I3344
Numerical Simulation & Modeling

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Tentative Syllabus

- ✓ Interpolation and extrapolation
  - ✓ Linear: linear regression y = ax + b; correlation, standard deviation, etc.
  - ✓ Non-linear: k nearest neighbors (KNN)
  - ✓ Validation of two models: k-fold cross validation method.
- **Solving a linear equation**
  - **Direct methods: Gauss and LU**
  - **Iterative methods: Jacobi and Gauss-Seidel**
- Approximation and modeling for the resolution of differential equations
- Derivation
  - Finite difference method (FDM): Euler and Runge-kutta
  - Application: speed and acceleration (transport system).
- Integration: surface estimation
  - Monte Carlo method.
  - Finite Element Method (FEM).
  - Comparison of two methods.
- Non-linear problems
  - Bisection method
- Introduction to the notion of parallel computing and underlying algorithms

Numerical Solutions of Linear Systems

- Numerical solution of linear systems Ax=b
  - Direct methods (use pre-determined number of operations)
    - Gauss elimination method
    - LU factorization methods
      - Doolittle's method
      - Cholesky's method
  - Indirect methods (use iterations, ideal for large systems)
    - Jacobi method
    - Gauss–Seidel method

Solution of Linear Systems of Equations

Iterative Methods

Introduction

- A successful iteration process (1) **starts with an initial vector** and (2) **generates a sequence of successive vectors** that eventually (3) **converges to the solution vector x**
- **Direct methods**: the total number of operations is known in advance
- **Iterative methods**: the number of operations required depends on:
  - How many **iteration steps** must be performed for **satisfactory convergence**
  - The **nature of the system** at hand.
- **Convergence**: the iteration must be terminated as soon as two successive vectors are close to one another
  - A **measure of the proximity** of two vectors is provided by a **vector norm**.

Vector Norms

- **Norm of a vector** v_{nx1} , denoted by $||v||$, provides a measure of how small or large v is, and has the following properties:
  - $||v|| \geq 0 \; \forall v$, and $||v|| = 0 \Leftrightarrow v = 0_{nx1}$
  - $||\alpha v|| = |\alpha| \; ||v||$, α scalar
  - $||v + w|| \leq ||v|| + ||w||$ for all vectors v and w

$$\mathbf{v} = \begin{Bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{Bmatrix}$$

Vector Norms (cont'd)

- Three commonly used vector norms
  - ***l_1-norm or $||v||_1$,***
    - $||v||_1 = |v_1| + |v_2| + \ldots + |v_n|$
  - ***l_∞-norm or $||v||_\infty$***
    - $||v||_\infty = max\{|v_1| , |v_2| , \ldots, |v_n|\}$
  - ***l2-norm or $||v||_2$***
    - $||v||_2 = (v_1^2 + v_2^2 + \ldots. + v_n^2)^{1/2}$

$$\mathbf{v} = \begin{Bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{Bmatrix}$$

Exercise

$$\mathbf{v} = \begin{Bmatrix} 8.3 \\ -2.9 \\ -12 \\ 6.7 \end{Bmatrix}$$

Matrix Norms

- **Norm of a matrix** A_{nxn} , denoted by **||A||** , is a nonnegative real number that provides a measure of how small or large A is
- It has the following properties:
  - $||A|| \geq 0 \; \forall \; A$, and $||A|| = 0 \Leftrightarrow A = 0_{nxn}$
  - $||\alpha A|| = |\alpha| \; ||A||$, α scalar
  - $||A + B|| \leq ||A|| + ||B||$ for all nxn matrices A and B
  - $||AB|| \leq ||A|| \; ||B||$ for all *nxn* matrices A and B

Matrix Norms (cont'd)

- Three common forms of matrix norms
  - ***1-norm*** **(column-sum norm)**

$$\|\mathbf{A}\|_1 = \max_{1\le j\le n}\left\{\sum_{i=1}^{n}|a_{ij}|\right\}$$

  - **Infinite-norm (row-sum norm)**

$$\|\mathbf{A}\|_\infty = \max_{1\le i\le n}\left\{\sum_{j=1}^{n}|a_{ij}|\right\}$$

  - **Euclidean norm (2-norm, Frobenius norm)**

$$\|\mathbf{A}\|_E = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^{\;2}\right]^{1/2}$$

