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DEGRADABLE TRANSPORTATION SYSTEMS: SENSITIVITY AND RELIABILITY ANALYSIS

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Abstract—This paper describes sensitivity analysis for a degradable transportation system, based on an integrated equilibrium model with elastic travel demand, to identify critical components and assist efforts to improve system reliability. A reliability model, involving practical measures of reliability, is also described. Algorithms for solving the reliability model are discussed. © 1997 Elsevier Science Ltd

1. INTRODUCTION

1.1. The integrated equilibrium model

Considering a transportation system as a degradable transportation system (DTS), in which the component state (arc capacities), system state and system performance can be degraded by a variety of events (e.g. earthquakes, floods, traffic accidents, industrial action, etc.), an integrated steady-state equilibrium model for predicting macroscopic traffic behaviour and evaluating the performance of a multi-modal DTS (including road, rail, water and air transport) has been developed (Du and Nicholson, 1993, 1996). This model allows for the network structure of a transportation system to change, with different degrees of arc capacity degradation, resulting in different degrees of system performance degradation.

The model assumes that each individual user chooses a path to minimize their total generalized cost of travel, in accordance with Wardrop's user-optimum principle. The model also assumes that the level of traffic flow between each origin–destination (or OD) pair is a direct outcome of the interaction between demand and supply (i.e. the traffic demand is elastic). The model formulation combines trip generation, trip distribution, mode split and traffic assignment, and is equivalent to a concave programming problem, involving finding the path flows for which the system surplus (the sum of the user and producer surpluses) is maximized.

The basic solution (the vector of path flows) is not unique, but is generally contained in a convex set. For the analysis of the socio-economic impacts of degradation, however, only the system state vector (comprising the OD flow and arc flow vectors) is essential, and this is unique under some weak assumptions about the monotonicity and differentiability of the traffic demand functions and travel time functions.

1.2. Sensitivity analysis

The modelling of degradable transportation systems has two goals, namely, identifying the effects on the performance of the system of both events (e.g. earthquakes, floods, etc.) and changes to the characteristics of the transportation system. The first goal involves assessing system performance for various combinations of component capacity degradation.

One option for identifying the effects of component degradation on the system state and system performance is to solve the integrated equilibrium model for numerous combinations of the component state variables, but this would require a substantial computational effort for anything other than a very small network. An alternative is to use an analytical sensitivity analysis, involving obtaining expressions for the derivatives of the system state vector and system performance index with respect to the component state vector (Frank, 1978).

The existing results of sensitivity analysis for general nonlinear programming (Fiacco, 1976, 1983) cannot be used directly, because the decision variables (the path flows) in the concave programming problem are not unique. Dafermos and Nagurney (1984a,b) studied the problem of predicting the direction of the change in the traffic pattern and the incurred travel costs, resulting from changes in the travel cost functions and the travel demands, but they did not obtain the relative derivatives. Tobin and Friesz (1988) studied sensitivity analysis for equilibrium network flow, assuming fixed demand and calculating the sensitivity of the arc flow vector, and their approach is extended to the analysis of the sensitivity of the system state vector and the system performance index with respect to the component state vector, based on the integrated equilibrium model with elastic demands.

1.3. Reliability analysis

In reliability engineering, the reliability of an individual component is generally equated to the probability of its surviving an event, with the system reliability being the probability if the system surviving (a function of the component reliabilities). For a DTS, there are generally more than two component states to be considered. In addition, components generally have some excess capacity, so that some capacity degradation will not affect noticeably the performance of the component and the system. The definitions of component and system reliability for a DTS need to take account of these matters.

With the classical graph theory approach, the focus is upon arc and OD pair connectivity, and the network flows are not considered. For the usual flow network reliability models with fixed demand, the focus is upon arc capacity and whether it is below some threshold (i.e. the arc has failed). When analysing the reliability of a DTS, however, the focus is upon the effects of arc degradation on the arc, OD and system flows. Particular emphasis is placed upon the OD flows, as the benefits of travel are associated with successfully completing trips, and Neuburger (1971) argues that one should consider complete journeys from particular origins to particular destinations, when assessing the socio-economic effects of changes to the transportation system.

A major issue in reliability analysis is how to calculate the network reliability when the probabilities of component capacity degradation are known. There are numerous network reliability algorithms (Hwang *et al.*, 1981; Politof and Satyanarayana, 1986), which can be divided into two broad categories: exact and approximate methods.

Exact methods include enumeration, simulation (Kumamoto *et al.*, 1977), Boolean algebra (Fratta and Montanari, 1973), fault trees (Page and Perry, 1986) and decomposition (Rushdi, 1984), but many of these methods have been proven or are suspected to lead to computationally complex problems (Ball, 1986). For example, it is required to identify and evaluate every minimal path (or cut-set) or to obtain flows for all the possible component state vectors when the enumeration approach is used to calculate the network reliability. The exact methods are efficient only for those networks which have very regular structures or are very small.

One of the main emphases in the development of approximate methods has been to compute the upper and lower bounds of the reliability measure (Provan, 1986). For example, one may study a subset of the component state vector space, so as to cover a large fraction of the component state probability space and obtain bounds on the reliability measure.

2. SENSITIVITY ANALYSIS

2.1. Selection of a unique path flow vector

The integrated equilibrium model involves finding the path flow vector $\boldsymbol{q}$ which minimizes the system surplus, which is a function of the OD flow vector $\boldsymbol{f}$, the arc flow vector $\boldsymbol{v}$ and the component state vector $\boldsymbol{x}$. The OD flow and arc flow vectors are functions of the path flow vector $\boldsymbol{q}$, which is a function of the component state vector $\boldsymbol{x}$, and the system surplus is therefore generally denoted as $SS(\boldsymbol{q},\boldsymbol{x})$.

