

Chapter 0. Preparation

0.1 Categories

0.1.1 Definition. A *category* \mathcal{C} consists of:

1. $\text{Obj}(\mathcal{C})$: the class of the objects (which may not be a set);
2. $\text{Mor}(\mathcal{C})$: the class of morphisms. For each morphism $f \in \text{Mor}(\mathcal{C})$, it has a **source** $s(f)$ and a **target** $t(f)$, where both $s(f)$ and $t(f)$ are elements of $\text{Obj}(\mathcal{C})$. Let X be the source of f and Y be the target of f we can denote f as $f : X \rightarrow Y$. And we define $\text{Hom}_{\mathcal{C}}(X, Y)$ (also $\text{Hom}(X, Y)$ for short) as the class of morphisms with source X and target Y ; that is $\text{Hom}_{\mathcal{C}}(X, Y) := s^{-1}(X) \cap t^{-1}(Y)$.

Additionally, objects and morphisms should satisfy these properties:

1. $\forall X, Y, Z \in \text{Obj}(\mathcal{C})$, there is a **composition** $\circ : \text{Hom}(Y, Z) \times \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z)$, $(g, f) \mapsto g \circ f$. We also abbreviate $g \circ f$ as gf .

And we can use a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g \circ f & \downarrow g \\ & & Z \end{array}$$

to describe.

2. The composition we defined above satisfies the **associative law**; that is $\forall X, Y, Z, T \in \text{Obj}(\mathcal{C})$ and $f : X \rightarrow Y, g : Y \rightarrow Z, h : Z \rightarrow T$, we have

$$(h \circ g) \circ f = h \circ (g \circ f).$$

And we can also use a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \circ f \downarrow & & \downarrow h \circ g \\ Z & \xrightarrow{h} & T \end{array}$$

to describe.

3. $\forall X \in \text{Obj}(\mathcal{C})$, there is an **identity** $1_X \in \text{Hom}(X, X)$, such that for all $Y, Z \in \text{Obj}(\mathcal{C})$ and $f : X \rightarrow Y, g : Z \rightarrow X$, we have

$$f \circ 1_X = f, 1_X \circ g = g.$$

0.1.2 Example.

1. Let's begin with a simple example, the category of sets, denoted as **Set**. $\text{Obj}(\text{Set})$ is the class of all sets (as we all know, it can't be a set because of the Russell's paradox). $\text{Hom}_{\text{Set}}(A, B)$ are all maps from A to B . It's easy to check **Set** satisfies the concept of category.
2. The category of topological spaces, denoted as **Top**. The objects of **Top** are all topological spaces, and the morphisms are continuous maps.
3. The category of groups, denoted as **Grp**, in which objects are all groups and the morphisms are group homomorphisms; similarly, the category of abelian groups, denoted as **Ab**, in which objects are all abelian groups and the morphisms are group homomorphisms.
4. Let k be a field. The category of the vector spaces on k is denoted as **Vect** $_k$, in which objects are all

vector spaces on k and morphisms are linear maps.

5. Let R be a ring (which may not be commutative). The category of the left modules on R is denoted as ${}_R\mathbf{Mod}$, in which objects are all modules on R and morphisms are R -module homomorphisms. Similarly, we have the category of right modules \mathbf{Mod}_R .
6. The category of topological spaces with basepoints, denoted as \mathbf{Top}^* . Objects of \mathbf{Top}^* are like (X, x_0) , where X is a topological space and $x_0 \in X$. A morphism $f : (X, x_0) \rightarrow (Y, y_0)$ is a continuous map from X to Y with $f(x_0) = y_0$.

0.1.3 Definition. Let \mathcal{C} be a category, and $f : X \rightarrow Y$ a morphism. Then,

- call f a **monomorphism** if $f\alpha_1 = f\alpha_2 \Rightarrow \alpha_1 = \alpha_2$ for all objects Z and morphisms $\alpha_1, \alpha_2 : Z \rightarrow X$;
- call f an **epimorphism** if $\beta_1 f = \beta_2 f \Rightarrow \beta_1 = \beta_2$ for all objects Z and morphisms $\beta_1, \beta_2 : Y \rightarrow Z$;
- call f an **isomorphism** if there is $g : Y \rightarrow X$ such that $gf = 1_X$ and $fg = 1_Y$.



Note A mono and epi morphism may not be an isomorphism.

