

1 Introduction

Sorting algorithms on modern computers have been playing an important role in Computer Science and can date back to the early 50s and 60s. It is the great efforts that humans have put into the field of developing faster sorting algorithms that shapes the vital parts that formed all the significant infrastructures of the vast computer world like databases, data centers, cluster networks etc. In the realm of sorting algorithms, Quicksort, introduced by computer scientist Tony Hoare [1] to the public, has made its way into the libc of the GNU/Linux system and stood the test of time as one of the most classical and widely used methods for sorting. Its divide-and-conquer strategy has been widely used and taken into research to fully utilize the performance of sorting. The basic Quicksort picks an arbitrary pivot element, partitions the array into 2 segments, one with elements less than the pivot and the other greater. It then recursively sorts these segments using the same method. While the original Quicksort method has proven effective, the quest of further optimizations and the endless pursuit of efficiency has led to many variations to explore, one of which is the Multi-Pivot Quicksort.

This thesis is my shallow attempt to unveil the benefits of Multi-Pivot Quicksort, exploring the effects more pivots can bring, followed by some potential implementations and the analysis on the benefits that arise from leveraging multiple pivots in the Quicksort process.

1.1 History and Related work

When Quicksort was first invented a great number of variations and modifications were put into research over the years, such as sorting with a better partition method and a more optimal strategy to select the arbitrary pivot element. But the innovative approach of Multi-pivot Quicksort, exemplified by Vladimir Yaroslavskiy [2] in 2009, Bentley and Bloch's algorithm, which was also highlighted by Wild and Nebel, adopted in Sun's Java 7 runtime library, introduces the use of two pivots. Contrary to initial expectations, studies have revealed that employing multiple pivots can enhance the performance, challenging prior assumptions derived from the dual-pivot proposal by Sedgewick [3] in 1978 where he analyzed a potential dual-pivot approach, which was deemed inferior to the classical Quicksort in his research. Furthermore, Kushagra presented a 3-pivot Quicksort [4] algorithm at ALENEX, has attracted significant attention, further pushing the researches of multi-pivot strategies.

One common pitfall in the performance downgrade of the conventional Quicksort is the degenerated cases, which could often arise when the input array is already mostly sorted or when certain patterns in the data cause the algorithm to consistently choose poor pivots that fail to split the data into evenly sized partitions. In such cases, the partitioning process may result in highly unbalanced partitions, leading to suboptimal performance and potentially degrading the time complexity to $\mathcal{O}(n^2)$ instead of the expected $\mathcal{O}(n \log n)$ for Quicksort. Wiser choices of pivot selections could decrease the possibilities of such events from happening, such as the median-of-3 method and its variations of median-of-5 and median-of-7. But on

mostly ascending or descending arrays, the median-of-3 method still has a chance of choosing the smallest or largest element as the pivot, which is not the best ideal approach to prevent the worst case from happening. It also brings to our concerns that in real world applications, duplicated elements are not rare, and the median-of-3 method could be easily affected by the duplicated elements, which could lead to the same worst case as the original Quicksort. There are some other variations of Quicksort that have been proposed to address these specific issues, such as the Introsort algorithm [5] by David Musser, which switches to Heapsort when the recursion depth exceeds a certain threshold. Thus the worst running time is guaranteed to be $\mathcal{O}(n \log n)$ by doing so.

And this is where Multi-pivot quicksort comes into play. Assuming we are dealing with randomly generated arrays, since we don't know the exact size of each partition, the more pivots we choose, the more partitions we divide the array into, the less chance it will be that we encounter imbalanced sizes of partitions. Moreover, choosing more pivots will significantly reduce the chance we access the same elements again and the maximum depth of recursion, which can translate into better running time. With n pivots we are dividing the array into $n+1$ partitions, the same things happen on the sub-partitions in the next level of recursion and we can conclude the maximum depth of recursion would be $\log_{n+1} L$ given n as the number of pivots and L for the length of array. Greater the n is, less the depth. Better memory behaviors can bring effects that can not be achieved by any of other traditional means such as choosing a better pivot value among samples extracted from different parts of the array, but the drawbacks are also obvious, branch misprediction could be terrible as the different outcomes of comparisons can largely impact on the run-time performance. A delay of 14 to 19 stages on a typical Intel desktop CPU or 20 to 25 stages on AMD's Zen CPUs could lead to huge efficiency loss for our algorithm, and this number only grows as the number of pivots increases and on more complicated architectures. These are the trade-offs we have to consider when we are choosing the number of pivots, although these side effects can be mitigated by the use of branchless comparisons and modern CPUs are able to run conditional move operations (cmov) to avoid branches, these overheads still remain as concerns. Apart from that, it could be really tricky when it comes to moving the elements to the right places among multiple partitions, especially when the number of pivots is large. Additionally, comparisons are the second expensive things next to swaps we are dealing with, and the more pivots we choose, the more comparisons we have to make. Even for the case of using only one pivot, the average number of comparisons remain unoptimal, being $1.38n \log n + \mathcal{O}(n)$ compared with the $n \log n + \mathcal{O}(n)$ of the Merge Sort, but the overall instructions are much less than the average of Merge Sort and with proper pivot selection strategy, it can still be reduced to a certain extent.

Fortunately, an efficient approach has been put forward by Stefan Edelkamp and Armin Weiß, in their thesis of BlockQuickSort [6] which evens the odds of both branch mis-prediction and cache misses which uses one or more buffer blocks of constant sizes on the stack to store the index offsets of out-of-order elements, swap them altogether in a second pass to rearrange them in order after scanning a large chunk of array with the size of the block. Their approach realized a significant speedup up to 80% faster compared with the GCC implementation of std sort with only one

single pivot. The BlockQuickSort has been adapted to the Multi-Pivot Quicksort and the results are also promising, the improvements on contiguous memory access significantly improves the spatial locality and reduces both the cache misses and branch mis-predictions. Branchless comparisons are also adapted which plays an important role in contributing to the huge performance boost by reducing the overhead when increasing the counter of elements stored in the block buffer(s). This novel approach has also developed variations that could effectively make the use of different partitioning methods and multiple pivots selected.

