Chapter 0. Preparation

0.1 Categories

0.1.1 Definition. *A **category** $\mathscr{C}$ consists of:*

1. $\mathrm{Obj}(\mathscr{C})$*: the class of the objects (which may not be a set);*
2. $\mathrm{Mor}(\mathscr{C})$*: the class of morphisms. For each morphism* $f \in \mathrm{Mor}(\mathscr{C})$*, it has a **source*** $s(f)$ *and a **target*** $t(f)$*, where both* $s(f)$ *and* $t(f)$ *are elements of* $\mathrm{Obj}(\mathscr{C})$*. Let* X *be the source of* f *and* Y *be the target of* f *we can denote* f *as* $f : X \to Y$*. And we define* $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ *(also* $\mathrm{Hom}(X, Y)$ *for short) as the class of morphisms with source* X *and target* Y*; that is* $\mathrm{Hom}_{\mathscr{C}}(X, Y) := s^{-1}(X) \cap t^{-1}(Y)$.

Additionally, objects and morphisms should satisfy these properties:

1. $\forall X, Y, Z \in \mathrm{Obj}(\mathscr{C})$*, there is a **composition*** $\circ : \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z)$, $(g, f) \mapsto g \circ f$*. We also abbreviate* $g \circ f$ *as* gf.

   And we can use a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

   to describe.

2. *The composition we defined above satisfies the **associative law**; that is* $\forall X, Y, Z, T \in \mathrm{Obj}(\mathscr{C})$ *and* $f : X \to Y, g : Y \to Z, h : Z \to T$*, we have*

$$(h \circ g) \circ f = h \circ (g \circ f).$$

   And we can also use a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g\circ f"'] & Y \arrow[d, "h\circ g"] \\ Z \arrow[r, "h"'] & T \end{tikzcd}$$

   to describe.

3. $\forall X \in \mathrm{Obj}(\mathscr{C})$*, there is an **identity*** $1_X \in \mathrm{Hom}(X, X)$*, such that for all* $Y, Z \in \mathrm{Obj}(\mathscr{C})$ *and* $f : X \to Y, g : Z \to X$*, we have*

$$f \circ 1_X = f, 1_X \circ g = g.$$

0.1.2 Example.

1. Let's begin with a simple example, the category of sets, denoted as $\mathbf{Set}$. $\mathrm{Obj}(\mathbf{Set})$ is the class of all sets (as we all know, it can't be a set because of the Russell's paradox). $\mathrm{Hom}_{\mathbf{Set}}(A, B)$ are all maps from A to B. It's easy to check $\mathbf{Set}$ satisfies the concept of category.
2. The category of topological spaces, denoted as $\mathbf{Top}$. The objects of $\mathbf{Top}$ are all topological spaces, and the morphisms are continuous maps.
3. The category of groups, denoted as $\mathbf{Grp}$, in which objects are all groups and the morphisms are group homomorphisms; similarly, the category of abelian groups, denoted as $\mathbf{Ab}$, in which objects are all abelian groups and the morphisms are group homomorphisms.
4. Let k be a field. The category of the vector spaces on k is denoted as $\mathbf{Vect}_k$, in which objects are all

vector spaces on k and morphisms are linear maps.

5. Let R be a ring (which may not be commutative). The category of the left modules on R is denoted as ${}_R\mathbf{Mod}$, in which objects are all modules on R and morphisms are R-module homomorphisms. Similarly, we have the category of right modules $\mathbf{Mod}_R$.
6. The category of topological spaces with basepoints, denoted as $\mathbf{Top}^*$. Objects of $\mathbf{Top}^*$ are like (X, x_0), where X is a topological space and $x_0 \in X$. A morphism $f : (X, x_0) \to (Y, y_0)$ is a continuous map from X to Y with $f(x_0) = y_0$.

0.1.3 Definition. *Let $\mathscr{C}$ be a category, and $f : X \to Y$ a morphism. Then,*

- *call f a **monomorphism** if $f\alpha_1 = f\alpha_2 \Rightarrow \alpha_1 = \alpha_2$ for all objects Z and morphisms $\alpha_1, \alpha_2 : Z \to X$;*
- *call f an **epimorphism** if $\beta_1 f = \beta_2 f \Rightarrow \beta_1 = \beta_2$ for all objects Z and morphisms $\beta_1, \beta_2 : Y \to Z$;*
- *call f an **isomorphism** if there is $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$.*

Note *A mono and epi morphism may not be an isomorphism.*

0.1.4 Definition. *Let $\mathscr{C}$ be a category. Define a category $\mathscr{C}^{\text{op}}$ as follows:*

- $\text{Obj}(\mathscr{C}^{\text{op}}) = \text{Obj}(\mathscr{C})$;
- $\text{Hom}_{\mathscr{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathscr{C}}(Y, X)$.
- $f \circ^{\text{op}} g = g \circ f, \forall g \in \text{Hom}_{\mathscr{C}^{\text{op}}}(X, Y), f \in \text{Hom}_{\mathscr{C}^{\text{op}}}(Y, Z)$.