Exercise

$$\mathbf{A} = \begin{bmatrix} 3 & 1.26 & -2 & 5 \\ -1 & 0 & 5.4 & 4.8 \\ 0.93 & -4 & 1 & 3.6 \\ -2 & -4.5 & 6 & 10 \end{bmatrix}$$

General Iterative Method

- General idea behind iterative methods to solve $Ax = b$
  - **Split the coefficient matrix** as $A = Q - P$
    - with the provision that **<u>Q is non-singular</u>** so that Q^{-1} exists
  - **Substitute into** $Ax = b$: $[Q - P]x = b \iff Qx = Px + b$
    - This system cannot be solved in its present form (the solution vector x appears on both sides)
  - **Solve the system by iterations**
    - Choose an initial vector $x^{(0)}$ and solve the following system for the new vector $x^{(1)}$: $Qx^{(1)} = Px^{(0)} + b$
    - use $x^{(1)}$ to find the new vector $x^{(2)}$: $Qx^{(2)} = Px^{(1)} + b$
    - Sequence of vectors is generated: $Qx^{(k+1)} = Px^{(k)} + b, \; k = 0,1,\ldots$

$$\mathbf{x^{(k+1)} = Q^{-1}Px^{(k)} + Q^{-1}b \,,\; k = 0,1,2,\ldots}$$

Convergence of General Iterative Method

- The **sequence of vectors** obtained **<u>converges</u>** if: **the sequence of <u>error vectors</u>** associated to **the iteration steps <u>approaches the zero vector</u>**
- The error vector at iteration k is defined as:
  - $\boldsymbol{e^{(k)} = x^{(k)} - x_a}$, $\boldsymbol{x_a}$= Actual solution vector
- But actual solution x_a is unknown!!

$$Ax_a = b \Leftrightarrow [\,Q - P]\,x_a = b$$

$$Qx^{(k+1)} = Px^{(k)} + [Q - P]x_a \Rightarrow$$

$$Q[x^{(k+1)} - x_a] = P\,[x^{(k)} - x_a\,] \Rightarrow$$

$$Qe^{(k+1)} = Pe^{(k)} \Rightarrow e^{(k+1)} = Q^{-1}Pe^{(k)} = Me^{(k)} \;\; (M=Q^{-1}P)$$

Convergence of General Iterative Method (cont'd)

$$e^{(k+1)} = Q^{-1}Pe^{(k)} = Me^{(k)} \ (M=Q^{-1}P)$$

$$e^{(1)} = Me^{(0)}, e^{(2)} = Me^{(1)} = M^2e^{(0)}, \ \ldots,$$

$$e^{(k)} = M^k e^{(0)}$$

$$||e^{(k)}||_\infty \leq (||M||_\infty)^k \, ||e^{(0)}||_\infty$$

$$if \ ||M||_\infty < 1: ||e^{(k)}||_\infty \rightarrow 0 \ _{as} \ as \ k \rightarrow \infty$$

- $M = Q^{-1}P$ plays a key role in the convergence of iterative schemes
  - Matrices Q and P must be selected so that $||M||_\infty < 1$

Jacobi Method

Introduction

- Let D, L, and U be the **diagonal**, **lower**, and **upper** triangular portions of matrix $A = [a_{ij}]_{n\times n}$

$$\mathbf{D} = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \cdots & \\ & & & a_{nn} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & & a_{2n} \\ \cdots & & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$A = Q - P = D + [L + U]$$

$$Q = D$$

$$P = -[L + U]$$

Overview of Jacobi Method

✳ $Qx^{(k+1)} = Px^{(k)} + b, \; k = 0,1,\ldots$

$$Dx^{(k+1)} = -[L + U]x^{(k)} + b, \; k = 0, 1, 2, \ldots$$

- For D^{-1} to exist, the diagonal entries of D (=> A), must all be nonzero.

  - *If a zero entry appears in a diagonal slot, the responsible equation in the original system must be switched with another equation so that no zero entry appears along the diagonal in the resulting coefficient matrix.*

$$x^{(k+1)} = D^{-1}\{-[L + U]x^{(k)} + b\}, \; k = 0,1,2,\ldots$$

L + U is exactly matrix A but with zero diagonal entries,

the diagonal entries of D^{-1} are $1/a_{ii}$ for i = 1,2, … , n.

Overview of Jacobi Method (cont'd)

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left\{ -\sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij} x_j^{(k)} + b_i \right\}, \quad i = 1,2,\ldots,n \qquad \textit{(Jacobi iteration method)}$$

- The Jacobi iteration matrix: $M_J = -D^{-1}\,[L + U]$
- A sufficient condition for Jacobi iteration to converge is that $\|M_J\|_\infty < 1$.

