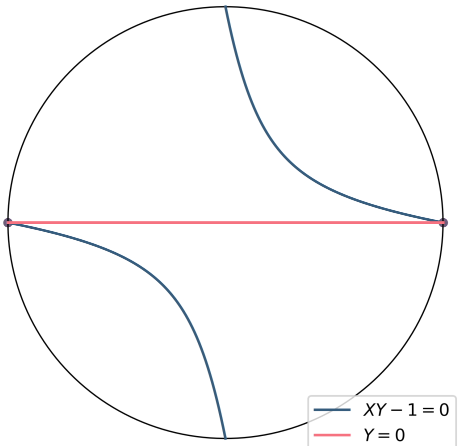
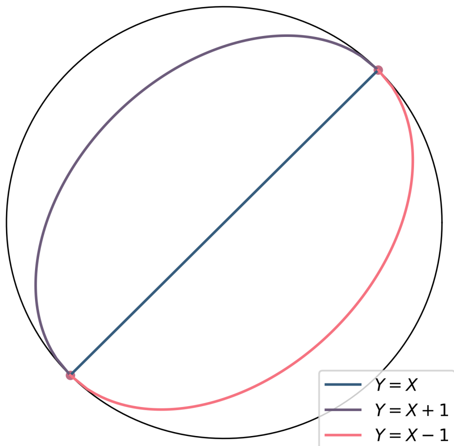
Chapter 2. Projective Algebraic Sets

Throughout this chapter, k will be a commutative field.

In affine spaces, two exceptional cases must always be accounted for when addressing intersection problems: parallelism and asymptoticity. However, projective spaces bypass the necessity of such case distinctions. Figure 2.1 visualizes the real projective plane, where both parallel curves and asymptotic curves acquire distinct intersection points.



(a) Asymptoticity



(b) Parallelism

Figure 2.1: $\mathbb{P}^2(\mathbb{R})$

2.1 Projective Space

Let n be an non-negative integer and let E be a k-vector space of dimension $n+1$.

2.1.1 Definition. *Define an equivalence relation $\mathcal{R}$ on $E - \{0\}$:*

$$x\mathcal{R}y \iff \exists\lambda \in k^*, y = \lambda x.$$

*The **projective space** associated to E, denoted by $\mathbb{P}(E)$, is the quotient of $E - \{0\}$ by the relation $\mathcal{R}$. When $E = k^{n+1}$, we write $\mathbb{P}(E) = \mathbb{P}^n(k)$ and we call this space **standard n-dimensional projective space**.*

2.1.2 Definition. *Let F be a subspace in E of dimension $m+1$. The image of $F - \{0\}$ in $\mathbb{P}(E)$ is called a projective subspace of dimension m, denoted by $\overline{F}$.*

2.1.3 Proposition. *Let V, W be two projective subspaces of $\mathbb{P}(E)$ of dimensions r and s such that $r+s-n \geq 0$. Then $V \cap W$ is a projective subspace of dimension $\geq r+s-n$. (In particular, $V \cap W$ is non-empty.)*

2.1.4 Definition. *If E is a vector space, then the linear group $\mathrm{GL}(E)$ acts on E. We consider $u \in \mathrm{GL}(E)$; since u is injevtive and preserves collinearity, u induces a bijection $\overline{u}$ of $\mathbb{P}(E)$. Such bijection of $\mathbb{P}(E)$ like $\overline{u}$ is called a **homography**. The set of homographies form a group $\mathrm{PGL}(E) = \mathrm{GL}(E)/k^*$.*

2.2 Projective Algebraic Sets

2.2.1 Proposition-Definition. *Consider $F \in k[X_0, \ldots, X_n]$ and $\overline{x} \in \mathbb{P}^n$. We say that $\overline{x}$ is a **zero** of F if $F(x) = 0$ for any system of homogeneous coordinates x for $\overline{x}$. We then write either $F(x) = 0$ or $F(\overline{x}) = 0$. If F is homogeneous, it is enough to check that $F(x) = 0$ for any system of homogeneous coordinates. If $F = F_0 + F_1 + \cdots + F_r$, where F_i is homogeneous of degree i, then it is necessary and sufficient that $F_i(x) = 0$ for all i.*