The integrated equilibrium model also involves a mapping from the component state vector space to the system state vector space, and then to the system performance index space. For a given component state vector $\boldsymbol{x}$, the OD flow vector $\boldsymbol{f}(\boldsymbol{x})$, the arc flow vector $\boldsymbol{v}(\boldsymbol{x})$ and the system surplus $SS(\boldsymbol{x})=SS[\boldsymbol{q}(\boldsymbol{x}),\boldsymbol{x}]$ are unique, so that while the path flow vector $\boldsymbol{q}(\boldsymbol{x})$ is not unique, the alternative path flow vectors lead to the same OD flow and arc flow vectors and system surplus (Nicholson and Du, 1997). Hence, it does not matter which path flow vector is used when determining the sensitivity of the system state vector and the system performance index with respect to the component state vector. All that is required is a method for selecting a path flow vector uniquely, and this can be done by solving the following quadratic programming problem: find $\boldsymbol{q}(\boldsymbol{x}) = [q_h(\boldsymbol{x})|h \in \boldsymbol{H}]$ to minimize

$$SEN[\mathbf{q}(\mathbf{x})] = \sum_{h \in \mathbf{H}} [q_h(\mathbf{x})]^2$$

such that

$$\mathbf{q}(\mathbf{x}) \in \Gamma(\mathbf{x}) = [\mathbf{q}(\mathbf{x})|\mathbf{Zq}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \mathbf{q}(\mathbf{x}) = \pi \mathbf{v}(\mathbf{x}), \mathbf{q}(\mathbf{x}) \geq \mathbf{0}]$$

where $\mathbf{Z}$ and Ξ are the OD-path and arc-path incidence matrices, respectively, and

$$\pi = diag[\rho_1 \mu_1, \ldots, \rho_A \mu_A],$$

where μ_A and ρ_A are the average vehicle capacity and average loading ratio for arc a, respectively. Since $SEN[\boldsymbol{q}(\boldsymbol{x})]$ is continuous and strictly convex on the closed convex set $\Gamma(\boldsymbol{x})$, then the solution $\boldsymbol{q}^*(\boldsymbol{x})$ is unique, and is a solution of the integrated equilibrium model (Nicholson and Du, 1997).

2.2. Stability and unchangeability of zero-elements of selected path flow vector

The system state vector and the system surplus are both continuous functions of the component state vector, and they are thus stable with respect to small perturbations in the component state vector. To obtain the derivatives of the system state and system performance index with respect to the component state, it is necessary that the selected path flow vector $\boldsymbol{q}^*(\boldsymbol{x})$ be stable with respect to small perturbations in the component state vector. It can be shown (Du and Nicholson, 1993) that $\boldsymbol{q}^*(\boldsymbol{x})$ is a continuous function of the component state vector $\boldsymbol{x}$, and thus satisfies the stability requirement.

Since the integrated equilibrium model involves traffic assignment in accordance with the user equilibrium principle (Wardrop, 1952), it follows that if two index sets are defined as follows:

$$\mathbf{Y}_+(\mathbf{x}) = [h|q_h^*(\mathbf{x}) > 0] \quad \text{and} \quad \mathbf{Y}_0(\mathbf{x}) = [h|q_h^*(\mathbf{x}) = 0]$$

where

$$\mathbf{Y}_+(\mathbf{x}) \cup \mathbf{Y}_0(\mathbf{x}) = \mathbf{H},$$

then

$$\mathbf{Y}_+(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{Y}_+(\mathbf{x}) \quad \text{and} \quad \mathbf{Y}_0(\mathbf{x} + \Delta\mathbf{x} = \mathbf{Y}_0(\mathbf{x}),$$

showing the unchangeability of the zero elements of the selected path flow vector $\boldsymbol{q}^*(\boldsymbol{x})$ under infinitesimal perturbation of the component state vector $\boldsymbol{x}$.

2.3. Partial derivatives of selected path flow vector

Since the elements in the index set $\boldsymbol{Y}_0(\boldsymbol{x})$ are unchangeable under infinitesimal perturbation of $\boldsymbol{x}$, the partial derivatives of those elements with respect to $\boldsymbol{x}$ are zero, that is $\partial q_h^*/\partial x_a = 0$ for $h \in \boldsymbol{Y}_0(\boldsymbol{x})$ and $a \in \boldsymbol{A}$. Since $\boldsymbol{q}^*(\boldsymbol{x})$ maximizes the system surplus $SS(\boldsymbol{q},\boldsymbol{x})$, then for elements in the index set $\boldsymbol{Y}_+(\boldsymbol{x})$, the derivatives of the system surplus with respect to $\boldsymbol{q}^*(\boldsymbol{x})$ must be zero. Hence it can be shown (Du and Nicholson, 1993) that the partial derivatives $\partial q_h^*/\partial x_a$ for $h \in \boldsymbol{Y}_+(\boldsymbol{x})$ and $a \in \boldsymbol{A}$ are given by the matrix equation

$$\{\nabla_{\mathbf{x}} \mathbf{q}^*(\mathbf{x})\}_{Q \times A} = -\{\nabla_{\mathbf{q}}^2 SS[\mathbf{q}^*(\mathbf{x}), \mathbf{x}]\}_{Q \times Q}^{-1} \{\nabla_{\mathbf{qx}}^2 SS[\mathbf{q}^*(\mathbf{x}), \mathbf{x}]\}_{Q \times A} \{\nabla_{\mathbf{x}} \mathbf{x}\}_{A \times A}, \tag{1}$$

