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Availability, reliability and downtime of systems with repairable components

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Abstract

Closed-form expressions are derived for the steady-state availability, mean rate of failure, mean duration of downtime and lower bound reliability of a general system with randomly and independently failing repairable components. Component failures are assumed to be homogeneous Poisson events in time and repair durations are assumed to be exponentially distributed. The results are expressed in terms of the mean rates of failure and mean durations of repair of the individual components. Closed-form expressions are also derived for the rates of change of the various probabilistic system performance measures with respect to the mean rate of failure and the mean duration of repair of each component. These expressions provide a convenient framework for identifying important components within the system and for decision-making aimed at upgrading the system availability or reliability, or reducing the mean duration of system downtime. Example applications to an electrical substation system demonstrate the use of the formulas developed in the paper.

1. Introduction

Consideration of systems with randomly failing repairable components is of interest in many engineering fields (see, e.g., [1–5]). A computer network consisting of servers, hubs, routers and workstations; a power distribution system consisting of generation plants, transmission lines, substations and local distribution lines; and a highway transportation network consisting of roadways, tunnels and bridges are examples of such systems. Of interest is not only the availability of the system for operation at any given time or the reliability for operation during a specified interval of time, but also a measure of how quickly the system can be put back into service after each failure. Furthermore, one is often interested in identifying critical components within a system, particularly in the context of upgrading the system availability or reliability, or reducing the duration of its downtimes. In a complex system with numerous components, identification of critical components is not straightforward.

In this paper, we consider the class of coherent systems composed of randomly failing repairable components. We assume components fail independently of each other and with constant mean failure rates; the repair of each component commences immediately after its failure and has a random duration, independent of the states of other components, with an exponential distribution. Furthermore, stationary conditions are assumed so that the probabilistic characteristics of the components and the system are invariant of a translation in time. With constant mean component failure and repair rates, stationary conditions are achieved after the effect of the initial system state has died out. Thus, the derived results are appropriate for consideration of the steady-state or limiting state of the system [1], i.e., for a future time period far removed from the present time at which the state of the system may be known. Under these assumptions and conditions, closed-form expressions are derived for the system availability (the probability that the system is operating at a given time), the mean rate of system failures, the mean duration of system downtime after each failure, and a lower bound to the

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system reliability (the probability that the system will be operating during a specified interval of time) as functions of the mean rate of failure and the mean duration of repair of each component. The closed-form solutions are used to derive rates of change of these system performance measures with respect to the component parameters, which are then used to formulate a set of component importance measures in the context of upgrading the performance of the system.

In the classical system reliability theory (e.g. [1,2,4]), problems of the type described above are usually handled by a Markovian model. For the availability function, a differential equation is formed by using the total probability theorem considering the state of the system at a preceding increment of time. By letting time go to infinity in the solution of the differential equation, the limiting expression for availability is obtained, which corresponds to the stationary condition mentioned above. In the approach developed in this paper, we use the homogeneous Poisson model for the initiation of the failure and repair events and directly obtain the stationary solution without considering the initial state of the system. As a result, we are able to obtain closed-form expressions for the four system performance measures defined in the preceding paragraph.

2. Representation of a component

Consider a system consisting of n two-state components. Assume the components experience random failures in time, independently of each other, and each failure entails a random duration of repair before the component is put back into service. We assume the duration of repair of the failed component is independent of the states of other components. Let ν_i denote the mean rate of failure of component i relative to the total time, i.e., including any repair durations. Assuming ergodicity, this value may be estimated by dividing the number of failures of the component in a long period of time by the length of the period, including any repair times. Also let μ_i denote the mean duration of repair of the component after each failure. Again assuming ergodicity, this value may be estimated by dividing the sum of repair times by the number of failures of the component over a long period of time. In view of the stationarity and ergodicity assumptions, $p_i = \nu_i\mu_i$ describes the fraction of time that the component is unavailable. Hence, for any component i, we must have $\nu_i\mu_i \leqslant 1$ and for most realistic components of high reliability we must have $\nu_i\mu_i \ll 1$.

Most classical reliability texts (e.g. [1,2,4]) define the mean failure rate relative to the operation time of the component, i.e., excluding repair periods. Here, ν_i is defined with respect to the total time, including repair periods. As we will see later, this definition leads to simpler expressions for the system performance measures. In terms of the adopted notation, the mean rate of failure during the operation time alone is given by $\hat{\nu}_i = \nu_i/(1-\nu_i\mu_i)$. This relation is obtained by equating the well known expression for the limiting unavailability, $p_i = \mu_i/(\mu_i + 1/\hat{\nu}_i)$ [1], with the expression $p_i = \nu_i\mu_i$ given above.

Due to the random nature of the component failures, the Poisson process is an appropriate model to employ. Consider a homogenous Poisson process $N_i(t)$ with the mean rate λ_i, hereafter denoted *the underlying Poisson process* for component i. Assume each occurrence in this Poisson process is followed by a "down" or "up" state of the component with probabilities p_i and $1-p_i$, respectively. Also assume the states after successive occurrences in the Poisson process are statistically independent. A realization of this process may appear as in Fig. 1, where the underlying Poisson events are marked by open or closed dots and the down and up states are shown with gray rectangles below and above the time axis, respectively. We define a succession of up states as a single operation period, and a succession of down states as a single repair period of the component (see Fig. 1). The Poisson events with closed dots, therefore, mark the failure events of the component.

Consider a succession of down states forming a single repair period. The number X of such consecutive down states has the geometric distribution $\Pr(X=x) = (1-p_i)p_i^{x-1}$, $x = 1, 2, \ldots$. Furthermore, since the duration of each individual down state is exponentially distributed with parameter λ_i, the repair period consisting of x successive down states has the gamma distribution with parameters x and λ_i. Thus, by the total probability rule, the unconditional probability density function of a single repair duration is given by

$$
\begin{aligned}
f_T(t) &= \sum_{x=1}^{\infty} \frac{\lambda_i^x}{(x-1)!} t^{x-1} \exp(-\lambda_i t)(p_i)^{x-1}(1-p_i) \\
&= \lambda_i(1-p_i)\exp[-\lambda_i(1-p_i)t], \quad 0<t.
\end{aligned} \tag{1}
$$

This is an exponential distribution with the mean $1/[\lambda_i(1-p_i)]$. By symmetry, a single operation period has the exponential distribution

$$
f_S(t) = \lambda_i p_i \exp(-\lambda_i p_i t), \quad 0<t. \tag{2}
$$

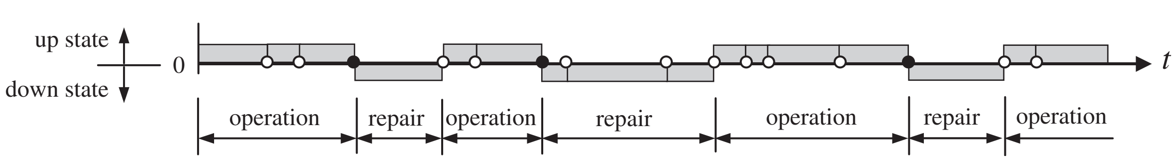

Fig. 1. A typical realization of the underlying Poisson process.

We also note that the availability of the component at any given time is equal to the probability of the up state, which is $1-p_i$.

In order for the model to correctly describe the mean repair duration and the mean failure rate, we set

$$\frac{1}{\lambda_i(1-p_i)} = \mu_i, \tag{3}$$

$$\lambda_i p_i = \hat{v}_i = \frac{v_i}{1-v_i\mu_i}. \tag{4}$$

The solution yields $p_i = v_i\mu_i$, which agrees with the expression for the component unavailability mentioned earlier, and $\lambda_i = 1/[\mu_i(1-v_i\mu_i)]$. With these parameter values, the Poisson model correctly describes the random failures and repairs of the component in time. It is noteworthy that the model is completely described by the two parameters v_i and μ_i. In the following sections, we derive closed-form solutions for the unavailability, mean failure rate and mean downtime of parallel, series and general systems in terms of the individual component parameters v_i and μ_i.

