Chapter 1. Affine Algebraic Sets

The aim of our seminar is to study the properties of the zero locus of a polynomial or several polynomials. The first task for us is to study algebraic sets, the common zeros of some polynomials.

1.1 Affine Algebraic Sets

Let n be a positive integer, and k any field. Consider the affine space k^n. If $x = (x_1, \ldots, x_n)$ is a point in k^n and $P(X_1, \ldots, X_n)$ is a polynomial, we denote $P(x_1, \ldots, x_n)$ by $P(x)$.

1.1.1 Definition. *Let S be a subset of $k[X_1, \ldots, X_n]$. We set*

$$V(S) := \{x \in k^n \mid \forall P \in S, P(x) = 0\},$$

*in other words, the $x \in V(S)$ are the common zeros of all the polynomials in S. We call $V(S)$ the **affine algebraic set** defined by S. When the set S is finite, we will often write $V(F_1, \ldots, F_r)$ instead of $V(\{F_1, \ldots, F_r\})$.*

*In particular, if $F \in k[X_1, \ldots, X_n]$, we call $V(F)$ the **hypersurface** defined by F. And if $\deg F = 1$, we call $V(F)$ a **hyperplane**.*

1.1.2 Proposition.

1. *The function V is decreasing: If $S \subset S'$, then $V(S') \subset V(S)$.*
2. *If $S \subset k[X_1, \ldots, X_n], I = (S)$, we have $V(S) = V(I)$.*
3. *Since $k[X_1, \ldots, X_n]$ is Noetherian, every ideal is finitely generated: $I = (f_1, \ldots, f_r)$, and hence every affine algebraic set is defined by a finite number of equations $V(I) = V(f_1) \cap \cdots \cap V(f_r)$.*
4. *A point of k^n is an affine algebraic set: if $a = (a_1, \ldots, a_n) \in k^n$, then $\{a\} = V(X_1 - a_1, \ldots, X_n - a_n)$.*
5. *An arbitrary intersection of affine algebraic sets is an affine algebraic set:*
$$\bigcap_j V(S_j) = V(\bigcup_j S_j).$$
6. *A finite union of affine algebraic sets is an affine algebraic set: if I, J are ideals, then*
$$V(I) \cup V(J) = V(IJ) = V(I \cap J).$$

1.1.3 Definition. *By 1.1.2, we can find that the affine algebraic sets verify the axioms for closed sets that define a topology. The **Zariski topology** on k^n is the topology whose closed sets are the affine algebraic sets. Of course, any subset X of k^n inherits an induced topology (again called the Zariski topology) whose closed sets are the sets of the form $X \cap V(I)$; in particular, if X is an affine algebraic set, then the closed sets of X are affine algebraic sets contained in X.*

1.1.4 Definition. *Consider $f \in k[X_1, \ldots, X_n]$. Define $D(f) := k^n - V(f)$, which is a Zariski open set of k^n and is called a **standard open set**. The standard open sets are a basis for this topology; more precisely, any open set U is a finite union of standard open sets.*

1.1.5 Definition. *Let V be a subset of k^n. The set*

$$I(V) := \{f \in k[X_1, \ldots, X_n] \mid \forall x \in V, f(x) = 0\}$$

*is called **the ideal of** V.*

In other words, $I(V)$ is the set of polynomial functions which vanish on V. To show that it is indeed an ideal, wo consider the ring homomorphism

$$r : k[X_1, \ldots, X_n] \to \mathcal{F}(V, k),$$

with image in the ring of all k-valued functions on V associating to a polynomial the restriction of the associated polynomial function to V. Then we can find that $I(V) = \ker r$. The image of r, which is isomorphic to $k[X_1, \ldots, X_n]/I(V)$, is denoted by $\Gamma(V)$. The ring $\Gamma(V)$ is called the ***affine algebra*** *of V, which is a k-algebra of finite type.*

1.1.6 Proposition.

1. *The map I is decreasing: If $V \subset V'$, then $I(V') \subset I(V)$.*
2. *If V is an affine algebraic set, then $V(I(V)) = V$. It follows that the map $V \mapsto I(V)$ is injective, and hence if $V \subsetneq W$, then there exists a polynomial which vanishes on V and does not vanish on W.*
3. *$I \subset I(V(I))$.*

1.1.7 Proposition. *Assume that k is infinite. Then $I(k^n) = 0$.*

Proof.