where $\{\nabla_{\mathbf{x}} \mathbf{q}^*(\mathbf{x})\}_{Q \times A}$, $\{\nabla_{\mathbf{q}}^2 SS[\mathbf{q}^*(\mathbf{x}), \mathbf{x}]\}_{Q \times Q}$, $\{\nabla_{\mathbf{qx}}^2 SS[\mathbf{q}^*(\mathbf{x}), \mathbf{x}]\}_{Q \times A}$ and $\{\nabla_{\mathbf{x}} \mathbf{x}\}_{A \times A}$ are matrices of $\partial q_h^*(\mathrm{x})/\partial \mathrm{x_a}$, $\partial^2 SS(\mathbf{q}, \mathrm{x})/(\partial \mathrm{q_r} \partial \mathrm{q_h})$, $\partial^2 SS(\mathbf{q}, \mathrm{x})/(\partial \mathrm{q_h} \partial \mathrm{x_a})$, and $\partial x_b/\partial x_a$ (for $\boldsymbol{q} = \boldsymbol{q}^*(\boldsymbol{x})$, $r,h \in \boldsymbol{Y}_+(\boldsymbol{x})$ and $a,b \in \boldsymbol{A}$), respectively, and Q and A are the number of elements in $\boldsymbol{Y}_+(\boldsymbol{x})$ and $\boldsymbol{A}$, respectively.

2.4. Partial derivatives of system state and performance index

Du and Nicholson have also shown that the partial derivatives of the system state vector and system performance index with respect to the component state vector are

$$\frac{\partial f_k(\mathbf{x})}{\partial x_a}=\sum_{h\in \mathbf{Y}_+(\mathbf{x})}\zeta_{kh}\frac{\partial q_h^*(\mathbf{x})}{\partial x_a}\quad k\in \mathbf{K}, a\in \mathbf{A} \tag{2}$$

$$\frac{\partial v_b(\mathbf{x})}{\partial x_a}=\frac{1}{\rho_b\mu_b}\sum_{h\in \mathbf{Y}_+(\mathbf{x})}\xi_{bh}\frac{\partial q_h^*(\mathbf{x})}{\partial x_a}\quad b\in \mathbf{A}, a\in \mathbf{A} \tag{3}$$

$$\frac{\partial SS(\mathbf{x})}{\partial x_a}=-\varphi\sum_{b\in \mathbf{A}}\rho_b\mu_b\frac{\partial x_b}{\partial x_a}\int_0^{v_b(\mathbf{x})}\frac{\partial t_b(v_b,x_b)}{\partial x_b}\,\mathrm{d}v_b\quad a\in \mathbf{A}, \tag{4}$$

where ζ_{kh} and ξ_{bh} are the elements of $\mathbf{Z}$ and Ξ, respectively.

Equations (1–4) for the partial derivatives all contain the partial derivative of x_b with respect to x_a. This partial derivative will be:

1. zero if the component state variables x_b and x_a are completely independent (i.e. x_b is not a function of x_a), except when a equals b, when it will be unity;
2. non-zero if the component state variables x_b and x_a are related (the value will depend upon the precise form of the relationship), and unity when a equals b.

Hence the component state variables can be treated as either independent or inter-related.

First-order approximations of the system state and performance index after a small change $\Delta\mathbf{x}$ in the component state vector x can readily be obtained as follows:

$$f_k(\mathbf{x}+\Delta\mathbf{x})\doteq f_k(\mathbf{x})+\sum_{b\in \mathbf{A}}\frac{\partial f_k(\mathbf{x})}{\partial x_b}\Delta x_b\quad k\in \mathbf{K} \tag{5}$$

$$v_a(\mathbf{x}+\Delta\mathbf{x})\doteq v_a(\mathbf{x})+\sum_{b\in \mathbf{A}}\frac{\partial v_a(\mathbf{x})}{\partial x_b}\Delta x_b\quad a\in \mathbf{A} \tag{6}$$

$$SS(\mathbf{x}+\Delta\mathbf{x})\doteq SS(\mathbf{x})+\sum_{b\in \mathbf{A}}\frac{\partial SS(\mathbf{x})}{\partial x_b}\Delta x_b. \tag{7}$$

3. RELIABILITY ANALYSIS

3.1. OD sub-system reliability

The component state vector x, comprising the arc capacities, can vary within the component state vector space X, which includes the non-degraded component state vector x_0. Then comparing $f_k(x)$ and $f_k(x_0)$, which are the traffic flows in the k^{th} OD sub-system for the component state vector x and x_0 respectively, the decrement rate of traffic flow is defined as

$$y_k(\mathbf{x})=\frac{f_k(\mathbf{x}_0)-f_k(\mathbf{x})}{f_k(\mathbf{x}_0)}\quad k\in \mathbf{K}. \tag{8}$$

The flow decrement rate for the k^{th} OD sub-system can vary between zero (i.e. no degradation) and unity (i.e. degradation is so severe that the sub-system flow is zero).

If the maximum acceptable flow decrement rate for the k^{th} OD sub-system is θ_k $(0<\theta_k<1)$, then the operating/failed function is

$$z_k(\theta_k,\mathbf{x})=\begin{cases}1 \text{ if } y_k(\mathbf{x})\le\theta_k\\ 0 \text{ if } y_k(\mathbf{x})>\theta_k\end{cases}\quad k\in \mathbf{K}. \tag{9}$$

That is, the k^{th} OD sub-system is considered operating for component state vector x if $z_k(\theta_k,x)=1$ and failed for $z_k(\theta_k,x)=0$.