Meanwhile, in the pursuit of advancing the efficiency of the Quicksort algorithm, my investigation led me to explore various modifications, with a particular focus on the pattern-defeating Quicksort (PDQSort) proposed by Orson Peters. PDQSort leverages the average-case performance of randomized Quicksort with the rapid worst-case execution of heapsort, while concurrently achieving linear time complexity for slices exhibiting specific patterns. Notably, PDQSort employs a judicious application of randomization to prevent the degenerated case from happening. When it does, PDQSort will strategically shuffle or reverse the entire array to simplify the sorting and increase the speed. If all the counter measures fail, it also has the final backup plan of switching to Heapsort when bad choices of pivots or the depth of recursions exceed the limit. In terms of pivot selection, it uses a fixed seed when generating a pseudo-random index to ensure the reproducibility and deterministic behavior. Furthermore, in instances where randomly selected elements from the original array appear in a descending order. The combination of these optimizations has demonstrated a significant enhancement in performance, showcasing PDQSort as up to twice faster than the original Quicksort algorithm. This outcome illuminates the transformative impact that a comprehensive analysis on the pattern of arrays has on the speed of sorting algorithms, not to mention it is a successful combination of BlockQuickSort's partition method, Heap sort when dealing with worst cases and Insertion sort when it comes to small sub-arrays of sizes less than the cache line size. Inspired by PDQSort's success, I dive into the diverse variations of Quicksort, motivating a search for novel patterns that may be leveraged to enhance the existing Quicksort algorithms.

1.2 Contributions

- Present the variants of Multi-Pivot QuickSort and implement a 4-Pivots Quicksort and evaluate N-Pivots Quicksort from $N = 1, 2, 3, 4$ experimentally. Comparing the implementations, times, instructions, cache misses and branch mis-predictions with a duration of 3 seconds warm-up time.
- Present a new layout of Multi-Pivot BlockQuickSort that could potentially reduce the memory access and increase the block buffer usage rate. Analyze this method with all other available variants with block partitioning and compare the pros and cons.

2 Preliminaries

Algorithm 1 QuickSort with Hoare Partition

```
1: procedure QUICKSORT( $A, l, r$ )
2:   if  $l < r$  then
3:      $p \leftarrow \text{HOAREPARTITION}(A, l, r)$ 
4:     QUICKSORT( $A, l, p-1$ )
5:     QUICKSORT( $A, p+1, r$ )
6:   end if
7: end procedure
8: procedure HOAREPARTITION( $A, l, r$ )
9:    $P \leftarrow A[l]$  ▷ Pivot
10:   $i \leftarrow l - 1$ 
11:   $j \leftarrow r + 1$ 
12:  while  $i < j$  do
13:    repeat
14:       $j \leftarrow j - 1$ 
15:    until  $A[j] < P$ 
16:    repeat
17:       $i \leftarrow i + 1$ 
18:    until  $A[i] > P$ 
19:    SWAP( $A[i], A[j]$ )
20:  end while
21:  SWAP( $A[l], A[j]$ ) ▷ Put pivot in the middle
22:  return  $j$ 
23: end procedure
```

For Quicksorts with single pivot value, the partitioning procedure splits the array into 2 parts where the left part are elements less than the pivot and the right part being greater, separated by the pivot value in the middle. Then the main body recursively sorts the left and right part to rearrange all the elements in the correct order. The hoare's partition method uses double pointers starting at the leftmost and the rightmost element in the array and scanning towards the middle until they meet. Each pointer scans for mis-placed elements which belong to the other partition by comparing the current element to the pivot. Out-of-order elements are swapped in pairs to the right place and the scanning procedure continues until all elements are moved to the places where they belong.

Algorithm 2 QuickSort with Lomuto Partition

```
1: procedure QUICKSORT( $A, l, r$ )
2:   if  $l < r$  then
3:      $p \leftarrow \text{LOMUTOPARTITION}(A, l, r)$ 
4:     QUICKSORT( $A, l, p-1$ )
5:     QUICKSORT( $A, p+1, r$ )
6:   end if
7: end procedure
8: procedure LOMUTOPARTITION( $A, l, r$ )
9:    $P \leftarrow A[r]$  ▷ Pivot
10:   $i \leftarrow l$ 
11:  for  $j \leftarrow l$  to  $r-1$  do
12:    if  $A[j] < P$  then
13:      SWAP( $A[i], A[j]$ )
14:       $i \leftarrow i + 1$ 
15:    end if
16:  end for
17:  SWAP( $A[i], A[r]$ ) ▷ Put pivot in the middle
18:  return  $i$ 
19: end procedure
```

While the Lomuto partition method uses another way, it scans the array in sequential order from left to right using one bare pointer, ascending the index by 1 each time. and swaps all the elements less than the pivot to the left side regardless, thus duplicated swaps are introduced due to not checking if the elements are already in the right places. Both of these 2 partition methods are straightforward, and their main structures are similar: scan, find mis-placed elements, swap and recursively repeat the same procedure until the whole array is sorted.

Algorithm 3 Dual-Pivot QuickSort

```
1: procedure DUALPIVOTPARTITION( $A, l, r$ )
2:    $P1, P2 \leftarrow A[l], A[r]$  ▷ Pivots
3:   if  $P1 > P2$  then
4:     SWAP( $P1, P2$ )
5:   end if
6:    $less \leftarrow l + 1$ 
7:    $greater \leftarrow r - 1$ 
8:    $k \leftarrow l$ 
9:   while  $k \leq greater$  do
10:    if  $A[k] < P1$  then ▷ If the current element is less than P1
11:      SWAP( $A[k], A[less]$ ) ▷ Shift the element to the leftmost partition
12:       $less \leftarrow less + 1$ 
13:    else if  $A[k] > P2$  then ▷ If the current element is greater than P2
14:      while  $A[greater] > P2$  and  $k < greater$  do
15:         $greater \leftarrow greater - 1$  ▷ Find the first element less than P2
16:      end while
17:      SWAP( $A[k], A[greater]$ ) ▷ Swap to the rightmost partition
18:       $greater \leftarrow greater - 1$ 
19:      if  $A[j] < P1$  then ▷ Double check if another swap is needed
20:        SWAP( $A[k], A[less]$ )
21:         $less \leftarrow less + 1$ 
22:      end if
23:    end if
24:     $k \leftarrow k + 1$  ▷ Move to the next element
25:  end while
26:   $less \leftarrow less - 1$ 
27:   $greater \leftarrow greater + 1$ 
28:  SWAP( $A[l], A[less]$ ) ▷ Put Pivot1 in the middle
29:  SWAP( $A[r], A[greater]$ ) ▷ Put Pivot2 in the middle
30:  return  $less, greater$ 
31: end procedure
```

With one more pivot added, the two-pivot Quicksort method is introduced by Vladimir Yaroslavskiy in 2009. It uses 2 pivots selected from each end to split the array into 3 parts. k stands for the current element being scanned, it will be swapped to the left part, which is accumulated by the $less$ pointer, if it is less than the first pivot or do nothing if it is in the middle part. Otherwise, if it is greater than the second pivot, it will be swapped to the right part, which is accumulated by the $greater$ pointer. We will double check the current element at index k after the swap to see if it is less than the first pivot, if so, we will swap it to the left part and move the $less$ pointer to the right. Hereby, we can see that the Dual-Pivot Quicksort is a generalization of the original Quicksort, and the partitioning method is also a generalization of the original Hoare's partition method but with 2 pivots.

