1 Introduction

Sorting algorithms on modern computers have been playing an important role in Computer Science and can date back to the early 50s and 60s. It is the great efforts that humans have put into the field of developing faster sorting algorithms that shapes the vital parts that formed all the significant infrastructures of the vast computer world like databases, data centers, cluster networks etc. In the realm of sorting algorithms, Quicksort, introduced by computer scientist Tony Hoare [1] to the public, has made its way into the libc of the GNU/Linux system and stood the test of time as one of the most classical and widely used methods for sorting. Its divide-and-conquer strategy has been widely used and taken into research to fully utilize the performance of sorting. The basic Quicksort picks an arbitrary pivot element, partitions the array into 2 segments, one with elements less than the pivot and the other greater. It then recursively sorts these segments using the same method. While the original Quicksort method has proven effective, the demand of further optimizations and the endless pursuit of efficiency has led to many variations to explore, one of which is the Multi-Pivot Quicksort.

This thesis is my shallow attempt to unveil the benefits of Multi-Pivot Quicksort, exploring the effects more pivots can bring, followed by some leveraged implementations of Multi-Pivot QuickSort and other variations alike and derived, analysis on the pros and cons that arise from multiple pivots.

1.1 History and Related work

When Quicksort was first brought to the fore a great number of variations and modifications were put into research over the years, such as sorting with a better partition method and a more optimal strategy to select the arbitrary pivot element. The innovative approach of Multi-pivot Quicksort bloomed, exemplified by Vladimir Yaroslavskiy [2] in 2009, Bentley and Bloch's algorithm (YBB Alogorithm), which was also highlighted by Wild and Nebel, adopted in Sun's Java 7 runtime library, introduces the use of two pivots. Contrary to initial expectations, studies have revealed that employing multiple pivots can enhance the performance, challenging prior assumptions derived from the dual-pivot proposal by Sedgewick [3] in 1978 where he analyzed a dual-pivot approach, which was deemed inferior to the classical Quicksort in his research. Furthermore, Kushagra presented a 3-pivot Quicksort [4] algorithm at ALENEX, has attracted significant attention, further pushing the researches to broaden the boundaries of multi-pivot strategy. But how does it deal with the worst cases and how does it perform in practice?

One common pitfall in the performance downgrade of the conventional Quicksort is the degenerated case, which could often arise when the input array is already mostly sorted or when certain patterns in the data cause the algorithm to consistently choose poor pivots that fail to split the data into evenly sized partitions. In such cases, the partitioning process may result in highly imbalanced partitions where the size of one partition is significantly larger than the other, leading to suboptimal performance and potentially degrading the time complexity to $\mathcal{O}(n^2)$ instead of the expected $\mathcal{O}(n \log n)$ for Quicksort. This can be a serious issue when it comes to scalability on

large datasets, thus wiser choices of pivot selections could decrease the possiblities of such case from happening, such as the median-of-3 method and its variations of median-of-5 and median-of-7. But on mostly ascending or descending arrays, whichever pivot we choose doesn't matter, the worst case is inevitable, and it might not the best ideal approach to prevent the worst case from happening. Real world applications also bring concerns to us that duplicated elements are not rare, and the median-of-n method could be easily affected by the duplicated elements, which could lead to the same worst case as the original Quicksort. Of course we can set a threshold to set the minium length to use median-of-n method, but it doesn't fix the problem once and for all. There are some other variations of Quicksort that have been proposed to address these specific issues, such as the Introsort algorithm [5] by David Musser, which switches to Heapsort when the recursion depth exceeds a certain threshold. Thus the worst running time is guaranteed to be $\mathcal{O}(n \log n)$ by doing so.

And this is where Multi-pivot quicksort comes into play. Assuming we are dealing with randomly generated arrays, since we don't know the exact size of each partition, the more pivots we choose, the more partitions we divide the array into, the less chance it will be that we encounter imbalanced sizes of partitions. Moreover, choosing more pivots will significantly reduce the chance we access the same elements again and the maximum depth of recursion, which can translate into better running time. With n pivots we are dividing the array into n+1 partitions, the same things happen on the sub-partitions in the next level of recursion and we can conclude the maximum depth of recursion would be $\log_{n+1} L$ given **n** as the number of pivots and **L** for the length of array. Greater the n is, less the depth. Better memory behaviors can bring effects that can not be achieved by any of other traditional means such as choosing a better pivot value among samples extracted from different parts of the array, but the drawbacks are also obvious, branch misprediction could be terrible as the different outcomes of comparisons can largely impact on the runtime performance.

This phenomenon was first observed by Kaligosi and Sanders (2006) on a Pentium 4 Prescott CPU, a processor with an extremely long pipeline, where the branch misprediction penalty was so high that a very skewed pivot choice outperformed the typically optimal median pivot, even though the latter leads to much fewer executed instructions in total. The effect was not reproducible, however, on the slightly different Pentium 4 Willamette. Here two effects counteract: a biased pivot makes branches easier to predict but also gives unbalanced partition sizes. Brodal and Moruz have shown that this is a general trade-off in comparison-based sorting: One can only save comparisons at the price of additional branch misses. [6]

This could also cause a delay of 14 to 19 stages on a typical Intel desktop CPU or 20 to 25 stages on AMD's Zen CPUs could lead to huge efficiency loss for our algorithm, and this number only grows as the number of pivots increases and on more complicatad architectures. These are the trade-offs we have to consider when we are choosing the number of pivots, although these side effects can be mitigated by the use of branchless comparisons and modern CPUs are able to run conditional move operations (CMOVE on x86) to avoid branches, these overheads still remain as concerns, but it could relief the conflicting situations between total comparisons and the cost of branch mispredictions in some ways. Apart from that, it could be really

tricky when it comes to moving the elements to the right places among multiple partitions, especially when the number of pivots is large. Additionally, comparisons are the second expensive things next to swaps we are dealing with, and the more pivots we choose, the more comparisons we have to make. Even for the case of using only one pivot, the average number of comparisons remain unoptimial, being $1.38n\log n+\mathcal{O}(n)$ compared with the $n\log n+\mathcal{O}(n)$ of the Merge Sort, but the overall instructions are much less than the average of Merge Sort and with proper pivot selection strategy, it can still be reduced to a certain extent.

Fortunately, an efficient approach has been put forward by Stefan Edelkamp and Armin Weiß, in their thesis of BlockQuickSort [7] which evens the odds of both branch mis-prediction and cache misses which uses one or more buffer blocks of constant sizes on the stack to store the index offsets of out-of-order elements, swap them altogether in a second pass to rearrange them in order after scanning a large chunk of array with the size of the block. It has eliminated almost all the if branches during comparisons, except inevitable branches produced by terminating situation of for loops which traverse the whole array. Their approach realized a significant speedup up to 80% faster compared with the GCC implementation of std sort with only one single pivot. The BlockQuickSort has been adapted to the Multi-Pivot Quicksort and the results are also promising, the improvements on contiguous memory access significantly improves the spatial locality and reduces both the cache misses and branch mis-predictions. Branchless comparisons are also adapted which plays an important role in contributing to the huge performance boost by reducing the overhead when increasing the counter of elements stored in the block buffer(s). This novel approach has also developed variations that could effectively make the use of different partitioning methods and multiple pivots selected.

Meanwhile, in the pursuit of advancing the efficiency of the Quicksort algorithm, my investigation led me to explore various modifications, with a particular focus on the pattern-defeating Quicksort (PDQSort) proposed by Orson Peters. PDQSort leverages the average-case performance of randomized Quicksort with the rapid worst-case execution of heapsort, while concurrently achieving linear time complexity for slices exhibiting specific patterns. Notably, PDQSort employs a judicious application of randomization to prevent the degenerated case from happening. When it does, PDQSort will strategically shuffle or reverse the entire array to simplify the sorting and potentially could increase the speed. If all the counter measures fail, it also has the final backup plan of switching to Heapsort when bad choices of pivots or the depth of recursions exceed the limit, learning the lesson from Introsort. In terms of pivot selection, it uses a fixed seed when generating a pseudo-random index to ensure the reproducibility and deterministic behavior. Furthermore, in instances where randomly selected elements from the original array appear in a descending order. The combination of these optimizations has demonstrated a significant enhancement in performance, showcasing PDQSort as up to twice faster than the original Quicksort algorithm. This outcome illuminates the transformative impact that a comprehensive analysis on the pattern of arrays has on the speed of sorting algorithms, not to mention it is a successful combination of BlockQuickSort's partition method, Heap sort when dealing with worst cases and Insertion sort when it comes to small sub-arrays of sizes less than the cache line size. Inspired by PDQSort's success, I dive

into the diverse variations of Quicksort, motivating a search for novel patterns finely tweaked to enhance the existing Quicksort algorithms.

1.2 Contributions

- Present the variants of Multi-Pivot QuickSort and implement a 4-Pivots Quicksort and evaluate N-Pivots Quicksort from N = 1, 2, 3, 4 experimentally. Comparing the implementations, times, instructions, cache misses and branch mis-predictions with a duration of 3 seconds warm-up time.
- Present a new layout of Multi-Pivot BlockQuickSort that could potentially reduce the memory access and increase the block buffer usage rate. Analyze this method with all other available variants with block partitioning and compare the pros and cons.

2 Preliminaries

2.1 Branch Misses

Branch mispredictions are one of the most common performance bottlenecks in modern CPU architectures. When the CPU encounters a conditional branch instruction like if statement, while and for loops, etc., it has to predict the outcome of the branch before the condition is evaluated. A typical modern CPU will decide which branch to follow, prefetch the instructions and data from the memory, decode the subsequent instructions and execute them in advance to reduce the latency of the execution. Because of the speculative execution, the CPU can execute the instructions in parallel, and if the prediction is correct, a lot of time can be saved, otherwise it has to stall the pipeline, flush the instructions and start over. The branch misprediction penalty is usually around 10 to 20 cycles, depending on the stages' length of pipelines, which can cause a huge overhead.

2.2 How to avoid branch mispredictions

One way to avoid branch mispredictions is to use branchless comparisons. The branchless comparison is a technique that uses bitwise operations to compare two values and generate a mask that represents the result of the comparison. For example, to compare two integers a and b, we can use the following code:

```
int mask = (a < b) - (a > b);
```

The mask will be -1 if a < b, 0 if a == b, and 1 if a > b. We can then use the mask to perform conditional operations without using branches. This technique can be used to avoid branch mispredictions in sorting algorithms, where comparisons are the most common operations. Another way to go around is the CMOVE instruction, which is a conditional move instruction that is available on modern CPUs. In x86 assembly, the CMOVE instruction can be used to conditionally move a value from one register to another based on a mask similar to the one generated by the branchless

comparison. An example of using it to swap two integers a and b if a < b is shown below:

```
mov eax, a
mov ebx, b
cmp eax, ebx
cmovl eax, ebx
cmovl ebx, eax
```

which in readable C code would be:

```
if (a < b) {
    int temp = a;
    a = b;
    b = temp;
}
```

but without the branch misprediction overhead, this technique is particularly useful when optimizing sorting algorithms that rely heavily on comparisons. A straightforward variation could be casting boolean values to integers and add them up to the accumulator variable, which could be used to determine how many elements are less than the pivot. The pseudo code is given as:

```
int less = 0;
for (int i = 0; i < n; i++) {
    less += (A[i] < pivot);
}
```

Here the less variable will add 1 if A[i] is less than the pivot, and 0 otherwise, and this could be used to determine the index where the pivot should be placed in the branch-free way.

2.3 Cache Misses

Caches, which are small and fast memory units located on the CPU, are used to store frequently accessed data and instructions to reduce the latency of memory access. When the CPU needs to access data or instructions, it will first check the cache, if the data is found in the cache, it is called a cache hit, otherwise, it is called a cache miss. Closer it is to the CPU, the more expensive the manufacturing cost is, and the smaller the size is. The cache is usually divided into several levels, L1, L2, L3, etc. L1 and L2 caches are usually private to each core, while L3 cache is shared among all the cores in the same CPU. When the CPU needs to access data, it will first check the L1 cache, if the data is not found, it will check the L2 cache, and then the L3 cache, and finally the main memory, accelerating the memory access. With the context of sorting algorithms, it is a convention to switch to insertion sort when it comes to arrays with sizes less than a cache line size, the exact threshold could be around 9 to 30 elements, varying on the architecture.

2.4 Classical QuickSort

Algorithm 1 QuickSort with Hoare Partition

```
1: procedure QUICKSORT(A, l, r)
2:     if l < r then
3:         p ← HOAREPARTITION(A, l, r)
4:         QUICKSORT(A, l, p-1)
5:         QUICKSORT(A, p+1, r)
6:     end if
7: end procedure
8: procedure HOAREPARTITION(A, l, r)
9:     P ← A[l]                                   ▷ Pivot
10:    i ← l − 1
11:    j ← r + 1
12:    while i < j do
13:        repeat
14:            j ← j − 1
15:        until A[j] < P
16:        repeat
17:            i ← i + 1
18:        until A[i] > P
19:        SWAP(A[i], A[j])
20:    end while
21:    SWAP(A[l], A[j])                           ▷ Put pivot in the middle
22:    return j
23: end procedure
```

For Quicksorts with single pivot value, the partitioning procedure splits the array into 2 parts where the left part are elements less than the pivot and the right part being greater, separated by the pivot value in the middle. Then the main body recursively sorts the left and right part to rearrange all the elements in the correct order. The hoare's partition method uses double pointers starting at the leftmost and the rightmost element in the array and scanning towards the middle until they meet. Each pointer scans for mis-placed elements which belong to the other partition by comparing the current element to the pivot. Out-of-order elements are swapped in pairs to the right place and the scanning procedure continues until all elements are moved to the places where they belong.

Algorithm 2 QuickSort with Lomuto Partition

```
1: procedure QUICKSORT(A, l, r)
2:     if l < r then
3:         p ← LOMUTOPARTITION(A, l, r)
4:         QUICKSORT(A, l, p-1)
5:         QUICKSORT(A, p+1, r)
6:     end if
7: end procedure
8: procedure LOMUTOPARTITION(A, l, r)
9:     P ← A[r]                                   ▷ Pivot
10:    i ← l
11:    for j ← l to r − 1 do
12:        if A[j] < P then
13:            SWAP(A[i], A[j])
14:            i ← i + 1
15:        end if
16:    end for
17:    SWAP(A[i], A[r])                           ▷ Put pivot in the middle
18:    return i
19: end procedure
```

While the Lomuto partition method uses another way, it scans the array in sequential order from left to right using one single pointer, incrementing the index by 1 each time. and swaps all the elements less than the pivot to the left side discriminately, resulting in rebundant swaps as it fails to check if the elements are already in their correct positions. Despite its simplicity, both of these 2 partition methods are straightforward, and share the same idea of partitioning with similar structures: scanning, identifying misplaced elements, swapping and recursively repeating the procedure until the whole array is sorted.

