

$2^1 - 1 = 3$  prim

$2^3 - 1 = 7$  prim

$2^5 - 1 = 31$  prim

$2^7 - 1 = 127$  prim

$2^{11} - 1 = 2047 = 23 \cdot 89$

# The CrypTool Book: Learning and Experiencing Cryptography with CrypTool and SageMath

Prof. Bernhard Esslinger  
and the Development Team  
of the Open-Source Software CrypTool

Edition 12 (2016)

The CrypTool Book:

# Learning and Experiencing Cryptography with CrypTool and SageMath

Background reading for CrypTool  
the free e-learning crypto program  
(Cryptography, Mathematics, and More)

12th edition – **draft version** (19:08:44)

Prof. Bernhard Esslinger (co-author and editor)  
and the CrypTool Team, 1998-2016

[www.cryptool.org](http://www.cryptool.org)

Thursday 12<sup>th</sup> January, 2017

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This also includes the code of the SageMath samples in this document.

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# Overview about the Content of the CrypTool Book

The rapid spread of the Internet has led to intensified research in the technologies involved, especially within the area of cryptography where a good deal of new knowledge has arisen.

In this *book accompanying the CrypTool programs* you will find predominantly mathematically oriented information on using cryptographic procedures. Also included are many sample code pieces written in the computer algebra system **SageMath** (see appendix A.7). The main chapters have been written by various **authors** (see appendix A.8) and are therefore independent from one another. At the end of most chapters you will find references and web links. The sections have been enriched with many *footnotes*. Within the footnotes you can see where the described functions can be called in the different CrypTool versions.

The **first chapter** explains the principles of symmetric and asymmetric **encryption** and list definitions for their resistibility.

Because of didactic reasons the **second chapter** gives an exhaustive overview about **paper and pencil encryption methods**.

Big parts of this book are dedicated to the fascinating topic of **prime numbers** (chap. 3). Using numerous examples, **modular arithmetic** and **elementary number theory** (chap. 4) are introduced. Here, the features of the **RSA procedure** are a key aspect.

By reading chapter 5 you'll gain an insight into the mathematical ideas and concepts behind **modern cryptography**.

Chapter 6 gives an overview about the status of attacks against modern **hash algorithms** and is then shortly devoted to **digital signatures**, which are an essential component of e-business applications.

Chapter 7 describes **elliptic curves**: They could be used as an alternative to RSA and in addition are extremely well suited for implementation on smartcards.

Chapter 8 introduces **Boolean algebra**. Boolean algebra is the foundation for most modern, symmetric encryption algorithms as these operate on bit streams and bit groups. Principal construction methods are described and implemented in SageMath.

Chapter 9 describes **homomorphic crypto functions**: They are a modern research topic which got especial attention in the course of cloud computing.

Chapter 10 describes **Current Results for Solving Discrete Logarithms and Factoring**. It provides a broad picture and comparison about the currently best algorithms for (a) computing discrete logarithms in various groups, for (b) the status of the factorization problem, and for (c) elliptic curves. This survey was put together as a reaction to a provocative talk at the Black Hat Conference 2013 which caused some uncertainty by incorrectly extrapolating progress at finite fields of small characteristics to the fields used in real world.

The last chapter **Crypto2020** discusses threats for currently used cryptographic methods and introduces alternative research approaches (post-quantum crypto) to achieve long-term security of cryptographic schemes.

Whereas the CrypTool *e-learning programs* motivate and teach you how to use cryptography in practice, the *book* provides those interested in the subject with a deeper understanding of the mathematical algorithms used – trying to do it in an instructive way.

Within the **appendices A.1, A.2, A.3, and A.4** you can gain a fast overview about the functions delivered by the different CrypTool variants via:

- the function list and the [menu tree of CrypTool 1 \(CT1\)](#),
- the function list and the [templates in CrypTool 2 \(CT2\)](#),
- the [function list of JCrypTool \(JCT\)](#), and
- the [function list of CrypTool-Online \(CTO\)](#).

The authors would like to take this opportunity to thank their colleagues in the particular companies and at the universities of Bochum, Darmstadt, Frankfurt, Gießen, Karlsruhe, Lausanne, Paris, and Siegen.

As with the e-learning program CrypTool, the quality of the book is enhanced by your suggestions and ideas for improvement. We look forward to your feedback.

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# Preface to the 12th Edition of the CrypTool Book

The book's goal is to explain some mathematical topics from cryptology in exact detail, nevertheless in a way which is easy to understand.

This book was delivered since the year 2000 – together with the CrypTool 1 (CT1) package in version 1.2.01. Since then it has been expanded and revised in almost every new version of CT1 and CT2.

Topics from mathematics and cryptography have been meaningfully split up and for each topic an extra chapter has been written which can be read on its own. This enables developers/authors to contribute independently of each other. Naturally there are many more interesting topics in cryptography which could be discussed in greater depth – therefore this selection is only one of many possible ways.

The later editorial work in LaTeX added footnotes and cross linkages between different sections, harmonized the index entries, and made some corrections.

Compared to edition 11, this edition completely updated the TeX sources of the document (e.g. one single bibtex file for all chapters and both languages), and of course, the content of the book was amended, corrected, and updated with many topics, for instance:

- the largest prime numbers (chap. 3.4),
- the list of movies or novels, in which cryptography or number theory played a major role (see appendix A.5),
- the overviews of all functions in [CrypTool 2 \(CT2\)](#), [JCrypTool \(JCT\)](#), and [CrypTool-Online \(CTO\)](#) (see appendix),
- further SageMath scripts for cryptography, and the appendix A.7 about using the computer algebra system SageMath,
- the section about the Goldbach conjecture (see 3.8.3) and about twin primes (see 3.8.4),
- the section about shared primes in RSA modules used in reality (see 4.11.5.4),
- the “[Introduction to Bitblock and Bitstream Ciphers](#)” is completely new (see chapter 8),
- the “[Survey on Current Academic Results for Solving Discrete Logarithms and for Factoring](#)” is completely new too (see chapter 10). It's a phantastic in-depth summary about the limits of the according current cryptanalytical methods.

## Acknowledgment

At this point I'd like to thank explicitly the following people who particularly contributed to the CrypTool project. They applied their very special talents and showed really great engagement:

- Mr. Henrik Koy
- Mr. Jörg-Cornelius Schneider
- Mr. Florian Marchal
- Dr. Peer Wichmann
- Mr. Dominik Schadow
- Staff of Prof. Johannes Buchmann, Prof. Claudia Eckert, Prof. Alexander May, Prof. Torben Weis, and especially Prof. Arno Wacker.

Also I want to thank all the many people not mentioned here for their hard work (mostly carried out in their spare time).

Thanks also to the readers who sent us feedback. And especial thanks for the free proof reading of this edition done by Helmut Witten and Prof. Ralph-Hardo Schulz.

I hope that many readers have fun with this book and that they get out of it more interest and greater understanding of this modern but also very ancient topic.

Bernhard Esslinger

Heilbronn/Siegen (Germany), August 2016

PS:

We'd be glad if new authors would show up to improve existing chapters or to add further chapters, e.g. about

- the Riemann Zeta function,
- hash functions and password attacks,
- lattice-based cryptography,
- random numbers, or
- the design and attack of crypto protocols (like SSL).

PPS:

Todos to be dealt with to make edition 12 of this CTB a release (till then we still call it a draft):

- Update all information about SageMath (chap. 7.9.2 and appendix) and run the code against the newest SageMath (version 7.x) – both command line and SageMathCloud notebook.
- Update the function lists of the four CT versions (in the appendix).

# Introduction – How do the Book and the Programs Play together?

## This CrypTool book

This document is delivered together with the open-source programs of the CrypTool project. You can also download it directly from the website of the CT portal: <https://www.cryptool.org/en/ctp-documentation>.

The chapters in this book are largely self-contained and can also be read independently of the CrypTool programs.

Chapters 5 (“Modern Cryptography”), 7 (“Elliptic Curves”), 8 (“Bitblock and Bitstream Ciphers”), 9 (“Homomorphic Ciphers”), and 10 (“Results for Solving Discrete Logarithms and for Factoring”) require a deeper knowledge in mathematics, while the other chapters should be understandable with a school leaving certificate.

The [authors](#) have attempted to describe cryptography for a broad audience – without being mathematically incorrect. We believe that this didactic pretension is the best way to promote the awareness for IT security and the readiness to use standardized modern cryptography.

## The programs CrypTool 1, CrypTool 2, and JCrypTool

CrypTool 1 (CT1) is an educational program enabling you to use and analyze cryptographic procedures within a unified graphical user interface. The comprehensive online help in CrypTool 1 contains both instructions how to use the program and explanations about the methods itself (both not as detailed and in another structure as in the CT book).

CrypTool 1 and the successor versions CrypTool 2 (CT2) and JCrypTool (JCT) are used world-wide for training in companies and teaching at schools and universities.

## CrypTool-Online

Another part of the CT project is the website CrypTool-Online (CTO) (<http://www.cryptool-online.org>), where you can apply crypto methods within a browser or on a smartphone. The scope of CTO is far below from the standalone programs CT1, CT2 and JCT. However, this is what people more and more use as the first contact, so we currently redesign the whole backbone system with most modern web technology to have a fast and consistent look&feel.

## MTC3

MTC3 is the abbreviation for MysteryTwister C3, an international cryptography contest (<http://www.mysterytwisterc3.org>), which is also based on the CT project. Here you can find cryptographic riddles in four categories, a high-score list and a moderated forum. As of

2016-06-16 more than 7000 users participate, and more than 200 challenges are offered (162 of them are solved by at least one participant).

## **The Computer Algebra System SageMath**

SageMath is a comprehensive open-source CAS package which can be used to easily program the mathematical methods explained in this book. A speciality of this CAS is, that the scripting language is Python (currently version 2.x). So in a Sage script, you have after an import statement all functions from the Python language at your disposal.

SageMath becomes more and more the standard CAS system at universities.

## **The Pupil's Crypto Courses**

Within this initiative, one and two 2 day courses in cryptology are offered for pupils and teachers in order to show how attractive MINT subjects like mathematics, computer science and especially cryptology are. The course agenda is a virtual secret agent story.

In the meantime, these courses took place for several years in Germany in different towns.

All course material is freely available at <http://www.cryptoool.org/schuelerkrypto/>.

All software used is free software (using mostly CT1 and CT2).

As all course material is currently available only in German – we'd be happy if someone could translate the course material and build an according course in English.

## **Acknowledgment**

I am deeply grateful to all the people helping with their impressive commitment who have made this global project so successful.

Bernhard Esslinger

Heilbronn/Siegen (Germany), August 2016

# Chapter 1

## Security Definitions and Encryption Procedures

([Bernhard Esslinger](#), [Joerg-Cornelius Schneider](#), May 1999; Updates Dec 2001, Feb 2003, Jun 2005, Jul 2007, Jan 2010, Mar 2013, Aug 2016)

This chapter introduces the topic in a more descriptive way without using too much mathematics.

The purpose of encryption is to change data in such a way that only an authorized recipient is able to reconstruct the plaintext. This allows us to transmit data without worrying about it getting into unauthorized hands. Authorized recipients possess a piece of secret information – called the key – which allows them to decrypt the data while it remains hidden from everyone else.<sup>1</sup>

For explanations in the following we use the notation from Figure 1.1:

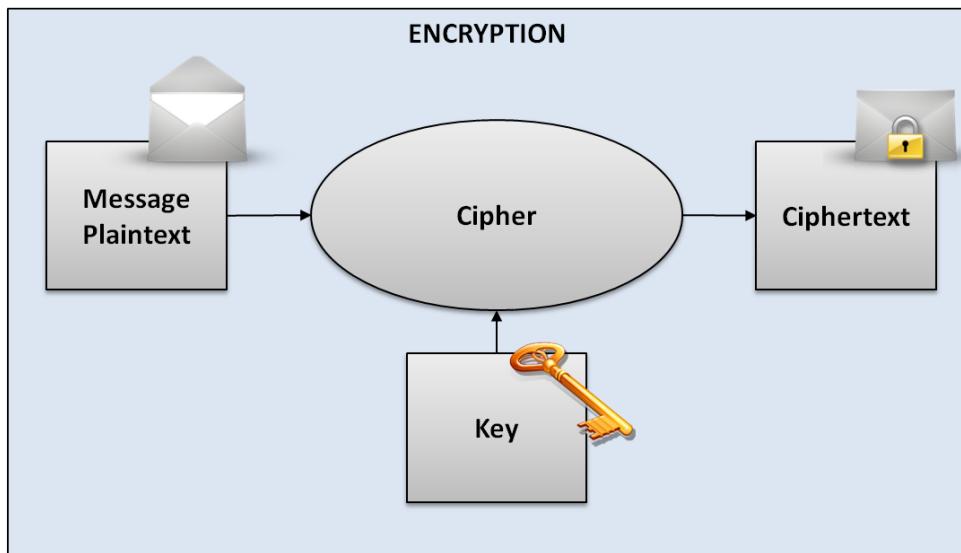


Figure 1.1: Common notations when using ciphers

<sup>1</sup>However, an attacker still can disturbance the connection or tap metadata (like who is communicating with whom).

Explain it to me, I will forget it.  
Show it to me, maybe I will remember it.  
Let me do it, and I will be good at it.

Quote 1: Saying from India

## 1.1 Security definitions and the importance of cryptology

First we present the ideas how the security of cryptosystems is defined.

Modern cryptography is heavily based on mathematical theory and computer science practice. Cryptographic algorithms are designed around computational hardness assumptions, making such algorithms hard to break in practice by any adversary.

Depending on the adversary's capabilities there are mainly two basic notations of security distinguished in literature (see e.g. *Contemporary Cryptography* [Opp11]):

- **Computational, conditional or practical security**

A cipher is *computationally* secure if it is theoretically possible to break such a system but it is infeasible to do so by any known practical means. Theoretical advances (e.g., improvements in integer factorization algorithms) and faster computing technology require these solutions to be continually adapted.

Even using the best known algorithm for breaking it will require so much resources (e.g., 1,000,000 years) that practically the cryptosystem is secure.

So this concept is based on assumptions of the adversary's limited computing power and the current state of science.

- **Information-theoretical or unconditional security**

A cipher is considered *unconditionally* secure if its security is guaranteed no matter how much resources (time, space) the attacker has – so even in the case where the adversary has unlimited resources for breaking a cipher. Even with unlimited resources an adversary is unable to gain any meaningful data from a ciphertext.

There exist information-theoretically secure schemes that provably cannot be broken even with unlimited computing power – an example is the *one-time pad* (OTP).

As the OTP is information-theoretically secure it derives its security solely from information theory and is secure even with unlimited computing power at the adversary's disposal. However, OTP has several practical disadvantages (the key used must be used only once, randomly selected and must be at least as long as the message being protected), which means that it is hardly used except in closed environments such as for the hot wire between Moscow and Washington.

Two more concepts are sometimes used:

- **Provable security** This means that breaking such a cryptographic system is as difficult as solving some supposedly difficult problem e.g. discrete logarithm computation, discrete square root computation, very large integer factorization.

Example: Currently we know that RSA is at most as difficult as factorization, but we cannot prove that its exactly as difficult as factorization. So RSA has no proven minimum

security. Or in other words: We cannot prove, that if RSA (the cryptosystem) is broken, that then factorization (the hard mathematical problem) can be solved.

The Rabin cryptosystem was the first cryptosystem which could be proven to be computationally equivalent to a hard problem.

- **Ad-hoc security** A cryptographic system has this security feature if it is not worth to try to break into such a system because of inadequate price of data with comparison to price of work needed to do so. Or an attack can't be done in sufficiently short time (see *Handbook of Applied Cryptography* [MvOV01]).

Example: This may apply if a message relevant for the stock market will be published tomorrow and you need a year to break it.

For good procedures used today the time taken to break them is so long that it is practically impossible to do so, and these procedures can therefore be considered (practically) secure – from a pure algorithm's point of view.<sup>2</sup>

We basically distinguish between symmetric (see chapter 1.3) and asymmetric (see chapter 1.4) encryption procedures. The books of Bruce Schneier [Sch96b] and Klaus Schmeh [Sch16a] also offer a very good overview of the different encryption algorithms.<sup>3</sup>

With the use of the internet and wireless communication, encryption technologies are used (mostly transparently) by everyone. However, they have been in use since centuries by governments, military, and diplomats. The side which had a better command of these technologies could exert big influence on **politics** and war with the help of secret services. This book only touches history when introducing the earlier cipher methods in chapter 2. You can gain an impression, how important cryptology was and still is, by considering the following two examples: the educational film “War of the letters” (German: Krieg der Buchstaben)<sup>4</sup> and the debates around the so called crypto wars<sup>5</sup>.

---

<sup>2</sup>Especially after the knowledge gathered by Edward Snowden there were many discussions, whether encryption is secure. In [ESS14] is the result of an evaluation, which cryptographic algorithms can be relied on – according to current knowledge. The article investigates: Which crypto systems can – despite the reveal of the NSA/GCHQ attacks – still be considered as secure? Where have systems been intentionally weakened? How can we create a secure cryptographic future? Where is the difference between maths and implementation?

<sup>3</sup>A compact overview about what is used where, which methods are secure, where you have to anticipate problems and where the construction areas will be in the future (incl. the lengthy procedures of the standardization) can be found in the German article [Sch16b].

<sup>4</sup>See [http://bscw.schule.de/pub/bscw.cgi/d1269787/Krieg\\_der\\_Buchstaben.pdf](http://bscw.schule.de/pub/bscw.cgi/d1269787/Krieg_der_Buchstaben.pdf).

A supporting quote from Denis Smyth, professor in the Department of History and the International Relations Programme at the University of Toronto: “Secret intelligence has long been regarded as the “missing dimension” of international relations.” (from <http://www.secretintelligencefiles.com>)

<sup>5</sup>See [https://en.wikipedia.org/wiki/Crypto\\_Wars](https://en.wikipedia.org/wiki/Crypto_Wars).

“One cannot not communicate.”

Quote 2: Paul Watzlawick<sup>6</sup>

## 1.2 Influences on encryption methods

Here, we just want to mention two aspects often neglected and dealt with too late:

- **Random based**

Algorithms can be divided up into deterministic and heuristic methods. Most students only learnt deterministic methods, where the output is uniquely determined by the input. On the other hand, heuristic methods make decisions using random values. Modern methods of machine learning are also part of them.

Random looms large in cryptographic methods. Keys have to be selected randomly, which means that at least for the key generation “random” is necessary. In addition, some methods, especially from cryptanalysis, are heuristic.

- **Constant based**

Many modern methods (especially hash methods and symmetric encryption) use numeric constants. Their values should be plausible and they shouldn't contain back doors. Numbers fulfilling this requirement are called nothing-up-my-sleeve numbers.<sup>7</sup>

---

<sup>6</sup>Paul Watzlawick, Janet H. Beavin, and Don D. Jackson, “Pragmatics of human communication; a study of interactional patterns, pathologies, and paradoxes”, Norton, 1967, The first of five axioms of their human communications theory.

<sup>7</sup>[http://en.wikipedia.org/wiki/Nothing\\_up\\_my\\_sleeve\\_number](http://en.wikipedia.org/wiki/Nothing_up_my_sleeve_number)

The following figure 1.2 intends to give an idea, that it is impossible to determine the correct plaintext from a OTP (if the OTP method has been applied correctly and if all keys have the same likelihood).

The example in this figure uses an 8 character long given ciphertext: 11 1B 1E 18 00 04 0A 15. There are many meaningful words with 8 letters and for each there is an according key. So an attacker can not determine alone from the ciphertext, which is the correct key and which is the correct plaintext word.

Also see figure 8.19 in chapter 8.3.2, where an according example with text strings is build with SageMath.

Key (hex)	Revealed plaintext (hex)	Revealed plaintext (txt)		
52 49 47 48 54 4B 45 59	43 52 59 50 54 4F 4F 4C	CRYPTOOL		
41 5A 50 57 52 45 47 54	50 41 4E 4F 52 41 4D 41	PANORAMA		
54 77 7B 68 68 65 64 61	45 4C 45 50 48 41 4E 54	ELEPHANT		
50 55 5B 55 4F 4A 4F 46	41 4E 45 4D 4F 4E 45 53	ANEMONES		
52 53 5F 55 50 4D 45 5B	43 48 41 4D 50 49 4F 4E	CHAMPION		
42 58 5B 56 41 56 43 5A	53 43 45 4E 41 52 49 4F	SCENARIO		

Figure 1.2: Illustration for the information-theoretically secure OTP scheme<sup>8</sup>

---

<sup>8</sup>Picture source: Free pictures from <https://pixabay.com/>

“Transparency. That’s the best one can hope for in a technologically advanced society ... otherwise you will just be manipulated.”

Quote 3: Daniel Suarez<sup>9</sup>

### 1.3 Symmetric encryption<sup>10</sup>

For *symmetric* encryption sender and recipient must be in possession of a common (secret) key which they have exchanged before actually starting to communicate. The sender uses this key to encrypt the message and the recipient uses it to decrypt it.

This is visualized in Figure 1.3:

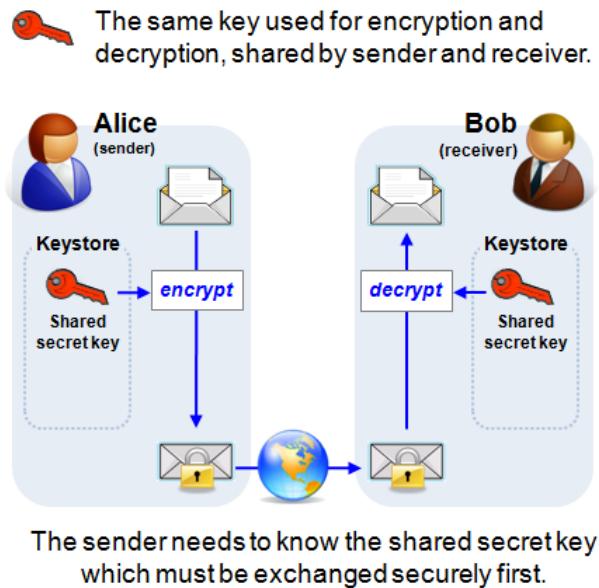


Figure 1.3: Symmetric or secret-key encryption

All classical ciphers are of the type symmetric. Examples can be found within the CT programs, in chapter 2 (“[Paper and Pencil Encryption Methods](#)”) of this script, or in [Nic96]. In this section however, we want to consider only modern symmetric mechanisms.

The advantages of symmetric algorithms are the high speed with which data can be encrypted and decrypted. One disadvantage is the need for key management. In order to communicate with

<sup>9</sup>Daniel Suarez, “Freedom”, Dutton Adult, 2010, Chapter 5, “Getting with the Program”, p. 63, Price.

<sup>10</sup>With CrypTool 1 (**CT1**) you can execute the following modern symmetric encryption algorithms (using the menu path **Crypt \ Symmetric (modern)**):

IDEA, RC2, RC4, DES (ECB), DES (CBC), Triple-DES (ECB), Triple-DES (CBC), MARS (AES candidate), RC6 (AES candidate), Serpent (AES candidate), Twofish (AES candidate), Rijndael (official AES algorithm).

With CrypTool 2 (**CT2**) you can execute the following modern symmetric encryption algorithms (using in the Startcenter **Templates \ Cryptography \ Modern \ Symmetric**):

AES, DES, PRESENT, RC2, RC4, SDES, TEA, Triple-DES, Twofish.

In JCrypTool (**JCT**) you can execute the following modern symmetric encryption algorithms:

AES, Rijndael, Camellia, DES, Dragon, IDEA, LFSR, MARS, Misty1, RC2, RC5, RC6, SAFER+, SAFER++, Serpent, Shacal, Shacal2, Twofish.

one another confidentially, sender and recipient must have exchanged a key using a secure channel before actually starting to communicate. Spontaneous communication between individuals who have never met therefore seems virtually impossible. If everyone wants to communicate with everyone else spontaneously at any time in a network of  $n$  subscribers, each subscriber must have previously exchanged a key with each of the other  $n - 1$  subscribers. A total of  $n(n - 1)/2$  keys must therefore be exchanged.

### 1.3.1 AES (Advanced Encryption Standard)<sup>11</sup>

Before AES, the most well-known modern symmetric encryption procedure was the DES algorithm. The DES algorithm has been developed by IBM in collaboration with the National Security Agency (NSA), and was published as a standard in 1975. Despite the fact that the procedure is relatively old, no effective attack on it has yet been detected. The most effective way of attacking consists of testing (almost) all possible keys until the right one is found (*brute-force-attack*). Due to the relatively short key length of effectively 56 bits (64 bits, which however include 8 parity bits), numerous messages encrypted using DES have in the past been broken. Therefore, the procedure can not be considered secure any longer. Alternatives to the DES procedure include IDEA, Triple-DES (TDES) and especially AES.

Up-to-the-minute procedure for symmetric ciphers is the AES. The associated Rijndael algorithm was declared winner of the AES award on October 2nd, 2000 and thus succeeds the DES procedure.

An introduction and further references about the AES algorithms and the AES candidates of the last round can be found i.e. within the online help of CrypTool<sup>12</sup> oder in Wikipedia<sup>13</sup>.

---

<sup>11</sup>In CT1 you can find 3 visualizations for this cipher via the menu **Indiv. Procedures \ Visualization of Algorithms \ AES**.

In CT2 you can find a template performing AES step-by-step (by entering the search string “AES” in the Startcenter).

<sup>12</sup>CrypTool 1 online help: The index head-word **AES** leads to the 3 help pages: **AES candidates**, **The AES winner Rijndael** and **The Rijndael encryption algorithm**

A comprehensive description of AES including C code can be found in [Haa08].

<sup>13</sup>[https://en.wikipedia.org/wiki/Advanced\\_Encryption\\_Standard](https://en.wikipedia.org/wiki/Advanced_Encryption_Standard)

The two screenshots 1.4 and 1.5 are taken from one of three AES visualizations in CT1.

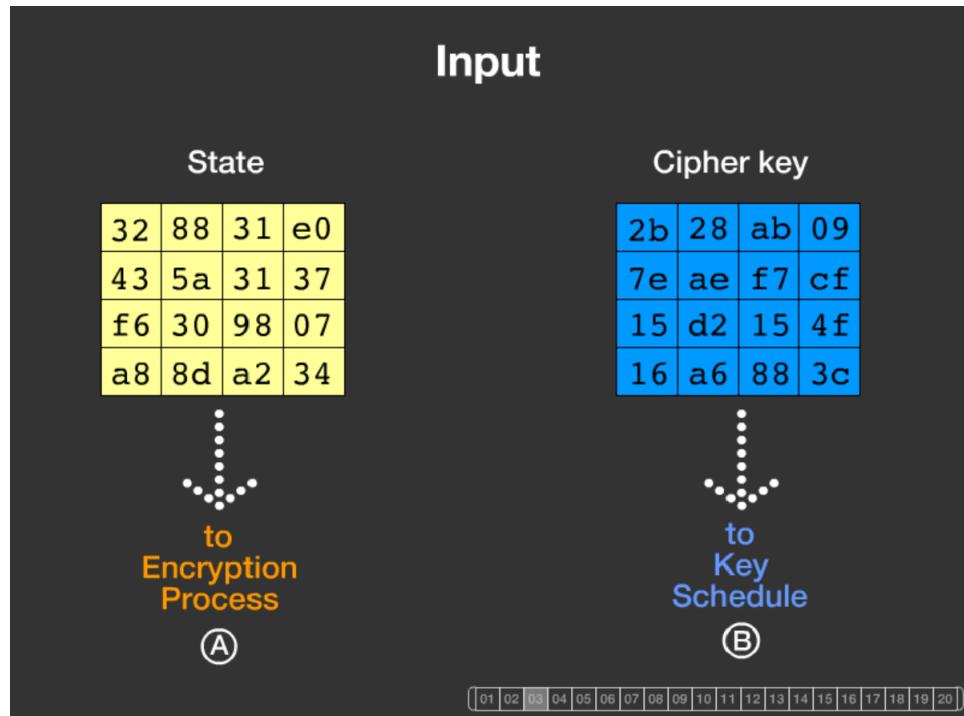


Figure 1.4: AES visualization by Enrique Zabala from CT1 (part 1)

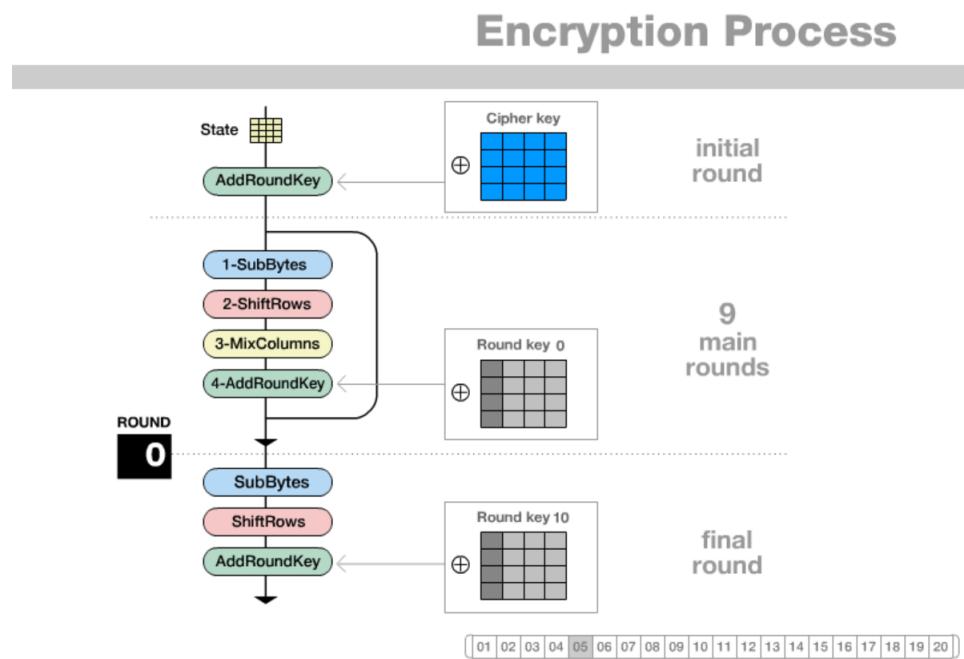


Figure 1.5: AES visualization by Enrique Zabala from CT1 (part 2)

Now we want to encrypt with AES in CBC mode a 128-bit block of plaintext. Of the resulting ciphertext we are only interested in the 1st block (if there is more, it would be padding, here null-padding). For demonstration we do it once with CT2 and once with OpenSSL.

Figure 1.6 shows the encryption of one block in **CT2**.

The plaintext “AESTEST1USINGCT2” is converted to hex (41 45 53 54 45 53 54 31 55 53 49 4E 47 43 54 32). Using this and the key 3243F6A8885A308D313198A2E0370734 the AES component creates the ciphertext. It is in hex:

B1 13 D6 47 DB 75 C6 D8 47 FD 8B 92 9A 29 DE 08

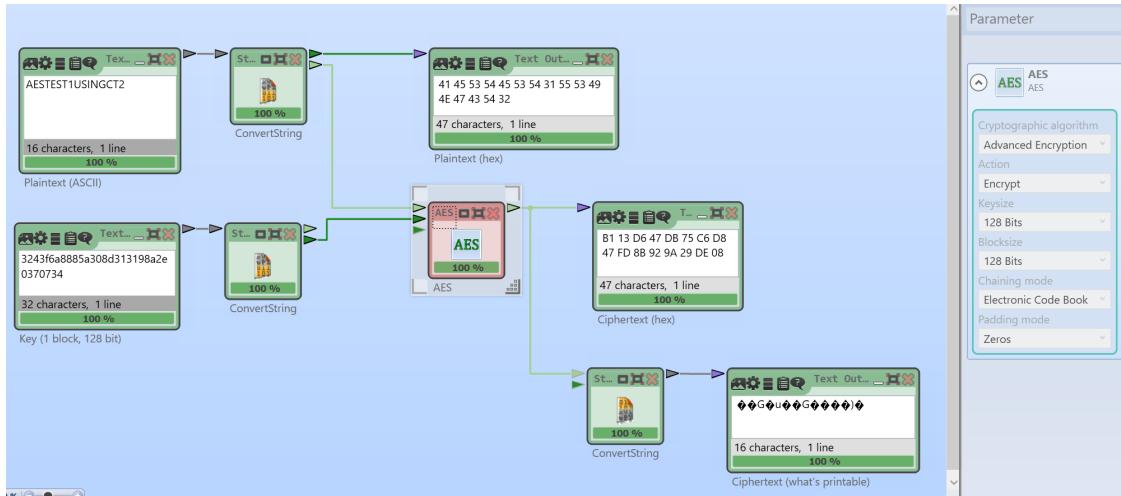


Figure 1.6: AES encryption (of exactly 1 block and without padding) in CT2

The same result can be achieved with **OpenSSL**<sup>14</sup> from the commandline:

---

**OpenSSL sample 1.1** AES encryption (of exactly one block and without padding) in OpenSSL

```
>openssl enc -e -aes-128-cbc
-K 3243F6A8885A308D313198A2E0370734
-iv 00000000000000000000000000000000
-in klartext-1.hex -out klartext-1.hex.enc

>dir
06.07.2016 12:43           16 key.hex
20.07.2016 20:19           16 klartext-1.hex
20.07.2016 20:37           32 klartext-1.hex.enc
```

---

<sup>14</sup>OpenSSL is a very widespread free open-source crypto library, used by many applications, for instance to implement the TLS protocol. Part of OpenSSL is the commandline tool openssl, which can be used to test the functionality directly on many operating systems and to request, create and manage certificates.

Contrarily to the also very widespread and very good commandline tool gpg from GNU Privacy Guard ([https://en.wikipedia.org/wiki/GNU\\_Privacy\\_Guard](https://en.wikipedia.org/wiki/GNU_Privacy_Guard)), openssl also allows calls with many details. The gpg has its focus on the practically applied ciphersuites. As far as we know, it is not possible to encrypt just one block without padding with the commandline tool gpg.

Also see <https://en.wikipedia.org/wiki/OpenSSL>.

### 1.3.1.1 Results about theoretical cryptanalysis of AES

Below you will find some results, which have recently called into question the security of the AES algorithm – from our point of view these doubts practically still remain unfounded . The following information is based particularly on the original papers and the articles [Wob02] and [LW02].

AES with a minimum key length of 128 bit is still in the long run sufficiently secure against brute-force attacks – as long as the quantum computers aren't powerful enough. When announced as new standard AES was immune against all known cryptanalytic attacks, mostly based on statistical considerations and earlier applied to DES: using pairs of clear and cipher texts expressions are constructed, which are not completely at random, so they allow conclusions to the used keys. These attacks required unrealistically large amounts of intercepted data.

Cryptanalysts already label methods as “academic success” or as “cryptanalytic attack” if they are theoretically faster than the complete testing of all keys (brute force analysis). In the case of AES with the maximal key length (256 bit) exhaustive key search on average needs  $2^{255}$  encryption operations. A cryptanalytic attack needs to be better than this. At present between  $2^{75}$  and  $2^{90}$  encryption operations are estimated to be performable only just for organizations, for example a security agency.

In their 2001-paper Ferguson, Schroeppel and Whiting [FSW01] presented a new method of symmetric codes cryptanalysis: They described AES with a closed formula (in the form of a continued fraction) which was possible because of the “relatively” clear structure of AES. This formula consists of around 1000 trillion terms of a sum - so it does not help concrete practical cryptanalysis. Nevertheless curiosity in the academic community was awakened. It was already known, that the 128-bit AES could be described as an over-determined system of about 8000 quadratic equations (over an algebraic number field) with about 1600 variables (some of them are the bits of the wanted key) – equation systems of that size are in practice not solvable. This special equation system is relatively sparse, so only very few of the quadratic terms (there are about 1,280,000 are possible quadratic terms in total) appear in the equation system.

The mathematicians Courtois and Pieprzyk [CP02] published a paper in 2002, which got a great deal of attention amongst the cryptology community: The pair had further developed the XL-method (eXtended Linearization), introduced at Eurocrypt 2000 by Shamir et al., to create the so called XSL-method (eXtended Sparse Linearization). The XL-method is a heuristic technique, which in some cases manages to solve big non-linear equation systems and which was till then used to analyze an asymmetric algorithm (HFE). The innovation of Courtois and Pieprzyk was, to apply the XL-method on symmetric codes: the XSL-method can be applied to very specific equation systems. A 256-bit AES could be attacked in roughly  $2^{230}$  steps. This is still a purely academic attack, but also a direction pointer for a complete class of block ciphers. The major problem with this attack is that until now nobody has worked out, under what conditions it is successful: the authors specify in their paper necessary conditions, but it is not known, which conditions are sufficient. There are two very new aspects of this attack: firstly this attack is not based on statistics but on algebra. So attacks seem to be possible, where only very small amounts of ciphertext are available. Secondly the security of a product algorithm<sup>15</sup>

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<sup>15</sup>A ciphertext can be used as input for another encryption algorithm. A cascade cipher is build up as a composition of different encryption transformations. The overall cipher is called product algorithm or cascade cipher (sometimes depending whether the used keys are statistically dependent or not).

Cascading does not always improve the security.

This process is also used *within* modern algorithms: They usually combine simple and, considered at its own, cryptologically relatively insecure single steps in several rounds into an efficient overall procedure. Most block ciphers (e.g. DES, IDEA) are cascade ciphers.

does not exponentially increase with the number of rounds.

Currently there is a large amount of research in this area: for example Murphy and Robshaw presented a paper at Crypto 2002 [MR02b], which could dramatically improve cryptanalysis: the burden for a 128-bit key was estimated at about  $2^{100}$  steps by describing AES as a special case of an algorithm called BES (Big Encryption System), which has an especially “round” structure. But even  $2^{100}$  steps are beyond what is achievable in the foreseeable future. Using a 256 bit key the authors estimate that a XSL-attack will require  $2^{200}$  operations.

More details can be found in the Web links section at “[AES or Rijndael cryptosystem](#)”.

So for AES-256 the attack is much more effective than brute-force but still far away from any computing power which could be accessible in the short-to-long term.