We proceed by induction on n.

$(n = 1)$.

Since a non-zero polynomial has only a finite number of roots, the proposition holds.

$(n > 1)$.

Let $P \in k[X_1, \ldots, X_n]$ be a non-zero polynomial. We can write $P(X_1, \ldots, X_n) = a_r(X_1, \ldots, X_{n-1})X_n^r + \cdots$. And by the inductive hypothesis, there is $(x_1, \ldots, x_{n-1}) \in k^{n-1}$ such that $a_r(x_1, \ldots, x_{n-1}) \neq 0$. Then $P(x_1, \ldots, x_{n-1}, X_n)$ has at most r roots; and hence there is $x_n \in k$ such that $P(x_1, \ldots, x_n) \neq 0$. $\square$

1.1.8 Example.

1. $I(\emptyset) = k[X_1, \ldots, X_n]$.
2. $I(\{(a_1, \ldots, a_n)\}) = (X_1 - a_1, \ldots, X_n - a_n)$.
   Let $P \in I(\{(a_1, \ldots, a_n)\})$, we can write $P = (X_1 - a_1)Q_1 + \cdots + (X_n - a_n)Q_n + c$ where $c \in k$. Since $P(a_1, \ldots, a_n) = 0$, we have $c = 0$.
3. Let us assume that k is infinite and calculate the ideal
   $$I(V) = I(V(Y^2 - X^3))$$
   in $k[X, Y]$. It's clear that $(Y^2 - X^3) \subset I(V)$. Conversely, we know that any point in V can be written as (t^2, t^3), with $t \in k$ (if $x \neq 0$, we take $t = y/x$ and if $x = 0$, we take $t = 0$). Suppose that $P \in I(V)$. We divide P by $Y^2 - X^3$ with respect to the variable Y:
   $$P = (Y^2 - X^3)Q(X, Y) + a(X)Y + b(X).$$
   It follows that for any $t \in k$, $P(t^2, t^3) = 0 = a(t^2)t^3 + b(t^2)$. Since k is infinite, we deduce that $a(T^2)T^3 + b(T^2) = 0$ in $k[T]$. Separating the terms of odd and even degrees we get $a = b = 0$ and we have hence proved that $I(V) = (Y^2 - X^3)$.

1.2 Irreducibility

1.2.1 Definition. *Let X be a non-empty topological space. The following are equivalent:*

1. *If we can write X in the form $X = F \cup G$, where F and G are closed sets in X, then $X = F$ or $X = G$.*
2. *If U, V are two open sets of X and $U \cap V = \emptyset$, then $U = \emptyset$ or $V = \emptyset$.*
3. *Any non-empty open set of X is dense in X.*

Under these conditions we say that X is ***irreducible****.*

1.2.2 Proposition.

1. *If X is irreducible and U is an open subset of X, then U is irreducible.*
2. *If X is of the form $U_1 \cup U_2$, where U_i are open and irreducible, and $U_1 \cap U_2 \neq \emptyset$, then X is irreducible.*

1.2.3 Theorem. *Let V be an affine algebraic set equipped with its Zariski topology. Then, V irreducible $\Leftrightarrow$ $I(V)$ prime $\Leftrightarrow \Gamma(V)$ integral.*

Proof.

We only need to show V irreducible $\Leftrightarrow I(V)$ prime.

If $IJ \subset I(V)$ where I, J are ideals, we have $V \subset V(I) \cup V(J)$. Thus, $V \subset V(I)$ or $V \subset V(J)$. And hence, $I \subset I(V)$ or $J \subset I(V)$; that is $V(I)$ is prime.