It is generally believed that Hoare's partition method is more efficient than the Lomuto's, primarily due to its reduced number of swaps, enhanced partitioning strategy and memory efficiency. While Hoare's partition scheme minimizes unnecessary swaps compared to Lomuto's method, which tends to perform rebundant swaps for arrays with many duplicated elements or already sorted elements. Nevertheless, Lomuto's scheme boasts simpler implementuses with less auxiliary variables used, contributing to wide adoption in many programming languages and libraries. However, these theoretical analysis must be validated through empirical experiments and brought into practice to see if the assumptions hold true, as the performance of the two partition methods could vary depending on the dataset's characteristics and the size of the array. Empirical studies are crucial to verify the assumptions and to determine the most suitable method for certain scenarios and contexts.

2.5 Block Partitioning

The following pseudocode is extracted from the BlockQuickSort paper by Stefan Edelkamp and Armin Weiß [7] who hold the copyrights of the original code. This method introduces a parallel-looking and branch misprediction friendly version of the classical QuickSort algorithm that makes use of a block buffer of constant size to store the index offsets of out-of-order elements on both sides of the array, and swaps them in the second pass to rearrange them. In the actual implementation I used the block size of 128 suggested by examples with least time costs in the paper, as the index offsets in this case would be 0 to 127 (**0b1111 1111**) as 1 byte for each cell in the buffer. A block size of 128, which in turn would be 128 bytes in total, neatly fits into 2 cache lines, each line 64 bytes on the test machine. Notably, it is also reflected in their paper that the block size of over 256 bytes will bring down the performance and is less stack memory friendly, especially on large data structures like 10-dimensional array of 64-bit vectors and 'records', referring to 21-dimensional array of 32-bit integers. An advantageous aspect of selecting a power-of-2 block size is it be easily optimized by the compiler to use bitwise operations to calculate the offsets. Loop unrolling inside the LLVM compiler could also give us a hand to optimize the code further and reduce the overhead of the loop termination condition check. Furthermore, they are placed on stack to get rid of the extra memory allocation and deallocation overheads on heap. Let's dive into the details of its implementation by following the pseudocode.

Algorithm 3 Block Partition Hoare

```
1: procedure BlockPartitionHoare(A[ℓ,...,r], pivot)
2:     uint offsets_L[0,...,B−1], offsets_R[0,...,B−1]
3:     uint start_L, start_R, num_L, num_R ← 0
4:     while r − ℓ + 1 > 2B do                                   ▷ start main loop
5:         if num_L = 0 then                                     ▷ if left buffer is empty, refill it
6:             start_L ← 0
7:             for i = 0,...,B−1 do
8:                 offsets_L[num_L] ← i                          ▷ Will this regardless write slows it down?
9:                 num_L += (pivot ≥ A[ℓ + i])                   ▷ branchless comparison to add counter
10:            end for
11:        end if
12:        if num_R = 0 then                                     ▷ if right buffer is empty, refill it
13:            start_R ← 0
14:            for i = 0,...,B−1 do
15:                offsets_R[num_R] ← i
16:                num_R += (pivot ≤ A[r − i])                   ▷ branchless comparison to add counter
17:            end for
18:        end if
19:        uint num = min(num_L, num_R)
20:        temp ← A[ℓ + offsets_L[start_L]]                      ▷ rearrangement using cyclic permutation
21:        A[ℓ + offsets_L[start_L]] ← A[r − offsets_R[start_R]]
22:        for j = 1,...,num−1 do
23:            A[r − offsets_R[start_R + j − 1]] ← A[ℓ + offsets_L[start_L + j]]
24:            A[ℓ + offsets_L[start_L + j]] ← A[r − offsets_R[start_R + j]]
25:        end for
26:        A[r − offsets_R[start_R + num − 1]] ←temp
27:        num_L, num_R −= num; start_L, start_R += num          ▷ update the counters for buffers
28:        if (num_L = 0) then ℓ += B end if                     ▷ if left buffer is empty, move the left pointer
29:        if (num_R = 0) then r −= B end if                     ▷ Vice versa
30:    end while                                                 ▷ end main loop
31:    rearrange remaining elements with the standard Hoare's method
32: end procedure
```

In the presented pseudocode, we witness a notable demonstration of how branchless comparisons are incorporated and function. The counter of elements less than the pivot is incremented by 1 if the condition is met, achieved by converting the result into a boolean value and adding it to the counter. This elegant approach eliminates the occurrences of branching on counter manipulations. The same technique is applied to the right buffer simultaneously, and the minimum number of elements in both buffers is calculated to ensure the correct number of elements are swapped, also in a branchless manner.

The block partitioning method represents a novel approach that not only reduces the number of swaps but also improves the spatial locality of both the memory access and cache access. I replaced the original swap function in the rearrangement phase with a cyclic permutation method also outlined in the referenced thesis, a finely tuned way to rearrange the elements in the buffer blocks.

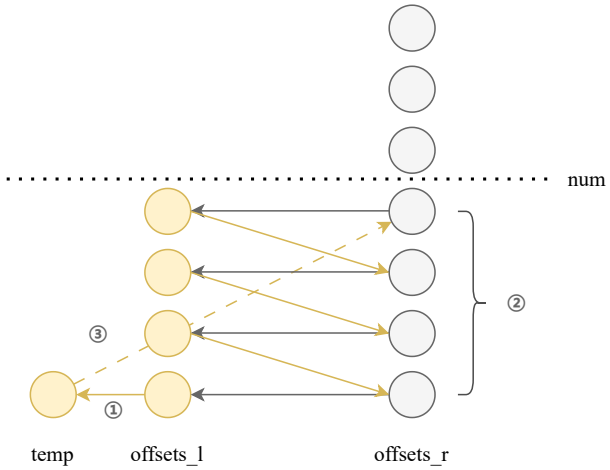
For better understanding, we can compare this cyclic permutation to a rotate-n

operation where n stands for the number of elements that need to be swapped to the right places. Consequently, there may be elements left in one of the buffer blocks, but we will just shift the other block towards the middle and rearrange the elements left behind in the next iteration.

For testing machine with 32MB L3 cache, the block size of 128 bytes would be a suitable choice as not only it would fit into 2 cache lines, but for the maximum testing size of 10^7 elements, the maximum recursion depth would be $\lceil \log \rceil_2 10^7 = 24$, the maximum stack usage would be around $24 \times 128 bytes = 3Kb$, a small amount for both memory and cache even with little extra space required by RIP, RBP registers and other local variables, ensuring its cache friendliness and cache efficiency.

Another thing we might concern is the regardless write of the offsets in the buffer blocks, which overwrites the previous value unless the offset index is incremented by 1 in the last step of scan. While this operation doesn't incur much overhead thanks to the write-back policy of the cache line supported by modern CPUs and is crucial for them, which replaces the old value with the new one in the cache line and write back to main memory only if the value is final, to get rid of multiple expensive direct main memory manipulation and stalls the entire pipeline. This action is penalty-free and the cost is much less than write-through operations that updates the main memory every time a write operation occurs. Eventually, the buffer blocks will be written back to the main memory eventually, in a deterministic manner. This is ensured by the write-back policy of the cache line, and is a common stable feature in modern CPUs to avoid excessive memory access. To offer a clearer visual comprehension of the cyclic permutation, a simple figure illustrating how it works is provided right below‘.

Figure 1: Cyclic Permutation



Algorithm 5 One-Pivot Block Partition Lomuto

procedure BlockLomuto1($A[1..n]$)
Require: $n > 1$, Pivot in $A[n]$
1: $\mathtt{p} \leftarrow A[n]$;
2: **integer** block$[0,\ldots,B-1]$, $\mathtt{i},\mathtt{j} \leftarrow 1, \mathtt{num} \leftarrow 0$
3: **while** $\mathtt{j} < n$ **do**
4: $\quad \mathtt{t} \leftarrow \min(B, n-\mathtt{j})$;
5: $\quad$ **for** $\mathtt{c} \leftarrow 0; \mathtt{c} < \mathtt{t}; \mathtt{c} \leftarrow \mathtt{c}+1$ **do**
6: $\quad\quad$ block[num] $\leftarrow \mathtt{c}$;
7: $\quad\quad \mathtt{num} \leftarrow \mathtt{num} + (\mathtt{p} > A[\mathtt{j}+\mathtt{c}])$;
8: $\quad$ **end for**
9: $\quad$ **for** $\mathtt{c} \leftarrow 0; \mathtt{c} < \mathtt{num}; \mathtt{c} \leftarrow \mathtt{c}+1$ **do**
10: $\quad\quad$ Swap $A[\mathtt{i}]$ and $A[\mathtt{j} + \text{block}[\mathtt{c}]]$
11: $\quad\quad \mathtt{i} \leftarrow \mathtt{i}+1$
12: $\quad$ **end for**
13: $\quad \mathtt{num} \leftarrow 0$;
14: $\quad \mathtt{j} \leftarrow \mathtt{j}+\mathtt{t}$;
15: **end while**
16: Swap $A[\mathtt{i}]$ and $A[n]$;
17: **return** i;

Algorithm 4 Cyclic Permutation

temp $\leftarrow A\big[\ell + \text{offsets}_L[\text{start}_L]\big]$ ▷ (1) preserve the last element in temp variable
$A\big[\ell + \text{offsets}_L[\text{start}_L]\big] \leftarrow A\big[r - \text{offsets}_R[\text{start}_R]\big]$ ▷ (2) start with the first one
for $j = 1,\ldots,\text{num}-1$ **do** ▷ main cyclic for loop
$\quad A\big[r - \text{offsets}_R[\text{start}_R + j - 1]\big] \leftarrow A\big[\ell + \text{offsets}_L[\text{start}_L + j]\big]$
$\quad A\big[\ell + \text{offsets}_L[\text{start}_L + j]\big] \leftarrow A\big[r - \text{offsets}_R[\text{start}_R + j]\big]$
end for
$A\big[r - \text{offsets}_R[\text{start}_R + \text{num} - 1]\big] \leftarrow \text{temp}$ ▷ (3) put the preserved element back to the last place

Here we mapped the pseudocode in parts to the steps in the figure, the first step is to preserve the last element in the left buffer block in a temporary variable, then we doing the cyclic permutation in the main loop, shifting misplaced elements from buffers on one side to the other to make sure they are well partitioned, and finally we put the preserved element back to the last place in the right buffer block. This rotate-n like method is basically made up of multiple copys from pointers in order to reduce the number of swaps to the minimum, as swaps will constantly overwrite the value in the temporary variable. It is a great improvement over the original Hoare's partition.

We also have adaptions of the Block Lomuto's partition method. Comprehension won't be hard on basis of our fundamental understandings of the classical Lomuto's partition method.

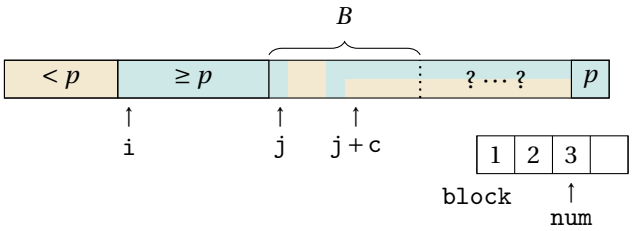


Figure 2: BlockQuicksort with Lomuto's partitioning scheme.

Partition left to the index j is the sub-result of last iteration, we are currently scanning on the block which covers the range of $A[j..j+B-1]$, and the block buffer is used to store the index offsets of the misplaced elements inside. The first element with index 0 is neglected for being blue – greater than the pivot, and the yellow elements with index 1 and 2 – less than the pivot, are stored in the buffer. The third blue element with index 2, temporarily stored in the buffer, will be replaced soon by the next misplaced item in the next iteration that deals with unknown values that can be either less or greater than the pivot.

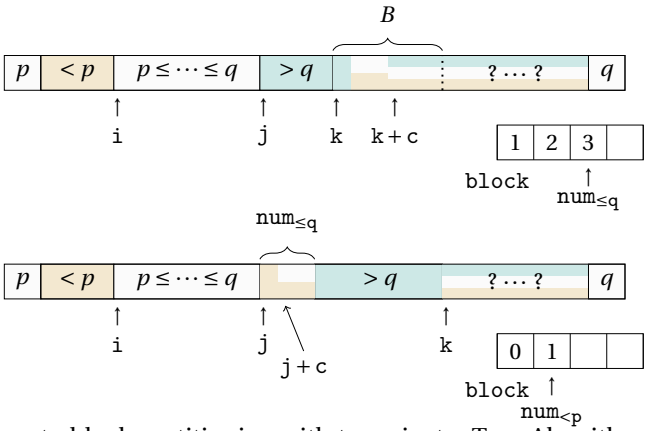


Figure 3: Lomuto block partitioning with two pivots. Top: Algorithm 6 compares elements with q and is on Line 8 with c = 3, cf. Figure 2. Bottom: Algorithm compares the $\texttt{num}_{\leq \texttt{q}}$ elements at most as large as q to p. Algorithm is on Line 14 with c = 1.

We can observe that there exists one more pass for 2-Pivot Block Lomuto's partition method than the 1-Pivot one. The first pass compares with the larger pivot q and the second pass compares with the smaller pivot p. The intermediate state between the 2 passes shows a mixture of elements that are less than p and at most as large as q swapped to the exchange area in the first pass. The second pass will then move the elements that are less than p to the left side of the array.