The discussion is very controversial at the moment: Don Coppersmith (one of the inventors of DES) for example queries the practicability of the attack because XLS would provide no solution for AES [Cop02]. This implies that then the optimization of Murphy and Robshaw [MR02a] would not work.

In 2009 Biryukov and Khovratovich [BK09] published another theoretical attack on AES. This attack uses different methods from the ones described above. They applied methods from hash function cryptanalysis (local collisions and boomerang switching) to construct a related-key attack on AES-256. I. e. the attacker not only needs to be able to encrypt arbitrary data (chosen plain text), in addition he needs to be able to manipulate the unknown key (related-key).

Based on those assumptions, the effort to find a AES-256 key is reduced to  $2^{119}$  time and  $2^{77}$  memory (considering asymmetric complexity). In the case of AES-192 the attack is even less practical, for AES-128 the authors do not provide an attack.

### 1.3.2 Algebraic or algorithmic cryptanalysis on symmetric algorithms

There are different modern methods attacking the structure of a problem directly or after a transformation of the problem. One of the attack methods is based on the satisfiability problem (SAT)<sup>16</sup>.

#### Description of a SAT solver

An old and well-studied problem in computer science is called the SAT problem. Here, for a given Boolean formula, it's the task to find out whether there is an assignment of the variables, so that the evaluation result of the formula is 1.

Example: The Boolean formula “A AND B” evaluates to 1, if and only if A=B=1. For the formula “A AND NOT(A)” there exists no assignment of its variable A, so that the formula is evaluated to the value 1.

For larger Boolean formulas, it is not easy to determine if an assignment exists for which the formula can be evaluated to 1 (this problem belongs to the NP-complete problems). Therefore specific tools have been developed to solve this problem for general Boolean formulas, so called SAT solvers<sup>17</sup>. As has been found, SAT solvers can also be used to attack cryptographic systems.

#### SAT solver based cryptanalysis

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Also serial usage of the same cipher with different keys (like with Triple-DES) is called cascade cipher.

<sup>16</sup>[http://en.wikipedia.org/wiki/Boolean\\_satisfiability\\_problem](http://en.wikipedia.org/wiki/Boolean_satisfiability_problem)

<sup>17</sup>With CT2 you can execute a SAT solver – using in the Startcenter **Templates \ Mathematics \ SAT Solver (Text Input)** and **SAT Solver (File Input)**.

The general approach to use SAT solvers in cryptanalysis is very straightforward: First, the cryptographic problem, e.g. finding the symmetric key or an inversion of a hash function, is translated into a SAT problem. Then, the SAT solver can be used to find a solution to the SAT problem. The solution of the SAT problem then also solves the original cryptographic problem. The paper by Massacci [MM00] describes the first known usage of a SAT solver in this context. Unfortunately, very soon it turned out that such a general approach cannot be used efficiently in practice. This is due to the fact that the cryptographic SAT problems are very complex and the runtime of a SAT solver increases exponentially with the problem size. Therefore, in modern approaches SAT solvers are used only for solving partial problems of cryptanalysis. A good example for this is described in the paper by Mironov and Zhang [MZ06]. They demonstrate the usage of a SAT solver in an attack on hash functions, where the SAT solver is used to solve some partial problems in a very efficient way.

### 1.3.3 Current status of brute-force attacks on symmetric algorithms

The current status of brute-force attacks on symmetric encryption algorithms can be explained with the block cipher RC5.

Brute-force (exhaustive search, trial-and-error) means to completely examine all keys of the key space: so no special analysis methods have to be used. Instead, the ciphertext is decrypted with all possible keys<sup>18</sup> and for each resulting text it is checked, whether this is a meaningful clear text<sup>19</sup>. A key length of 64 bit means at most  $2^{64} = 18,446,744,073,709,551,616$  or about 18 trillion (GB) / 18 quintillion (US) keys to check.

Companies like RSA Security provided so-called cipher challenges in order to quantify the security offered by well-known symmetric ciphers as DES, Triple-DES or RC5.<sup>20</sup> They offered prizes for those who managed to decipher ciphertexts, encrypted with different algorithms and different key lengths, and to unveil the symmetric key (under controlled conditions). So theoretical estimates can be confirmed.

It is well-known, that the “old” standard algorithm DES with a fixed key length of 56 bit is no more secure: This was demonstrated already in January 1999 by the Electronic Frontier Foundation (EFF). With their specialized computer Deep Crack they cracked a DES encrypted message within less than a day.<sup>21</sup>

The current record for strong symmetric algorithms unveiled a key 64 bit long. The algorithm used was RC5, a block cipher with variable key size.

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<sup>18</sup>With CT1 you can also perform brute-force attacks of modern symmetric algorithms (using the menu path **Analysis \ Symmetric Encryption (modern)**): Here the weakest knowledge of an attacker is assumed, he performs a ciphertext-only attack.

With CT2 you can also perform brute-force attacks (using the templates under **Cryptanalysis \ Modern**). Highly powerful is the KeySearcher component, which can be used to distribute the calculations to many different computers.

<sup>19</sup>If the cleartext is written in a natural language and at least 100 B long, this check also can be performed automatically.

To achieve a result in an appropriate time with a single PC you should mark not more than 24 bit of the key as unknown.

<sup>20</sup><https://www.emc.com/emc-plus/rsa-labs/historical/the-rsa-laboratories-secret-key-challenge.htm>  
Unfortunately, in May 2007 RSA Inc announced that they will not confirm the correctness of the not yet solved RC5-72 challenge.

There are also cipher challenges for asymmetric algorithms (please see chapter 4.11.4).

A wide spectrum of both simple and complex, both symmetric and asymmetric crypto riddles are included in the international cipher contest **MysteryTwister C3**: <http://www.mysterytwisterc3.org>.

<sup>21</sup><https://www.emc.com/emc-plus/rsa-labs/historical/des-challenge-iii.htm>

The RC5-64 challenge has been solved in July 2002 by the distributed.net team after 5 years.<sup>22</sup> In total 331,252 individuals co-operated over the internet to find the key.<sup>23</sup> More than 15 trillion (GB) / 15 quintillion (US) keys were checked, until they found the right key.<sup>24</sup>

So, symmetric algorithms (even if they have no cryptographic weakness) using keys of size 64 bit are no more appropriate to keep sensible data private.

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<sup>22</sup>[http://www.distributed.net/Pressroom\\_press-rc5-64](http://www.distributed.net/Pressroom_press-rc5-64)  
[http://www.distributed.net/images/9/92/20020925\\_-\\_PR\\_-\\_64\\_bit\\_solved.pdf](http://www.distributed.net/images/9/92/20020925_-_PR_-_64_bit_solved.pdf)

<sup>23</sup>An overview of current distributed computing projects can be found here:  
<http://distributedcomputing.info/>

<sup>24</sup>CT2 started to experiment with a general infrastructure for distributed computing called CrypCloud (both peer-to-peer and centralized). So in the future, CT2 will be able to distribute the calculations on many computers. What could be achieved after the components are made ready for parallelization showed a cluster for distributed cryptanalysis of DES and AES: Status on March 21st, 2016 is, that an AES brute-force attack (distributed keysearching) worked on 50 i5 PCs, each with 4 virtual CPU cores. These 200 virtual “worker threads” achieved to test about 350 million AES keys/sec. The “cloud” processed a total amount of about 20 GB/sec of data. CrypCloud is a peer-to-peer-based cloud system that enables CT2 users to voluntarily join distributed computing jobs.

## 1.4 Asymmetric encryption<sup>25</sup>

In the case of *asymmetric* encryption each subscriber has a personal pair of keys consisting of a *secret* key and a *public* key. The public key, as its name implies, is made public – e.g. in a key directory on the Internet (this kind of “bill-board” is also called just directory or public key ring) or within a so-called public-key certificate.

The asymmetric encryption is visualized in Figure 1.7:

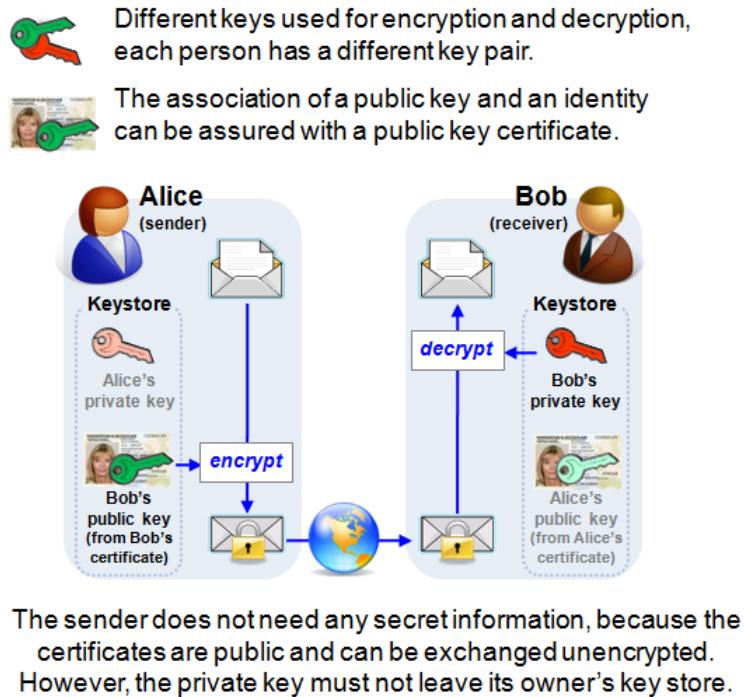


Figure 1.7: Asymmetric or public-key encryption

If Alice<sup>26</sup> wants to communicate with Bob, then she looks for Bob's public key and uses it to encrypt her message to him. She then sends this ciphertext to Bob, who is then able to decrypt it again using his secret key. As only Bob knows his secret key, only he can decrypt messages addressed to him. Even Alice who sends the message cannot restore plaintext from the (encrypted) message she has sent. Of course, you must first ensure that the public key cannot be used to derive the private key.

Such a procedure can be demonstrated using a series of thief-proof letter boxes. If I have

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<sup>25</sup>The RSA cryptosystem can be executed in many variations with CT1 (using the menu path **Individual Procedures \ RSA Cryptosystem \ RSA Demonstration**).

With CT1 you can execute RSA encryption and decryption (using the menu path **Crypt \ Asymmetric**). In both cases you must select a RSA key pair. Only in the case of decryption the secret RSA key is necessary – so here you are asked to enter the PIN.

With CT2 you can also perform asymmetric methods (using the templates under **Cryptography \ Modern**). JCT offers asymmetric methods like RSA both within the **Visuals** menu of the Default Perspective as well as within the Algorithm Perspective.

<sup>26</sup>In order to describe cryptographic protocols participants are often named Alice, Bob, ... (see [Sch96b, p. 23]). Alice and Bob perform all 2-person-protocols. Alice will initiate all protocols and Bob answers. The attackers are named Eve (eavesdropper) and Mallory (malicious active attacker).

composed a message, I then look for the letter box of the recipient and post the letter through it. After that, I can no longer read or change the message myself, because only the legitimate recipient has the key for the letter box.

The advantage of asymmetric procedures is the easier key management. Let's look again at a network with  $n$  subscribers. In order to ensure that each subscriber can establish an encrypted connection to each other subscriber, each subscriber must possess a pair of keys. We therefore need  $2n$  keys or  $n$  pairs of keys. Furthermore, no secure channel is needed before messages are transmitted, because all the information required in order to communicate confidentially can be sent openly. In this case, you simply<sup>27</sup> have to pay attention to the accuracy (integrity and authenticity) of the public key. Disadvantage: Pure asymmetric procedures take a lot longer to perform than symmetric ones.

The most well-known asymmetric procedure is the RSA algorithm<sup>28</sup>, named after its developers Ronald Rivest, Adi Shamir and Leonard Adleman. The RSA algorithm was published in 1978.<sup>29</sup> The concept of asymmetric encryption was first introduced by Whitfield Diffie and Martin Hellman in 1976. Today, the ElGamal procedures also play a decisive role, particularly the Schnorr variant in the DSA (Digital Signature Algorithm).

Attacks against asymmetric ciphers are touched in

- chapter [4: Elementary Number Theory](#),
- chapter [5: Modern Cryptography](#),
- chapter [7: Elliptic Curves](#), and
- chapter [10: Current Results for Solving Discrete Logarithms And Factoring](#).

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<sup>27</sup>That this is also not trivial is explained e.g. in chapter [4.11.5.4](#). Besides the requirements for the key generation it has to be considered that nowadays also (public-key) infrastructures itself are targets of cyber attacks.

<sup>28</sup>The RSA algorithm is extensively described in chapter [4.10](#) and later within this script. The topical research results concerning RSA are described in chapter [4.11](#).

<sup>29</sup>Hints about the history of RSA and its publication which didn't amuse the NSA can be found within the series *RSA & Co. at school: Modern cryptology, old mathematics, and subtle protocols*. Unfortunately these are currently only available in German. See [[WS06](#)], pp 55 ff ("Penible Lämmergeier").

## 1.5 Hybrid procedures<sup>30</sup>

In order to benefit from the advantages of symmetric and asymmetric techniques together, hybrid procedures are usually used (for encryption) in practice.

In this case the bulk data is encrypted using symmetric procedures: The key used for this is a secret session key generated by the sender randomly<sup>31</sup> that is only used for this message.

This session key is then encrypted using the asymmetric procedure, and transmitted to the recipient together with the message.

Recipients can determine the session key using their private keys and then use the session key to encrypt the message.

In this way, we can benefit from the easy key management of asymmetric procedures (using public/private keys) and we benefit from the efficiency of symmetric procedures to encrypt large quantities of data (using secret keys).

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<sup>30</sup>Within CT1 you can find this technique using the menu path **Crypt \ Hybrid**: There you can follow the single steps and its dependencies with concrete numbers. The variant with RSA as the asymmetric algorithm is graphically visualized; the variant with ECC uses the standard dialogs. In both cases AES is used as the symmetric algorithm.

JCT offers hybrid methods like ECIES within the Algorithm Perspective under **Algorithms \ Hybrid Ciphers**.

<sup>31</sup>An important part of cryptographically secure techniques is to generate random numbers.

- Within CT1 you can check out different random number generators (PRNGs) using the menu path **Indiv. Procedures \ Generate Random Numbers**. Using the menu path **Analysis \ Analyze Randomness** you can apply different test methods for random data to binary documents.
- In CT2 you can find templates using PRNGs (by entering the search string “random” in the Startcenter). The PRNGs internally use for instance Keccak or the Linear Congruential Generator (LCG), and they are used e.g. for key generation or decimalization.
- JCT offers **pseudo** random number generators both within the Default Perspective in the menu **Algorithms \ Random Number Generator** as well as within the Algorithm Perspective.

There is an old saying inside the US National Security Agency (NSA):  
“Attacks always get better; they never get worse.”

Quote 4: IETF<sup>32</sup>

## 1.6 Cryptanalysis and symmetric ciphers for educational purposes<sup>33</sup>

Compared to public-key ciphers based on mathematics like RSA, the structure of AES and most other modern symmetric ciphers (like DES, IDEA or Present), is very complex and cannot be explained as easily as RSA.

So simplified variants of modern symmetric ciphers were developed for educational purposes in order to allow beginners to perform encryption and decryption by hand and gain a better understanding of how the algorithms work in detail. These simplified variants also help to understand and apply the according cryptanalysis methods.

The most well-known of these variants are SDES (Simplified DES)<sup>34</sup> and S-AES (Simplified-AES) by Prof. Ed Schaefer and his students<sup>35</sup>, and Mini-AES (see chapter 1.8.1 “Mini-AES”):

- Edward F. Schaefer: *A Simplified Data Encryption Standard Algorithm* [Sch96a].
- Raphael Chung-Wei Phan: *Mini Advanced Encryption Standard (Mini-AES): A Testbed for Cryptanalysis Students* [Pha02].
- Raphael Chung-Wei Phan: *Impossible differential cryptanalysis of Mini-AES* [Pha03].
- Mohammad A. Musa, Edward F. Schaefer, Stephen Wedig: *A simplified AES algorithm and its linear and differential cryptanalyses* [MSW03].
- Nick Hoffman: *A SIMPLIFIED IDEA ALGORITHM* [Hof06].
- S. Davod. Mansoori, H. Khaleghei Bizaki: *On the vulnerability of Simplified AES Algorithm Against Linear Cryptanalysis* [MB07].

## 1.7 Further information

Beside the information you can find in the following chapters, in many other books and on a good number of websites, the online help of all CrypTool variants also offer very many details about the symmetric and asymmetric encryption methods.

<sup>32</sup><http://tools.ietf.org/html/rfc4270>

<sup>33</sup>A very good starting point to learn cryptanalysis is the book from Mark Stamp [SL07]. Also good, but very high-level and concentrating on analyzing symmetric block ciphers only, is the article from Bruce Schneier [Sch00]. Several of the cipher challenges at “MysteryTwister C3” (<http://www.mysterytwisterc3.org>) are also well suitable for educational purposes.

<sup>34</sup>If you double-click on the title of the icon of the SDES component in CT2 you can see a visualization of the SDES algorithm, showing how the bits of the given data flow through the whole algorithm. An according screenshot: <https://www.facebook.com/CrypTool2/photos/a.505204806238612.1073741827.243959195696509/597354423690316>

<sup>35</sup>See the article “Devising a Better Way to Teach and Learn the Advanced Encryption Standard” at <http://math.scu.edu/~eschaefer/getfile.pdf>

## 1.8 Appendix: Examples using SageMath

Below is SageMath source code related to contents of the chapter 1.6 (“[Cryptanalysis and symmetric ciphers for educational purposes](#)”).

Further details concerning cryptosystems within SageMath (e.g. about the Simplified Data Encryption Standard SDES) can be found e.g. in the thesis of Minh Van Nguyen [[Ngu09a](#)].

### 1.8.1 Mini-AES

The SageMath module `crypto/block_cipher/miniaes.py` supports Mini-AES to allow students to explore the inner working of a modern block cipher.

Mini-AES, originally described at [[Pha02](#)], is a simplified variant of the Advanced Encryption Standard (AES) to be used for cryptography education.

How to use Mini-AES is exhaustively described at the this SageMath reference page:

[http://doc.sagemath.org/html/en/reference/cryptography/sage/crypto/block\\_cipher/miniaes.html](http://doc.sagemath.org/html/en/reference/cryptography/sage/crypto/block_cipher/miniaes.html).

The following SageMath code [1.1](#) is taken from the release tour of SageMath 4.1<sup>36</sup> and calls the implementation of the Mini-AES.

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<sup>36</sup>See <http://mvngu.wordpress.com/2009/07/12/sage-4-1-released/>.

Further example code for Mini-AES can be found in [[Ngu09b](#), chap. 6.5 and appendix D].

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**SageMath sample 1.1** Encryption and decryption with Mini-AES

---

```
# We can encrypt a plaintext using Mini-AES as follows:
sage: from sage.crypto.block_cipher.miniaes import MiniAES
sage: maes = MiniAES()
sage: K = FiniteField(16, "x")
sage: MS = MatrixSpace(K, 2, 2)
sage: P = MS([K("x^3 + x"), K("x^2 + 1"), K("x^2 + x"), K("x^3 + x^2")]); P
[x^3 + x  x^2 + 1]
[x^2 + x  x^3 + x^2]
sage: key = MS([K("x^3 + x^2"), K("x^3 + x"), K("x^3 + x^2 + x"), K("x^2 + x + 1")]); key
[x^3 + x^2  x^3 + x]
[x^3 + x^2 + x  x^2 + x + 1]
sage: C = maes.encrypt(P, key); C
[x  x^2 + x]
[x^3 + x^2 + x  x^3 + x]

# Here is the decryption process:
sage: plaintext = maes.decrypt(C, key)
sage: plaintext == P
True

# We can also work directly with binary strings:
sage: from sage.crypto.block_cipher.miniaes import MiniAES
sage: maes = MiniAES()
sage: bin = BinaryStrings()
sage: key = bin.encoding("KE"); key
0100101101000101
sage: P = bin.encoding("Encrypt this secret message!")
sage: C = maes(P, key, algorithm="encrypt")
sage: plaintext = maes(C, key, algorithm="decrypt")
sage: plaintext == P
True

# Or work with integers n such that 0 <= n <= 15:
sage: from sage.crypto.block_cipher.miniaes import MiniAES
sage: maes = MiniAES()
sage: P = [n for n in xrange(16)]; P
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]
sage: key = [2, 3, 11, 0]; key
[2, 3, 11, 0]
sage: P = maes.integer_to_binary(P)
sage: key = maes.integer_to_binary(key)
sage: C = maes(P, key, algorithm="encrypt")
sage: plaintext = maes(C, key, algorithm="decrypt")
sage: plaintext == P
True
```

---

### 1.8.2 Further symmetric crypto algorithms in SageMath

The Reference for SageMath v7.2 lists e.g. the following further cryptographic functions:<sup>37</sup>

- Linear feedback shift register (LFSR),
- Blum-Blum-Shub (BBS): pseudo-random generator (to be found with streams),
- Lattice-based functions.

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<sup>37</sup>See <http://doc.sagemath.org/html/en/reference/sage/crypto/index.html>,  
<http://doc.sagemath.org/html/en/reference/cryptography/index.html>, and  
<http://combinat.sagemath.org/doc/reference/cryptography/sage/crypto/stream.html>

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[http://bscw.schule.de/pub/bscw.cgi/d404410/RSA\\_u\\_Co\\_NF2.pdf](http://bscw.schule.de/pub/bscw.cgi/d404410/RSA_u_Co_NF2.pdf).

All links have been confirmed at July 10, 2016.

# Web Links

1. AES discussion groups at NIST (archive page provided for historical purposes, last update on Feb 28th, 2001)  
<http://csrc.nist.gov/archive/aes/>
2. AES or Rijndael cryptosystem (page maintained by Nicolas T. Courtois, last update on Aug 24th, 2007)  
<http://www.cryptosystem.net/aes>
3. distributed.net: “RC5-64 has been solved”  
[http://www.distributed.net/Pressroom\\_press-rc5-64](http://www.distributed.net/Pressroom_press-rc5-64)
4. RSA Labs (former RSA Security): “The RSA Secret Key Challenge”  
<https://www.emc.com/emc-plus/rsa-labs/historical/the-rsa-laboratories-secret-key-challenge.htm>
5. RSA Labs (former RSA Security): “DES Challenge”  
<https://www.emc.com/emc-plus/rsa-labs/historical/des-challenge-iii.htm>
6. Further links can be found at the CrypTool homepage  
<http://www.cryptool.org>

All links have been confirmed at July 10, 2016.

## Chapter 2

# Paper and Pencil Encryption Methods

(Christine Stoetzel, Apr 2004; Updates: B.+C. Esslinger, Jun 2005; Updates: Minh Van Nguyen and Bernhard Esslinger, Nov 2009, Jun 2010; Bernhard Esslinger, May 2013, Aug 2016)

Few persons can be made to believe that it is not quite an easy thing to invent a method of secret writing which shall baffle investigation. Yet it may be roundly asserted that human ingenuity cannot concoct a cipher which human ingenuity cannot resolve.

Quote 5: Edgar Allan Poe: A Few Words on Secret Writing, 1841

The following chapter provides a broad overview of paper and pencil methods<sup>1</sup> each with references to deeper information. All techniques that people can apply manually to en- and decipher a message are embraced by this term. These methods were and still are especially popular with secret services, as a writing pad and a pencil – in contrast to electronic aids – are totally unsuspicious.

The first paper and pencil methods already arose about 3000 years ago, but new procedures were developed during the past century, too. All paper and pencil methods are a matter of symmetric methods. Even the earliest encryption algorithms use the basic principles such as transposition, substitution, block construction and their combinations. Hence it is worthwhile to

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<sup>1</sup>The footnotes of this chapter describe how these methods can be performed using the offline programs CrypTool 1 (**CT1**), CrypTool 2 (**CT2**), and JCrypTool (**JCT**). See the appendices [A.1](#), [A.2](#), and [A.3](#).

Many of the methods can also be performed within a browser, e.g. on the website CrypTool-Online (**CTO**) (<http://www.cryptool-online.org>). See the appendix [A.4](#) in this book.

While the CrypTool websites and programs offer both classic and modern ciphers, there are several sites related to the American Cryptogram Association (ACA) (<http://www.cryptogram.org/>) which focus very deeply and detailed only on classic ciphers.

For instance, <https://sites.google.com/site/bionspot/> and <https://encode-decode.appspot.com/> from Bion.

An attractive new site for performing classic ciphers is from Phil Pilcrow ([www.cryptoprograms.com](http://www.cryptoprograms.com)). It also includes descriptions and examples for each type. His FAQ stated at July 23rd, 2016: “The site is designed for creating classical cipher types, not machine based or modern ones but if you want another cipher type added let me know.”

Additionally the last sub chapter ([2.5](#)) of this chapter contains according example code for the computer-algebra system **SageMath**.

closely consider this “ancient” methods especially under didactic aspects.

Methods to be successful and wide-spread had to fulfill some attributes which are equally required for modern algorithms:

- Exhaustive description, almost standardization (including special cases, padding, etc.).
- Good balance between security and usability (because methods being too complicated were error-prone or unacceptably slow).

## 2.1 Transposition ciphers

Encrypting a message by means of transposition does not change the original characters of this message, only their order is modified (transposition = exchange).<sup>2</sup>

### 2.1.1 Introductory samples of different transposition ciphers

- **Rail fence cipher**<sup>3</sup> [Sin99]: The characters of a message are alternately written in two (or more) lines, creating a zigzag pattern. The resulting ciphertext is read out line by line. This is more a children's method.

See table 2.1.

Plaintext<sup>4</sup>: an example of transposition

n	x	m	l	o	t	a	s	o	i	i	n
a	e	a	p	e	f	r	n	p	s	t	o

Table 2.1: Rail Fence cipher

Ciphertext<sup>5</sup>: NXMLO TASOI INAEA PEFRN PSTO

- **Scytale**<sup>6</sup> [Sin99]: This method was probably used since 600 B.C. – a description of how it operated is not known from before Plutarch (50-120 B.C.).  
A long strip of paper is wrapped around a wooden cylinder and then the message is written along the length of this strip. After unwinding the strip contains the ciphertext. For decryption the recipient needs to have a – previously agreed – cylinder of the same size and with the same number of edges.
- **Grille cipher** [Goe14]: Both parties use identical stencils. Line by line, their holes are filled with plaintext that is read out column by column to produce the ciphertext. If there is plaintext left, the procedure is repeated.<sup>7</sup>

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<sup>2</sup>Sometimes, the name permutation is used to describe how characters, groups of characters or columns of the plaintext are exchanged, e.g.  $(1, 2, 3, 4, 5) \Leftrightarrow (3, 4, 2, 1, 5)$ .

<sup>3</sup>This method can directly be found in CT1 at the menu item **Encrypt/Decrypt \ Symmetric (classic) \ Scytale / Rail Fence**. You can simulate this method also under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Permutation**: For a Rail Fence with 2 lines use as key "B,A" and accept the default settings (only one permutation, where your input is done line-by-line and the output is taken column-by-column). Using the key "A,B" would start the zigzag pattern below in the way, that the first letter is written into the first line instead of the second line.

<sup>4</sup>Convention: If the alphabet uses only 26 letters, we write from now onwards the plaintext in small letters and the ciphertext in capital letters.

<sup>5</sup>The letters of the plaintext are – as used historically – grouped within blocks of 5 letters. It does not matter if a different (constant) block length is used or if there is no separation by blanks.

<sup>6</sup>This method can directly be found in CT1 at the menu item **Encrypt/Decrypt \ Symmetric (classic) \ Scytale / Rail Fence**. As this method is a special case of a simple columnar transposition, you also can simulate it in CT1 under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Permutation**: For the Scytale within the dialog box only the first permutation is used. If the wood has e.g. 4 angles use as key "1,2,3,4". This is equivalent to write the text horizontally in blocks of 4 letters in a matrix and to read it out vertically. Because the key is in an ascending order, the Scytale is denoted as an identical permutation. And because writing and read-out is done only once it is a simple (and no double) permutation.

In CT2 you can find the Scytale within the templates **Cryptography \ Classical**.

<sup>7</sup>This method cannot be simulated with a pure column transposition.

- **Turning grille** [Sav99]: The German army used turning grilles during WW1.<sup>8</sup> A square grille serves as a stencil, a quarter of its fields being holes. The first part of the message is written on a piece of paper through these holes, then the grille is rotated by 90 degrees and the user can write down the second part of the message, etc. But this method does only work, if the holes are chosen carefully: Every field has to be used, and no field may be used twice, either. The ciphertext is read out line by line.

In the example for a turning grille in table 2.2 you can write 4 times 16 characters of the plaintext on a piece of paper:

O	-	-	-	-	O	-	-
-	-	-	-	O	O	-	-
-	-	-	-	O	-	-	O
-	-	-	O	-	-	-	-
-	-	-	-	-	O	-	-
O	-	O	-	-	-	O	-
-	O	-	-	-	-	-	O
-	-	-	O	O	-	-	-

Table 2.2: 8x8 turning grille

### 2.1.2 Column and row transposition<sup>9</sup>

- **Simple columnar transposition** [Sav99]: First of all, a keyword is chosen, that is written above the columns of a table. This table is filled with the text to be encrypted line by line. Then the columns are rearranged by sorting the letters of the keyword alphabetically. Afterwards the columns are read out from left to right to build the ciphertext.<sup>10</sup>

See table 2.3.

Plaintext: an example of transposition

Transposition key: K=2; E=1; Y=3.

Ciphertext: NALFA PIOAX PORSS IEMET NOTN

- **AMSCO cipher** [ACA02]: The characters of the plaintext are written in alternating groups of one respectively two letters into a grille. Then the columns are swapped and the text can be read out.
- **Double column transposition (DCT)** [Sav99] : Double columnar transposition was frequently used during WW2 and during the Cold War. Two simple columnar transpositions with different keys are executed successively.<sup>11</sup>

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<sup>8</sup>The turning grille was already invented in 1881 by Eduard Fleissner von Wostrowitz.

A good visualization can be found under [www.turning-grille.com](http://www.turning-grille.com).

In JCT you can find it in the default perspective via the menu item **Visuals \ Grille**.

<sup>9</sup>Most of the following methods can be simulated in CT1 under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Permutation**.

<sup>10</sup>Using CT1: Choose a key for the 1st permutation, input line by line, permute and output column by column.

In CT2 you can find the transposition within the templates **Cryptography \ Classical**. This component also visualizes how the text is put into and taken off the matrix and how the columns are permuted.

<sup>11</sup>Using CT1: Choose a key for the 1st permutation, input line by line, permute and output column by column. Then choose a (different) key for the 2nd permutation, input line by line, permute and output column by column.

K	E	Y
a	n	e
x	a	m
p	l	e
o	f	t
r	a	n
s	p	o
s	i	t
i	o	n

Table 2.3: Simple columnar transposition

If the keys are different and long enough (at least each 20 characters), then this is even for today's computer a real challenge.<sup>12</sup>

- **Column transposition, General Luigi Sacco** [Sav99]: The columns of a table are numbered according to the letters of the keyword. The plaintext is entered line by line, in the first line up to column number one, in the second line up to column number two, etc. Again, the ciphertext is read out in columns.

See table 2.4.

Plaintext: an example of transposition

C 1	O 5	L 2	U 6	M 3	N 4
a					
n	e	x			
a	m	p	l	e	
o	f	t	r	a	n
s	p		t		
o	s	i			
i	o	n			

Table 2.4: Columnar transposition (General Luigi Sacco)

Ciphertext: ANAOS OIEMF PSOXP TINLR TEAN

- **Column transposition, French army in WW1** [Sav99]: After executing a simple columnar transposition, diagonal rows are read out.
- **Row transposition** [Sav99]: The plaintext is divided into blocks of equal length and a keyword is chosen. Now the letters of the keyword are numbered and permutation is done only within each block according to this numbering.<sup>13</sup>

<sup>12</sup>MTC3 offers according challenges, for instance

<http://www.mysterytwisterc3.org/en/challenges/level-x/double-column-transposition> and  
<https://www.mysterytwisterc3.org/en/challenges/level-iii/double-column-transposition-reloaded-part-1>

<sup>13</sup>Using CT1: Choose a key for 1st permutation, input line by line, permute column by column and output line by

### 2.1.3 Further transposition algorithm ciphers

- **Geometric figures** [Goe14]: Write the message into a grille following one pattern and read it out using another.
- **Union Route Cipher** [Goe14]: The Union Route Cipher derives from Civil War. This method does not rearrange letters of a given plaintext, but whole words. Particularly sensitive names and terms are substituted by codewords which are recorded in codebooks together with the existing routes. A route determines the size of a grille and the pattern that is used to read out the ciphertext. In addition, a number of filler words is defined.
- **Nihilist Transposition** [ACA02]: Insert the plaintext into a square grille and write the same keyword above the columns and next to the lines. As this keyword is sorted alphabetically, the contents of the grille are rearranged, too. Read out the ciphertext line by line.

See table 2.5.

Plaintext: an example of transposition

	W	O	R	D	S		D	O	R	S	W	
W	a	n	e	x	a		D	s	p	o	i	s
O	m	p	l	e	o		O	e	p	l	o	m
R	f	t	r	a	n		R	a	t	r	n	f
D	s	p	o	s	i		S	n	i	o	-	t
S	t	i	o	n	-		W	x	n	e	a	a

Table 2.5: Nihilist transposition<sup>14</sup>

Ciphertext: SPOIS EPLOM ATRNF NIOTX NEAA

- **Cadenus cipher** [ACA02]: Cadenus is a form of columnar transposition that uses two keywords.  
The 1st keyword is used to swap columns.  
The 2nd keyword is used to define the initial letter of each column: this 2nd keyword is a permutation of the used alphabet. This permutation is written on the left of the first column. Afterwards, each column is moved (wrap-around) so that it begins with the letter, which is in the same line as the key letter of the first keyword within the second keyword.  
Ciphertext is read out line by line.

See table 2.6.

Plaintext: cadenus is a form of columnar transposition using a keyword

Ciphertext:

SAASR PIFIU LONNS KTGWN EDOOA TDNNU IISFA OMYOC ROUCM AERRS

---

line.

In CT2 you can find the transposition within the templates **Cryptography \ Classical**. This component also visualizes the row wise transposition.

<sup>14</sup>After filling the matrix with the plaintext you get the left block. After switching rows and columns you get the right block

<sup>15</sup>Within the 2nd block of three chars those chars are printed bold which are at the top of the 3rd block after applying the 2nd key word.

	K	E	Y	E	K	Y	E	K	Y
A	c	a	d	a	c	d	<b>s</b>	a	a
D	e	n	u	n	e	u	s	r	p
X	s	i	s	i	s	s	i	f	i
K	a	f	o	f	<b>a</b>	o	u	l	o
C	r	m	o	m	r	o	n	n	s
W	f	c	o	c	f	o	k	t	g
N	l	u	m	u	l	m	w	n	e
S	n	a	r	a	n	r	d	o	o
Y	t	r	a	r	t	<b>a</b>	a	t	d
<b>E</b>	n	<b>s</b>	p	<b>s</b>	n	p	n	n	u
D	o	s	i	s	o	i	i	i	s
T	t	i	o	i	t	o	f	a	o
U	n	u	s	u	n	s	m	y	o
B	i	n	g	n	i	g	c	r	o
R	a	k	e	k	a	e	u	c	m
G	y	w	o	w	y	o	a	e	r
H	r	d	-	d	r	-	r	s	-

Table 2.6: Cadenus cipher<sup>15</sup>

## 2.2 Substitution ciphers

### 2.2.1 Monoalphabetic substitution ciphers

Monoalphabetic substitution assigns one character of the ciphertext alphabet to each plaintext character. This mapping remains unchanged during the whole process of encryption.

- **General monoalphabetic substitution / Random letter pairs<sup>16</sup>** [Sin99]: The substitution occurs by a given assignment of single letters.
- **Atbash cipher<sup>17</sup>** [Sin99]: Replace the first letter of the alphabet by the last letter of the alphabet, the second one by the last but one, etc.
- **Shift cipher, for example Caesar cipher<sup>18</sup>** [Sin99]: Plaintext alphabet and ciphertext alphabet are shifted against each other by a determined number of letters. Using the Caesar cipher means shifting letters about three positions.

Plaintext: three positions to the right

Ciphertext: WKUHH SRVLWLRQV WR WKH ULJKW

- **Affine cipher<sup>19</sup>**: This is a generalization of the shift cipher. A plaintext character is first substituted by another character and then the result is encrypted using the shift cipher. The name “affine cipher” was chosen because its encryption and decryption can be described as affine or linear function.
- **Substitution with symbols** [Sin99], for instance the so-called “freemason cipher”: Each letter is replaced with a symbol.
- **Variants**: Fill characters, intentional mistakes [Sin99].
- **Nihilist substitution<sup>20</sup>** [ACA02]: Insert the alphabet into a 5x5-matrix to assign each letter the number built from row and column number. A keyword is chosen and placed above the columns of a second matrix (grille). The plaintext is written row by row into the grille. The ciphertext results from adding the numbers of the plaintext and the numbers of the keyword. Numbers between 100 and 110 are transformed to numbers between 00 and 10, so that each letter is represented by a two-digit number.

See table 2.7.

Plaintext: an example of substitution

Ciphertext: 58 53 85 88 54 96 78 72 85 56 63 65 47 44 65 49 46 68 47 55 69 56 53

---

<sup>16</sup>This cipher can be simulated in CT1 under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Substitution / Atbash**.

In CT2 you can find these methods within the templates **Cryptography \ Classical**. According analyzers can be found within the templates **Cryptanalysis \ Classical**.

<sup>17</sup>This cipher can be simulated in CT1 under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Substitution / Atbash**.

<sup>18</sup>In CT1 this method can be found at three different places in the menu tree:

- **Encrypt/Decrypt \ Symmetric (classic) \ Caesar / ROT13**
- **Analysis \ Symmetric Encryption (classic) \ Ciphertext only \ Caesar**
- **Indiv. Procedures \ Visualization of Algorithms \ Caesar**.

<sup>19</sup>Some according SageMath samples are implemented at [2.5.2.3](#).

<sup>20</sup>An animation of this Nihilist method can be found in CT1 at the menu item **Indiv. Procedures \ Visualization of Algorithms \ Nihilist**.