Convergence of Jacobi Iteration Method

- Convergence of the Jacobi method relies on a special class of matrices known as **diagonally dominant.**
- An *nxn* matrix A is diagonally dominant ⇔

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad i = 1, 2, \ldots, n$$

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{|a_{ij}|}{|a_{ii}|} < 1, \quad i = 1, 2, \ldots, n$$

Convergence of Jacobi Iteration Method (Cont'd)

- Let A be diagonally dominant
  - the linear system Ax = b has **a unique solution** x_a
  - the sequence of vectors generated by Jacobi iteration **converges to** x_a
  - **regardless** of the choice of the initial vector $x^{(0)}$

Jacobi Iteration Method - Example

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} 4 & 1 & -1 \\ -2 & 5 & 0 \\ 2 & 1 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ -7 \\ 13 \end{Bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \underset{\text{Initial vector}}{\mathbf{x}^{(0)}} = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

- Verify that the coefficient matrix A is diagonally dominant

$$4 > 1 + |-1| \,,\ 5 > |-2| \,,\ 6 > 2 + 1$$

$$\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \mathbf{L} + \mathbf{U} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

Jacobi Iteration Method - Example

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} 4 & 1 & -1 \\ -2 & 5 & 0 \\ 2 & 1 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 1 \\ -7 \\ 13 \end{Bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \underset{\text{Initial vector}}{\mathbf{x}^{(0)}} = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad \mathbf{L} + \mathbf{U} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\mathbf{x}^{(1)} = \mathbf{D}^{-1}\left\{-[\mathbf{L} + \mathbf{U}]\mathbf{x}^{(0)} + \mathbf{b}\right\} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} \left\{ -\begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 1 \\ -7 \\ 13 \end{Bmatrix} \right\} = \begin{Bmatrix} 0.25 \\ -1.4 \\ 2 \end{Bmatrix}$$

Jacobi Iteration Method - Example

$$\mathbf{A}\mathbf{x}=\mathbf{b},\quad \mathbf{A}=\begin{bmatrix}4 & 1 & -1\\ -2 & 5 & 0\\ 2 & 1 & 6\end{bmatrix},\quad \mathbf{b}=\begin{Bmatrix}1\\ -7\\ 13\end{Bmatrix},\quad \mathbf{x}=\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix},\quad \underset{\text{Initial vector}}{\mathbf{x}^{(0)}}=\begin{Bmatrix}0\\ 1\\ 1\end{Bmatrix}$$

$$\mathbf{D}=\begin{bmatrix}4 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{bmatrix},\quad \mathbf{L}+\mathbf{U}=\begin{bmatrix}0 & 1 & -1\\ -2 & 0 & 0\\ 2 & 1 & 0\end{bmatrix}$$

$$x_i^{(1)}=\frac{1}{a_{ii}}\left\{-\sum_{\substack{j=1\\ j\neq i}}^{3} a_{ij}x_j^{(0)}+b_i\right\},\quad i=1,2,3$$

$$x_1^{(1)}=\frac{1}{a_{11}}\left\{-\left[a_{12}x_2^{(0)}+a_{13}x_3^{(0)}\right]+b_1\right\}=\tfrac{1}{4}\left\{-[(1)(1)+(-1)(1)]+1\right\}=0.25$$

$$x_2^{(1)}=\frac{1}{a_{22}}\left\{-\left[a_{21}x_1^{(0)}+a_{23}x_3^{(0)}\right]+b_2\right\}=\tfrac{1}{5}\left\{-[(-2)(0)+(0)(1)]+(-7)\right\}=-1.4$$

$$x_3^{(1)}=\frac{1}{a_{33}}\left\{-\left[a_{31}x_1^{(0)}+a_{32}x_2^{(0)}\right]+b_3\right\}=\tfrac{1}{6}\left\{-[(2)(0)+(1)(1)]+13\right\}=2$$

Exercise

- For each linear system find the components of the first two vectors generated by the Jacobi method.