What's attracting is that there was once a debate on whether multiple pivots

could enhance the performance, as Sedgewick, who analyzed a dual-pivot approach with an average of $\frac{32}{15}n \ln n + \mathcal{O}n$ comparisons in 1978, in contrast to the $2n \ln n + \mathcal{O}n$ of the standard Quick Sort, considered this prototype to be inferior to the classical Quicksort. However, the Yaroslavskiy's Dual-Pivot Quicksort has proven to be a success, with the experimental results showing that it has a less amount of $1.9n \ln n + \mathcal{O}n$ amortized comparisons, but more swaps, approximately $0.6n \ln n + \mathcal{O}n$ which is much greater than the $0.33n \ln n + \mathcal{O}n$ of the original one, is faster than the original Quicksort in most cases. It is surprising that even with asymptotically only 5% of less comparisons and nearly 80% more swaps, the Dual-Pivot Quicksort still outperforms the original Quicksort in practice. What makes this more mysterious, is the variation of Yaroslavskiy's partition method that always compares to the larger pivot, by Martin Aumüller and Martin Dietzfelbinger 2013 in their thesis of 'Optimal Partitioning for Dual Pivot Quicksort' [7]. The method shows no difference in terms of both comparison and swaps with the original Dual-Pivot Quicksort, but it turned out that it is as fast as the Yaroslavskiy's partition, and even faster than another method along, named 'Countering Strategy C', who claimed to have the least estimations of $1.8n \ln n + \mathcal{O}(n)$ comparisons and $1.6n \ln n + \mathcal{O}n$ swaps. This is a great example of how the real-world performance could be different from the theoretical analysis. In fact, in their following tests, the 'Countering Strategy C' method was proven to be the slowest among all the 3 methods.

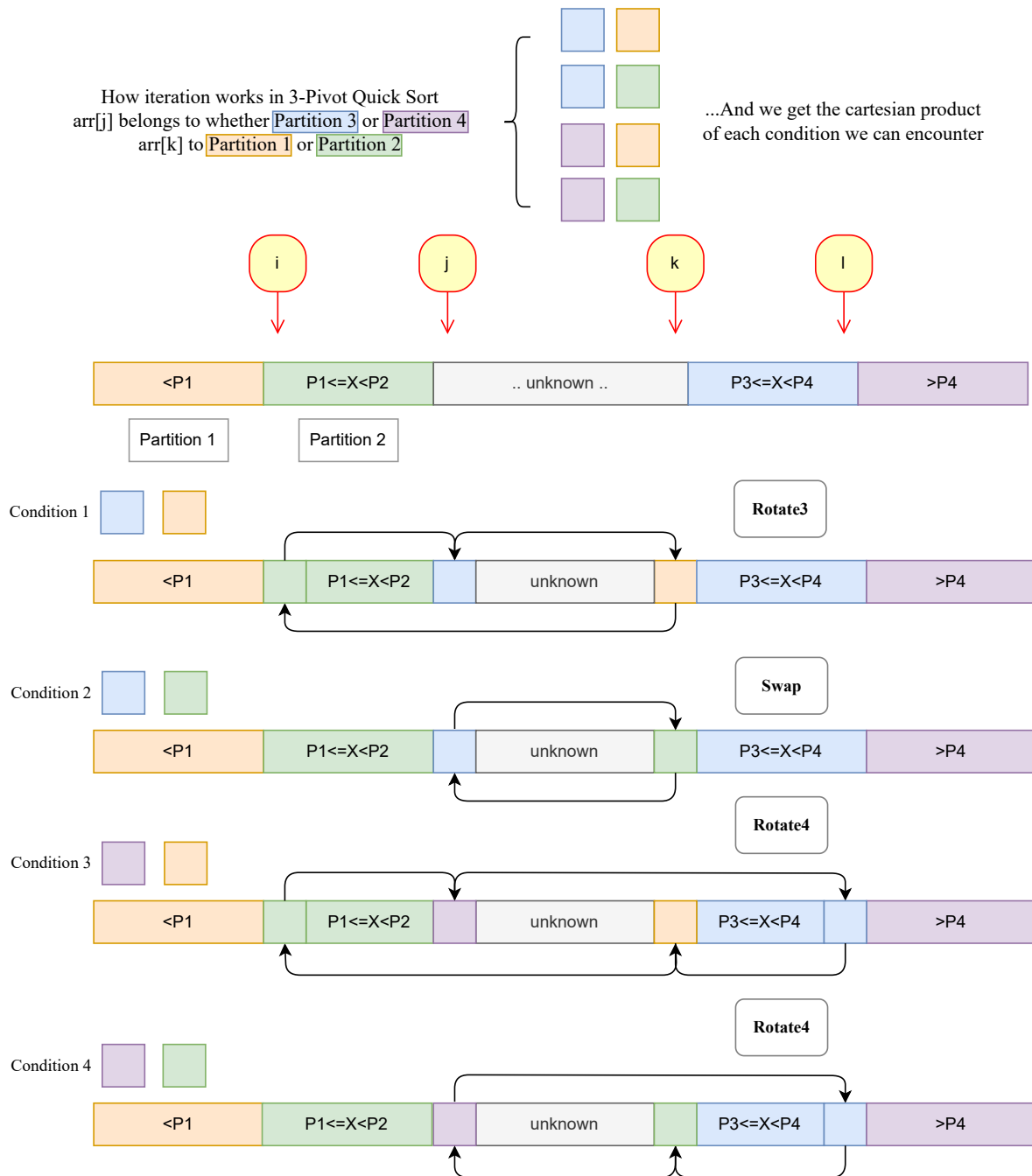
It is noteworthy that there still lies a huge gap between the theoretical analysis and the real-world performance. It is time we broaden our horizon and look for more factors that could contribute more to the change of the performance of the Quicksort algorithm. The cache misses, branch mis-predictions, and the number of instructions are the things we should take into consideration when we are analyzing on mutated versions of the Quicksort algorithm. Apparently, the number of pivots plays a significant role, and is one of the factors that could potentially bring a significant change. Let's dive deeper to see the world of 3-Pivots Quicksort.

