I3344
Numerical Simulation & Modelling

Ahmad Fadlallah

Lebanese University / Faculty of Science - I

Solving a Linear Equation

Solution of Linear System of Equations

Linear System of Equations

- A linear system of n algebraic equations in n unknowns x_1, x_2 , ..., x_n is in the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n = b_2 \\ ... \\ a_{n1}x_1 + a_{n2}x_2 + ...a_{nn}x_n = b_n \end{cases}$$

- a_{ij} (i, j = 1, 2, ..., n) and b_k (k = 1, 2,..., n) are known constants
- a_{ij}s are the coefficients
- If every b_k is zero, the system is **homogeneous**, otherwise it is **non-homogeneous**

Linear System of Equations (cont'd)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots a_{nn}x_n = b_n \end{cases} \quad \Leftrightarrow \quad Ax = b$$

$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ & \dots & \dots & \dots & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}_{n \times n}}_{\textbf{Coefficient Matrix}}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{bmatrix}_{n \times 1}, b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_n \end{bmatrix}_{n \times 1}$$

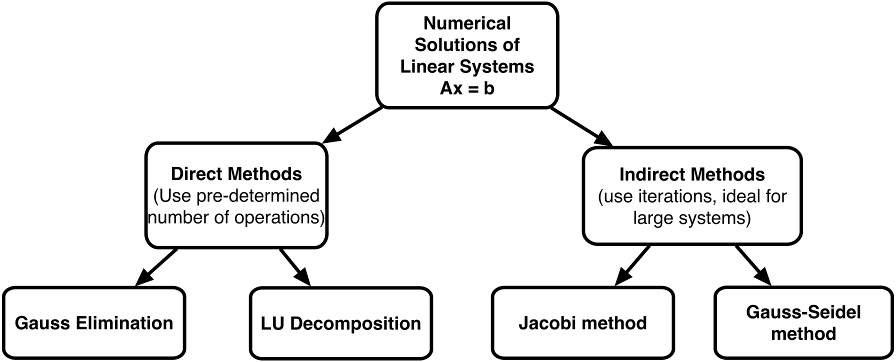
Linear System of Equations (cont'd)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n = b_n \end{cases} \quad \Leftrightarrow \quad Ax = b$$

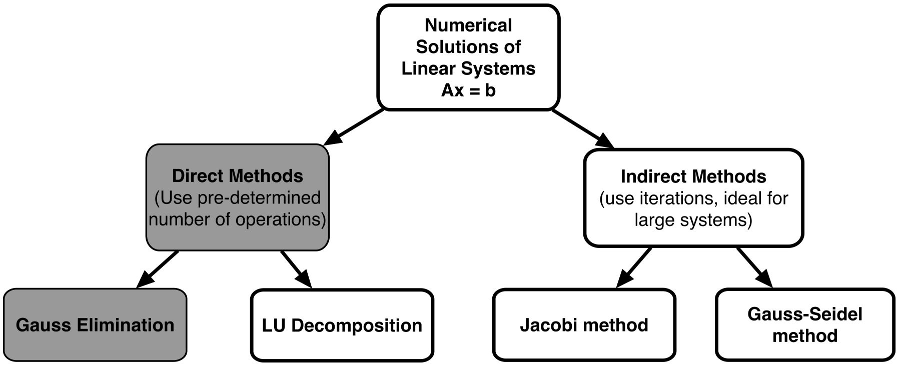
$$A = [A|b] = \left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} & b_n \end{array}\right]_{n\times(n+1)}$$

Augmented Matrix

Numerical Solutions of Linear Systems



Numerical Solutions of Linear Systems



Gauss Elimination Method

Introduction

- **Gauss elimination** is a procedure that **transforms** a **linear system of equations** (Ax = b) into **upper-triangular form**
  - The solution of the transformed system is found by **Back Substitution**.
- The **modifications must be applied to the augmented matrix [A | b]** and **not matrix A alone**
- The transformation into upper-triangular form is achieved by **using elementary row operations**

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_n \end{array}\right]$$

$\Downarrow$ (*Forward Elimination*)

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a'_{22} & a'_{23} & b'_2 \\ & & a''_{33} & b''_3 \end{array}\right]$$