3. Parallel systems

First consider a parallel system of two components i and j with their respective underlying Poisson processes $N_i(t)$ and $N_j(t)$ having the mean rates $\lambda_i = 1/[\mu_i(1-v_i\mu_i)]$ and $\lambda_j = 1/[\mu_j(1-v_j\mu_j)]$. Since the component states are statistically independent, the probability that at any given time both components are failed and the system is down, i.e., the system unavailability, is $P_{ij} = p_ip_j = v_i\mu_iv_j\mu_j$. The mean rate of joint failures of the two components (relative to the total time) can be derived as follows: Let $S_{i1}, S_{i2}, \ldots, S_{in}, \ldots$ denote the sequence of consecutive operation times and $T_{i1}, T_{i2}, \ldots, T_{in}, \ldots$ denote the sequence of consecutive repair times of component i. Also let N^i_{jr} denote the number of times component j fails during the time T_{ir}, where $r = 1, 2, \ldots$. Based on the law of large numbers, the mean rate of failures of component j while component i is in repair is given by

$$\begin{aligned} v^i_j &= \lim_{n\to\infty} \frac{N^i_{j1}+N^i_{j2}+\cdots+N^i_{jn}}{S_{i1}+T_{i1}+S_{i2}+T_{i2}+\cdots+S_{in}+T_{in}} \\ &= \lim_{n\to\infty}\left(\frac{N^i_{j1}+N^i_{j2}+\cdots+N^i_{jn}}{T_{i1}+T_{i2}+\cdots+T_{in}}\right. \\ &\qquad \left.\times\frac{T_{i1}+T_{i2}+\cdots+T_{in}}{S_{i1}+T_{i1}+S_{i2}+T_{i2}+\cdots+S_{in}+T_{in}}\right) \\ &= \lim_{n\to\infty}\frac{N^i_{j1}+N^i_{j2}+\cdots+N^i_{jn}}{T_{i1}+T_{i2}+\cdots+T_{in}} \\ &\qquad \times\lim_{n\to\infty}\frac{T_{i1}+T_{i2}+\cdots+T_{in}}{S_{i1}+T_{i1}+S_{i2}+T_{i2}+\cdots+S_{in}+T_{in}} \\ &= v_jp_i \\ &= v_jv_i\mu_i, \end{aligned} \tag{5}$$

where the limits are in the sense of convergence with probability 1. In a similar way, $v^j_i = v_iv_j\mu_j$. It follows that the mean rate of joint failures of components i and j relative to the total time, denoted v_{ij}, is

$$\begin{aligned} v_{ij} &= v^j_i + v^i_j \\ &= v_iv_j(\mu_i+\mu_j). \end{aligned} \tag{6}$$

The mean duration of joint failure events, i.e., the mean system downtime, is obtained by dividing the fraction of time that the system is unavailable, $P_{ij} = v_iv_j\mu_i\mu_j$, by the mean rate of joint failures, $v_{ij} = v_iv_j(\mu_i+\mu_j)$. The result is $\mu_{ij} = 1/(1/\mu_i + 1/\mu_j)$.

For a parallel system with three components i, j and k, following the same reasoning, we have the unavailability as $P_{ijk} = v_iv_jv_k\mu_i\mu_j\mu_k$, the mean rate of joint failures as $v_{ijk} = v_iv_jv_k(\mu_j\mu_k + \mu_i\mu_k + \mu_i\mu_j)$ and the mean duration of downtimes as $\mu_{ijk} = 1/(1/\mu_i + 1/\mu_j + 1/\mu_k)$. The expression for v_{ijk} is obtained as the sum $v^{ij}_k + v^{jk}_i + v^{ki}_j$, where v^{ij}_k is the mean rate of failures of component k while both components i and j are in repair. The latter is obtained from (5) by replacing S_{ir} and T_{ir} by S_{ijr} and T_{ijr}, the operation and repair times of the two-component parallel system, and N^i_{jr} by N^{ij}_{kr}, the number of failures of component k during the joint repair times of components i and j. The result is $v^{ij}_k = v_kP_{ij} = v_iv_jv_k\mu_i\mu_j$. The sum of the three terms then gives the result for v_{ijk} given above. The expression for μ_{ijk} is obtained by dividing P_{ijk} by v_{ijk}.

Generalizing these results for a parallel system with n components, we obtain for the unavailability, mean rate of failure and mean duration of downtime, respectively,

$$P_{1\cdots n} = \prod_{i=1}^{n} v_i\mu_i, \tag{7}$$

$$v_{1\cdots n} = \left(\prod_{i=1}^{n} v_i\mu_i\right)\left(\sum_{i=1}^{n}\frac{1}{\mu_i}\right), \tag{8}$$

$$\mu_{1\cdots n} = \frac{1}{\sum_{i=1}^{n} 1/\mu_i}. \tag{9}$$

The subscript in the form of product of component indices in the above expressions signifies that these quantities relate to the joint failures of the indicated components.

4. Series systems

Now consider a series system of two components. The system fails whenever either component fails. Since the component states are statistically independent, the probability that the system is unavailable at any given time is given by $P_{i\cup j} = 1-(1-p_i)(1-p_j) = 1-(1-v_i\mu_i)(1-v_j\mu_j)$. The mean rate of the system failures (relative to the total time) is $v_{i\cup j} = v_i + v_j - v_{ij}$, where we have subtracted the mean rate of joint component failures from the sum of the individual failure rates because at such events the system is already failed and counting the joint failure as a new failure occurrence would result in double counting. The subscript $i \cup j$ in these expressions signifies that these quantities relate to the union of failure events of the two components. The mean duration of each system failure is given by the ratio $\mu_{i\cup j} = P_{i\cup j}/v_{i\cup j}$, which does not simplify.

For a series system with three components, these relations become $P_{i\cup j\cup k} = 1 - (1 - v_i\mu_i)(1 - v_j\mu_j)(1- v_k\mu_k)$, $v_{i\cup j\cup k} = v_i + v_j + v_k - v_{ij} - v_{jk} - v_{ki} + v_{ijk}$, where the inclusion–exclusion rule has been used to avoid over- or under-counting of the system failure events, and $\mu_{i\cup j\cup k} = P_{i\cup j\cup k}/v_{i\cup j\cup k}$. Generalizing these results for a series system of n components leads to the following formulas, respectively, for the unavailability, the mean failure rate and the mean duration of downtime of the system:

$$P_{1\cup\cdots\cup n} = 1 - \prod_{i=1}^{n}(1 - v_i\mu_i), \tag{10}$$

$$v_{1\cup\cdots\cup n} = \sum_{i=1}^{n} v_i - \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} v_{ij} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n} v_{ijk} - \cdots + (-1)^{n-1}v_{1\cdots n}, \tag{11}$$

$$\mu_{1\cup\cdots\cup n} = \frac{P_{1\cup\cdots\cup n}}{v_{1\cup\cdots\cup n}}. \tag{12}$$

