{1}{|C_{ik}|}$$

For $\mu_{ikc}^{[0]}, \sigma_{ikc}^{2[0]}$:

$$\mu_{ikc}^{[0]} = \begin{cases} \mu_{ik}^{[0]} + a\sigma_{ik}^{[0]}, & |C_{ik} \geq 2| \\ \mu_{ik}^{[0]}, & |C_{ik} = 1| \end{cases}, \left(\sigma_{ikc}^{[0]}\right)^2 = \frac{\left(\sigma_{ik}^{[0]}\right)^2}{|C_{ik}|}$$

where

$$\mu_{ik}^{[0]} = \frac{\sum_x I_i(x) p\left(A_{kx}\right)}{\sum_x p\left(A_{kx}\right)}, \text{ and } \left(\sigma_{ik}^{[0]}\right)^2 = \frac{\sum_x \left(I_i(x) - \mu_{ik}^{[0]}\right)^2 p\left(A_{kx}\right)}{\sum_x p\left(A_{kx}\right)}$$

Registration in MvMM

Two types of misalignment in Myocardial Segmentation:

- Inter-slice: motion shift.
- Misalignment between atlas and common space.

For the slice reg, we model it as a rigid transformation

$$P(I_i(x) \mid c_{ik}, \theta, G_{i,s}) = \Phi_{ikc}(I_i(G_{i,s}(x)))$$

where $G_{i,s}$ the transformations for correcting.

The atlas deformation, modeled as a FFD, denoted as D,

$$P_A(S(x) = k \mid D) = P_A(s(D(x)) = k) = A_k(D(x))$$

So the prior is $\pi_{kx|D} = P(s(x) = k \mid D)$.

The likelihood:

$$\begin{aligned} L(\theta, D, \{G_{i,s}\}) &= \prod_{x \in \Omega} \sum_{k \in K} \pi_{kx|D} P(I(x) \mid s(x) = k, \theta, \{G_{i,s}\}) \\ &= \prod_{x \in \Omega} \sum_{k \in K} \pi_{kx|D} \prod_{i \in N_I} P(I_i(x) \mid s(x) = k, \theta, G_{i,s}) \\ &= \prod_{x \in \Omega} \sum_{k \in K} \pi_{kx|D} \prod_{i \in N_I} \sum_{c_{ik}} \tau_{ikc} \Phi_{ikc}(I_i(G_{i,s}(x))) \end{aligned}$$

So the log-likelihood:

$$LL(\theta, D, \{G_{i,s}\}) = \log L(\theta, D, \{G_{i,s}\}) = \sum_{x \in \Omega} \log\{\sum_{k \in K} \pi_{kx|D} \prod_{i \in N_I} \sum_{c_{ik}} \tau_{ikc} \Phi_{ikc}(I_i(G_{i,s}(x)))\}$$

We use ICM (Iterative Conditional Mode) Optimization, optimize some while fixing others.

Here we fix segmentation parameters θ, and optimize the transformation $\{G\}, D$.

$$\begin{aligned}\frac{\partial LL}{\partial G_{i,s}} &= \sum_{x\in\Omega}\frac{1}{LH(x)}\sum_{k\in K}\prod_{j\neq i}P(I_j(x)\mid s(x)=k,\theta,G_{j,s})\\ &\cdot\sum_c\tau_{ikc}\frac{\partial\Phi_{ikc}(I_i(G_{i,s}(x)))}{\partial(I_i(G_{i,s}(x)))}\cdot\frac{\partial(I_i(G_{i,s}(x)))}{\partial G_{i,s}(x)}\cdot\frac{\partial G_{i,s}(x)}{\partial G_{i,s}}\\ &= \sum_{x\in\Omega}\frac{1}{LH(x)}\sum_{k\in K}\prod_{j\neq i}P(I_j(x)\mid s(x)=k,\theta,G_{j,s})\\ &\cdot\sum_c\tau_{ikc}\Phi'_{ikc}\nabla I_i(G_{i,s}(x))\nabla G_{i,s}(x)\end{aligned}$$

For the deformation D

$$\frac{\partial LL}{\partial D} = \sum_{x\in\Omega}\frac{1}{LH(x)}\sum_{k\in K}\frac{\partial\pi_{kx|D}}{\partial D}P(I\mid s(x),\theta,\{G_{i,s}\})$$

Where $\frac{\partial\pi_{kx|D}}{\partial D} = \frac{\partial A_k(D(x))}{\partial D}$,

$$\frac{\partial A_k(D(x))}{\partial D} = \nabla A_k\big|_{y=[y_1,y_2,y_3]}\times\left[\frac{\partial y_1}{\phi_d},\frac{\partial y_2}{\phi_d},\frac{\partial y_3}{\phi_d}\right]^{\mathrm{T}}$$

Where $y = D(x), \{\phi_d\}$ are parameters of FFD.

So, the update is,

$$\begin{aligned}G_{i,s}^{[k+1]} &= G_{i,s}^{[k]} + l_G\cdot\frac{\partial LL}{\partial G_{i,s}}\\ D^{[k+1]} &= D^{[k]} + l_D\cdot\left(\frac{\partial LL}{\partial D}+\lambda\cdot\frac{\partial C_{\text{smooth}}}{\partial D}\right)\end{aligned}$$

where $\dfrac{\partial C_{\text{smooth}}}{\partial D}$ is a regularization term.

Hetero-Coverage Multi-Modality Images (HCMMI)

Introduction of HCMMI: owing to each modality may have different coverage of the subject, so we employ multiple MvMMs to address this problem.

First, The whole common space is divided into several spaces:

$$\Omega = \cup_{v=1}^{N_{sr}}\Omega_v$$

where Ω_v expresses the v_{th} non-overlapping region.

So our likelihood is the summation of each region Ω_v,

$$LL_{HC} = \sum_{v=1}^{N_{sr}}LL_{\Omega_v} = \sum_{v=1}^{N_{sr}}\sum_{x\in\Omega_v}\log\sum_{k\in K}\pi_{kx}\prod_{i\in N_v}(\sum_{c\in C_{ik}}\Phi_{ikc}(I_i(x)))$$

The parameter update is simliar above, but $x \in \Omega_v$ instead of $x \in \Omega$ and $\prod_{i \in N_v}$ instead of $\prod_{i \in N_I}$. Specially, if $\|\Omega_v\| = 1$, it's a GMM. If $\|\Omega_v\| \geq 2$, it's MvMM.