Algorithm 4 3-Pivot QuickSort

```
1: procedure THREEPIVOTPARTITION( $A, l, r$ )
2:    $P1, P2, P3 \leftarrow A[l], A[l+1], A[r]$   $\triangleright$  And sort 3 Pivots in ascending order
3:    $i, j \leftarrow l+2$ 
4:    $k, l \leftarrow r-1$ 
5:   while  $j \leq k$  do
6:     while  $A[j] < P2$  do  $\triangleright$  Put left side elements less than P2 in order
7:       if  $A[j] < P1$  then
8:         SWAP( $A[i], A[j]$ )
9:          $i \leftarrow i+1$ 
10:      end if
11:       $j \leftarrow j+1$ 
12:    end while
13:    while  $A[k] > P2$  do  $\triangleright$  Put right side elements greater than P2 in order
14:      if  $A[k] > P3$  then
15:        SWAP( $A[k], A[l]$ )
16:         $l \leftarrow l-1$ 
17:      end if
18:       $k \leftarrow k-1$ 
19:    end while
20:    if  $j \leq k$  then
21:      if  $A[j] > P3$  then  $\triangleright$  Deal with 2 branches when  $A[j]$  is greater than P3
22:        if  $A[k] < P1$  then
23:          ROTATE3( $A, [j, i, k]$ )
24:           $i \leftarrow i+1$ 
25:        else
26:          SWAP( $A[j], A[k]$ )  $\triangleright A[j] > P3$  and  $P2 > A[k] \geq P1$ 
27:        end if
28:        SWAP( $A[k], A[l]$ )
29:         $l \leftarrow l-1$ 
30:      else  $\triangleright$  Deal with 2 branches when  $A[j]$  is less than or equal to P3
31:        if  $A[k] < P1$  then
32:          ROTATE3( $A, [j, i, k]$ )
33:           $i \leftarrow i+1$ 
34:        else
35:          SWAP( $A[j], A[k]$ )
36:        end if
37:      end if
38:       $j \leftarrow j+1, k \leftarrow k-1$ 
39:    end if
40:  end while
41:   $i \leftarrow i-1, j \leftarrow j-1, k \leftarrow k+1, l \leftarrow l+1$ 
42:  ROTATE3( $A, [left+1, i, j]$ )
43:  SWAP( $A[left], A[i]$ )
44:  SWAP( $A[right], A[l]$ )
45:  return  $i, j, l$ 
46: end procedure
```

The three-pivot Quicksort, introduced by Shrinu Kushagra, uses the similar hoare-like partition method, is a further generalization of the Dual-Pivot Quicksort. In the original implementation, the 3 pivots are selected from the leftmost, the second element and the rightmost element of the array, and are sorted in ascending order. The scanning procedure is similar to the Dual-Pivot Quicksort, but with 3 pivots, the array is divided into 4 parts, and is repeated until all elements are set. The i , j , k , l pointers are used to scan the array and the elements are swapped to the right places according to the comparison results with the 3 pivots, where i stands for the leftmost part where elements less than Pivot 1 belongs, j stands for the middle-left part, k stands for the middle-right part and l stands for the rightmost part. Between j and k stands the unknown region where elements are not yet compared with any of the pivots. If the element at index j is less than Pivot 2, it is whether swapped to the leftmost part if it is less than Pivot 1, or do nothing otherwise, since the middle-left part is for elements greater than Pivot 1 and less than or equal to Pivot 2. Same logic applies to the rightmost part, where elements are greater than Pivot 2 and less than or equal to Pivot 3. These two pre-conditions are checked before the main body of the swapping and rotating procedure, which corresponds to the 2 while loops applied at the top of the main scanning loop. Afterwards, the real swapping and rotating procedure is applied to the elements at index j and k , which are on the left and right boundaries of the unknown region, need to go to the other side of the array and moved to the right places. We can simplify this part into a 2 if-else nested block, where the outer if-else block is for the case at index j and the inner one is for the case at index k (vice versa, but we will not go into details here). Both of them have 2 branches, one for the case where the current element belongs to the leftmost or the rightmost part, and the other for the case where it belongs to the middle-left or the middle-right part. It is not hard to tell there are $2 \times 2 = 4$ branches in total, and the swapping and rotating procedure is applied to each of them. The following chart * illustrates the 4 branches and the corresponding actions.