The reliability of the k^{th} OD sub-system is then defined as

$$R_k(\theta_k) = Pr[\mathbf{x}|\mathbf{x} \in \mathbf{X}, z_k(\theta_k, \mathbf{x}) = 1] \quad k \in \mathbf{K}. \tag{10}$$

3.2. System reliability

If the total traffic flow in the whole system for the component state vector x is

$$F(\mathbf{x}) = \sum_{k\in\mathbf{K}} f_k(\mathbf{x}) \tag{11}$$

then comparing the traffic flows in the system for the component state vectors x and x_0, the flow decrement rate for the system is defined as

$$y(\mathbf{x}) = \frac{F(\mathbf{x}_0) - F(\mathbf{x})}{F(\mathbf{x}_0)}. \tag{12}$$

The flow decrement rate for the system can also vary between zero (i.e. no degradation) and unity (i.e. degradation is so severe that the system flow is zero).

It can be shown that the system flow decrement rate is a flow-weighted average of the sub-system flow decrement rates, that is,

$$y(\mathbf{x}) = \sum_{k\in\mathbf{K}} \nu_k(\mathbf{x}_0) y_k(\mathbf{x}), \tag{13}$$

where

$$\nu_k(\mathbf{x}_0) = \frac{f_k(\mathbf{x}_0)}{F(\mathbf{x}_0)} \quad k \in \mathbf{K}. \tag{14}$$

If the maximum acceptable system flow decrement rate is a flow-weighted average of the maximum acceptable sub-system flow decrement rates, that is,

$$\theta = \sum_{k\in\mathbf{K}} \nu_k(\mathbf{x}_0)\theta_k, \tag{15}$$

then the operating/failed function is

$$z(\theta, \mathbf{x}) = \begin{cases} 1 & \text{if } y(\mathbf{x}) \le \theta \\ 0 & \text{if } y(\mathbf{x}) > \theta \end{cases} \tag{16}$$

That is, the system is considered operating for component state vector x if $z(\theta,x)=1$ and failed for $z(\theta,x)=0$.

The reliability of the system is then defined as

$$R(\theta) = Pr[\mathbf{x}|\mathbf{x} \in \mathbf{X}, z(\theta, \mathbf{x}) = 1]. \tag{17}$$

3.3. An exact algorithm

The integrated equilibrium model and sensitivity analysis are based on the component state vector space X being continuous, which in reality it is, as the capacity of each arc can vary continuously. For a reliability analysis, however, it is impractical to consider X as continuous, and it is approximated by a discrete component vector space; the greater the number of discrete component states, the better the approximation will be.

If p_s is the probability of the s^{th} component state vector x_s and W is the number of component state vectors in the discrete component state vector space X, then

$$R_k(\theta_k) = \sum_{s=0}^{W} p_s z_k(\theta_k, \mathbf{x}_s) \quad k \in \mathbf{K} \tag{18}$$

and

$$R(\theta) = \sum_{s=0}^{W} p_s z(\theta, \mathbf{x}_s). \tag{19}$$

These reliabilities can be calculated by first enumerating all possible component state vectors x_s and probabilities p_s, $(s = 0,\cdots,W)$ and then computing the traffic flows $f_k(x_s)$ and $F(x_s)$ by solving the integrated equilibrium model for each x_s. The decrement rates $y_k(x_s)$ and $y(x_s)$, and the operating/failed functions $z_k(\theta_k,x_s)$ and $z(\theta,x_s)$, should then be calculated for each x_s. The reliabilities $R_k(\theta_k)$ and $R(\theta)$ should then be computed.

3.4. An approximating algorithm

When the number of component states is large the exact algorithm is impractical, as it entails solving the integrated equilibrium model a very large number of times. In addition, some of the component state vectors may have very small probabilities, and will contribute very little to the reliability estimates. If the component state vectors x_s, $s = 0,\cdots,W$ are arranged such that

$$p_{s_0} \geq p_{s_1} \geq p_{s_2} \geq \cdots \geq p_{s_W},$$

then the cumulative probability

$$p(J) = \sum_{j=0}^{J-1} p_{s_j} \quad J \in [1, \ldots, W+1]$$

is the largest sum of probabilities of any J component state vectors in the space X.

Using the method of Li and Silvester (1984), the reliabilities for a given J are within the ranges

$$\begin{cases} \underline{B}_{R_k(\theta_k)}(J) \leq R_k(\theta_k) \leq \overline{B}_{R_k(\theta_k)}(J) & k \in \mathbf{K} \\ \underline{B}_{R(\theta)}(J) \leq R(\theta) \leq \overline{B}_{R(\theta)}(J) \end{cases}$$

where the lower and upper bounds for the k^{th} sub-system are

$$\begin{cases} \underline{B}_{R_k(\theta_k)}(J) = \sum_{j=0}^{J-1} p_{s_j} z_k(\theta_k, \mathbf{x}_{s_j}) + [1 - p(J)] z_k(\theta_k, \mathbf{x}_W) \\ \overline{B}_{R_k(\theta_k)}(J) = \sum_{j=0}^{J-1} p_{s_j} z_k(\theta_k, \mathbf{x}_{s_j}) + [1 - p(J)] z_k(\theta_k, \mathbf{x}_0) \end{cases} \quad k \in \mathbf{K}$$

and for the whole system are

$$\begin{cases} \underline{B}_{R(\theta)}(J) = \sum_{j=0}^{J-1} p_{s_j} z(\theta, \mathbf{x}_{s_j}) + [1 - p(J)] z(\theta, \mathbf{x}_W) \\ \overline{B}_{R(\theta)}(J) = \sum_{j=0}^{J-1} p_{s_j} z(\theta, \mathbf{x}_{s_j}) + [1 - p(J)] z(\theta, \mathbf{x}_0) \end{cases}$$