2.6 rotate-n operations

The rotate-n operation is a common operation in the Quicksort algorithm, where we shift the elements by their indices in batch to get rid of the duplicated swaps, and to

Algorithm 6 Two-Pivot Block Partition Lomuto

procedure BlockLomuto2($A[1..n]$)
Require: $n > 1$, Pivots in $A[1] \le A[n]$
1: $\mathtt{p} \leftarrow A[1]$; $\mathtt{q} \leftarrow A[n]$;
2: **integer** block$[0,\ldots,B-1]$
3: $\mathtt{i},\mathtt{j},\mathtt{k} \leftarrow 2, \mathtt{num}_{<\mathtt{p}}, \mathtt{num}_{\le \mathtt{q}} \leftarrow 0$
4: **while** $\mathtt{k} < n$ **do**
5: $\quad \mathtt{t} \leftarrow \min(B, n-\mathtt{k})$;
6: $\quad$ **for** $\mathtt{c} \leftarrow 0; \mathtt{c} < \mathtt{t}; \mathtt{c} \leftarrow \mathtt{c}+1$ **do**
7: $\quad\quad$ block$[\mathtt{num}_{\le \mathtt{q}}] \leftarrow \mathtt{c}$;
8: $\quad\quad \mathtt{num}_{\le \mathtt{q}} \leftarrow \mathtt{num}_{\le \mathtt{q}} + (\mathtt{q} \ge A[\mathtt{k}+\mathtt{c}])$;
9: $\quad$ **end for**
10: $\quad$ **for** $\mathtt{c} \leftarrow 0; \mathtt{c} < \mathtt{num}_{\le \mathtt{q}}; \mathtt{c} \leftarrow \mathtt{c}+1$ **do**
11: $\quad\quad$ Swap $A[\mathtt{j}+\mathtt{c}]$ and $A[\mathtt{k}+\text{block}[\mathtt{c}]]$
12: $\quad$ **end for**
13: $\quad \mathtt{k} \leftarrow \mathtt{k}+\mathtt{t}$;
14: $\quad$ **for** $\mathtt{c} \leftarrow 0; \mathtt{c} < \mathtt{num}_{\le \mathtt{q}}; \mathtt{c} \leftarrow \mathtt{c}+1$ **do**
15: $\quad\quad$ block$[\mathtt{num}_{<\mathtt{p}}] \leftarrow \mathtt{c}$;
16: $\quad\quad \mathtt{num}_{<\mathtt{p}} \leftarrow \mathtt{num}_{<\mathtt{p}} + (\mathtt{p} > A[\mathtt{j}+\mathtt{c}])$;
17: $\quad$ **end for**
18: $\quad$ **for** $\mathtt{c} \leftarrow 0; \mathtt{c} < \mathtt{num}_{<\mathtt{p}}; \mathtt{c} \leftarrow \mathtt{c}+1$ **do**
19: $\quad\quad$ Swap $A[\mathtt{i}]$ and $A[\mathtt{j}+\text{block}[\mathtt{c}]]$
20: $\quad\quad \mathtt{i} \leftarrow \mathtt{i}+1$
21: $\quad$ **end for**
22: $\quad \mathtt{j} \leftarrow \mathtt{j} + \mathtt{num}_{\le \mathtt{q}}$;
23: $\quad \mathtt{num}_{<\mathtt{p}}, \mathtt{num}_{\le \mathtt{q}} \leftarrow 0$;
24: **end while**
25: Swap $A[\mathtt{i}-1]$ and $A[1]$;
26: Swap $A[\mathtt{j}]$ and $A[n]$;
27: **return** $(\mathtt{i}-1, \mathtt{j})$;

make the partitioning process more plain and simple to understand. This improves slightly the performance of the Quicksort algorithm, and is widely used in tweaked Multi-Pivot QuickSort. The rotate-n operation is defined as follows:

```
rotaten(A, [index1, index2, ... indexn]);
```

which is equivalent to the following code:

```
T temp = A[index1];
A[index1] = A[index2];
...
A[indexn] = temp;
```

In Rust I implemented the rotate-n operation as follows:

```
#[inline(always)]
pub unsafe fn rotate_$n<T>(arr: *mut T, idx: [usize; $n]) {
    let tmp = std::ptr::read(arr.add(idx[0]));
```

```
    for i in 1..$n {
        ptr::copy_nonoverlapping(arr.add(idx[i]), arr.add(idx[i - 1]), 1);
    }
    ptr::copy_nonoverlapping(&tmp, arr.add(idx[$n - 1]), 1);
}
```

Using direct pointer manipulation, we can avoid the overhead of the Rust borrow checker and array boundary checking to maximize the performance.

2.7 Multi-Pivot QuickSort

Algorithm 7 Dual-Pivot QuickSort

procedure DUALPIVOTPARTITION(A, l, r)
  $P1, P2 \leftarrow A[l], A[r]$ ▷ Pivots
  if $P1 > P2$ **then**
    SWAP($P1, P2$)
  end if
  $less \leftarrow l + 1$
  $greater \leftarrow r - 1$
  $k \leftarrow l$
  while $k \leq greater$ **do**
    if $A[k] < P1$ **then** ▷ If the current element is less than P1
      SWAP($A[k], A[less]$) ▷ Shift the element to the leftmost partition
      $less \leftarrow less + 1$
    else if $A[k] > P2$ **then** ▷ If the current element is greater than P2
      while $A[greater] > P2$ **and** $k < greater$ **do**
        $greater \leftarrow greater - 1$ ▷ Find the first element less than P2
      end while
      SWAP(A[k], A[greater]) ▷ Swap to the rightmost partition
      $greater \leftarrow greater - 1$
      if $A[j] < P1$ **then** ▷ Double check if another swap is needed
        SWAP($A[k], A[less]$)
        $less \leftarrow less + 1$
      end if
    end if
    $k \leftarrow k + 1$ ▷ Move to the next element
  end while
  $less \leftarrow less - 1$
  $greater \leftarrow greater + 1$
  SWAP($A[l], A[less]$) ▷ Put Pivot1 in the middle
  SWAP($A[r], A[greater]$) ▷ Put Pivot2 in the middle
  return $less, greater$
end procedure

With one more pivot added, the two-pivot Quicksort method is introduced by Vladimir Yaroslavskiy in 2009. It uses 2 pivots selected from each end to split the array into 3 parts. K stands for the current element being scanned, it will be swapped to the left part, which is accumulated by the less pointer, if it is less than the first pivot or do nothing if it is in the middle part. Otherwise, if it is greater than the second pivot, it will be swapped to the right part, which is accumulated by the greater pointer. We will double check the current element at index k after the swap to see if it is less than the first pivot, if so, we will swap it to the left part and move the less pointer to the right. Hereby, we can see that the Dual-Pivot Quicksort is a generalization of the original Quicksort, and the partitioning method is also a generalization of the original Hoare's partition method but with 2 pivots.

What's attracting is that there was once a debate on whether multiple pivots could enhance the performance, as Sedgewick, who analyzed a dual-pivot approach with an average of $\frac{32}{15} n \ln n + \mathcal{O} n$ comparisons in 1978, in constrast to the $2n \ln n + \mathcal{O} n$ of the standard Quick Sort, considered this prototype to be inferior to the classical Quicksort. However, the Yaroslavskiy's Dual-Pivot Quicksort has proved a success, with the experimental results showing that it has a less amount of $1.9n \ln n + \mathcal{O} n$ amortized comparisons, but more swaps, approximately $0.6n \ln n + \mathcal{O} n$ which is much greater than the $0.33n \ln n + \mathcal{O} n$ of the original one, is faster than the original Quicksort in most cases. It is surprising that even with asymptotically only 5% of less comparisons and nearly 80% more swaps, the Dual-Pivot Quicksort still outperforms the original Quicksort in practice. What makes this more mysterious, is the variation of Yaroslavskiy's partition method that always compares to the larger pivot, by Martin Aumüller and Martin Dietzfelbinger 2013 in their thesis of 'Optimal Partitioning for Dual Pivot Quicksort' [8]. The method shows no difference in terms of both comparison and swaps with the original Dual-Pivot Quicksort, but it turned out that it is as fast as the Yaroslavskiy's partition, and even faster than another method along with it, named 'Countering Strategy C', who claimed to have the least estimations of $1.8n \ln n + \mathcal{O}(n)$ comparisons and $1.6n \ln n + \mathcal{O} n$ swaps. This is a great example of how the real-world performance could be different from the theoretical analysis. In fact, in their following tests, the 'Countering Strategy C' method was proven to be the slowest among all the 3 methods.

It is noteworthy the there still lies a huge gap between the theoretical analysis and the real-world performance. It is time we broaden our horizon and look for more factors that could contribute more to the change of the performance of the Quicksort algorithm. The cache misses, branch mis-predictions, and the number of instructions are the things we should take into consideration when we are analyzing on mutated versions of the Quicksort algorithm. Apparently, the number of pivots plays a significant role, and is one of the factors that could potentially bring a significant change. Let's dive deeper to see the world of 3-Pivots Quicksort.

Algorithm 8 3-Pivot QuickSort

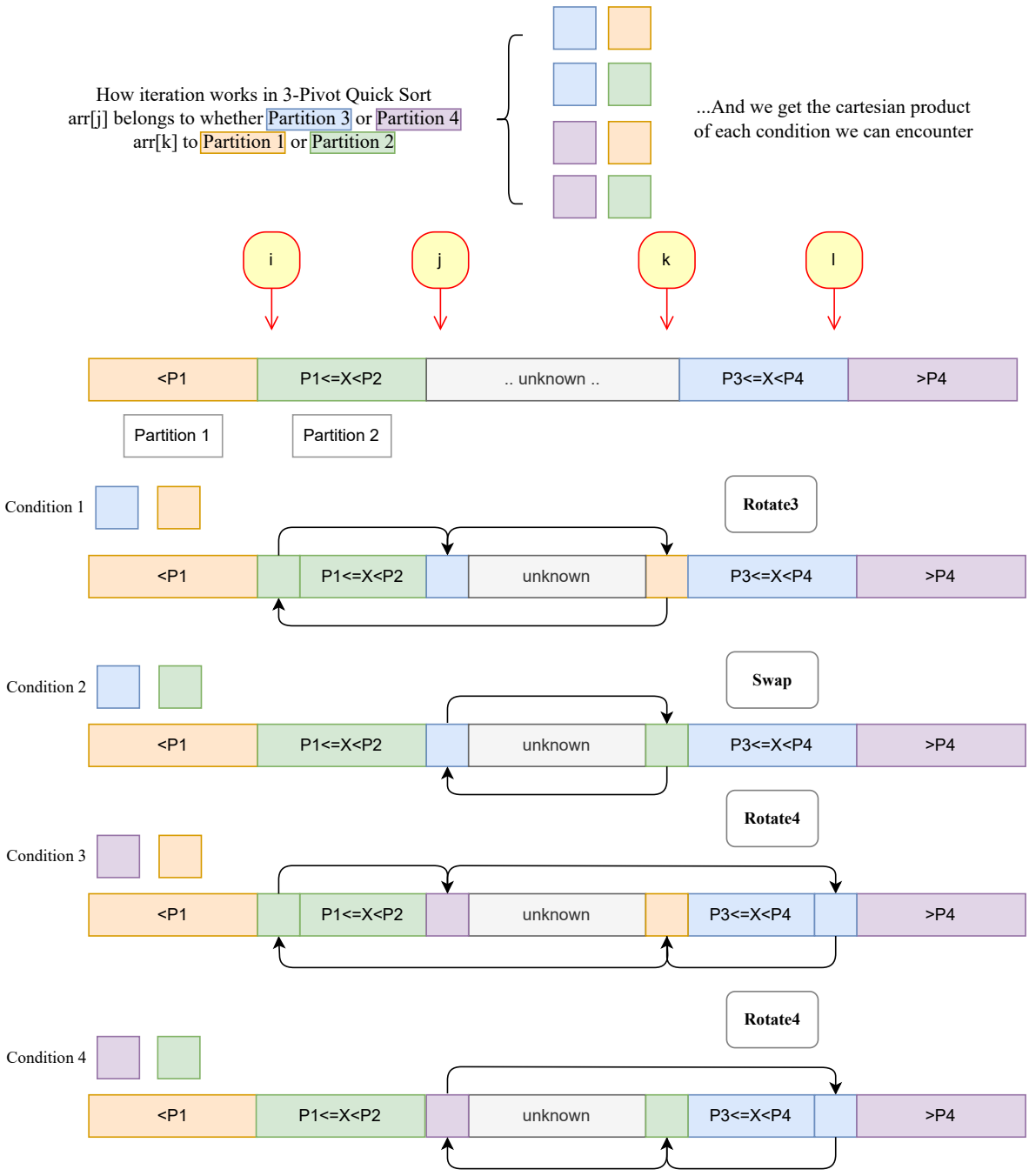
```
1: procedure ThreePivotPartition(A, l, r)
2:     P1, P2, P3 ← A[l], A[l+1], A[r]          ▷ And sort 3 Pivots in ascending order
3:     i, j ← l+2
4:     k, l ← r−1
5:     while j ≤ k do
6:         while A[j] < P2 do                    ▷ Put left side elements less than P2 in order
7:             if A[j] < P1 then
8:                 Swap(A[i], A[j])
9:                 i ← i+1
10:            end if
11:            j ← j+1
12:        end while
13:        while A[k] > P2 do                    ▷ Put right side elements greater than P2 in order
14:            if A[k] > P3 then
15:                Swap(A[k], A[l])
16:                l ← l−1
17:            end if
18:            k ← k−1
19:        end while
20:        if j ≤ k then
21:            if A[j] > P3 then                 ▷ Deal with 2 branches when A[j] is greater than P3
22:                if A[k] < P1 then
23:                    Rotate3(A, [j, i, k])
24:                    i ← i+1
25:                else
26:                    Swap(A[j], A[k])          ▷ A[j] > P3 and P2 > A[k] ≥ P1
27:                end if
28:                Swap(A[k], A[l])
29:                l ← l−1
30:            else                              ▷ Deal with 2 branches when A[j] is less than or equal to P3
31:                if A[k] < P1 then
32:                    Rotate3(A, [j, i, k])
33:                    i ← i+1
34:                else
35:                    Swap(A[j], A[k])
36:                end if
37:            end if
38:            j ← j+1, k ← k−1
39:        end if
40:    end while
41:    i ← i−1, j ← j−1, k ← k+1, l ← l+1
42:    Rotate3(A, [left+1, i, j])
43:    Swap(A[left], A[i])
44:    Swap(A[right], A[l])
45:    return i, j, l
46: end procedure
```

The three-pivot Quicksort, introduced by Shrinu Kushagra, uses the similar hoare-like partition method, is a further generalization of the Dual-Pivot Quicksort. In the original implementation, the 3 pivots are selected from the leftmost, the second element and the rightmost element of the array, and are sorted in ascending order. The scanning procedure is similar to the Dual-Pivot Quicksort, but with 3 pivots, the array is divided into 4 parts, and is repeated until all elements are set. The i, j, k, l pointers are used to scan the array and the elements are swapped to the right places according to the comparison results with the 3 pivots, where i stands for the leftmost part where elements less than Pivot 1 belongs, j stands for the middle-left part, k stands for the middle-right part and l stands for the rightmost part. Between j and k stands the unknown region where elements are not yet compared with any of the pivots. If the elementa at index j is less than Pivot 2, it is whether swapped to the leftmost part if it is less than Pivot 1, or do nothing otherwise, since the middle-left part is for elements greater than Pivot1 and less than or equal to Pivot 2. Same logic applies to the rightmost part, where elements are greater than Pivot 2 and less than or equal to Pivot 3. These two pre-conditions are checked before the main body of the swapping and rotating procedure, which corresponds to the 2 while loops applied at the top of the main scanning loop. Afterwards, the real swapping and rotating procedure is applied to the elements at index j and k, which are on the left and right boundaries of the unknown region, need to go to the other side of the array and moved to the right places.

We can simplify this part into a 2 if-else nested block, where the outter if-else block is for the case at index j and the inner one is for the case at index k (vice versa, but we will not go into details here). Both of them have 2 branches, one for the case where the current element belongs to the leftmost or the rightmost part, and the other for the case where it belongs to the middle-left or the middle-right part. It is not hard to tell there are $2 \times 2 = 4$ branches in total, and the swapping and rotating procedure is applied to each of them. The following chart illustrates the 4 branches and their corresponding actions.