5. General systems

We now consider a general system of n components. Let C_j, $j = 1, \ldots, r$, denote the set of minimum cut sets of the system, each containing a minimum set of indices of components whose joint failure constitutes failure of the system. Note that due to possible sharing of components, the cut set events in general are dependent. As is well known, the system can be represented as a series system of parallel sub-systems, with each parallel sub-system representing a cut set [1]. The system failure event is now the union of failure events of the cut sets, with the failure of each cut set being the intersection of the failure events of its constituent components. Using the inclusion–exclusion rule, and owing to the statistical independence between the component states, the probability that the system is in the failed state at any given time, i.e., the unavailability, is given by

$$P_{\text{sys}} = \sum_{j=1}^{r}\left(\prod_{i\in C_j} p_i\right) - \sum_{j=1}^{r-1}\sum_{k=j+1}^{r}\left(\prod_{i\in C_j\cup C_k} p_i\right) + \sum_{j=1}^{r-2}\sum_{k=j+1}^{r-1}\sum_{l=k+1}^{r}\left(\prod_{i\in C_j\cup C_k\cup C_l} p_i\right) - \cdots + (-1)^{r-1}\left(\prod_{i\in C_1\cup\cdots\cup C_r} p_i\right)$$

$$= \sum_{j=1}^{r}\left(\prod_{i\in C_j} v_i\mu_i\right) - \sum_{j=1}^{r-1}\sum_{k=j+1}^{r}\left(\prod_{i\in C_j\cup C_k} v_i\mu_i\right) + \sum_{j=1}^{r-2}\sum_{k=j+1}^{r-1}\sum_{l=k+1}^{r}\left(\prod_{i\in C_j\cup C_k\cup C_l} v_i\mu_i\right) - \cdots + (-1)^{r-1}\left(\prod_{i\in C_1\cup\cdots\cup C_r} v_i\mu_i\right). \tag{13}$$

Since each cut set is a parallel system, we can determine its mean failure rate (relative to total time) from (8). Thus, let

$$v_{C_j} = \left(\prod_{i\in C_j} v_i\mu_i\right)\left(\sum_{i\in C_j}\frac{1}{\mu_i}\right) \tag{14}$$

denote the mean rate of failure of the jth cut set. Furthermore, intersections of cut sets are also parallel systems and (8) can be used to compute their mean rates of occurrence. All we need to do is replace C_j in (14) with the appropriate set of cut sets. Thus, for example, the mean rate of joint occurrences of cut sets C_1, C_2 and C_3 is given by

$$v_{C_1C_2C_3} = \left(\prod_{i\in C_1\cup C_2\cup C_3} v_i\mu_i\right)\left(\sum_{i\in C_1\cup C_2\cup C_3}\frac{1}{\mu_i}\right), \tag{15}$$

where the subscript notation $C_1C_2C_3 = C_1 \cap C_2 \cap C_3$ has been used. Now observe that in terms of the cut sets, the system is a series system. Thus (11) can be used to determine the mean rate of system failures (relative to total time) in terms of $v_{C_j}v_{C_jC_k}$, etc. Specifically,

$$v_{\text{sys}} = \sum_{j=1}^{r} v_{C_j} - \sum_{j=1}^{r-1}\sum_{k=j+1}^{r} v_{C_jC_k} + \sum_{j=1}^{r-2}\sum_{k=j+1}^{r-1}\sum_{l=k+1}^{r} v_{C_jC_kC_l} - \cdots + (-1)^{r-1}v_{C_1\cdots C_r}. \tag{16}$$

Finally, the mean duration of system unavailability after each failure is given by the ratio

$$\mu_{\text{sys}} = \frac{P_{\text{sys}}}{v_{\text{sys}}}. \tag{17}$$

An alternative solution for the general system can be derived in terms of the minimum link sets. Let L_j, $j = 1, \ldots, s$, denote the set of minimum link sets of the system, each containing a minimum set of indices of components whose joint survival constitutes survival of the system. Note that due to possible sharing of components, link set events in general are dependent. As is well known, the system can be represented as a parallel system of series sub-systems, with each series sub-system representing a link set [1]. Thus, the system failure event is the intersection of the failure events of the link sets. Alternatively, using de Morgan's rule, the system failure event is equal to the complement of the union of the survival events of the link sets, with the survival of each link set being the intersection of the survival events of its constituent components. Using the inclusion–exclusion rule on the complement of the failure event, results in

$$\begin{aligned}P_{\rm sys} = 1 &- \sum_{j=1}^{s}\left[\prod_{i\in L_j}(1-p_i)\right] + \sum_{j=1}^{s-1}\sum_{k=j+1}^{s}\left[\prod_{i\in L_j\cup L_k}(1-p_i)\right] \\ &- \sum_{j=1}^{s-2}\sum_{k=j+1}^{s-1}\sum_{l=k+1}^{s}\left[\prod_{i\in L_j\cup L_k\cup L_l}(1-p_i)\right] \\ &+ \cdots - (-1)^{s-1}\left[\prod_{i\in L_1\cup\cdots\cup L_s}(1-p_i)\right] \\ = 1 &- \sum_{j=1}^{s}\left[\prod_{i\in L_j}(1-v_i\mu_i)\right] + \sum_{j=1}^{s-1}\sum_{k=j+1}^{s}\left[\prod_{i\in L_j\cup L_k}(1-v_i\mu_i)\right] \\ &- \sum_{j=1}^{s-2}\sum_{k=j+1}^{s-1}\sum_{l=k+1}^{s}\left[\prod_{i\in L_j\cup L_k\cup L_l}(1-v_i\mu_i)\right] \\ &+ \cdots - (-1)^{s-1}\left[\prod_{i\in L_1\cup\cdots\cup L_s}(1-v_i\mu_i)\right]. \end{aligned} \tag{18}$$

It is shown in Appendix A that, in terms of the link sets, the mean rate of system failures is given by

$$\begin{aligned}v_{\rm sys} = \sum_{j=1}^{s} v_{L_j} - \sum_{j=1}^{s-1}\sum_{k=j+1}^{s} v_{L_j\cup L_k} + \sum_{j=1}^{s-2}\sum_{k=j+1}^{s-1}\sum_{l=k+1}^{s} v_{L_j\cup L_k\cup L_l} \\ - \cdots + (-1)^{s-1} v_{L_1\cup L_2\cup\cdots\cup L_s},\end{aligned} \tag{19}$$

where the terms v_{L_j}, $v_{L_j\cup L_k}$, $v_{L_j\cup L_k\cup L_l}$, etc., represent mean rates of failure of any of the indicated link sets. Since the union of link sets is a series system, these terms can be computed by using (11). Specifically, $v_{L_j} = v_{\bigcup_{i\in L_j} i}$, $v_{L_j\cup L_k} = v_{\bigcup_{i\in L_j\cup L_k} i}$, $v_{L_j\cup L_k\cup L_l} = v_{\bigcup_{i\in L_j\cup L_k\cup L_l} i}$, etc., where, consistent with the notation used in (11), the subscript $\bigcup_{i\in S} i$ signifies that these quantities relate to the union of the failure events of all components in the indicated set S. Having determined $P_{\rm sys}$ and $v_{\rm sys}$ from (18) and (19), respectively, the mean duration of system failure is obtained from (17).

For important systems, even a single failure could be catastrophic. In that case, one is interested in the reliability of the system over a future planned period $(t, t+\tau)$ of operation, which is the probability that no system failures will occur during that period. Due to statistical dependence between successive system failure events, an exact expression for the system reliability is not available. Instead, we opt for a lower bound to the system reliability. It is well known that the probability of one or more events during an interval $(t, t+\tau)$ is less than the mean number of events during the same interval. Thus, considering the stationary assumption for the system failures and accounting for the probability of system failure at time t, one can write for the reliability of the system [6]

$$R(\tau) \geqslant 1 - P_{\rm sys} - v_{\rm sys}\tau. \tag{20}$$

This result is obviously useful only for highly reliable systems, for which $P_{\rm sys} + v_{\rm sys}\tau < 1$.