In CT2 you can find nihilist within the templates **Cryptography \ Classical**.

	1	2	3	4	5	
Matrix	1	S	U	B	T	I
	2	O	N	A	C	D
	3	E	F	G	H	K
	4	L	M	P	Q	R
	5	V	W	X	Y	Z

	K (35)	E (31)	Y (54)
a	n	e	
(58)	(53)	(85)	
x	a	m	
(88)	(54)	(96)	
p	l	e	
(78)	(72)	(85)	
o	f	s	
(56)	(63)	(65)	
u	b	s	
(47)	(44)	(65)	
t	i	t	
(49)	(46)	(68)	
u	t	i	
(47)	(55)	(69)	
o	n		
(56)	(53)		

Table 2.7: Nihilist substitution

- **Codes** [Sin99]: In the course of time, codebooks were used again and again. A codebook assigns a codeword, a symbol or a number to every possible **word** of a message. Only if both parties hold identical codebooks and if the assignment of codewords to plaintext words is not revealed, a successful and secret communication can take place.
- **Nomenclator** [Sin99]: A nomenclator refers to techniques that combine the use of a cipher algorithm with a codebook. Often the encryption system is based upon a ciphertext alphabet. This alphabet is used to encrypt (via substitution) the bigger part of the message. Particularly frequent or top-secret words are replaced by a limited number of codewords existing besides the ciphertext alphabet.
- **Map cipher** [Thi99]: This method constitutes a combination of substitution and steganography<sup>21</sup>. Plaintext characters are replaced by symbols which are arranged in a map following certain rules.
- **Straddling Checkerboard** [Goe14]: A 3x10 matrix is filled with the letters of the used alphabet and two arbitrary digits or special characters as follows: The different letters of a keyword and the remaining characters are written into the grille. The columns are numbered 0 to 9, the second and the third line are numbered 1 and 2. Each plaintext character is replaced by the corresponding digit, respectively the corresponding pair of

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<sup>21</sup>Instead of encrypting a message, pure steganography tries to conceal its existence.

digits. As “1” and “2” are the first digits of the possible two-digit-numbers, they are not used as single digits.

See table 2.8.

Plaintext: an example of substitution

	0	1	2	3	4	5	6	7	8	9
0	K	-	-	E	Y	W	O	R	D	A
1	B	C	F	G	H	I	J	L	M	N
2	P	Q	S	T	U	V	X	Z	.	/

Table 2.8: Straddling checkerboard with password “Keyword”

Ciphertext: 91932 69182 01736 12222 41022 23152 32423 15619

Besides, “1” and “2” are the most commonly used digits, but this feature is removed by the following technique.

It is ostentatious, how often the numbers 1 and 2 appear, but this will be fixed with the following version.

- **Straddling Checkerboard, variant [Goe14]:** This variant of the straddling checkerboard was developed by Soviet spies during WW2. Ernesto (Ché) Guevara and Fidel Castro allegedly used this cipher for their secret communication.<sup>22</sup> A grille is filled with the alphabet (number of columns = length of keyword), and two arbitrary digits are chosen as reserved to indicate the second and third line of a 3x10-matrix (see above). Now the grille is traversed column by column and the single letters are transferred row by row into the matrix: For a faster encryption, the eight most common letters (ENIRSATO) are assigned the digits from 0 to 9, the reserved 2 digits are not assigned. The remaining letters are provided with combinations of digits one after another and are inserted into the grille.

See table 2.9.

Plaintext: an example of substitution

Ciphertext: 04271 03773 33257 09343 29181 34185 4

- **Ché Guevara cipher:** A special variant is the cipher used by Ché Guevara (with an additional substitution step and a slightly changed checkerboard):
  - \* The seven most frequent letters in Spanish are distributed in the first row.
  - \* Four instead of three rows are used.
  - \* So one could encrypt  $10 * 4 - 4 = 36$  different characters.
- **Tri-Digital cipher [ACA02]:** A keyword with ten letters is used to create a numeric key by numbering its letters corresponding to their alphabetical order. This key is written

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<sup>22</sup>In addition, Ché Guevara used for his communication with Fidel Castro also a one-time pad.

See part 3 of the series *RSA & Co. at school: Modern cryptology, old mathematics, and subtle protocols*: [WLS99], p. 52.

Unfortunately these are currently only available in German.

Grille	K	E	Y	W	O	R	D				
	A	B	C	F	G	H	I				
	J	L	M	N	P	Q	S				
	T	U	V	X	Z	.	/				
Matrix		0	1	2	3	4	5	6	7	8	9
		A	T	E	-	N	O	R	-	I	S
	3	K	J	B	L	U	Y	C	M	V	W
	7	F	X	G	P	Z	H	Q	.	D	/

Table 2.9: Variant of the straddling checkerboard

above the columns of the 3x10-matrix. This matrix is filled line by line with the alphabet as follows: The different letters of a keyword are inserted first, followed by the remaining letters. The last column is left out. Plaintext characters are substituted with numbers, the number of the last column is used to separate words.

- **Baconian cipher** [ACA02]: Assign a five-digit binary code to every letter and to 6 numbers or special characters (for example 00000 = A, 00001 = B, etc.) and replace the plaintext characters with this binary code. Now use a second, unsuspicious message to hide the ciphertext inside of it. This may happen by upper and lower case or italicized letters: e.g. all letters of the unsuspicious message below a binary “1” are capitalized. Overall this is obtrusive.

See table 2.10.

message	F	I	G	H	T
ciphertext	00101	01000	00110	00111	10011
unsuspicious message	it is w	a rman	the s u	n i s s h	i n i n g
Baconian Cipher	itIsW	aRman	thESu	niSSH	IniNG

Table 2.10: Baconian cipher

## 2.2.2 Homophonic substitution ciphers

Homophonic methods constitute a special form of monoalphabetic substitution. Each character of the plaintext alphabet is assigned several ciphertext characters.

- **Homophonic monoalphabetic substitution**<sup>23</sup> [Sin99]: Each language has a typical frequency distribution of letters. To conceal this distribution, each plaintext letter is assigned several ciphertext characters. The number of ciphertext characters assigned depends on the frequency of the letter to be encrypted.
- **Beale cipher** [Sin99]: The Beale cipher is a book cipher that numbers the words of a keytext. These numbers replace the plaintext letters by the words’ initial letters.

<sup>23</sup>This cipher can be simulated in CT1 under the menu **Encrypt/Decrypt \Symmetric (classic) \ Homophone**.

- **Grandpré cipher** [Sav99]: A square grille with 10 columns (other layouts are possible, too) is filled with ten words. The initial letters should result in an eleventh word. As columns and rows are numbered from 0 to 9, letters can be replaced by two-digit numbers. It is obvious that with the table having a hundred fields, most letters can be represented by more than one number. You should keep in mind that those ten words have to contain all letters of the plaintext alphabet.
- **Book cipher:** The words of a message are substituted by triples “page-line-position”. This method requires a detailed agreement of which book to use, especially regarding the edition (layout, error correction, etc.).

### 2.2.3 Polygraphic substitution ciphers

Polygraphic techniques do not work by replacing single characters, but by replacing whole groups of characters. In most cases, these groups are diagrams, trigrams or syllables.

- **“Great Chiffre”** [Sin99]: This cipher was used by Louis XIV. and was not solved until the end of the nineteenth century. Cryptograms consisted of 587 different numbers, every number representing a syllable. The inventors of the “Great Chiffre” (Rossignol, father and son) constructed additional traps to increase security. For example, a number could assign a different meaning to or delete the preceding one.
- **Playfair cipher**<sup>24</sup> [Sin99]: A 5x5-matrix is filled with the plaintext characters. For example, the different letters of a keyword are inserted first, followed by the remaining letters. The plaintext is divided into pairs, these digraphs are encrypted using the following rules:
  1. If both letters can be found in the same column, they are replaced by the letters underneath.
  2. If both letters can be found in the same row, take the letters to their right.
  3. If both letters of the digraph are in different columns and rows, the replacement letters are obtained by scanning along the row of the first letter up to the column where the other letter occurs and vice versa.
  4. Double letters are treated by special rules, if they appear in one digraph. They can be separated by a filler, for example.

See table 2.11.

Unformated Plaintext: plaintext letters are encrypted in pairs

Formated Plaintext: pl ai nt ex tl et te rs ar ee ncrypted in pairs

Formated Plaintext: pl ai nt ex tl et te rs ar ex en cr yp te di np ai rs

Ciphertext: SHBHM UWUZF KUUKC MBDWU DURDA VUKBG PQBHC M

- **Trigraphic Playfair:** A 5x5-matrix is filled with the alphabet (see above) and the plaintext is divided into trigraphs. Trigraphs are encrypted according to the following rules:

---

<sup>24</sup>In CT1 you can call this method under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Playfair**.

In CT2 you can find the Playfair within the templates **Cryptography \ Classical**.

In JCT you can find it in the default perspective via the menu item **Algorithms \ Classic \ Playfair**.

K	E	Y	W	O
R	D	A	B	C
F	G	H	I	L
M	N	P	Q	S
T	U	V	X	Z

Table 2.11: 5x5 Playfair matrix with password “Keyword”

1. Three equal letters are substituted by three equal letters. It is the letter on the right underneath the original letter.
  2. A trigraph with two different letters is encrypted like a digraph in Playfair.
  3. If a trigraph contains three different characters, very complex rules come into effect. See [Sav99]
- **Substituting digraphs by symbols** [Sav99]: Giovanni Battista della Porta, 15th century. He created a 20x20-matrix that contained one symbol for every possible combination of letters (his alphabet did not comprise more than twenty letters).
  - **Four square cipher** [Sav99]: This method is similar to Playfair, because it is based on a system of coordinates whose four quadrants are each filled with the alphabet. The layout of letters can differ from quadrant to quadrant. To encipher a message, act in the following way: Look up the first plaintext letter in the first quadrant and the second one in the third quadrant. These two letters are opposite corners of a rectangle and the ciphertext letters can be found in quadrant number two and four.

See table 2.12.

Plaintext: plaintext letters are encrypted in pairs

d	w	x	y	m	E	P	T	O	L
r	q	e	k	i	C	V	I	Q	Z
u	v	h	<b>p</b>	s	R	<b>M</b>	A	G	U
a	l	b	z	n	F	W	Y	H	S
g	c	o	f	t	B	N	D	X	K
Q	T	B	L	E	v	q	i	p	g
Z	H	N	D	X	s	t	u	o	h
P	M	I	Y	C	n	r	d	x	y
V	S	K	<b>W</b>	O	b	<b>I</b>	w	m	f
U	A	F	R	G	c	z	k	a	e

Table 2.12: Four square cipher

Ciphertext: MWYQW XQINO VNKGC ZWPZF FGZPM DIICC GRVCS

- **Two square cipher** [Sav99]: The two square cipher resembles the four square cipher, but the matrix is reduced to two quadrants. Are both letters of the digraph part of the same row, they are just exchanged. Otherwise, the plaintext letters are considered as opposite corners of a rectangle and substituted by the other vertices. Quadrants can be arranged horizontal and vertical.

- **Tri square cipher** [ACA02]: Three quadrants are filled with the same alphabet. The first plaintext letter is looked up in the first quadrant and can be encrypted with every letter of that column. The second plaintext letter is looked up in the second quadrant (diagonally across) and can be encrypted with every letter of that row. Between these two ciphertext characters, the letter at the intersection point is set.
- **Dockyard cipher** [Sav99]: Used by the German navy during WW2.

#### 2.2.4 Polyalphabetic substitution ciphers

Concerning polyalphabetic substitution, the assignment of ciphertext characters to plaintext characters is not static, but changes during the process of encryption (depending on the key).

- **Vigenère<sup>25</sup>** [Sin99]: Each plaintext character is encrypted with a different ciphertext alphabet that is determined by the characters of a keyword (the so-called Vigenère tableau serves auxiliary means). If the plaintext is longer than the key, the latter is repeated.

See table 2.13.

Plaintext:	the	alphabet	is	changing
Key:	KEY	KEYKEYKE	YK	EYKEYKEY
Ciphertext:	DLC	KPNREZOX	GC	GFKRESRE

-	A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
A	A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
B	B C D E F G H I J K L M N O P Q R S T U V W X Y Z A
C	C D E F G H I J K L M N O P Q R S T U V W X Y Z A B
D	D E F G H I J K L M N O P Q R S T U V W X Y Z A B C
E	E F G H I J K L M N O P Q R S T U V W X Y Z A B C D
F	F G H I J K L M N O P Q R S T U V W X Y Z A B C D E
G	G H I J K L M N O P Q R S T U V W X Y Z A B C D E F
H	H I J K L M N O P Q R S T U V W X Y Z A B C D E F G
I	I J K L M N O P Q R S T U V W X Y Z A B C D E F G H
J	J K L M N O P Q R S T U V W X Y Z A B C D E F G H I
K	K L M N O P Q R S T U V W X Y Z A B C D E F G H I J
...	....

Table 2.13: Vigenère tableau

- **Interrupted key:** The key is not repeated continuously, but starts again with every new word of the message.
- **Autokey<sup>26</sup>** [Sav99]: After using the agreed key, use the message itself as a key. See table 2.14.

<sup>25</sup>In CT1 you can call this method under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Vigenère**.  
In CT2 you can find this method within the templates **Cryptography \ Classical**.

In JCT you can find it in the default perspective via the menu item **Algorithms \ Classic \ Vigenère**.

<sup>26</sup>In CT2 you can find this method within the templates **Cryptography \ Classical**.  
In JCT you can find it in the default perspective via the menu item **Algorithms \ Classic \ Autokey Vigenère**.

Plaintext:	the	alphabet	is	changing
Key:	KEY	THEALPHA	BE	TISCHANG
Ciphertext:	DLC	TSTHLQLT	JW	VPSPNIAM

Table 2.14: Autokey variant of Vigenère

- **Progressive key** [Sav99]: The key changes during the process of encryption. With every repetition, the characters of the keyword are shifted about one position. “KEY” becomes “LFZ”.
- **Gronsfeld** [Sav99]: Variant of Vigenère that uses a numeric key.
- **Beaufort** [Sav99]: Variant of Vigenère, the key is subtracted, not added. The ciphertext alphabets may be written backwards.
- **Porta** [ACA02]: Variant of Vigenère with only 13 alphabets. As a consequence, two letters of the keyword are assigned the same ciphertext alphabet and the first and the second half of the alphabet are reciprocal.
- **Slidefair** [ACA02]: This method can be used as a variant of Vigenère, Gronsfeld or Beaufort. Slidefair does encrypt digraphs according to the following rules: Look up the first letter in the plaintext alphabet above the tableau. Then look up the second one in the row belonging to the corresponding keyword letter. These two letters make up opposite corners of an imaginary rectangle. The letters at the two remaining corners substitute the digraph.
- **One-time pad (OTP)**<sup>27,28</sup>: This is a major concept: A sequence of bytes is XORed byte-by-byte to the plaintext. This is a generalization of Vigenère’s mechanism and it was the first information theoretically secure scheme (see chapter 1.1 “Security definitions and the importance of cryptology”).

To fulfill this claim the pad must be random and it must be used only once (to eliminate any semblance of pattern from the ciphertext).

Reason: Given ciphertext C, plaintext P, pad K and two plaintexts encrypted with the same key:  $C_1 = P_1 \oplus K$ ;  $C_2 = P_2 \oplus K$ ;

Thus,  $C_1 \oplus C_2 = (P_1 \oplus K) \oplus (P_2 \oplus K) = P_1 \oplus P_2$ ;  
which effectively could leak the plaintexts.<sup>29</sup>

- **Superposition** (some variants of the OTP)

- **Running-key cipher**: A keytext (for example out of a book) is added to the plaintext.
- **Superposition with numbers**: A sequence or a number of sufficient length (for example pi) is added.

---

<sup>27</sup>On a big scale OTPs have been successfully analyzed by Americans and British during the “Venona” project – because of wrong usage by the soviet spies. See [https://en.wikipedia.org/wiki/Venona\\_project](https://en.wikipedia.org/wiki/Venona_project).

<sup>28</sup>CT1 offers this method under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Vernam / OTP**.

In CT2 you can find this method within the templates **Cryptography \ Classical**.

In JCT you can find it in the default perspective via the menu item **Algorithms \ Classic \ XOR**.

In chapter 8.3.1 you find a detailed description of the OTP as bitstream cipher and its implementation in SageMath.

<sup>29</sup>In JCT you can play with an automatic cryptanalysis of running-key ciphertexts using the Viterbi analysis under the menu **Visuals \ Viterbi**. You can see how astonishing it is, if you get little by little from XORed ciphertexts or XORed plaintexts both original plaintexts.

- **Phillips cipher** [ACA02]: The alphabet is filled into a square table with 5 columns. Seven more tables are generated by first shifting the first row one position towards the bottom, then shifting the second row towards the bottom. The plaintext is divided into blocks of five which are encrypted with one matrix each. Letters are substituted by the ones on their right and underneath.
- **Ragbaby cipher** [ACA02]: Construct an alphabet with 24 characters. Then number the plaintext characters, starting the numeration of the first word with “1”, the numeration of the second one with “2” and so forth. Number 25 corresponds to number 1. Each letter of the message is encrypted by shifting it the corresponding positions to the right.

See table 2.15.

alphabet: KEYWORDABCFGHILMNPSTUVXZ

Plaintext:	t h e	a l p h a b e t	i s	c h a n g i n g
Numbering:	1 2 3	2 3 4 5 6 7 8 9	3 4	4 5 6 7 8 9 10 11
Ciphertext:	U L O	C P V P I M C O	N X	I P I Z T X Y X

Table 2.15: Ragbaby cipher

## 2.3 Combining substitution and transposition

In the history of cryptography one often comes across combinations of the previous mentioned methods.

- **ADFG(V)X**<sup>30</sup> [Sin99]: ADFG(V)X-encryption was developed in Germany during WW1. The alphabet is filled into a 5x5 or 6x6 matrix, and columns and rows are marked with the letters ADFGX and V, depending on the size of the grille. Each plaintext character is substituted by the corresponding pair of letters. Finally, a (row-) transposition cipher is performed on the resulting text.
- **Fractionation** [Sav99]: Generic term for all kinds of methods that encrypt one plaintext character by several ciphertext characters and then apply a transposition cipher to this ciphertext so that ciphertext characters originally belonging to each other are separated.
  - **Bifid/Polybius square/checkerboard** [Goe14]: Bifid encryption is the basic form of fractionation. A 5x5 matrix is filled with the plaintext alphabet (see Playfair encryption), rows and columns are numbered, so that each plaintext character can be substituted by a pair of digits. Mostly the plaintext is divided into blocks of equal length. The length of blocks (here 5) is another configuration parameter of this cipher. Block-by-block all line numbers are read out first, followed by all numbers naming the columns. To obtain the ciphertext, the digits are pairwise transformed into letters again. The numbers can be any permutation of (1,2,3,4,5), which is one key of configuration parameter of this cipher. Instead of numbering rows and columns, a keyword can be used, too.

See table 2.16.

	2	4	5	1	3
1	K	E	Y	W	O
4	R	D	A	B	C
2	F	G	H	I	L
3	M	N	P	Q	S
5	T	U	V	X	Z

Plaintext:	combi	nings	ubsti	tutio	nandt	ransp	ositi
Rows:	41342	32323	54352	55521	34345	44333	13252
Columns:	33211	41443	41321	24213	45442	25435	33121

Table 2.16: Bifid cipher

41342 32323 54352 55521 34345 44333 13252 33211 41443 41321 24213 45442 25435  
33121

Ciphertext: BNLLL UPHVI NNUCS OHLMW BDNOI GINUR HCZQI

- **Trifid** [Sav99]: 27 characters (alphabet + 1 special character) may be represented by a triple consisting of the digits 1 to 3. The message to be encrypted is divided into

<sup>30</sup>In CT1 you can call this method under the menu **Encrypt/Decrypt \ Symmetric (classic) \ ADFGVX**. In CT2 you can find this method within the templates **Cryptography \ Classical**.

blocks of three and the relevant triple is written underneath each plaintext character as a column. The resulting numbers below the plaintext blocks are read out line by line and are substituted with the corresponding characters.

- **Bazeries cipher [ACA02]:** The plaintext alphabet is filled into a 5x5-matrix column by column, a second matrix is filled line by line with a keyword (a number smaller than a million) followed by the remaining letters of the alphabet. Then the message is divided into blocks of arbitrary length and their characters' order is inverted. Finally, each letter is substituted – according to its position in the original matrix – by its counterpart in the second matrix.

See table 2.17.

Plaintext: combining substitution and transposition

Keyword: 900.004 (nine hundred thousand and four)

a	f	l	q	v	N	I	E	H	U
b	g	<b>m</b>	r	w	D	R	<b>T</b>	O	S
c	h	n	s	x	A	F	B	C	G
d	i	o	t	y	K	L	M	P	Q
e	k	p	u	z	V	W	X	Y	Z

com	bini	ngs	ub	stitu	tiona	ndt	ran	sposi	ti	on
moc	inib	sgn	bu	utits	anoit	tdn	nar	isops	it	no
TMA	LBLD	CRB	DY	YPLPC	NBMLP	PKB	BNO	LCMXC	LP	BM

Table 2.17: Bazeries cipher

- **Digrafid cipher [ACA02]:** To substitute digraphs, the following table is used (to simplify matters, the alphabet is used in its original form). Look up the first letter of the digraph in the horizontal alphabet and write down the column number. Then look up the second letter in the vertical alphabet and write down the corresponding line number. Between these two numbers, the number at the intersection point is set. Afterwards, the triple are written vertically underneath the digraphs that are arranged in groups of three. The three digit numbers arising horizontally are transformed back into digraphs.

**Remark:** This cipher only works with complete blocks of 3 pairs of plaintext characters. For a complete description, it is necessary to explain how sender and receiver handle texts which fill in the last block only 1-5 characters. The possibilities range from ignoring a last and incomplete block to padding it with random characters or with characters predefined in advance.

See table 2.18.

1	2	<b>3</b>	4	5	6	7	8	9			
A	B	C	D	E	F	G	H	I	1	<b>2</b>	3
J	K	L	M	N	O	P	Q	R	4	5	6
S	T	U	V	W	X	Y	Z	.	7	8	9
									A	J	S
									B	K	T
									C	L	U
									D	M	V
									E	N	W
									F	<b>O</b>	X
									G	P	Y
									H	Q	Z
									I	R	.
											9

co mb in	in gs ub	st it ut	io na nd	tr an sp	os it io
3 4 9	9 7 3	1 9 3	9 5 5	2 1 1	6 9 9
2 4 2	2 3 7	9 3 9	2 4 4	8 2 8	6 3 2
6 2 5	5 1 2	2 2 2	6 1 4	9 5 7	1 2 6
LI KB FN	.C BY EB	SU I. BK	RN KD FD	BA HQ RP	X. FT AO

Table 2.18: Digrafid cipher

- **Nicodemus cipher** [ACA02]: First of all, a simple columnar transposition is carried out. Before reading out the columns, the message is encrypted additionally by Vigenère (all letters of a column are enciphered with the corresponding keyword letter). The ciphertext is read out in vertical blocks.

See table 2.19.

Plaintext: combining substitution and transposition

Ciphertext: SMRYX MLSCC KLEZG YSRVW JSKDX RLBYN WMYDG N

K	E	Y	E	K	Y	E	K	Y
c	o	m	o	c	m	S	M	K
b	i	n	i	b	n	M	L	L
i	n	g	n	i	g	R	S	E
s	u	b	u	s	b	Y	C	Z
s	t	i	t	s	i	X	C	G
t	u	t	u	z	t	Y	J	R
i	o	n	o	i	n	S	S	L
a	n	d	n	a	d	R	K	B
t	r	a	r	t	a	V	D	Y
n	s	p	s	n	p	W	X	N
o	s	i	s	o	i	W	Y	G
t	i	o	i	t	o	M	D	N

Table 2.19: Nicodemus cipher

- **Double column transposition (DCT) / “Granit E160”** [Dro15] : Granit is a 2-step cipher. The second step is the double column transposition; in the first step before, the cleartext is substituted by a sequence of digits using a codebook and a matrix (a variant of the Polybius square).

The Granit cipher was used for instance by the spy Guenter Guillaume for his communication with the Ministry of State Security of the former GDR until about 1960.<sup>31</sup>

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<sup>31</sup>MTC3 offers according challenges. If you enter at <https://www.mysterytwisterc3.org/en/challenges/the-four-levels?showAll=1> in your browser the search item “Granit”, you’ll find 6 challenges about it. A detailed 20-page description about the Granit cipher can be found at:  
<https://www.mysterytwisterc3.org/images/challenges/mtc3-drobick-01-doppelwuerfel-01-en.pdf>

## 2.4 Further methods

- “**Pinprick encryption**” [Sin99]: For centuries, this simple encryption method has been put into practice for different reasons (actually steganography). During the Victorian Age, for example, small holes underneath letters in newspaper articles marked the characters of a plaintext, as sending a newspaper was much more cheaper than the postage on a letter.
- **Stencil**: Stencils (Cardboard with holes) are also known as “Cardinal-Richelieu-Key”. Sender and receiver have to agree upon a text. Above this text, a stencil is laid and the letters that remain visible make up the ciphertext.
- **Card games** [Sav99]: The key is created by means of a pack of cards and rules that are agreed upon in advance. All methods mentioned in this paragraph are designed as paper and pencil methods, i.e. they are applicable without electronic aid. A pack of cards is unsuspicious to outsiders, shuffling the deck provides a certain amount of coincidence, cards can be transformed into numbers easily and a transposition cipher can be carried out without any further aid.

- **Solitaire cipher (Bruce Schneier)**<sup>32</sup> [Sch99a]: Sender and receiver have to own a deck of cards shuffled in the same manner. A key stream is generated that has to consist of as many characters as the message to be encrypted.

The algorithm to generate the key is based on a shuffled deck of 54 cards (Ace, 2 - 10, jack, queen, king in four suits and two jokers). The pack of cards is held face up:

1. Swap the first joker with the card beneath it.
2. Move the second joker two cards down.
3. Now swap the cards above the first joker with those below the second one.
4. Look at the bottom card and convert it into a number from 1 to 53 (bridge order of suits: clubs, diamonds, hearts, spades; joker = 53). Write down this number and count down as many cards starting with the top card. These cards are swapped with the remaining cards, only the bottom card remains untouched.
5. Look at the top card and convert it into a number, too. Count down as many cards starting with the top card.
6. Write down the number of the following card. This card is converted into your first keystream character. As we need numbers from 1 to 26 to match the letters of our alphabet, clubs and hearts correspond to the numbers 1 to 13, diamonds and spades to 14 to 26. If your output card is a joker, start again.

For each keystream character you like to generate, these six steps have to be carried out. This procedure is – manually – very lengthy (4 h for 300 characters, dependent on your exercise) and requires high concentration.

Encryption takes place by addition modulo 26. Encryption is relatively fast compared to the key stream generation.

This P&P cipher creates a key stream which is so good, that even nowadays it is hard to crack the cipher if you don't know the originally sorted card deck (ciphertext-only attack).

---

<sup>32</sup>In CT1 you can call this method under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Solitaire**.  
In CT2 you can find this method within the templates **Cryptography \ Classical and Cryptanalysis \ Classical**.

- **Mirdek cipher (Paul Crowley)** [[Cro00](#)]: Even though this method is quite complicated, the author provides a very good example to illustrate the procedure.
- **Playing Card cipher (John Savard)** [[Sav99](#)]: This algorithm uses a shuffled deck of 52 cards (no joker). Separate rules describe how to shuffle the deck. A keystream is created via the following steps:
  1. The pack of cards lies in front of the user, top down. Cards are turned up and dealt out in a row until the total of the cards is 8 or more.
  2. If the last card dealt out is a J, Q or K, write down its value, otherwise write down the sum of the cards dealt out (a number between 8 and 17). In a second row, deal out that number of cards.
  3. The remaining cards are dealt out in rows under the second row. The first one ends under the lowest card of the top row, the second one under the next lowest card, and so on. If there are two identical cards, red is lower than black.
  4. The cards dealt out under step 3 are collected column by column, starting with the column under the lowest card. The first card that is picked up becomes the bottom card (face up).
  5. The cards dealt out in step 1 and 2 are picked up, beginning with the last card.
  6. The deck is turned over, the top card is now the bottom card (face down). Afterwards, steps 1 to 6 are repeated twice.

To generate a keystream character, write down the first card not being J, Q or K. Count down that number of cards. The card selected has to be between 1 and 10. Now repeat these steps beginning with the last card. These two numbers are added and the last digit of the sum is your keystream character.

- **VIC cipher** [[Sav99](#)]: This is a highly complicated but relatively secure paper and pencil method. It has been developed and applied by Soviet spies. Amongst other things, the user had to create ten pseudo-random numbers out of a date, the first words of a sentence and any five-digit number. A straddling checkerboard is part of the encryption, too. A detailed description can be found under [[Sav99](#)].

## 2.5 Appendix: Examples using SageMath

In the following section some classic ciphers are implemented using the open source computer algebra system **SageMath**.<sup>33</sup> The code was tested with SageMath version 5.3. All ciphers are explained in chapter 2 (“Paper and Pencil Encryption Methods”).

To make the sample code<sup>34</sup> of the ciphers easier to understand, we used the structure and the naming conventions shown in the Figure 2.1 below:

- Encryption consists of the two steps encoding and enciphering.
  - Encoding adapts the letters in the given plaintext  $P$  to the case defined in the given alphabet, and all non-alphabet characters are filtered out.
  - Enciphering creates the ciphertext  $C$ .
- Decryption also consists of two steps: deciphering and decoding.
  - Decoding is only necessary if the symbols in the alphabet are not ASCII characters.

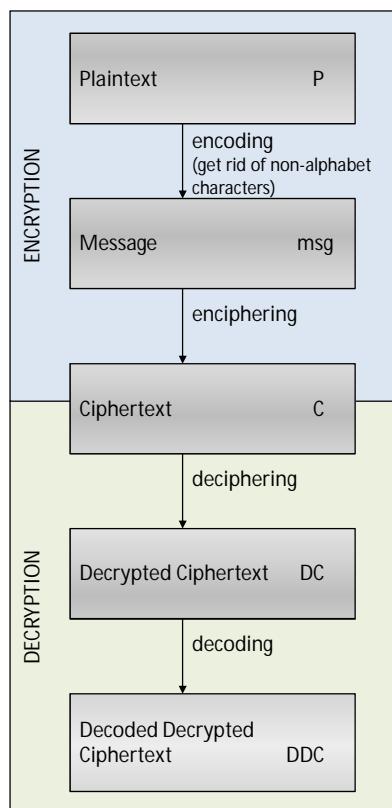


Figure 2.1: Structure and naming convention of the SageMath cipher code examples

<sup>33</sup>A first introduction to the CAS SageMath can be found in the appendix A.7.

<sup>34</sup>Further examples with SageMath concerning classic crypto methods can be found e.g.:

- as PDF in <http://doc.sagemath.org/pdf/en/reference/cryptography/cryptography.pdf>
- as HTML under <http://doc.sagemath.org/html/en/reference/cryptography/index.html>
- at <http://doc.sagemath.org/html/en/reference/cryptography/sage/crypto/classical.html>
- in the thesis of Minh Van Nguyen [[Ngu09a](#)]

### 2.5.1 Transposition ciphers

Transposition ciphers are implemented in the SageMath class

```
sage.crypto.classical.TranspositionCryptosystem
```

To construct and work with a transposition cipher, we first need to determine the alphabet that contains the symbols used to build the space of our plaintext and ciphertext. Typically, this alphabet will be the upper-case letters of the English alphabet, which can be accessed via the function

```
sage.monoids.string_monoid.AlphabeticStrings
```

We then need to decide on the block length of a block permutation, which is the length of the row vector to be used in the simple columns transposition. This row vector is our key, and it specifies a permutation of a plaintext.

The following first example of transposition ciphers has block length 14, and the key is build in a way, that every letter in the plaintext is shifted to the right by two characters, with wrap around at the end of the block. That is the encryption process. The decryption process is shifting each letter of the ciphertext to the left by  $14 - 2 = 12$ .

---

**SageMath sample 2.1** Simple transposition by shifting (key and inverse key explicitly given)

```
sage: # transposition cipher using a block length of 14
sage: T = TranspositionCryptosystem(AlphabeticStrings(), 14)
sage: # given plaintext
sage: P   = "a b c d e f g h i j k l m n"
sage: # encryption key
sage: key = [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 1, 2]
sage:
sage: # encode plaintext (get rid of non-alphabet chars, convert lower-case to upper-case)
sage: msg = T.encoding(P)
sage: # encrypt plaintext by shifting to the left by 2 letters (do it in two steps)
sage: E   = T(key)
sage: C   = E(msg); C
CDEFGHIJKLMNOPAB
sage:
sage: # decrypt ciphertext by shifting to the left by 12 letters
sage: keyInv = [13, 14, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
sage: D   = T(keyInv)
sage: D(C)
ABCDEFGHIJKLMN
sage:
sage: # Representation of key and inverse key as permutations
sage: E
(1,3,5,7,9,11,13)(2,4,6,8,10,12,14)
sage: D
(1,13,11,9,7,5,3)(2,14,12,10,8,6,4)
```

---

The second example of transposition ciphers is also a simple shifting column transposition. But now the code is a little bit more automated: The keys are generated from the shift parameter.

---

**SageMath sample 2.2** Simple transposition by shifting (key and inverse key constructed with “range”)

---

```
sage: # transposition cipher using a block length of 14, code more variable
sage: keylen = 14
sage: shift = 2
sage: A = AlphabeticStrings()
sage: T = TranspositionCryptosystem(A, keylen)
sage:
sage: # construct the plaintext string from the first 14 letters of the alphabet plus blanks
sage: # plaintext   = "A B C D E F G H I J K L M N"
sage: A.gens()
(A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z)
sage: P=''
sage: for i in range(keylen): P=P + " " + str(A.gen(i))
....:
sage: P
'A B C D E F G H I J K L M N'
sage:
sage: # encryption key
sage: # key = [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 1, 2]
sage: key = [(i+shift).mod(keylen) + 1 for i in range(keylen)]; key
[3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 1, 2]
sage:
sage: # encode plaintext (get rid of non-alphabet chars)
sage: msg = T.encoding(P)
sage: # encrypt plaintext by shifting to the left by 2 letters (do it in one step)
sage: C = T.enciphering(key, msg); C
CDEFGHIJKLMNOPAB
sage:
sage: # decrypt ciphertext by shifting to the left by 12 letters
sage: # keyInv = [13, 14, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
sage: shiftInv=keylen-shift;
sage: keyInv = [(i+shiftInv).mod(keylen) + 1 for i in range(keylen)]; keyInv
[13, 14, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
sage: DC = T.enciphering(keyInv, C); DC
ABCDEFHIJKLMNOP
sage:
sage: # decryption using the "deciphering method with key" instead of "enciphering with keyInv"
sage: # using the deciphering method requires to change the type of the variable key
sage: DC = T.deciphering(T(key).key(), C); DC
ABCDEFHIJKLMNOP
sage:
sage: # representation of key and inverse key as permutations
sage: T(key)
(1,3,5,7,9,11,13)(2,4,6,8,10,12,14)
sage: T(key).key()
(1,3,5,7,9,11,13)(2,4,6,8,10,12,14)
sage: T(keyInv)
(1,13,11,9,7,5,3)(2,14,12,10,8,6,4)
```

---

In the third example of transposition ciphers we use an arbitrary permutation as key in the encryption and decryption processes in order to scramble the characters within each block (block length = number of columns in a simple column transposition). If the block length is  $n$ , then the key must be a permutation on  $n$  symbols. The following example uses the method `random_key()` of the class `TranspositionCryptosystem`. Each call to `random_key()` produces a different key. Note that therefore your results (key and ciphertext) may be different from the following example.

---

**SageMath sample 2.3** Simple column transposition with randomly generated (permutation) key

---

```
sage: # Remark: Enciphering here requires, that the length of msg is a multiple of keylen
sage: keylen = 14    # length of key
sage: A = AlphabeticStrings()
sage: T = TranspositionCryptosystem(A, keylen); T
Transposition cryptosystem on Free alphabetic string monoid on A-Z of block length 14
sage:
sage: P = "a b c d e f g h i j k l m n o p q r s t u v w x y z a b"
sage: key = T.random_key(); key
(1,2,3,13,6,5,4,12,7)(11,14)
sage: msg = T.encoding(P); msg
ABCDEFGHIJKLMNPQRSTUVWXYZAB
sage: C = T.enciphering(key, msg); C
BCMLDEAHIJNGFKPQAZRSOVWXBYT
sage: # decryption using the "deciphering method with key" instead of "enciphering with keyInv"
sage: DC = T.deciphering(key, C); DC
ABCDEFGHIJKLMNPQRSTUVWXYZAB
sage:
sage: # Just another way of decryption: Using "enciphering" with the inverse key
sage: keyInv = T.inverse_key(key); keyInv
(1,7,12,4,5,6,13,3,2)(11,14)
sage: DC = T.enciphering(keyInv, C); DC
ABCDEFGHIJKLMNPQRSTUVWXYZAB
sage:
sage: # Test correctness of decryption
sage: msg == DC
True
```

---

The fourth example of transposition ciphers additionally shows the key space of a simple column transposition.

---

**SageMath sample 2.4** Simple column transposition (showing the size of the key space)

```
sage: keylen = 14 # length of key
sage: A = AlphabeticStrings()
sage: T = TranspositionCryptosystem(A, keylen); T
Transposition cryptosystem on Free alphabetic string monoid on A-Z of block length 14
sage: T.key_space()
Symmetric group of order 14! as a permutation group
sage: # Remark: The key space is not quite correct as also permutations shorter than keylen are counted.
sage:
sage: P = "a b c d e f g h i j k l m n o p q r s t u v w x y z a b"
sage: key = T.random_key(); key
(1,2,7)(3,9)(4,5,10,12,8,13,11)(6,14)
sage: msg = T.encoding(P); msg
ABCDEFGHIJKLM NOPQRSTUVWXYZAB
sage:
sage: # enciphering in one and in two steps
sage: C = T.enciphering(key, msg); C
BGIEJNAMCLDHKFPUWSXBOAQZRVYT
sage:
sage: enc = T(key); enc.key()
(1,2,7)(3,9)(4,5,10,12,8,13,11)(6,14)
sage: C = enc(msg); C
BGIEJNAMCLDHKFPUWSXBOAQZRVYT
sage:
sage: # deciphering
sage: DC = T.deciphering(key, C); DC
ABCDEFGHIJKLM NOPQRSTUVWXYZAB
```

---

## 2.5.2 Substitution ciphers

Substitution cryptosystems are implemented in SageMath in the class

```
sage.crypto.classical.SubstitutionCryptosystem
```

The following code sample uses SageMath to construct a substitution cipher with a random key. A random key can be generated using the method `random_key()` of the class `SubstitutionCryptosystem`. Different keys determine different substitution ciphers: With each call to `random_key()` a different result is returned.