The following figure tags elements belonging to the leftmost, the middle-left, the middle-right and the rightmost part with different colors, and the arrows indicate how they get swapped to be in order.

Figure 4: 3-Pivot Quick Sort



Algorithm 9 4-Pivot QuickSort

```
1: procedure FourPivotPartition(A, l, r)
2:     P1, P2, P3, P4 ← A[l], A[l+1], A[r−1], A[r]          ▷ And sort 4 Pivots in order
3:     i, j, k, l, m ← l+2, l+2, l+2, r−2, r−2
4:     while k ≤ l do
5:         while A[k] < P3 do                    ▷ Put left side elements less than P3 in order
6:             if A[k] < P1 then
7:                 Rotate3(A, [k, j, i])
8:                 i ← i+1, j ← j+1
9:             else if A[k] < P2 then
10:                Swap(A[k], A[j])
11:                j ← j+1
12:            end if
13:            k ← k+1
14:        end while
15:        while A[l] > P3 do          ▷ Put right side elements greater than P3 in order
16:            if A[l] > P4 then
17:                Swap(A[l], A[m])
18:                m ← m−1
19:            end if
20:            l ← l−1
21:        end while
22:        if k ≤ l then    ▷ Start manipulation on the boundaries of unknown region
23:            if A[k] < P4 then        ▷ Deal with 3 branches when A[k] is less than P4
24:                if A[l] < P1 then
25:                    Rotate4(A, [k, j, i, l])
26:                    i ← i+1, j ← j+1
27:                else if A[l] < P2 then
28:                    Rotate3(A, [k, j, l])
29:                    j ← j+1
30:                else
31:                    Swap(A[k], A[l])
32:                end if
33:            else                    ▷ Deal with 3 branches when A[k] is greater than P4
34:                if A[l] > P2 then
35:                    Rotate3(A, [k, l, m])
36:                else if A[l] > P1 then
37:                    Rotate4(A, [k, j, l, m])
38:                    j ← j+1
39:                else
40:                    Rotate5(A, [k, j, i, l, m])
41:                    i ← i+1, j ← j+1
42:                end if
43:                m ← m−1
44:            end if
45:            k ← k+1, l ← l−1
46:        end if
```

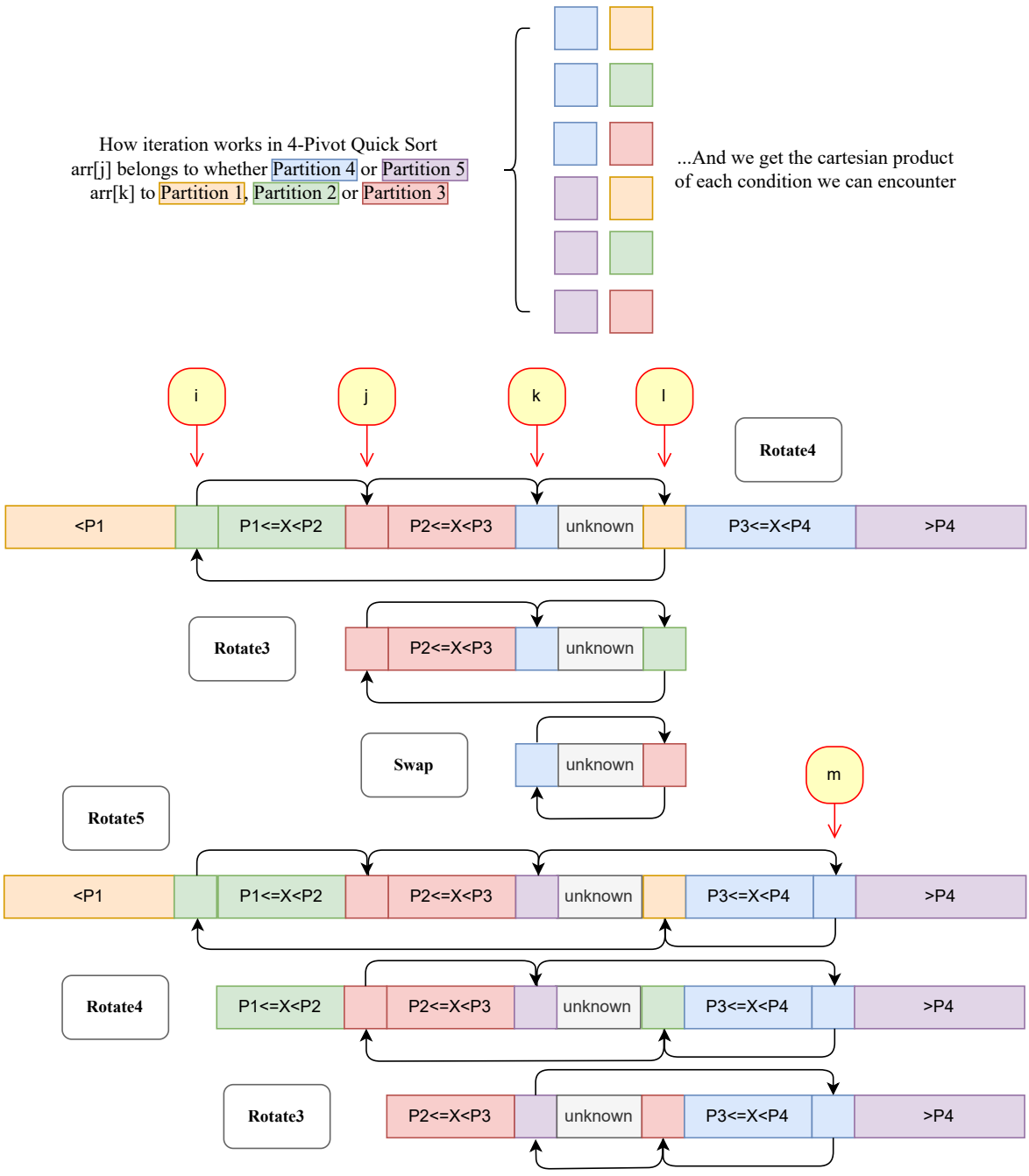
Algorithm 10 4-Pivot QuickSort (Continued)

```
47:     end while
48:     i ← i − 1, j ← j − 1, k ← k − 1, l ← l + 1, m ← m + 1
49:     ROTATE3(A, [left + 1, i, j])
50:     i ← i − 1
51:     SWAP(A[left], A[i])
52:     ROTATE3(A, [right − 1, m, l])
53:     m ← m + 1
54:     SWAP(A[right], A[m])
55: end procedure
```

As long as the number of pivots increases, the number of branches will grow linearly, and the complexity of the partitioning method will also increase. Thankfully, for 4-Pivot QuickSort, we are just adding two more branches (As showed in the blue-red and purple-red branches in the chart below) compared with the $2 \times 2 = 4$ branches for 3-Pivots into considerations and the branching logic is still under control. We declare this as the balanced partitionlayout that we are using, where each side of the array is filled with $\frac{N+1}{2}$ partitions for odd number N pivots, or one side with $\frac{N}{2}$ partitions and $\frac{N}{2}+1$ on the other. The comparisons take place like looking for a value in a binary search tree (BST), for instance when there are 3 pivots in total, comparing with the Pivot 2 first, then the Pivot 1 or Pivot 3. This search tree is of depth 2 and if we using a unbalanced BST that puts Pivot 1 or Pivot 3 on top, we would have a tree with depth 3, which is not efficient. Bearing this premise in mind, it is not hard to tell that we are going to have 9 branches for 5-Pivot QuickSort (3 partitions on each side, $3 \times 3 = 9$) and 12 branches for 6-Pivot QuickSort (3 partitions on one side and 4 on the other, $3 \times 4 = 12$). The complexity of the partitioning method is growing way too far and thus only 4-Pivot QuickSort will be discussed in this thesis. In the [10]'how good is multi-pivot quicksort' paper, the author sketched an extremal tree for seven pivots in fig 10. It seems to differ from the balanced BST I was expecting and due to the packed time I was given, this question should be given to the future work to analyze whether it is the optimal scheme for 7-Pivots or not.

Right after finishing the 4-Pivots implementation, I surprisingly found that similar layout has been depicted in the subsection 7.4 of the thesis *'How good is Multi-Pivot Quicksort'* [10] on page 23, where the odd number of pivots are analyzed. It uses the same rotate-n operation to rearrange the elements in large sizes, and is a key in the partitioning method that iteratively shrink the middle unknown region and expand the left and right parts. For better understanding, a completed visualization on the iterations will be provided right below with the chart of 4-Pivot QuickSort.

Figure 5: 4-Pivot Quick Sort



2.8 Analysis

The analysis on the results of Branch Misses, Cache Misses, Instructions and Time will be conducted below with regard to the Multi-Pivot QuickSorts with traditional partitions. The details regarding to the testing machine, environment and compiler settings [1] are as follows:

CPU	Cache		
R7 3700x @4.3GHz Overclocked	L1 8x32KB	L2 8x512KB	L3 2x16MB

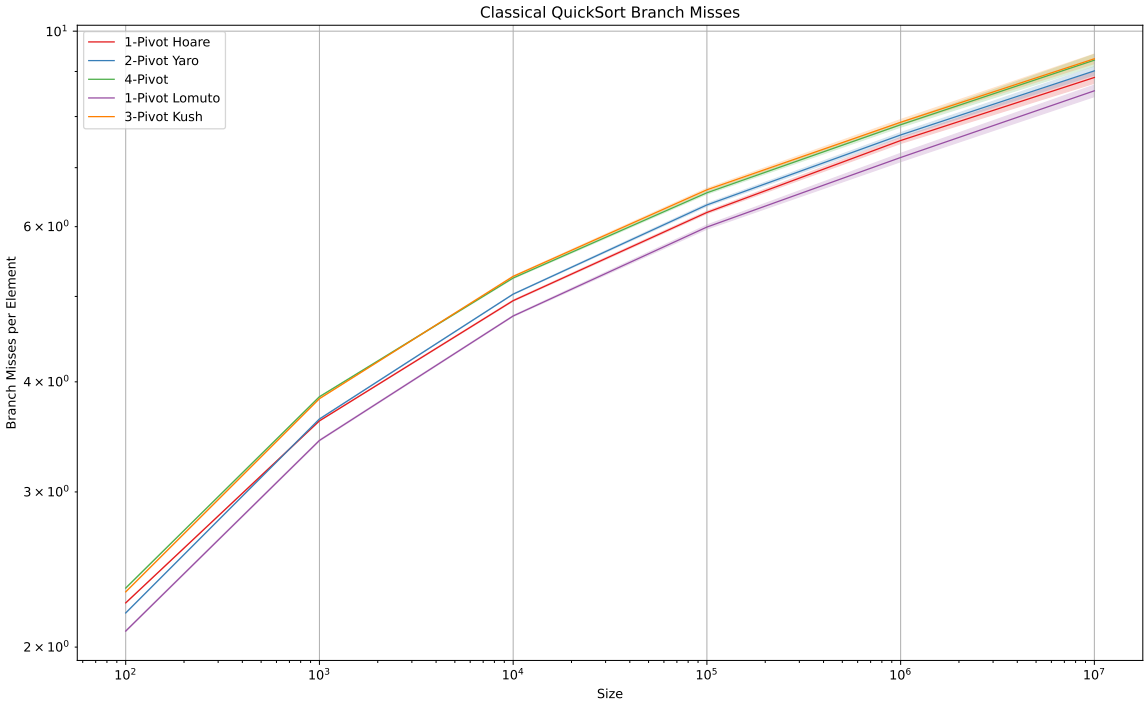
RAM	OS	Compiler
16GB	GNU/Linux Ubuntu 22.04	Rustc 1.78.0 +Nightly
Rustc Flags	opt-level=3,codegen-units=1,lto="thin"	

All the tests above are benchmarked using the criterion library, inspired by the Haskell criterion library, which is a powerful and easy-to-use benchmarking tool for Rust. Tests are put into groups and each group is a collection of benchmarks that are run together, with the exact same environment and the same settings. Each test runs in a black box, which blocks the compiler from doing optimizations that are way too aggressive, such as removing instructions it thinks are redundant. The criterion-perf-events plugin is also adapted to capture the hardware events of the CPU, with the help of the well-known profiling tool $perf$ on Linux. Why not cachegrind? Because cachegrind is a simulation tool on the cache misses and introduces a huge overhead on the runtime, in previous tests with perf, the overall runtime appeared to have at least 200% difference with the real-world performance, so it was rejected in the end. Comparably, perf is a more lightweight tool that suits our needs and the given results are more reliable. Previous tests with cachegrind ‘ are given in the appendix.

The compiler flags passed in are the same for all the tests, and the tests are conducted on the same machine with the same environment, with 3 seconds of warm-up time and 5 seconds of measurement time for each test. The results are then calculated over regression tests with 100, 1000, 10000, 100000, 1000000 and 10000000 elements, filtering out the outliers and the each item is collected with the mean, standard deviation and median. The results are shown in the figures below. The x-axis is uniformed to the size of the array, and the y-axis is the average number of events captured by the perf tool or the average running time in µs per element. Standard deviations will be showed as transparent areas around the mean which follow the colors of their corresponding lines.

[1] 'opt-level=3' is equivalent to '-O3', 'codegen-units=1' is equivalent to '-fno-merge-all-constants', 'lto="thin"' is equivalent to '-flto=thin' in GCC.

Figure 6: Classical QuickSort Branch Misses



From the global picture, after the short rocket for sizes below 10^4, the growth of branch misses seem to level off for all the partition methods afterwards. The 4-Pivot partition and 1-Pivot Hoare partition seem to fall behind the 3-Pivot Kushagra's partition method and 2-Pivot Yaroslavskiy's method, but soon catch up and surpass them when the size of the array exceeds 10k.