Conversely, if $V \subset U \cup W$ where U, W are closed sets, we have $I(U)I(W) \subset I(V)$. Hence, $I(U) \subset I(V)$ or $I(W) \subset I(V)$. Therefore, $V \subset U$ or $V \subset W$ and V is irreducible. □

1.2.4 Corollary. *Assume that k is infinite. Then the affine space k^n is irreducible.*

1.2.5 Proposition. *Let X be a topological space and Y a subspace of X. Then if Y is irreducible, so is its closure $\overline{Y}$. If U is an open set of X, then the maps $Y \mapsto \overline{Y}$ and $Z \mapsto Z \cap U$ are mutually inverse bijections between the irreducible closed sets Y in U and the irreducible closed sets Z in X which meet U.*

1.2.6 Theorem-Definition. *Let V be a non-empty affine algebraic set. We can write V uniquely (up to permutation) in the form $V = V_1 \cup \cdots \cup V_r$, where sets V_i are irreducible affine algebraic sets and $V_i \not\subset V_j$ for $i \neq j$. The sets V_i are called the* ***irreducible components*** *of V.*

Proof.

(Existence).

We proceed by contradiction. Assume there exist non-decomposable affine algebraic sets and we pick one whose ideal is maximal amongst all such sets. (Such a V exists since the ring $k[X_1, \ldots, X_n]$ is Noetherian.) Since V is not irreducible, we can write $V = U \cup W$, where $U, W \neq V$ and U, V are closed.

It follows by injectivity and of I that $I(V) \subsetneq I(U), I(W)$. By maximality of $I(V)$, it follows that U and W are composable: $U = U_1 \cup \cdots \cup U_r, W = W_1 \cup \cdots \cup W_s$, but V is decomposable, which gives us a contradiction.

(Uniqueness).

Assume given two expressions: $V = V_1 \cup \cdots \cup V_r = W_1 \cup \cdots W_s$. We have $V_i = V \cap V_i = (W_1 \cap V_i) \cup \cdots \cup (W_s \cap V_i)$. Since V_i is irreducible, there is a j such that $V_i = W_j \cap V_i$; that is $V_i \subset W_j$. Likewise, there is a l such that $W_j \cup V_l$, and hence $V_i \subset V_l$, which implies by hypothesis that $i = k$ and hence $V_i = W_j$. □

1.3 The Nullstellensatz

This is one of the first fundamental theorems of algebraic geometry. It controls the correspondence between affine algebraic sets and ideals; in particular, it enables us to calculate $I(V(I))$. **In this section, we assume that k is algebraically closed.**

Note *Algebraically closed fields are always infinite.*

1.3.1 Theorem. (Weak Nullstellensatz) *Let $I \subset k[X_1, \ldots, X_n]$ be a proper ideal. Then $V(I)$ is non-empty.*

Proof.

Since every ideal is contained in a maximal ideal and the map V is decreasing, we can assume that I is maximal. Then let $L = k[X_1, \ldots, X_n]/I$, which is both a field and a k-algebra of finite type. By the Theorem 0.5.6, it follows that $\partial_k L = \dim_K L = 0$, which implies that L is algebraic over k. Since k is algebraically closed, then $L = k$. Therefore, for all X_i, there is $a_i \in k$ such that $X_i - a_i \in I$. It follows by $(X_1 - a_1, \dots, X_n - a_n)$ is a maximal ideal that $I = (X_1 - a_1, \dots, X_n - a_n)$ and $(a_1, \dots, a_n) \in V(I)$. □

1.3.2 Theorem. (Nullstellensatz) *Let I be an ideal of $k[X_1, \dots, X_n]$. Then*

$$I(V(I)) = \sqrt{I}.$$

Proof.