0.1.4 Definition. Let \mathcal{C} be a category. Define a category \mathcal{C}^{op} as follows:

- $\text{Obj}(\mathcal{C}^{\text{op}}) = \text{Obj}(\mathcal{C})$;
- $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$.
- $f \circ^{\text{op}} g = g \circ f, \forall g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y), f \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z)$.

The category is called **opposite category** of \mathcal{C} .

0.1.5 Definition. Let \mathcal{C}, \mathcal{D} be two categories. A **(covariant) functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is a map satisfying

1. $\forall X \in \text{Obj}(\mathcal{C}), F(X) \in \text{Obj}(\mathcal{D})$;
2. $\forall f \in \text{Hom}_{\mathcal{C}}(X, Y), F(f) \in \text{Hom}_{\mathcal{D}}(F(X), F(Y))$;
3. $F(g \circ f) = F(g) \circ F(f)$;
4. $F(1_X) = 1_{F(X)}$.

It can be considered that this functor preserves commutative diagrams.

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 & \searrow g \circ f & \downarrow g \\
 & & Z
 \end{array}
 \xrightarrow{\quad F \quad}
 \begin{array}{ccc}
 FX & \xrightarrow{F(f)} & FY \\
 & \searrow F(g \circ f) & \downarrow F(g) \\
 & & FZ
 \end{array}$$

A **contravariant functor** $G : \mathcal{C} \rightarrow \mathcal{D}$ is a map satisfying

1. $\forall X \in \text{Obj}(\mathcal{C}), G(X) \in \text{Obj}(\mathcal{D})$;
2. $\forall f \in \text{Hom}_{\mathcal{C}}(X, Y), G(f) \in \text{Hom}_{\mathcal{D}}(G(Y), G(X))$;
3. $G(g \circ f) = G(f) \circ G(g)$;
4. $G(1_X) = 1_{G(X)}$.

It can also be regarded as a covariant functor from \mathcal{C}^{op} to \mathcal{D} .

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 & \searrow g \circ f & \downarrow g \\
 & & Z
 \end{array}
 \xrightarrow{\quad G \quad}
 \begin{array}{ccc}
 GX & \xleftarrow{G(f)} & GY \\
 & \nwarrow G(g \circ f) & \uparrow G(g) \\
 & & GZ
 \end{array}$$

0.1.6 Example.

1. The forgetful functor $F : \mathbf{Top} \rightarrow \mathbf{Set}$ assigns to each topological space its underlying set and to each continuous map itself ("forgetting" its continuity). Similarly, there are forgetful functors $\mathbf{Grp} \rightarrow \mathbf{Set}$, $\mathbf{Ab} \rightarrow \mathbf{Grp}$, $\mathbf{Ab} \rightarrow \mathbf{Set}$, and so on.
2. The functor $\pi_1 : \mathbf{Top}^* \rightarrow \mathbf{Grp}$ assigns to each topological space its fundamental group at the basepoint and to each continuous map to its induced homomorphism between fundamental groups.

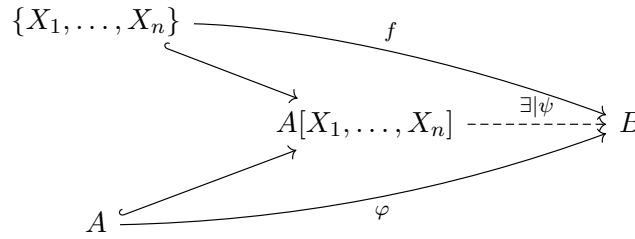
0.1.7 Definition. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then,

- say F is **faithful** if $F : \text{Hom}(X, Y) \rightarrow \text{Hom}(FX, FY)$ is injective for all objects X, Y in \mathcal{C} ;
- say F is **fully faithful** if $F : \text{Hom}(X, Y) \rightarrow \text{Hom}(FX, FY)$ is bijective for all objects X, Y in \mathcal{C} ;
- say F is **essentially surjective** if for each $Z \in \text{Obj}(\mathcal{D})$, there is $X \in \text{Obj}(\mathcal{C})$ such that $FX \cong Z$.

0.2 Rings and Ideals

Unless explicitly noted otherwise, all rings considered in this seminar are commutative rings with a unit. Note that the trivial ring 0 is within our consideration.

0.2.1 Proposition. Let A, B be rings. The polynomial ring $A[X_1, \dots, X_n]$ has a universal property: for all ring homomorphism $\varphi : A \rightarrow B$ and a map $f : \{X_1, \dots, X_n\} \rightarrow B$, there is a unique ring homomorphism $\psi : A[X_1, \dots, X_n] \rightarrow B$ such that the following diagram



commutes.