---

### SageMath sample 2.5 Monoalphabetic substitution with randomly generated key

---

```
sage: # plaintext/ciphertext alphabet
sage: A = AlphabeticStrings()
sage: S = SubstitutionCryptosystem(A)
sage:
sage: P = "Substitute this with something else better."
sage: key = S.random_key(); key
INZDHFUXJPATQOYLKSWGVEMR
sage:
sage: # method encoding can be called from A or from T
sage: msg = A.encoding(P); msg
SUBSTITUTETHISWITHSOMETHINGELSEBETTER
sage: C = S.enciphering(key, msg); C
WVNWGJGVGHGXJWCJGXWYQHGXJOUHTWHNHGGHS
sage:
sage: # We now decrypt the ciphertext to recover our plaintext.
sage:
sage: DC = S.deciphering(key, C); DC
SUBSTITUTETHISWITHSOMETHINGELSEBETTER
sage: msg == DC
True
```

---

### 2.5.2.1 Caesar cipher

The following example uses SageMath to construct a Caesar cipher.

---

**SageMath sample 2.6** Caesar (substitution by shifting the alphabet; key explicitly given, step-by-step approach)

---

```
sage: # plaintext/ciphertext alphabet
sage: A = AlphabeticStrings()
sage: P = "Shift the alphabet three positions to the right."
sage:
sage: # construct Caesar cipher
sage: S = SubstitutionCryptosystem(A)
sage: key = A([3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, \
....:           20, 21, 22, 23, 24, 25, 0, 1, 2])
sage:
sage: # encrypt message
sage: msg      = A.encoding(P); msg
SHIFTTHEALPHABETTHREEPOSITIONSTOTHERIGHT
sage: encrypt = S(key); encrypt
DEFGHIJKLMNOPQRSTUVWXYZABC
sage: C      = encrypt(msg); C
VKLIWWKHOSKDEHWWKUHHSRVLWLRQVWRWKHULJKW
sage:
sage: # Next, we recover the plaintext.
sage: # decrypt message
sage: keyInv = A([23, 24, 25, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \
....:           14, 15, 16, 17, 18, 19, 20, 21, 22])
sage: decrypt = S(keyInv); decrypt
XYZABCDEFGHIJKLMNOPQRSTUVWXYZ
sage: DC      = decrypt(C); DC
SHIFTTHEALPHABETTHREEPOSITIONSTOTHERIGHT
sage: msg == DC
True
```

---

The second Caesar sample does the same, but the code is more sophisticated/automated/flexible.

---

**SageMath sample 2.7** Caesar (substitution by shifting the alphabet; substitution key generated)

---

```
sage: # plaintext/ciphertext alphabet
sage: A = AlphabeticStrings()
sage: keylen = len(A.gens()); keylen
26
sage: shift = 3
sage: P = "Shift the alphabet three positions to the right."
sage:
sage: # construct Caesar cipher
sage: S = SubstitutionCryptosystem(A)
sage: S
Substitution cryptosystem on Free alphabetic string monoid on A-Z
sage: # key = A([3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, \
sage: #           20, 21, 22, 23, 24, 25, 0, 1, 2])
sage: key = [(i+shift).mod(keylen) for i in range(keylen)];
sage: key = A(key); key
DEFGHIJKLMNOPQRSTUVWXYZABC
sage: len(key)
26
sage:
sage: # encrypt message
sage: msg      = A.encoding(P); msg
SHIFTTHEALPHABETTHREEPOSITIONSTOTHERIGHT
sage: C       = S.enciphering(key, msg); C
VKLIWWKHOSKDEHWWKUHHSRVLWLRQVWRWKHULJKW
sage:
sage: # Next, we recover the plaintext.
sage: # decrypt message
sage: # keyInv = A([23, 24, 25, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, \
sage: #           14, 15, 16, 17, 18, 19, 20, 21, 22])
sage: shiftInv=keylen-shift;
sage: keyInv = [(i+shiftInv).mod(keylen) for i in range(keylen)];
sage: keyInv = A(keyInv); keyInv
XYZABCDEFGHIJKLMNOPQRSTUVWXYZ
sage: DC      = S.enciphering(keyInv, C); DC
SHIFTTHEALPHABETTHREEPOSITIONSTOTHERIGHT
sage:
sage: # Just another way of decryption: Using "deciphering" with the key
sage: DC      = S.deciphering(key, C); DC
SHIFTTHEALPHABETTHREEPOSITIONSTOTHERIGHT
sage:
sage: msg == DC
True
```

---

### 2.5.2.2 Shift cipher

The shift cipher can also be thought of as a generalization of the Caesar cipher. While the Caesar cipher restricts us to shift exactly three positions along an alphabet, the shift cipher allows us to shift any number of positions along the alphabet.

In the above samples we applied the `SubstitutionCryptosystem` and build Caesar as a special kind of substitution. In contrast here Caesar can be build as a special kind of the shift cipher.

The shift cipher is implemented directly in the SageMath class

```
sage.crypto.classical.ShiftCryptosystem
```

In the following example, we construct a shift cipher over the capital letters of the English alphabet. We then encrypt a plaintext P by shifting it 12 positions along the alphabet. Finally, we decrypt the ciphertext C and make sure that the result (DC) is indeed the original plaintext. Shifting is a special way of substitution.

---

#### SageMath sample 2.8 A shift cipher over the upper-case letters of the English alphabet

---

```
sage: # construct Shift cipher directly
sage: shiftcipher = ShiftCryptosystem(AlphabeticStrings()); shiftcipher
Shift cryptosystem on Free alphabetic string monoid on A-Z
sage: P = shiftcipher.encoding("Shift me any number of positions."); P
SHIFTMEANYNUMBEROFPOSITIONS
sage: key = 12  # shift can be any integer number
sage:
sage: # shift the plaintext by 12 positions to get the ciphertext
sage: C = shiftcipher.enciphering(key, P); C
ETURFYQMZKZGYNQDARBAEUFUAZE
sage:
sage: # decrypt the ciphertext and ensure that it is the original plaintext
sage: DC = shiftcipher.deciphering(key, C); DC
SHIFTMEANYNUMBEROFPOSITIONS
sage: DC == P
True
```

---

The Caesar cipher is simply a shift cipher whose shifting key is 3. In the next example, we use the shift cipher to create a Caesar cipher over the capital letters of the English alphabet.

---

#### SageMath sample 2.9 Constructing the Caesar cipher using the shift cipher

---

```
sage: # create a Caesar cipher
sage: caesarcipher = ShiftCryptosystem(AlphabeticStrings())
sage: P = caesarcipher.encoding("Shift the alphabet by three positions to the right."); P
SHIFTTHEALPHABETBYTHREEPOSITIONSTOTHERIGHT
sage:
sage: key = 3  # shift the plaintext by exactly 3 positions
sage: C = caesarcipher.enciphering(key, P); C
VKLIWWKHDKOSKDEHWEBWKUHHSRVLWLRQVWRWKHULJKW
sage:
sage: # decrypt the ciphertext and ensure that it is the original plaintext
sage: DC = caesarcipher.deciphering(key, C); DC
SHIFTTHEALPHABETBYTHREEPOSITIONSTOTHERIGHT
sage: DC == P
True
```

---

### 2.5.2.3 Affine cipher

The affine cipher is implemented in the SageMath class

```
sage.crypto.classical.AffineCryptosystem
```

In the following example, we construct an affine cipher  $c_i = b * p_i + a$  with key (3, 13) and use this key to encrypt a given plaintext  $P = (p_1, p_2, \dots, p_n)$ . The plaintext is then decrypted and the result DC is compared to the original plaintext.

---

**SageMath sample 2.10** An affine cipher with key (3, 13)

```
sage: # create an affine cipher
sage: affineCipher = AffineCryptosystem(AlphabeticStrings()); affineCipher
Affine cryptosystem on Free alphabetic string monoid on A-Z
sage: P = affineCipher.encoding("The affine cryptosystem.")
sage: P
THEAFFINECRYPTOSYSTEM
sage:
sage: # encrypt the plaintext using the key (3, 13)
sage: a, b = (3, 13)
sage: C = affineCipher.enciphering(a, b, P)
sage: C
SIZNCCLAZTMHGSDPHPSZK
sage:
sage: # decrypt the ciphertext and make sure that it is equivalent to the original plaintext
sage: DC = affineCipher.deciphering(a, b, C)
sage: DC
THEAFFINECRYPTOSYSTEM
sage: DC == P
True
```

---

We can also construct a shift cipher using the affine cipher. To do so, we need to restrict keys of the affine cipher be of the form  $(1, b)$  where  $b$  is any non-negative integer. For instance, we can work through SageMath example 2.8 on page 56 as follows:

---

**SageMath sample 2.11** Constructing a shift cipher using the affine cipher

```
sage: # construct a shift cipher
sage: shiftcipher = AffineCryptosystem(AlphabeticStrings()); shiftcipher
Affine cryptosystem on Free alphabetic string monoid on A-Z
sage: P = shiftcipher.encoding("Shift me any number of positions.")
sage: P
SHIFTMEANYNUMBEROFPPOSITIONS
sage:
sage: # shift the plaintext by 12 positions to get the ciphertext
sage: a, b = (1, 12)
sage: C = shiftcipher.enciphering(a, b, P)
sage: C
ETURFYQMZKZGYNQDARBAEUFUAZE
sage:
sage: # decrypt the ciphertext and ensure that it is the original plaintext
sage: DC = shiftcipher.deciphering(a, b, C); P
SHIFTMEANYNUMBEROFPPOSITIONS
sage: DC == P
True
```

---

We can also use the affine cipher to create the Caesar cipher. To do so, the encryp-

tion/decryption key must be  $(1, 3)$ . In the next example, we work through SageMath example 2.9 on page 56 using the affine cipher.

---

**SageMath sample 2.12** Constructing the Caesar cipher using the affine cipher

---

```
sage: # create a Caesar cipher
sage: caesarcipher = AffineCryptosystem(AlphabeticStrings())
sage: P = caesarcipher.encoding("Shift the alphabet by three positions to the right.")
sage: P
SHIFTTHEALPHABETBYTHREEPOSITIONSTOTHERIGHT
sage:
sage: # shift the plaintext by 3 positions
sage: a, b = (1, 3)
sage: C = caesarcipher.enciphering(a, b, P)
sage: C
VKLIWWKHOSKDEHWEBKUHHHSRVLWLRQVWRWKHULJKW
sage:
sage: # decrypt the ciphertext and ensure that it is the original plaintext
sage: DC = caesarcipher.deciphering(a, b, C)
sage: DC
SHIFTTHEALPHABETBYTHREEPOSITIONSTOTHERIGHT
sage: DC == P
True
```

---

#### 2.5.2.4 Substitution with symbols

In the following SageMath example the symbols are from the binary number system. A monoalphabetic substitution cipher with a binary alphabet has very little security: Because the plaintext/ciphertext alphabet has only the two elements 0 and 1, there are only two keys possible: (0 1) and (1 0).

Remark: The key of a general substitution cipher contains all symbols of the alphabet exactly once.

---

#### SageMath sample 2.13 Monoalphabetic substitution with a binary alphabet

---

```
sage: # the plaintext/ciphertext alphabet
sage: B = BinaryStrings()
sage: # substitution cipher over the alphabet B; no keylen argument possible
sage: S = SubstitutionCryptosystem(B); S
Substitution cryptosystem on Free binary string monoid
sage: # To get a substitute for each symbol, key has always the length of the alphabet
sage: key = S.random_key(); key
10
sage: len(key)
2
sage: P = "Working with binary numbers."
sage: # encryption
sage: msg = B.encoding(P); msg
010101101101110111001001101011011010010110111001100111001000000111011101101\
00101110100011010000010000001100010011010010110111001100001011100100111100100\
1000000110111001110101011011010110001001100101011100100111001100101110
sage: C = S.enciphering(key, msg); C
1010100010010000100011011001010010010110100100011001100011011111000100010010\
110100010111001011110111110011101100101101001000110011110100011011000011011\
0111110010001100010101001001010011101101010001101100011001100100110010001
sage: # decryption
sage: DC = S.deciphering(key, C); DC
010101101101110111001001101011011010010110111001100111001000000111011101101\
00101110100011010000010000001100010011010010110111001100001011100100111100100\
100000011011100111010101101100010011001010111001001110011001100101110
sage: msg == DC
True
```

---

Remark: Currently `S` has no attribute `key`, and I found no way to transform the binary sequence `DC` back to ASCII.

The second sample of a monoalphabetic substitution with symbols uses a larger alphabet as plaintext/ciphertext space as the first sample. Here the hexadecimal number system is used as substitution alphabet.

---

**SageMath sample 2.14** Monoalphabetic substitution with a hexadecimal alphabet (and decoding in Python)

---

```
sage: A = HexadecimalStrings()
sage: S = SubstitutionCryptosystem(A)
sage: key = S.random_key(); key
2b56a4e701c98df3
sage: len(key)
16
sage: # Number of possible keys
sage: factorial(len(key))
20922789888000
sage: P   = "Working with a larger alphabet."
sage:
sage: msg = A.encoding(P); msg
576f726b696e6720776974682061206c617267657220616c7068616265742e
sage: C   = S.enciphering(key, msg); C
47e375e9e1efe75277e17ae052eb52e8eb75e7e47552ebe872e0ebe5e47a5f
sage: DC  = S.deciphering(key, C); DC
576f726b696e6720776974682061206c617267657220616c7068616265742e
sage: msg == DC
True
sage:
sage: # Conversion hex back to ASCII:
sage: # - AlphabeticStrings() and HexadecimalStrings() don't have according methods.
sage: # - So we used Python directly.
sage: import binascii
sage: DDC = binascii.a2b_hex(repr(DC)); DDC
'Working with a larger alphabet.'
sage:
sage: P == DDC
True
```

---

### 2.5.2.5 Vigenère cipher

The Vigenère cipher is implemented in the SageMath class

```
sage.crypto.classical.VigenereCryptosystem
```

For our ciphertext/plaintext space, we can work with the upper-case letters of the English alphabet, the binary number system, the octal number system, or the hexadecimal number system. Here is an example using the class `AlphabeticStrings`, which implements the English capital letters.

---

#### SageMath sample 2.15 Vigenère cipher

---

```
sage: # construct Vigenere cipher
sage: keylen = 14
sage: A = AlphabeticStrings()
sage: V = VigenereCryptosystem(A, keylen); V
Vigenere cryptosystem on Free alphabetic string monoid on A-Z of period 14
sage:
sage: # alternative could be a given key: key = A('ABCDEFGHIJKLMN'); key
sage: key = V.random_key(); key
WSSSEEGVVAARUD
sage: len(key)
14
sage:
sage: # encoding
sage: P = "The Vigener cipher is polyalphabetic."
sage: len(P)
38
sage: msg = V.encoding(P); msg      # alternative: msg = A.encoding(P); msg
THEVIGENERECIPHERISPOLYALPHABETIC
sage:
sage: # encryption [2 alternative ways (in two steps or in one): both work]
sage: # encrypt = V(key); encrypt
sage: # C = encrypt(msg); C
sage: C   = V.enciphering(key, msg); C
PZWMKIZRETCSDWJAWTUGTALGBDXLAG
sage:
sage: # decryption
sage: DC  = V.deciphering(key, C); DC
THEVIGENERECIPHERISPOLYALPHABETIC
sage: msg == DC
True
```

---

### 2.5.3 Hill cipher

The Hill [Hil29, Hil31] or matrix cipher<sup>35</sup> is mathematically more sophisticated than the other ciphers mentioned in this chapter. The encryption/decryption key of this cipher is an invertible square matrix (here called *key*). Plaintext and ciphertext are vectors ( $P$  and  $C$ ). The encryption and decryption processes use matrix operations modulo 26, here it is  $C = P * \text{key} \pmod{26}$ .

The Hill cipher is implemented in the SageMath class

```
sage.crypto.classical.HillCryptosystem
```

In the following example our plaintext/ciphertext space is the capital letters of the English alphabet. In the Hill cipher, each letter of this alphabet is assigned a unique integer modulo 26. The size of the key matrix (also called its dimension) is not restricted by the cipher.

**Remark:** Comparing the Hill implementation in CrypTool v1.4.31 and in SageMath version 5.3:

- SageMath offers fast command-line operations; CT1 offers its functionality within a GUI.
- SageMath offers for the key matrix any dimension; CT1 is restricted to a matrix size between 1 and 10.
- SageMath allows negative numbers in the key matrix, and converts them automatically into appropriate non-negative numbers; CT1 doesn't allow negative numbers in the key matrix.
- SageMath always sets the first alphabet character to 0;  
SageMath only allows the 26 capital letters as alphabet;  
and it uses only the multiplication variant plaintext row vector \* key matrix:  
 $C = P * \text{key}$ .
- CT1 offers to choose also 1 as value for the first alphabet character; you can combine your alphabet within the text options dialog; and it also offers to use a reverse multiplication variant:  $C = \text{key} * P$ .

---

<sup>35</sup>In CT1 you can call this method under the menu **Encrypt/Decrypt \ Symmetric (classic) \ Hill**.  
In CT2 you can find this method within the templates **Cryptography \ Classical** and **Cryptanalysis \ Classical**.

---

**SageMath sample 2.16** Hill cipher with randomly generated key matrix

---

```
sage: # construct a Hill cipher
sage: keylen = 19      # An Alternative could be: Use a non-random small key (e.g. keylen = 3)
sage: A = AlphabeticStrings(); H = HillCryptosystem(A, keylen); H
Hill cryptosystem on Free alphabetic string monoid on A-Z of block length 19
sage:
sage: # Alternative: Here, HKS is necessary in addition [H.key_space() isn't enough].
sage: # HKS = H.key_space(); key = HKS([[1,0,1],[0,1,1],[2,2,3]]); key
sage:
sage: # Random key creation
sage: key = H.random_key(); key
[10  7  5  2  0  6 10 23 15  7 17 19 18  2  9 12  0 10 11]
[23  1  1 10  4  9 21  1 25 22 19  8 17 22 15  8 12 25 22]
[ 4 12 16 15  1 12 24  5  9 13  5 15  8 21 23 24 22 20  6]
[ 5 11  6  7  3 12  8  9 21 20  9  4 16 18 10  3  2 23 18]
[ 8 22 14 14 20 13 21 19  3 13  2 11 13 23  9 25 25  6  8]
[24 25  8 24  7 18  3 20  6 11 25  5  6 19  7 24  2  4 10]
[15 25 11  1  4  7 11 24 20  2 18  4  9  8 12 19 24  0 12]
[14  6  2  9 11 20 13  4 10 11  4 23 14 22 14 16  9 12 18]
[12 10 21  5 21 15 16 17 19 20  1  1 15  5  0  2 23  4 14]
[21 15 15 16 15 20  4 10 25  7 15  4  7 12 24  9 19 10  6]
[25 15  2  3 17 23 21 16  8 18 23  4 22 11 15 19  6  0 15]
[14 23  9  3 18 15 10 18  7  5 12 23 11  9 22 21 20  4 14]
[ 3  6  8 13 20 16 11  1 13 10  4 21 25 15 12  3  0 11 18]
[21 25 14  6 11  3 21  0 19 17  5  8  5  4  9  2 23 19 15]
[ 8 11  9 11 20 15  6  1  3 18 18 22 16 17  6  3 15 11  2]
[21 15  5 22  2  9  0  4 22 10  2 10 19 19 17 19  1 21  4]
[ 7 17  9  2 15  5 14  3  6  9 12 12 22 15  8  4 21 14 19]
[19 14 24 19  7  5 22 22 13 14  7 18 17 19 25  2  1 23  6]
[ 2  6 14 22 17  7 23  6 22  7 13 20  0 14 23 17  6  1 12]
sage:
sage: # encoding and encryption
sage: P = "Hill or matrix cipher uses matrix operations."; len(P)
45
sage: # implementation requires: Length of msg is a multiple of matrix dimension (block_length)
sage: msg = H.encoding(P); msg; len(msg)
HILLORMATRIXCIPHERUSES MATRIXOPERATIONS
38
sage:
sage: # encryption (the length of msg must be a multiple of keylen).
sage: C = H.enciphering(key, msg); C
CRWCKPRVYXNBZRZNCTQWFWSWBCHABGMNEHVP
sage:
sage: # decryption
sage: DC = H.deciphering(key, C); DC; msg == DC
HILLORMATRIXCIPHERUSES MATRIXOPERATIONS
True
sage:
sage: # alternative decryption using inverse matrix
sage: keyInv = H.inverse_key(key); keyInv
[ 6 23  1 23  3 12 17 22  6 16 22 14 18  3  1 10 21 16 20]
[18 23 15 25 24 23  7  4 10  7 21  7  9  0 13 22  5  5 23]
...
[10 11 12  6 11 17 13  9 19 16 14 24  4  8  5 16 18 20  1]
[19 16 16 21  1 19  7 12  3 18  1 17  7 10 24 21  7 16 11]
sage: DC      = H.enciphering(keyInv, C); DC
HILLORMATRIXCIPHERUSES MATRIXOPERATIONS
```

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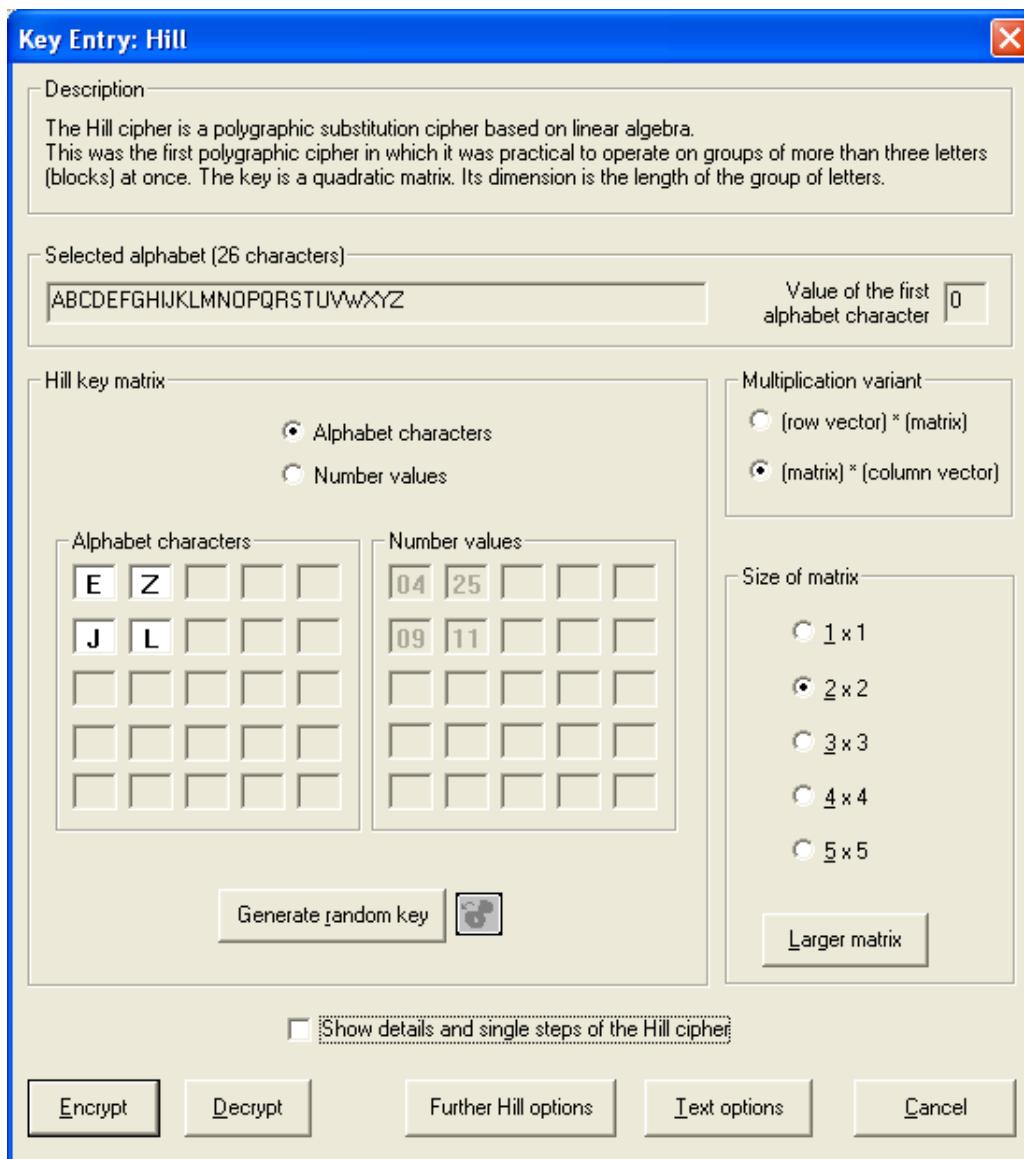


Figure 2.2: Hill dialog in CT1 with the operations and options available

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All links have been confirmed at July 11, 2016.

# Chapter 3

## Prime Numbers

([Bernhard Esslinger](#), May 1999; Updates: Nov 2000, Dec 2001, Jun 2003, May 2005, Mar 2006, Jun 2007, Jan 2010, Aug 2013, Jul 2016)

Progress requires exchange of knowledge.

Quote 6: Albert Einstein<sup>1</sup>

### 3.1 What are prime numbers?

Prime numbers are whole, positive numbers greater than or equal to 2 that can only be divided by 1 and themselves. All other natural numbers greater than or equal to 4 are composite numbers, and can be formed by multiplying prime numbers.

The *natural* numbers  $\mathbb{N} = \{1, 2, 3, 4, \dots\}$  thus comprise

- the number 1 (the unit value)
- the primes and
- the composite numbers.

Prime numbers are particularly important for three reasons:

- In number theory, they are considered to be the basic components of natural numbers, upon which numerous brilliant mathematical ideas are based.
- They are of extreme practical importance in modern cryptography (public key cryptography). The most common public key procedure, invented at the end of the 1970's, is RSA encryption. Only using (large) prime numbers for particular parameters can you guarantee that an algorithm is secure, both for the RSA procedure and for even more modern procedures (e.g. elliptic curves).
- The search for the largest known prime numbers does not have any practical usage known to date, but requires the best computers, is an excellent benchmark (possibility for determining the performance of computers) and leads to new calculation methods on many computers (see also: <http://www.mersenne.org/prime.htm>).

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<sup>1</sup>Albert Einstein, German physicist and Nobel Prize winner, Mar 14, 1879 – Apr 14, 1955.

Many people have been fascinated by prime numbers over the past two millennia. Ambition to make new discoveries about prime numbers has often resulted in brilliant ideas and conclusions. The following section provides an easily comprehensible introduction to the basics of prime numbers. We will also explain what is known about the distribution (density, number of prime numbers in particular intervals) of prime numbers and how prime number tests work.

## 3.2 Prime numbers in mathematics

Every whole number has a factor. The number 1 only has one factor, itself, whereas the number 12 has the six factors 1, 2, 3, 4, 6, 12. Many numbers can only be divided by themselves and by 1. With respect to multiplication, these are the “atoms” in the area of numbers. Such numbers are called prime numbers.

In mathematics, a slightly different (but equivalent) definition is used.

**Definition 3.2.1.** *A whole number  $p \in \mathbf{N}$  is called prime if  $p > 1$  and  $p$  only possesses the trivial factors  $\pm 1$  and  $\pm p$ .*

By definition, the number 1 is not a prime number. In the following sections,  $p$  will always denote a prime number.

The sequence of prime numbers starts with

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, ⋯.

The first 100 numbers include precisely 25 prime numbers. After this, the percentage of primes constantly decreases. Prime numbers can be factorized in a uniquely *trivial* way:

$$5 = 1 \cdot 5, \quad 17 = 1 \cdot 17, \quad 1013 = 1 \cdot 1013, \quad 1,296,409 = 1 \cdot 1,296,409.$$

All numbers that have 2 or more factors not equal 1 are called *composite* numbers. These include

$$4 = 2 \cdot 2, \quad 6 = 2 \cdot 3$$

as well as numbers that *look like primes*, but are in fact composite:

$$91 = 7 \cdot 13, \quad 161 = 7 \cdot 23, \quad 767 = 13 \cdot 59.$$

The following table gives a first impression how primes are distributed between natural numbers. There are many graphical forms of representation (the most well-known is the Ulam spiral). However till now, these graphical forms gained no new insights, but for some people they created the impression that there are at least local patterns within the random distribution.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180
181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210
211	212	213	214	215	216	217	218	219	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240
241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260	261	262	263	264	265	266	267	268	269	270
271	272	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288	289	290	291	292	293	294	295	296	297	298	299	300
301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319	320	321	322	323	324	325	326	327	328	329	330
331	332	333	334	335	336	337	338	339	340	341	342	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360
361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	378	379	380	381	382	383	384	385	386	387	388	389	390

Figure 3.1: Primes within the first 390 integers – marked with color<sup>2</sup>

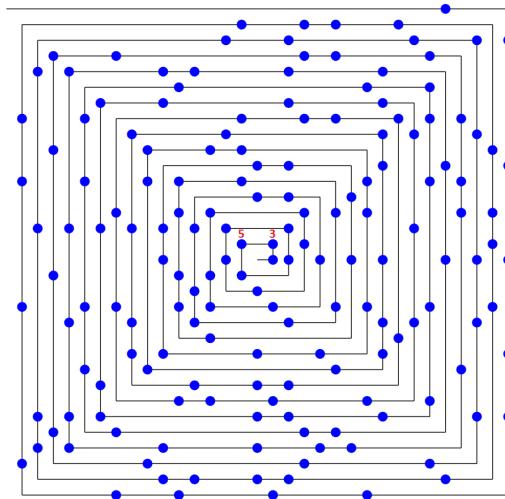


Figure 3.2: Primes within the first 999 integers – as Ulam spiral<sup>3</sup>

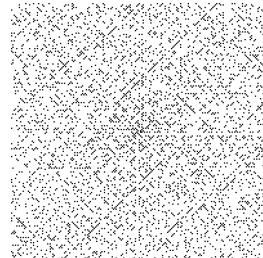


Figure 3.3: Primes within the first 4000 integers – as Ulam spiral<sup>4</sup>

<sup>2</sup>Graphics from <http://mathforum.org/mathimages/index.php/Image:Irisprime.jpg>, 30\*13 rectangle

<sup>3</sup>Graphics from CT2, menu Crypto Tutorials, World of Primes, Distribution of primes, Ulam's spiral; 32\*32 points.

**Theorem 3.2.1.** *Each whole number  $m$  greater than 1 possesses a lowest factor greater than 1. This is a prime number  $p$ . Unless  $m$  is a prime number itself, then:  $p$  is less than or equal to the square root of  $m$ .*

All whole numbers greater than 1 can be expressed as a product of prime numbers — in a unique way. This is the claim of the **1st fundamental theorem of number theory** (= fundamental theorem of arithmetic = fundamental building block of all positive integers).

**Theorem 3.2.2.** *Each element  $n$  of the natural numbers greater than 1 can be written as the product  $n = p_1 \cdot p_2 \dots p_m$  of prime numbers. If two such factorizations*

$$n = p_1 \cdot p_2 \cdot \dots \cdot p_m = p'_1 \cdot p'_2 \cdot \dots \cdot p'_{m'}$$

*are given, then they can be reordered such that  $m = m'$  and for all  $i$ :  $p_i = p'_i$ . ( $p_1, p_2, \dots, p_m$  are called the prime factors of  $n$ ).*

In other words: each natural number other than 1 can be written as a product of prime numbers in precisely one way, if we ignore the order of the factors. The factors are therefore unique (the *expression as a product of factors* is unique)! For example,

$$60 = 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3^1 \cdot 5^1.$$

And this — other than changing the order of the factors — is the only way in which the number 60 can be factorized. If you allow numbers other than primes as factors, there are several ways of factorizing integers and the **uniqueness** is lost:

$$60 = 1 \cdot 60 = 2 \cdot 30 = 4 \cdot 15 = 5 \cdot 12 = 6 \cdot 10 = 2 \cdot 3 \cdot 10 = 2 \cdot 5 \cdot 6 = 3 \cdot 4 \cdot 5 = \dots$$

The following section is aimed more at those familiar with mathematical logic: The 1st fundamental theorem only appears to be obvious . We can construct numerous other sets of numbers (i.e. other than positive whole numbers greater than 1), for which numbers in the set cannot be expressed uniquely as a product of the prime numbers of the set: In the set  $M = \{1, 5, 10, 15, 20, \dots\}$  there is no equivalent to the fundamental theorem under multiplication. The first five prime numbers of this sequence are 5, 10, 15, 20, 30 (note: 10 is prime, because 5 is not a factor of 10 in this set — the result is not an element of the given basic set  $M$ ). Because the following applies in  $M$ :

$$100 = 5 \cdot 20 = 10 \cdot 10$$

and 5, 10, 20 are all prime numbers in this set, the expression as a product of prime factors is not unique here.

### 3.3 How many prime numbers are there?

For the natural numbers, the primes can be compared to elements in chemistry or the elementary particles in physics (see [Blu99, p. 22]).

Although there are only 92 natural chemical elements, the number of prime numbers is unlimited. Even the Greek, Euclid<sup>5</sup> knew this in the third century B.C.

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<sup>4</sup>Graphics from [http://mathforum.org/mathimages/index.php/Image:Ulam\\_spiral.png](http://mathforum.org/mathimages/index.php/Image:Ulam_spiral.png), 200\*200 Ulam spiral

<sup>5</sup>Euclid, a Greek mathematician of 4th and 3rd century B.C. He worked at the Egyptian academy of Alexandria and wrote “The Elements”, the most well known systematically textbook of the Greek mathematics.

**Theorem 3.3.1** (Euclid<sup>6</sup>). *The sequence of prime numbers does not discontinue. Therefore, the quantity of prime numbers is infinite.*

His proof that there is an infinite number of primes is still considered to be a brilliant mathematical consideration and conclusion today (**proof by contradiction**). He assumed that there is only a finite number of primes and therefore a largest prime number. Based on this assumption, he drew logical conclusions until he obtained an obvious contradiction. This meant that something must be wrong. As there were no mistakes in the chain of conclusions, it could only be the assumption that was wrong. Therefore, there must be an infinite number of primes!

### Proof according to Euclid (proof by contradiction)

**Assumption:** There is a *finite* number of primes.

**Conclusion:** Then these can be listed  $p_1 < p_2 < p_3 < \dots < p_n$ , where  $n$  is the (finite) number of prime numbers.  $p_n$  is therefore the largest prime. Euclid now looks at the number  $a = p_1 \cdot p_2 \cdots p_n + 1$ . This number cannot be a prime number because it is not included in our list of primes. It must therefore be divisible by a prime, i.e. there is a natural number  $i$  between 1 and  $n$ , such that  $p_i$  divides the number  $a$ . Of course,  $p_i$  also divides the product  $a - 1 = p_1 \cdot p_2 \cdots p_n$ , because  $p_i$  is a factor of  $a - 1$ . Since  $p_i$  divides the numbers  $a$  and  $a - 1$ , it also divides the difference of these numbers. Thus:  $p_i$  divides  $a - (a - 1) = 1$ .  $p_i$  must therefore divide 1, which is impossible.

**Contradiction:** Our assumption was false.

Thus there is an *infinite* number of primes (Cross-reference: [overview](#) under 3.9 of the number of prime numbers in various intervals).  $\square$

Here we should perhaps mention yet another fact which is initially somewhat surprising. Namely, in the prime numbers sequence  $p_1, p_2, \dots$ , gaps between prime numbers can have an individually determined length  $n$ . It is undeniable that under the  $n$  succession of natural numbers

$$(n+1)! + 2, \dots, (n+1)! + (n+1),$$

none of them is a prime number since in order, the numbers  $2, 3, \dots, (n+1)$  are comprised respectively as real divisors. ( $n!$  means the product of the first  $n$  natural numbers therefore  $n! = n * (n-1) * \dots * 2 * 1$ ).

## 3.4 The search for extremely large primes

The largest prime numbers known today have several millions digits, which is too big for us to imagine. The number of elementary particles in the universe is estimated to be “only” a 80-digit decimal number ([see the overview under 3.11 of various orders of magnitude / dimensions](#)).

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<sup>6</sup>The common usage of the term does not denote Euclid as the inventor of the theorem rather; the true inventor is merely not as prominent. The theorem has already been distinguished and proven in Euclid's Elements (Book IX, theorem 20). The phraseology is remarkable due to the fact that the word infinite is not used. The text reads as followed

*Oἱ πρῶτοι ἀριθμοὶ πλείους εἰσὶ παντὸς τοῦ προτεύντος πλήν οὓς πρώτων ἀριθμῶν,*

the English translation of which is: the prime numbers are more than any previously existing amount of prime numbers.