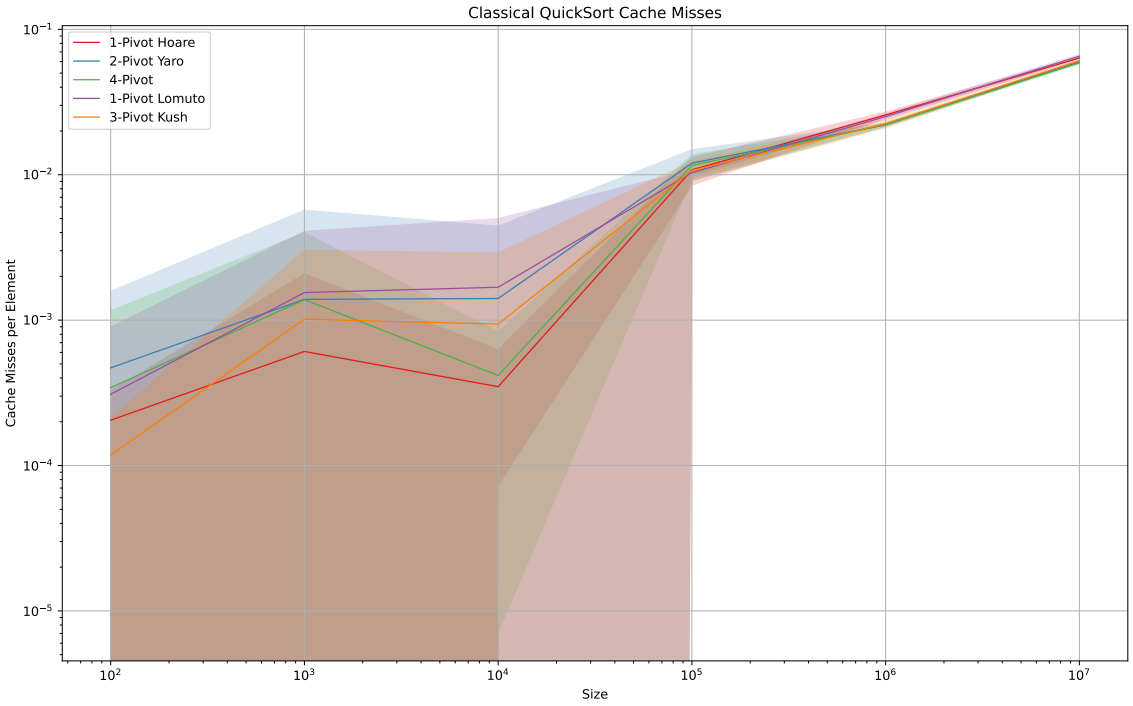
On a smaller scale, the traditional means don't show much difference between the 1-Pivot Hoare and Lomuto partition methods, it makes sense that the 1-Pivot Hoare's partition method tend to always have a slightly higher mean of branch misses than the 1-Pivot Lomuto's partition method when Hoare's partition method seeks 2 out-of-order elements to swap, while only 1 for Lomuto's. We are bound to have some more branch misses in the former case. But the 2-Pivot Yaro's partition method has a slightly higher mean of branch misses. The introduction of the 2nd pivot has brought more branches into concerns, the complexities grow linearly with the number of pivots but the standard deviation grows rapidly as the size of array exceeds 10k, and explodes after 1 million. Down follows the 3-Pivot Kush's partition method

has a slightly higher mean of branch misses than the 2-Pivot Yaroslavskiy's partition method.

While things go beyond our expectations, the 4-Pivot partition shows better statistics of branch misses than the 3-Pivot partition method for larger arrays, which is a surprise to us, even though it has slightly higher means of branch misses than the 2-Pivot Yaroslavskiy's partition method in all, the standard deviation remains at a high level indicating the possible instabilities of the 4-Pivot partition method brought by 2 more branches ' that 1 more pivot introduced. It is still worth our notice, that it can happen when more pivots could reduce the branch misses instead of adding more burdon to the CPU. It might be that the layout of 'balanced partition' we are using is more friendly to the CPU's branch predictor, but yet to explain why 2 more branches could bring better effects. This could be a topic in future works, further expand the research on the Multi-Pivot Quicksort algorithm.

Apart from the branch misses, the cache misses could also play a significant factor that could affect a lot, which is what we are going to shift our focus to.

Figure 7: Classical QuickSort Cache Misses



The global trend shows the outburst of cache misses' means when scaling from

10^4 to 10^5, possibly due to the drain of the L1 cache. Before that the cache misses are relatively low by their means even though the messy standard deviations are still there. CPU will run out of L1 cache for size 10^5 as $10^5 \times 4bytes/element = 4 \times 10^5 bytes$, exceeding the total L1 cache size of about $8 \times 32Kb = 2.62144 \times 10^5 bytes$ while the spatial locality of the elements might still struggle to keep the L2 cache misses at a low level. Since L2 misses have a much larger latency than L1 misses, the performance of the algorithm is seriously affected by having to wait for the main memory, this amplifies the differences between these algorithms that converge to a stable level after 10^5, where 2-Pivot Yaroslavskiy, 3-Pivot Kushagra and 4-Pivot partition take slow and steady increase of cache misses, while 1-Pivot Hoare and Lomuto partition methods still going a little bit upwards. But all the partition methods seem to reach about the same cache misses for 10^7 elements. This could be the magic of modern CPU's cache management, where the L3 cache is shared among cores, in my case among CCDs (Core Complex Die) and the memory controller, so the cache misses are not that sensitive to the size of the array. Changes in rapid growth of standard deviation is also noteworthy, indicating the dramatic influences caused by the cache misses when the size of the array under 10^4.

In 1-Pivot tests, Lomuto shows much higher cache misses than Hoare for smaller arrays, could be due to the fact that Lomuto's partition method has more overlapping swaps than Hoare's, where most swaps take on the elements that are already in the right place. As the number of pivots grow, the cache misses seem to decrease, appeared in the 4-Pivot version, might help to support the claim that 'less memory access and fewer cache misses' given the declaration above 'choosing more pivots will significantly reduce the chance we access the same elements again', but the high level of standard deviation still remains a concern.

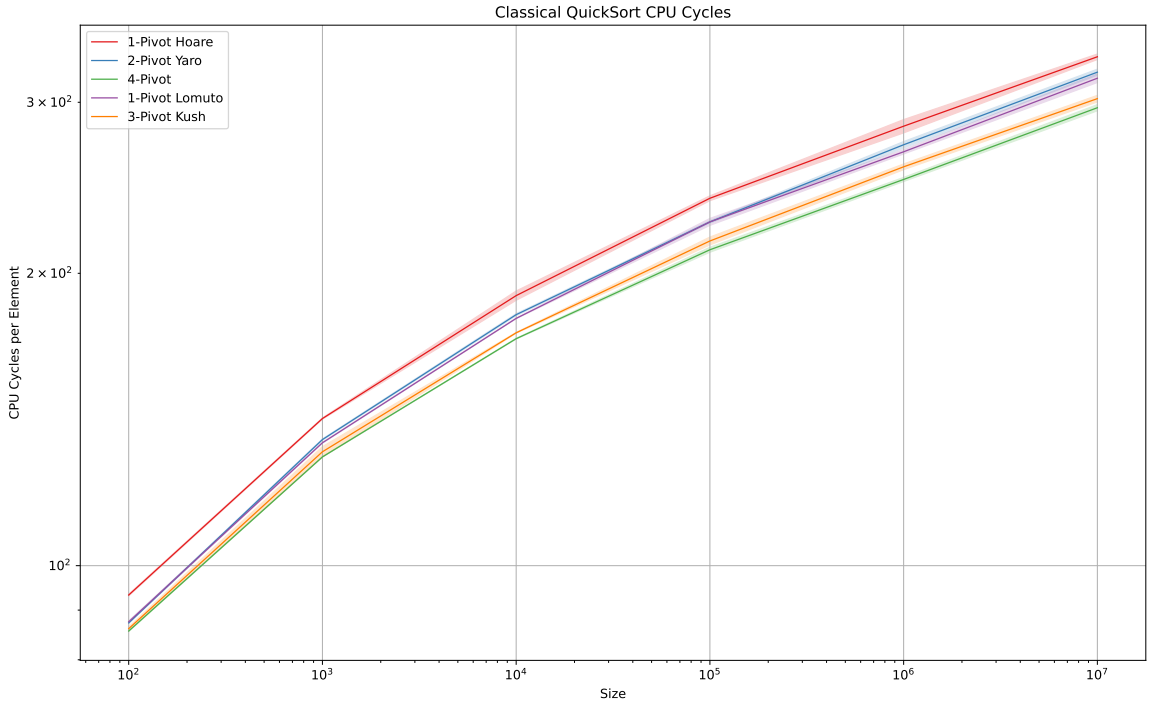
Among all variations, the 2-Pivot Yaroslavskiy's partition method seems to have the lowest mean of both cache misses and branch misses combined together in most of the larger array tests, with stable standard deviation. Comparably, 1-Pivot Hoare's and Lomuto's partitions have less branch misses, but more cache misses, while the opposite for 3 and 4-Pivots. This is a trade-off and it seems Yaroslavskiy's scheme strikes a proper balance between, it shows decent deterministic behavior, a potential good sign for the optimistic expectations on performance of the algorithm.

For 3-Pivots, the cache behavior didn't seem to fully comply with the theoretical analysis that it is cache friendlier than classical and Yaroslavskiy's 2-Pivot QuickSorts, the cache misses are higher than the 2-Pivot Yaroslavskiy's partition method when the array size is larger than 1 million. The bound between cache misses and the number of pivots is not yet clear. Contrary to the conclusion reported on experiments carried out by Kushagra's team, which supports the improvements on Multi-Pivots QuickSort are due to better cache behaviors [4], the results of the tests conducted in this thesis show that the cache misses are not always decreasing with more pivots. For instance, the 3-Pivot Kush's has more observed cache misses than the 2-Pivot Yaroslavskiy's, shows better performance than 2-Pivots during runtime tests. These effects are not yet fully understood and need further discussions, as I'm using Rust to implement the Multi-Pivot QuickSorts, whose compiler is based on top of LLVM (Native LLVM as the Rust's bootstrapped backend codegen component Cranelift is far from being mature on x86 architecture), that might have optimized the code in a different way that the

cache behaviors are not well reflected in the results, although I enabled the most aggressive optimizations flags in the compiler settings and they are similar as GCC and Clang's. They also reported that their 7-Pivot QuickSort has worse performance than the 3-Pivot QuickSort as a result of poor cache behaviors, which is not yet tested in this thesis. But with the counter examples we have, we need to be more cautious about leveraging how much propotions of the performance improvements are due to better cache behaviors. It is still a topic for further research. We will discuss about the complicated relationships further when we finish up the analysis on the instructions and time.

In the parts below, we will continue to seek more exceptions that could shift our opinions on the connections between cache and algorithm efficiency.

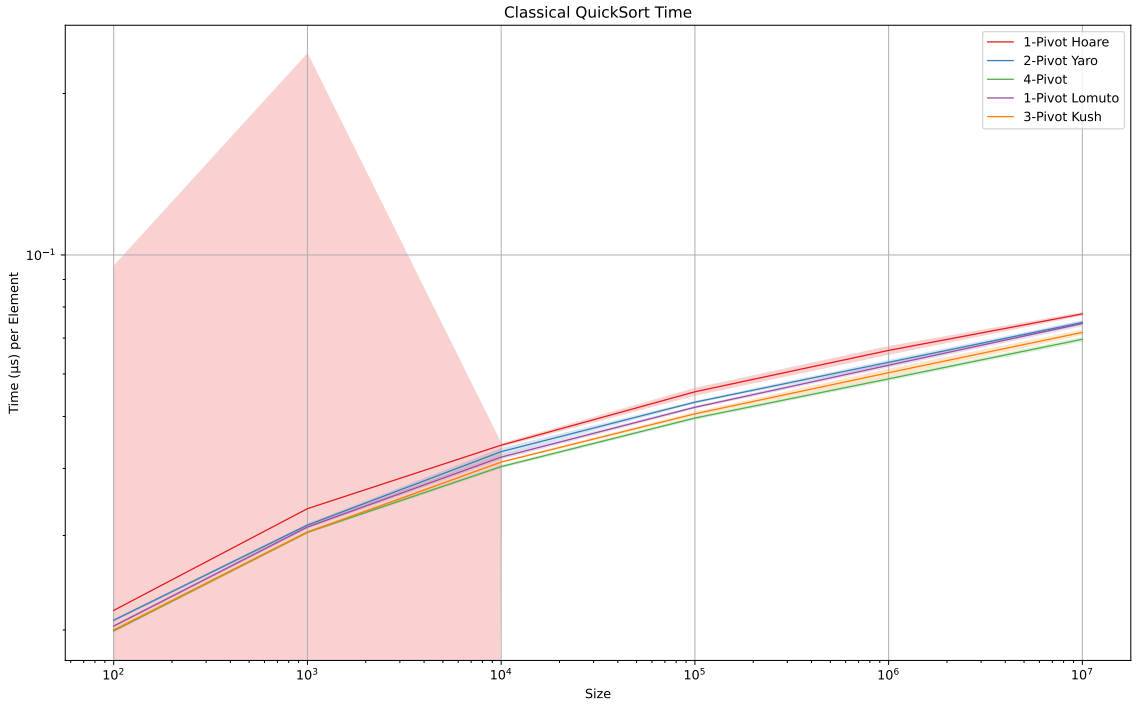
Figure 8: Classical QuickSort CPU Cycles



Now it is time to deal with the total CPU cycles, the global trend is as the pivots increase, the number of CPU cycles falls. But the 2-Pivot Yaroslavskiy's method exposed its weakness with respect to the 1-Pivot Lomuto's method.

The 4-Pivot shows the least amount of cycles and least deviations. Combining all the factors together, the final runtime performance is reveiled as follows.

Figure 9: Classical QuickSort Time



Contrary to our expectations, 2-Pivot Yaroslavskiy's method was defeated by the 1-Pivot Lomuto's method in terms of time, even though it outperformed in the cache misses. The running time does show the correlation with the branch misses, but the cache misses don't seem to have a significant impact, where Yaroslavskiy's method has the lower cache misses but more cost time than Lomuto's method. It also has lower cache misses than Kushagra's 3-Pivot method, but its time cost is still much higher for arrays with over 1 million elements than the 3-Pivot method. It seems the past analysis on the cache misses are more or less limited to the size of the array, and perhaps the old CPU architecture's deficiency on large chunk of data. What should be noted that the testing CPU, R7 3700x, has a $2 \times 8MB = 16MB$ L3 cache, which has 8MB of cache per CCD, that is, 4 cores share 8MB L3 cache, but L3 caches are not shared between the 2 CCDs. This issue was later solved in the Zen 3 architecture, where all the L3 cache is shared thanks to the high-performance ring buses that connect the CCDs and significantly reduce the latency of cross-CCD communication.

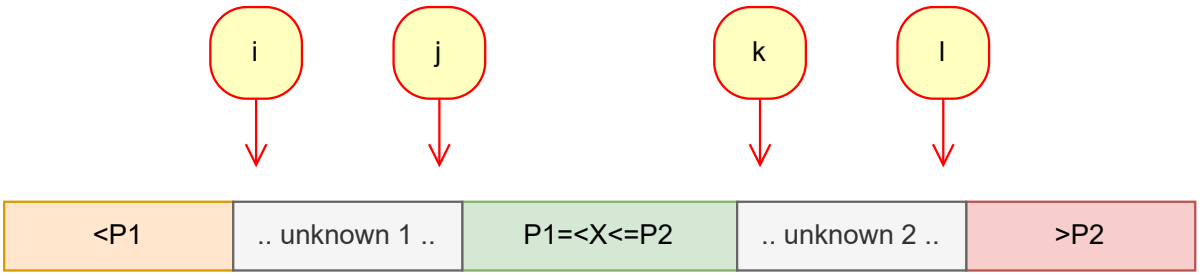
The final running time also supports the exceptional cases above, where cache misses don't necessarily have a direct impact on the final performance of the algorithm. Hoare partition, with 2 while loops scanning for the out-of-order elements

on both sides, introduced extra overheads on branch misses and didn't manage to compete with other algorithms. This could be the actual pain point that the 1-Pivot Hoare's method has to face, the branch misses appear to be the bottleneck, but we will soon find out the way to overcome it in the following sections. What's enlightening is that the 4-Pivot method has the least amount of time along with less branch misses and cache misses, the improvements are made in small steps but the overall performance is promising, although how the 4-Pivot method has more predicted branches but less actual branch misses still requires further researches.