0.2.2 Definition. Let A be a ring, and $\{I_\lambda\}_{\lambda \in \Lambda}$ a family of ideals. Define the **sum** of $\{I_\lambda\}$ by

$$\sum_{\lambda \in \Lambda} I_\lambda := \left\{ \sum_{\lambda \in \Lambda} a_\lambda x_\lambda \mid a_\lambda \in A, x_\lambda \in I_\lambda, \text{ and there is finite } a_\lambda \neq 0 \right\}.$$

And if Λ is finite, define their **product** by

$$\prod_{\lambda \in \Lambda} I_\lambda := \left\{ \prod_{\lambda \in \Lambda} x_\lambda \mid x_\lambda \in I_\lambda \right\}.$$

It's trivial to check their sum and product are both ideals. Let I, J be two ideals. Then we have

$$IJ \subset I \cap J \subset I + J.$$

0.2.3 Proposition. Let A be a ring, and I be an ideal of A . Then we have

- A/I is an integral domain if and only if I is a prime ideal;
- A/I is a field if and only if I is a maximal ideal.

0.2.4 Proposition. Let $f : A \rightarrow B$ be a ring homomorphism, \mathfrak{p} a ideal in B , and $\mathfrak{q} = f^{-1}(\mathfrak{p})$. Then, if \mathfrak{p} is prime, \mathfrak{q} is prime; the converse holds if f is surjective.

0.2.5 Proposition-Definition. A ring A is said to be **Noetherian** if it satisfies the following three equivalent properties:

1. Any ideal in A is finitely generated.
2. Any increasing sequence of ideals in A is eventually stable.
3. Any non-empty set of ideals in A has a maximal element for the inclusion relation.

0.2.6 Theorem. (Hilbert Basis) If A is a Noetherian ring, then the polynomial ring $A[X]$ is a Noetherian ring.

Proof.

Let I be an ideal in $A[X]$. We need to prove I is finitely generated.

If $F = a_0 + a_1X + \dots + a_nX^n \in A[X]$ with $a_n \neq 0$, we call a_n the leading coefficient of F .

Let J be the set of leading coefficients of all polynomials in I . It's easy to check that J is an ideal in A .

Since A is Noetherian, there are polynomials $F_1, \dots, F_r \in I$ whose leading coefficients generate J .

Take an integer N larger than the degree of each F_i . For each $m \leq N$, let J_m be the ideal in A consisting of all leading coefficients of all polynomials $F \in I$ such that $\deg F \leq m$. Let $\{F_{mj}\}$ be a finite set of polynomials in I of degree $\leq m$ whose leading coefficients generate J_m .

Let I' be the ideal generated by F_i 's and all the F_{mj} 's. It suffices to show that $I = I'$.

Suppose I' were smaller than I ; let G be an element of I of lowest degree that is not in I' .

If $\deg G > N$ we can find polynomials Q_i such that $\sum Q_i F_i$ and G have the same leading term. But then $\deg(G - \sum Q_i F_i) < \deg G$, so $G - \sum Q_i F_i \in I'$; and hence $G \in I'$, contradicts.

Similarly, if $\deg G = m \leq N$, we can lower the degree by subtracting off $\sum Q_j F_{mj}$ for some Q_j , which will also make a contradiction.

Therefore, $I' = I$. □

0.2.7 Definition. Let A be a ring. An **A -algebra** B is a ring equipped with a homomorphism $f : A \rightarrow B$ (which is often but not always injective). It is said to be of **finite-type** if it is generated as an algebra by a finite number of elements x_1, \dots, x_n of B , i.e., if every element of B is a polynomial function of the elements x_i with coefficients in A .

0.2.8 Definition. Let $f : A \rightarrow B$ be an A -algebra and consider $x \in B$. We say that x is **integral** over A if it satisfies a unitary equation

$$x^n + f(a_{n-1})x^{n-1} + \dots + f(a_0) = 0,$$

where $a_i \in A$. (If f is the inclusion of A in B , we omit f .)

If b is integral over A for all $b \in B$, we say B is **integral** over A .

0.2.9 Definition. Let A be a ring. Its **Jacobson radical** $\text{rad}(A)$ is defined to be the intersection of all its maximal ideals.