### 3.4.1 The 30+ largest known primes (as of June 2016)

The following table contains the biggest, currently known primes and a description of its particular number type<sup>7</sup>:

	<b>Definition</b>	<b>Decimal Digits</b>	<b>When</b>	<b>Description</b>
1	$2^{74,207,281} - 1$	22,338,618	2016	Mersenne, 49th known
2	$2^{57,885,161} - 1$	17,425,170	2013	Mersenne, 48th known
3	$2^{43,112,609} - 1$	12,978,189	2008	Mersenne, 47th known
4	$2^{42,643,801} - 1$	12,837,064	2009	Mersenne, 46th known
5	$2^{37,156,667} - 1$	11,185,272	2008	Mersenne, 45th known
6	$2^{32,582,657} - 1$	9,808,358	2006	Mersenne, M-44
7	$2^{30,402,457} - 1$	9,152,052	2005	Mersenne, M-43
8	$2^{25,964,951} - 1$	7,816,230	2005	Mersenne, M-42
9	$2^{24,036,583} - 1$	7,235,733	2004	Mersenne, M-41
10	$2^{20,996,011} - 1$	6,320,430	2003	Mersenne, M-40
11	$2^{13,466,917} - 1$	4,053,946	2001	Mersenne, M-39
12	$19,249 \cdot 2^{13,018,586} + 1$	3,918,990	2007	Generalized Mersenne <sup>8</sup>
13	$3 \cdot 2^{11,895,718} - 1$	3,580,969	2015	Generalized Mersenne
14	$3 \cdot 2^{11,731,850} - 1$	3,531,640	2015	Generalized Mersenne
15	$3 \cdot 2^{11,484,018} - 1$	3,457,035	2014	Generalized Mersenne
16	$3 \cdot 2^{10,829,346} + 1$	3,259,959	2014	Generalized Mersenne
17	$475,856^{524,288} + 1$	2,976,633	2012	Generalized Fermat
18	$356,926^{524,288} + 1$	2,911,151	2012	Generalized Fermat
19	$341,112^{524,288} + 1$	2,900,832	2012	Generalized Fermat
20	$27,653 \cdot 2^{9,167,433} + 1$	2,759,677	2005	Generalized Mersenne
21	$90,527 \cdot 2^{9,162,167} + 1$	2,758,093	2010	Generalized Mersenne
22	$2,038 \cdot 366^{1,028,507} - 1$	2,636,562	2016	Generalized Fermat
23	$75,898^{524,288} + 1$	2,558,647	2011	Generalized Fermat
24	$28,433 \cdot 2^{7,830,457} + 1$	2,357,207	2004	Generalized Mersenne
25	$502,573 \cdot 2^{7,181,987} - 1$	2,162,000	2014	Generalized Mersenne
26	$402,539 \cdot 2^{7,173,024} - 1$	2,159,301	2014	Generalized Mersenne
27	$161,041 \cdot 2^{7,107,964} + 1$	2,139,716	2015	Generalized Mersenne
28	$3 \cdot 2^{7,033,641} + 1$	2,117,338	2011	Generalized Mersenne
29	$33,661 \cdot 2^{7,031,232} + 1$	2,116,617	2007	Generalized Mersenne
30	$2^{6,972,593} - 1$	2,098,960	1999	Mersenne, M-38
...				
328	$1,372,930^{131,072} + 1$	804,474	2003	Generalized Fermat <sup>9</sup>
...				
342	$342,673 \cdot 2^{2,639,439} - 1$	794,556	2007	Generalized Mersenne

Table 3.1: The 30+ largest known primes and its particular number types (as of June 2016)

<sup>7</sup>An up-to-date version can be found in the internet at <https://primes.utm.edu/primes/search.php?Number=1000>, at <http://primes.utm.edu/mersenne/index.html>, and at <http://www.mersenne.org/primes/>.

The development over time is shown in figure 3.4. It starts with M21701 found in 1978. Note the vertical scale.

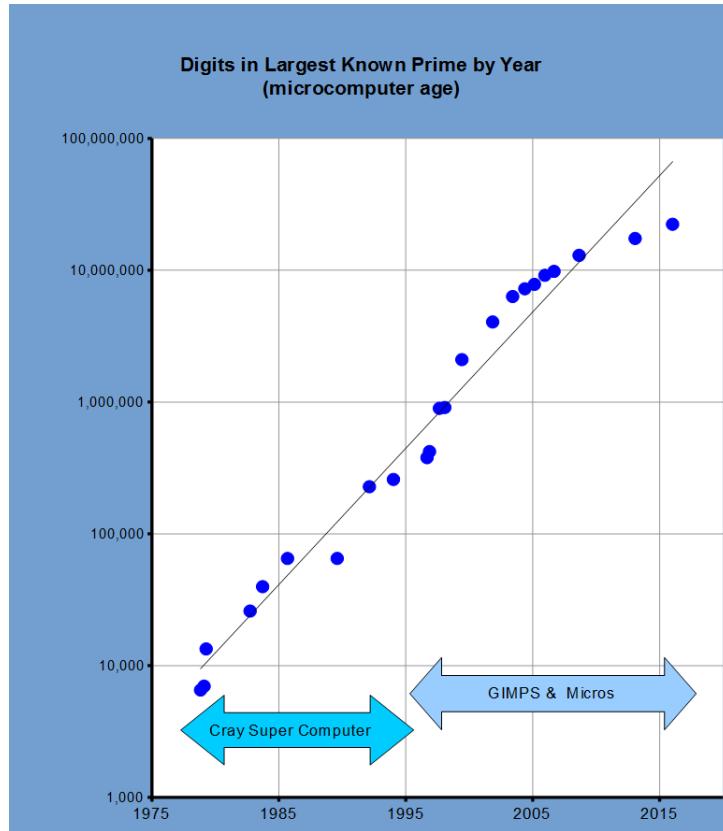


Figure 3.4: Number of digits of largest known prime by year (as of July 2016)<sup>10</sup>

The largest currently known prime is a Mersenne prime, found by the [GIMPS project](#) (chapter 3.4.2). Within the largest known primes there are also numbers of the type [generalized Mersenne number](#) (chapter 3.6.2) and [generalized Fermat numbers](#) (chapter 3.6.5).

<sup>8</sup>This number was found within the distributed computing project “Seventeen or Bust” (SoB) ([https://en.wikipedia.org/wiki/Seventeen\\_or\\_Bust](https://en.wikipedia.org/wiki/Seventeen_or_Bust)) at March 26, 2007. While the well known [GIMPS project](#) (chapter 3.4.2) searches for bigger and bigger of the infinitely many primes, there is a chance, that the SoB project could have been completed its task sometime.

The SoB project tries to prove computationally, that the number  $k = 78,557$  is the smallest Sierpinski number (John Selfridge proved in 1962, that 78,557 is a Sierpinski number).

The famous Polish mathematician Waclaw Sierpinski (1882 to 1969) proved in 1960, that there exist infinitely many odd integers  $k$ , which fulfill the following property: For all Sierpinski numbers  $k$  it is true: All numbers  $N = k \cdot 2^n + 1$  are composite for all integers  $n \geq 1$  (Sierpinski’s Composite Number Theorem, <http://mathworld.wolfram.com/SierpinskisCompositeNumberTheorem.html>).

When the project started in 2002 there have been 17 possible candidates  $< 78557$  (this is the reason for the project’s name “Seventeen or Bust”). It is sufficient to find one single counter-example, to exclude a candidate  $k$ , which means to find a single  $n \geq 1$ , where  $N = k \cdot 2^n + 1$  is prime. So it is only a byproduct of this task that this also generates new monster primes.

As of about April 19, 2016, the main SoB server is down and the future of the project unknown.

<sup>9</sup>Generalized Fermat number:  $1,372,930^{131,072} + 1 = 1,372,930^{(2^{17})} + 1$

<sup>10</sup>Source: Chris Caldwell, [http://primes.utm.edu/notes/by\\_year.html](http://primes.utm.edu/notes/by_year.html)

### 3.4.2 Special number types – Mersenne numbers and Mersenne primes

Almost all known huge prime numbers are special candidates, called *Mersenne numbers*<sup>11</sup> of the form  $2^p - 1$ , where  $p$  is a prime. Not all Mersenne numbers are prime:

$$\begin{array}{ll} 2^2 - 1 = 3 & \Rightarrow \text{prime} \\ 2^3 - 1 = 7 & \Rightarrow \text{prime} \\ 2^5 - 1 = 31 & \Rightarrow \text{prime} \\ 2^7 - 1 = 127 & \Rightarrow \text{prime} \\ 2^{11} - 1 = 2,047 = 23 \cdot 89 & \Rightarrow \text{NOT prime!} \end{array}$$

Even Mersenne knew that not all Mersenne numbers are prime (see exponent  $p = 11$ ). A prime Mersenne number is called Mersenne prime number.

However, he is to be thanked for the interesting conclusion that a number of the form  $2^n - 1$  cannot be a prime number if  $n$  is a composite number:

**Theorem 3.4.1** (Mersenne). *If  $2^n - 1$  is a prime number, then  $n$  is also a prime number.*

#### Proof

The theorem of Mersenne can be proved by contradiction. We therefore assume that there exists a composite natural number  $n$  (with real factorization)  $n = n_1 \cdot n_2$ , with the property that  $2^n - 1$  is a prime number.

From

$$\begin{aligned} (x^r - 1)((x^r)^{s-1} + (x^r)^{s-2} + \cdots + x^r + 1) &= ((x^r)^s + (x^r)^{s-1} + (x^r)^{s-2} + \cdots + x^r) \\ &\quad - ((x^r)^{s-1} + (x^r)^{s-2} + \cdots + x^r + 1) \\ &= (x^r)^s - 1 = x^{rs} - 1, \end{aligned}$$

we conclude

$$2^{n_1 n_2} - 1 = (2^{n_1} - 1)((2^{n_1})^{n_2-1} + (2^{n_1})^{n_2-2} + \cdots + 2^{n_1} + 1).$$

Because  $2^n - 1$  is a prime number, one of the above two factors on the right-hand side must be equal to 1. This is the case if and only if  $n_1 = 1$  or  $n_2 = 1$ . But this contradicts our assumption. Therefore the assumption is false. This means that there exists no composite number  $n$ , such that  $2^n - 1$  is a prime.  $\square$

Unfortunately this theorem only applies in one direction (the inverse statement does not apply, no equivalence): That means that there exist prime exponent for which the Mersenne number is **not** prime (see the above example  $2^{11} - 1$ , where 11 is prime, but  $2^{11} - 1$  not).

Mersenne claimed that  $2^{67} - 1$  is a prime number. There is also a mathematical history behind this claim: It first took over 200 years before Edouard Lucas (1842-1891) proved that this number is composite. However, he argued indirectly and did not name any of the factors. In 1903, Cole<sup>12</sup> showed which factors make up this composite number:

$$2^{67} - 1 = 147,573,952,589,676,412,927 = 193,707,721 \cdot 761,838,257,287.$$

---

<sup>11</sup>Marin Mersenne, French priest and mathematician, Sep 08, 1588 – Sep 01, 1648.

<sup>12</sup>Frank Nelson Cole, American mathematician, Sep. 20, 1861 – May 26, 1926.

He admitted to having worked 20 years on the **factorization** (dissection as a product of prime factors)<sup>13</sup> of this 21-digit decimal number!

Due to the fact that the exponents of the Mersenne numbers do not use all natural numbers, but only the primes, the *experimental space* is limited considerably. The currently known 49 Mersenne prime numbers have the following exponents:<sup>14</sup>

2; 3; 5; 7; 13; 17; 19; 31; 61; 89; 107; 127; 521; 607; 1,279; 2,203; 2,281; 3,217;  
4,253; 4,423; 9,689; 9,941, 11,213; 19,937; 21,701; 23,207; 44,497; 86,243; 110,503;  
132,049; 216,091; 756,839; 859,433; 1,257,787; 1,398,269; 2,976,221; 3,021,377;  
6,972,593; 13,466,917; 20,996,011; 24,036,583; 25,964,951; 30,402,457;  
32,582,657; 37,156,667; 42,643,801, 43,112,609, 57,885,161, 74,207,281.

The 19th number with the exponent 4,253 was the first with at least 1,000 digits in decimal system (the mathematician Samuel Yates coined the expression *titanic* prime for this; it was discovered by Hurwitz in 1961); the 27th number with the exponent 44,497 was the first with at least 10,000 digits in the decimal system (Yates coined the expression *gigantic* prime for this. These names are now long outdated).

For the first 44 Mersenne prime numbers we know that this list is complete. The exponents until the 49th Mersenne prime number have not yet been checked completely.<sup>15</sup>

As of 2016-07-19 all prime exponents smaller than 36,058,573 have been tested and double-checked<sup>16</sup>: So we can be certain, that this is really the 41th Mersenne prime number and that

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<sup>13</sup>Using CT1 you can factorize numbers in the following way: menu **Indiv. Procedures \ RSA Cryptosystem \ Factorization of a Number**.

CT1 can factorize with the quadratic sieve (QS) on a single PC in a reasonable time numbers no longer than 250 bit. Numbers bigger than 1024 bits are anyway currently not accepted by CT1.

CT2 has a GeneralFactorizer component based on YAFU. This is faster than the implemented functions in CT1. So CT2 offers the following factoring methods:

- brute-force with small primes
- Fermat
- Shanks square forms factorization (squfof)
- Pollard rho
- Pollard p-1
- Williams p+1
- Lenstra elliptic curve method (ECM)
- self-initializing quadratic sieve (SIQS)
- multiple polynomial quadratic sieve (MPQS)
- special number field sieve (SNFS)
- general number field sieve (GNFS).

CT2 started to experiment with a general infrastructure for distributed computing called CrypCloud (both peer-to-peer and centralized). So in the future, CT2 will be able to distribute the calculations on many computers. What could be achieved after the components are made ready for parallelization showed a cluster for distributed cryptanalysis of DES and AES: Status on March 21st, 2016 is, that an AES brute-force attack (distributed keysearching) worked on 50 i5 PCs, each with 4 virtual CPU cores. These 200 virtual “worker threads” achieved to test about 350 million AES keys/sec. The “cloud” processed a total amount of about 20 GB/sec of data. CrypCloud is a peer-to-peer-based cloud system that enables CT2 users to voluntarily join distributed computing jobs.

The current factorization records are listed in chapter [4.11.4](#).

<sup>14</sup>Landon Curt Noll lists all known Mersenne primes including its date of discovery and its value as number and as word: <http://www.isthe.com/chongo/tech/math/prime/mersenne.html>. Also see: <http://www.utm.edu/>.

<sup>15</sup>The current status of the check can be found at: <http://www.mersenne.org/primenet/>.

Hints, how the primality of a number can be checked, are in chapter [3.5](#), prime number tests.

<sup>16</sup>See home page of the GIMPS project: [http://www.mersenne.org/report\\_milestones](http://www.mersenne.org/report_milestones).

there are no smaller undiscovered Mersenne primes (it is common usage to use the notation  $M-n$  not until it is proven, that the  $n$ -th known Mersenne prime is really the  $n$ -th Mersenne prime). Here are some examples in more detail:

### **M-37 – January 1998**

The 37th Mersenne prime,

$$2^{3,021,377} - 1$$

was found in January 1998 and has 909,526 digits in the decimal system, which corresponds to 33 pages in the newspaper!

### **M-38 – June 1999**

The 38th Mersenne prime, called M-38,

$$2^{6,972,593} - 1$$

was discovered in June 1999 and has 2,098,960 digits in the decimal system (that corresponds to around 77 pages in the newspaper).

### **M-39 – December 2001**

The 39th Mersenne prime, called M-39,

$$2^{13,466,917} - 1,$$

was published at December 6, 2001 – more exactly, the verification of this number, found at November 14, 2001 by the Canadian student Michael Cameron, was successfully completed. This number has about 4 million decimal digits (exactly 4,053,946 digits). Trying only to print this number

$$(924947738006701322247758 \dots 1130073855470256259071)$$

would require around 200 pages in the Financial Times.

## **GIMPS**

The GIMPS project (Great Internet Mersenne Prime Search) was founded in 1996 by George Woltman to search for new largest Mersenne primes (<http://www.mersenne.org>). Further explanations about this number type can be found under [Mersenne numbers](#) and [Mersenne primes](#).

Right now the GIMPS project has discovered 15 largest Mersenne primes so far, including the largest known prime number at all.

Table 3.2 contains these Mersenne record primes.<sup>17,18</sup>

Richard Crandall discovered the advanced transform algorithm used by the GIMPS program. George Woltman implemented Crandall's algorithm in machine language, thereby producing a

<sup>17</sup>An up-to-date version can be found in the internet at <http://www.mersenne.org/history.htm>.

<sup>18</sup>Always, when a new record is published in the respective forums the same and often ironic discussions start: Does this kind of research have a deeper sense? Can this result be applied for anything useful? The answer is, that we don't know it yet. In fundamental research one cannot see at once whether and how it brings mankind forward.

Definition	Decimal Digits	When	Who
$2^{74,207,281} - 1$	22,338,618	January 7, 2016	Curtis Cooper
$2^{57,885,161} - 1$	17,425,170	January 25, 2013	Curtis Cooper
$2^{43,112,609} - 1$	12,978,189	August 23, 2008	Edson Smith
$2^{42,643,801} - 1$	12,837,064	April 12, 2009	Odd Magnar Strindmo
$2^{37,156,667} - 1$	11,185,272	September 6, 2008	Hans-Michael Elvenich
$2^{32,582,657} - 1$	9,808,358	September 4, 2006	Curtis Cooper/Steven Boone
$2^{30,402,457} - 1$	9,152,052	December 15, 2005	Curtis Cooper/Steven Boone
$2^{25,964,951} - 1$	7,816,230	February 18, 2005	Martin Nowak
$2^{24,036,583} - 1$	7,235,733	May 15, 2004	Josh Findley
$2^{20,996,011} - 1$	6,320,430	November 17, 2003	Michael Shafer
$2^{13,466,917} - 1$	4,053,946	November 14, 2001	Michael Cameron
$2^{6,972,593} - 1$	2,098,960	June 1, 1999	Nayan Hajratwala
$2^{3,021,377} - 1$	909,526	January 27, 1998	Roland Clarkson
$2^{2,976,221} - 1$	895,932	August 24, 1997	Gordon Spence
$2^{1,398,269} - 1$	420,921	November 1996	Joel Armengaud

Table 3.2: The largest primes found by the GIMPS project (as of January 2016)

prime-search program of unprecedented efficiency, and that work led to the successful GIMPS project.

On June 1st, 2003 a possible Mersenne prime was reported to the GIMPS server, which was checked afterwards as usual, before it was to be published. Unfortunately mid June the initiator and GIMPS project leader George Woltman had to tell, that two independent verification runs proved the number was composite. This was the first false positive report of a client in 7 years.

Now more than 130,000 volunteers, amateurs and experts, participate in the GIMPS project. They connect their computers into the so called “PrimeNet”, originally organized by the company Entropia from Scott Kurowski.

### 3.4.3 Challenge of the Electronic Frontier Foundation (EFF)

This search is also spurred on by a competition started by the non-profit organization EFF (Electronic Frontier Foundation) using the means of an unknown donor. The participants are rewarded with a total of 500,000 USD if they find the longest prime number. In promoting this project, the unknown donor is not looking for the quickest computer, but rather wants to draw people’s attention to the opportunities offered by *cooperative networking*  
<http://www.eff.org/awards/coop>

The discoverer of M-38 received 50,000 USD from the EFF for discovering the first prime with more than 1 million decimal digits.

For the next prize of 100,000 USD offered by EFF for a proven prime with more than 10 million decimal digits, Edson Smith qualified, who found the number  $2^{43,112,609} - 1$  within the GIMPS project.

According to the EFF rules for their prizes they offer in the next stage 150,000 USD for a proven prime with more than 100 million decimal digits.

Edouard Lucas (1842-1891) held the record for the longest prime number for over 70 years

by proving that  $2^{127} - 1$  is prime. No new record is likely to last that long.

### 3.5 Prime number tests<sup>19,20</sup>

In order to implement secure encryption procedures we need extremely large prime numbers (numbers in the region of  $2^{2,048}$  have more than 600 digits in the decimal system).

If we look for the prime factors in order to decide whether a number is prime, then the search takes too long, if even the smallest prime factor is enormous. Factorizing numbers using systematic computational division or using the [sieve of Eratosthenes](#) is only feasible using current computers for numbers with up to around 20 digits in the decimal system. The biggest number factorized into its 2 almost equal prime factors has 232 digits (see [RSA-768](#) in chapter [4.11.4](#)).

However, if we know something about the *construction* of the number in question, there are extremely highly developed procedures that are much quicker. These procedures can determine the primality attribute of a number, but they cannot determine the prime factors of a number, if it is compound.

In the 17th century, Fermat<sup>21</sup> wrote to Mersenne that he presumed that all numbers of the form

$$f(n) = 2^{2^n} + 1$$

are prime for all whole numbers  $n \geq 0$  (see below, chapter [3.6.4](#)).

As early as in the 19th century, it was discovered that the 29-digit number

$$f(7) = 2^{2^7} + 1$$

is not prime. However, it was not until 1970 that Morrison/Billhart managed to factorize it.

$$\begin{aligned} f(7) &= 340,282,366,920,938,463,463,374,607,431,768,211,457 \\ &= 59,649,589,127,497,217 \cdot 5,704,689,200,685,129,054,721 \end{aligned}$$

Despite Fermat was wrong with this supposition, he is the originator of an important theorem in this area: Many rapid prime number tests are based on the (little) Fermat theorem put forward by Fermat in 1640 (see chapter [4.8.3](#)).

**Theorem 3.5.1** (“little” Fermat). *Let  $p$  be a prime number and  $a$  be any whole number, then for all  $a$*

$$a^p \equiv a \pmod{p}.$$

*This could also be formulated as follows:*

*Let  $p$  be a prime number and  $a$  be any whole number that is not a multiple of  $p$  (also  $a \not\equiv 0 \pmod{p}$ ), then  $a^{p-1} \equiv 1 \pmod{p}$ .*

---

<sup>19</sup>A didactical article about the different prime number tests emphasizing the Miller-Rabin test can be found within the series *RSA & Co. at school: Modern cryptology, old mathematics, and subtle protocols*: see NF part 5 [[WS10b](#)]. Unfortunately these are currently only available in German.

<sup>20</sup>With the educational tool for number theory **NT** you can apply the tests of Fermat and of Miller-Rabin: See NT learning units 3.2 and 3.3, pages 3-11/11.

NT can be called in CT1 via the menu path **Indiv. Procedures \ Number Theory Interactive \ Learning Tool for Number Theory**. See also appendix [A.6](#).

CT2 contains a visualisation of these methods within the tutorial “**World of Primes**”.

<sup>21</sup>Pierre de Fermat, French mathematician, Aug 17, 1601 – Jan 12, 1665.

If you are not used to calculate with remainders (modulo), please simply accept the theorem or first read [chapter 4 “Introduction to Elementary Number Theory with Examples”](#). What is important here is that this sentence implies that if this equation is not met for any whole number  $a$ , then  $p$  is not a prime! The tests (e.g. for the first formulation) can easily be performed using the *test basis*  $a = 2$ .

This gives us a criterion for non-prime numbers, i.e. a negative test, but no proof that a number  $a$  is prime. Unfortunately Fermat’s theorem does not apply — otherwise we would have a simple proof of the prime number property (or to put it in other words, we would have a simple prime number criterion).

## Pseudo prime numbers

Numbers  $n$  that have the property

$$2^n \equiv 2 \pmod{n}$$

but are not prime are called *pseudo prime numbers* (i.e. the exponent is not a prime). The first pseudo prime number is

$$341 = 11 \cdot 31.$$

## Carmichael numbers

There are pseudo prime numbers  $n$  that pass the Fermat test

$$a^{n-1} \equiv 1 \pmod{n}$$

with all bases  $a$  which are relatively prime to  $n$  [ $\gcd(a, n) = 1$ ], despite these numbers  $n$  are not prime: These numbers are called *Carmichael numbers*. The first of these is

$$561 = 3 \cdot 11 \cdot 17.$$

Sample: The number to be tested is 561. Because  $561 = 3 \cdot 11 \cdot 17$  it is:

The test condition  $a^{560} \pmod{561} = 1$  is satisfied for  $a = 2, 4, 5, 7, \dots$ ,

but not for  $a = 3, 6, 9, 11, 12, 15, 17, 18, 21, 22, \dots$

This means the test condition must not be satisfied for multiples of the prime factors 3, 11 or 17.

The test applied for  $a = 3$  results in:  $3^{560} \pmod{561} = 375$ .

The test applied for  $a = 5$  results in:  $5^{560} \pmod{561} = 1$ .

## Strong pseudo prime numbers

A stronger test is provided by Miller/Rabin<sup>22</sup>: It is only passed by so-called *strong pseudo prime numbers*. Again, there are strong pseudo prime numbers that are not primes, but this is

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<sup>22</sup>In 1976 an efficient probabilistic primality test was published by Prof. Rabin, based on a number theoretic result of Prof. Miller from the year before.

Prof. Rabin worked at the Harvard and Hebrew University. Michael Oser Rabin was born 1931 in Breslau. His family emigrated in 1935 to Palestine because of the Nazi politics.

Prof. Miller worked at the Carnegie-Mellon University, School of Computer Science.

The functionality of the Miller-Rabin test is explained in detail in [WS10b] using a Python program.

much less often the case than for (simple) pseudo prime numbers or for Carmichael numbers. The smallest strong pseudo prime number base 2 is

$$15,841 = 7 \cdot 31 \cdot 73.$$

If you test all 4 bases, 2, 3, 5 and 7, you will find only one strong pseudo prime number up to  $25 \cdot 10^9$ , i.e. a number that passes the test and yet is not a prime number.

More extensive mathematics behind the Rabin test delivers the probability that the number examined is prime (such probabilities are currently around  $10^{-60}$ ).

Detailed descriptions of tests for finding out whether a number is prime can be found on Web sites such as:

<http://www.utm.edu/research/primes/mersenne.shtml>  
<http://www.utm.edu/research/primes/prove/index.html>

## 3.6 Special types of numbers and the search for a formula for primes

There are currently no useful, open (i.e. not recursive) formulae known that only deliver prime numbers (recursive means that in order to calculate the function the same function is used with a smaller variable). Mathematicians would be happy if they could find a formula that leaves gaps (i.e. does not deliver all prime numbers) but does not deliver any composite (non-prime) numbers.

Ideally, we would like, for the number  $n$ , to immediately be able to obtain the  $n$ -th prime number, i.e. for  $f(8) = 19$  or for  $f(52) = 239$ .

Ideas for this can be found at

[http://www.utm.edu/research/primes/notes/faq/p\\_n.html](http://www.utm.edu/research/primes/notes/faq/p_n.html).

Cross-reference: the table under 3.10 contains the precise values for the  $n$ th prime numbers for selected  $n$ .

For “prime number formulae” usually very special types of numbers are used. The following enumeration contains the most common ideas for “prime number formulae”, and what our current knowledge is about very big elements of the number series: Is their primality proven? If their are compound numbers could their prime factors be determined?

### 3.6.1 Mersenne numbers $f(n) = 2^n - 1$ for $n$ prime

As shown above, this formula seems to deliver relatively large prime numbers but - as for  $n = 11$  [ $f(n) = 2,047$ ] - it is repeatedly the case that the result even with prime exponents is not prime. Today, all the Mersenne primes having less than around 4,000,000 digits are known (M-39):

<http://yves.gallot.pagesperso-orange.fr/primes/index.html>

### 3.6.2 Generalized Mersenne numbers $f(k, n) = k \cdot 2^n \pm 1$ for $n$ prime and $k$ small prime / Proth numbers<sup>23</sup>

This first generalization of the Mersenne numbers creates the so called Proth numbers. There are (for small  $k$ ) extremely quick prime number tests (see [Knu98]). This can be performed in practice using software such as the Proths software from Yves Gallot:

<http://www.prothsearch.net/index.html>.

### 3.6.3 Generalized Mersenne numbers $f(b, n) = b^n \pm 1$ / The Cunningham project

This is another possible generalisation of the Mersenne numbers. The **Cunningham project** determines the factors of all composite numbers that are formed as follows:

$$f(b, n) = b^n \pm 1 \quad \text{for } b = 2, 3, 5, 6, 7, 10, 11, 12$$

( $b$  is not equal to multiples of bases already used, such as 4, 8, 9).

Details of this can be found at:

<http://www.cerias.purdue.edu/homes/ssw/cun>

### 3.6.4 Fermat numbers<sup>24</sup> $f(n) = 2^{2^n} + 1$

As mentioned above in chapter 3.5, Fermat wrote to Mersenne regarding his assumption, that all numbers of this type are primes. This assumption was disproved by Euler (1732). The prime 641 divides  $f(5)$ .<sup>25</sup>

$f(0) = 2^{2^0} + 1 = 2^1 + 1 = 3$	$\mapsto$ prime
$f(1) = 2^{2^1} + 1 = 2^2 + 1 = 5$	$\mapsto$ prime
$f(2) = 2^{2^2} + 1 = 2^4 + 1 = 17$	$\mapsto$ prime
$f(3) = 2^{2^3} + 1 = 2^8 + 1 = 257$	$\mapsto$ prime
$f(4) = 2^{2^4} + 1 = 2^{16} + 1 = 65,537$	$\mapsto$ prime
$f(5) = 2^{2^5} + 1 = 2^{32} + 1 = 4,294,967,297 = 641 \cdot 6,700,417$	$\mapsto$ NOT prime!
$f(6) = 2^{2^6} + 1 = 2^{64} + 1 = 18,446,744,073,709,551,617$	$= 274,177 \cdot 67,280,421,310,721 \mapsto$ NOT prime!
$f(7) = 2^{2^7} + 1 = 2^{128} + 1 = (\text{see page } 77)$	$\mapsto$ NOT prime!

Within the project “Distributed Search for Fermat Number Dividers” offered by Leonid Durman there is also progress in finding new monster primes:

<http://www.fermatsearch.org/>

This website links to other web pages in Russian, Italian and German.

The discovered factors can be compound integers or primes.

On February 22, 2003 John Cosgrave discovered

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<sup>23</sup>Their names come from the French farmer François Proth (1852-1879). More famous as the Proth primes is the related Sierpinski problem: Find all numbers  $k$ , so that  $k * 2^n + 1$  is composite for all  $n > 0$ . See table 3.1.

<sup>24</sup>The Fermat prime numbers play a role in circle division. As proven by Gauss a regular  $p$ -edge can only be constructed with the use of a pair of compasses and a ruler, when  $p$  is a Fermat prime number.

<sup>25</sup>Surprisingly this number can easily be found by using Fermat’s theorem (see e.g. [Sch06, p. 176])

- the largest composite Fermat number to date and
- the largest prime non-simple Mersenne number so far with 645,817 decimal digits.

The Fermat number

$$f(2, 145, 351) = 2^{(2^{2,145,351})} + 1$$

is divisible by the prime

$$p = 3 * 2^{2,145,353} + 1$$

At that time this prime  $p$  was the largest known prime generalized Mersenne number and the 5th largest known prime number at all.

This work was done using NewPGen from Paul Jobling's, PRP from George Woltman's, Proth from Yves Gallot's programs and also the Proth-Gallot group at St. Patrick's College, Dublin.

More details are in

[http://www.fermatsearch.org/history/cosgrave\\_record.htm](http://www.fermatsearch.org/history/cosgrave_record.htm)

### 3.6.5 Generalized Fermat numbers<sup>26</sup> $f(b, n) = b^{2^n} + 1$

Generalized Fermat numbers are more numerous than Mersenne numbers of equal size and many of them are waiting to be discovered to fill the big gaps between the Mersenne primes already found or still undiscovered. Progress in number theory made it possible that numbers, where the representation is not limited to the base 2, can be tested at almost the same speed than a Mersenne number.

Yves Gallot wrote the program Proth.exe to investigate generalized Fermat numbers.

Using this program at February 16, 2003 Michael Angel discovered the largest of them till then with 628,808 digits, which at that time became the 5th largest known prime number:

$$b^{2^{17}} + 1 = 62,722^{131,072} + 1.$$

More details are in

<http://primes.utm.edu/top20/page.php?id=12>

### 3.6.6 Carmichael numbers

As mentioned [above](#) in chapter 3.5 not all Carmichael numbers are prime.

### 3.6.7 Pseudo prime numbers

See [above](#) in chapter 3.5.

### 3.6.8 Strong pseudo prime numbers

See [above](#) in chapter 3.5.

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<sup>26</sup>The base of this power is no longer restricted to 2. Even more generic would be:  $f(b, c, n) = b^{c^n} \pm 1$

### 3.6.9 Idea based on Euclid's proof $p_1 \cdot p_2 \cdots p_n + 1$

This idea is based on [Euclid's proof](#) (see chapter 3.3), that there are infinite many prime numbers.

$2 \cdot 3 + 1$	$= 7$	$\mapsto$ prime
$2 \cdot 3 \cdot 5 + 1$	$= 31$	$\mapsto$ prime
$2 \cdot 3 \cdot 5 \cdot 7 + 1$	$= 211$	$\mapsto$ prime
$2 \cdot 3 \cdots 11 + 1$	$= 2,311$	$\mapsto$ prime
$2 \cdot 3 \cdots 13 + 1$	$= 59 \cdot 509$	$\mapsto$ NOT prime!
$2 \cdot 3 \cdots 17 + 1$	$= 19 \cdot 97 \cdot 277$	$\mapsto$ NOT prime!

### 3.6.10 As above but $-1$ except $+1$ : $p_1 \cdot p_2 \cdots p_n - 1$

$2 \cdot 3 - 1$	$= 5$	$\mapsto$ prime
$2 \cdot 3 \cdot 5 - 1$	$= 29$	$\mapsto$ prime
$2 \cdot 3 \cdots 7 - 1$	$= 11 \cdot 19$	$\mapsto$ NOT prime!
$2 \cdot 3 \cdots 11 - 1$	$= 2,309$	$\mapsto$ prime
$2 \cdot 3 \cdots 13 - 1$	$= 30,029$	$\mapsto$ prime
$2 \cdot 3 \cdots 17 - 1$	$= 61 \cdot 8,369$	$\mapsto$ NOT prime!

### 3.6.11 Euclidean numbers $e_n = e_0 \cdot e_1 \cdots e_{n-1} + 1$ with $n \geq 1$ and $e_0 := 1$

$e_{n-1}$  is not the  $(n-1)$ th prime number, but the number previously found here. Unfortunately this formula is not open but recursive. The sequence starts with

$e_1 = 1 + 1$	$= 2$	$\mapsto$ prime
$e_2 = e_1 + 1$	$= 3$	$\mapsto$ prime
$e_3 = e_1 \cdot e_2 + 1$	$= 7$	$\mapsto$ prime
$e_4 = e_1 \cdot e_2 \cdot e_3 + 1$	$= 43$	$\mapsto$ prime
$e_5 = e_1 \cdot e_2 \cdots e_4 + 1$	$= 13 \cdot 139$	$\mapsto$ NOT prime!
$e_6 = e_1 \cdot e_2 \cdots e_5 + 1$	$= 3,263,443$	$\mapsto$ prime
$e_7 = e_1 \cdot e_2 \cdots e_6 + 1$	$= 547 \cdot 607 \cdot 1,033 \cdot 31,051$	$\mapsto$ NOT prime!
$e_8 = e_1 \cdot e_2 \cdots e_7 + 1$	$= 29,881 \cdot 67,003 \cdot 9,119,521 \cdot 6,212,157,481$	$\mapsto$ NOT prime!

Also  $e_9, \dots, e_{17}$  are composite, which means that this formula is not particularly useful.

#### Comment:

However, what is special about these numbers is that any pair of them does not have a common factor other than 1<sup>27</sup>. Therefore they are *relatively prime*.

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<sup>27</sup>This can easily be shown via the following rule for the *greatest common divisor gcd* with  $\gcd(a, b) = \gcd(b - \lfloor b/a \rfloor \cdot a, a)$ .

We have for  $i < j$ :

$$\gcd(e_i, e_j) \leq \gcd(e_1 \cdots e_i \cdots e_{j-1}, e_j) = \gcd(e_j - e_1 \cdots e_i \cdots e_{j-1}, e_1 \cdots e_i \cdots e_{j-1}) = \gcd(1, e_1 \cdots e_i \cdots e_{j-1}) = 1.$$

See page [180](#).

### 3.6.12 $f(n) = n^2 + n + 41$

This sequence starts off very *promisingly*, but is far from being a proof.

$f(0) = 41$	$\mapsto$ prime
$f(1) = 43$	$\mapsto$ prime
$f(2) = 47$	$\mapsto$ prime
$f(3) = 53$	$\mapsto$ prime
$f(4) = 61$	$\mapsto$ prime
$f(5) = 71$	$\mapsto$ prime
$f(6) = 83$	$\mapsto$ prime
$f(7) = 97$	$\mapsto$ prime
$\vdots$	
$f(33) = 1,163$	$\mapsto$ prime
$f(34) = 1,231$	$\mapsto$ prime
$f(35) = 1,301$	$\mapsto$ prime
$f(36) = 1,373$	$\mapsto$ prime
$f(37) = 1,447$	$\mapsto$ prime
$f(38) = 1,523$	$\mapsto$ prime
$f(39) = 1,601$	$\mapsto$ prime
$f(40) = 1681 = 41 \cdot 41$	$\mapsto$ NOT prime!
$f(41) = 1763 = 41 \cdot 43$	$\mapsto$ NOT prime!

The first 40 values are prime numbers (which have the obvious regularity that their difference starts with 2 and increases by 2 each time), but the 41th and 42th values are not prime numbers. It is easy to see that  $f(41)$  cannot be a prime number:  $f(41) = 41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \cdot 43$ .

### 3.6.13 $f(n) = n^2 - 79 \cdot n + 1,601$

This function delivers prime numbers for all values from  $n = 0$  to  $n = 79$ .<sup>28</sup> Unfortunately  $f(80) = 1,681 = 11 \cdot 151$  is not a prime number. To this date, no function has been found that delivers more prime numbers in a row. On the other hand, each prime occurs twice (first in the decreasing then in the increasing sequence), which means that the algorithm delivers a total of 40 different prime values (these are the same ones as delivered by the function in chapter 3.6.12)<sup>29</sup>.

---

<sup>28</sup>See chapter 3.14, “Appendix: Examples using SageMath” for the source code to compute the table using SageMath.

<sup>29</sup>Another quadratic polynom, which delivers these primes, is:  $f(n) = n^2 - 9 \cdot n + 61$ .

Among the first 1000 sequence elements more than 50% are prime (See chapter 3.14, “Appendix: Examples using SageMath”).