In conclusion, there exist few things that go out of what we have expected, the 2-Pivot Yaroslavskiy's method has the lowest cache misses but deficient performance, the 4-Pivot method has the least amount of time, somewhat partially due to its decent cache behavior, but more importantly, it has more predicted branches but less branch misses in field experiments. This phenomenon could bring us to a new direction for explorations.

With the help of my supervisor Dr. David Gregg I reached out to new layouts that might shift our perspectives on classical QuickSorts. Unfortunately they proved to be too complicated to implement or just not possible to be implemented by far. One of the layouts is the 2 unknown segments for 2-Pivot QuickSort.

Figure 10: 2-Pivot New Layout

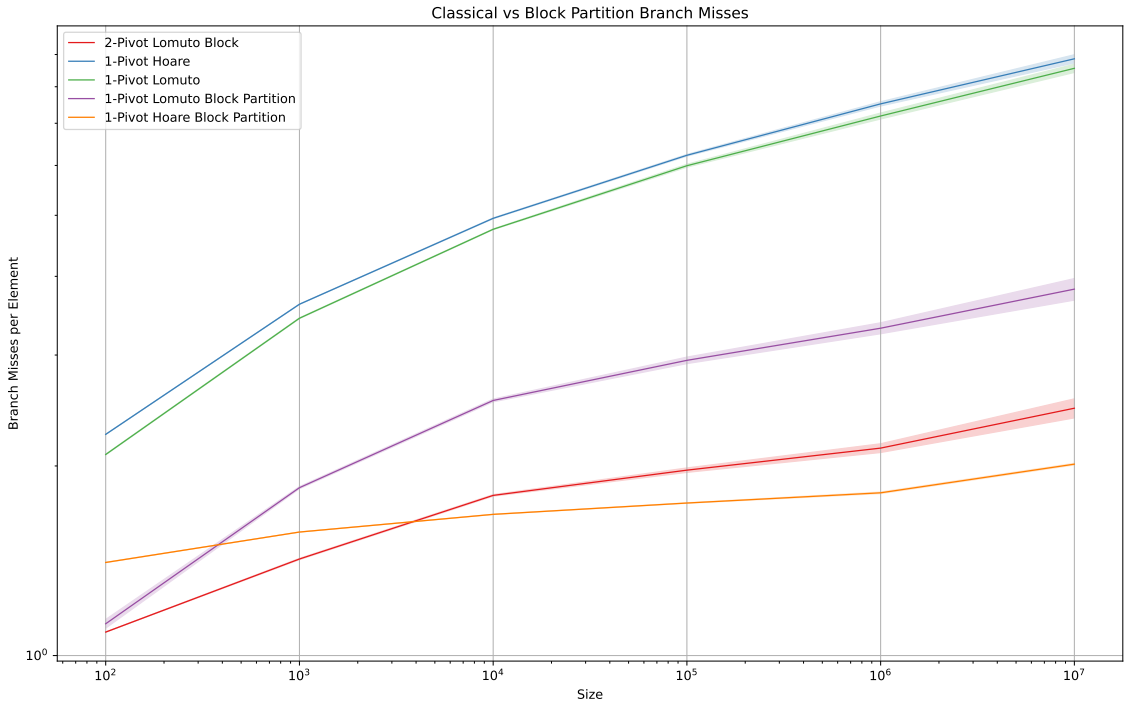


Obviously, this will dramatically add the complexities to our algorithms when introducing one more unknown segment, which means 2 more unknown elements at the boundaries of the unknown region. Worst cases is, the comparison procedure will be consist of 4 if-else nested branches, each deciding the final destination of one of the 4 unknown elements at index i, j, k, l. This could be a disaster for branch prediction, and the cache misses will be increased as well, as the elements are more likely to be swapped between the unknown regions, contributing the form of vicious cycle that will cause the performance degradation. This early exploration has been abandoned due to the complexity and the potential performance issues it might

bring.

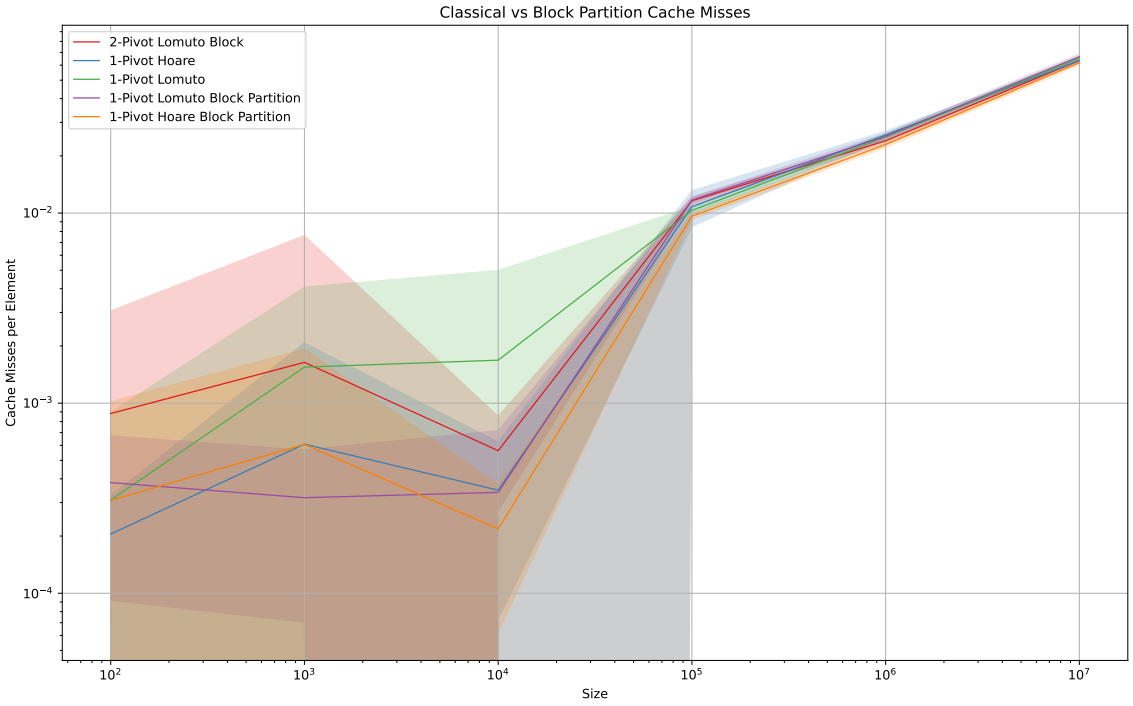
While searching for the solutions to the branch misses, I have found that Block-QuickSort could refresh us with a new vision of partitioning, it's easily adaptable to most of the partition methods and its direct impact on the performance is beyond amazing according to what has been observed in the tests below.

Figure 11: Block QuickSort Branch Misses



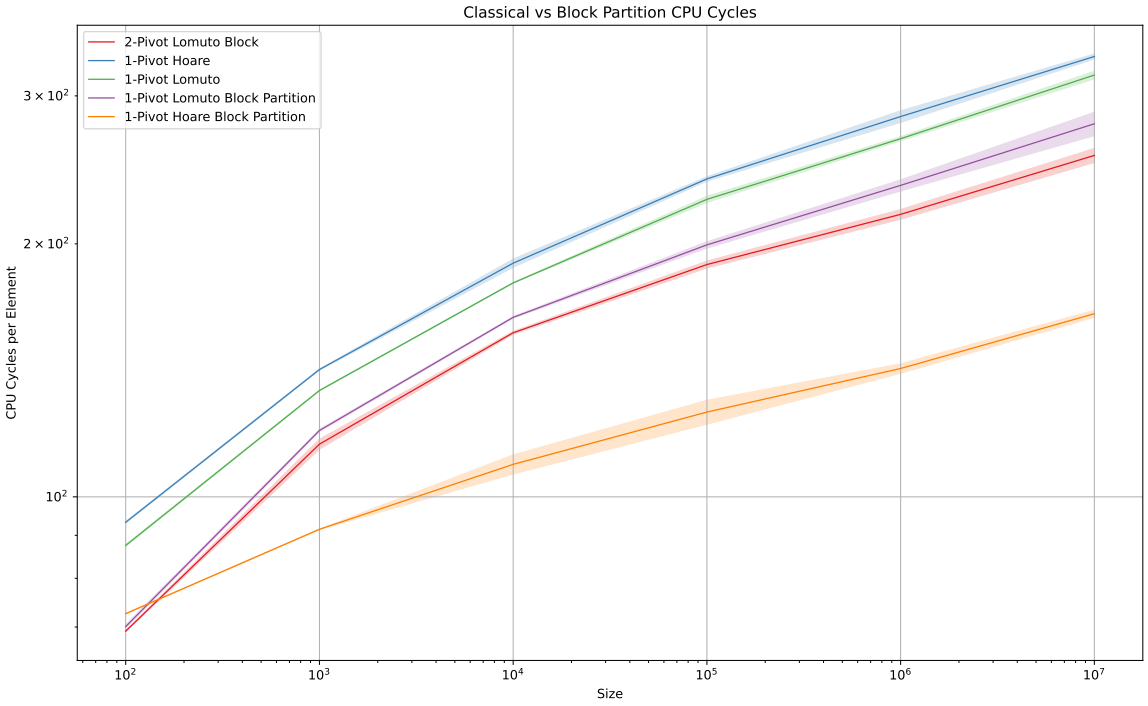
With the help of Block Partition together with branchless comparisons and branchless additions on counters, we reduced about 70% of the branch misses in the 1-Pivot Hoare's method, and 40% in the 1-Pivot Lomuto's method. These results will only increase as the size of the array grows. On gigantic arrays of size 10 million, we achieved 77% and 55% branch miss reductions respectively as a result. Full tables of detailed statistics could be found in the final appendix. That's a huge success so far, though it doesn't show superior performance on smaller arrays, but the average branch misses barely even increase as the size grows exponentially.

Figure 12: Block QuickSort Cache Misses



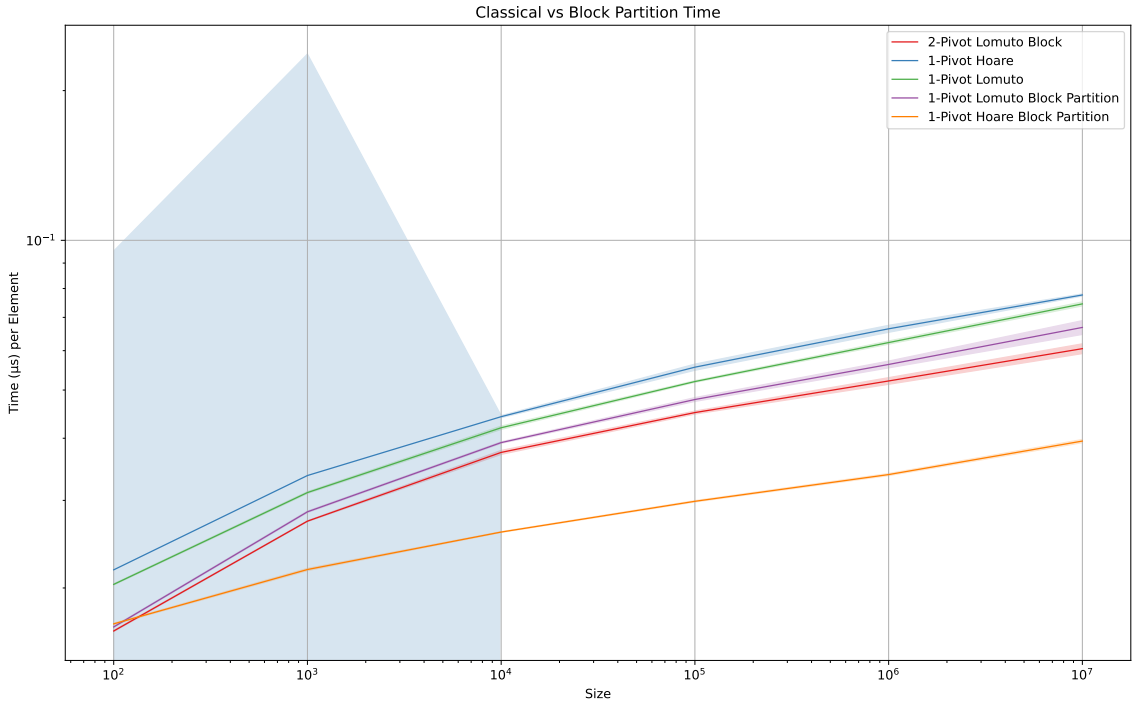
Again to our surprise, the changes in cache miss isn't as obvious as the branch misses, the tiny improvements on Hoare's Block Partition and 2-Pivot Lomuto's Block Partition seem to be only exceptions. Considering the traditional partition schemes only require very small number of constant size, comparing to the large chunk of memory required by the Block Partition, the Block Partition's revision on Lomuto's method adds more cache misses instead. But as we observe the CPU instructions count and the time cost, we will find out that the Block Partition's method is still the best choice for the 1-Pivot Hoare's method, and the 2-Pivot Lomuto's method.

Figure 13: Block QuickSort CPU Cycles



It seems the Hoare's Block Partition has the most significant improvements on the CPU cycles, with about half of the reductions on total cycles. The block partition doesn't seem to improve much with the Lomuto's scheme, which is still a mystery to me together with its dramatic standard deviation, but the 2-Pivot Lomuto's Block Partition has a decent amount of improvements. Still, the cache misses for small arrays seem completely random, so are the standard deviations. There seems no way that we could analyse these results. But the time cost will give us a clear answer.