$\Downarrow$ (*Back Substitution*)

$$x_3 = b''_3 / a''_{33}$$
$$x_2 = (b'_2 - a'_{23}x_3)/a'_{22}$$
$$x_1 = (b_1 - a_{12}x_2 - a_{13}x_3)/a_{11}$$

Introduction (cont'd)

- There are three types of elementary row operations:
    1. **Multiply a row** of the augmented matrix by a non-zero constant
    2. **Interchange any two rows** of the augmented matrix
    3. **Multiply the i^{th} row** of the augmented matrix by a constant $\alpha \neq 0$ and **add the result to the k^{th} row**, then **replace the k^{th} row with the outcome**
    - The i^{th} row is called the **pivot row**
- The nature of a linear system is preserved under elementary row operations
    - The new system and the original one are called **row-equivalent**

Gauss Elimination - How it works?

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + ...a_{nn}x_n = b_n \end{cases}$$

- The *first objective* is to **eliminate x_1 in all equations below the first**, thus the first row is the **pivot row**
- The entry that plays the most important role here is a_{11} (known as the **pivot**)
- If $a_{11} = 0$, the first row must be interchanged with another row to ensure that x_1 has a non-zero coefficient; This is called **Partial Pivoting**.
- **Partial pivoting** can be done also when the pivot is very small

Gauss Elimination - How it works?

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n = b_2 \\ \dots \\ a_{n1}x_1 + a_{n2}x_2 + ...a_{nn}x_n = b_n \end{cases}$$

- The *next step* is to **focus on the coefficient of x_2 in the second row** of this new system.
  - If it is nonzero, and not very small, we use it as the **pivot** and **eliminate x_2 in all the lower-level equations**.
  - The second row is the **pivot row** and remains unchanged.
- This continues until an **upper-triangular system is formed**.
- Finally, **Back Substitution** is used to find the solution.

Gauss Elimination - Example

$$\boxed{\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{array}\right] (1)} \quad \Rightarrow \quad \boxed{\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 2 & 2 & 8 \end{array}\right] (2)}$$

$$\boxed{\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] (3)} \quad \Rightarrow \quad \boxed{\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] (4)}$$

Exercise

- Using Gauss elimination, find the solution x_1, x_2, x_3, x_4 of the system whose augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 2 & 3 & 1 & 3 \\ 2 & -4 & 1 & 2 & -1 \\ -3 & 8 & 4 & -1 & 6 \\ 1 & 4 & 7 & -2 & -4 \end{array}\right]$$

Exercise - Solution

$$1 \quad -3 \quad 2 \quad \left[\begin{array}{cccc|c} -1 & 2 & 3 & 1 & 3 \\ 2 & -4 & 1 & 2 & -1 \\ -3 & 8 & 4 & -1 & 6 \\ 1 & 4 & 7 & -2 & -4 \end{array}\right] \text{Pivot row} \longrightarrow \left[\begin{array}{cccc|c} -1 & 2 & 3 & 1 & 3 \\ 0 & 0 & 7 & 4 & 5 \\ 0 & 2 & -5 & -4 & -3 \\ 0 & 6 & 10 & -1 & -1 \end{array}\right]$$

$$-3 \quad \left[\begin{array}{cccc|c} -1 & 2 & 3 & 1 & 3 \\ 0 & 2 & -5 & -4 & -3 \\ 0 & 0 & 7 & 4 & 5 \\ 0 & 6 & 10 & -1 & -1 \end{array}\right] \text{Pivot row} \longrightarrow -25/7 \quad \left[\begin{array}{cccc|c} -1 & 2 & 3 & 1 & 3 \\ 0 & 2 & -5 & 34 & -3 \\ 0 & 0 & 7 & 4 & 5 \\ 0 & 0 & 25 & 11 & 8 \end{array}\right] \text{Pivot row} \longrightarrow \left[\begin{array}{cccc|c} -1 & 2 & 3 & 1 & 3 \\ 0 & 2 & -5 & -4 & -3 \\ 0 & 0 & 7 & 4 & 5 \\ 0 & 0 & 0 & -\frac{23}{7} & -\frac{69}{7} \end{array}\right]$$