Figure 1: 3-Pivot Quick Sort



Algorithm 5 4-Pivot QuickSort

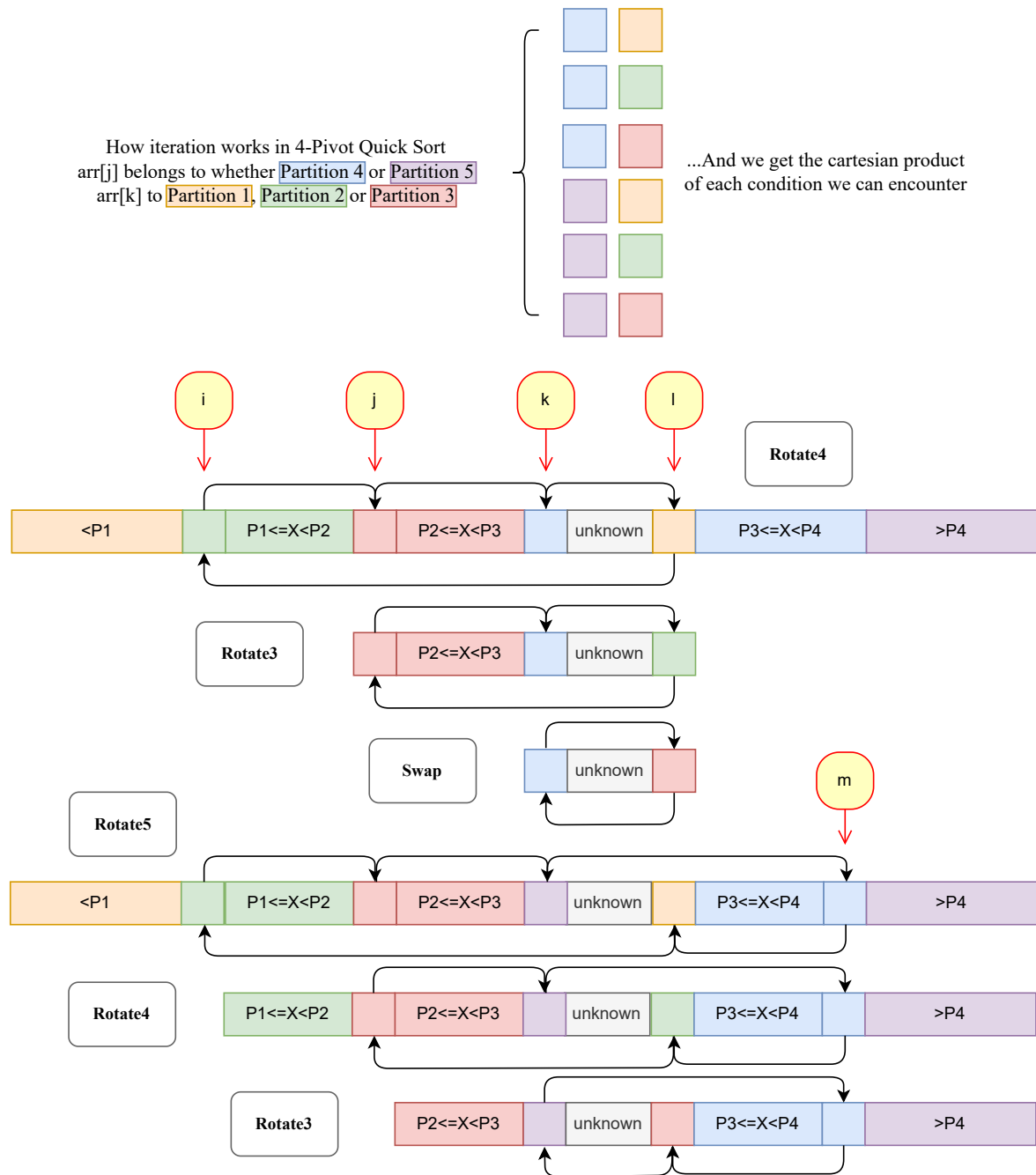
```
1: procedure FOURPIVOTPARTITION( $A, l, r$ )
2:    $P1, P2, P3, P4 \leftarrow A[l], A[l+1], A[r-1], A[r]$        $\triangleright$  And sort 4 Pivots in order
3:    $i, j, k, l, m \leftarrow l+2, l+2, l+2, r-2, r-2$ 
4:   while  $k \leq l$  do
5:     while  $A[k] < P3$  do       $\triangleright$  Put left side elements less than P3 in order
6:       if  $A[k] < P1$  then
7:         ROTATE3( $A, [k, j, i]$ )
8:          $i \leftarrow i+1, j \leftarrow j+1$ 
9:       else if  $A[k] < P2$  then
10:        SWAP( $A[k], A[j]$ )
11:         $j \leftarrow j+1$ 
12:      end if
13:       $k \leftarrow k+1$ 
14:    end while
15:    while  $A[l] > P3$  do       $\triangleright$  Put right side elements greater than P3 in order
16:      if  $A[l] > P4$  then
17:        SWAP( $A[l], A[m]$ )
18:         $m \leftarrow m-1$ 
19:      end if
20:       $l \leftarrow l-1$ 
21:    end while
22:    if  $k \leq l$  then  $\triangleright$  Start manipulation on the boundaries of unknown region
23:      if  $A[k] < P4$  then       $\triangleright$  Deal with 3 branches when  $A[k]$  is less than P4
24:        if  $A[l] < P1$  then
25:          ROTATE4( $A, [k, j, i, l]$ )
26:           $i \leftarrow i+1, j \leftarrow j+1$ 
27:        else if  $A[l] < P2$  then
28:          ROTATE3( $A, [k, j, l]$ )
29:           $j \leftarrow j+1$ 
30:        else
31:          SWAP( $A[k], A[l]$ )
32:        end if
33:      else       $\triangleright$  Deal with 3 branches when  $A[k]$  is greater than P4
34:        if  $A[l] > P2$  then
35:          ROTATE3( $A, [k, l, m]$ )
36:        else if  $A[l] > P1$  then
37:          ROTATE4( $A, [k, j, l, m]$ )
38:           $j \leftarrow j+1$ 
39:        else
40:          ROTATE5( $A, [k, j, i, l, m]$ )
41:           $i \leftarrow i+1, j \leftarrow j+1$ 
42:        end if
43:         $m \leftarrow m-1$ 
44:      end if
45:       $k \leftarrow k+1, l \leftarrow l-1$ 
46:    end if
```

Algorithm 6 4-Pivot QuickSort (Continued)

```
47:   end while
48:    $i \leftarrow i - 1, j \leftarrow j - 1, k \leftarrow k - 1, l \leftarrow l + 1, m \leftarrow m + 1$ 
49:   ROTATE3( $A, [left + 1, i, j]$ )
50:    $i \leftarrow i - 1$ 
51:   SWAP( $A[left], A[i]$ )
52:   ROTATE3( $A, [right - 1, m, l]$ )
53:    $m \leftarrow m + 1$ 
54:   SWAP( $A[right], A[m]$ )
55: end procedure
```

As long as the number of pivots increases, the number of branches will grow linearly, and the complexity of the partitioning method will also increase. Thankfully, for 4-Pivot QuickSort *, we are just adding two more branches (As showed in the blue-red and purple-red branches in the chart below) into considerations and the branching logic is still under control. When we are using a balanced partition method, where each side of the array is filled with $\frac{N+1}{2}$ partitions for odd number N of pivots, or one side with $\frac{N}{2}$ partitions and $\frac{N}{2} + 1$ on the other, we are going to have 9 branches for 5-Pivot QuickSort (3 partitions on each side, $3 \times 3 = 9$) and 12 branches for 6-Pivot QuickSort (3 partitions on one side and 4 on the other, $3 \times 4 = 12$). The complexity of the partitioning method is growing way too far and thus only 4-Pivot QuickSort will be discussed in this thesis.