Before closing this section, we make two remarks relative to the computational aspects of the above formulas. As we have seen, the computation of the unavailability and the mean rate of system failures involves the inclusion–exclusion formula, specifically in (11), (13), (16) and (18). When the number of components, cut sets or link sets of the system are large, these expressions may involve a very large number of terms. For example, (11) involves $2^n - 1$ terms, which for a system with $n = 100$ components equals the enormously large number 1.268×10^{30}. This computational problem is resolved by noting that if a sum of ordered terms with alternating signs and never increasing in absolute value is truncated after a negative term, then a lower bound to the sum is obtained, and if it is truncated after a positive term, then an upper bound to the sum is obtained. The summation terms with alternating signs in the inclusion–exclusion formula satisfy this property [7]. Thus, as the summation terms in each of the formulas in (11), (13), (16) or (18) are computed, the difference between the last pair of summation terms is monitored. When this difference is sufficiently small, the computation is terminated. In our experience, it is seldom necessary to compute more than 5–6 summation terms in the inclusion–exclusion formula. The expression in (19), on the other hand, does not satisfy this property since its alternating terms may increase in absolute value. A computationally more convenient form of this expression is described in Appendix A.

The second remark relates to the identification of the cut sets or link sets of a system. Although methods for automatic generation of these sets are available (see, e.g., [8]), it may not be practical or feasible to identify all the minimum cut sets or the minimum link sets of the system. For any partial list of the minimum cut sets, (13) and (16) provide lower bound estimates of the system unavailability and the mean failure rate. On the other hand, for any partial list of the minimum link sets, (18) and (19) provide upper bound estimates of the same quantities. Thus, by using both sets of the formulas, one can obtain bounds on the system unavailability and mean failure rate for any partial lists of the minimum cut sets and the minimum link sets. The following example confirms the validity of this and the earlier remark.

Example: Consider the two transmission-line electrical substation system in Fig. 2, which is adopted from Song and Der Kiureghian [9]. The system is composed of 12 components as defined in the figure. It is assumed that the system fails when no connection exists between either of the input lines and either of the output lines. Referring to the component numbers indicated in the figure, the system has the 25 minimum cut sets $\{(1,2), (4,5), (4,7), (4,9), (5,6), (6,7), (6,9), (5,8), (7,8), (8,9), (11,12), (1,3,5), (1,3,7), (1,3,9), (2,3,4), (2,3,6), (2,3,8), (4,10,12), (6,10,12), (8,10,12), (5,10,11), (7,10,11), (9,10,11), (1,3,10,12), (2,3,10,11)\}$ and the eight minimum link sets $\{(1,4,6,8,11),$

(2, 5, 7, 9, 12), (1, 3, 5, 7, 9, 12), (2, 3, 4, 6, 8, 11), (1, 4, 6, 8, 10, 12), (2, 5, 7, 9, 10, 11), (1, 3, 5, 7, 9, 10, 11), (2, 3, 4, 6, 8, 10, 12)}. We consider the mean component failure rates relative to operation time alone and the mean repair durations as listed in Table 1.

The parameters $v_i = \hat{v}_i/(1 + \hat{v}_i\mu_i)$ and μ_i for each component are used in (13) and (16) to compute the system unavailability and mean failure rate in terms of the minimum cut sets, and in (18) and (19) to compute the same in terms of the minimum link sets. The mean duration of system downtime is computed from (17) for both approaches. Figs. 3–5 show the results for P_{sys}, v_{sys} and μ_{sys}, respectively, as functions of the number of cut sets and link sets included in the computation. As can be seen, any pair of subsets of the cut sets and link sets provides bounds on the system unavailability and mean failure rate. Furthermore, estimates for all three system performance measures tend to quickly converge with increasing number of the considered cut and link sets. The converged numerical results obtained from the two formulations coincide and are $P_{\text{sys}} = 0.00224$, $v_{\text{sys}} = 0.0849\,\text{yr}^{-1}$ and $\mu_{\text{sys}} = 0.0264\,\text{yr}$.

To examine the efficiency of the computation with the cut set formulation, in Fig. 6 we show the relative errors in the estimations of P_{sys}, v_{sys} and μ_{sys} from (13), (16) and (17), respectively, as functions of the number of summation terms used in the inclusion–exclusion formulas in (13) and (16). It is evident that the relative error is exceedingly small after just a few summation terms. Furthermore, each sequential pair of terms bounds all three measures.

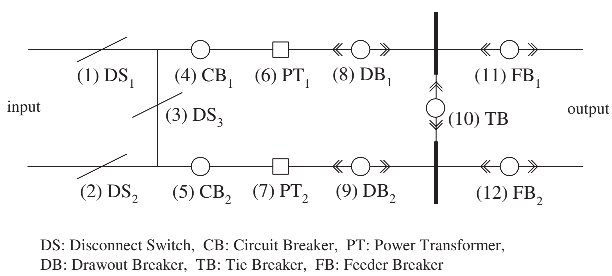

Fig. 2. A two-transmission-line substation system.

Table 1
Component mean failure rates and mean durations of repair

Component	$\hat{v}_i$ (per year)	μ_i (years)
DS_1, DS_2, DS_3	0.2	1/52
CB_1, CB_2	0.4	3/52
PT_1, PT_2	0.1	5/52
DB_1, DB_2	0.4	2/52
TB	0.2	2/52
FB_1, FB_2	0.2	2/52

6. System sensitivities

An important objective in system analysis is to find ways to improve or upgrade the performance of the system in terms of its availability, reliability or duration of downtime. One way to achieve these objectives is to reduce the mean failure rates or the mean durations of repair of selected components. If the objective is to increase the availability of the system, then one tries to minimize P_{sys}. If the role of the system is such that short periods of

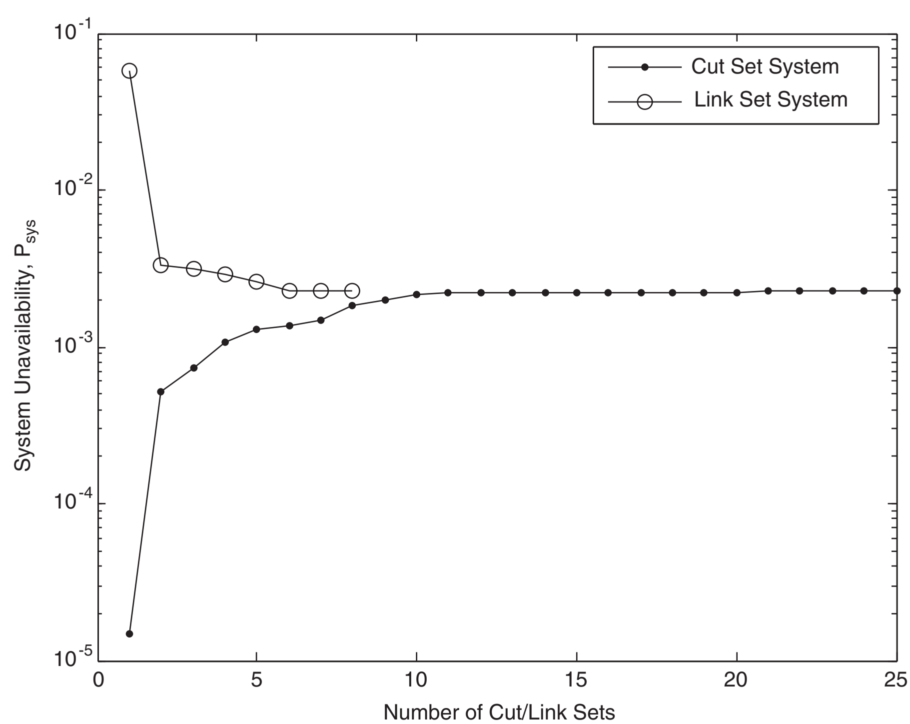

Fig. 3. System unavailability as a function of number of cut/link sets.