$$-\tfrac{23}{7}x_4 = -\tfrac{69}{7} \quad \Rightarrow \quad x_4 = 3 \quad \Rightarrow$$

$$x_3 = \tfrac{1}{7}(5 - 4x_4) = -1$$

$$x_2 = \tfrac{1}{2}(5x_3 + 4x_4 - 3) = 2$$

$$x_1 = 2x_2 + 3x_3 + x_4 - 3 = 1$$

Exercises

- Use Gaussian elimination to solve the following systems of linear equations

$$\begin{cases} 2x_1 + x_3 = -8 \\ x_1 - 2x_2 - 3x_3 = 0 \\ -x_1 + x_2 + 2x_3 = 3 \end{cases}$$

$$\begin{cases} x_1 - 2x_2 - 6x_3 = 12 \\ 2x_1 + 4x_2 + 12x_3 = -17 \\ x_1 - 4x_2 - 12x_3 = 22 \end{cases}$$

Choosing the Pivot Row
Partial Pivoting with Row Scaling

- When using **Partial Pivoting**, in the first step of the elimination process, it is common to **choose as the pivot row the row in which x_1 has the largest (in absolute value) coefficient**.
  - The subsequent steps are treated in a similar manner.

- This is mainly to **handle round-off error** while dealing with large matrices.

Choosing the Pivot Row
Partial Pivoting with Row Scaling (cont'd)

- Assume A is $n \times n$
- In each row i of A, find the **entry with the largest absolute value**; call it M_i
- In each row i, find the **ratio of the absolute value of the coefficient of** x_1 **to the absolute value of** M_i: $r_i = |a_{i1}| \,/\, |M_i|$
- Among r_i (i = 1, 2, ... , n) **pick the largest**.
  - Whichever row is responsible for this maximum value is picked as the **pivot row**.
  - Eliminate x_1 to obtain a new system.
- In the new system, consider the $(n-1) \times (n-1)$ sub-matrix of the coefficient matrix occupying the lower right corner.
  - In this matrix, use the same logic as above to choose the pivot row to eliminate x_2, and so on.

Choosing the Pivot Row
Partial Pivoting with Row Scaling (cont'd)

Example

$$[\mathbf{A}|\ \mathbf{b}] = \left[\begin{array}{ccc|c} -4 & -3 & 5 & 0 \\ 6 & 7 & -3 & 2 \\ 2 & -1 & 1 & 6 \end{array}\right]$$

$$r_1 = \frac{|-4|}{|5|} = \frac{4}{5}, \quad r_2 = \frac{|6|}{|7|} = \frac{6}{7}, \quad r_3 = \frac{|2|}{|2|} = 1$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 6 \\ 6 & 7 & -3 & 2 \\ -4 & -3 & 5 & 0 \end{array}\right] \xrightarrow{\text{Eliminate } x_1} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 6 \\ 0 & 10 & -6 & -16 \\ 0 & -5 & 7 & 12 \end{array}\right]$$

B

Choosing the Pivot Row
Partial Pivoting with Row Scaling (cont'd)

Example (cont'd)

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 6 \\ 6 & 7 & -3 & 2 \\ -4 & -3 & 5 & 0 \end{array}\right] \xrightarrow{\text{Eliminate } x_1} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 6 \\ 0 & 10 & -6 & -16 \\ 0 & -5 & 7 & 12 \end{array}\right]$$

B

$$\frac{|10|}{|10|} = 1 \;, \quad \frac{|-5|}{|7|} = \frac{5}{7}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 6 \\ 0 & 10 & -6 & -16 \\ 0 & 0 & 4 & 4 \end{array}\right] \quad \Rightarrow x_3 = 1, x_2 = -1, x_1 = 2$$

Exercises

Solve by Gauss elimination with partial pivoting. Show all steps of the computation

$$\begin{cases} 2x_1 - 6x_2 - x_3 = -38 \\ -3x_1 - x_2 + 7x_3 = -34 \\ -8x_1 + x_2 - 2x_3 = -20 \end{cases}$$

Exercises

- Solve by Gauss elimination with partial pivoting. Show all steps of the computation

$$\begin{cases} x_1 + x_2 - x_3 = -3 \\ 6x_1 + 2x_2 + 2x_3 = 2 \\ -3x_1 + 4x_2 + x_3 = 1 \end{cases}$$

Gauss-Jordan Method

- The **Gauss-Jordan method** is a variation of Gauss elimination.
- The major difference is that when an unknown is eliminated in the Gauss-Jordan method, **it is eliminated from all other equations** rather than just the subsequent ones.