Figure 2: 4-Pivot Quick Sort



2.1 Analysis

The analysis on the results of Branch Misses, Cache Misses, Instructions and Time will be conducted below with regard to the Multi-Pivot QuickSorts with traditional partitions. The details regarding to the testing machine, environment and compiler settings ¹ are as follows:

CPU		Cache		
R7 3700x @4.3GHz Overclocked		L1 8x32KB	L2 8x512KB	L3 2x16MB
RAM	OS		Compiler	
16GB	GNU/Linux Ubuntu 22.04		Rustc 1.78.0 +Nightly	
Rustc Flags	opt-level=3,codegen-units=1,lto="thin"			

The compiler flags passed in are the same for all the tests, and the tests are conducted on the same machine with the same environment, with 3 seconds of warm-up time and 5 seconds of measurement time for each test. The results are calculated over regression tests with 100, 1000, 10000, 100000, 1000000 and 10000000 elements, filtering out the outliers and the each item is collected with the mean, standard deviation and median. The results are shown in the tables below.

¹'opt-level=3' is equivalent to '-O3', 'codegen-units=1' is equivalent to '-fno-merge-all-constants', 'lto="thin"' is equivalent to '-flto=thin' in GCC.

Branch Misses	Size	Mean	SD	Median
1-Pivot Hoare	100	224.4847	0.0588	224.4906
	1000	3612.1521	2.2119	3612.3146
	10000	49451.5958	72.9468	49464.5783
	100000	622681.4604	1879.7192	622662.0813
	1000000	7514512.0625	57015.6085	7528135.1250
	10000000	88601386.6500	1316610.4428	88910896.5000
1-Pivot Lomuto	100	208.5500	0.2674	208.4572
	1000	3431.9977	4.9888	3432.6774
	10000	47510.1674	62.4065	47492.9219
	100000	599369.5910	3094.0668	599127.2778
	1000000	7188431.3250	77244.2181	7204133.7500
	10000000	85572728.6000	1273266.3095	85589444.5000
2-Pivot Yaro	100	218.6639	0.0935	218.6437
	1000	3629.7245	1.7137	3630.2813
	10000	50301.2096	38.3586	50300.6213
	100000	635045.2987	2126.8544	634416.1873
	1000000	7624533.4500	45160.5385	7614424.8750
	10000000	90144952.0000	955908.5187	90243746.5000
3-Pivot Kush	100	231.0457	0.0996	231.0266
	1000	3828.0046	1.7692	3827.5547
	10000	52692.9469	57.0170	52699.8674
	100000	660656.7803	2410.3154	660486.3048
	1000000	7881473.7500	55041.5020	7885420.7000
	10000000	93039500.9000	1138513.6270	93254088.5000
4-Pivot	100	233.3581	0.1961	233.3429
	1000	3847.1970	1.7934	3847.1812
	10000	52466.7069	69.7884	52466.1749
	100000	655540.9742	2248.3174	655697.4468
	1000000	7831009.1900	49027.0168	7835433.7000
	10000000	92702517.3500	1393247.4531	93065712.5000

The traditional means don't show much difference between the 1-Pivot Hoare and Lomuto partition methods, it makes sense that the 1-Pivot Hoare's partition method has a slightly higher mean of branch misses than the 1-Pivot Lomuto's partition method when Hoare's partition method seeks 2 out-of-order elements to swap, while only 1 for Lomuto's. We are bound to have some more branch misses in the former case. But the 2-Pivot Yaro's partition method has a slightly higher mean of branch misses. The introduction of the 2nd pivot has brought more branches into concerns, the complexities grow linearly with the number of pivots but the standard deviation grows rapidly as the size of array exceeds 10k, and explodes after 1 million. Down follows the 3-Pivot Kush's partition method has a slightly higher mean of branch misses than the 2-Pivot Yaroslavskiy's partition method. But things go beyond our expectations, the 4-Pivot partition shows better statistics of branch misses than the 3-Pivot partition method, which is a surprise to us, even though it has slightly higher means of branch misses than the 2-Pivot Yaroslavskiy's partition

method, the standard deviation remains at a high level indicating the possible instabilities of the 4-Pivot partition method. It is still worth our notice that it happens when more pivots could reduce the branch misses.

Apart from the branch misses, the cache misses are also a significant factor that could affect a lot, which is what we are going to shift our focus to.

Cache Misses	Size	Mean	SD	Median
1-Pivot Hoare	100	0.0205	0.0120	0.0188
	1000	0.6087	1.4656	0.2268
	10000	3.4867	2.7533	2.7844
	100000	1079.6629	230.8850	996.8944
	1000000	25764.5000	1166.9931	25475.1250
	10000000	634608.9000	13770.3278	633085.5000
1-Pivot Lomuto	100	0.0309	0.0585	0.0114
	1000	1.5477	2.5235	0.7386
	10000	16.8345	33.0704	6.4018
	100000	1033.2760	46.1483	1016.9857
	1000000	25119.6125	1033.3887	24916.1250
	10000000	653495.7500	17064.0432	653582.5000
2-Pivot Yaro	100	0.0470	0.1104	0.0155
	1000	1.3900	4.3104	0.3234
	10000	14.0858	30.1847	6.7674
	100000	1200.8078	284.2979	1131.0076
	1000000	22058.5875	528.7325	21870.3750
	10000000	598454.2000	6634.4607	597553.5000
3-Pivot Kush	100	0.0118	0.0093	0.0095
	1000	1.0156	1.9968	0.5050
	10000	9.3897	19.5556	4.8184
	100000	1071.0240	157.6619	1032.5982
	1000000	22472.1100	1444.4493	22098.3000
	10000000	608203.3000	7020.9750	607993.0000
4-Pivot	100	0.0343	0.0814	0.0120
	1000	1.3838	2.5992	0.5333
	10000	4.1651	4.0935	3.1953
	100000	1158.2534	209.3988	1090.7000
	1000000	22077.3200	600.9565	22069.7000
	10000000	587479.3000	11805.1672	582773.5000