$f(0) = 1.601 \rightarrow \text{prim}$	$f(26) = 223 \rightarrow \text{prim}$
$f(1) = 1.523 \rightarrow \text{prim}$	$f(27) = 197 \rightarrow \text{prim}$
$f(2) = 1.447 \rightarrow \text{prim}$	$f(28) = 173 \rightarrow \text{prim}$
$f(3) = 1.373 \rightarrow \text{prim}$	$f(29) = 151 \rightarrow \text{prim}$
$f(4) = 1.301 \rightarrow \text{prim}$	$f(30) = 131 \rightarrow \text{prim}$
$f(5) = 1.231 \rightarrow \text{prim}$	$f(31) = 113 \rightarrow \text{prim}$
$f(6) = 1.163 \rightarrow \text{prim}$	$f(32) = 97 \rightarrow \text{prim}$
$f(7) = 1.097 \rightarrow \text{prim}$	$f(33) = 83 \rightarrow \text{prim}$
$f(8) = 1.033 \rightarrow \text{prim}$	$f(34) = 71 \rightarrow \text{prim}$
$f(9) = 971 \rightarrow \text{prim}$	$f(35) = 61 \rightarrow \text{prim}$
$f(10) = 911 \rightarrow \text{prim}$	$f(36) = 53 \rightarrow \text{prim}$
$f(11) = 853 \rightarrow \text{prim}$	$f(37) = 47 \rightarrow \text{prim}$
$f(12) = 797 \rightarrow \text{prim}$	$f(38) = 43 \rightarrow \text{prim}$
$f(13) = 743 \rightarrow \text{prim}$	$f(39) = 41 \rightarrow \text{prim}$
$f(14) = 691 \rightarrow \text{prim}$	$f(40) = 41 \rightarrow \text{prim}$
$f(15) = 641 \rightarrow \text{prim}$	$f(41) = 43 \rightarrow \text{prim}$
$f(16) = 593 \rightarrow \text{prim}$	$f(42) = 47 \rightarrow \text{prim}$
$f(17) = 547 \rightarrow \text{prim}$	$f(43) = 53 \rightarrow \text{prim}$
$f(18) = 503 \rightarrow \text{prim}$	$\dots$
$f(19) = 461 \rightarrow \text{prim}$	$f(77) = 1.447 \rightarrow \text{prim}$
$f(20) = 421 \rightarrow \text{prim}$	$f(78) = 1.523 \rightarrow \text{prim}$
$f(21) = 383 \rightarrow \text{prim}$	$f(79) = 1.601 \rightarrow \text{prim}$
$f(22) = 347 \rightarrow \text{prim}$	$f(80) = 41 \cdot 41 \rightarrow \text{NOT prim!}$
$f(21) = 383 \rightarrow \text{prim}$	$f(81) = 41 \cdot 43 \rightarrow \text{NOT prim!}$
$f(22) = 347 \rightarrow \text{prim}$	$f(82) = 1.847 \rightarrow \text{prim}$
$f(23) = 313 \rightarrow \text{prim}$	$f(83) = 1.933 \rightarrow \text{prim}$
$f(24) = 281 \rightarrow \text{prim}$	$f(84) = 43 \cdot 47 \rightarrow \text{NOT prim!}$
$f(25) = 251 \rightarrow \text{prim}$	

### 3.6.14 Polynomial functions $f(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x^1 + a_0$ ( $a_i$ in $\mathbb{Z}$ , $n \geq 1$ )

There exists no such polynomial that for all  $x$  in  $\mathbb{Z}$  only delivers prime values. For a proof of this, please refer to [Pad96, p. 83 f.], where you will also find further details about prime number formulae.

This means there is no hope in looking for further formulae (functions) similar to that in chap. 3.6.12 or chap. 3.6.13.

### 3.6.15 Catalan's Mersenne conjecture<sup>30</sup>

Catalan conjectured<sup>31</sup> that  $C_4$  and any further term in this sequence is a prime:

$$\begin{aligned} C_0 &= 2, \\ C_1 &= 2^{C_0} - 1, \\ C_2 &= 2^{C_1} - 1, \\ C_3 &= 2^{C_2} - 1, \\ C_4 &= 2^{C_3} - 1, \dots \end{aligned}$$

This sequence is defined recursively and increases extremely quickly. Does it only consist of primes?

$C(0) = 2$	$\mapsto$ prime
$C(1) = 2^2 - 1 = 3$	$\mapsto$ prime
$C(2) = 2^3 - 1 = 7$	$\mapsto$ prime
$C(3) = 2^7 - 1 = 127$	$\mapsto$ prime
$C(4) = 2^{127} - 1 = 170,141,183,460,469,231,731,687,303,715,884,105,727$	$\mapsto$ prime

It is not (yet) known whether  $C_5 = 2^{C_4} - 1$  and all higher elements are prime. In any case, it has not been proved that this formula delivers only primes.

It seems very unlikely that  $C_5$  (or many of the larger terms) would be prime.

So this could be another example of Guy's "law of small numbers"<sup>32</sup>.

### 3.6.16 Double Mersenne primes

The above Catalan-Mersenne numbers are from  $C_2$  a subset of the double Mersenne primes.<sup>33</sup> A double Mersenne prime is a Mersenne prime of the form

$$M_{M_p} = 2^{2^p - 1} - 1$$

where  $p$  is a Mersenne prime exponent and  $M_p$  ist a prime Mersenne number.

The first values of  $p$  for which  $M_p$  is prime are  $p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, \dots$  (see [above](#)).

$M_{M_p}$  is known to be prime for  $p = 2, 3, 5, 7$ , and there has the according values: 7, 127, 2.147.483.647, 170.141.183.460.469.231.731.687.303.715.884.105.727.<sup>34</sup>

<sup>30</sup>Eugene Charles Catalan, Belgian mathematician, May 5, 1814–Feb 14, 1894.

After him, the so-called *Catalan numbers*  $A(n) = (1/(n+1)) * (2n)!/(n!)^2$

= 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, ... are named.

<sup>31</sup>See <http://primes.utm.edu/mersenne/index.html> under "Conjectures and Unsolved Problems".

<sup>32</sup><http://primes.utm.edu/glossary/page.php?sort=LawOfSmall>

<sup>33</sup>[http://en.wikipedia.org/wiki/Catalan%20%93Mersenne\\_number](http://en.wikipedia.org/wiki/Catalan%20%93Mersenne_number)

<sup>34</sup>SageMath calculates this with:

```
sage: for p in (2,3,5,7): Mp=(2^p)-1; MMp=(2^Mp)-1; B=is_prime(MMp); print p, Mp, MMp, B;
....:
2 3 7  True
3 7 127  True
5 31 2147483647  True
7 127 170141183460469231731687303715884105727  True
```

For  $p = 11, 13, 17, 19$ , and  $31$ , the corresponding double Mersenne numbers are not prime.

The next candidate for the next double Mersenne prime is  $M_{M_{61}} = 2^{2305843009213693951} - 1$ . Being approximately  $1.695 * 10^{694,127,911,065,419,641}$ , this number — like  $C_5$  — is far too large for any currently known primality test to be successfully applied.

### 3.7 Density and distribution of the primes

As Euclid discovered, there is an infinite number of primes. However, some infinite sets are *denser* than others.

Within the set of natural numbers, there is an infinite number of even, uneven and square numbers. How to compare the “density” of two infinite sets is shown with even and square numbers.

The following proves that the even numbers are distributed more densely than square ones.<sup>35</sup>

- the size of the  $n$ th element:

The  $n$ th element of the even numbers is  $2n$ ; the  $n$ th element of the square numbers is  $n^2$ . Because for all  $n > 2$ :  $2n < n^2$ , the  $n$ th even number occurs much earlier than the  $n$ th square number.

- the numbers of values that are less than or equal to a certain *maximum value*  $x$  in  $\mathbb{R}$  are: There are  $\lfloor x/2 \rfloor$  such even numbers and  $\lfloor \sqrt{x} \rfloor$  square numbers. Because for all  $x > 6$  the value  $\lfloor x/2 \rfloor$  is greater than the largest integer smaller or equal to the square root of  $x$ , the even numbers are distributed more densely.

#### The value of the $n$ -th prime $P(n)$

**Theorem 3.7.1.** *For large  $n$ : The value of the  $n$ -th prime  $P(n)$  is asymptotic to  $n \cdot \ln(n)$ , i.e. the limit of the relation  $P(n)/(n \cdot \ln n)$  is equal to 1 if  $n$  tends to infinity.*

For  $n > 5$ ,  $P(n)$  lies between  $2n$  and  $n^2$ . This means that there are fewer prime numbers than even natural numbers, but more prime numbers than square numbers.<sup>36</sup>

#### The number of prime numbers $PI(x)$

The definition for the number<sup>37</sup>  $PI(x)$  is similar: It is the number of all primes that do not exceed the maximum value  $x$ .

**Theorem 3.7.2.**  *$PI(x)$  is asymptotic to  $x/\ln(x)$ .*

---

<sup>35</sup>Whereas in colloquial language you often can hear, that “there are more” even numbers than square ones, mathematicians say, that from both there are infinitely many, that their sets are equivalent to  $\mathbb{N}$  (so both are infinite and countable, i.e. one can assign to each even number and to each square number an integer), but that the set of even numbers is denser than the set of square numbers.

<sup>36</sup>Please refer to [the table 3.10](#)

<sup>37</sup>Often, instead of  $PI(x)$  the convention  $\Pi(x)$  is used.

This is the **prime number theorem**. It was put forward by Legendre<sup>38</sup> and Gauss<sup>39</sup> but not proved until over 100 years later.<sup>40</sup>

The tables under 3.9 show the number of prime numbers in various intervals.

The distribution is graphically presented within Figure 3.9 at page 107 within the appendix 3.13.

The formulae for the prime number theorem only apply when  $n$  tends to infinity. The formula of Gauss can be replaced by more precise formulae. For  $x \geq 67$ :

$$\ln(x) - 1,5 < x/PI(x) < \ln(x) - 0,5$$

Given that we know  $PI(x) = x/\ln x$  only for very large  $x$  ( $x$  tending towards infinity), we can create the following overview:

$x$	$\ln(x)$	$x/\ln(x)$	$PI(x)(\text{counted})$	$PI(x)/(x/\ln(x))$
$10^3$	6.908	144	168	1.160
$10^6$	13.816	72,386	78,498	1.085
$10^9$	20.723	48,254,942	50,847,534	1.054

For a binary number<sup>41</sup>  $x$  of the length of 250 bits ( $2^{250}$  is approximately  $= 1.809251 * 10^{75}$ ) it is:

$$PI(x) = 2^{250}/(250 \cdot \ln 2) \text{ is approximately } = 2^{250}/173.28677 = 1.045810 \cdot 10^{73}.$$

We can therefore expect that the set of numbers with a bit length of less than 250 contains approximately  $10^{73}$  primes (a reassuring result?!).

We can also express this as follows: Let us consider a *random* natural number  $n$ . Then the probability that this number is prime is around  $1/\ln(n)$ . For example, let us take numbers in the region of  $10^{16}$ . Then we must consider  $16 \cdot \ln 10 = 36,8$  numbers (on average) until we find a prime. A precise investigation shows: There are 10 prime numbers between  $10^{16} - 370$  and  $10^{16} - 1$ .

Under the heading *How Many Primes Are There at*

<http://primes.utm.edu/howmany.html>

you will find numerous other details.

Using the following Web site:

<https://primes.utm.edu/nthprime/>

you can easily determine  $PI(x)$ .

The **distribution** of primes<sup>42</sup> displays several irregularities for which no “system” has yet been found: On the one hand, many occur closely together, like 2 and 3, 11 and 13, 809 and 811, on the other hand large gaps containing no primes also occur. For example, no primes lie between 113 and 127, 293 and 307, 317 and 331, 523 and 541, 773 and 787, 839 and 853 as well as between 887 and 907.

For details, please see:

<http://primes.utm.edu/notes/gaps.html>

To discover the secrets of these irregularities is precisely part of what motivates mathematicians.

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<sup>38</sup>Adrien-Marie Legendre, French mathematician, Sep 18, 1752 – Jan 10, 1833.

<sup>39</sup>Carl Friedrich Gauss, German mathematician and astronomer, Apr 30, 1777–Feb 23, 1855.

<sup>40</sup>A didactical article about the prime number theorem and its application to the RSA algorithm can be found within the series *RSA & Co. at school: Modern cryptology, old mathematics, and subtle protocols*: see NF part 4 [WS10a]. Unfortunately these are currently only available in German.

<sup>41</sup>Number written in the binary system consists only of the digits 0 and 1.

<sup>42</sup>Some visualizations (plots) of the quantity of primes in different number dimensions can be found in chapter 3.13, “Appendix: Visualization of the quantity of primes in higher ranges”.

## Sieve of Eratosthenes

An easy way of calculating all  $PI(x)$  primes less than or equal to  $x$  is to use the sieve of Eratosthenes. In the 3rd century B.C., he found an extremely easy, automatic way of finding this out. To begin with, you write down all numbers from 2 to  $x$ , circle 2, then cross out all multiples of 2. Next, you circle the lowest number that hasn't been circled or crossed out (now 3) and again cross out all multiples of this number, etc.

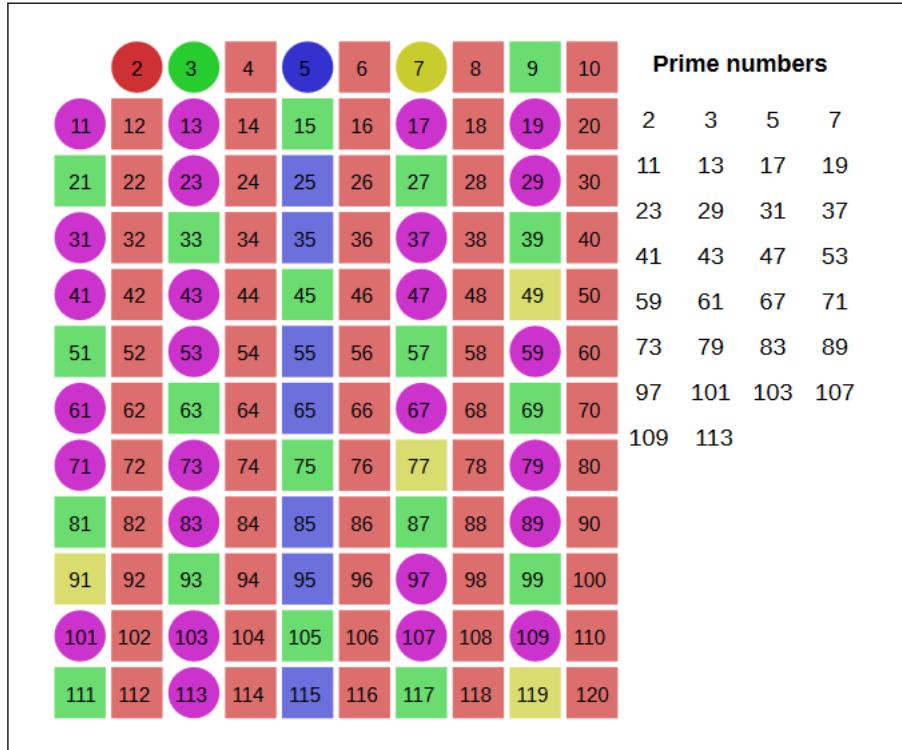


Figure 3.5: The sieve of Eratosthenes, applied to the first 120 numbers<sup>43</sup>

You need to continue only until you reach the largest number whose square is less than or equal to  $x$  (here up to 10, as  $11^2$  is already  $> 120$ ).<sup>44</sup>

Apart from 2, prime numbers are never even. Apart from 2 and 5, prime numbers never end in 2, 5 or 0. So you only need to consider numbers ending in 1, 3, 7, 9 anyway (there are infinite primes ending in these numbers; see [Tie73, vol. 1, p. 137]).

Nowadays, you can now find large databases that contain either a large number of primes or the factorization of numerous composite numbers.

## Further interesting topics regarding prime numbers

This chapter 3 didn't consider other number theory topics such as divisibility rules, modulus

<sup>43</sup>Graphics from [https://upload.wikimedia.org/wikipedia/commons/0/0b/Sieve\\_of\\_Eratosthenes\\_animation.svg](https://upload.wikimedia.org/wikipedia/commons/0/0b/Sieve_of_Eratosthenes_animation.svg)

<sup>44</sup>With the educational tool for number theory NT you can apply the sieve of Eratosthenes in a computer-aided and guided way: Enter your own number and do the sieving step by step: See the NT learning unit 1.2, pages 6/21 and 7/21.

NT can be called in CT1 via the menu path **Indiv. Procedures \ Number Theory Interactive \ Learning Tool for Number Theory**. See appendix A.6.

CT2 contains a visualisation of this method within the tutorial “World of Primes”.

calculation, modular inverses, modular powers, modular roots, Chinese remainder theorem, Euler Phi function or perfect numbers. Some of these topics are considered in the [next chapter](#) (chapter 4).

## 3.8 Notes about primes

The following notes list some interesting theorems, conjectures and open questions about primes, but also some quaint things and overviews.

### 3.8.1 Proven statements / theorems about primes

- For each number  $n$  in  $\mathbf{N}$  there are  $n$  consecutive natural numbers that are not primes. A proof of this can be found in [Pad96, p. 79].
- Paul Erdős<sup>45</sup> proved: Between each random number not equal to 1 and its double, there is at least one prime. He was not the first to prove this theorem, but proved it in a much simpler manner than those before him.
- There is a real number  $a$  such that the function  $f : \mathbf{N} \rightarrow \mathbb{Z}$  where  $n \mapsto \lfloor a^{3^n} \rfloor$  only delivers primes<sup>46</sup> for all  $n$  (see [Pad96, p. 82]). Unfortunately, problems arise when we try to determine  $a$  (see chapter 3.8.2).<sup>47</sup>
- There are arithmetic prime sequences of arbitrary length.<sup>48,49</sup>

In 1923 the famous British mathematician Hardy<sup>50</sup> compiled the conjecture, that there are arithmetic sequences of arbitrary length, which consist of primes only. This conjecture was proven in 2004 by two young American mathematicians.

At some point every school child learns about arithmetic number series. These are sequences of numbers, for which the difference between any 2 consecutive numbers is equal or constant (an arithmetic sequence must have at least three elements but can also have indefinitely many). In the sample sequence 5, 8, 11, 14, 17, 20 the difference between the series's elements is 3 and the length of the sequence is 6.

Arithmetic series have been known for millennia and one would think they have no more secrets. They get more interesting again, if we impose additional constraints on the serie's elements - as the prime example shows.

E.g. 5, 17, 29, 41, 53 is an arithmetic prime series which consists of 5 elements and the difference between the elements is always 12.

The sequence is not extendable - the next would be 65, but 65 is not prime (65 is the product of 5 and 13).

How many elements are possible within an arithmetic prime number sequence? Around 1770 the French Joseph-Louis Lagrange and the British Edward Waring investigated

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<sup>45</sup>Paul Erdős, Hungarian mathematician, Mar 26, 1913–Sep 20, 1996.

<sup>46</sup>The Gauss bracket  $\lfloor x \rfloor$  of a real number  $x$  is defined via:  $\lfloor x \rfloor$  is the next integer less or equal  $x$ .

<sup>47</sup>If someone knows how to prove this, we are very interested to learn about this. Friedhelm Padberg told us, he doesn't have his prove any more.

<sup>48</sup>Sources:

- <http://primes.utm.edu/glossary/page.php?sort=ArithmetSequence> Original source
- [http://en.wikipedia.org/wiki/Primes\\_in\\_arithmetic\\_progression](http://en.wikipedia.org/wiki/Primes_in_arithmetic_progression)
- [http://en.wikipedia.org/wiki/Problems\\_involving\\_arithmetic\\_progressions](http://en.wikipedia.org/wiki/Problems_involving_arithmetic_progressions)
- [http://en.wikipedia.org/wiki/Cunningham\\_chain](http://en.wikipedia.org/wiki/Cunningham_chain)
- German magazine GEO 10 / 2004: "Experiment mit Folgen"

<sup>49</sup><http://www.faz.net> "Hardys Vermutung – Primzahlen ohne Ende" by Heinrich Hemme (July 06, 2004)

<sup>50</sup>Arithmetic sequences with  $k$  primes are called prime arithmetic progressions and therefore their abbreviation is PAP- $k$  or AP- $k$ .

<sup>50</sup>Godfrey Harold Hardy, British mathematician, Feb 7, 1877–Dec 1, 1947.

this question. In 1923 the famous British mathematician Godfrey Harold Hardy and his colleague John Littlewood theorized, that there is no upper limit for the number of elements. But they could not prove this. In 1939 more progress was achieved: The Dutch mathematician Johannes van der Corput was able to prove that there are infinitely many different arithmetic prime number sequences with exactly three elements. Two examples are 3, 5, 7 and 47, 53, 59.

The longest arithmetic prime number sequence known today contains 25 elements. The table 3.3 lists the longest currently known arithmetic prime number sequences with minimal difference<sup>51</sup>.

As a team, the two young<sup>52</sup> mathematicians Ben Green and Terence Tao, were able in 2004 to prove Hardy's conjecture, which had puzzled mathematicians for over 80 years: It states, that for any arbitrary length there exists an arithmetic prime number series. Additionally they managed to prove, that for any given length there are infinitely many different series.

Green and Tao intended to proof that there are infinitely many arithmetic sequences of length four. For this they considered sets of numbers consisting of primes and so called "near primes". These are numbers with a small set of divisors like numbers which are the product of two primes - these numbers are called "half primes". Thus they managed to considerably simplify their work because about near primes there already existed a lot of useful theorems. Finally they discovered that the results of their theorem were far more reaching than they had assumed and so they were able to prove Hardy's conjecture.

Any one who believes that it is easy to use Green's and Tao's 49 page proof to compute arithmetic prime number series of arbitrary length will soon become disappointed, because the proof is non-constructive. It is a so called proof of existence. This means that these mathematicians have shown "only" that these series exist, but not how to find them in practice.

This means that in the set of the natural numbers there is e.g. a series of one billion primes, which all have the same distance; and there are infinitely many of them. But these sequences lie extremely far beyond the numbers we usually use ("far outside").

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<sup>51</sup>In the opposite, [http://en.wikipedia.org/wiki/Primes\\_in\\_arithmetic\\_progression](http://en.wikipedia.org/wiki/Primes_in_arithmetic_progression) lists the "largest known AP-k". Therefore, there the last sequence element is a prime as big as possible.

However, table 3.3 lists the sequences which have the smallest known difference for a given length.

<sup>52</sup>Hardy wrote in his memoirs in 1940, that mathematics - more than all other arts and sciences - is an activity for young people.

At that time, 27-years-old Ben Green from the University of British Columbia in Vancouver and 29-year-old Terence Tao from the University of California in Los Angeles seem to confirm Hardy.

Elements	First element	Distance	When Digits	Discovered by
3	3	2	1	
4	5	6	2	
5	5	6	2	
6	7	30	1909 3	G. Lenaire
7	7	150	1909 3	G. Lenaire
.....				
21	28,112,131,522,731,197,609	9,699,690 = 19#	2008 20	Jaroslaw Wroblewski
22	166,537,312,120,867	96,599,212,710 = 9,959·19#	2006 15	Markus Frind
23	403,185,216,600,637	2,124,513,401,010 = 9,523·23#	2006 15	Markus Frind,
24	515,486,946,529,943	30,526,020,494,970 = 136,831·23#	2008 16	Raanan Chermoni, Jaroslaw Wroblewski
25	6,171,054,912,832,631	81,737,658,082,080 = 366,384·23#	2008 16	Raanan Chermoni, Jaroslaw Wroblewski

Table 3.3: Arithmetic prime number sequences with minimal difference (as of Aug. 2012)

If someone wants to discover such sequences the following thoughts may be helpful. The length of a sequence determines the minimal common distance between the single primes of the sequence. Given a sequence with 6 elements the distance between them has to be 30 or a multiple of 30. The number 30 here results from the product of all primes smaller than the length of the sequence (here 6):  $6\# = 5\# = 2 * 3 * 5 = 30$ . Another example:  $10\# = 7\# = 2 * 3 * 5 * 7 = 210$ . If you look for a sequence with 15 elements, then the common distance is at least  $15\# = 13\# = 2 * 3 * 5 * 7 * 11 * 13 = 30,030$ .

This means that the length of an arithmetic prime sequence can be arbitrary big, but the distance between the elements cannot be any arbitrary number. E.g. there is no arithmetic prime sequence with the distance 100, because 100 cannot be divided by 3.

k	k#
2	2
3	6
5	30
7	210
11	2,310
13	30,030
17	510,510
19	9,699,690
23	223,092,870

Table 3.4: Products of the first primes  $\leq k$  (called k primorial or  $k\#$ )

### Further restriction:

If you look at arithmetic prime sequences, which fulfill the *additional* requirement, that all primes are *consecutive*<sup>53</sup>, then it's getting even more complicated. At the website of Chris Caldwell<sup>54</sup> you can find further details: The longest known arithmetic prime sequence, consisting only of directly consecutive primes (as of Aug. 2012), has a length of 10, the common distance is

$$10\# = 7\# = 2 * 3 * 5 * 7 = 210$$

and starts with the 93-digit prime

100 9969724697 1424763778 6655587969 8403295093 2468919004 1803603417 7589043417  
0334888215 9067229719

---

<sup>53</sup>They are also called consecutive prime arithmetic progressions and therefore their abbreviation is CPAP-k or CAP-k.

<sup>54</sup><http://primes.utm.edu/glossary/page.php?sort=ArithmeticSequence>

### 3.8.2 Unproven statements/ conjectures/ open questions about primes<sup>55</sup>

- Goldbach<sup>56</sup> conjectured: Every even natural number greater than 2 can be represented as the sum of two prime numbers.<sup>57</sup>
- Riemann<sup>58</sup> put forward an important, but still unproved hypothesis<sup>59</sup> about the location of the nontrivial zeros of the Riemann zeta function. A consequence is an improved estimation within the prime number theorem (distribution of primes).
- Benford's law<sup>60,61</sup> does not apply to primes.

According to Benford's law, also called the first-digit law, the single digits in lists of numbers from many (but not all) real-life sources of data, are distributed in a non-uniform way. Especially the leading digit is much more often the digit 1 than any other digit.

Which empirical data applies to this “law” is not completely clear yet. Timo Eckhardt analyzed in his thesis in 2008 extensively attributes of prime numbers. For example, all primes until 7,052,046,499 were described with different bases of the positional notation.

Comparing the bases 3 to 10 the deviation from Benford's law was lowest with base 3. Comparing the first digit for base 10 all digits are almost equally distributed. Analyzing bigger bases showed strong differences.

- The proof (mentioned above in chapter 3.8.1) of the function  $f : N \rightarrow Z$  with  $n \mapsto \lfloor a^{3^n} \rfloor$  only guarantees the existence of such a number  $a$ . How can we determine this number  $a$  and will it have a value, making the function also of some practical interest?
- Is there an infinite number of Mersenne prime numbers?
- Is there an infinite number of Fermat prime numbers?
- Does a polynomial time algorithm exist for calculating the prime factors of a number (see [KW97, p. 167])? This question can be divided into the three following questions:
  - Does a polynomial time algorithm exist that decides whether a number is prime?  
This question has been answered by the AKS algorithm (see chapter 4.11.5.3, “Primes in P”: Primality testing is polynomial).

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<sup>55</sup>Marcus du Sautoy, a professor of mathematics in Oxford, describes in his popular science book “The Music of the Primes”, how some of the most brilliant mathematicians looked at different aspects of prime numbers. He introduces these people (especially Gauß, Euler, Riemann, Ramanujan, Gödel, and Connes) and then focuses on the Riemann hypothesis. See [dS05].

Simple explanations and further references about the Riemann hypothesis can also be found in *RSA & Co. at school*, NF part 4 [WS10a].

<sup>56</sup>Christian Goldbach, German mathematician, Mar 18, 1690–Nov 20, 1764.

<sup>57</sup>Please compare chapter 3.8.3.

<sup>58</sup>Bernhard Riemann, German mathematician, Sep 17, 1826–Jul 20, 1866.

<sup>59</sup>[http://en.wikipedia.org/wiki/Riemann\\_hypothesis](http://en.wikipedia.org/wiki/Riemann_hypothesis)

<sup>60</sup>[http://en.wikipedia.org/wiki/Benford%27s\\_law](http://en.wikipedia.org/wiki/Benford%27s_law),  
[http://arxiv.org/PS\\_cache/arxiv/pdf/0906/0906.2789v1.pdf](http://arxiv.org/PS_cache/arxiv/pdf/0906/0906.2789v1.pdf).

<sup>61</sup>Two good didactical articles in German about applications of Benford's law are:

- Rüdeger Baumann: “Ziffernanalyse zwecks Betrugsaufdeckung — Beispiel für kompetenzorientierten und kontextbezogenen Informatikunterricht”,  
in LOGIN, Informatische Bildung und Computer in der Schule, No. 154/155, 2008, p. 68-72
- Norbert Hungerbühler: “Benfords Gesetz über führende Ziffern”, March 2007,  
[https://www.ethz.ch/content/dam/ethz/special-interest/dual/educeth-dam/documents/Unterrichtsmaterialien/mathematik/Benfords%20Gesetz%20%C3%BCber%20f%C3%BChrende%20Ziffern%20\(Artikel\)/benford.pdf](https://www.ethz.ch/content/dam/ethz/special-interest/dual/educeth-dam/documents/Unterrichtsmaterialien/mathematik/Benfords%20Gesetz%20%C3%BCber%20f%C3%BChrende%20Ziffern%20(Artikel)/benford.pdf)

- Does a polynomial time algorithm exist that calculates for a composite number from how many prime factors it is made up (without calculating these factors)?
- Does a polynomial time algorithm exist that calculates for a composite number  $n$  a non-trivial (i.e. other than 1 and  $n$ ) factor of  $n$ ?<sup>62</sup>

At the end of chapter 4.11.4, section RSA-200 you can see the limits for which numbers the current algorithms for primality testing and for factorization deliver good results.

### 3.8.3 The Goldbach conjecture

Here we will have a closer look at what is called the **Goldbach conjecture**<sup>63</sup>.

#### 3.8.3.1 The weak Goldbach conjecture<sup>64</sup>

Goldbach conjectured in 1742 within a letter to the mathematician Euler:

Every **odd** natural number greater than 5 can be represented as the sum of **exactly three** prime numbers.

Samples:  $7 = 3 + 2 + 2$  or  $27 = 19 + 5 + 3$  or  $27 = 17 + 5 + 5$

This conjecture is still (since more than 250 years) assumed unproven.

Computers have verified the **weak Goldbach conjecture** for all odd natural numbers up to:  $4 * 10^{18}$  (simple check at April 2012) and  $4 * 10^{17}$  (double check at May 2013).<sup>65</sup>

Previous results proved, that the weak Goldbach conjecture is true for all odd natural numbers larger  $e^{3100} \approx 2 \times 10^{1346}$ .

As with many famous conjectures in mathematics, there are also a number of purported proofs of the Goldbach conjecture, none accepted by the mathematical community yet.<sup>66</sup>

A preliminary work for a prove could be the prove published recently<sup>67</sup> by Terence Tao from the University of California. He proved that every **odd** natural number greater than 1 can be represented as the sum of **at most five** prime numbers.

Considerable work has been done on Goldbach's weak conjecture, culminating in a 2013 claim by Harald Helfgott to fully prove the conjecture for all odd integers greater than 7.<sup>68</sup>

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<sup>62</sup>Please compare chapters 4.11.5.1 and 4.11.4.

<sup>63</sup>[http://en.wikipedia.org/wiki/Goldbach%27s\\_conjecture](http://en.wikipedia.org/wiki/Goldbach%27s_conjecture)

<sup>64</sup>It is also known as the **odd** or **ternary** Goldbach conjecture.

<sup>65</sup>See <http://sweet.ua.pt/tos/goldbach.html> by Tomás Oliveira e Silva

<sup>66</sup>One of them is in the paper from Shan-Guang Tan, submitted on 16 Oct 2011 (v1), last revised 20 May 2016 (v19) which claims to even proof the strong Goldbach conjecture.

v1 has the title "A proof of the Goldbach conjecture".

v19 has the title "On the representation of even numbers as the sum and difference of two primes and the representation of odd numbers as the sum of an odd prime and an even semiprime and the distribution of primes in short intervals".

See <http://arxiv.org/abs/1110.3465>.

<sup>67</sup><http://arxiv.org/abs/1201.6656>, submitted on 31 Jan 2012 (v1), last revised 3 Jul 2012 (v4)

<sup>68</sup>Helfgott, H.A. (2013): "Major arcs for Goldbach's theorem". <http://arxiv.org/abs/1305.2897>.

v4 of his 79 pages paper (14 Apr 2014) states, that the proof works starting at  $10^{29}$  (instead of  $10^{30}$ ). And for all numbers up to  $10^{30}$  he claims, that he and David Platt proved at the computer that the weak Goldbach is valid.

### 3.8.3.2 The strong Goldbach conjecture<sup>69</sup>

Goldbach's strong prime number hypothesis was formulated by Euler after a mail exchange with Goldbach. This is now called **the** Goldbach conjecture:

Every **even** natural number greater than 2 can be represented as the sum of **exactly two** prime numbers.

Samples of Goldbach partitions:  $8 = 5 + 3$  or  $28 = 23 + 5$

Computers have once verified<sup>70</sup> the Goldbach conjecture for all even numbers up to  $4 * 10^{18}$  (as at May 2013), but no general proof has yet been found.<sup>71,72,73</sup>

As bigger an even number is, as more such binary Goldbach partitions can be found – in average: For 4 there is only one partition  $2 + 2$ ; for 16 there are two,  $3 + 13$  and  $5 + 11$ . With 100 there are six such partitions:  $3 + 97$ ,  $11 + 89$ ,  $17 + 83$ ,  $29 + 71$ ,  $41 + 59$ ,  $47 + 53$ .<sup>74</sup>

### 3.8.3.3 Interconnection between the two Goldbach conjectures

If the strong Goldbach conjecture holds, then also the weak is true (so the strong implies the weak conjecture).

The proof for this is relativly simple:

Prerequisite:  $u$  is an odd number bigger than 5.

Each such odd number  $u$  can be written as sum  $u = (u - 3) + 3$ . The first summand then is even

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<http://truthiscool.com/prime-numbers-the-271-year-old-puzzle-resolved>.

<http://www.newscientist.com/article/dn23535-proof-that-an-infinite-number-of-primes-are-paired.html#.UgwhOpLOEk0>.

<http://www.spiegel.de/wissenschaft/mensch/beweis-fuer-schwache-goldbachsche-vermutung-a-901111.html>.

<sup>69</sup>It is also known as the **even** or **binary** Goldbach conjecture.

<sup>70</sup>It is generally accepted today, that the **Goldbach conjecture** is true, i. e. valid for all even natural numbers greater than 2. In 1999, mathematician Jörg Richstein from the computer sciences institute at the University of Giessen, studied even numbers up to 400 billion ( $4 * 10^{14}$ ) and found no contradictory example ([Ric01]).

In the meantime further progress was made: See

<http://sweet.ua.pt/tos/goldbach.html> by Tomás Oliveira e Silva,

[http://en.wikipedia.org/wiki/Goldbach%27s\\_conjecture](http://en.wikipedia.org/wiki/Goldbach%27s_conjecture),

<http://primes.utm.edu/glossary/page.php?GoldbachConjecture.html>.

Nevertheless, this does not provide us with general proof.

The fact is that despite all efforts, Goldbach's conjecture has to date not been proven. This leads one to believe that since the pioneer work of the Austrian mathematician Kurt Gödel is well-known, not every true mathematical theorem is provable (see <https://www.mathematik.ch/mathematiker/goedel.php>). Perhaps Goldbach's conjecture was correct, but in any case the proof will never be found. Conversely, that will presumably also remain unproven.

<sup>71</sup>The English publisher *Faber* and the American publisher *Bloomsbury* issued in 2000 the 1992 published book "Uncle Petros and Goldbach's Conjecture" by Apostolos Doxiadis. It's the story of an old maths professor who fails to prove a more than 250 year old puzzle. To boost the sales figures the English and American publishers have offered a prize of 1 million USD, if someone can prove the conjecture – which should be published by 2004 in a well-known mathematical journal.

Surprisingly only British and American citizens are allowed to participate.

<sup>72</sup>The theorem which has come closest so far to Goldbach's conjecture was proved by Chen Jing-Run in 1966 in a way which is somewhat hard to understand: Each even integer greater than 2 is the sum of one prime and of the product of two primes. E.g.:  $20 = 5 + 3 * 5$ .

Most of the research about the Goldbach conjecture is collected in the book: "Goldbach Conjecture", ed. Wang Yuan, 1984, World scientific Series in Pure Maths, Vol. 4.

<sup>73</sup>Especilly this conjecture makes it clear, that even today we do not have a complete understanding of the deeper connections between addition and multiplication of natural numbers.

<sup>74</sup>Progress in research about the Goldbach conjectures is often a topic in mathematical press column, like <http://oumathclub.wordpress.com/>.

and  $\geq 4$ , so it fulfills the prerequisite of the strong Goldbach conjecture, and can be written as sum of two primes  $p_1$  und  $p_2$  ( $p_1$  and  $p_2$  are not necessarily different). So we found a partition of  $u$  into the three primes  $p_1, p_2$  and 3. This means, its always possible to find such a sum, where one of the primes is the number 3.

Similarly easy it can be shown, that the weak Goldbach conjecture implies the above mentioned conjecture from Terence Tao (both hold for odd numbers):

- For odd numbers  $u > 5$  directly the weak Goldbach conjecture implies that the sum consists at most of five primes.
- For the remaining odd numbers 3 and 5 you can directly check it:  
 $3 = 3$  (the “sum” has only one and therefore at most five prime summands);  
 $5 = 2 + 3$  (the sum has two and therefore at most five prime summands).

### 3.8.4 Open questions about twin primes and cousin primes

Twin primes are prime numbers whose difference is exactly 2. Examples include 5 and 7, or 101 and 103, or  $1,693,965 \cdot 2^{66,443} \pm 1$ .

The biggest known twin pair is

$$3,756,801,695,685 \cdot 2^{666,669} \pm 1$$

with 200,700 decimal digits (found in December 2011).<sup>75</sup>

Open questions are:

- What is the number of twin primes: Are there infinitely many or only a limited number?<sup>76,77,78</sup>
- Does a formula exist for calculating the number of twin primes per interval?