If the estimated reliabilities equal the true reliabilities plus estimation errors, that is

$$\begin{cases} M_{R_k(\theta_k)}(J) = R_k(\theta_k) + \varepsilon_{R_k(\theta_k)}(J) & k \in \mathbf{K} \\ M_{R(\theta)}(J) = R(\theta) + \varepsilon_{R(\theta)}(J) \end{cases}$$

then the estimated reliabilities and the upper bounds of the absolute errors can be calculated using the recursion formulae:

$$\begin{cases} M_{R_k(\theta_k)}(0) = \frac{1}{2}[1 + z_k(\theta_k, \mathbf{x}_W)] \\ M_{R_k(\theta_k)}(J) = M_{R_k(\theta_k)}(J-1) + p_{s_{J-1}}\left[z_k(\theta_k, \mathbf{x}_{s_{J-1}}) - M_{R_k(\theta_k)}(0)\right] \quad J = 1, \ldots, W+1 \end{cases} \quad k \in \mathbf{K} \tag{20}$$

$$\begin{cases} M_{R(\theta)}(0) = \frac{1}{2}[1 + z(\theta, \mathbf{x}_W)] \\ M_{R(\theta)}(J) = M_{R(\theta)}(J-1) + p_{s_{J-1}}\left[z_k(\theta, \mathbf{x}_{s_{J-1}}) - M_{R(\theta)}(0)\right] \quad J = 1, \ldots, W+1 \end{cases} \tag{21}$$

$$\begin{cases} \bar{\varepsilon}_{R_k(\theta_k)}(0) = \frac{1}{2}[1 - z_k(\theta_k, \mathbf{x}_W)] \\ \bar{\varepsilon}_{R_k(\theta_k)}(J) = \bar{\varepsilon}_{R_k(\theta_k)}(J-1) - p_{s_{J-1}}\bar{\varepsilon}_{R_k(\theta_k)}(0) \quad J = 1, \ldots, W+1 \end{cases} k \in \mathbf{K} \tag{22}$$

$$\begin{cases} \bar{\varepsilon}_{R(\theta)}(0) = \frac{1}{2}[1 - z(\theta, \mathbf{x}_W)] \\ \bar{\varepsilon}_{R(\theta)}(J) = \bar{\varepsilon}_{R(\theta)}(J-1) - p_{s_{J-1}}\bar{\varepsilon}_{R(\theta)}(0) \quad J = 1, .., W+1 \end{cases} \tag{23}$$

To compute the sub-system and system reliabilities, one should proceed as follows:

1. determine the probabilities of the various levels of degradation for each arc in the system, the maximum acceptable sub-system flow decrement rates θ_k for each sub-system (i.e. $k \in \boldsymbol{K}$) and the maximum acceptable values of the estimation errors for the sub-system and system reliabilities;
2. solve the integrated equilibrium model to obtain the OD flows $f_k(\boldsymbol{x}_0)$, $k \in \boldsymbol{K}$, and total system flow $F(\boldsymbol{x}_0)$ for the non-degraded component state vector $\boldsymbol{x}_0$, and calculate the maximum acceptable system flow decrement θ, from eqns (13–15);
3. determine the probability of each component state vector and rearrange the component state vectors in a descending order of probability;
4. starting with the most probable degraded component state vector $\boldsymbol{x}$, solve the integrated equilibrium model to obtain the OD flows $f_k(\boldsymbol{x})$, $k \in \boldsymbol{K}$;
5. compute the sub-system flow decrement rates $y_k(\boldsymbol{x})$ and the operating/failed functions $z_k(\theta_k, \boldsymbol{x})$, $k \in \boldsymbol{K}$, using eqns (8) and (9), respectively;
6. compute the system flow decrement rate $y(\boldsymbol{x})$ and the operating/failed function $z(\theta, \boldsymbol{x})$, using eqns (12) and (16), respectively;
7. by using eqns (20–24), obtain estimates of the reliabilities and the upper bounds of the absolute values of the estimation errors;
8. check that all of the error upper bounds (i.e. for all sub-systems and the whole system) are not greater than the corresponding maximum acceptable errors (step (1) above); if not, then return to step (4), using the second, third, etc. most probable component state vector, until all of the error upper bounds are acceptable.

This process can be shown to be convergent, but an increase in the precision of the reliability estimates can be achieved only with an increase in computation time, and the maximum acceptable errors should be set to achieve an appropriate compromise between precision and computation time.

4. A NUMERICAL EXAMPLE

4.1. *Problem description*

Consider the case of travel from one town (zone i) to the nearest other town in the region (zone j), where the towns are connected by two arcs, *1* and *2*, for which the travel time functions are of the form (Davidson, 1966)

$$t_a = t_a^0\left[\frac{1 - (1 - j_a)(v_a/x_a)}{1 - (v_a/x_a)}\right],$$

where j_a is a constant ($0 < j_a \leq 1$) and t_a^0 is the travel time at zero flow. Let the number of trips f_{ij} from zone i to zone j be elastic and depend upon the generalized cost c_{ij} according to the power function:

$$f_{ij} = \tau_{ij}(\alpha_{ij}/c_{ij})^{\beta} = L_{ij}/(c_{ij})^{\beta}$$

where τ_{ij} and α_{ij} are the maximum demand flow and the minimum generalized cost of travel from zone i to zone j, respectively, and β is the elasticity parameter ($\tau_{ij} \geq f_{ij} > 0$, $c_{ij} \geq \alpha_{ij} > 0$, $\beta > 0$). These travel time and demand functions satisfy all the assumptions of the integrated equilibrium model (Nicholson and Du, 1997).

Let $L_{ij} = 6\times 10^6$, $\beta = 2.5$, $\varphi = \$20/\text{h}$, and the link characteristics be as shown in Table 1, say, and let the average vehicle capacities (μ_1 and μ_2) and vehicle loading proportions (ρ_1 and ρ_2) be such that $\rho_1\mu_1 = \rho_2\mu_2 = 1$. The total flow f_{ij} for each of 36 combinations of arc capacities, calculated using the integrated equilibrium model, is shown in Table 2.