Branch Misses	Size	Mean	SD	Median
1-Pivot Hoare	100	224.4847	0.0588	224.4906
	1000	3612.1521	2.2119	3612.3146
	10000	49451.5958	72.9468	49464.5783
	100000	622681.4604	1879.7192	622662.0813
	1000000	7514512.0625	57015.6085	7528135.1250
	10000000	88601386.6500	1316610.4428	88910896.5000
Block Partition	100	140.5428	0.0876	140.5400
	1000	1570.7223	0.3938	1570.6199
	10000	16756.4235	7.9552	16757.5304
	100000	174650.0671	156.4744	174665.2439
	1000000	1813011.9915	5043.7407	1811789.7762
	10000000	20131908.6500	59866.7898	20134795.0000
1-Pivot Lomuto	100	208.5500	0.2674	208.4572
	1000	3431.9977	4.9888	3432.6774
	10000	47510.1674	62.4065	47492.9219
	100000	599369.5910	3094.0668	599127.2778
	1000000	7188431.3250	77244.2181	7204133.7500
	10000000	85572728.6000	1273266.3095	85589444.5000
Block Partition	100	112.3253	1.7694	112.7843
	1000	1847.2526	6.8858	1570.6199
	10000	25410.1772	123.7813	25350.7255
	100000	294244.7788	3465.6995	294878.4121
	1000000	3309146.6000	66124.1202	3313478.8000
	10000000	38173447.6500	1498913.4243	37931268.5000
2-Pivot Yaro	100	218.6639	0.0935	218.6437
	1000	3629.7245	1.7137	3630.2813
	10000	50301.2096	38.3586	50300.6213
	100000	635045.2987	2126.8544	634416.1873
	1000000	7624533.4500	45160.5385	7614424.8750
	10000000	90144952.0000	955908.5187	90243746.5000
Lomuto Block	100	108.9624	0.0537	108.9703
	1000	1423.2516	3.1800	1422.2247
	10000	17959.2518	49.0859	17951.8464
	100000	196951.7141	1583.5204	197172.6696
	1000000	2134863.0700	33965.1267	2119311.9000
	10000000	24700025.8000	856684.2381	24551996.5000
3-Pivot Kush	100	231.0457	0.0996	231.0266
	1000	3828.0046	1.7692	3827.5547
	10000	52692.9469	57.0170	52699.8674
	100000	660656.7803	2410.3154	660486.3048
	1000000	7881473.7500	55041.5020	7885420.7000
	10000000	93039500.9000	1138513.6270	93254088.5000
4-Pivot	100	233.3581	0.1961	233.3429
	1000	3847.1970	1.7934	3847.1812
	10000	52466.7069	69.7884	52466.1749
	100000	655540.9742	2248.3174	655697.4468
	1000000	7831009.1900	49027.0168	7835433.7000
	10000000	92702517.3500	1393247.4531	93065712.5000

Cache Misses	Size	Mean	SD	Median
1-Pivot Hoare	100	0.0205	0.0120	0.0188
	1000	0.6087	1.4656	0.2268
	10000	3.4867	2.7533	2.7844
	100000	1079.6629	230.8850	996.8944
	1000000	25764.5000	1166.9931	25475.1250
	10000000	634608.9000	13770.3278	633085.5000
Block Partition	100	0.0308	0.0710	0.0115
	1000	0.6074	1.3152	0.1710
	10000	2.1814	1.5526	1.6280
	100000	962.7563	36.5304	957.6906
	1000000	22939.3449	772.0659	22956.2750
	10000000	618067.7500	15870.0637	615660.5000
1-Pivot Lomuto	100	0.0309	0.0585	0.0114
	1000	1.5477	2.5235	0.7386
	10000	16.8345	33.0704	6.4018
	100000	1033.2760	46.1483	1016.9857
	1000000	25119.6125	1033.3887	24916.1250
	10000000	653495.7500	17064.0432	653582.5000
Block Partition	100	0.0382	0.0290	0.0314
	1000	0.3182	0.2477	0.2104
	10000	3.3952	3.7817	2.3033
	100000	1170.8087	38.3259	1170.6985
	1000000	25346.0800	820.9205	25164.7000
	10000000	664360.6000	21405.0063	659137.5000
2-Pivot Yaro	100	0.0470	0.1104	0.0155
	1000	1.3900	4.3104	0.3234
	10000	14.0858	30.1847	6.7674
	100000	1200.8078	284.2979	1131.0076
	1000000	22058.5875	528.7325	21870.3750
	10000000	598454.2000	6634.4607	597553.5000
Lomuto Block	100	0.0882	0.2168	0.0370
	1000	1.6396	5.9674	0.2243
	10000	5.6262	2.8924	5.2277
	100000	1160.9140	51.9771	1143.0456
	1000000	24010.0300	790.7382	24020.7000
	10000000	635062.3500	14272.0662	634228.5000
3-Pivot Kush	100	0.0118	0.0093	0.0095
	1000	1.0156	1.9968	0.5050
	10000	9.3897	19.5556	4.8184
	100000	1071.0240	157.6619	1032.5982
	1000000	22472.1100	1444.4493	22098.3000
	10000000	608203.3000	7020.9750	607993.0000
4-Pivot	100	0.0343	0.0814	0.0120
	1000	1.3838	2.5992	0.5333
	10000	4.1651	4.0935	3.1953
	100000	1158.2534	209.3988	1090.7000
	1000000	22077.3200	600.9565	22069.7000
	10000000	587479.3000	11805.1672	582773.5000