Gauss-Jordan - How it works?

- All rows are **normalized** by **dividing them by their pivot elements** $\Rightarrow$
- The **elimination step results in an identity matrix** rather than a triangular matrix $\Rightarrow$
- it is **not necessary to employ back substitution** to obtain the solution.

Gauss-Jordan - How it works? (cont'd)

$$\begin{cases} 3x_1 - 0.1x_2 - 0.2x_3 = 7.85 \\ 0.1x_1 + 7x_2 - 0.3x_3 = -19.3 \\ 0.3x_1 - 0.2x_2 + 10x_3 = 71.4 \end{cases} \Longrightarrow \begin{bmatrix} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

Normalize the first row by dividing it by the pivot element: 3

$$\Longrightarrow \begin{bmatrix} 1 & -0.03333 & -0.06667 & 2.61667 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{bmatrix}$$

Eliminate x_1 term from the second and third rows

$$\Longrightarrow \begin{bmatrix} 1 & -0.03333 & -0.06667 & 2.61667 \\ 0 & 7.00333 & -0.29333 & -19.5617 \\ 0 & -0.19000 & 10.02000 & 70.6150 \end{bmatrix}$$

Gauss-Jordan - How it works? (cont'd)

Normalize second row by dividing it by the pivot element: 7.00333

$$\Longrightarrow \begin{bmatrix} 1 & -0.03333 & -0.06667 & 2.61667 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & -0.19000 & 10.02000 & 70.6150 \end{bmatrix}$$

Reduction of the x_2 terms from the first and third equations

$$\Longrightarrow \begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 10.01200 & 70.0843 \end{bmatrix}$$

Normalize the third row by dividing it by 10.0120

$$\Longrightarrow \begin{bmatrix} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 11 & 7.000 \end{bmatrix}$$

Gauss-Jordan - How it works? (cont'd)

The x_3 terms can be reduced from the first and second equations

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3.0000 \\ 0 & 1 & 0 & -2.5000 \\ 0 & 0 & 1 & 7.0000 \end{bmatrix}$$

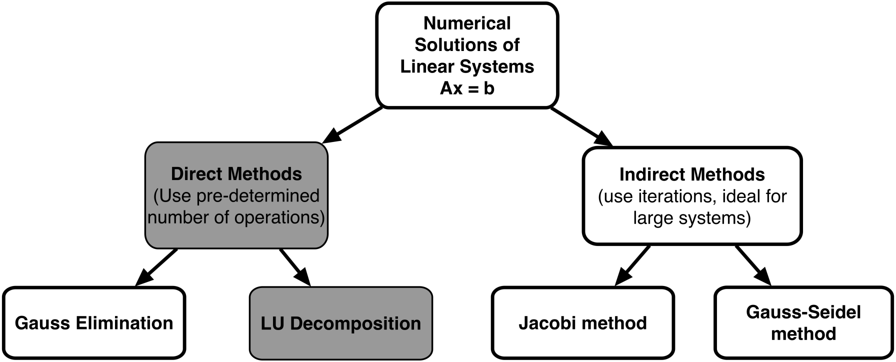
Exercises

- Use Gauss-Jordan elimination to solve each of the following linear systems of Equations

$$\begin{cases} 2x_1 + x_2 - x_3 = 1 \\ 5x_1 + 2x_2 + 2x_3 = -4 \\ 3x_1 + x_2 + x_3 = 5 \end{cases}$$

$$\begin{cases} x_1 + x_2 - x_3 = -3 \\ 6x_1 + 2x_2 + 2x_3 = 2 \\ -3x_1 + 4x_2 + x_3 = 1 \end{cases}$$

Numerical Solutions of Linear Systems



LU Decomposition

Introduction

- Main advantage of LU decomposition
  - Involves **only operations on the matrix of coefficients [A]**.
  - Well suited for those situations where **many right-hand-side vectors B must be evaluated** for a single value of [A].
    - Gauss Elimination becomes inefficient in such cases
- One motive for introducing LU decomposition is that it provides an **efficient means to compute the matrix inverse**.