4.2. Sensitivity estimation

For the above travel time function

$$\int_0^{v_a} \frac{\partial t_a}{\partial x_a}\, \mathrm{d}v_a = -t_a^0 j_a\left[\ln(1 - r_a) + \frac{r_a}{1 - r_a}\right] = -t_a^0 j_a\left[\frac{1}{2}r_a^2 + \frac{2}{3}r_a^3 + \frac{3}{4}r_a^4 + \ldots\right]$$

where $r_a = v_a/x_a$ $(0 \leq r_a \leq 1)$. Substituting into eqn (4), the partial derivative of the system surplus with respect to the component state is

$$\frac{\partial SS(\mathbf{x})}{\partial x_a} = \varphi \sum_{b\in \mathrm{A}} \rho_b \mu_b t_b^0 j_b\left[\frac{1}{2}r_b^2 + \frac{2}{3}r_b^3 + \frac{3}{4}r_b^4 + \ldots\right]\frac{\partial x_b}{\partial x_a} \quad \mathrm{A} = \{1, 2\}. \tag{24}$$

Clearly this partial derivative is positive, indicating that for the chosen travel time function, the system performance will always decline when the capacity of an arc used by traffic is degraded by a small amount, except when the arc capacities are related such that $\partial x_b/\partial x_a$ is negative. This is extremely unlikely, as it implies that the capacity of one arc increases as the capacity of the other arc decreases.

For the non-degraded state (i.e. $x_1 = 10{,}000$ and $x_2 = 5000$), the integrated equilibrium model gives a total flow of 12,294 veh/h, with arc flows v_1 and v_2 equal to 8304 and 3990 respectively, and arc flow/capacity ratios r_1 and r_2 equal to 0.8304 and 0.7980, respectively. Substituting into eqn (24) gives

$$\frac{\partial SS(\mathbf{x})}{\partial x_1} = 3.1219 + 4.7020\frac{\partial x_2}{\partial x_1},$$

$$\frac{\partial SS(\mathbf{x})}{\partial x_2} = 4.7020 + 3.1219\frac{\partial x_1}{\partial x_2},$$

and, if the component states are independent, then the terms $\partial x_2/\partial x_1$ and $\partial x_1/\partial x_2$ are zero and the partial derivatives become 3.1219 and 4.7020, respectively, indicating that the system surplus is

Table 1. Arc characteristics

	Arc 1	Arc 2
Toll ($)	2	0
Free-flow travel time (h)	0.25	0.2
Travel time function j-factor	0.2	0.5
Non-degraded capacity (veh/h)	10,000	5000

Table 2. OD flows for 36 component state vectors

Arc 2 capacity	Arc 1 capacity					
	10,000	8000	6000	4000	2000	0
5000	12,294	10,822	9294	7710	6070	4375
4000	11,596	10,099	8546	6936	5269	3544
3000	10,887	9366	7787	6149	4453	2697
2000	10,168	8621	7014	5348	3621	1830
1000	9437	7864	6229	4532	2770	936
0	8695	7093	5428	3698	1895	0

more sensitive to changes in the capacity of arc 2 than arc 1. If the component states are not independent, and $\partial x_2/\partial x_1 = 0.5$ while $\partial x_1/\partial x_2 = 0$ (i.e. x_2 depends on x_1, but not vice versa), then the sensitivity coefficient for x_1 will increase to 5.4729, above that for x_2.

The OD and arc flows and the system surplus after a change in the component state vector can be estimated using eqns (5–7). For instance, a reduction of 2000 in x_1 (from 10,000 to 8000) gives an estimated system surplus of 145,952, which is very close to the value of 145,452 given by solving the integrated equilibrium model, while a reduction of 1000 in x_1 (from 5000 to 4000) gives an estimated system surplus of 147,494, which is extremely close to the value of 147,388 given by solving the integrated equilibrium model. The first-order approximations seem very accurate, and could be used to avoid the computational effort of solving the integrated equilibrium model every time estimates of the OD and arc flows and the system surplus are required.

4.3. Reliability estimation

To estimate the reliability for the system, which consists of just one sub-system, an estimate of the probability of each component state vector is required. If the arc degradation probabilities are identical (see Table 3) and the arc capacities are independent, then the component state vector probabilities are as shown in Table 4.

The flow decrement rates (Table 5) can then be calculated using the OD flows (Table 2), obtained from the integrated equilibrium model, and eqns (8) and (12). The operating/failed functions for a maximum acceptable flow decrement rate of 0.6, obtained from eqns (9) and (16), are shown in

Table 3. Arc capacity reduction probabilities

Capacity reduction (%)	0	20	40	60	80	100
Estimated probability	0.1	0.15	0.25	0.25	0.15	0.1

Table 4. Probabilities for component state vectors

Arc 2 capacity	Arc 1 capacity					
	10,000	8000	6000	4000	2000	0
5000	0.0100	0.0150	0.0250	0.0250	0.0150	0.0100
4000	0.0150	0.0225	0.0375	0.0375	0.0225	0.0150
3000	0.0250	0.0375	0.0625	0.0625	0.0375	0.0250
2000	0.0250	0.0375	0.0625	0.0625	0.0375	0.0250
1000	0.0150	0.0225	0.0375	0.0375	0.0225	0.0150
0	0.0100	0.0150	0.0250	0.025	0.0150	0.0100

Table 5. Flow decrement rates for component state vectors

Arc 2 capacity	Arc 1 capacity					
	10,000	8000	6000	4000	2000	0
5000	0.00	0.12	0.24	0.37	0.51	0.64
4000	0.06	0.18	0.30	0.44	0.57	0.71
3000	0.11	0.24	0.37	0.50	0.64	0.78
2000	0.17	0.30	0.43	0.56	0.71	0.85
1000	0.23	0.36	0.49	0.63	0.77	0.92
0	0.29	0.42	0.56	0.70	0.85	1.00

Table 6. Operating/failed function values

Arc 2 capacity	Arc 1 capacity					
	10,000	8000	6000	4000	2000	0
5000	1	1	1	1	1	0
4000	1	1	1	1	1	0
3000	1	1	1	1	0	0
2000	1	1	1	1	0	0
1000	1	1	1	0	0	0
0	1	1	1	0	0	0

Table 6. It can be seen that only 12 of the 36 arc capacity combinations will lead to failure. As the threshold flow decrement rate decreases, the proportion of capacity combinations which will lead to failure will increase (e.g. there will be 21 failure combinations for a threshold of 0.4).