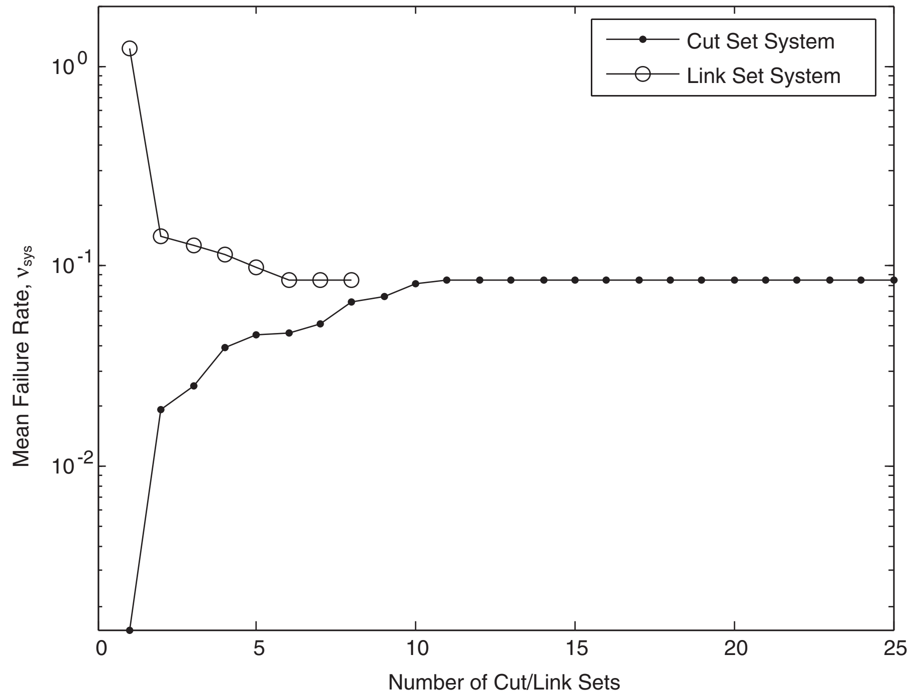

Fig. 4. System mean failure rate as a function of number of cut/link sets.

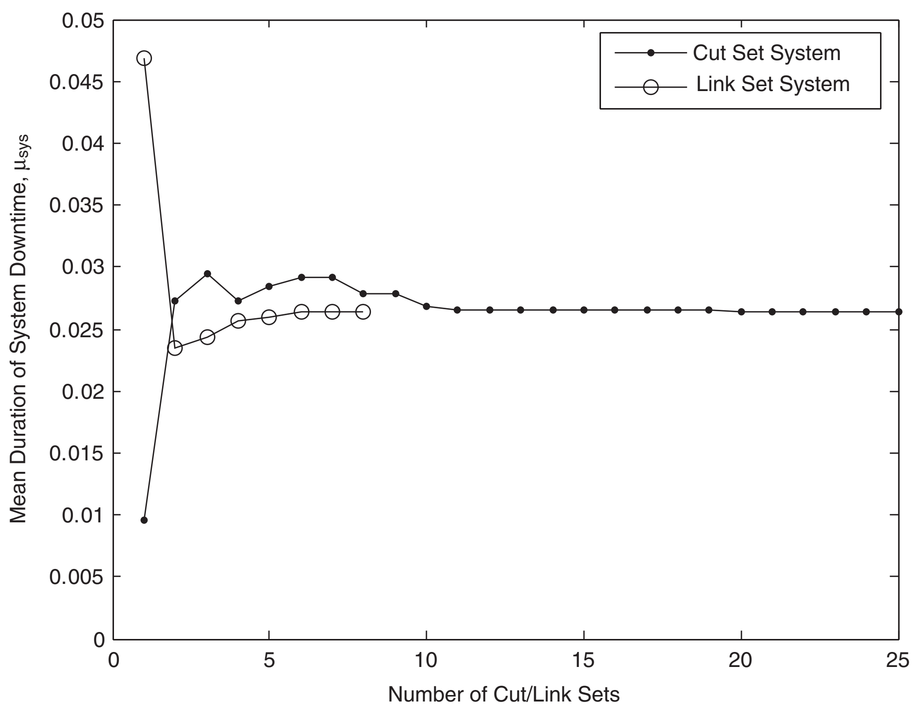

Fig. 5. System mean downtime as a function of number of cut/link sets.

unavailability can be tolerated, but long durations of downtime cannot be tolerated, then one tries to minimize μ_{sys}. On the other hand, if even one downtime is intolerable, then one must minimize $P_{\mathrm{sys}} + \nu_{\mathrm{sys}}\tau$ in order to maximize the lower bound reliability in (20). Of course these quantities are related through (17). Thus, for example, minimizing P_{sys} is equivalent to minimizing the product $\nu_{\mathrm{sys}}\mu_{\mathrm{sys}}$. Clearly, the best strategy for improving one of these measures may not be the best strategy for improving either of the other two.

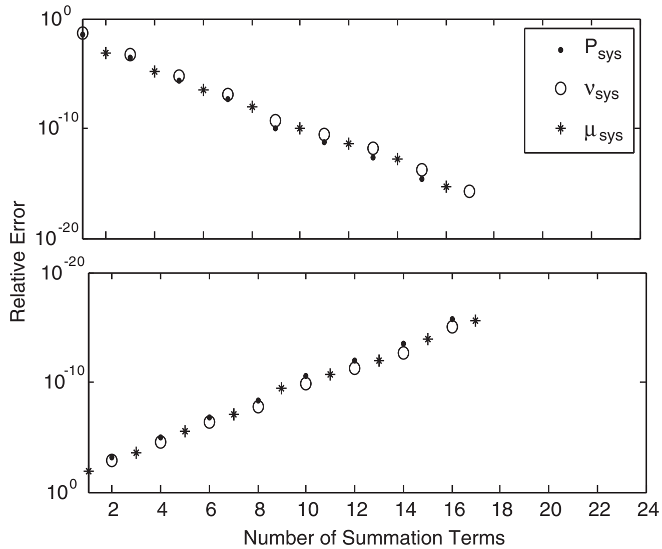

Fig. 6. Relative error as a function of number of summation terms in the inclusion–exclusion formula.

Another objective in system analysis is to reduce operational costs. This objective may be achieved by using cheaper components that have higher failure rates, or by reducing the size of repair crews, which may result in longer repair durations. One is then interested in determining cost-cutting alternatives, which have the least adverse influence on the system performance measures.

If the objectives of enhancing system performance or reducing operational costs are to be achieved by altering the mean failure rates or the mean repair durations of the individual components, the question then arises as to which components to choose for such alteration and how much change to apply to each component parameter, with due consideration of the applicable cost increases or reductions. An important ingredient for such decision making is the rate of change of the performance measures $P_{\rm sys}$, $\nu_{\rm sys}$ and $\mu_{\rm sys}$ with respect to the component parameters ν_i and μ_i, which are commonly known as sensitivities. Since we have closed-form expressions for the system performance measures in terms of these parameters, the sensitivities can be easily derived.

A careful examination of (13), which is based on the cut-set formulation, reveals that, for fixed values of all other parameters, $P_{\rm sys}$ is an affine function of each of the parameters ν_i and μ_i, $i = 1, \ldots, n$. Close examination of (14)–(16) reveals the same to be true for $\nu_{\rm sys}$. On the other hand, it is clear from (17) that $\mu_{\rm sys}$ is a nonlinear function of ν_i and μ_i. Below, we derive expressions for the rates of change of all three measures.