$$\begin{bmatrix} -3 & 1 & 2 \\ 2 & 4 & -1 \\ 1 & -2 & 4 \end{bmatrix} \mathbf{x} = \begin{Bmatrix} 24 \\ -5 \\ 12 \end{Bmatrix}, \quad \mathbf{x}^{(0)} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

$$\begin{bmatrix} -5 & 4 & 0 \\ 2 & 6 & -3 \\ -1 & 2 & 3 \end{bmatrix} \mathbf{x} = \begin{Bmatrix} 18 \\ 11 \\ 3 \end{Bmatrix}, \quad \mathbf{x}^{(0)} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

Gauss–Seidel Iteration Method

Introduction

- Considered a **refinement of Jacobi method**
- In Jacobi method, every component of $x^{(k+1)}$ is calculated entirely from $x^{(k)}$ of the previous iteration.
  - To access $x^{(k+1)}$, the k^{th} iteration has to be completed so that $x^{(k)}$ is entirely available.
- Performance of Jacobi iteration can be improved if the **most updated components of a vector are utilized**, <u>as soon as they are available</u>, to compute the subsequent components of the same vector.

Overview of Gauss–Seidel Method

$$\mathbf{x}^{(k)} = \begin{Bmatrix} x_1^{(k)} \\ \dots \\ x_p^{(k)} \\ x_{p+1}^{(k)} \\ \dots \\ x_n^{(k)} \end{Bmatrix}, \quad \mathbf{x}^{(k+1)} = \begin{Bmatrix} x_1^{(k+1)} \\ \dots \\ x_p^{(k+1)} \\ x_{p+1}^{(k+1)} \\ \dots \\ x_n^{(k+1)} \end{Bmatrix}, \quad \mathbf{x}_u = \begin{Bmatrix} x_1 \\ \dots \\ x_p \\ x_{p+1} \\ \dots \\ x_n \end{Bmatrix}$$

- $x_p^{(k+1)}$ is expected to be a better estimate of x_p than $x_p^{(k)}$ is
- using $x_p^{(k+1)}$ instead of $x_p^{(k)}$ should lead to a better approximation of the next component, $x_{p+1}^{(k+1)}$ in the current vector.

Gauss–Seidel Iteration Method (cont'd)

$$A = Q - P = [D + L] + U$$

$$Q = D + L,\ P = -U$$

$$[D + L]\, x^{(k+1)} = -Ux^{(k)} + b,\ k = 0, 1, 2, \ldots$$

- D+L is a lower-triangular matrix whose diagonal entries are those of A
  - $[D+L]^{-1}$ exists if A has nonzero diagonal entries
- What if a diagonal entry is zero?
  - the responsible equation in the original system must be switched with another equation so that no zero entry appears along the diagonal in the resulting coefficient matrix

$$x^{(k+1)} = [D + L]^{-1} \{-Ux^{(k)} + b\}\ k = 0, 1, 2, \ldots$$

(Gauss–Seidel iteration method)

Gauss–Seidel Iteration Method (cont'd)

$$\mathbf{x}^{(k)} = \begin{Bmatrix} x_1^{(k)} \\ \dots \\ x_p^{(k)} \\ x_{p+1}^{(k)} \\ \dots \\ x_n^{(k)} \end{Bmatrix} \qquad x_i^{(k+1)} = \frac{1}{a_{ii}} \left\{ -\sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} + b_i \right\}, \quad i = 1, 2, \dots, n$$

$$\mathbf{M}_{GS} = -[\mathbf{D} + \mathbf{L}]^{-1}\mathbf{U} \qquad \textit{(Gauss–Seidel iteration matrix)}$$

- sufficient condition for the Gauss–Seidel iteration to converge is $\|M_{GS}\|_\infty < 1$.

Convergence of Gauss - Seidel Method

- if **A is diagonally dominant**, the Gauss–Seidel iteration is also **guaranteed to converge, and faster than the Jacobi**
- If A is not diagonally dominant, the convergence of the Gauss–Seidel method relies on another special class of matrices known as **symmetric, positive definite**
  - Symmetric: $A^T = A$
  - positive definite: all determinants > 0
- Let A be symmetric, positive definite => the linear system $Ax = b$ has a unique solution x_a , and the sequence of vectors generated by the Gauss–Seidel iteration, converges to x_a regardless of the choice of the initial vector $x^{(0)}$.