In the following two major milestones are explained which may allow to come closer to the riddle.

#### 3.8.4.1 GPY 2003

A big step towards the solution of this problem was made by Dan Goldston, János Pintz and Cem Yıldırım in 2003.<sup>79</sup> The three mathematicians were investigating the distribution of prime

<sup>75</sup><http://primes.utm.edu/primes>, <http://www.primegrid.com/download/twin-666669.pdf>

<sup>76</sup>Remark: **Triplet** primes, however, only occur once: 3, 5, 7. For all other sets of three consecutive odd numbers, one of them is always divisible by 3 and thus not a prime.

<sup>77</sup>The conjecture that there are infinite many twin primes is not obvious. It's known, that for large numbers in average the expected gap between primes is constantly growing and circa 2.3 times the number of decimal digits. For example, among 100-digit decimal numbers the expected gap between primes is in average 230. But this statement is true just on average – often the gap is much bigger, often much smaller.

<sup>78</sup>[http://en.wikipedia.org/wiki/Twin\\_Prime\\_Conjecture](http://en.wikipedia.org/wiki/Twin_Prime_Conjecture)

<sup>79</sup>D. A. Goldston: “Gaps Between Primes”

<http://www.math.sjsu.edu/~goldston/OberwolfachAbstract.pdf>

See also:

- D. A. Goldstone: “Are There Infinitely Many Twin Primes?”  
<http://www.math.sjsu.edu/~goldston/twinprimes.pdf>
- K. Soundararajan: “Small Gaps Between Prime Numbers: The Work Of Goldston-Pintz-Yıldırım”  
<http://www.ams.org/bull/2007-44-01/S0273-0979-06-01142-6/S0273-0979-06-01142-6.pdf>

numbers. They could proof, that

$$\liminf_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{\log p_n} = 0,$$

where  $p_n$  denotes the  $n$ -th prime number.

This means that the smallest limit point ( $\liminf$ ) of the sequence  $\frac{p_{n+1} - p_n}{\log p_n}$  equals zero. A point is called limit point of a sequence, if there lie in any arbitrarily small neighbourhood of that point infinitely many elements of the sequence.

$\log p_n$  is about the average distance between the prime  $p_n$  and the next prime  $p_{n+1}$ . Hence, the term above implies, that there are infinitely many consecutive primes with a gap between them which is arbitrarily small compared to the expected average gap.

Moreover, it was proofed, that

$$p_{n+1} - p_n < (\log p_n)^{8/9}$$

holds true for infinitely many primes<sup>80</sup>.

### 3.8.4.2 Zhang 2013

In May 2013 the results of Yitang Zhang became known.<sup>81</sup> Zhang proved, that there are infinitely many “cousin primes”, or more concretely that there is some number  $H$  smaller than 70 million such that there are infinitely many pairs of primes that differ by  $H$ .<sup>82,83,84</sup>

Those results could be the basis for the proof, that infinitely many twin primes exist.

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<sup>80</sup>c’t magazine 2003, no. 8, page 54

<sup>81</sup>Erica Klarreich (May 19, 2013): “Unheralded Mathematician Bridges the Prime Gap”

<https://www.simonsfoundation.org/quanta/20130519-unheralded-mathematician-bridges-the-prime-gap/>

<sup>82</sup>Whereas the gap between the primes of a twin prime is exactly 2, cousin primes do denote two primes, which have a gap between them, which has a value of a bigger, even, but finite number  $H$ .

<sup>83</sup>This is close to the conjecture stated in 1849 by the French mathematician Alphonse de Polignac that there are infinitely many prime pairs for any possible even finite gap, not just 2.

<sup>84</sup>In the meantime this minimal gap  $H$  of 70 millions was improved in further work. The according progress is documented in the Polymath8 project (massively collaborative online mathematical projects): “Bounded gaps between primes”. The best known value of  $H$  was 4680 (as of August 2013) and is till now (as of April 2014) 2460 – this is a good progress compared to 70 million, but far away from 2.

See [http://michaelnielsen.org/polymath1/index.php?title=Bounded\\_gaps\\_between\\_primes](http://michaelnielsen.org/polymath1/index.php?title=Bounded_gaps_between_primes)

### 3.8.5 Quaint and interesting things around primes<sup>85</sup>

Primes are not only a very active and serious research area in mathematics. Also a lot of people think about them in their free time and outside the scientific research.

#### 3.8.5.1 Recruitment at Google in 2004

In summer 2004 the company Google used the number  $e$  to attract potential employees.<sup>86,87</sup>

On a prominent billboard in California's Silicon Valley on July 12 there appeared the following mysterious puzzle:

*(first 10 digit prime in consecutive digits of  $e$ ).com*

Finding the first 10 digit prime in the decimal expansion of  $e$  is not easy, but with various software tools, one can determine that the answer is

7,427,466,391

Then if you visited the website [www.7427466391.com](http://www.7427466391.com), you were presented with an even more difficult puzzle. Figuring this second puzzle out took you to a web page that asks you, to submit your CV to Google. The ad campaign got high attention.

Presumably Google's conceit was that if you're smart enough to solve the puzzles, you're smart enough to work for them. Of course some days after the launch, anyone who really wanted to discover the answers without incurring a headache could merely do a Google search for them, since many solvers immediately posted their solutions online.<sup>88</sup>

#### 3.8.5.2 Contact [movie, 1997] – Primes helping to contact aliens

The movie, directed by Robert Zemeckis, originated from Carl Sagan's book with the same title.

After years of unavailing search the radio astronomer Dr. Ellie Arroway (Jodie Foster) discovers signals from the solar system Vega, 26 light years away. These signals contain the primes in the right order and without a gap. This makes the hero confident, that this message is different from the radio signals which permanently hit earth and which are random and of cosmic origin (radio galaxies, pulsars). In an unmasking scene a politician asks her after that, why these intelligent aliens didn't just speak English ...

Doing communication with absolute strange and unknown beings from deep space is very hard especially because of 2 reasons: First, the big distance and therefore the long transfer

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<sup>85</sup>Further curious things about primes may be found at:

- <http://primes.utm.edu/curios/home.php>  
- <http://www.primezahlen.de>.

<sup>86</sup>The base of the natural logarithm  $e$  is approximately 2.718 281 828 459. This is one of the most important numbers in all of mathematics like complex analysis, finance, physics and geometry. Now it was used the first time – as far as I know – for marketing or recruitment.

<sup>87</sup>Most of this information is taken from the article “e-number crunching” by John Allen Paulos in TheGuardian, Sept. 30, 2004, and from the web:

- <https://mkaz.tech/google-billboard-problems.html>  
- <http://epramono.blogspot.com/2004/10/7427466391.html>  
- <http://mathworld.wolfram.com/news/2004-10-13/google/>.

<sup>88</sup>The second level of the puzzle, which involved finding the 5th term of a given number sequence had nothing to do with primes any more.

time make it impossible to exchange within an average lifetime more than one message in each direction. Secondly the first contact must give the receiver of the radio signals a good chance to notice the message and to categorize it as something from intelligent beings. Therefore the aliens send numbers at the beginning of their message, which can be considered as the easiest part of any higher language, and which are not too trivial: So they chose the sequence of primes. These special numbers play such a fundamental role in mathematics that one can assume that they are well known to each species who has the technical know-how to receive radio waves.

The aliens then send a plan to build a mysterious machine ...

### 3.9 Appendix: Number of prime numbers in various intervals

Ten-sized intervals		Hundred-sized intervals		Thousand-sized intervals	
Interval	Number	Interval	Number	Interval	Number
1-10	4	1-100	25	1-1000	168
11-20	4	101-200	21	1001-2000	135
21-30	2	201-300	16	2001-3000	127
31-40	2	301-400	16	3001-4000	120
41-50	3	401-500	17	4001-5000	119
51-60	2	501-600	14	5001-6000	114
61-70	2	601-700	16	6001-7000	117
71-80	3	701-800	14	7001-8000	107
81-90	2	801-900	15	8001-9000	110
91-100	1	901-1000	14	9001-10000	112

Table 3.5: How many primes exist within the first intervals of tens, of hundreds and of thousands?

Dimension	Interval	Number	Average number per 1000
4	1 - 10,000	1,229	122.900
5	1 - 100,000	9,592	95.920
6	1 - 1,000,000	78,498	78.498
7	1 - 10,000,000	664,579	66.458
8	1 - 100,000,000	5,761,455	57.615
9	1 - 1,000,000,000	50,847,534	50.848
10	1 - 10,000,000,000	455,052,512	45.505

Table 3.6: How many primes exist within the first intervals of dimensions?

A visualization of the number of primes in higher intervals of powers of 10 can be found in chapter [3.13](#) at page [107](#).

### 3.10 Appendix: Indexing prime numbers ( $n$ -th prime number)

Index	Precise value	Rounded value	Comment
1	2	2	
2	3	3	
3	5	5	
4	7	7	
5	11	11	
6	13	13	
7	17	17	
8	19	19	
9	23	23	
10	29	29	
100	541	541	
1,000	7,917	7,917	
664,559	9,999,991	9.99999E+06	All prime numbers up to 1E+07 were known at the beginning of the 20th century.
1E+06	15,485,863	1.54859E+07	
6E+06	104,395,301	1.04395E+08	This prime was discovered in 1959.
1E+07	179,424,673	1.79425E+08	
1E+09	22,801,763,489	2.28018E+10	
1E+12	29,996,224,275,833	2.99962E+13	

Table 3.7: List of particular  $n$ -th prime numbers  $P(n)$

#### Comment:

With gaps, extremely large prime numbers were discovered at an early stage.

#### Web links:

<https://primes.utm.edu/nthprime/>  
[https://primes.utm.edu/notes/by\\_year.html](https://primes.utm.edu/notes/by_year.html).

### 3.11 Appendix: Orders of magnitude / dimensions in reality

In the description of cryptographic protocols and algorithms, numbers occur that are so large or so small that they are inaccessible to our intuitive understanding. It may therefore be useful to provide comparative numbers from the real world around us so that we can develop a feeling for the security of cryptographic algorithms. Some of the numbers listed below originate from [Sch96c] and [Sch96b, p.18].

Probability that you will be hijacked on your next flight	$5.5 \cdot 10^{-6}$
Annual probability of being hit by lightning	$10^{-7}$
Probability of 6 correct numbers in the lottery	$7.1 \cdot 10^{-8}$
Risk of being hit by a meteorite	$1.6 \cdot 10^{-12}$
Time until the next ice age (in years)	$14,000 = (2^{14})$
Time until the sun dies (in years)	$10^9 = (2^{30})$
Age of the earth (in years)	$10^9 = (2^{30})$
Age of the universe (in years)	$10^{10} = (2^{34})$
Number of molecules within one waterdrop	$10^{20} = (2^{63})$
Number of bacteria living on earth	$10^{30.7} = (2^{102})$
Number of the earth's atoms	$10^{51} = (2^{170})$
Number of the sun's atoms	$10^{57} = (2^{190})$
Number of atoms in the universe (without dark material)	$10^{77} = (2^{265})$
Volume of the universe (in $cm^3$ )	$10^{84} = (2^{280})$

Table 3.8: Likelihoods and dimensions from physics and everyday life

### 3.12 Appendix: Special values of the binary and decimal system

These values can be used to evaluate from a key length in bit the corresponding number of possible keys and the search effort (if assumed, that e.g. one million keys can be tested within one second).

Binary system	Decimal system
$2^{10}$	1024
$2^{40}$	$1.09951 \cdot 10^{12}$
$2^{56}$	$7.20576 \cdot 10^{16}$
$2^{64}$	$1.84467 \cdot 10^{19}$
$2^{80}$	$1.20893 \cdot 10^{24}$
$2^{90}$	$1.23794 \cdot 10^{27}$
$2^{112}$	$5.19230 \cdot 10^{33}$
$2^{128}$	$3.40282 \cdot 10^{38}$
$2^{150}$	$1.42725 \cdot 10^{45}$
$2^{160}$	$1.46150 \cdot 10^{48}$
$2^{192}$	$6.27710 \cdot 10^{57}$
$2^{250}$	$1.80925 \cdot 10^{75}$
$2^{256}$	$1.15792 \cdot 10^{77}$
$2^{320}$	$2.13599 \cdot 10^{96}$
$2^{512}$	$1.34078 \cdot 10^{154}$
$2^{768}$	$1.55252 \cdot 10^{231}$
$2^{1024}$	$1.79769 \cdot 10^{308}$
$2^{2048}$	$3.23170 \cdot 10^{616}$

Table 3.9: Special values of the binary and decimal systems

Such tables can easily be calculated using computer algebra systems. Here is a code sample for SageMath:

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**SageMath sample 3.1** Special values of the binary and decimal systems

---

```
E = [10, 40, 56, 64, 80, 90, 112, 128, 150, 160, 192, 256, 1024, 2048]
for e in E:
    # print "2^" + str(e), "---", 1.0*(2^e)
    print "2^%4d" % e , " --- ", RR(2^e).n(24)
....:
2^ 10   ---  1024.00
2^ 40   ---  1.09951e12
2^ 56   ---  7.20576e16
2^ 64   ---  1.84467e19
2^ 80   ---  1.20893e24
2^ 90   ---  1.23794e27
2^ 112  ---  5.19230e33
2^ 128  ---  3.40282e38
2^ 150  ---  1.42725e45
2^ 160  ---  1.46150e48
2^ 192  ---  6.27710e57
2^ 256  ---  1.15792e77
2^1024  ---  1.79769e308
2^2048  ---  3.23170e616
```

---

## 3.13 Appendix: Visualization of the quantity of primes in higher ranges

### The distribution of primes

Between 1 and 10 there are 4 primes. Between  $10^3$  and  $10^4$  there are already 1061 primes. In the interval  $[10^9, 10^{10}]$  lie  $404,204,977 \approx 4 \cdot 10^8$  primes, and in the interval from  $10^{19}$  to  $10^{20}$  there are  $1,986,761,935,284,574,233 \approx 1,9 \cdot 10^{18}$  primes.<sup>89</sup>

Why is the difference between the number of primes in the different intervals so big, although the boundaries of the intervals differ only by value 1 of the exponent of the power of 10?

### The prime number theorem

The number  $\text{PI}(x)$  of primes up to a given number  $x$  can approximately be determined by a formula, derived from the so called prime number theorem (see chapter 3.7).  $\text{PI}(x)$  denotes the number of primes which are smaller or equal to  $x$ . Then the formula is

$$\text{PI}(x) \sim \frac{x}{\ln x}.$$

Note, that this formula only gives an approximation of the number of primes smaller or equal to  $x$ . It's getting more exact as the number  $x$  increases.

In the following we are using the prime number theorem to examine the distribution of primes.

To understand, why the number of primes is growing so rapidly, although the boundaries of the intervals only differ by the exponent 1, let's have a closer look to both components of the right side of the formula:  $x$  and  $\ln x$ .

### The functions $x$ and $10^x$

The function  $x$  is a straight line. It is shown in figure 3.6a on page 106.

In the next step the function of the boundaries of the intervals are drawn in figure 3.6b on page 106. To get an idea of how the functions look like, the domain of definition was chosen to be from 0 to  $10^{10}$  and from 0 to 10, respectively. You can see, that with increasing exponent  $x$  the numbers grow stronger.

### The function $\ln x$

In comparison to that we consider the function  $\ln x$ . The left picture of figure 3.7 on page 106 shows the graph with the domain of definition from 1 to 100. On the right picture the domain of definition was chosen between 1 and  $10^{10}$ .

One can see that the values of the function  $\ln x$  grow slowly compared to the growth of the function  $x$ . This is visualized by the graph of both functions in one picture shown in figure 3.8 on page 106. In addition to that the graph of the function  $\frac{x}{\ln x}$  was drawn in the same figure.

<sup>89</sup>[http://en.wikipedia.org/wiki/Prime\\_number\\_theorem](http://en.wikipedia.org/wiki/Prime_number_theorem)

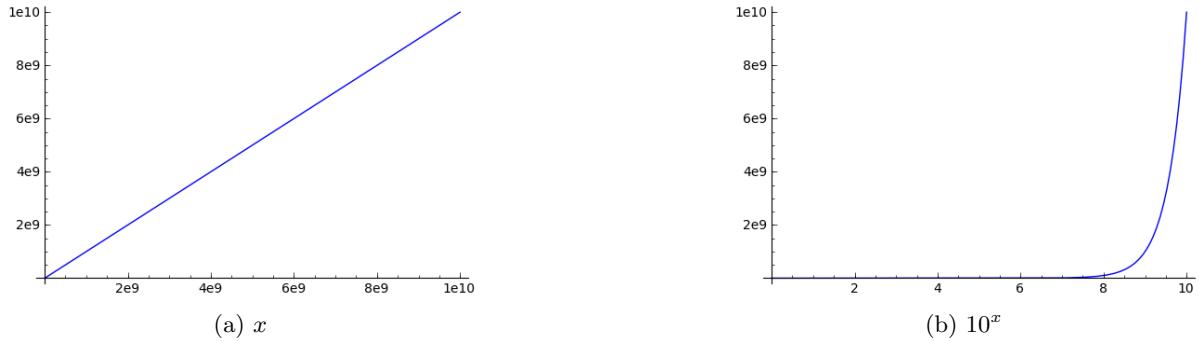


Figure 3.6: Graph of the functions  $x$  and  $10^x$

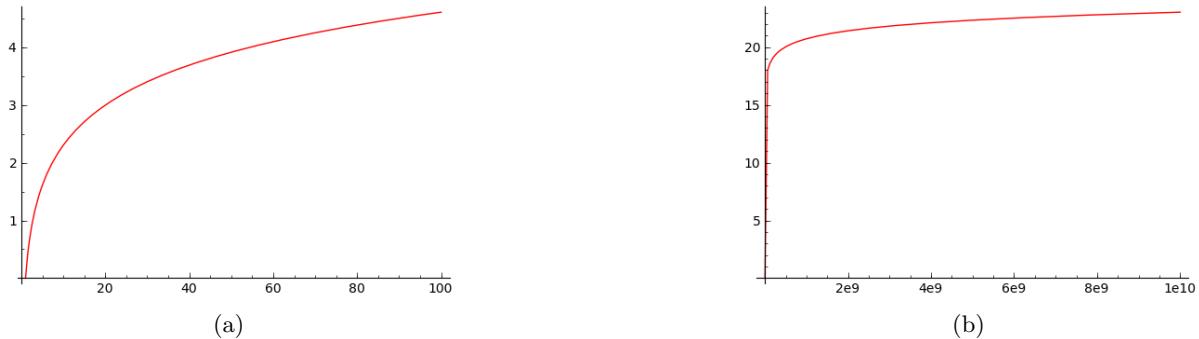


Figure 3.7: Graph of the function  $\ln x$  till 100 and till  $10^{10}$

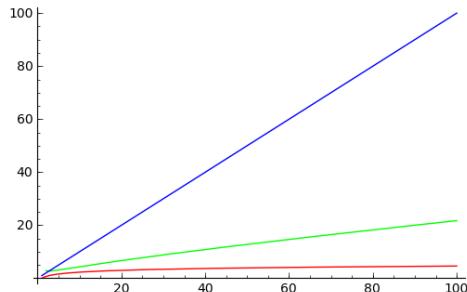


Figure 3.8: The functions  $x$  (blue),  $\ln x$  (red) and  $\frac{x}{\ln x}$  (green)

**The function  $PI(x) = \frac{x}{\ln x}$**

The function  $\frac{x}{\ln x}$  consists of the function  $x$  as the numerator and the function  $\ln x$  in the denominator, which, in comparison to  $x$ , increases very slowly. Compared to the number  $x$  itself, the number of primes less or equal to  $x$  is small. But still,  $\frac{x}{\ln x}$  is an increasing function as you can see in figure 3.8 on page 106.

### The number of primes in the different intervals

Figure 3.9 visualizes how the number of primes in the intervals  $[1, 10^x]$  and  $[10^{x-1}, 10^x]$  behave. To calculate it faster, the result of the approximation function is used (not the exact numbers like in the tables in chapter 3.9).

Here for each base 10 exponent two bars are drawn:  $\frac{10^x}{\ln 10^x}$  and  $\frac{10^x}{\ln 10^x} - \frac{10^{x-1}}{\ln 10^{x-1}}$ : The left chart shows the values for the exponents  $x$  from 1 to 5, and the right one for  $x$  from 1 to 10, where  $x$  is the base 10 exponent.

The blue bars represent the overall number of primes up to  $10^x$ . The red bars show how many primes accrue in the interval  $[10^{x-1}, 10^x]$ , respectively. This makes clear, that the number of primes in intervals of higher exponents keeps growing quite fast.

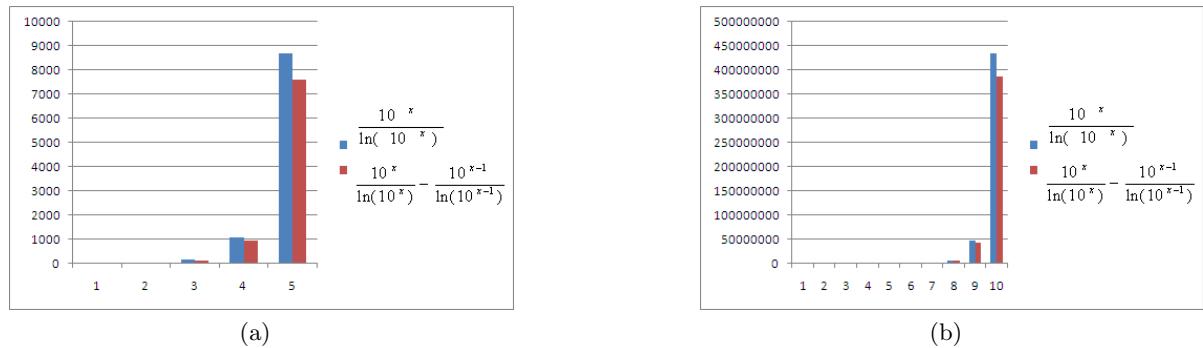


Figure 3.9: Numbers of primes in the interval  $[1, 10^x]$  (blue) and in the interval  $[10^{x-1}, 10^x]$  (red) for different exponents  $x$

A table containing the number of primes in some dedicated intervals can be found in chapter 3.9 at page 101: For example, within the interval  $[1, 10^4]$  there are 1229 primes; thereof are in the interval  $[10^3, 10^4]$   $1229 - 168 = 1061$  primes.

Theory about the prime number theorem and the function  $\text{PI}(x)$  can be found in chapter 3.7.

---

**SageMath sample 3.2** Generation of the graphs of the three functions  $x$ ,  $\log(x)$  and  $x/\log(x)$ 

---

```
# Definition of function f(x)=x and plots for the domains from 0 to 10^10 and 0 to 100
sage: def f(x):return x
....:
sage: F=plot(f,(0,10^10))
sage: F.plot()

sage: F2=plot(f,(1,100))
sage: F2.plot()

# Definition of function g(x)=10^x and plots for the domain from 0 to 10
sage: def g(x): return 10^x
....:
sage: G=plot(g,(0,10))
sage: G.plot()

# Definition of function h(x)=log(x) and plots for the domains from 1 to 100 and 1 to 10^10
sage: def h(x): return log(x)
....:
sage: H=plot(h,(1,100),color="red")
sage: H.plot()

sage: H2=plot(h,(1,10^10),color="red")
sage: H2.plot()

# Definition of function k(x)=x/log(x) and plots for the domain from 2 to 100
sage: def k(x): return x/log(x)
....:
sage: K=plot(k,(2,100),color="green")
sage: K.plot()

# Plots of the functions f, k and h for the domain of definition up to 100
sage: F2+K+H

# Generation of the data for the bar charts .....
# Determination of the number of primes in the interval [1,10]
sage: pari(10).primepi()-pari(1).primepi()
4

# Determination of the number of primes in the interval [10^3,10^4]
sage: pari(10**4).primepi()-pari(10**3).primepi()
1061

# Determination of the number of primes in the interval [10^8,10^9]
sage: pari(10**9).primepi()-pari(10**8).primepi()
45086079

# (for 10^10: OverflowError: long int too large to convert)
```

---

## 3.14 Appendix: Examples using SageMath

Below is SageMath source code related to contents of the chapter 3 (“Prime Numbers”).

### 3.14.1 Some basic functions about primes using SageMath

This part of the appendix contains SageMath code, to perform some simple computations about primes.<sup>90</sup>

---

#### SageMath sample 3.3 Some basic functions about primes

---

```
# primes (general commands)
# The set of prime numbers
sage: P=Primes(); P
Set of all prime numbers: 2, 3, 5, 7, ...

# Returns the next prime number
sage: next_prime(5)
7

# Returns how many primes <=x are there
sage: pari(10).primepi()
4

# Returns the first x primes
sage: primes_first_n(5)
[2, 3, 5, 7, 11]

# Returns the primes in an interval
sage: list(primes(1,10))
[2, 3, 5, 7]
```

---

<sup>90</sup> See the SageMath documentation about elementary number theory [http://doc.sagemath.org/html/en/constructions/number\\_theory.html](http://doc.sagemath.org/html/en/constructions/number_theory.html).

### 3.14.2 Check primality of integers generated by quadratic functions

The following SageMath code verifies the primality of integers generated by the function  $f(n) = n^2 - 9n + 61$ . The code defines a function called `quadratic_prime_formula()` that takes three arguments:

- `start` — An integer which is the lower bound for integers in the sequence `start, start + 1, start + 2, ..., end - 1, end`.
- `end` — An integer which is the upper bound for the integers in the sequence `start, start + 1, start + 2, ..., end - 1, end`.
- `verbose` — (default: `True`) a flag to signify whether to print a message indicating the primality of an integer generated by  $f(n)$ .

A meaningful modification of this code is to use another function, of which the primality of its function values should be checked.

---

#### SageMath sample 3.4 Verify the primality of integers generated by a quadratic function

---

```
def quadratic_prime_formula(start, end, verbose=True):
    print "N -- N^2 - 9*N + 61"
    P = 0 # the number of primes between start and end
    for n in xrange(start, end + 1):
        X = n^2 - 9*n + 61
        if is_prime(X):
            P += 1
            if verbose:
                print str(n) + " -- " + str(X) + " is prime"
        else:
            if verbose:
                print str(n) + " -- " + str(X) + " is NOT prime"
    print "Number of primes: " + str(P)
    print "Percentage of primes: " + str(float((P * 100) / (end - start + 1)))
```

---

With the following function call we compute the values of  $f(n) = n^2 - 9n + 61$  for  $n = 0, 1, 2, \dots, 50$  and verify the primality of the generated integers:

```
sage: quadratic_prime_formula(0, 50)
N -- N^2 - 9*N + 61
0 -- 61 is prime
1 -- 53 is prime
2 -- 47 is prime
3 -- 43 is prime
4 -- 41 is prime
5 -- 41 is prime
6 -- 43 is prime
7 -- 47 is prime
8 -- 53 is prime
9 -- 61 is prime
10 -- 71 is prime
11 -- 83 is prime
12 -- 97 is prime
13 -- 113 is prime
14 -- 131 is prime
15 -- 151 is prime
```

```

16 -- 173 is prime
17 -- 197 is prime
18 -- 223 is prime
19 -- 251 is prime
20 -- 281 is prime
21 -- 313 is prime
22 -- 347 is prime
23 -- 383 is prime
24 -- 421 is prime
25 -- 461 is prime
26 -- 503 is prime
27 -- 547 is prime
28 -- 593 is prime
29 -- 641 is prime
30 -- 691 is prime
31 -- 743 is prime
32 -- 797 is prime
33 -- 853 is prime
34 -- 911 is prime
35 -- 971 is prime
36 -- 1033 is prime
37 -- 1097 is prime
38 -- 1163 is prime
39 -- 1231 is prime
40 -- 1301 is prime
41 -- 1373 is prime
42 -- 1447 is prime
43 -- 1523 is prime
44 -- 1601 is prime
45 -- 1681 is NOT prime
46 -- 1763 is NOT prime
47 -- 1847 is prime
48 -- 1933 is prime
49 -- 2021 is NOT prime
50 -- 2111 is prime
Number of primes: 48
Percentage of primes: 94.1176470588

```

The last two lines of the output contain a small statistics. You can see that  $f(n)$  generates 48 primes when  $0 \leq n \leq 50$ , which is approximately 94% of the values generated by  $f(n)$ .

For larger sequences, it is impractical to print all single messages indicating the primality of integers. In the following SageMath session, only the statistics at the end is printed (by setting the verbose parameter to false): the overall number of primes and the percentage of primes, generated by  $f(n)$  where  $0 \leq n \leq 1000$ .

```

sage: quadratic_prime_formula(0, 1000, False)
N -- N^2 - 9*N + 61
Number of primes: 584
Percentage of primes: 58.3416583417

```

# Bibliography

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[http://bscw.schule.de/pub/nj\\_bscw.cgi/d864891/RSA\\_u\\_Co\\_NF4.pdf](http://bscw.schule.de/pub/nj_bscw.cgi/d864891/RSA_u_Co_NF4.pdf).
- [WS10b] Witten, Helmut and Ralph Hardo Schulz: *RSA & Co. in der Schule: Moderne Kryptologie, alte Mathematik, raffinierte Protokolle. NF Teil 5: Der Miller-Rabin-Primzahltest oder: Falltüren für RSA mit Primzahlen aus Monte Carlo.* LOG IN, (166/167):92–106, 2010.  
[http://bscw.schule.de/pub/nj\\_bscw.cgi/d864895/RSA\\_u\\_Co\\_NF5.pdf](http://bscw.schule.de/pub/nj_bscw.cgi/d864895/RSA_u_Co_NF5.pdf).

All links have been confirmed at July 11, 2016.

# Web links

1. GIMPS (Great Internet Mersenne Prime Search)  
www.mersenne.org is the home page of the GIMPS project,  
<http://www.mersenne.org/primes/>
2. The Proth Search Page with the Windows program by Yves Gallot  
<http://primes.utm.edu/programs/gallot/index.html>
3. Generalized Fermat Prime Search  
<http://primes.utm.edu/top20/page.php?id=12>
4. Distributed Search for Fermat Number Divisors  
<http://www.fermatsearch.org/>
5. The University of Tennessee hosts extensive research results about prime numbers.  
<http://www.utm.edu/>
6. The best overview about prime numbers is offered from my point of view by “The Prime Pages” from professor Chris Caldwell.  
<http://primes.utm.edu/>
7. Descriptions e.g. about prime number tests  
<http://www.utm.edu/research/primes/mersenne.shtml>  
<http://primes.utm.edu/prove/index.html>
8. Showing the  $n$ -th prime number  $P(n)$   
[https://primes.utm.edu/notes/by\\_year.html](https://primes.utm.edu/notes/by_year.html)  
<https://primes.utm.edu/nthprime/>
9. The supercomputer manufacturer SGI Cray Research not only employed brilliant mathematicians but also used the prime number tests as benchmarks for its machines.  
[http://www.isthe.com/chongo/tech/math/prime/prime\\_press.html](http://www.isthe.com/chongo/tech/math/prime/prime_press.html)
10. The Cunningham Project  
<http://www.cerias.purdue.edu/homes/ssw/cun/>
11. EFF Cooperative Computing Awards  
<http://www.eff.org/awards/coop>
12. Goldbach conjecture verification project by Tomás Oliveira e Silva,  
<http://sweet.ua.pt/tos/goldbach.html>
13. Kurt Gödel  
<https://www.mathematik.ch/mathematiker/goedel.php>

All links have been confirmed at July 11, 2016.

## **Acknowledgments**

I would like to take this opportunity to thank Mr. Henrik Koy and Mr. Roger Oyono for their very constructive proof-reading of the first versions of this article.

## Chapter 4

# Introduction to Elementary Number Theory with Examples

([Bernhard Esslinger](#), Jul 2001; Updates: Dec 2001, Jun 2002, May 2003, May 2005, Mar 2006, Jun 2007, Jul 2009, Jan 2010, Aug 2013, Jul 2016)

This “introduction” is for people with a mathematical interest. There is no more pre-knowledge necessary than what you learn in the secondary school.

We intentionally had “beginners” in mind; we did not take the approach of mathematical textbooks, called “introduction”, which cannot be understood at the first reading further than page 5 and which have the real purpose to deliver all information that special monographs can be read.

### 4.1 Mathematics and cryptography

A large proportion of modern, asymmetric cryptography is based on mathematical knowledge – on the properties (“laws”) of whole numbers, which are investigated in elementary number theory. Here, the word “elementary” means that questions raised in number theory are essentially rooted in the set of natural and whole numbers.

Further mathematical disciplines currently used in cryptography include (see [[Bau95](#), p. 2], [[Bau00](#), p. 3]) :

- Group theory
- Combination theory
- Complexity theory
- Stochastic (ergodic theory)
- Information theory.

Number theory or arithmetic (the emphasis here is more on the aspect of performing calculations with numbers) was established by Gauss<sup>1</sup> as a special mathematical discipline. Its

---

<sup>1</sup>Carl Friedrich Gauss, German mathematician and astronomer, Apr 30, 1777–Feb 23, 1855.

elementary features include the greatest common divisor<sup>2</sup> ( $\gcd$ ), congruence (remainder classes), factorization, the Euler-Fermat theorem and primitive roots. However, the most important aspect is prime numbers and their multiplicative operation.

For a long time, number theory was considered to be the epitome of pure research, the ideal example of research in the ivory tower. It delved into “the mysterious laws of the realm of numbers”, giving rise to philosophical considerations as to whether it described elements that exist everywhere in nature or whether it artificially constructed elements (numbers, operators and properties).

We now know that patterns from number theory can be found everywhere in nature. For example, the ratio of rotating counterclockwise and rotating clockwise spirals in a sunflower is equal to two consecutive Fibonacci numbers<sup>3</sup>, for example 21 : 34.

Also, at the latest when number theory was applied in modern cryptography, it became clear that a discipline that had been regarded as purely theoretical for centuries actually had a practical use. Today, experts in this field are in great demand on the job market.

Applications in (computer) security now use cryptography because this mathematical discipline is simply better and easier to prove than all other “creative” substitution procedures that have been developed over the course of time and better than all sophisticated physical methods such as those used to print bank notes [Beu96, p. 4].

This article explains the basics of elementary number theory in a way that you can easily understand. It provides numerous examples and very rarely goes into any proofs (these can be found in mathematical textbooks).

The goal is not to exhaustively explain the number theory findings, but to show the essential procedures. The volume of the content is so oriented that the reader can understand and apply the RSA method.

For this purpose we will use both theory and examples to explain how to perform calculations in finite sets and describe how these techniques are applied in cryptography. Particular attention will be paid to the traditional Diffie-Hellman (DH) and RSA public key procedures.<sup>4</sup>

It was important to me to make verifiable statements about the security of the RSA algorithm, and to add SageMath code for as much as possible examples.

---

<sup>2</sup>This article deals with the  $\gcd$  (greatest common divisor) in appendix 4.14.

<sup>3</sup>The sequence of Fibonacci numbers  $(a_i)_{i \in \mathbb{N}}$  is defined by the “recursive” rule  $a_1 := a_2 := 1$  and for all numbers  $n = 1, 2, 3, \dots$  we define  $a_{n+2} := a_{n+1} + a_n$ . This historical sequence can be found in many interesting forms in nature (for example, see [GKP94, p. 290 ff] or the website of [Ron Knott](#), which is devoted to Fibonacci numbers). A lot is known about the Fibonacci sequence and it is used today as an important tool in mathematics.

<sup>4</sup>The same intention has the series *RSA & Co. at school: New series*. Links to the articles plus a short abstract can be found at <https://www.cryptoportal.org/>, menu “Linksammlung”, catchword “rsa”. Unfortunately these are currently only available in German.

Mathematics is the queen of sciences and number theory is the queen of mathematics.

Quote 7: Carl Friedrich Gauss

## 4.2 Introduction to number theory<sup>5</sup>

Number theory arose from interest in positive whole numbers  $1, 2, 3, 4, \dots$ , also referred to as the set of natural numbers *natural numbers*  $\mathbb{N}$ . These are the first mathematical constructs used by human civilization. According to Kronecker<sup>6</sup>, they are a creation of God. In Dedekind's<sup>7</sup> opinion, they are a creation of the human intellect. Dependent upon one's ideology, this is an unsolvable contradiction or one and the same thing.

In ancient times, no distinction was made between number theory and numerology, which attributed a mystical significance to specific numbers. In the same way as astronomy and chemistry gradually detached themselves from astrology and alchemy during the Renaissance (from the 14th century), number theory also separated itself from numerology.

Number theory has always been a source of fascination – for both amateurs and professional mathematicians. In contrast to other areas of mathematics, many of the problems and theorems in number theory can be understood by non-experts. On the other hand, the solutions to these problems or the prove to the theorems often resisted to the mathematicians for a very long time. It is therefore one thing to pose good questions but quite another matter to find the answer. One example of this is what is known as Fermat's Last (or large) theorem.<sup>8</sup>

Up until the mid 20th century, number theory was considered to be the purest area of mathematics, an area that had no practical use in the real world. This changed with the development of computers and digital communication, as number theory was able to provide several unexpected solutions to real-life tasks. At the same time, advances in information technology allowed specialists in number theory to make huge progress in factorizing large numbers, finding new prime numbers, testing (old) conjectures and solving numerical problems that were previously impossible to solve. Modern number theory is made up of areas such as:

- Elementary number theory
- Algebraic number theory
- Analytic number theory
- Geometric number theory
- Combinatorial number theory
- Numeric number theory

---

<sup>5</sup>A didactical very well prepared article about the elementary number theory can be found within the series *RSA & Co. at school: Modern cryptology, old mathematics, and subtle protocols*: see NF part 3 [WS08]. Unfortunately these are currently only available in German.

<sup>6</sup>Leopold Kronecker, German mathematician, Dec 7, 1823 – Dec 29, 1891

<sup>7</sup>Julius Wilhelm Richard Dedekind, German mathematician, Oct 6, 1831 – Feb 12, 1916.

<sup>8</sup>One of the things we learn in mathematics at school is Pythagoras' theorem, which states the following for a right-angle triangle:  $a^2 + b^2 = c^2$ , where  $a$  and  $b$  are the lengths of the sides containing the right angle and  $c$  is the length of the hypotenuse.