The reliability of such a simple system can readily be calculated exactly using eqns (18,19), giving 0.725 and 0.385 for threshold flow decrement rates of 0.6 and 0.4, respectively. Using the approximating algorithm and the recursion equations (eqns 20–23), the estimated reliability M and upper bound absolute errors $\bar{\varepsilon}$ can be calculated for the system. These are shown in Fig. 1, for a 0.6 threshold flow decrement rate.

It can be seen that the best estimate quickly converges as the number of component states J increases, with the value after $J=6$ being relatively close to the correct value (0.725), and the range between the lower and upper bounds decreases in width as J increases, reaching zero at $J=36$, but the width remains substantial until well after $J=6$. If the termination criterion for the recursion estimation process was simply the rate of change of the best estimate, then one could stop at $J=6$, which means that the integrated equilibrium model needs solving only six times, rather than 36 times if the exact algorithm is used; this highlights the major computational advantage of the recursion method. If one wished to have a very precise estimate of the system reliability (i.e. a small error range), then the computational advantage of the recursion method is reduced substantially.

5. DISCUSSION

5.1. *Sensitivity analysis*

The sensitivity analysis is based on an integrated equilibrium model, but since the basic solution (the path flow vector) of the equivalent concave programming problem is generally not unique, the essential assumptions of existing sensitivity analysis methods for general nonlinear programming are not satisfied. This difficulty has been overcome by selecting a unique path flow vector, to analytically obtain the partial derivatives of the system state vector and system performance index with respect to the component state vector, through the well-defined partial derivatives of the selected path flow vector with respect to the component state vector.

The partial derivatives can be used to predict the direction of change in the system state vector and the system performance index, resulting from changes in the component state vector. Once they are calculated, first-order numerical approximations of the new system state vector and the corresponding system performance index can easily be estimated for any combination of element perturbations in the component state vector.

The derivatives can also be used to identify the important components, whose degradation probabilities can then be assessed, to identify those that are also weak and are thus critical components; such components should be the focus of efforts to improve system reliability. Once the critical components have been identified, the derivatives can be used in system design, to assess how the system performance will change as a result of design changes to those components. That is, the results of sensitivity analysis provide information for increasing the reliability of a DTS.

In addition, the relevant sensitivity data can be used to identify those input data to which the system state vector and system performance index are most sensitive, and special effort can be given to obtaining accurate estimates for those data. This is especially important for the study of a

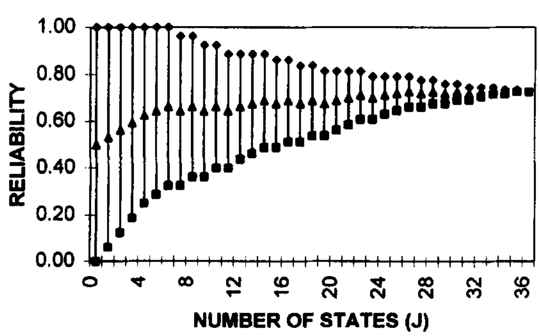

Fig. 1. Estimated reliability (with lower and upper bounds).

DTS, since estimates of the degradation probabilities given some event (e.g. an earthquake or storm) are likely to be difficult and expensive to obtain.

5.2. Reliability analysis

A network reliability model specifically for a DTS is required, as the existing network reliability models are either for pure networks and completely ignore the flow constraints in networks, or are for flow networks in which the demand is fixed and maximum possible flow through the networks is ensured by a network manager.

The proposed DTS reliability model involves defining the reliability of individual sub-systems (connecting pairs of origins and destinations) as the probability that given some event, the proportional reduction of flow in the sub-system is less than some threshold value. This approach takes account of the fact that there is usually excess capacity in a network, and some degradation can be tolerated before the DTS performance degradation reaches an unacceptable level, where 'failure' might be considered to have occurred. Those threshold values can vary between sub-systems, to reflect the variations in spare capacity that commonly exist in a real transportation system.

The DTS reliability model does not involve defining arc reliabilities, unlike for the classical graph theory approach. In addition, the classical graph theory approach gives an explicit and direct relationship between the reliability for the network and the arc reliabilities (Nicholson and Du, 1997), while for the DTS reliability model the sub-system and system reliabilities are related to the arc degradation probabilities, but the relationship is not explicit. The arc capacity reduction probabilities are used to estimate the probabilities of component state vectors, and these probabilities, in conjunction with the OD flows obtained from the integrated equilibrium model, determine the OD sub-system and system reliabilities, based upon the effects of arc capacity degradations on network flows. If the arc characteristics change so that the arc capacity reduction probabilities change, then there will be consequential changes in the component state vector probabilities and hence changes in the sub-system and system reliabilities.

The reliability of the whole system is defined as the probability that the reduction in flow in the whole system is less than a threshold, which is a weighted sum of the OD sub-system thresholds, where the weights are the ratios of the flows between the OD pairs to the total flow, for the non-degraded state. That is, the emphasis is upon the performance and reliability of the whole DTS.