*The category is called **opposite category** of $\mathscr{C}$.*

0.1.5 Definition. *Let $\mathscr{C}, \mathscr{D}$ be two categories. A **(covariant) functor** $F : \mathscr{C} \to \mathscr{D}$ is a map satisfying*

1. $\forall X \in \text{Obj}(\mathscr{C}), F(X) \in \text{Obj}(\mathscr{D})$;
2. $\forall f \in \text{Hom}_{\mathscr{C}}(X, Y), F(f) \in \text{Hom}_{\mathscr{D}}(F(X), F(Y))$;
3. $F(g \circ f) = F(g) \circ F(f)$;
4. $F(1_X) = 1_{F(X)}$.

It can be considered that this functor preserves commutative diagrams.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \quad \overset{F}{\rightsquigarrow} \quad \begin{tikzcd} FX \arrow[r, "F(f)"] \arrow[dr, "F(g\circ f)"'] & FY \arrow[d, "F(g)"] \\ & FZ \end{tikzcd}$$

*A **contravariant functor** $G : \mathscr{C} \to \mathscr{D}$ is a map satisfying*

1. $\forall X \in \text{Obj}(\mathscr{C}), G(X) \in \text{Obj}(\mathscr{D})$;
2. $\forall f \in \text{Hom}_{\mathscr{C}}(X, Y), G(f) \in \text{Hom}_{\mathscr{D}}(G(Y), G(X))$;
3. $G(g \circ f) = G(f) \circ G(g)$;
4. $G(1_X) = 1_{G(X)}$.

It can also be regarded as a covariant functor from $\mathscr{C}^{\text{op}}$ to $\mathscr{D}$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd} \quad \overset{G}{\rightsquigarrow} \quad \begin{tikzcd} GX & GY \arrow[l, "G(f)"'] \\ & GZ \arrow[u, "G(g)"'] \arrow[ul, "G(g\circ f)"] \end{tikzcd}$$

0.1.6 Example.

1. The forgetful functor $F : \mathbf{Top} \to \mathbf{Set}$ assigns to each topological space its underlying set and to each continuous map itself("forgetting" its continuity). Similarly, there are forgetful functors $\mathbf{Grp} \to \mathbf{Set}$, $\mathbf{Ab} \to \mathbf{Grp}$, $\mathbf{Ab} \to \mathbf{Set}$, and so on.
2. The functor $\pi_1 : \mathbf{Top}^* \to \mathbf{Grp}$ assigns to each topological space its fundamental group at the basepoint and to each continuous map to its induced homomorphism between fundamental groups.

0.1.7 Definition. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then,*

- *say F is **faithful** if $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(FX, FY)$ is injective for all objects X, Y in $\mathscr{C}$;*
- *say F is **fully faithful** if $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(FX, FY)$ is bijective for all objects X, Y in $\mathscr{C}$;*
- *say F is **essentially surjective** if for each $Z \in \mathrm{Obj}(\mathscr{D})$, there is $X \in \mathrm{Obj}(\mathscr{C})$ such that $FX \cong Z$.*

0.2 Rings and Ideals

Unless explicitly noted otherwise, all rings considered in this seminar are commutative rings with a unit. Note that the trivial ring 0 is within our consideration.

0.2.1 Proposition. *Let A, B be rings. The polynomial ring $A[X_1, \dots, X_n]$ has a universal property: for all ring homomorphism $\varphi : A \to B$ and a map $f : \{X_1, \dots, X_n\} \to B$, there is a unique ring homomorphism $\psi : A[X_1, \dots, X_n]$ such that the following diagram*

$$\begin{tikzcd}
\{X_1, \dots, X_n\} \arrow[dr, hook] \arrow[drr, bend left, "f"] & & \\
& A[X_1, \dots, X_n] \arrow[r, dashed, "\exists!\psi"] & B \\
A \arrow[ur, hook] \arrow[urr, bend right, "\varphi"'] & &
\end{tikzcd}$$

commutes.