For an index set S, define

$$S^{(a)} = \begin{cases} \emptyset, & a \notin S, \\ S, & a \in S. \end{cases} \tag{21}$$

Also, for an indexed set of nonzero parameters x_i, $i = 1, 2, \ldots$, let $\prod_{i\in\emptyset} x_i = \sum_{i\in\emptyset} x_i = 0$. Using (13), we can then write

$$\begin{aligned}\frac{\partial P_{\rm sys}}{\partial \nu_a} = \frac{1}{\nu_a}\Bigg[&\sum_{j=1}^{r}\Bigg(\prod_{i\in(C_j)^{(a)}} \nu_i\mu_i\Bigg) - \sum_{j=1}^{r-1}\sum_{k=j+1}^{r}\Bigg(\prod_{i\in(C_j\cup C_k)^{(a)}} \nu_i\mu_i\Bigg) \\ &+ \sum_{j=1}^{r-2}\sum_{k=j+1}^{r-1}\sum_{l=k+1}^{r}\Bigg(\prod_{i\in(C_j\cup C_k\cup C_l)^{(a)}} \nu_i\mu_i\Bigg) - \cdots + (-1)^{r-1} \\ &\times \Bigg(\prod_{i\in(C_1\cup\cdots\cup C_r)^{(a)}} \nu_i\mu_i\Bigg)\Bigg], \quad a = 1, \ldots, n,\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\partial P_{\rm sys}}{\partial \mu_a} = \frac{1}{\mu_a}\Bigg[&\sum_{j=1}^{r}\Bigg(\prod_{i\in(C_j)^{(a)}} \nu_i\mu_i\Bigg) - \sum_{j=1}^{r-1}\sum_{k=j+1}^{r}\Bigg(\prod_{i\in(C_j\cup C_k)^{(a)}} \nu_i\mu_i\Bigg) \\ &+ \sum_{j=1}^{r-2}\sum_{k=j+1}^{r-1}\sum_{l=k+1}^{r}\Bigg(\prod_{i\in(C_j\cup C_k\cup C_l)^{(a)}} \nu_i\mu_i\Bigg) - \cdots + (-1)^{r-1} \\ &\times \Bigg(\prod_{i\in(C_1\cup\cdots\cup C_r)^{(a)}} \nu_i\mu_i\Bigg)\Bigg], \quad a = 1, \ldots, n.\end{aligned} \tag{23}$$

Observe that ν_a or μ_a in the denominator of the factor in front of the square brackets cancels out with a similar term in each product so that the resulting derivatives are independent of ν_a and μ_a, respectively. Furthermore, $\partial P_{\rm sys}/\partial\mu_a = (\nu_a/\mu_a) \times (\partial P_{\rm sys}/\partial\nu_a)$, which greatly simplifies the computation. For the mean failure rate, using (16), we have

$$\begin{aligned}\frac{\partial \nu_{\rm sys}}{\partial x} = &\sum_{j=1}^{r}\frac{\partial \nu_{C_j}}{\partial x} - \sum_{j=1}^{r-1}\sum_{k=j+1}^{r}\frac{\partial \nu_{C_jC_k}}{\partial x} + \sum_{j=1}^{r-2}\sum_{k=j+1}^{r-1}\sum_{l=k+1}^{r} \\ &\times \frac{\partial \nu_{C_jC_kC_l}}{\partial x} - \cdots + (-1)^{r-1}\frac{\partial \nu_{C_1\cdots C_r}}{\partial x},\end{aligned} \tag{24}$$

where $x = \nu_a$ or μ_a, $a = 1, \ldots, n$, and a typical partial derivative on the right-hand side is given as follows:

$$\frac{\partial \nu_{C_1\cdots C_r}}{\partial \nu_a} = \begin{cases} 0, & a \notin C_1\cup\cdots\cup C_r, \\ \dfrac{\nu_{C_1\cdots C_r}}{\nu_a}, & a \in C_1\cup\cdots\cup C_r, \end{cases} \tag{25}$$

$$\frac{\partial \nu_{C_1\cdots C_r}}{\partial \mu_a} = \begin{cases} 0, & a \notin C_1\cup\cdots\cup C_r, \\ \dfrac{\nu_{C_1\cdots C_r}}{\mu_a}\left(1 - \dfrac{1}{\mu_a}\dfrac{1}{\sum_{i\in C_1\cup\cdots\cup C_r} 1/\mu_i}\right), & a \in C_1\cup\cdots\cup C_r. \end{cases} \tag{26}$$

$$\frac{\partial P_{sys}}{\partial v_a} = \frac{\mu_a}{1 - v_a\mu_a}\Bigg\{\sum_{j=1}^{s}\Bigg[\prod_{i\in(L_j)^{(a)}}(1 - v_i\mu_i)\Bigg] - \sum_{j=1}^{s-1}\sum_{k=j+1}^{s}\Bigg[\prod_{i\in(L_j\cup L_k)^{(a)}}(1 - v_i\mu_i)\Bigg] + \sum_{j=1}^{s-2}\sum_{k=j+1}^{s-1}\sum_{l=k+1}^{s}\Bigg[\prod_{i\in(L_j\cup L_k\cup L_l)^{(a)}}(1 - v_i\mu_i)\Bigg] - \cdots + (-1)^{s-1}\Bigg[\prod_{i\in(L_1\cup\cdots\cup L_s)^{(a)}}(1 - v_i\mu_i)\Bigg]\Bigg\}, \tag{27}$$

$$\frac{\partial P_{sys}}{\partial \mu_a} = \frac{v_a}{1 - v_a\mu_a}\Bigg\{\sum_{j=1}^{s}\Bigg[\prod_{i\in(L_j)^{(a)}}(1 - v_i\mu_i)\Bigg] - \sum_{j=1}^{s-1}\sum_{k=j+1}^{s}\Bigg[\prod_{i\in(L_j\cup L_k)^{(a)}}(1 - v_i\mu_i)\Bigg] + \sum_{j=1}^{s-2}\sum_{k=j+1}^{s-1}\sum_{l=k+1}^{s}\Bigg[\prod_{i\in(L_j\cup L_k\cup L_l)^{(a)}}(1 - v_i\mu_i)\Bigg] - \cdots + (-1)^{s-1}\Bigg[\prod_{i\in(L_1\cup\cdots\cup L_s)^{(a)}}(1 - v_i\mu_i)\Bigg]\Bigg\} \tag{28}$$

and using (19), we obtain

$$\frac{\partial v_{sys}}{\partial x} = \sum_{j=1}^{s}\frac{\partial v_{L_j}}{\partial x} - \sum_{j=1}^{s-1}\sum_{k=j+1}^{s}\frac{\partial v_{L_j\cup L_k}}{\partial x} + \sum_{j=1}^{s-2}\sum_{k=j+1}^{s-1}\sum_{l=k+1}^{s} \times \frac{\partial v_{L_j\cup L_k\cup L_l}}{\partial x} - \cdots + (-1)^{s-1}\frac{\partial v_{L_1\cup L_2\cup\cdots\cup L_s}}{\partial x}, \tag{29}$$

where $x = v_a$ or μ_a, $a = 1, \ldots, n$, and using (11) a typical partial derivative on the right-hand side of (29) is given as follows:

$$\frac{\partial v_{L_1\cup\cdots\cup L_s}}{\partial v_a} = \begin{cases} 0, & a \notin L_1\cup\cdots\cup L_s, \\ \dfrac{1}{v_a}\Bigg(v_a - \sum\limits_{\substack{j\in L_1\cup\cdots\cup L_s \\ j\neq a}} v_{aj} + \sum\limits_{\substack{j,k\in L_1\cup\cdots\cup L_s \\ j\neq k\neq a}} v_{ajk} - \cdots + (-1)^{m-1} v_{\bigcap_{j\in L_1\cup\cdots\cup L_s} j}\Bigg), & a \in L_1\cup\cdots\cup L_s, \end{cases} \tag{30}$$

$$\frac{\partial v_{L_1\cup\cdots\cup L_s}}{\partial \mu_a} = \begin{cases} 0, & a \notin L_1\cup\cdots\cup L_s, \\ v_a\Bigg(-\sum\limits_{\substack{j\in L_1\cup\cdots\cup L_s \\ j\neq a}} v_{j} + \sum\limits_{\substack{j,k\in L_1\cup\cdots\cup L_s \\ j\neq k\neq a}} v_{jk} - \cdots + (-1)^{m-1} v_{\bigcap_{j\in L_1\cup\cdots\cup L_s,\ j\neq a} j}\Bigg), & a \in L_1\cup\cdots\cup L_s, \end{cases} \tag{31}$$

where m is the number of components in the set $L_1 \cup \cdots \cup L_s$. Consistent with the notation used in (7), subscripts of the form $\bigcap_{j\in S} j$ in the above expressions signify that the corresponding mean rates describe the joint failures of the components indicated in the set S.