Overview of LU Decomposition

- Simple Presentation (without Pivoting)

$$[A]\{x\}-\{B\}=0 \underset{e.g.,\textbf{Gauss Elimination}}{\Rightarrow} \underbrace{\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}}_{[U]} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}}_{\{X\}} = \underbrace{\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix}}_{\{D\}}$$

$$\Rightarrow \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} - \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = 0 \Rightarrow [U]\{X\}-\{D\}=0$$

Assume that there is a lower diagonal matrix with 1's on the diagonal

$$[L]=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \Big/ [L]\{[U]\{X\}-\{D\}\}=[A]\{X\}-\{B\} \Rightarrow \begin{cases} [L][U]=[A] \\ [L]\{D\}=\{B\} \end{cases}$$

LU Decomposition - Example

- How to factorize/ Decompose A?

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12} + U_{22} & L_{21}U_{13} + U_{23} \\ L_{31}U_{11} & L_{31}U_{12} + L_{32}U_{22} & L_{31}U_{13} + L_{32}U_{23} + U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

LU Decomposition - Example (cont'd)

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U12+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_33 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$$\boxed{U_{11} = 1}\ ,\ \boxed{U_{12} = 2}\ ,\ \boxed{U_{13} = 4}$$

$$L_{21}U_{11} = 3 \Rightarrow L_{21} \times 1 = 3 \Rightarrow \boxed{L_{21} = 3}$$

$$L_{21}U_{12} + U_{22} = 8 \Rightarrow 3 \times 2 + U_{22} = 8 \Rightarrow \boxed{U_{22} = 2}$$

$$L_{21}U_{13} + U_{23} = 14 \Rightarrow 3 \times 4 + U_{23} = 14 \Rightarrow \boxed{U_{23} = 2}$$

$$L_{31}U_{11} = 2 \Rightarrow L_{31} \times 1 = 2 \Rightarrow \boxed{L_{31} = 2}$$

$$L_{31}U_{12} + L_{32}U_{22} = 6 \Rightarrow 2 \text{ x } 2 + L_{32} \text{ x } 2 = 6 \Rightarrow \boxed{L_{32} = 1}$$

$$L_{31}U_{13} + L_{32}U_{23} + U_{33} = 13 \Rightarrow (2 \text{ x } 4) + (1 \text{ x } 2) + U_{33} = 13 \Rightarrow \boxed{U_{33} = 3}$$

LU Decomposition - Example (cont'd)

$$A = \begin{bmatrix} 1 & 2 & 34 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

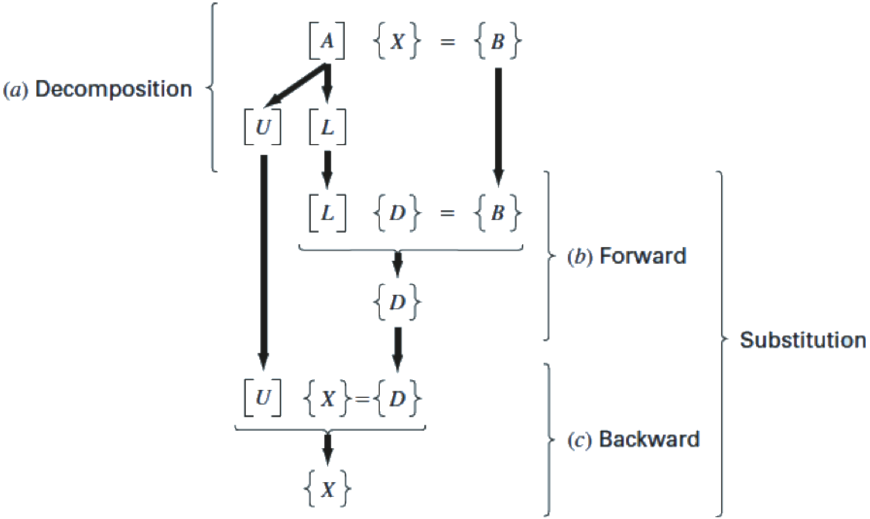
Exercises

- Find an LU decomposition of $\begin{bmatrix} 3 & 1 \\ -6 & -4 \end{bmatrix}$
- Find an LU decomposition of $\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}$

