Linear Systems

*Gauss Elimination:

- transform to upper triangular $\left(\begin{array}{c|c} A & b \end{array}\right) \rightarrow \left(\begin{array}{c|c} \ddots & \vdots \end{array}\right)$ with forward elimination
- find values of $x_1 \ldots x_n$ using Back Substitution.

examples

pivot →

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ \boxed{1} & 1 & -1 & 1 \\ \boxed{3} & 11 & 6 & 36 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & \boxed{2} & 3 & 9 \end{array}\right] \xrightarrow{R_3 = R_3 + R_2} \begin{array}{cccc} x_1 & x_2 & x_3 & b \\ \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & -2 & -2 & -8 \\ 0 & 0 & \boxed{1} & \boxed{1} \end{array}\right] \rightarrow x_3 = 1$$

first we must turn these into zeros

now we repeat the process to turn this 2 into zero, using R2 as a pivot.

$$x_2 = \frac{-8+2}{-2} = 3$$

$$x_1 = 9 - 1 - 9 = -1$$

+Partial Pivoting:

- formula to choose which row should be the pivot.
- optional and won't affect results; makes our work easier.

$$r = \frac{\text{first entry in row}}{\text{largest absolute entry}}$$

ex:

$$\left[\begin{array}{ccc|c} -4 & -3 & 5 & 0 \\ 6 & 7 & -3 & 2 \\ 2 & -1 & 1 & 6 \end{array}\right] \quad \begin{array}{l} r_1 = 4/5 \\ r_2 = 6/7 \\ r_3 = 2/2 = 1 \end{array}$$

=> R3 is the pivot

=> interchange R_1 & R_2.

*Gauss-Jordan:

- Variant of Jacobi where we eliminate the unknown from equations above our pivot in addition to those below it.
- Doesn't require back substitution.

examples

$$\left[\begin{array}{ccc|c} \boxed{1} & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 6 & 36 \end{array}\right] \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 0 & \boxed{-2} & -2 & -8 \\ 0 & 2 & 3 & 9 \end{array}\right] \xrightarrow{R_2 = -R_2/2} \left[\begin{array}{ccc|c} 1 & \boxed{3} & 1 & 9 \\ 0 & 1 & 1 & 4 \\ 0 & \boxed{2} & 3 & 9 \end{array}\right]$$

first element of pivot is 1 => no need to normalize

must be normalized

$$\xrightarrow[R_1 = R_1 - 3R_2]{R_3 = R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & \boxed{-2} & -3 \\ 0 & 1 & \boxed{1} & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_1 = R_1 + 2R_3]{R_2 = R_2 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}$$

Note: Just like in GE, we can also do Partial Pivoting.

* LU Decomposition: step 1: set $A = \overset{\text{Lower}}{L}\overset{\text{Upper}}{U}$

$$\overset{A}{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}} = \overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix}} \overset{U}{\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}U_{11} & L_{21}U_{12}+U_{22} & L_{21}U_{13}+U_{23} \\ L_{31}U_{11} & L_{31}U_{12}+L_{32}U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{bmatrix}$$

step 2: $AX = B$

$LUX = B$ let $Y = UX$

solve $LY = B \therefore Y = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$ By Forward Substitution

solve $UX = Y \therefore X = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$ By Backward Substitution.

Note: to check if a matrix has a LU decomposition, all leading sub matricies: $\left[\left[\left[\cdot\right] \; \cdot \; \right] \; \cdot \; \right]$ must not have a determinant of zero.

Determinants reminder: • $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (ad)-(bc)$

• $\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a.\overset{\det}{\begin{bmatrix} e & f \\ h & i \end{bmatrix}} - b.\overset{\det}{\begin{bmatrix} d & f \\ g & i \end{bmatrix}} + c\overset{\det}{\begin{bmatrix} d & e \\ g & h \end{bmatrix}}$

If found to be impossible, we can always reorder the rows.