0.2.2 Definition. *Let A be a ring, and $\{I_\lambda\}_{\lambda \in \Lambda}$ a family of ideals. Define the **sum** of $\{I_\lambda\}$ by*

$$\sum_{\lambda \in \Lambda} I_\lambda := \{\sum_{\lambda \in \Lambda} a_\lambda x_\lambda \mid a_\lambda \in A, x_\lambda \in I_\lambda, \text{and there is finite } a_\lambda \neq 0\}.$$

*And if Λ is finite, define their **product** by*

$$\prod_{\lambda \in \Lambda} I_\lambda := \{\prod_{\lambda \in \Lambda} x_\lambda \mid x_\lambda \in I_\lambda\}.$$

It's trivial to check their sum and product are both ideals. Let I, J be two ideals. Then we have

$$IJ \subset I \cap J \subset I + J.$$

0.2.3 Proposition. *Let A be a ring, and I be an ideal of A. Then we have*

- *A/I is an integral domain if and only if I is a prime ideal;*
- *A/I is a field if and only if I is a maximal ideal.*

0.2.4 Proposition. *Let $f : A \to B$ be a ring homomorphism, $\mathfrak{p}$ a ideal in B, and $\mathfrak{q} = f^{-1}(\mathfrak{p})$. Then, if $\mathfrak{p}$ is prime, $\mathfrak{q}$ is prime; the converse holds if f is surjective.*

0.2.5 Proposition-Definition. *A ring A is said to be **Noetherian** if it satisfies the following three equivalent properties:*

1. *Any ideal in A is finitely generated.*
2. *Any increasing sequence of ideals in A is eventually stable.*
3. *Any non-empty set of ideals in A has a maximal element for the inclusion relation.*

0.2.6 Theorem. (Hilbert Basis) *If A is a Noetherian ring, then the polynomial ring $A[X]$ is a Noetherian ring.*

Proof.

Let I be an ideal in $A[X]$. We need to prove I is finitely generated.

If $F = a_0 + a_1 X + \cdots + a_n X^n \in A[X]$ with $a_n \neq 0$, we call a_n the leading coefficient of F.

Let J be the set of leading coefficients of all polynomials in I. It's easy to check that J is an ideal in A.

Since A is Noetherian, there are polynomials $F_1, \ldots, F_r \in I$ whose leading coefficients generate J.

Take an integer N larger than the degree of each F_i. For each $m \leq N$, let J_m be the ideal in A consisting of all leading coefficients of all polynomials $F \in I$ such that $\deg F \leq m$. Let $\{F_{mj}\}$ be a finite set of polynomials in I of degree $\leq m$ whose leading coefficients generate J_m.

Let I' be the ideal generated by F_i's and all the F_{mj}'s. It suffices to show that $I = I'$.

Suppose I' were smaller than I; let G be an element of I of lowest degree that is not in I'.

If $\deg G > N$ we can find polynomials Q_i such that $\sum Q_i F_i$ and G have the same leading term. But then $\deg(G - \sum Q_i F_i) < \deg G$, so $G - \sum Q_i F_i \in I'$; and hence $G \in I'$, contradicts.

Similarly, if $\deg G = m \leq N$, we can lower the degree by subtracting off $\sum Q_j F_{mj}$ for some Q_j, which will also make a contradiction.

Therefore, $I' = I$. □

0.2.7 Definition. *Let A be a ring. An A-**algebra** is a ring equipped with a homomorphism $f : A \to B$ (which is often but not always injective). It is said to be of **finite-type** if it is generated as an algebra by a finite number of elements $x_1, \ldots, x_n$ of B, i.e., if every element of B is a polynomial function of the elements x_i with coefficients in A.*

0.2.8 Definition. *Let $f : A \to B$ be an A-algebra and consider $x \in B$. We say that x is **integral** over A if it satisfies a unitary equation*

$$x^n + f(a_{n-1})x^{n-1} + \cdots + f(a_0) = 0,$$

where $a_i \in A$. (If f is the inclusion of A in B, we omit f.)

*If b is integral over A for all $b \in B$, we say B is **integral** over A.*

0.2.9 Definition. *Let A be a ring. Its **Jacobson radical** $\mathrm{rad}(A)$ is defined to be the intersection of all its maximal ideals.*

0.2.10 Proposition. *Let A be a ring, I an ideal, $x \in A$, and $u \in A^\times$. Then $x \in \mathrm{rad}(A)$ if and only if $u - xy \in A^\times$ for all $y \in A$. In particular, the sum of an element of $\mathrm{rad}(A)$ and a unit is a unit, and $I \subset \mathrm{rad}(A)$ if $1 - I \subset A^\times$.*

Proof.