0.2.10 Proposition. Let A be a ring, I an ideal, $x \in A$, and $u \in A^\times$. Then $x \in \text{rad}(A)$ if and only if $u - xy \in A^\times$ for all $y \in A$. In particular, the sum of an element of $\text{rad}(A)$ and a unit is a unit, and $I \subset \text{rad}(A)$ if $1 - I \subset A^\times$.

Proof.

(\Rightarrow). Suppose there were y such that $u - xy$ is not a unit. Then we have $(u - xy)$ is a proper ideal. Hence, there is maximal ideal $\mathfrak{m} \supset (u - xy)$. Since $x \in \text{rad}(A) \subset \mathfrak{m}$, we have $u \in \mathfrak{m}$, contradicts.

(\Leftarrow). Suppose there were a maximal ideal \mathfrak{m} such that $x \notin \mathfrak{m}$. Then we have $(x) + \mathfrak{m} = A$, that is $\exists y \in A, m \in \mathfrak{m}$ such that $xy + m = u$. Consequently, $m = u - xy$ is a unit, contradicts. □

0.2.11 Definition. Let A be a ring, and I an ideal. Define the **radical** of I by

$$\sqrt{I} := \{f \in A \mid f^n \in I \text{ for some } n\}.$$

And we call $\sqrt{(0)}$ the **nilradical**, and denote it by $\text{nil}(A)$. If $\text{nil}(A) = 0$, we call A a **reduced** ring.

0.3 Modules

0.3.1 Definition. Let A be a ring. An A -module M is an Abelian group, written additively, with a **scalar multiplication**, $A \times M \rightarrow M, (a, m) \mapsto am$, which satisfies ($a, b \in A$ and $m, n \in M$)

1. $a(m + n) = am + an$;
2. $(a + b)m = am + bm$;

3. $a(bm) = (ab)m$;
4. $1 \cdot m = m$.

A **submodule** N of M is a subgroup that is closed under scalar multiplication.

0.3.2 Definition. Let A be a ring and M, N A -modules. A (**A -module**) **homomorphism** (or **A -linear map**) $f : M \rightarrow N$ is a homomorphism between abelian group which satisfies $f(am) = af(m)$, $\forall a \in A, m \in M$.

Similar to abelian groups, we have the fundamental homomorphism theorem,

$$M / \ker f \cong \operatorname{im} f.$$

0.3.3 Definition. A (finite or infinite) sequence of module homomorphisms

$$\cdots \rightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \rightarrow \cdots$$

is said to be **exact at** M_i if $\ker f_i = \operatorname{im} f_{i-1}$. The sequence is said to be **exact** if it is exact at every M_i , except an initial source or final target.

0.3.4 Proposition.

- The sequence $0 \rightarrow L \xrightarrow{f} M$ is exact if and only if f is injective.
- The sequence $L \xrightarrow{f} M \rightarrow 0$ is exact if and only if f is surjective.

0.3.5 Proposition-Definition. Let $f : M' \rightarrow M, g : M \rightarrow M''$ be module homomorphisms.

We call f a **retraction** if there is a homomorphism $\alpha : M \rightarrow M'$ such that $\alpha f = 1_{M'}$.

We call g a **section** if there is a homomorphism $\beta : M'' \rightarrow M$ such that $g\beta = 1_{M''}$.

If there is an exact sequence

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0,$$

the following three propositions are equivalent:

1. f is a retraction;
2. g is a section;
3. There is an isomorphism $\varphi : M \xrightarrow{\sim} M' \oplus M''$ such that φf is the inclusion and $g\varphi^{-1}$ is the projection.

Under these conditions, we say the sequence **splits**.

0.3.6 Definition. Let A be a ring, M, N, P modules. We call a map $f : M \times N \rightarrow P$ **A -bilinear** if it is linear in each variable; that is, given $m \in M$ and $n \in N$, the maps

$$m' \mapsto f(m', n), n' \mapsto f(m, n'),$$

are both A -linear.

0.3.7 Definition. Let A be a ring, M, N modules. The **tensor product** $M \otimes_A N$ (or simply, $M \otimes N$) of M and N is a module equipped with a A -bilinear map $- \otimes - : M \times N \rightarrow M \otimes N$, $(m, n) \mapsto m \otimes n$, and it satisfies, for all A -bilinear map $f : M \times N \rightarrow P$, there is a unique A -linear map $\varphi : M \otimes N \rightarrow P$ such that the diagram

$$\begin{array}{ccc} M \times N & & \\ \downarrow - \otimes - & \searrow f & \\ M \otimes N & \xrightarrow[\exists! \varphi]{} & P \end{array}$$

commutes.