Finally, using (17), the derivatives of μ_{sys} are given by

$$\frac{\partial \mu_{sys}}{\partial x} = \frac{1}{v_{sys}}\frac{\partial P_{sys}}{\partial x} - \frac{P_{sys}}{(v_{sys})^2}\frac{\partial v_{sys}}{\partial x}, \tag{32}$$

where $x = v_a$ or μ_a, $a = 1, \ldots, n$.

It is noted that the above sensitivity equations are computed in conjunction with the expressions for P_{sys} and v_{sys} with only a minor additional effort. This effort is far less than that required for computing the sensitivities by finite-difference calculations.

The above sensitivities represent changes in the system performance measures for independent variations in the component parameters v_i and μ_i. In reality, since v_i is defined relative to the total time (including repair durations), independent variations in these parameters cannot be implemented. What we need are the sensitivities for independent variations in the mean failure rate relative to the operation time, $\hat{v}_i$, and μ_i. Using the chain rule of differentiation with the relation $\hat{v}_i = v_i/(1 - v_i\mu_i)$, the desired sensitivities are obtained as

$$\left.\frac{\partial F_{sys}}{\partial \hat{v}_i}\right|_{\mu_i \text{ fixed}} = (1 - v_i\mu_i)^2 \left.\frac{\partial F_{sys}}{\partial v_i}\right|_{\mu_i \text{ fixed}}, \tag{33}$$

$$\left.\frac{\partial F_{sys}}{\partial \mu_i}\right|_{\hat{v}_i \text{ fixed}} = \left.\frac{\partial F_{sys}}{\partial \mu_i}\right|_{v_i \text{ fixed}} - v_i^2 \left.\frac{\partial F_{sys}}{\partial v_i}\right|_{\mu_i \text{ fixed}}, \tag{34}$$

where F_{sys} denotes any of the system performance measures P_{sys}, v_{sys} and μ_{sys}. The derivatives on the right-hand side of (33) and (34) are the ones given in (22)–(32). Hereafter we use the derivatives on the left-hand side while omitting the indicated conditions.

Having determined the sensitivities, a first-order approximation of the variation of each performance measure for any given set of variations $\Delta\hat{v}_i$ and $\Delta\mu_i$, $i = 1, \ldots, n$, in the component parameters is computed by the formula

$$\Delta F_{\mathrm{sys}} \cong \sum_{i=1}^{n} \left(\frac{\partial F_{\mathrm{sys}}}{\partial \hat{v}_i} \Delta\hat{v}_i + \frac{\partial F_{\mathrm{sys}}}{\partial \mu_i} \Delta\mu_i \right). \tag{35}$$

This estimate can be used to compare various upgrading or cost-cutting alternatives, or in a search algorithm to determine the optimal variations in the component parameters.

7. Component importance measures

It is often of interest to identify important components within a system. Various measures of component importance have been defined in the literature [10]. These often relate to the relative contribution of a component to the failure probability of the system. Having closed-form expressions for the rates of change of the three system performance measures with respect to the component parameters, here we define a new set of component importance measures that relate to the upgrading of system performance.

Let $\Delta\hat{v}_i$ and $\Delta\mu_i$, $i = 1, \ldots, n$, respectively, represent variations in the mean failure rate relative to the operation time alone and the mean repair duration of each component that can be achieved at a fixed cost increment. Negative values for these increments represent reductions in $\hat{v}_i$ and μ_i. Note that actions necessary to change $\hat{v}_i$ for a component usually are different from those necessary to change μ_i for the same component. Whereas the former involves changing the reliability of the component, the latter involves changing the processes of repair. So, we assume that actions taken to reduce one of these parameters for a component do not affect the other parameter. As measures of importance of a component, we consider the reductions in the system unavailability, system mean failure rate and system mean duration of downtime that can be achieved by the reductions $\Delta\hat{v}_i$ and $\Delta\mu_i$ in the parameters of the component. Hence, the following three pairs of importance measures are defined:

$$I_{P_{\mathrm{sys}},\hat{v}_i} = -\left(\frac{\partial P_{\mathrm{sys}}}{\partial \hat{v}_i}\right)\Delta\hat{v}_i, \quad I_{P_{\mathrm{sys}},\mu_i} = -\left(\frac{\partial P_{\mathrm{sys}}}{\partial \mu_i}\right)\Delta\mu_i, \tag{36}$$

$$I_{v_{\mathrm{sys}},\hat{v}_i} = -\left(\frac{\partial v_{\mathrm{sys}}}{\partial \hat{v}_i}\right)\Delta\hat{v}_i, \quad I_{v_{\mathrm{sys}},\mu_i} = -\left(\frac{\partial v_{\mathrm{sys}}}{\partial \mu_i}\right)\Delta\mu_i, \tag{37}$$

$$I_{\mu_{\mathrm{sys}},\hat{v}_i} = -\left(\frac{\partial \mu_{\mathrm{sys}}}{\partial \hat{v}_i}\right)\Delta\hat{v}_i, \quad I_{\mu_{\mathrm{sys}},\mu_i} = -\left(\frac{\partial \mu_{\mathrm{sys}}}{\partial \mu_i}\right)\Delta\mu_i. \tag{38}$$

Each of these measures accounts not only for the influence of the component on system performance, but also for the cost of improving the component performance. If the objective is to reduce the system unavailability, then the importance measures in (36) are relevant. If the objective is to increase the system reliability, then the importance measures in (36) and (37) are relevant. Similarly, if reducing the duration of system downtime is essential, then the importance measures in (38) are most relevant. For each objective, the larger the importance measure is, the more important the corresponding component is.

Example: Suppose for the system shown in Fig. 2, the reductions in the mean component failure rates relative to the operation time and mean durations of repair shown in Table 2 can be achieved for a fixed cost. Table 3 lists the importance measures computed by use of (36)–(38). The following observations are noteworthy:

For a fixed-cost upgrading of the system availability, the most important components are CB1 and CB2, closely followed by PT1 and PT2, if a reduction in the component mean failure rate is to be realized, and DB1 and DB2, if a reduction in the component mean repair durations is to be realized. If only one action can be taken, then reducing the mean failure rate of CB1 or CB2 is most profitable.