($\Rightarrow$). Suppose there were y such that $u - xy$ is not a unit. Then we have $(u - xy)$ is a proper ideal. Hence, there is maximal ideal $\mathfrak{m} \supset (u - xy)$. Since $x \in \mathrm{rad}(A) \subset \mathfrak{m}$, we have $u \in \mathfrak{m}$, contradicts.

($\Leftarrow$). Suppose there were a maximal ideal $\mathfrak{m}$ such that $x \notin \mathfrak{m}$. Then we have $(x) + \mathfrak{m} = A$, that is $\exists y \in A, m \in \mathfrak{m}$ such that $xy + m = u$. Consequently, $m = u - xy$ is a unit, contradicts. □

0.2.11 Definition. *Let A be a ring, and I an ideal. Define the **radical** of I by*

$$\sqrt{I} := \{f \in A \mid f^n \in I \text{ for some } n\}.$$

*And we call $\sqrt{(0)}$ the **nilradical**, and denote it by $\mathrm{nil}(A)$.*

0.3 Modules

0.3.1 Definition. *Let A be a ring. An A-module M is an Abelian group, written additively, with a **scalar multiplication**, $A \times M \to M, (a, m) \mapsto am$, which satisfies ($a, b \in A$ and $m, n \in M$)*

1. *$a(m + n) = am + an$;*
2. *$(a + b)m = am + bm$;*

3. $a(bm) = (ab)m$;
4. $1 \cdot m = m$.

*A **submodule** N of M is a subgroup that is closed under scalar multiplication.*

0.3.2 Definition. *Let A be a ring and M, N A-modules. A (**A-module) homomorphism (or A-linear map)** $f : M \to N$ is a homomorphism between abelian group which satisfies $f(am) = af(m), \forall a \in A, m \in M$.*

Similar to abelian groups, we have the fundamental homomorphism theorem,

$$M/\ker f \cong \operatorname{im} f.$$

0.3.3 Definition. *A (finite or infinite) sequence of module homomorphisms*

$$\cdots \to M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \to \cdots$$

*is said to be **exact at** M_i if $\ker f_i = \operatorname{im} f_{i-1}$. The sequence is said to be **exact** if it is exact at every M_i, except an initial source or final target.*

0.3.4 Proposition.

- *The sequence $0 \to L \xrightarrow{f} M$ is exact if and only if f is injective.*
- *The sequence $L \xrightarrow{f} M \to 0$ is exact if and only if f is surjective.*

0.3.5 Proposition-Definition. *Let $f : M' \to M, g : M \to M''$ be module homomorphisms.*

*We call f a **retraction** if there is a homomorphism $\alpha : M \to M'$ such that $\alpha f = 1_{M'}$.*

*We call g a **section** if there is a homomorphism $\beta : M'' \to M$ such that $g\beta = 1_{M''}$.*

If there is an exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0,$$

the following three propositions are equivalent:

1. *f is a retraction;*
2. *g is a section;*
3. *There is an isomorphism $\varphi : M \xrightarrow{\sim} M' \oplus M''$ such that φf is the inclusion and $g\varphi^{-1}$ is the projection.*

*Under these conditions, we say the sequence **splits**.*

0.3.6 Definition. *Let A be a ring, M, N, P modules. We call a map $f : M \times N \to P$ **A-bilinear** if it is linear in each variable; that is, given $m \in M$ and $n \in N$, the maps*

$$m' \mapsto f(m', n), n' \mapsto f(m, n'),$$

are both A-linear.

0.3.7 Definition. *Let A be a ring, M, N modules. The **tensor product** $M \otimes_A N$ (or simply, $M \otimes N$) of M and N is a module equipped with a A-bilinear map $- \otimes - : M \times N \to M \otimes N, (m, n) \mapsto m \otimes n$, and it satisfies, for all A-bilinear map $f : M \times N \to P$, there is a unique A-linear map $\varphi : M \otimes N \to P$ such that the diagram*

$$\begin{tikzcd}
M \times N \arrow[d, "- \otimes -"'] \arrow[dr, "f"] & \\
M \otimes N \arrow[r, dashed, "\exists | \varphi"'] & P
\end{tikzcd}$$

commutes.