Figure 14: Block QuickSort Time



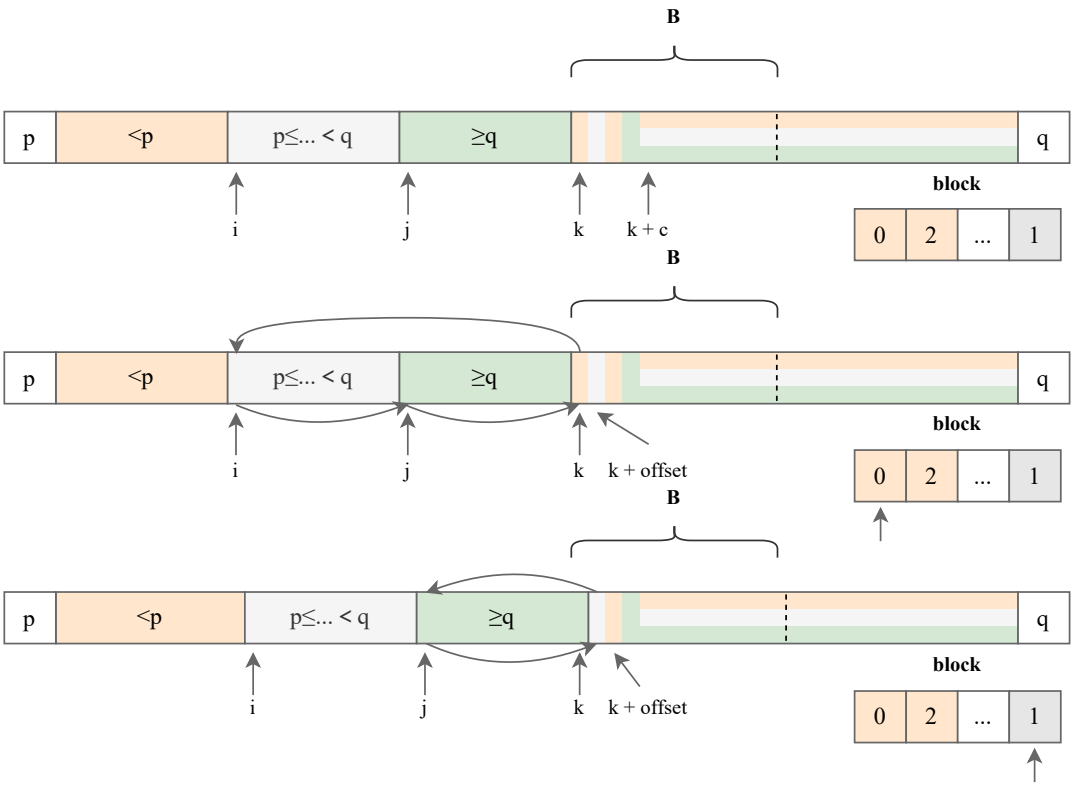
The most intriguing result has been given by the 1-Pivot Hoare' BlockQuickSort variation with least amount of time cost, nearly the half as the original, as a result of less than 3% improvement on cache misses and approximately 70% improvement on branch misses. It is already of much progress I would have to say, but the 2-Pivot Lomuto's BlockQuickSort variation also has a decent amount of improvements, with up to 18% less time cost than the original. The former example has proven a success, but future works will also be carried out to improve the cache misses, even though there exists very limited things we could do, the modern CPU architecture has already done a great job on the overall cache hit rate, which can be supported by the cachegrind tests that all the tested algorithms have around less than 10% cache misses. Although this won't be the deterministic evidence to say that the cache misses are not the bottleneck, but it is still a good sign that the cache misses are not the main issue that we have to deal with.

The tests above are carried out using constant pivot selection method, choosing half start and half end of the array as the pivots as the most common practice goes, but as mentioned earlier in the preliminaries part ', the pivot selection is a crucial part of the QuickSort algorithm, and the skewed pivot selection, anti-tuitionally

as it seems, could bring better performance than the random pivot selection. It doesn't bring us to the conclusion that the skewed pivot selection is always better other methods, even it makes sense to a static branch predictor where it assumes all branches will follow the correct majority path, for instance, if our skewed pivot could divide the array into x% and $(100-x)$% parts, the branch predictor will always predict the $max(x\%, 100-x\%)$ path, we can use less time at current stage to finish the partitioning process, but the imbalanced partitions will pose future challenges to maintain the efficiency of overall algorithm. Luckily, skewed pivot always proves unnecessary in the BlockQuickSort, free from the concerns on branch misses' costs, proved in the experiments of [7]. With skew factor 2 (the middlest element is chosen as the pivot), the BlockQuickSort has the least amount of time cost, and larger skew factors will only increase the runtime. Skew factor of 6 appeared to have the best performance in the experiments for classical QuickSorts as a balance between the stalls caused by branch misses and total CPU Instructions.

The 2-Pivot Lomuto's Block Partition has less branch misses for smaller arrays, but fail to compete as the size extends, as the duplicated swaps have been amplified when the size of the array grows, Lacking as a key feature to Lomuto's block scheme is the cyclic swaps that relief a lot of duplicated swaps and constant cache to the temp variable. This method has adapted well to the Hoare's scheme, for it starts on both ends and the duplicated swaps are less likely to happen as long as misplaced elements are spotted on both sides, a prerequisite for the cyclic swaps to take place. The cyclic swaps are not only reducing the duplicated swaps, but also reducing the cache misses by the help of write-back policy. The 2-Pivot Lomuto's Block Partition has failed to adapt to the massive cyclic swaps that eliminates the biggest headache of small swaps' costs on frequently changed temp variable, plus the duplicated swaps are not reduced, but amplified, which might be the meat of the issue for the performance degradation compared with Hoare's block partition. Mimicing the layout of Lomuto's Block Partition, I implemented another 2-Pivots Lomuto's Block Partition which requires only one phase of rearranging, but with more rotate-n moves to shift elements to their correct partitions. This could introduce more complexity as in a chaotic block with 2 types of items less than Pivot 1 or greater than Pivot 1 but less than Pivot 2, simply dumping their offsets to the correct partitions regardless will not work, for in each step we have to deal with smaller offset first to make sure our algorithm behaves the same as the classical version which scans the array in one-way order, but this extra min() operation could be a huge drawback for this scheme. Results for this is even slower than the classical QuickSort, and it is not recommended to be used in practice, only for academic purposes instead of industrial applications.

Figure 15: Block QuickSort New 2-Pivot Lomuto



The cache misses also suffer from the new layout, constantly swapping side to read and compare the next minium offset also poses a challenge to the spatial locality of cache, causing the general time cost being also higher than the classical QuickSort. In my fuzzy tests of cachegrind , the data read (Dr) rate 28.98% and data write (Dw) rate 19.41% all rank the top, and are both much higher than the classical QuickSort even though it has low cache misses. My assumption would be these constant memory accesses are caused by the cache write-back policy due to inconsistent spatial locality, though the actual relationship is yet to be confirmed by further tests in the future. It reveals from another perspective that the block partition is restricted by the number of pivots, as the experiments in this thesis only shows that the the pattern of comparing with one pivot in one pass is the only way to go, otherwise, like the new layout illustrated above, if we get too greedy to compare with 2 pivots simultaneously, the offsets have to be stored in separated chunks and it will introduce the memory accesses as the most expensive overheads that stall the CPU. This puts forward another trade-off between the number of pivots and the block partition, and it is not yet clear how to leverage the best of both worlds. The

BlockQuickSort is still in its infancy, despite the fact that PDQSort proves a great success in the real-world applications, and the future works will be carried out to explore more possibilities and to find more optimal layout for the BlockQuickSort. As to the new layout, for your interest the detailed statistics will be attached in the appendix section.

For the sake of convenience, the sorting functions are exported to Python using the Foreign Function Interface (FFI) provided by PyO3, which is a safe and easy way to call Rust functions from Python. The sorting functions are then tested on the same machine with the same environment with supported types 32 bits IEEE-754 floats and unsigned 32 bits integers. The original Rust results are provided in the appendix in fully elaborated tables. The Python results are consistent with the Rust tests, reliable and stable. Only extra overhead would be the Python interpreter and ffi calls, which is not significant in this case, and is negligible.

3 Full Tables

Branch Misses	Size	Mean	SD	Median
1-Pivot Hoare	100	224.4847	0.0588	224.4906
	1000	3612.1521	2.2119	3612.3146
	10000	49451.5958	72.9468	49464.5783
	100000	622681.4604	1879.7192	622662.0813
	1000000	7514512.0625	57015.6085	7528135.1250
	10000000	88601386.6500	1316610.4428	88910896.5000
Block Partition	100	140.5428	0.0876	140.5400
	1000	1570.7223	0.3938	1570.6199
	10000	16756.4235	7.9552	16757.5304
	100000	174650.0671	156.4744	174665.2439
	1000000	1813011.9915	5043.7407	1811789.7762
	10000000	20131908.6500	59866.7898	20134795.0000
1-Pivot Lomuto	100	208.5500	0.2674	208.4572
	1000	3431.9977	4.9888	3432.6774
	10000	47510.1674	62.4065	47492.9219
	100000	599369.5910	3094.0668	599127.2778
	1000000	7188431.3250	77244.2181	7204133.7500
	10000000	85572728.6000	1273266.3095	85589444.5000
Block Partition	100	112.3253	1.7694	112.7843
	1000	1847.2526	6.8858	1570.6199
	10000	25410.1772	123.7813	25350.7255
	100000	294244.7788	3465.6995	294878.4121
	1000000	3309146.6000	66124.1202	3313478.8000
	10000000	38173447.6500	1498913.4243	37931268.5000
2-Pivot Yaro	100	218.6639	0.0935	218.6437
	1000	3629.7245	1.7137	3630.2813
	10000	50301.2096	38.3586	50300.6213
	100000	635045.2987	2126.8544	634416.1873
	1000000	7624533.4500	45160.5385	7614424.8750
	10000000	90144952.0000	955908.5187	90243746.5000
Lomuto Block	100	108.9624	0.0537	108.9703
	1000	1423.2516	3.1800	1422.2247
	10000	17959.2518	49.0859	17951.8464
	100000	196951.7141	1583.5204	197172.6696
	1000000	2134863.0700	33965.1267	2119311.9000
	10000000	24700025.8000	856684.2381	24551996.5000
3-Pivot Kush	100	231.0457	0.0996	231.0266
	1000	3828.0046	1.7692	3827.5547
	10000	52692.9469	57.0170	52699.8674
	100000	660656.7803	2410.3154	660486.3048
	1000000	7881473.7500	55041.5020	7885420.7000
	10000000	93039500.9000	1138513.6270	93254088.5000
4-Pivot	100	233.3581	0.1961	233.3429
	1000	3847.1970	1.7934	3847.1812
	10000	52466.7069	69.7884	52466.1749
	100000	655540.9742	2248.3174	655697.4468
	1000000	7831009.1900	49027.0168	7835433.7000
	10000000	92702517.3500	1393247.4531	93065712.5000

Cache Misses	Size	Mean	SD	Median
1-Pivot Hoare	100	0.0205	0.0120	0.0188
	1000	0.6087	1.4656	0.2268
	10000	3.4867	2.7533	2.7844
	100000	1079.6629	230.8850	996.8944
	1000000	25764.5000	1166.9931	25475.1250
	10000000	634608.9000	13770.3278	633085.5000
Block Partition	100	0.0308	0.0710	0.0115
	1000	0.6074	1.3152	0.1710
	10000	2.1814	1.5526	1.6280
	100000	962.7563	36.5304	957.6906
	1000000	22939.3449	772.0659	22956.2750
	10000000	618067.7500	15870.0637	615660.5000
1-Pivot Lomuto	100	0.0309	0.0585	0.0114
	1000	1.5477	2.5235	0.7386
	10000	16.8345	33.0704	6.4018
	100000	1033.2760	46.1483	1016.9857
	1000000	25119.6125	1033.3887	24916.1250
	10000000	653495.7500	17064.0432	653582.5000
Block Partition	100	0.0382	0.0290	0.0314
	1000	0.3182	0.2477	0.2104
	10000	3.3952	3.7817	2.3033
	100000	1170.8087	38.3259	1170.6985
	1000000	25346.0800	820.9205	25164.7000
	10000000	664360.6000	21405.0063	659137.5000
2-Pivot Yaro	100	0.0470	0.1104	0.0155
	1000	1.3900	4.3104	0.3234
	10000	14.0858	30.1847	6.7674
	100000	1200.8078	284.2979	1131.0076
	1000000	22058.5875	528.7325	21870.3750
	10000000	598454.2000	6634.4607	597553.5000
Lomuto Block	100	0.0882	0.2168	0.0370
	1000	1.6396	5.9674	0.2243
	10000	5.6262	2.8924	5.2277
	100000	1160.9140	51.9771	1143.0456
	1000000	24010.0300	790.7382	24020.7000
	10000000	635062.3500	14272.0662	634228.5000
3-Pivot Kush	100	0.0118	0.0093	0.0095
	1000	1.0156	1.9968	0.5050
	10000	9.3897	19.5556	4.8184
	100000	1071.0240	157.6619	1032.5982
	1000000	22472.1100	1444.4493	22098.3000
	10000000	608203.3000	7020.9750	607993.0000
4-Pivot	100	0.0343	0.0814	0.0120
	1000	1.3838	2.5992	0.5333
	10000	4.1651	4.0935	3.1953
	100000	1158.2534	209.3988	1090.7000
	1000000	22077.3200	600.9565	22069.7000
	10000000	587479.3000	11805.1672	582773.5000

CPU Cycles	Size	Mean	SD	Median
1-Pivot Hoare	100	9327.4759	18.6229	9321.5095
	1000	141738.7953	210.1901	141718.3450
	10000	1896946.0648	20737.2369	1891616.5700
	100000	23894324.6266	136138.2225	23938976.7268
	1000000	283465722.4250	4456168.7125	282439687.8750
	10000000	3341781994.300	21642481.3347	3341179100.5000
Block Partition	100	7261.6214	3.9023	7261.0766
	1000	91497.6734	44.2229	91503.7709
	10000	1093098.0648	28276.9430	1087067.0555
	100000	12615251.1810	408397.6682	12525100.7167
	1000000	142142979.8257	1816742.9514	141750443.9091
	10000000	1651547110.850	15522352.0233	1651945749.0000
1-Pivot Lomuto	100	8753.9251	19.1779	8747.1730
	1000	133820.5575	121.5330	133786.0877
	10000	1797257.9908	1206.3404	1797317.9663
	100000	22592563.3050	184018.1868	22560236.2500
	1000000	266780288.5625	1370978.1629	266496379.5000
	10000000	3176006233.550	31963176.2929	3178944155.5000
Block Partition	100	7008.9387	28.2391	7005.8331
	1000	119936.7616	332.2665	120045.5206
	10000	1635196.2005	3174.6739	1635068.8220
	100000	19944485.2959	165665.3166	19999479.0516
	1000000	234756806.7100	3512955.3454	233878365.1000
	10000000	2779056745.450	88165361.1430	2783135055.5000
2-Pivot Yaro	100	8731.0535	13.1483	8733.6726
	1000	134846.2398	206.7811	134824.3757
	10000	1813160.2271	4167.9340	1812861.3252
	100000	22590180.7687	50252.7800	22586713.9725
	1000000	271262015.0750	2037510.3443	271168330.2500
	10000000	3222076486.750	22825580.2990	3228776030.0000
Lomuto Block	100	6921.2645	7.4863	6920.0053
	1000	115531.6098	1531.6089	115046.6171
	10000	1567567.6369	5508.4648	1567416.7897
	100000	18897769.3384	175912.1206	18848614.8083
	1000000	216844641.5500	2817917.2648	216536644.3000
	10000000	2548418986.850	48249924.4893	2559985148.0000
3-Pivot Kush	100	8612.3759	35.5221	8603.2238
	1000	130997.1678	1591.6522	130493.5023
	10000	1736859.7051	1691.3828	1736487.8719
	100000	21593228.2474	199876.9076	21493032.9813
	1000000	257443837.9800	1733684.7359	256993526.2000
	10000000	3026119949.450	23471563.5615	3025067078.5000
4-Pivot	100	8563.8790	9.0442	8562.7883
	1000	129414.4258	60.9929	129430.0027
	10000	1712788.0590	1490.2721	1712866.7067
	100000	21148019.6577	123063.0849	21104913.9275
	1000000	249836540.5700	1201917.2970	249647638.0000
	10000000	2961500698.750	21436898.4187	2957871275.5000