We set

$$R = k[X_1, \dots, X_n], \quad I = (P_1, \dots, P_r) \quad \text{and} \quad V = V(I).$$

Since $F^m(x) = 0 \Rightarrow F(x) = 0$, then $\sqrt{I} \subset I(V(I))$. It remains to show $I(V(I))$; that is, take $F \in I(V(I))$, show $F \in \sqrt{I}$. Since we have

$$\begin{aligned} F \in \sqrt{I} &\Leftrightarrow \exists m, F^m \in I \\ &\Leftrightarrow \exists m, F^m \in IR_F \\ &\Leftrightarrow IR_F = (1), \end{aligned}$$

we only need to show $IR_F = (1)$. But the ring R_F is isomorphic to $k[X_1, \dots, X_n, T]/(1 - TF)$. To show $IR_F = (1)$, it's sufficient to show $J = k[X_1, \dots, X_n, T]$, where $J := (P_1, \dots, P_r, 1 - TF)$.

Assume $J \neq k[X_1, \dots, X_n, T]$. By the Weak Nullstellensatz, it follows that $V(J) \neq \emptyset$. Take a point $(x_1, \dots, x_n, t) \in V(J)$. Then, for all $1 \le i \le r$, $P_i(x_1, \dots, x_n) = 0$. Thus, $(x_1, \dots, x_n) \in V$ and $F(x_1, \dots, x_n) = 0$. Hence, $1 - TF$ can't vanish at $(x_1, \dots, x_n, t)$, which leads a contradiction. □

1.3.3 Corollary. *Consider $F \in k[X_1, \dots, X_n]$, $F = F_1^{\alpha_1} \cdots F_r^{\alpha_r}$, where the F are irreducible and non-associated and $\alpha_i > 0$. We then have:*

1. *$I(V(F)) = (F_1 \cdots F_r)$. In particular, if F is irreducible, then $I(V(F)) = (F)$.*
2. *The decomposition of $V(F)$ into irreducible components is given by $V(F) = V(F_1) \cup \cdots \cup V(F_r)$. In particular, if F is irreducible, then $V(F)$ is as well.*

1.3.4 Corollary. *Let V be an affine algebraic set. We associate to V its ideal $I(V)$ and its affine algebra $\Gamma(V) \simeq k[X_1, \dots, X_n]/I(V)$, which is a reduced k-algebra of finite type.*

1.3.5 Corollary. *There is a decreasing bijection $W \mapsto I(W)$, whose inverse is $I \mapsto V(I)$, between affine algebraic sets in k^n and radical ideals in $k[X_1, \dots, X_n]$. Moreover, we have:*

1. *W irreducible $\Leftrightarrow I(W)$ prime $\Leftrightarrow \Gamma(W)$ integral;*
2. *W is a point $\Leftrightarrow I(W)$ maximal $\Leftrightarrow \Gamma(W) = k$.*

1.3.6 Proposition. *V is finite $\Leftrightarrow \Gamma(V)$ is a finite-dimensinal k-vector space.*

Proof.

($\Rightarrow$). Consider the ring homomorphism r in 1.1.5. When V is finite, $\mathscr{F}(V, k)$ is a finite-dimensinal k-vector space; and hence, $\Gamma(V) = \operatorname{im} r$ is finite-dimensinal.

($\Leftarrow$). Let $\overline{X_i}$ be the image image of X_i in $\Gamma(V)$. The elements $1, \overline{X_i}, \overline{X_i}^2, \dots$ are linear dependent, and hence in $\Gamma(V)$ there is an identity

$$a_s \overline{X_i}^s + \cdots + a_1 \overline{X_i} + a_0 = 0,$$

where $a_j \in k$ and $a_s \neq 0$. If $u = (x_1, \dots, x_n)$ is an arbitrary point of V, it follows that we also have

$$a_s x_i^s + \cdots + a_1 x_i + a_0 = 0$$

and hence there are only a finite number of possible values for the ith coordinate of u and hence also for u. □

Then, let's fix an arbitrary affine algebraic set V and consider algebraic sets contained in V.

1.3.7 Definition. *If W is an algebraic affine set contained in V, then $I(V) \subset I(W)$ and $I(W)$ determine an ideal $I_V(W)$ of the ring $\Gamma(V)$ (namely its image, which is simply the set of $f \in \Gamma(V)$ which vanish on W). We have an isomorphism $\Gamma(V) \backslash I_V(W) \simeq \Gamma(W)$, from which it follows that this ideal is also radical. We note that if I is an ideal of $\Gamma(V)$, then we can define $V(I)$ either as the set of zeros of functions of I on V:*

$$V(I) := \{x \in V \mid \forall f \in I, f(x) = 0\}$$

or, which amounts to the same thing, by setting $V(I) = V(r^{-1}(I))$, where r is the canonical projection of $k[X_1, \ldots, X_n]$ onto $\Gamma(V)$.