2.2.2 Definition. *Let S be a subset of $k[X_0, \ldots, X_n]$. We set*

$$V_p(S) := \{x \in \mathbb{P}^n \mid \forall F \in S, F(x) = 0\}.$$

*We say that $V_p(S)$ is the **projective algebraic set** defined by S. When there is no risk of confusion, we denote this set by $V(S)$.*

Similarly, an arbitrary intersection or finite union of projective algebraic sets is a projective algebraic set, so there is a Zariski topology on $\mathbb{P}^n$ whose closed sets are the projective algebraic sets.

2.2.3 Definition. *Let V be a subset of $\mathbb{P}^n$. We define the ideal of V by the formula*

$$I_p(V) = \{F \in k[X_0, \ldots, X_n] \mid \forall x \in V, F(x) = 0\}.$$

2.2.4 Theorem. (Projective Nullstellensatz) *Assume that k is algebraically closed. Let I be a homogeneous ideal (an ideal generated by homogeneous polynomials) of $k[X_1, \ldots, X_n]$ and set $V = V_p(I)$.*

1. $V_p(I) = \emptyset \iff \exists N$ *such that* $(X_0 \ldots, X_n)^N \subset I$
   $\iff (X_0, \ldots, X_n) = R^+ \subset \sqrt{I}$.
2. *If* $V_p(I) \neq \emptyset$, *then* $I_p(V_p(I)) = \sqrt{I}$.

2.3 Graded Rings

2.3.1 Definition. *A k-algebra R is said to be **graded** if it can be written as a direct sum (of vector spaces)*

$$R = \bigoplus_{n \in \mathbb{N}} R_n,$$

*where the subspaces R_n of R satisfy $R_p R_q \subset R_{p+q}$. The elements of R_p are said to be **homogeneous** of degree p and this condition is the usual rule for the degree of a product.*

2.3.2 Proposition-Definition. *Let R be a graded k-algebra and let I an ideal of R. The following are equivalent.*

1. *I is generated by homogeneous elements.*
2. *If $f \in I$ and $f = \sum_{i=0}^{r} f_i$ and f_i is homogeneous of degree i, then $f_i \in I$ for every i.*

*Such an ideal is said to be **homogeneous**.*

2.3.3 Proposition. *Let R be a graded k-algebra and let I be a homogeneous ideal of R. Let S ne the quotient k-algebra $S = R/I$ and p the canonical projection. Then S has a natural grading given by $S_i = p(R_i)$.*

2.3.4 Definition. *Let R be a graded k-algebra. An R-module M is said to be **graded** if it can be written as a direct sum (of vector spaces)*

$$M = \bigoplus_{n \in \mathbb{Z}} M_n,$$

*where the k-subspaces M_n of M satisfy $R_p M_q \subset M_{p+q}$ for all $p \in \mathbb{N}$ and $q \in \mathbb{Z}$. A homomorphism $\varphi : M \to N$ between two graded R-modules is said to be **homogeneous** of degree d if, for all n, $\varphi(M_n) \subset N_{d+n}$.*

Let V be a projective algebraic set and let $I_p(V)$ be its ideal. Since $I_p(V)$ is homogeneous, the quotient ring

$$\Gamma_h(V) := k[X_0, \ldots, X_n]/I_p(V)$$

is graded.

2.3.5 Proposition-Definition. *Let V be a projective algebraic set and consider a homogeneous element $f \in \Gamma_h(V)$ of degree > 0. We set*

$$D^+(f) := \{x \in V \mid f(x) \neq 0\}.$$

It is clear that the sets $D(f)$ are open sets of V and every non-empty open set of V is a finite union of open sets of the form $D^+(f)$.