0.4 Localization

0.4.1 Definition. Let S be a subset of ring A . S is said to be **multiplicative** if $1 \in S$ and $xy \in S, \forall x, y \in S$. And if $xy \in S \Rightarrow x \in S$ and $y \in S$, we call S is **saturated**. Let T be a subset of A . We call $\{x \in A \mid \exists y \in A, xy \in T\}$ the **saturation** of T . It's easy to check the saturation of T is a saturated multiplicative set.

0.4.2 Example. Let A be a ring.

- For some $f \in A$, $S_f := \{f^n \mid n \in \mathbb{Z}_{\geq 0}\}$ is a multiplicative subset.
- For some prime ideal $\mathfrak{p} \subset A$, the complement set $A \setminus \mathfrak{p}$ is a multiplicative subset.

0.4.3 Definition. Let S be a multiplicative subset of ring A . Define a relation on $A \times S$ by $(x, s) \sim (y, t)$ if there is $u \in S$ such that $u(xt - ys) = 0$. We can find that this relation is an equivalence relation. Denote by $S^{-1}A$ or A_S the set of equivalence classes, and by x/s the class of (x, s) .

Define $x/s \cdot y/t := (xy)/(st)$ and $x/s + y/t := (tx + sy)/(st)$. It's easy to check these sum and product are well-defined, and under them, $S^{-1}A$ forms a ring. It is called the **ring of fractions** with respect to S or the **localization** at S .

There is a natural homomorphism $i : A \rightarrow S^{-1}A, a \mapsto a/1$, and the image under i of an element in S is invertible.



Note i is not guaranteed to be injective or surjective.

We can find $S^{-1}A$ has a universal property: for all ring homomorphism $f : A \rightarrow B$ where elements in $f(S)$ are all invertible, there is a unique ring homomorphism $\varphi : S^{-1}A \rightarrow B$ such that the diagram

$$\begin{array}{ccc} A & & \\ \downarrow i & \searrow f & \\ S^{-1}A & \xrightarrow{\exists! \varphi} & B \end{array}$$

commutes.

0.4.4 Definition. Let A be a ring, $f \in A$, and \mathfrak{p} a prime ideal. We always denote $S_f^{-1}A$ by A_f and $(A \setminus \mathfrak{p})^{-1}A$ by $A_{\mathfrak{p}}$.

0.4.5 Proposition. Let S be a multiplicative subset of A , and S' be the saturation of S . Then we have $S^{-1}A$ and $S'^{-1}A$ are canonical isomorphism.

0.4.6 Proposition. Let A be a Noetherian ring, and S a multiplicative subset of A . Then we have $S^{-1}A$ is Noetherian.

Proof.

Let $i : A \rightarrow S^{-1}A$ be the natural homomorphism. It's sufficient to prove if ideals $I_1, I_2 (I_1 \subseteq I_2)$ in $S^{-1}A$ with $i^{-1}(I_1) = i^{-1}(I_2)$, then we have $I_1 = I_2$. Let $a/s \in I_2$. We have $a/1 = (s/1)(a/s) \in I_2$; and hence $a \in i^{-1}(I_2) = i^{-1}(I_1)$. Therefore, $a/1 \in I_1$ and then $a/s = (1/s)(a/1) \in I_1$. \square

0.4.7 Proposition. Let S be a multiplicative subset of A . Then $S^{-1}A = 0$ if and only if $0 \in S$.

0.4.8 Theorem. (Scheinnullstellensatz) Let A be a ring, I an ideal. Then we have

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I \text{ is prime ideal}} \mathfrak{p}.$$

Proof.

For all $f \in \sqrt{I}$, we have $f^n \in I$ for some n ; and hence, for any prime ideal $\mathfrak{p} \supset I$, $f^n \in \mathfrak{p} \Rightarrow f \in \mathfrak{p}$. Thus, $\sqrt{I} \subset \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$.

Next, it's sufficient to show for all $f \in A \setminus \sqrt{I}$, there is a prime ideal $\mathfrak{p} \supset I$ such that $f \notin \mathfrak{p}$.

Consider the natural maps

$$A \xrightarrow{i_1} A/\sqrt{I} \xrightarrow{i_2} (A/\sqrt{I})_{\bar{f}},$$

where \bar{f} is the class of f in A/\sqrt{I} .