It can be shown that if all the OD sub-systems in a DTS are operating for a component state vector, then the DTS is also operating. However, the converse is not true. This can be illustrated by considering a DTS consisting of two OD sub-systems, for which the maximum acceptable sub-system flow decrement rates are $\theta_1 = 1/2$ and $\theta_2 = 1/2$, and the flow weightings (eqn (14)) are $v_1 = 1/3$ and $v_2 = 2/3$. It follows from eqn (15) that the maximum acceptable system flow decrement rate is $\theta = v_1\theta_1 + v_2\theta_2 = 1/2$. Now if the sub-system flow decrement rates (eqn (8)) are $y_1 = 0.3$ and $y_2 = 0.6$, then the system flow decrement rate (eqn (13)) is $y = v_1y_1 + v_2y_2 = 1/2$, and the system and first sub-system are deemed to be 'operating', but the second sub-system is deemed to be 'failed'.

That the system may be 'operating' even when one or more sub-systems may have 'failed' emphasizes the need for care in defining system reliability, but is consistent with a well-known result in reliability engineering that a system may have high reliability even though some sub-systems have low reliabilities.

An alternative would be to define the system reliability as the minimum of the sub-system reliabilities; this would mean that if the flow decrement for any sub-system exceeded the threshold for that sub-system (i.e. the sub-system has 'failed'), then the system would also be considered to have 'failed', even though the flow between the OD pair might be very small. Such a sub-system might be considered less important than those for which the flows are very large, but this approach implies no variation in the importance of sub-systems, unlike with the flow-weighted average approach, which implies the importance of sub-systems varies according to their flows.

The adoption of an operating/failed function based on the flow in a sub-system was for practical reasons. The system surplus was adopted as the performance measure for the integrated equilibrium model, to enable assessment of the socio-economic impact of system degradation. While it might seem sensible to define reliability in terms of the reduction in system surplus, until such time as more is known about the socio-economic impact of degradation, it is unlikely that such a reliability model could readily be used; it is likely that estimating maximum acceptable flow rate decrements would prove difficult enough. Such an approach would also mean focusing on the overall system performance and ignoring the performance of individual sub-systems.

Clearly, the values of society (in particular, the extent to which the performance of the transportation system is expected to meet the access and mobility needs of small, remote communities) need to be taken into account when deciding upon the measure of system reliability.

An exact algorithm for solving the DTS reliability model is impractical, except for a very small or very regular network, due to the massive computation effort involved. An approximating recursive algorithm, involving considering each component state vector in descending order of probability, maximizes the speed of convergence of estimates of reliability to the true value, and gives absolute lower and upper bounds for system reliability. The process can also be terminated once the precision of the estimate reaches the desired level, and appears to be a practicable approach for estimating the reliability of large transportation networks.

5.3. *Independence of component states*

For a DTS, component capacity degradations are not always statistically independent, that is, the elements of the component state vector may be inter-related (Nicholson and Du, 1997). Since the sensitivity analysis method developed in this paper is directly based on the partial derivatives among the elements of a component state vector, it can handle both independent and dependent degradations of component capacities. In addition, the approximate recursive algorithm for estimating system reliability, being based directly on the probabilities of the component state vectors, can also handle both independent and dependent degradations of component capacities.

The importance of allowing for inter-dependent component states is illustrated by the numerical example above (Section 4). Ignoring inter-relationships between arc states can lead to substantial error, especially when there is considerable interaction and the network is sparse, as may well be the case when studying a rural area.

5.4. *Temporal variations in component states*

The socio-economic impact of component degradation, and hence the importance of the component, will clearly increase as the time to repair or replace the component (i.e. the degradation duration) increases. The integrated equilibrium model, and thus both the sensitivity and reliability analyses based on that model, are for a steady-state condition, and it would be desirable to extend the component state vector space of a DTS to the time domain, and to take account of stochastic changes in the component state variables over time, using stochastic process techniques. This would enable differences in the degradation duration to be considered when estimating the socio-economic impacts, and identifying the critical components and the system reliability.

In addition to the time of occurrence of component degradations being random, the duration of component degradations can be considered a stochastic variable. The duration of degradation enables differentiation between non-repairable and repairable systems (Ascher and Feingold, 1984). The degradation duration in a non-repairable DTS is effectively infinite, that is, after degradation, the system cannot be (or is not worth being) repaired and is scrapped. The degradation duration in a repairable DTS is finite (i.e. the DTS can be and is restored), and the component and system states alternate between un-degraded and degraded. Different types of event can lead to different levels of component capacity degradation and involve different repair times (i.e. involve different distributions of degradation durations). Development of dynamic, stochastic, repairable-DTS models would also enable the analysis and evaluation of different repair strategies and their effects on the performance and reliability of DTSs.

6. CONCLUSION

A technique for overcoming the problem arising from the non-uniqueness of the path flow vector for the integrated equilibrium model has been found, so that a sensitivity analysis can be done. Expressions for the partial derivatives of the system state vector (comprising the origin–destination and arc flows) and the system performance index (the system surplus), with respect to the component state vector (comprising the arc capacities), have been obtained. These enable identification of the components, degradation of which will result in a substantial socio-economic impact.

The partial derivatives also allow for non-independence of component states, and it has been shown for a simple network that ignoring inter-relationships between component states can lead to substantial error in the estimate of the socio-economic impact.

A relatively simple reliability model has been proposed; nevertheless, it is very unlikely that an exact solution can be found, except for a very small or very regular network. The most practical approach would be to obtain an approximate solution, and a recursive algorithm seems the most promising method.

The reliability model has been applied to a simple network, and the exact and approximating recursive algorithms used to estimate the system reliability. This has shown how the component reliabilities influence the system reliability, as well as demonstrating that the recursive algorithm converges quickly to the correct value.

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