LU Decomposition to solve Linear Systems

- Given A, find L and U so that A = LU =¿ LUX = B.
- Let Y = UX =¿ LY = B
  - Solve this triangular system for Y .
- Solve the triangular system UX = Y for X.
- The benefit of this approach is that we only ever need to solve triangular systems.
- The cost is that we have to solve two of them.

LU Decomposition to solve Linear Systems (cont'd)



LU Decomposition to solve Linear Systems - Example

Find the solution of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *of* $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

LU Decomposition to solve Linear Systems - Example

Find the solution of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *of* $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

First step: calculate the LU decomposition of A

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \; U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

LU Decomposition to solve Linear Systems - Example

Find the solution of x $= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *of* $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

First step: calculate the LU decomposition of A

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Second step: solve LY = B

$$\text{LY} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

LU Decomposition to solve Linear Systems - Example

Find the solution of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *of* $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

First step: calculate the LU decomposition of A

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Second step: solve LY = B

$$LY = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

Third step: solve UX = Y

$$UX = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} = Y \Rightarrow X = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

Exercise

- Use the LU decomposition (previously calculated) to solve

$$\begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & 16 \\ 0 & 8 & -17 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 17 \end{bmatrix}$$

Conditions for LU decomposition

- Sometimes it is impossible to write a matrix in the form "lower triangular" $\times$ "upper triangular"
- An invertible matrix A has an LU decomposition provided that all its leading sub-matrices have non-zero determinants.
- The k^{th} leading sub-matrix of A is denoted A_k and is the $k \times k$ matrix found by looking only at the top k rows and leftmost k columns.

Conditions for LU decomposition - Example

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$$A_1 = 1, A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$|A_1| = 1$

$|A_2| = (1 \times 8) - (2 \times 3) = 2,$

$|A_3| = \begin{pmatrix} 8 & 14 \\ 6 & 13 \end{pmatrix} - 2 \begin{pmatrix} 3 & 14 \\ 2 & 13 \end{pmatrix} + 4 \begin{pmatrix} 3 & 8 \\ 2 & 6 \end{pmatrix} = 20 - (2 \times 11) + (4 \times 2) = 6$

A has an LU decomposition because none of these determinants is zero

Exercises

- Check if the following matrix has an LU decomposition

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$$

- Which, if any, of these matrices have an LU decomposition?

(i) $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, (ii) $\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$, (iii) A$= \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

Conditions for LU decomposition (cont'd)

- Can we get around this problem?
- It is always possible to re-order the rows of an invertible matrix so that all of the sub-matrices have non-zero determinants
- Example: Reorder the rows of A so that the reordered matrix has an LU decomposition.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 3 & 4 \end{bmatrix}$$

Exercise

- Reorder the rows of A so that the reordered matrix has an LU decomposition.

$$A = \begin{bmatrix} 1 & -3 & 7 \\ -2 & 6 & 1 \\ 0 & 3 & -2 \end{bmatrix}$$

Exercises

- Solve the following Linear Systems using LU decomposition

(a) $\begin{bmatrix} 2 & 1 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 1 & -4 \\ 2 & 1 & -2 \\ 6 & 3 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 7 \\ 41 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 8 & 5 \\ 1 & 11 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

Exercises

Consider $A = \begin{bmatrix} 1 & 6 & 2 \\ 2 & 12 & 5 \\ -1 & -3 & -1 \end{bmatrix}$

(a) Show that A does not have an LU decomposition

(b) Re-order the rows of A and find an LU decomposition of the new matrix

(c) Hence solve

$$\begin{cases} x_1 + 6x_2 + 2x_3 = 9 \\ 2x_1 + 12x_2 + 5x_3 = -4 \\ -x_1 - 3x_2 - x_3 = 17 \end{cases}$$