For a fixed-cost upgrading of the system reliability, the most important components are CB1 and CB2, closely

Table 2
Fixed-cost reductions in component mean failure rates and mean durations of repair

Component	$\Delta\hat{v}_i$ (per year)	$\Delta\mu_i$ (years)
DS_1, DS_2, DS_3	−0.100	−0.010
CB_1, CB_2	−0.100	−0.005
PT_1, PT_2	−0.050	−0.005
DB_1, DB_2	−0.100	−0.010
TB	−0.100	−0.010
FB_1, FB_2	−0.100	−0.010

Table 3
Component importance measures

Component	$I_{P_{\mathrm{sys}},\hat{v}_i}$	$I_{P_{\mathrm{sys}},\mu_i}$	$I_{v_{\mathrm{sys}},\hat{v}_i}$	$I_{v_{\mathrm{sys}},\mu_i}$	$I_{\mu_{\mathrm{sys}},\hat{v}_i}$	$I_{\mu_{\mathrm{sys}},\mu_i}$
DS_1, DS_2	7.62×10^{-6}	7.92×10^{-6}	7.97×10^{-4}	4.16×10^{-4}	-1.58×10^{-4}	-3.60×10^{-5}
DS_3	6.47×10^{-7}	6.73×10^{-7}	7.85×10^{-5}	4.65×10^{-5}	-1.68×10^{-5}	-6.56×10^{-6}
CB_1, CB_2	2.51×10^{-4}	8.69×10^{-5}	8.85×10^{-3}	1.53×10^{-3}	1.95×10^{-4}	5.47×10^{-4}
PT_1, PT_2	2.12×10^{-4}	2.20×10^{-5}	5.94×10^{-3}	3.87×10^{-4}	6.41×10^{-4}	1.39×10^{-4}
DB_1, DB_2	1.68×10^{-4}	1.75×10^{-4}	7.40×10^{-3}	3.08×10^{-3}	-3.23×10^{-4}	1.10×10^{-3}
TB	2.55×10^{-6}	1.32×10^{-6}	1.77×10^{-4}	5.72×10^{-5}	-2.50×10^{-5}	-2.20×10^{-6}
FB_1, FB_2	3.01×10^{-5}	1.57×10^{-5}	1.59×10^{-3}	4.14×10^{-4}	-1.39×10^{-4}	5.55×10^{-5}

followed by DB1 and DB2, if a reduction in the component mean failure rate is to be realized, and DB1 and DB2, if a reduction in the component mean repair durations is to be realized. If only one action can be taken, then reducing the mean failure rate of CB1 or CB2 is most profitable.

For the mean system downtime, several of the importance measures are negative. At first sight this might appear counter-intuitive. However, it is quite possible that a reduction in the mean failure rate or the mean duration of repair of a component result in elongating the mean duration of system downtime. This happens because changing the component parameter results in a different mix of the system downtimes with a larger mean. For example, reducing the mean failure rate of a component with short repair durations may eliminate a number of short-duration system downtimes, such that the remaining mix of system downtimes has a larger mean value. From the results in Table 3, it is clear that only reducing the mean failure rates of CB1, CB2, PT1 and PT2 will reduce the mean system downtime, and among these the latter two are more effective. If the mean component repair times are to be reduced, then components DB_1 and DB_2 are the most important in reducing the mean system downtime. Overall, if only one action can be taken, then reducing the mean repair time of component DB_1 or DB_2 is most profitable.

8. Summary and conclusions

Closed-form expressions are derived for the steady-state availability, mean rate of failure, mean duration of downtime and a lower bound of reliability of a general system with randomly and independently failing repairable components. Both cut-set and link-set formulations of the system are considered. The results are expressed in terms of the mean rates of failure and mean durations of repair of the individual components. Closed-form expressions are also derived for the rates of change of the various probabilistic system performance measures with respect to the mean rate of failure and the mean duration of repair of each component. These expressions provide a convenient framework for identifying important components within the system and for decision-making aimed at upgrading the system performance or reducing operational costs.

Example applications to an electrical substation system demonstrate the use of the formulas developed in the paper. It is shown that any subsets of the minimum cut sets and minimum link sets can be used to derive bounds on the system availability and mean failure rate. Furthermore, the required computations take advantage of the fact that sequential pairs of summation terms in the inclusion–exclusion formulas bound the exact result.

The expressions derived in this paper can be useful in many applications concerned with assessing the performance of systems with repairable components.

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Appendix A

Consider the inclusion–exclusion rule for n complementary events

$$P\left(\bigcup_{i=1}^{n}\bar{E}_i\right)=\sum_{i=1}^{n}P(\bar{E}_i)-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}P(\bar{E}_i\bar{E}_j)+\cdots+(-1)^{n-1}P\left(\bigcap_{i=1}^{n}\bar{E}_i\right). \tag{A.1}$$

Applying the rule of complement and De Morgan's rule to each probability term in (A.1), one obtains

$$\begin{aligned}1-P\left(\bigcap_{i=1}^{n}E_i\right)&=\left[\binom{n}{1}-\sum_{i=1}^{n}P(E_i)\right]-\left[\binom{n}{2}-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}P(E_i\cup E_j)\right]\\&\quad+\cdots+(-1)^{n-1}\left[\binom{n}{n}-P\left(\bigcup_{i=1}^{n}E_i\right)\right].\end{aligned} \tag{A.2}$$

Rearranging the terms,

$$\begin{aligned}P\left(\bigcap_{i=1}^{n}E_i\right)&=\left[\binom{n}{0}-\binom{n}{1}+\binom{n}{2}+\cdots-(-1)^{n-1}\binom{n}{n}\right]\\&\quad+\sum_{i=1}^{n}P(E_i)-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}P(E_i\cup E_j)+\cdots+(-1)^{n-1}P\left(\bigcup_{i=1}^{n}E_i\right)\\&=\sum_{i=1}^{n}P(E_i)-\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}P(E_i\cup E_j)+\cdots+(-1)^{n-1}P\left(\bigcup_{i=1}^{n}E_i\right),\end{aligned} \tag{A.3}$$

where the summation inside square brackets is zero. This expression, which is analogous to the inclusion–exclusion rule with the union and intersection operations exchanged, may be regarded as an "inverse" inclusion–exclusion rule.

Now observe that in terms of link sets, the system is a parallel system. Let $\nu_{L_1L_2\cdots L_k}$ denote the mean rate of joint failures of the link sets L_j, $j = 1, \dots, k$. Also observe that for an infinitesimal time interval Δt the product $v\Delta t$, where v denotes a constant mean rate of occurrence, can be treated as the probability of occurrence within Δt. Thus, the inverse inclusion–exclusion rule in (A.3) also applies to mean occurrence rates. For the system with s minimum link sets, this leads to (19) in the main body of the paper.

A computationally more convenient form of (19), which is motivated by the inclusion–exclusion formula in (A.2), is

$$\begin{aligned} v_{\text{sys}} = \frac{1}{\Delta t} - &\left[\binom{n}{1} - \Delta t \sum_{j=1}^{s} v_{L_j}\right] \\ &+ \left[\binom{n}{2} - \Delta t \sum_{j=1}^{s-1}\sum_{k=j+1}^{s} v_{L_j \cup L_k}\right] \\ &- \left[\binom{n}{3} - \Delta t \sum_{j=1}^{s-2}\sum_{k=j+1}^{s-1}\sum_{l=k+1}^{s} v_{L_j \cup L_k \cup L_l}\right] \\ &+ \cdots (-1)^{s-1}\left[\binom{n}{n} - \Delta t v_{L_1 \cup L_2 \cup \cdots \cup L_s}\right], \end{aligned} \tag{A.4}$$

where Δt is an arbitrary parameter. The bounding property of the inclusion–exclusion formula described in the paragraph following (20) applies to the square-bracketed terms in this expression. That is, truncation of the series at each sequential pair of bracketed terms bounds v_{sys}. For best convergence, it is appropriate to select Δt such that the very last term in the series vanishes, i.e., set $\Delta t = 1/v_{L_1 \cup L_2 \cup \cdots \cup L_s}$.

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