1.3.8 Proposition. *There are mutually inverse decreasing bijections $W \mapsto I_V(W)$ and $I \mapsto V(I)$ between affine algebraic subsets contained in V and radical ideals of $\Gamma(V)$. Moreover, we have:*

1. *W irreducible $\Leftrightarrow I_V(W)$ prime $\Leftrightarrow \Gamma(W)$ integral,*
2. *W is a point $\Leftrightarrow I_V(W)$ maximal $\Leftrightarrow \Gamma(W) = k$,*
3. *W is an irreducible component of $V \Leftrightarrow I_V(W)$ is a minimal prime ideal of $\Gamma(V)$.*

1.3.9 Definition. *To any $x \in V$, there corresponds a homomorphism of k-algebras $\chi_x : \Gamma(V) \to k, f \mapsto f(x)$ whose kernel is the maximal ideal*

$$\mathfrak{m}_x := I(\{x\}) = \{f \in \Gamma(V) \mid f(x) = 0\}.$$

*The k-algebra homorphisms $\chi : \Gamma(V) \to k$ are also called the **characters** of $\Gamma(V)$.*

1.3.10 Proposition. *The points of V are in bijective correspondence with the maximal ideal of $\Gamma(V)$, or, alternatively, with characters of $\Gamma(V)$.*

1.3.11 Proposition-Definition. *Let V be an affine algebraic set and let $f \in \Gamma(V)$ be non-zero. The set*

$$D_V(f) = V - V(f) = \{x \in V \mid f(x) \neq 0\}$$

*(which we denote by $D(f)$ when there is no risk of confusion) is called a **stand open set** of V. Every open set in V is a finite union of standard open sets.*

1.4 Intersection of Plane Curves

We will now show that the intersection of two plane curves without common components is finite. In this section, k is an arbitrary field.

Fix $F, G \in k[X, Y]$, which are non-zero polynomials without common factors.

1.4.1 Lemma. *There is a non-zero polynomial $d \in k[X]$ and polynomials $A, B \in k[X, Y]$ scuh that $d = AF + BG$. (In other words, $d \in (F, G)$.)*

Proof.

Regard F, G as polynomials in $k(X)[Y]$. Since F, G have no common factors, they are co-prime in $k(X)[Y]$.Then, by Bézout's identity, it follows that there are $A, B \in k(X)[Y]$ such that

$$AF + BG = 1.$$

Canceling denominators, we can get the identity in the lemma. □

1.4.2 Theorem. *The ring $k[X, Y]/(F, G)$ is a finite dimensional k-vector space.*

Proof.

By 1.4.1, there are polynomials $d_1 \in k[X], d_2 \in k[Y]$, such that $d_1, d_2 \in (F, G)$. Let $s = \deg(d_1), t = \deg(d_2)$. We have all the X^iY^j's$(i < s, j < t)$ consist a basis of $k[X, Y]/(F, G)$. □

1.4.3 Theorem. *$V(F) \cap V(G)$ is finite.*

Proof.

Since $(F,G) \subset I(V(F,G))$, we have a surjective homomorphism $k[X,Y]/(F,G) \to \Gamma(V(F,G))$, which is also a k-linear map. Therefore, $\Gamma(V(F,G))$ is finte-denmensional. And by 1.3.6, it follows that $V(F,G)$ is finite. □

1.5 Morphisms

In this section we will assume that the field k is infinite.

1.5.1 Definition. *Let $V \subset k^n$ and $W \subset k^m$ be two affine algebraic sets and let $\varphi : V \to W$ be a map which we can write in the form $\varphi = (\varphi_1, \dots, \varphi_m)$, where $\varphi_i : V \to k$. We say that f is* ***regular*** *(or a* ***morphism****) if its components f_i are polynomial (in other words, $f_i \in \Gamma(V)$). We denote the set of regular maps from V to W by* $\mathrm{Reg}(V,W)$.