Gauss-Seidel Method - Example

$$\mathbf{Ax}=\mathbf{b},\quad \mathbf{A}=\begin{bmatrix}4 & 1 & -1\\ -2 & 5 & 0\\ 2 & 1 & 6\end{bmatrix},\quad \mathbf{b}=\begin{Bmatrix}1\\ -7\\ 13\end{Bmatrix},\quad \mathbf{x}=\begin{Bmatrix}x_1\\ x_2\\ x_3\end{Bmatrix},\quad \underset{\text{Initial vector}}{\mathbf{x}^{(0)}}=\begin{Bmatrix}0\\ 1\\ 1\end{Bmatrix}$$

- Find $x^{(1)}$ using both forms of the Gauss–Seidel method

$$\mathbf{D}+\mathbf{L}=\begin{bmatrix}4 & 0 & 0\\ -2 & 5 & 0\\ 2 & 1 & 6\end{bmatrix},\quad \mathbf{U}=\begin{bmatrix}0 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\mathbf{x}^{(1)}=\left[\mathbf{D}+\mathbf{L}\right]^{-1}\left\{-\mathbf{U}\mathbf{x}^{(0)}+\mathbf{b}\right\}=\frac{1}{120}\begin{bmatrix}30 & 0 & 0\\ 12 & 24 & 0\\ -12 & -4 & 20\end{bmatrix}\left\{-\begin{bmatrix}0 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}0\\ 1\\ 1\end{Bmatrix}+\begin{Bmatrix}1\\ -7\\ 13\end{Bmatrix}\right\}=\begin{Bmatrix}0.25\\ -1.3\\ 2.3\end{Bmatrix}$$

Gauss-Seidel Method - Example

$$\mathbf{Ax}=\mathbf{b},\quad \mathbf{A}=\begin{bmatrix} 4 & 1 & -1 \\ -2 & 5 & 0 \\ 2 & 1 & 6 \end{bmatrix},\quad \mathbf{b}=\begin{Bmatrix} 1 \\ -7 \\ 13 \end{Bmatrix},\quad \mathbf{x}=\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix},\quad \underset{\text{Initial vector}}{\mathbf{x}^{(0)}}=\begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$$

- Find $x^{(1)}$ using both forms of the Gauss–Seidel method

$$x_i^{(1)}=\frac{1}{a_{ii}}\left\{-\sum_{j=1}^{i-1} a_{ij}x_j^{(1)}-\sum_{j=i+1}^{3} a_{ij}x_j^{(0)}+b_i\right\},\quad i=1,2,3$$

$$x_1^{(1)}=\frac{1}{a_{11}}\left[-a_{12}x_2^{(0)}-a_{13}x_3^{(0)}+b_1\right]=\frac{1}{4}\left[-(1)(1)-(-1)(1)+1\right]=0.25$$

$$x_2^{(1)}=\frac{1}{a_{22}}\left[-a_{21}x_1^{(1)}-a_{23}x_3^{(0)}+b_2\right]=\frac{1}{5}\left[-(-2)(0.25)-(0)(1)-7\right]=-1.3$$

$$x_3^{(1)}=\frac{1}{a_{33}}\left[-a_{31}x_1^{(1)}-a_{32}x_2^{(1)}+b_3\right]=\frac{1}{6}\left[-(2)(0.25)-(1)(-1.3)+13\right]=2.3$$

Gauss-Seidel Method - Example

- Positive definite symmetric matrix

$$\mathbf{Ax} = \mathbf{b}, \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 10 & 4 \\ -2 & 4 & 24 \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} 5.5 \\ 17.5 \\ -19 \end{Bmatrix}, \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \underset{\text{Initial vector}}{\mathbf{x}^{(0)}} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

- The coefficient matrix is symmetric since $A = A^T$.
- It is also positive definite because

$$1 > 0, \quad \begin{vmatrix} 1 & 1 \\ 1 & 10 \end{vmatrix} = 9 > 0, \quad \begin{vmatrix} 1 & 1 & -2 \\ 1 & 10 & 4 \\ -2 & 4 & 24 \end{vmatrix} = 144 > 0$$

Exercise

- For each linear system find the components of the first two vectors generated by the Gauss-Seidel method.

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 10 & 4 \\ -3 & 4 & 30 \end{bmatrix} \mathbf{x} = \begin{Bmatrix} 1 \\ 30 \\ 48 \end{Bmatrix}, \quad \mathbf{x}^{(0)} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 6 & 2 & 6 \\ 2 & 18 & 5 \\ 6 & 5 & 16 \end{bmatrix} \mathbf{x} = \begin{Bmatrix} -28 \\ 2 \\ -45 \end{Bmatrix}, \quad \mathbf{x}^{(0)} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$