Choose a maximal ideal \mathfrak{m} in $(A/\sqrt{I})_{\bar{f}}$ (For $f \notin \sqrt{I}$, we have \bar{f} is not nilpotent; that is $0 \notin S_{\bar{f}}$. Thus, we have $(A/\sqrt{I})_{\bar{f}} \neq 0$. The maximal ideal exists.) And let's show $i_1^{-1}i_2^{-1}(\mathfrak{m})$ is a prime ideal with $f \notin i_1^{-1}i_2^{-1}(\mathfrak{m})$.

Since \mathfrak{m} is prime, we have $i_1^{-1}i_2^{-1}(\mathfrak{m})$ is prime. For $\mathfrak{m} \neq (A/\sqrt{I})_{\bar{f}}$, we have $\bar{f}/1 \notin \mathfrak{m}$ and then $\bar{f} \notin i_2^{-1}(\mathfrak{m})$. Consequently, $f \notin i_1^{-1}i_2^{-1}(\mathfrak{m})$. \square

0.4.9 Definition. Let A be a ring, S a multiplicative subset, and M a module. Define a relation on $M \times S$ by $(m, s) \sim (n, t)$ if there is $u \in S$ such that $u(tm - sn) = 0$. It's easy to check this is an equivalence relation.

Denote by $S^{-1}M$ or M_S the set of equivalence classes, and by m/s the class of (m, s) . Then $S^{-1}M$ is an $S^{-1}A$ module with addition given by $m/s + n/t := (tm + sn)/st$ and scalar multiplication by $a/s \cdot m/t = (am)/(st)$. We call $S^{-1}M$ the **localization of M at S** .

It also has a universal property: Let $i : M \rightarrow S^{-1}M, m \mapsto m/1$, N a $S^{-1}A$ -module. For all A -linear map $f : M \rightarrow N$, there is a unique $S^{-1}A$ -linear map $\varphi : S^{-1}M \rightarrow N$ such that the diagram

$$\begin{array}{ccc} M & & \\ \downarrow i & \searrow f & \\ S^{-1}M & \xrightarrow[\exists! \varphi]{} & N \end{array}$$

commutes.

0.4.10 Definition. Let A be a ring, M a module, $f \in A$, and \mathfrak{p} a prime ideal. We always denote $S_f^{-1}M$ by M_f and $(A \setminus \mathfrak{p})^{-1}M$ by $M_{\mathfrak{p}}$.

0.4.11 Proposition. Let A be a ring, S a multiplicative subset, and M a module. Define a scalar multiplication

$$\begin{aligned} S^{-1}A \times (S^{-1}A \otimes_A M) &\rightarrow S^{-1}A \otimes_A M, \\ (a, b \otimes m) &\mapsto (ab) \otimes m, \end{aligned}$$

and then $S^{-1}A \otimes_A M$ can be also seen as an $S^{-1}A$ -module. As $S^{-1}A$ -modules, $S^{-1}M$ and $S^{-1}A \otimes_A M$ are canonical isomorphism.

0.5 Transcendence Bases and Krull Dimension

0.5.1 Definition. Let $K \subset L$ be a field extension. A subset B in L is said to be **algebraically free** over K (we also say that its elements are **algebraically independent**) if for any finite subset $\{x_1, \dots, x_n\} \subset B$ and any polynomial $P \in K[X_1, \dots, X_n]$, the equality $P(x_1, \dots, x_n) = 0$ implies $P = 0$. Otherwise, we say that the elements of B are **algebraically dependent**.

0.5.2 Definition. Let $K \subset L$ be a field extension. A subset B in L is said to be an **algebraic generating set** over K if L is algebraic over the subfield $K(B)$ generated by B .

0.5.3 Definition. Let $K \subset L$ be a field extension. A subset B in L is a **transcendence basis** for L over K if it both algebraically free and an algebraic generating set.

By Zorn's lemma, a transcendence basis always exists. These bases all have the same cardinality, called **transcendence degree** of L over K . We denote it by $\partial_K(L)$.

0.5.4 Definition. Let X be a set. A **chain** of subsets of X is a sequence $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$. Such a chain is said to be of **length** n .

0.5.5 Definition. The **Krull dimension** of A is the maximum of the lengths of chains of prime ideals of A . We denote it by $\dim_K A$.

0.5.6 Theorem. Let A be an integral domain which is a k -algebra of finite type. The Krull dimension of A is equal to the transcendence degree of $\text{Fr}(A)$ over k :

$$\dim_K A = \partial_k \text{Fr}(A).$$

($\text{Fr}(A)$ means the field of fractions from A .)