Note *It is clear that we obtain in this way a category: the identity is a morphism, as is the composition of two morphisms. We note that morphisms are continuous map for the Zariski topology (which is to say that the preimage of an algebraic set is again a algebraic set), but converse is false (for example, any bijective map from k to k is continuous for the Zariski topology but is not necessarily polynomial).*

1.5.2 Example.

1. The elements of $\Gamma(V)$, particularly the coordinate functions, are morphisms from V to k.
2. The bijective affine maps from k^n to itself are isomorphisms: they correspond to polynomials of degree 1.
3. Consider $V \subset k^n$. The projection f from V to $k^p (p \leq n)$, given by $\varphi(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_p})$, is a morphism.
4. Let V be the parabola $V(Y - X^2)$ and let f be the projection $\varphi : V \to k, (x,y) \mapsto x$. Then f is an isomorphism, whose inverse is given by $x \mapsto (x, x^2)$.
5. The map $\varphi : k \to V(X^3 + Y^2 - X^2)$, given by the parameterisation $x = t^2 - 1, y = t(t^2 - 1)$ (obtained by intersection with the line $Y = tX$), is a morphism but not an isomorphism (φ is note injective).
6. The map $\varphi : k \to V(Y^2 - X^3)$ given by the parameterisation $t \mapsto (t^2, t^3)$ is a bijective morphism, but we will see further on that it is not an isomorphism.

We have associated to an affine algebraic set V its affine algebra $\Gamma(V)$ and started to set up a dictionary allowing us to pass from one to the other. Of course, we will have to extend this correspondence to morphisms: in other words, we must show it is funcorial.

1.5.3 Proposition-Definition. *Let $\varphi : V \to W$ be a morphism. For any $f \in \Gamma(W)$, we set $\varphi^*(f) = f \circ \varphi$. Then φ^* is a morphism of k-algebras, $\varphi^* \Gamma(W) \to \Gamma(V)$.*

Now, we can regard Γ as a contravariant funtor from the category of affine algebraic sets with regular maps to the category of k-algebras with k-algebra morphisms.

We can calculate φ^* in the following way: let $V \subset k^n$ and $W \subset k^m$ be two affine algebraic sets and let $\varphi : V \to W$ be a morphism, written in the form $\varphi = (\varphi_1, \dots, \varphi_m)$, where $\varphi_i \in \Gamma(V)$. We denote by η_i the ith coordinate function on W, which is the image of Y_i in $\Gamma(W)$. Then $\varphi^*(\eta_i) = \varphi_i$. If the functions φ_i are restriction to V of polynomials $P_i(X_1, \dots, X_n)$, then the homomorphism

$$\varphi^* : k[Y_1, \dots, Y_m]/I(W) \to k[X_1, \dots, X_n]/I(V)$$

is given by $Y_i \mapsto \overline{P_i}(X_1, \ldots, X_n)$.

If $\varphi(x) = y$, then it is easily checked that $(\varphi^*)^{-1}(\mathfrak{m}_x) = \mathfrak{m}_y$.

1.5.4 Example.

1. If φ is the projection $V(F) \subset k^2 \to k$, where $\varphi(x, y) = x$, then φ^* is the map from $\Gamma(k) = k[X]$ to $k[X, Y]/(F)$ which associates X to X.
2. Consider the parameterisation of $V(Y^2 - X^3)$ by t^2, t^3. We have
$$\varphi^* : k[X, Y]/(Y^2 - X^3) \to k[T],$$
which is given by $\varphi^*(\overline{X}) = T^2$ and $\varphi^*(\overline{Y}) = T^3$.

Next, we will study the properties of the functor Γ.