0.4 Localization

0.4.1 Definition. *Let S be a subset of ring A. S is said to be **multiplicative** if $1 \in S$ and $xy \in S, \forall x, y \in S$. And if $xy \in S \Rightarrow x \in S$ and $y \in S$, we call S is **saturated**. Let T be a subset of A. We call $\{x \in A \mid \exists y \in A, xy \in T\}$ the **saturation** of T. It's easy to check the saturation of T is a saturated multiplicative set.*

0.4.2 Example. Let A be a ring.

- For some $f \in A$, $S_f := \{f^n \mid n \in \mathbb{Z}_{\geq 0}\}$ is a multiplicative subset.
- For some prime ideal $\mathfrak{p} \subset A$, the complement set $A \backslash \mathfrak{p}$ is a multiplicative subset.

0.4.3 Definition. *Let S be a multiplicative subset of ring A. Define a relation on $R \times S$ by $(x, s) \sim (y, t)$ if there is $u \in S$ such that $u(xt - ys) = 0$. We can find that this relation is an equivalence relation. Denote by $S^{-1}A$ or A_S the set of equivalence classes, and by x/s the class of (x, s).*

*Define $x/s \cdot y/t := (xy)/(st)$ and $x/s + y/t = (tx + st)/(st)$. It's easy to check these sum and product are well-defined, and under them, $S^{-1}A$ forms a ring. It is called the **ring of fractions** with respect to S or the **localization** at S.*

There is a natural homomorphism $i : A \to S^{-1}A, a \mapsto a/1$, and the image under i of an element in S is invertible.

Note *i is not guaranteed to be injective or surjective.*

We can find $S^{-1}A$ has a universal property: for all ring homomorphism $f : A \to B$ where elements in $f(S)$ are all invertible, there is a unique ring homomorphism $\varphi : S^{-1}A \to B$ such that the diagram

$$\begin{tikzcd} A \arrow[d, "i"'] \arrow[dr, "f"] & \\ S^{-1}A \arrow[r, dashed, "\exists|\varphi"'] & B \end{tikzcd}$$

commutes.

0.4.4 Definition. *Let A be a ring, $f \in A$, and $\mathfrak{p}$ a prime ideal. We always denote $S_f^{-1}A$ by A_f and $(A\backslash\mathfrak{p})^{-1}A$ by $A_{\mathfrak{p}}$.*

0.4.5 Proposition. *Let S be a multiplicative subset of A, and S' be the saturation of S. Then we have $S^{-1}A$ and $S'^{-1}A$ are canonical isomorphism.*

0.4.6 Proposition. *Let A be a Noetherian ring, and S a multiplicative subset of A. Then we have $S^{-1}A$ is Noetherian.*

Proof.

Let $i : A \to S^{-1}A$ be the natural homomorphism. It's sufficient to prove if ideals $I_1, I_2 (I_1 \subseteq I_2)$ in $S^{-1}A$ with $i^{-1}(I_1) = i^{-1}(I_2)$, then we have $I_1 = I_2$. Let $a/s \in I_2$. We have $a/1 = (s/1)(a/s) \in I_2$; and hence $a \in i^{-1}(I_2) = i^{-1}(I_1)$. Therefore, $a/1 \in I_1$ and then $a/s = (1/s)(a/1) \in I_1$. □

0.4.7 Proposition. *Let S be a multiplicative subset of A. Then $S^{-1}A = 0$ if and only if $0 \in S$.*

0.4.8 Theorem. (Scheinnullstellensatz) *Let A be a ring, I an ideal. Then we have*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supset I \text{ is prime ideal}} \mathfrak{p}.$$

Proof.

For all $f \in \sqrt{I}$, we have $f^n \in I$ for some n; and hence, for any prime ideal $\mathfrak{p} \supset I$, $f^n \in \mathfrak{p} \Rightarrow f \in \mathfrak{p}$. Thus, $\sqrt{I} \subset \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$.

Next, it's sufficient to show for all $f \in A\backslash I$, there is a prime ideal $\mathfrak{p} \supset I$ such that $f \notin \mathfrak{p}$.

Consider the natural maps

$$A \xrightarrow{i_1} A/\sqrt{I} \xrightarrow{i_2} (A/\sqrt{I})_{\bar{f}},$$

where $\bar{f}$ is the class of f in $A/\sqrt{I}$.