CPU Cycles	Size	Mean	SD	Median
1-Pivot Hoare	100	9327.4759	18.6229	9321.5095
	1000	141738.7953	210.1901	141718.3450
	10000	1896946.0648	20737.2369	1891616.5700
	100000	23894324.6266	136138.2225	23938976.7268
	1000000	283465722.4250	4456168.7125	282439687.8750
	10000000	3341781994.300	21642481.3347	3341179100.5000
	Block Partition	7261.6214	3.9023	7261.0766
	100	91497.6734	44.2229	91503.7709
	10000	1093098.0648	28276.9430	1087067.0555
	100000	12615251.1810	408397.6682	12525100.7167
Block Partition	1000000	142142979.8257	1816742.9514	141750443.9091
	10000000	1651547110.850	15522352.0233	1651945749.0000
1-Pivot Lomuto	100	8753.9251	19.1779	8747.1730
	1000	133820.5575	121.5330	133786.0877
	10000	1797257.9908	1206.3404	1797317.9663
	100000	22592563.3050	184018.1868	22560236.2500
	1000000	266780288.5625	1370978.1629	266496379.5000
	10000000	3176006233.550	31963176.2929	3178944155.5000
	Block Partition	7008.9387	28.2391	7005.8331
	100	119936.7616	332.2665	120045.5206
	10000	1635196.2005	3174.6739	1635068.8220
	100000	19944485.2959	165665.3166	19999479.0516
Block Partition	1000000	234756806.7100	3512955.3454	233878365.1000
	10000000	2779056745.450	88165361.1430	2783135055.5000
2-Pivot Yaro	100	8731.0535	13.1483	8733.6726
	1000	134846.2398	206.7811	134824.3757
	10000	1813160.2271	4167.9340	1812861.3252
	100000	22590180.7687	50252.7800	22586713.9725
	1000000	271262015.0750	2037510.3443	271168330.2500
	10000000	3222076486.750	22825580.2990	3228776030.0000
	Lomuto Block	6921.2645	7.4863	6920.0053
	100	115531.6098	1531.6089	115046.6171
	10000	1567567.6369	5508.4648	1567416.7897
	100000	18897769.3384	175912.1206	18848614.8083
Lomuto Block	1000000	216844641.5500	2817917.2648	216536644.3000
	10000000	2548418986.850	48249924.4893	2559985148.0000
3-Pivot Kush	100	8612.3759	35.5221	8603.2238
	1000	130997.1678	1591.6522	130493.5023
	10000	1736859.7051	1691.3828	1736487.8719
	100000	21593228.2474	199876.9076	21493032.9813
	1000000	257443837.9800	1733684.7359	256993526.2000
	10000000	3026119949.450	23471563.5615	3025067078.5000
4-Pivot	100	8563.8790	9.0442	8562.7883
	1000	129414.4258	60.9929	129430.0027
	10000	1712788.0590	1490.2721	1712866.7067
	100000	21148019.6577	123063.0849	21104913.9275
	1000000	249836540.5700	1201917.2970	249647638.0000
	10000000	2961500698.750	21436898.4187	2957871275.5000

Time	Size	Mean	SD	Median
1-Pivot Hoare	100	2.1751 μ s	7.3394 μ s	2.1724 μ s
	1000	33.659 μ s	203.14 μ s	33.553 μ s
	10000	442.01 μ s	2.1185 μ s	441.58 μ s
	100000	5.5583 ms	79.072 μ s	5.5294 ms
	1000000	66.392 ms	1.0688 ms	66.594 ms
	10000000	776.49 ms	3.7364 ms	776.07 ms
Block Partition	100	1.6938 μ s	4.7546 ns	1.6926 μ s
	1000	21.782 μ s	107.85 ns	21.743 μ s
	10000	259.09 μ s	490.99 ns	258.97 μ s
	100000	2.9872 ms	9.5993 μ s	2.9855 ms
	1000000	33.808 ms	159.14 μ s	33.788 ms
	10000000	394.79 ms	2.9162 ms	395.56 ms
1-Pivot Lomuto	100	2.0334 μ s	5.281 ns	2.0322 μ s
	1000	31.105 μ s	123.19 ns	31.062 μ s
	10000	419.93 μ s	3.2284 μ s	418.01 μ s
	100000	5.2008 ms	19.270 μ s	5.1963 ms
	1000000	62.292 ms	452.07 μ s	62.217 ms
	10000000	745.10 ms	6.7564 ms	743.02 ms
Block Partition	100	1.6698 μ s	4.2852 ns	1.6689 μ s
	1000	28.453 μ s	87.002 ns	28.424 μ s
	10000	392.02 μ s	1.3621 μ s	391.81 μ s
	100000	4.7857 ms	42.338 μ s	4.7824 ms
	1000000	56.288 ms	884.73 μ s	56.358 ms
	10000000	668.14 ms	21.283 ms	662.61 ms
2-Pivot Yaro	100	2.0848 μ s	6.1677 ns	2.0827 μ s
	1000	31.372 μ s	95.770 ns	31.347 μ s
	10000	429.91 μ s	5.6708 μ s	428.28 μ s
	100000	5.3164 ms	15.375 μ s	5.3128 ms
	1000000	63.092 ms	500.09 μ s	62.958 ms
	10000000	748.26 ms	4.8510 ms	749.60 ms
Lomuto Block	100	1.6387 μ s	4.3470 ns	1.6373 μ s
	1000	27.249 μ s	52.407 ns	27.249 μ s
	10000	374.62 μ s	3.3842 μ s	373.88 μ s
	100000	4.5048 ms	31.057 μ s	4.5022 ms
	1000000	52.149 ms	802.78 μ s	52.234 ms
	10000000	605.61 ms	13.676 ms	602.31 ms
3-Pivot Kush	100	1.9985 μ s	9.7010 ns	1.9943 μ s
	1000	30.422 μ s	80.286 ns	30.392 μ s
	10000	411.09 μ s	798.13 ns	410.84 μ s
	100000	5.0570 ms	28.975 μ s	5.0482 ms
	1000000	60.328 ms	906.06 μ s	60.771 ms
	10000000	717.27 ms	6.5670 ms	716.72 ms
4-Pivot	100	1.9923 μ s	6.2348 ns	1.9901 μ s
	1000	30.422 μ s	94.982 ns	30.400 μ s
	10000	403.15 μ s	1.7442 μ s	402.50 μ s
	100000	4.9661 ms	23.757 μ s	4.9566 ms
	1000000	58.758 ms	365.07 μ s	58.683 ms
	10000000	696.42 ms	5.1556 ms	696.82 ms

References

- [1] Hoare, C.A.R. (1962). Quicksort. *The Computer Journal*, 5, 10-15.
- [2] Yaroslavskiy, V. (2009). Dual-Pivot Quicksort. *Proceedings of the 6th International Conference on Computer Science and Education*, 13-27.
- [3] Sedgewick, R. (1978). Implementing Quicksort Programs. *Communications of the ACM*, 21(10), 847-857.
- [4] Shrinu Kushagra, Alejandro López-Ortiz, Aurick Qiao, and J. Ian Munro. *Multi-pivot quicksort: Theory and experiments*. In ALENEX, pages 47-60, 2014.
- [5] David R. Musser. *Introspective Sorting and Selection Algorithms*. Software: Practice and Experience, 27(8):983-993, 1997.
- [6] BlockQuicksort: How Branch Mispredictions don't affect Quicksort. *Stefan Edelkamp and Armin Weiß*. 2016.
- [7] Martin Aumüller and Martin Dietzfelbinger. 2013. *Optimal Partitioning for Dual Pivot Quicksort*. In Proc. of the 40th International Colloquium on Automata, Languages and Programming (ICALP'13). Springer, 33-44.