1.5.5 Proposition. *The functor Γ is fully faithful. In other words, the map $\gamma : \varphi \mapsto \varphi^*$ from* $\mathrm{Reg}(V, W)$ *to* $\mathrm{Hom}_{k\text{-}\mathbf{Alg}}(\Gamma(W), \Gamma(V))$ *is bijective.*

Proof.

We assume $V \subset k^n$ and $W \subset k^m$ are affine algebraic sets and the coordinate functions on W by η_i.

(Γ is faithful).

Let φ and ψ are two morphisms from V to W such that $\varphi^* = \psi^*$. Then we have $\varphi_i = \varphi^*(\eta_i) = \psi^*(\eta_i) = \psi_i$; and hence $\varphi = \psi$.

(Γ is fully faithful).

Let $\theta : \Gamma(W) \to \Gamma(V)$ be a homomorphism of k-algebras. We set $\varphi_i = \theta(\eta_i) \in \Gamma V$. We consider the map $\varphi : V \to k^m$ whose coordinates are the elements φ_i. It remains to show the image of φ is contained in W. Consider $F(Y_1, \ldots, Y_m) \in I(W)$ and $x \in V$. We have

$$\begin{aligned} F(\varphi_i(x)) &= F(\theta(\eta_1)(x), \ldots, \theta(\eta_n)(x)) \\ &= F(\theta(\eta_1), \ldots, \theta(\eta_n))(x) \\ &= \theta(F(\eta_1, \ldots, \eta_n)))(x) \\ &= 0. \end{aligned}$$

Therefore, φ is a morphism from V to W and Γ is fully faithful. □

1.5.6 Corollary. *Let $\varphi : V \to W$ be a morphism. Then φ is an isomorphism if and only if φ^* is an isomorphism. It follows that V and W are isomorphic if and only if $\Gamma(V)$ and $\Gamma(W)$ are isomorphic.*

1.5.7 Example. The morphism $\varphi : k \to V(Y^2 - X^3)$ given by $\varphi(t) = (t^2, t^3)$ is not an isomorphism.

1.5.8 Definition. *Let $\varphi : V \to W$ be a morphism. We say that φ is **dominant** if the closure of its image (in the Zariski topology) is equal to the whole of W, $\overline{\varphi(V)} = W$.*

1.5.9 Proposition. *Let $\varphi : V \to W$ be a morphism.*

1. *φ dominant $\Leftrightarrow$ φ^* injective.*
2. *Assume that φ is dominant and V is irreducible. Then W is irreducible.*

Proof.

(1). If φ is dominant and $f \in \ker \varphi^*$, then $f\varphi = 0$ and hence f vanishes on $\varphi(V)$. Since f is continuous, f vanishes on $\overline{\varphi(V)} = W$. Conversely, set $X = \overline{\varphi(V)}$. This is an affine algebraic set contained in W. Assume $X \neq W$. Then there exists a non-zero $f \in \Gamma(W)$ which vanishes on X. But then $f\varphi = \varphi^*(f) = 0$, which is a contradiction.

(2). It follows from (1) and 1.3.5. □

1.5.10 Theorem. *Assume that k is algebraically closed. The functor Γ is then an equivalence of categories* *between the category of affine algebraic sets with regular maps and the category of reduced k-algebras of finite type with homomorphisms of k-algebras. (This means that the functor is fully faithful and essentially surjective.)*

Proof.

Let A be a reduced k-algebra of finite type. Since A is of finite type, we can write $A \simeq k[X_1, \ldots, X_n]/I$, and since A is reduced, the ideal I is radical. We set $V = V(I)$. We have $I(V) = \sqrt{I} = I$ by the Nullstellensatz, and hence $A \simeq \Gamma(V)$. □

1.5.11 Definition. *Let V be an irreducible affine algebraic set, so the ring $\Gamma(V)$ is integral. The field of fractions of $\Gamma(V)$ is called the field of rational functions on V and is denoted by $K(V)$.*

Note *If $f \in K(V)$, then f can be written in the form $f = g/h$, where $g, h \in \Gamma(V)$ and $h \neq 0$. We can therefore consider f to be a function defined on the standard open set $D(h)$ defined by $h(x) \neq 0$.*