Choose a maximal ideal $\mathfrak{m}$ in $(A/\sqrt{I})_{\bar{f}}$ (For $f \notin \sqrt{I}$, we have $\bar{f}$ is not nilpotent; that is $0 \notin S_{\bar{f}}$. Thus, we have $(A/\sqrt{I})_{\bar{f}} \neq 0$. The maximal ideal exists.) And let's show $i_1^{-1}i_2^{-1}(\mathfrak{m})$ is a prime ideal with $f \notin i_1^{-1}i_2^{-1}(\mathfrak{m})$.

Since $\mathfrak{m}$ is prime, we have $i_1^{-1}i_2^{-1}(\mathfrak{m})$ is prime. For $\mathfrak{m} \neq (A/\sqrt{I})_{\bar{f}}$, we have $\bar{f}/1 \notin \mathfrak{m}$ and then $\bar{f} \notin i_2^{-1}(\mathfrak{m})$. Consequently, $f \notin i_1^{-1}i_2^{-1}(\mathfrak{m})$. $\square$

0.4.9 Definition. *Let A be a ring, S a multiplicative subset, and M a module. Define a relation on $M \times S$ by $(m,s) \sim (n,t)$ if there is $u \in S$ such that $u(tm - sn) = 0$. It's easy to check this is a equivalence relation.*

Denote by $S^{-1}M$ or M_S the set of equivalence classes, and by m/s the class of (m,s). Then $S^{-1}M$ is an $S^{-1}A$ module with addition given by $m/s + n/t := (tm + sn)/st$ and scalar multiplication by $a/s \cdot m/t = (am)/(st)$. We call $S^{-1}M$ the ***localization of** M **at** S.*

It also has a universal property: Let $i : M \to S^{-1}M, m \mapsto m/1$, N a $S^{-1}A$- module. For all A-linear map $f : M \to N$, there is a unique $S^{-1}A$-linear map $\varphi : S^{-1}M \to N$ such that the diagram

$$\begin{tikzcd} M \arrow[d, "i"'] \arrow[dr, "f"] & \\ S^{-1}M \arrow[r, dashed, "\exists|\varphi"'] & N \end{tikzcd}$$

commutes.

0.4.10 Definition. *Let A be a ring, M a module, $f \in A$, and $\mathfrak{p}$ a prime ideal. We always denote $S_f^{-1}M$ by M_f and $(A\backslash\mathfrak{p})^{-1}M$ by $M_{\mathfrak{p}}$.*

0.4.11 Proposition. *Let A be a ring, S a multiplicative subset, and M a module. Define a scalar multiplication*

$$S^{-1}A \times (S^{-1}A \otimes_A M) \to S^{-1}A \otimes_A M,$$

$$(a, b \otimes m) \mapsto (ab) \otimes m,$$

and then $S^{-1}A \otimes_A M$ can be also seen as an $S^{-1}A$-module. As $S^{-1}A$-modules, $S^{-1}M$ and $S^{-1}A \otimes_A M$ are canonical isomorphism.

0.5 Transcendence Bases and Krull Dimension

0.5.1 Definition. *Let $K \subset L$ be a field extension. A subset B in L is said to be* ***algebraically free*** *over K (we also say that its elements are* ***algebraically independent****) if for any finite subset $\{x_1, \ldots, x_n\} \subset B$ and any polynomial $P \in K[X_1, \ldots, X_n]$, the equality $P(x_1, \ldots, x_n)$ implies $P = 0$. Otherwise, we say that the elements of B are* ***algebraically dependent****.*

0.5.2 Definition. *Let $K \subset L$ be a field extension. A subset B in L is said to be an* ***algebraic generating set*** *over K if L is algebraic over the subfield $K(B)$ generated by B.*

0.5.3 Definition. *Let $K \subset L$ be a field extension. A subset B in L is a* ***transcendence basis*** *for L over K if it both algebraically free and an algebraic generating set.*

By Zorn's lemma, a transcendence basis always exists. These bases all have the same cardinality, called ***transcendence degree*** *of L over K. We denote it by $\partial_K(L)$.*

0.5.4 Definition. *Let X be a set. A **chain** of subsets of X is a sequence $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n$. Such a chain is said to be of **length** n.*

0.5.5 Definition. *The **Krull dimension** of A is the maximum of the lengths of chains of prime ideals of A. We denote it by $\dim_K A$.*

0.5.6 Theorem. *Let A be an integral domain which is a k-algebra of finite type. The Krull dimension of A is equal to the transcendence degree of $\operatorname{Fr}(A)$ over k:*

$$\dim_K A = \partial_k \operatorname{Fr}(A).$$

($\operatorname{Fr}(A)$ means the ring of fractions from A.)