Time	Size	Mean	SD	Median
1-Pivot Hoare	100	2.1751 µs	7.3394 µs	2.1724 µs
	1000	33.659 µs	203.14 µs	33.553 µs
	10000	442.01 µs	2.1185 µs	441.58 µs
	100000	5.5583 ms	79.072 µs	5.5294 ms
	1000000	66.392 ms	1.0688 ms	66.594 ms
	10000000	776.49 ms	3.7364 ms	776.07 ms
Block Partition	100	1.6938 µs	4.7546 ns	1.6926 µs
	1000	21.782 µs	107.85 ns	21.743 µs
	10000	259.09 µs	490.99 ns	258.97 µs
	100000	2.9872 ms	9.5993 µs	2.9855 ms
	1000000	33.808 ms	159.14 µs	33.788 ms
	10000000	394.79 ms	2.9162 ms	395.56 ms
1-Pivot Lomuto	100	2.0334 µs	5.281 ns	2.0322 µs
	1000	31.105 µs	123.19 ns	31.062 µs
	10000	419.93 µs	3.2284 µs	418.01 µs
	100000	5.2008 ms	19.270 µs	5.1963 ms
	1000000	62.292 ms	452.07 µs	62.217 ms
	10000000	745.10 ms	6.7564 ms	743.02 ms
Block Partition	100	1.6698 µs	4.2852 ns	1.6689 µs
	1000	28.453 µs	87.002 ns	28.424 µs
	10000	392.02 µs	1.3621 µs	391.81 µs
	100000	4.7857 ms	42.338 µs	4.7824 ms
	1000000	56.288 ms	884.73 µs	56.358 ms
	10000000	668.14 ms	21.283 ms	662.61 ms
2-Pivot Yaro	100	2.0848 µs	6.1677 ns	2.0827 µs
	1000	31.372 µs	95.770 ns	31.347 µs
	10000	429.91 µs	5.6708 µs	428.28 µs
	100000	5.3164 ms	15.375 µs	5.3128 ms
	1000000	63.092 ms	500.09 µs	62.958 ms
	10000000	748.26 ms	4.8510 ms	749.60 ms
Lomuto Block	100	1.6387 µs	4.3470 ns	1.6373 µs
	1000	27.249 µs	52.407 ns	27.249 µs
	10000	374.62 µs	3.3842 µs	373.88 µs
	100000	4.5048 ms	31.057 µs	4.5022 ms
	1000000	52.149 ms	802.78 µs	52.234 ms
	10000000	605.61 ms	13.676 ms	602.31 ms
3-Pivot Kush	100	1.9985 µs	9.7010 ns	1.9943 µs
	1000	30.422 µs	80.286 ns	30.392 µs
	10000	411.09 µs	798.13 ns	410.84 µs
	100000	5.0570 ms	28.975 µs	5.0482 ms
	1000000	60.328 ms	906.06 µs	60.771 ms
	10000000	717.27 ms	6.5670 ms	716.72 ms
4-Pivot	100	1.9923 µs	6.2348 ns	1.9901 µs
	1000	30.422 µs	94.982 ns	30.400 µs
	10000	403.15 µs	1.7442 µs	402.50 µs
	100000	4.9661 ms	23.757 µs	4.9566 ms
	1000000	58.758 ms	365.07 µs	58.683 ms
	10000000	696.42 ms	5.1556 ms	696.82 ms

3.1 New Layout of 2-Pivot Lomuto's Block Partition Tests

Table 1: Branch Misses

Size	Mean	SD	Median
100	150.9847	0.0653	150.9726
1000	2318.4835	9.9296	2317.9223
10000	31167.1394	80.6562	31184.6376
100000	371744.0274	2827.6261	372691.0625
1000000	4334773.9875	102158.1340	4335591.3750
10000000	51459230.9000	1557769.9690	51377714.5000

Table 2: Cache Misses

Size	Mean	SD	Median
100	0.0205	0.0339	0.0112
1000	0.2856	0.1491	0.2300
10000	15.8177	56.1246	2.8154
100000	1411.5398	82.2908	1392.3521
1000000	27392.0125	875.9860	27313.2500
10000000	670240.8000	15488.9557	668665.5000

Table 3: CPU Cycles

Size	Mean	SD	Median
100	9484.3716	120.3484	9448.6366
1000	156216.1596	128.3762	156217.6846
10000	2186118.3441	7013.3473	2184792.1616
100000	27635332.3081	167125.6873	27670210.5496
1000000	333242126.0500	4954054.1009	331742534.7500
10000000	3997343820.250	62970785.7480	3995304309.5000

Table 4: Time

Size	Mean	SD	Median
100	2.2327 µs	29.601 ns	2.2257 µs
1000	36.708 µs	142.53 ns	36.662 µs
10000	518.56 µs	2.1303 µs	518.11
100000	6.5765 ms	46.260 µs	6.5651 ms
1000000	79.235 ms	882.47 µs	79.284 ms
10000000	951.74 ms	16.173 ms	948.58 ms

3.2 Cachegrind Results

```
==31674== Cachegrind, a cache and branch-prediction profiler
==31674== Copyright (C) 2002-2017, and GNU GPL'd, by Nicholas Nethercote et al.
==31674== Using Valgrind-3.18.1 and LibVEX; rerun with -h for copyright info
==31674== Command: target/release/cache
==31674==
--31674-- warning: L3 cache found, using its data for the LL simulation.

u32
std sort 10m array cost: 5987503982ns
quick sort 1-pivot (hoare partition) 10m array cost: 6949218668ns
quick sort 1-pivot (lomuto partition) 10m array cost: 5893410786ns
quick sort 2-pivot 10m array cost: 6050136667ns
quick sort 3-pivot 10m array cost: 4945230962ns
quick sort 4-pivot 10m array cost: 5582964437ns
quick sort 1-pivot (hoare partition block) 10m array cost: 7087265527ns
quick sort 1-pivot (lomuto partition block) 10m array cost: 13095039428ns
quick sort 2-pivot (block partition) 10m array cost: 12781142680ns
quick sort 2-pivot (new block partition) 10m array cost: 21336123394ns

f32
std sort 10m array cost: 7047995328ns
quick sort 1-pivot (hoare partition) 10m array cost: 9384315610ns
quick sort 1-pivot (lomuto partition) 10m array cost: 8849787330ns
quick sort 2-pivot 10m array cost: 9187125786ns
quick sort 3-pivot 10m array cost: 6326447943ns
quick sort 4-pivot 10m array cost: 6182636001ns
quick sort 1-pivot (hoare partition block) 10m array cost: 11061601282ns
quick sort 1-pivot (lomuto partition block) 10m array cost: 16044714565ns
quick sort 2-pivot (block partition) 10m array cost: 16078338106ns
quick sort 2-pivot (new block partition) 10m array cost: 26167134034ns

==31674==
==31674== I   refs:      95,959,007,560
==31674== I1  misses:             2,469
==31674== LLi misses:             2,435
==31674== I1  miss rate:           0.00%
==31674== LLi miss rate:           0.00%
==31674==
==31674== D   refs:      22,415,640,761  (14,065,770,855 rd   + 8,349,869,906 wr)
==31674== D1  misses:       230,634,166  (   216,262,654 rd   +    14,371,512 wr)
==31674== LLd misses:        40,852,371  (    27,095,675 rd   +    13,756,696 wr)
==31674== D1  miss rate:            1.0% (           1.5%     +           0.2%  )
==31674== LLd miss rate:            0.2% (           0.2%     +           0.2%  )
==31674==
==31674== LL refs:          230,636,635  (   216,265,123 rd   +    14,371,512 wr)
==31674== LL misses:         40,854,806  (    27,098,110 rd   +    13,756,696 wr)
==31674== LL miss rate:             0.0% (           0.0%     +           0.2%  )
```

```
--------------------------------------------------------------------------------
Profile data file 'cachegrind.out.31674'
--------------------------------------------------------------------------------
I1 cache:         32768 B, 64 B, 8-way associative
D1 cache:         32768 B, 64 B, 8-way associative
LL cache:         33554432 B, 64 B, direct-mapped
Profiled target:  target/release/cache
Events recorded:  Ir I1mr ILmr Dr D1mr DLmr Dw D1mw DLmw
Events shown:     Ir I1mr ILmr Dr D1mr DLmr Dw D1mw DLmw
Event sort order: Ir I1mr ILmr Dr D1mr DLmr Dw D1mw DLmw
Thresholds:       99 0 0 0 0 0 0 0 0
Include dirs:
User annotated:
Auto-annotation:  on

--------------------------------------------------------------------------------
Ir                     I1mr           ILmr           Dr
--------------------------------------------------------------------------------
95,959,007,560 (100.0%) 2,469 (100.0%) 2,435 (100.0%) 14,065,770,855 (100.0%)

--------------------------------------------------------------------------------
D1mr                DLmr               Dw
--------------------------------------------------------------------------------
216,262,654 (100.0%) 27,095,675 (100.0%) 8,349,869,906 (100.0%)

--------------------------------------------------------------------------------
D1mw                DLmw
--------------------------------------------------------------------------------
14,371,512 (100.0%)  13,756,696 (100.0%)  PROGRAM TOTALS

--------------------------------------------------------------------------------
Ir                     I1mr         ILmr         Dr
--------------------------------------------------------------------------------
21,960,123,024 (22.88%)  47 ( 1.90%)  47 ( 1.93%) 4,076,573,635 (28.98%)
13,969,610,982 (14.56%)  19 ( 0.77%)  19 ( 0.78%) 1,491,698,621 (10.61%)
13,675,018,865 (14.25%)  33 ( 1.34%)  33 ( 1.36%) 1,735,170,012 (12.34%)
 8,149,209,024 ( 8.49%)  16 ( 0.65%)  16 ( 0.66%)   799,322,583 ( 5.68%)
 7,606,991,471 ( 7.93%)  54 ( 2.19%)  54 ( 2.22%) 1,139,116,007 ( 8.10%)
 6,967,578,920 ( 7.26%)  18 ( 0.73%)  18 ( 0.74%) 1,124,191,062 ( 7.99%)
 6,538,606,429 ( 6.81%)  15 ( 0.61%)  15 ( 0.62%)   994,198,806 ( 7.07%)
 4,862,994,232 ( 5.07%)  27 ( 1.09%)  27 ( 1.11%)   622,009,731 ( 4.42%)
 4,844,423,688 ( 5.05%)  44 ( 1.78%)  44 ( 1.81%)   611,828,274 ( 4.35%)
 4,281,356,909 ( 4.46%)  87 ( 3.52%)  87 ( 3.57%)   877,012,667 ( 6.24%)
 2,194,368,923 ( 2.29%) 161 ( 6.52%) 161 ( 6.61%)   471,403,998 ( 3.35%)


--------------------------------------------------------------------------------
D1mr                DLmr               Dw                         D1mw
--------------------------------------------------------------------------------
```

```
20,748,859 ( 9.59%)    889,063 ( 3.28%) 1,620,521,723 (19.41%)     99,933 ( 0.70%)
25,897,120 (11.97%)  1,090,334 ( 4.02%) 1,200,472,821 (14.38%)     79,644 ( 0.55%)
20,799,962 ( 9.62%)  1,090,123 ( 4.02%) 1,328,750,860 (15.91%)     78,459 ( 0.55%)
20,546,480 ( 9.50%)  1,250,024 ( 4.61%)   351,354,604 ( 4.21%)     15,190 ( 0.11%)
18,748,990 ( 8.67%)  1,291,970 ( 4.77%)   903,014,938 (10.81%)    119,029 ( 0.83%)
15,643,246 ( 7.23%)    937,966 ( 3.46%)   476,735,340 ( 5.71%)     20,062 ( 0.14%)
26,847,354 (12.41%)  1,048,761 ( 3.87%)   698,810,467 ( 8.37%)     36,939 ( 0.26%)
13,628,374 ( 6.30%)    750,565 ( 2.77%)   475,003,077 ( 5.69%)     24,112 ( 0.17%)
13,229,375 ( 6.12%)  1,596,439 ( 5.89%)   465,598,575 ( 5.58%)     22,453 ( 0.16%)
15,163,113 ( 7.01%)  1,423,422 ( 5.25%)   662,690,122 ( 7.94%)    118,975 ( 0.83%)
12,500,177 ( 5.78%)  3,224,176 (11.90%)    60,624,285 ( 0.73%)  1,250,113 ( 8.70%)

--------------------------------------------------------------------------------
    DLmw            file:function
--------------------------------------------------------------------------------
    1,038 ( 0.01%)  ???:rust_sorts::qsort::double_pivot_quicksort_new_partition_block
      738 ( 0.01%)  ???:rust_sorts::qsort::quick_sort_lomuto_partition_block
      569 ( 0.00%)  ???:rust_sorts::qsort::double_pivot_quicksort_lomuto_partition_block
      106 ( 0.00%)  ???:rust_sorts::qsort::quick_sort_hoare_partition
      926 ( 0.01%)  ???:rust_sorts::qsort::quick_sort_hoare_partition_block
      926 ( 0.01%)  ???:rust_sorts::qsort::quick_sort_hoare_partition_block
      168 ( 0.00%)  ???:rust_sorts::qsort::double_pivot_quicksort
      324 ( 0.00%)  ???:rust_sorts::qsort::quick_sort_lomuto_partition
      311 ( 0.00%)  ???:rust_sorts::qsort::triple_pivot_quicksort
      261 ( 0.00%)  ???:rust_sorts::qsort::quad_pivot_quicksort
    1,050 ( 0.01%)  ???:core::slice::sort::recurse
1,250,113 ( 9.09%)  ???:cache::main
```

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