Fermat famously proposed that  $a^n + b^n \neq c^n$  for  $a, b, c \in \mathbb{N}$  and whole-number exponents  $n > 2$ . Unfortunately, the border of his book from Diophant where he made the claim did not have enough space for him to prove it. The theorem was not proven until over 300 years later [Wil95, p. 433-551].

- Probability theory.

All of the different areas are concerned with questions regarding whole numbers (both positive and negative whole numbers plus zero). However, they each have different methods of dealing with them.

This article only deals with the area of elementary number theory.

#### 4.2.1 Convention

Unless stated otherwise:

- The letters  $a, b, c, d, e, k, n, m, p, q$  are used to present whole numbers.
- The letters  $i$  and  $j$  represent natural numbers.
- The letters  $p$  always represents a prime number.
- The sets  $\mathbb{N} = \{1, 2, 3, \dots\}$  and  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$  are the *natural* and *whole* numbers respectively.

This isn't magic – it's logic – a puzzle. A lot of the greatest wizards haven't got an ounce of logic.

Quote 8: Joanne K. Rowling<sup>9</sup>

## 4.3 Prime numbers and the first fundamental theorem of elementary number theory

Many of the problems in elementary number theory are concerned with prime numbers (see chapter 3).

Every whole number has divisors or factors. The number 1 has just one – itself, whereas the number 12 has the six factors 1, 2, 3, 4, 6 and 12.<sup>10</sup> Many numbers are only divisible by themselves and by 1. When it comes to multiplication, these can be regarded as the “atoms” in the realm of numbers.

**Definition 4.3.1.** Prime numbers are natural numbers greater than 1 that can only be divided by 1 and themselves.

By definition, 1 is not a prime number.

If we write down the prime numbers in ascending order (prime number sequence), then we get:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, ⋯ .

The first 100 numbers include precisely 25 prime numbers. After this, the percentage of primes decreases, but never reaches zero.

We come across whole numbers that are prime fairly often. In the last decade only, three years were prime: 1993, 1997 and 1999. If they were rare, cryptography would not be able to work with them to the extent it does.

Prime numbers can be factorized in a unique (“trivial”) way:

$$\begin{aligned} 5 &= 1 * 5 \\ 17 &= 1 * 17 \\ 1013 &= 1 * 1013 \\ 1,296,409 &= 1 * 1,296,409. \end{aligned}$$

**Definition 4.3.2.** Natural numbers greater than 1 that are not prime are called composite numbers. These have at least two factors other than 1.

Examples of the decomposition (the dissection of a number into its prime factors is called

<sup>9</sup> Joanne K. Rowling, “Harry Potter and the Philosopher’s Stone”, Bloomsbury, (c) 1997, chapter “Through the trapdoor”, p. 307, by Hermine.

<sup>10</sup> Due to the fact that 12 has so many factors, this number – and multiples of this number – is often found in everyday life: the 12-hour scale on clocks, the 60 minutes in an hour, the 360-degree scale for measuring angles, etc. If we divide these scales into segments, the segments often turn out to be whole numbers. These are easier to use in mental arithmetic than fractions.

factorization) of such numbers into prime factors:

$$\begin{aligned}
 4 &= 2 * 2 \\
 6 &= 2 * 3 \\
 91 &= 7 * 13 \\
 161 &= 7 * 23 \\
 767 &= 13 * 59 \\
 1029 &= 3 * 7^3 \\
 5324 &= 22 * 11^3.
 \end{aligned}$$

**Theorem 4.3.1.** *Each composite number  $a$  has a lowest factor greater than 1. This factor is a prime number  $p$  and is less than or equal to the square root of  $a$ .*

All whole numbers greater than 1 can be expressed as a product of prime numbers — in a *unique* way.

This is the claim of the 1st *fundamental theorem of number theory* (= fundamental theorem of arithmetic = fundamental building block of all positive integers). This was formulated precisely for the first time by Carl Friedrich Gauss in his *Disquisitiones Arithmeticae* (1801).

**Theorem 4.3.2. Gauss 1801** *Every even natural number greater than 1 can be written as the product of prime numbers. Given two such decompositions  $a = p_1 * p_2 * \dots * p_n = q_1 * q_2 * \dots * q_m$ , these can be resorted such that  $n = m$  and, for all  $i$ ,  $p_i = q_i$ .*

In other words: Each natural number other than 1 can be written as a product of prime numbers in precisely one way, if we ignore the order of the factors. The factors are therefore unique (the “expression as a product of factors” is unique)!

For example,  $60 = 2 * 2 * 3 * 5 = 2^2 * 3 * 5$ . And this — other than changing the order of the factors — is the only way in which the number 60 can be factorized.

If you allow numbers other than primes as factors, there are several ways of factorizing integers and the *uniqueness* is lost:

$$60 = 1 * 60 = 2 * 30 = 4 * 15 = 5 * 12 = 6 * 10 = 2 * 3 * 10 = 2 * 5 * 6 = 3 * 4 * 5 = \dots$$

The 1st fundamental theorem only appears to be obvious. We can construct numerous other sets of numbers<sup>11</sup> for which numbers in the set *cannot* be expressed uniquely as a product of the prime numbers of the set.

In order to make a mathematical statement, therefore, it is important to state not only the operation for which it is defined but also the basic set on which the operation is defined.

For more details on prime numbers (e.g. how “Fermat’s Little Theorem” can be used to test extremely large numbers to determine whether they are prime), please refer to the article on prime numbers, chapter 3 in this script.

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<sup>11</sup>These sets are formed especially from the set of natural numbers. An example of this can be found in this script on page 70 at the end of chapter 3.2.

## 4.4 Divisibility, modulus and remainder classes<sup>12</sup>

If whole numbers are added, subtracted or multiplied, the result is always another whole number.

The division of two whole numbers does not always result in a whole number. For example, if we divide 158 by 10 the result is the decimal number 15.8, which is not a whole number!

If, however, we divide 158 by 2 the result 79 is a whole number. In number theory we express this by saying that 158 is *divisible* by 2 but not by 10. In general, we say:

**Definition 4.4.1.** A whole number  $n$  is **divisible** by a whole number  $d$  if the quotient  $n/d$  is a whole number  $c$  such that  $n = c * d$ .

$n$  is called a *multiple* of  $d$ , whereas  $d$  is called a *divisor* or *factor* of  $n$ .

The mathematical notation for this is  $d|n$  (read “ $d$  divides  $n$ ”). The notation  $d\nmid n$  means that  $d$  does not divide the number  $n$ .

In our example therefore:  $10|158$  but  $2\nmid158$ .

### 4.4.1 The modulo operation – working with congruence

When we investigate divisibility, it is only the remainder of the division that is important. When dividing a number  $n$  by  $m$ , we often use the following notation:

$$\frac{n}{m} = c + \frac{r}{m},$$

where  $c$  is a whole number and  $r$  is a number with the values  $0, 1, \dots, m - 1$ . This notation is called division with remainder, whereby  $c$  is called the whole-number “quotient” and  $r$  is the “remainder” of the division.

**Example:**

$$\frac{19}{7} = 2 + \frac{5}{7} \quad (m = 7, \quad c = 2, \quad r = 5)$$

What do the numbers 5, 12, 19, 26, … have in common for division by 7? The remainder is always  $r = 5$ . For division by 7, only the following remainders are possible:

$$r = 0, 1, 2, \dots, 6$$

If  $r = 0$ , then:  $m|n$  (“ $m$  divides  $n$ ”).

The numbers that result in the same remainder  $r$  when divided by 7 are combined to form the “remainder class  $r$  modulo 7”. Two numbers  $a$  and  $b$  belonging to the same remainder class modulo 7 are said to be “congruent modulo 7”. Or in general:

**Definition 4.4.2.** The **remainder class  $r$  modulo  $m$**  is the set of all whole numbers  $a$  that have the same remainder  $r$  when divided by  $m$ .

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<sup>12</sup>With the educational tool for number theory NT you can have a playful view at the calculation with congruences, discussed in this and the next chapter (see NT learning unit 2.1, pages 2-9/40).

NT can be called in CrypTool via the menu path **Indiv. Procedures \ Number Theory Interactive \ Learning Tool for Number Theory**. See appendix [A.6](#).

CT2 contains a visualisation of these methods within the tutorial “**World of Primes**”.

**Example of remainder classes RC:**

$$\text{RC } 0 \bmod 4 = \{x|x = 4 * n; n \in \mathbb{Z}\} = \{\dots, -16, -12, -8, -4, 0, 4, 8, 12, 16, \dots\}$$

$$\text{RC } 3 \bmod 4 = \{x|x = 4 * n + 3; n \in \mathbb{Z}\} = \{\dots, -13, -9, -5, -1, 3, 7, 11, 15, \dots\}$$

As only the remainders  $0, 1, 2, \dots, m-1$  are possible for division modulo  $m$ , modular arithmetic works with finite sets. For each modulo  $m$  there are precisely  $m$  remainder classes.

The result of the modulo operation can be formulated as:  $a \bmod m = a - m * \lfloor a/m \rfloor$

**Definition 4.4.3.** Two numbers  $a, b \in \mathbb{N}$  are said to be congruent modulo  $m \in \mathbb{N}$  if and only if they have the same remainder when divided by  $m$ .

We write:  $a \equiv b \pmod{m}$  (read  $a$  is congruent  $b$  modulo  $m$ ), which means that  $a$  and  $b$  belong to the same remainder class. The modulo is therefore the divisor. This notation was introduced by Gauss. Although the divisor is usually positive,  $a$  and  $b$  can also be any whole numbers.

**Example:**

$19 \equiv 12 \pmod{7}$ , because the remainders are equal:  $19/7 = 2$  remainder 5 and  $12/7 = 1$  remainder 5.

$23103 \equiv 0 \pmod{453}$ , because  $23103/453 = 51$  remainder 0 and  $0/453 = 0$  remainder 0.

**Theorem 4.4.1.**  $a \equiv b \pmod{m}$  if and only if, the difference  $(a - b)$  is divisible by  $m$ , i.e. if  $q \in \mathbb{Z}$  exists with  $(a - b) = q * m$ .<sup>13</sup>

In other words:  $a \equiv b \pmod{m} \iff m|(a - b) \iff (a - b) \equiv 0 \pmod{m}$

Therefore: If  $m$  divides the difference, there exists a whole number  $q$  such that:  $a = b + q * m$ . As an alternative to the congruence notation, we can also use the divisibility notation:  $m|(a - b)$ .

**Example of equivalent statements:**

$35 \equiv 11 \pmod{3} \iff 35 - 11 \equiv 0 \pmod{3}$ , where  $35 - 11 = 24$  is divisible by 3 without remainder while  $35 : 3$  and  $11 : 3$  leave the remainder 2.

We can apply the above equivalence in theorem 4.4.1 if we need a quick and easy method of determining whether large numbers are divisible by a certain number.

**Example:** Is 69,993 divisible by 7?

The number can be written in the form of a difference in which it is clear that each operand is divisible by 7:  $69,993 = 70,000 - 7$ . Therefore, the difference is also divisible by 7.

Although these considerations and definitions may seem to be rather theoretical, we are so familiar with them in everyday life that we no longer think about the formal procedure. For example, the 24 hours on a clock are represented by the numbers  $1, 2, \dots, 12$ . We obtain the hours after 12 noon as the remainder of a division by 12 and know immediately that 2 o'clock in the afternoon is the same as 14.00.

---

<sup>13</sup>The above equivalence does apply only to the difference  $(a - b)$ , not to the sum  $(a + b)$ !

**Example:**

$11 \equiv 2 \pmod{3}$ , therefore  $11 - 2 = 9 \equiv 0 \pmod{3}$ ; but  $11 + 2 = 13$  is not divisible by 3.

The statement in theorem 4.4.1 does not even apply to sums in one direction. It is correct for sums only if the remainder is 0 and only in the following direction: If a divisor divides both summands with no remainder, it also divides the sum with no remainder.

The “modular” arithmetic (based on division remainders) forms the basis of asymmetric encryption procedures. Cryptographic calculations are therefore not based on real numbers, as the calculations you performed mostly at school, but rather on character strings with a limited length, in other words on positive whole numbers that cannot exceed a certain value. This is one of the reasons why we choose a large number  $m$  and “calculate modulo  $m$ ”. That is, we ignore whole-number multiples of  $m$  and, rather than working with a number, we only work with the remainder when this number is divided by  $m$ . The result is that all results are in the range 0 to  $m - 1$ .

## 4.5 Calculations with finite sets

### 4.5.1 Laws of modular calculations

From algebra theorems it follows that essential parts of the conventional calculation rules are kept when we proceed to modular calculations over a basic set  $\mathbb{Z}$ . For example, addition remains commutative. The same goes for multiplication modulo  $m$ . The result of a division<sup>14</sup> is not a fraction but rather a whole number between 0 and  $m - 1$ .

The known laws apply:

#### 1. Associative law:

$$\begin{aligned} ((a + b) + c) \pmod{m} &\equiv (a + (b + c)) \pmod{m}. \\ ((a * b) * c) \pmod{m} &\equiv (a * (b * c)) \pmod{m}. \end{aligned}$$

#### 2. Commutative law:

$$\begin{aligned} (a + b) \pmod{m} &\equiv (b + a) \pmod{m}. \\ (a * b) \pmod{m} &\equiv (b * a) \pmod{m}. \end{aligned}$$

The associative law and the commutative law apply to both addition and multiplication.

#### 3. Distributive law:

$$(a * (b + c)) \pmod{m} \equiv (a * b + a * c) \pmod{m}.$$

#### 4. Reducibility:

$$\begin{aligned} (a + b) \pmod{m} &\equiv (a \pmod{m} + b \pmod{m}) \pmod{m}. \\ (a * b) \pmod{m} &\equiv (a \pmod{m} * b \pmod{m}) \pmod{m}. \end{aligned}$$

When adding or multiplying the order in which the modulo operation is performed does not matter.

#### 5. Existence of an identity (neutral element):

$$\begin{aligned} (a + 0) \pmod{m} &\equiv (0 + a) \pmod{m} \equiv a \pmod{m}. \\ (a * 1) \pmod{m} &\equiv (1 * a) \pmod{m} \equiv a \pmod{m}. \end{aligned}$$

#### 6. Existence of an inverse element<sup>15</sup>:

##### – Additive inverse

For all whole numbers  $a$  and  $m$  there exists a whole number  $-a$  such that:

$$(a + (-a)) \pmod{m} \equiv 0 \pmod{m}$$

##### – Multiplicative inverse modulo a prime p

For each whole number  $a$  (with  $a \not\equiv 0 \pmod{p}$  and  $p$  prime) there exists a whole number  $a^{-1}$ , such that:  $(a * a^{-1}) \pmod{p} \equiv 1 \pmod{p}$

##### – Multiplicative inverse modulo a compound number m<sup>16</sup>

For all whole numbers  $a$  and  $m$  (with  $a \not\equiv 0 \pmod{m}$  and  $ggT(a, m) = 1$ ) there exists a whole number  $a^{-1}$ , such that:  $(a * a^{-1}) \pmod{m} \equiv 1 \pmod{m}$

<sup>14</sup>The division modulo  $m$  is only defined for numbers co-prime to  $m$  because other numbers have the same property as zero (this means there is no inverse number). See law number 6 **existence of an inverse element**. See footnote 20 in chapter 4.6.1 and table 4.3 in chapter 4.6.2.

<sup>15</sup>An inverse element only exists, if it is unique for the given operation.

<sup>16</sup>As  $8 \equiv 3 \pmod{5}$  and  $3 * 2 \equiv 1 \pmod{5}$ , then  $2 = 3^{-1} = 8^{-1}$  is a (unique) inverse for 3 and 8.

A multiple of  $p$  or  $m$  has no inverse mod  $p$  or mod  $m$ :  $5 \equiv 10 \equiv 0 \pmod{5}$ .

**7. Closeness**<sup>17</sup>:

$$\begin{aligned} a, b \in G &\implies (a + b) \in G. \\ a, b \in G &\implies (a * b) \in G. \end{aligned}$$

**8. Transitivity:**

$$[a \equiv b \pmod m, b \equiv c \pmod m] \implies [a \equiv c \pmod m].$$

#### 4.5.2 Patterns and structures

In general mathematicians investigate “Structures”. They ask e.g. at  $a * x \equiv b \pmod m$ , which values  $x$  can take for given values of  $a, b, m$ .

Especially the case is investigated, where the result  $b$  of this operation is the neutral element. Then  $x$  is the inverse of  $a$  regarding this operation.

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<sup>17</sup>The property of closeness is always defined in relation to an operation in a set. See chapter 4.15 “Appendix: Forming closed sets”.

The way of theory is long — it is short and effective by examples.

Quote 9: Seneca<sup>18</sup>

## 4.6 Examples of modular calculations

As we have already seen:

For two natural numbers  $a$  and  $m$ ,  $a \bmod m$  denotes the remainder obtained when we divide  $a$  by  $m$ . This means that  $a \pmod{m}$  is always a number between 0 and  $m - 1$ .

For example,  $1 \equiv 6 \equiv 41 \equiv 1 \pmod{5}$  because the remainder is always 1. Another example is:  $2000 \equiv 0 \pmod{4}$  because 4 divides 2000 with no remainder.

Modular arithmetic only contains a limited quantity of non-negative numbers. The number of these is specified by a modulus  $m$ . If the modulo is  $m = 5$ , then only the 5 numbers in the set  $\{0, 1, 2, 3, 4\}$  are used.

A calculation result larger than 4 is then reduced “modulo 5”. In other words, it is the remainder when the result is divided by 5. For example,  $2 * 4 \equiv 8 \equiv 3 \pmod{5}$  because 3 is the remainder when we divide 8 by 5.

### 4.6.1 Addition and multiplication

The following shows two tables:

- the addition table<sup>19</sup>  $\pmod{5}$  (table 4.1) and
- the multiplication tables<sup>20</sup> for mod 5 (table 4.2) and mod 6 (table 4.3).

#### Example of an addition table:

The result when we add 3 and 4  $\pmod{5}$  is determined as follows: Calculate  $3 + 4 = 7$  and keep subtracting 5 from the result until the result is less than the modulo:  $7 - 5 = 2$ . Therefore:  $3 + 4 \equiv 2 \pmod{5}$ .

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Table 4.1: Addition table modulo 5

<sup>18</sup>Lucius Annaeus Seneca, philosophical writer and poet, 4 B. C. – 65 A. D.

<sup>19</sup>Comment on subtraction modulo 5:

$2 - 4 = -2 \equiv 3 \pmod{5}$ .

So it is not true modulo 5 that  $-2 = 2$  (see chapter 4.16 “Appendix: Comments on modulo subtraction”).

<sup>20</sup>Comment on modulo division:

Due to the special role of zero as the identity for addition, division by zero is not permitted.

For all  $a$  it is  $a * 0 = 0$ , because  $a * 0 = a * (0 + 0) = a * 0 + a * 0$ . Obviously 0 has no inverse regarding the multiplication, because if there would be one, it must be  $0 = 0 * 0^{-1} = 1$ . Also see footnote 14 in chapter 4.5.1.

### Example of a multiplication table:

The result of the multiplication  $4 * 4 \pmod{5}$  is determined as follows: Calculate  $4 * 4 = 16$  and subtract 5 until the result is less than the modulus.

$$16 - 5 = 11; 11 - 5 = 6; 6 - 5 = 1$$

The table directly shows that  $4 * 4 \equiv 1 \pmod{5}$  because  $16 : 5 = 3$  remainder 1.

Remark: Multiplication is defined on the set  $\mathbb{Z}$  excluding 0 (as  $0 * x$  is always 0, and 0 has no inverse).

*	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Table 4.2: Multiplication table modulo 5

### 4.6.2 Additive and multiplicative inverses

You can use the tables to read the inverses for each number in relation to addition and multiplication.

The inverse of a number is the number that gives the result 0 when the two numbers are added, and 1 when they are multiplied. Thus, the inverse of 4 for addition mod 5 is 1, and the inverse of 4 for multiplication mod 5 is 4 itself, because

$$\begin{aligned} 4 + 1 &= 5 \equiv 0 \pmod{5}; \\ 4 * 4 &= 16 \equiv 1 \pmod{5}. \end{aligned}$$

The inverse of 1 for multiplication mod 5 is 1, while the inverse modulo 5 of 2 is 3 and, since multiplication is commutative, the inverse of 3 is again 2.

If we take a random number and add or multiply another number (here 4) and then add<sup>21</sup> or multiply the corresponding inverse (1 or 4) to the interim result (1 or 3), then the end result is the same as the initial value.

#### Example:

$$\begin{aligned} 2 + 4 &\equiv 6 \equiv 1 \pmod{5}; & 1 + 1 &\equiv 2 \equiv 2 \pmod{5}, \\ 2 * 4 &\equiv 8 \equiv 3 \pmod{5}; & 3 * 4 &\equiv 12 \equiv 2 \pmod{5}. \end{aligned}$$

In the set  $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$  for the addition, and in the set  $\mathbb{Z}_5^*$  for the multiplication, all numbers have a **unique** inverse modulo 5.

In the case of modular addition, this is true for every modulo (not just for 5).

However, this is not the case for modular multiplication (important theorem):

---

<sup>21</sup>In general  $x + y + (-y) \equiv x \pmod{m}$  [ $(-y)$  = additive inverse of  $y \pmod{m}$ ].

**Theorem 4.6.1.** A natural number  $a$  from the set  $\{1, \dots, m-1\}$  has one modular multiplicative inverse if and only if this number and the modulo  $m$  are co-prime<sup>22</sup>, in other words if  $a$  and  $m$  have no common prime factors.

Since  $m = 5$  is prime, the numbers 1 to 4 are relatively prime to 5 and each of these numbers has a multiplicative inverse in mod 5.

Table 4.3 shows as a counterexample the multiplication table for mod 6 (since the modulus  $m = 6$  is not prime, not all elements from  $\mathbb{Z}_6 \setminus \{0\}$  are relatively prime to 6).

*	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Table 4.3: Multiplication table modulo 6

In addition to 0, also for the numbers 2, 3, and 4 there exists no other factor, so that the product equals 1 mod 6. We can say these numbers have **no** inverse.

The numbers 2, 3 and 4 have the factor 2 or 3 in common with the modulus 6. Only the numbers 1 and 5, which are relatively prime to 6, have multiplicative inverses, namely themselves.

The number of numbers that are relatively prime to the modulus  $m$  is the same as the number of numbers that have a multiplicative inverse (see the Euler function  $\phi(m)$  in chapter 4.8.2).

For the two moduli 5 and 6 used in the multiplication tables, this means: the modulus 5 is a prime number itself. In mod 5, therefore, there are exactly  $\phi(5) = 5 - 1 = 4$  numbers that are relatively prime to the modulus, that is all numbers from 1 to 4.

Since 6 is not a prime number, we write it as a product of its factors:  $6 = 2 * 3$ . In mod 6, therefore, there are exactly  $\phi(6) = (2 - 1) * (3 - 1) = 1 * 2 = 2$  numbers that have a multiplicative inverse, that is 1 and 5.

Although it may seem difficult to calculate the table of multiplicative inverses for large moduli (this only applies to the areas of the table shaded dark grey), we can use Fermat's Little Theorem to create a simple algorithm for this [Pfl97, p. 80]. Quicker algorithms are described, for instance, in [Knu98].<sup>23</sup>

Cryptographically not only the unique nature of the inverse is important, but also that the set of possible values has been exhausted.

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<sup>22</sup>Two whole numbers  $a$  and  $b$  are co-prime if and only if  $\gcd(a, b) = 1$ .

If  $p$  is prime and  $a$  is a random whole number that is not a multiple of  $p$ , then  $p$  and  $a$  are co-prime. Further name to the topic co-prime (with  $a_i \in \mathbb{Z}, i = 1, \dots, n$ ):

1.  $a_1, a_2, \dots, a_n$  are *relatively prime*, if  $\gcd(a_1, \dots, a_n) = 1$ .
2. An even stronger request for more than two numbers is :  
 $a_1, \dots, a_n$  are *in pairs relatively prime*, if for all  $i = 1, \dots, n$  and  $j = 1, \dots, n$  with  $i \neq j$ :  $\gcd(a_i, a_j) = 1$ .

**Example:**

2, 3, 6 are relatively prime, because  $\gcd(2, 3, 6) = 1$ . They are not in pairs relatively prime, because  $\gcd(2, 6) = 2 > 1$ .

<sup>23</sup>Using Euclid's extended theorem (extended gcd), we can calculate the multiplicative inverse and determine whether numbers have an inverse (see appendix 4.14). Alternatively, we can also use the primitive roots.

**Theorem 4.6.2.** For  $a, i \in \{1, \dots, m-1\}$  with  $\gcd(a, m) = 1$ , then the product  $a * i \bmod m$  takes for a certain number  $a$  all values from  $\{1, \dots, m-1\}$  (exhaustive permutation of the length  $m-1$ ).<sup>24</sup>

The following three examples<sup>25</sup> illustrate the properties of multiplicative inverses (here only the lines for the factors 5 und 6 are listed; not the complete multiplication table).

Table 4.4 (Multiplication table mod 17) was calculated for  $i = 1, 2, \dots, 18$ :

$$(5 * i) / 17 = a \text{ remainder } r \text{ and highlighted } 5 * i \equiv 1 \pmod{17},$$

$$(6 * i) / 17 = a \text{ remainder } r \text{ and highlighted } 6 * i \equiv 1 \pmod{17}.$$

We need to **find** the  $i$  for which the product remainder  $a * i \bmod 17$  with  $a = 5$  or  $a = 6$  has the value 1 (i.e. the multiplicative inverse of  $a * i$ ).

i $\Rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
5 * i remainder	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
5 10 15 3 8 13 <b>1</b> 6 11 16 4 9 14 2 7 12 0 5	5	10	15	3	8	13	<b>1</b>	6	11	16	4	9	14	2	7	12	0	5
6 * i remainder	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108
6 12 <b>1</b> 7 13 2 8 14 3 9 15 4 10 16 5 11 0 6	6	12	<b>1</b>	7	13	2	8	14	3	9	15	4	10	16	5	11	0	6

Table 4.4: Multiplication table modulo 17 (for  $a = 5$  and  $a = 6$ )

Between  $i = 1, \dots, m$ , all values between  $0, \dots, m-1$  occur for the remainders, because both 5 and 6 are also relatively prime to the modulus  $m = 17$ .

**The multiplicative inverse of 5 (mod 17) is 7, while the inverse of 6 (mod 17) is 3.**

Table 4.5 (Multiplication table mod 13) calculates the remainders of the products  $5 * i$  and  $6 * i$ :

i $\Rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
5 * i remainder	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
5 10 2 7 12 4 9 <b>1</b> 6 11 3 8 0 5 10 2 7 12	5	10	2	7	12	4	9	<b>1</b>	6	11	3	8	0	5	10	2	7	12
6 * i remainder	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108
6 12 5 11 4 10 3 9 2 8 <b>1</b> 7 0 6 12 5 11 4	6	12	5	11	4	10	3	9	2	8	<b>1</b>	7	0	6	12	5	11	4

Table 4.5: Multiplication table modulo 13 (for  $a = 5$  and  $a = 6$ )

Between  $i = 1, \dots, m$ , all values between  $0, \dots, m-1$  occur for the remainders, because both 5 and 6 are relatively prime to the modulus  $m = 13$ .

**The multiplicative inverse of 5 (mod 13) is 8, while the inverse of 6 (mod 13) is 11.**

Table 4.6 contains an example, where the modulus  $m$  and the number  $a = 6$  are *not* relatively prime.

We have calculated  $(5 * i) \pmod{12}$  and  $(6 * i) \pmod{12}$ . Between  $i = 1, \dots, m$ , not all values between  $0, \dots, m-1$  occur and 6 does not have an inverse mod 12, because 6 and the modulus  $m = 12$  are not co-prime.

<sup>24</sup>See also theorem 4.9.1 in chapter 4.9, Multiplicative order and primitive roots.

<sup>25</sup>See chapter 4.19.1 “Multiplication table modulo m” for [the source code to compute the tables using SageMath](#).

i $\Rightarrow$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
5 * i	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
remainder	5	10	3	8	1	6	11	4	9	2	7	0	5	10	3	8	1	6
6 * i	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108
remainder	6	0	6	0	6	0	6	0	6	0	6	0	6	0	6	0	6	0

Table 4.6: Multiplication table modulo 12 (for  $a = 5$  and  $a = 6$ )

The multiplicative inverse of 5 (mod 12) is 5. The number 6 has no inverse (mod 12).

#### 4.6.3 Raising to the power

In modular arithmetic, raising to the power is defined as repeated multiplication – as usual. With small exceptions we can even apply the usual rules, such as:

$$\begin{aligned} a^{b+c} &= a^b * a^c, \\ (a^b)^c &= a^{b*c} = a^{c*b} = (a^c)^b \end{aligned}$$

Modular powers work in the same way as modular addition and modular multiplication:

$$3^2 = 9 \equiv 4 \pmod{5}.$$

Even consecutive powers work in the same way:

**Example 1:**

$$(4^3)^2 = 64^2 \equiv 4096 \equiv 1 \pmod{5}.$$

(1) We can speed up<sup>26</sup> the calculation by reducing the **interim results** modulo 5 but we need to take care because *not* everything will then work in the same way as in standard arithmetic.

$$\begin{aligned} (4^3)^2 &\equiv (4^3 \pmod{5})^2 \pmod{5} \\ &\equiv (64 \pmod{5})^2 \pmod{5} \\ &\equiv 4^2 \pmod{5} \\ &\equiv 16 \equiv 1 \pmod{5}. \end{aligned}$$

(2) In standard arithmetic, consecutive powers can be reduced to a single power by multiplying the exponents:

$$(4^3)^2 = 4^{3*2} = 4^6 = 4096.$$

This is not quite as simple in modular arithmetic because this would give:

$$(4^3)^2 \equiv 4^{3*2} \pmod{5} \equiv 4^6 \pmod{5} \equiv 4^1 \equiv 4 \pmod{5}.$$

---

<sup>26</sup>The time required to calculate the multiplication of two numbers normally depends on the length of the numbers. We can observe this if we use the school method to calculate, for instance,  $474 * 228$ . The time required increases in a quadratic square manner, because we need to multiply  $3 * 3$  numbers. The numbers become considerably smaller if we reduce the interim result.

But as we saw above, the correct result is 1 !

(3) Therefore, the rule is slightly different for consecutive powers in modular arithmetic:  
We do not multiply the exponents in  $(\text{mod } m)$  but rather in  $(\text{mod } \phi(m))$ .

Using  $\phi(5) = 4$  gives:

$$(4^3)^2 \equiv 4^{3*2} \pmod{\phi(5)} \equiv 4^6 \pmod{4} \equiv 4^2 \equiv 16 \equiv 1 \pmod{5}.$$

This delivers the correct result.

**Theorem 4.6.3.**  $(a^b)^c \equiv a^{b*c} \pmod{\phi(m)}$  ( $\text{mod } m$ ).

**Example 2:**

$$3^{28} = 3^{4*7} \equiv 3^{4*7} \pmod{10} \equiv 3^8 \equiv 6561 \equiv 5 \pmod{11}.$$

#### 4.6.4 Fast calculation of high powers

RSA encryption and decryption<sup>27</sup> entails calculating high powers modulo  $m$ . For example, the calculation  $(100^5) \pmod{3}$  exceeds the 32-bit long integer number range provided we calculate  $a^n$  by actually multiplying  $a$  with itself  $n$  times in line with the definition. In the case of extremely large numbers, even a fast computer chip would take longer than the age of the universe to calculate a single exponential. Luckily, there is an extremely effective shortcut for calculating exponentials (but not for calculating logarithms).

If the expression is divided differently using the rules of modular arithmetic, then the calculation does not even exceed the 16-bit short integer number range:

$$(a^5) \equiv (((a^2 \pmod{m})^2 \pmod{m}) * a) \pmod{m}.$$

We can generalize this by representing the exponent as a binary number. For example, the naive method would require 36 multiplications in order to calculate  $a^n$  for  $n = 37$ . However, if we write  $n$  in the binary representation as  $100101 = 1 * 2^5 + 1 * 2^2 + 1 * 2^0$ , then we can rewrite the expression as:  $a^{37} = a^{2^5+2^2+2^0} = a^{2^5} * a^{2^2} * a^1$

**Example 3:**  $87^{43} \pmod{103}$ .

Since  $43 = 32 + 8 + 2 + 1$ ,  $103$  is prime,  $43 < \phi(103)$

and the squares ( $\text{mod } 103$ ) can be calculated beforehand

$$\begin{aligned} 87^2 &\equiv 50 \pmod{103}, \\ 87^4 &\equiv 50^2 \equiv 28 \pmod{103}, \\ 87^8 &\equiv 28^2 \equiv 63 \pmod{103}, \\ 87^{16} &\equiv 63^2 \equiv 55 \pmod{103}, \\ 87^{32} &\equiv 55^2 \equiv 38 \pmod{103}. \end{aligned}$$

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<sup>27</sup>See chapter 4.10 (“Proof of the RSA procedure with Euler-Fermat”) and chapter 4.13 (“The RSA procedure with actual numbers”).

We have<sup>28</sup>:

$$\begin{aligned} 87^{43} &\equiv 87^{32+8+2+1} \pmod{103} \\ &\equiv 87^{32} * 87^8 * 87^2 * 87 \pmod{103} \\ &\equiv 38 * 63 * 50 * 87 \equiv 85 \pmod{103}. \end{aligned}$$

The powers  $(a^2)^k$  can be determined easily by means of repeated squaring. As long as  $a$  does not change, a computer can calculate them beforehand and – if enough memory is available – save them. In order to then find  $a^n$  in each individual case, it now only needs to multiply those  $(a^2)^k$  for which there is a one in the  $k$ -th position of the binary representation of  $n$ . The typical effort is then reduced from  $2^{600}$  to  $2 * 600$  multiplications! This frequently used algorithm is called “Square and Multiply”.

#### 4.6.5 Roots and logarithms

The inverses of the powers modulo  $m$  are also defined. The roots and logarithms are again whole numbers. Yet in contrast to the usual situation, they are not only difficult to calculate but, in the case of large numbers, cannot be calculated at all within a reasonable amount of time.

Let us take the equation  $a \equiv b^c \pmod{m}$ .

##### a) Taking the logarithm (determining $c$ ) — Discrete logarithm problem<sup>29</sup>:

If we know  $a$  and  $b$  of the three numbers  $a$ ,  $b$  and  $c$  that meet this equation, then every known method of finding  $c$  is approximately just as time-consuming as trying out all  $m$  possible values for  $c$  one after the other. For a typical  $m$  of the order of magnitude of  $10^{180}$  for 600-digit binary numbers, this is a hopeless task. More precisely, for suitably large numbers  $m$ , the time required according to current knowledge is proportional to  $\exp(C * (\log m [\log \log m]^2)^{1/3})$  with a constant  $C > 1$ .

##### b) Calculating the root (determining $b$ ):

The situation is similar if  $b$  is the unknown variable and we know the values of  $a$  and  $c$ : If we know the Euler function<sup>30</sup>  $\phi(m)$ , then we can easily<sup>31</sup> calculate  $d$  with  $c * d \equiv 1 \pmod{\phi(m)}$  and use theorem 4.6.3 to obtain:

$$a^d \equiv (b^c)^d \equiv b^{c*d} \equiv b^{c*d} \pmod{\phi(m)} \equiv b^1 \equiv b \pmod{m}$$

the  $c$ -th root  $b$  of  $a$ .

If  $\phi(m)$  cannot be determined<sup>32</sup>, it is difficult to calculate the  $c$ -th root. This forms the basis for the security assumption used by the RSA encryption system (see chapter 4.10 or chapter 5.3.1).

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<sup>28</sup>See chapter 4.19.2 “Fast exponentiation” for source code implementing the square and multiply method in SageMath, which can be used to reproduce the calculations above.

<sup>29</sup>Further details about the discrete logarithm problem can be found in chapter 5.4.

<sup>30</sup>See chapter 4.8.2, “The Euler phi function”.

<sup>31</sup>See chapter 4.14, “Appendix: gcd and the two algorithms of Euclid”.

<sup>32</sup>According to the first fundamental theorem of number theory and theorem 4.8.4, we can determine  $\phi(m)$  by reducing  $m$  to prime factors.

The time required for inverting addition and multiplication, on the other hand, is simply proportional to  $\log m$  or  $(\log m)^2$ . Powers (for a number  $x$  calculate  $x^a$  with  $a$  fixed) and exponents (for a number  $x$  calculate  $a^x$  with  $a$  fixed) are therefore typical one way functions (compare chapters 5.1 and 4.12.1).

## 4.7 Groups and modular arithmetic in $\mathbb{Z}_n$ and $\mathbb{Z}_n^*$

Mathematical “groups” play a decisive role in number theory and cryptography. We only talk of groups if, for a defined set and a defined relation (an operation such as addition or multiplication), the following properties are fulfilled:

- The set is closed
- A neutral element exists
- An inverse element exists for each element
- The associative law applies.

The abbreviated mathematical notation is  $(G, +)$  or  $(G, *)$ .

**Definition 4.7.1.**  $\mathbb{Z}_n$ :

$\mathbb{Z}_n$  comprises all numbers from 0 to  $n - 1$ :  $\mathbb{Z}_n = \{0, 1, 2, \dots, n - 2, n - 1\}$ .

$\mathbb{Z}_n$  is an often used finite group of the natural numbers. It is sometimes also called the *remainder set R modulo n*.

For example, 32-bit computers (standard PCs) only directly work with whole numbers in a finite set, that is the value range  $0, 1, 2, \dots, 2^{32} - 1$ .

This value range is equivalent to the set  $\mathbb{Z}_{2^{32}}$ .

### 4.7.1 Addition in a group

If we define the operation mod+ on such a set where

$$a \text{ mod+ } b := (a + b) \pmod{n},$$

then the set  $\mathbb{Z}_n$  together with the relation mod+ is a group because the following properties of a group are valid for all elements in  $\mathbb{Z}_n$ :

- $a \text{ mod+ } b$  is an element of  $\mathbb{Z}_n$  (the set is closed),
- $(a \text{ mod+ } b) \text{ mod+ } c \equiv a \text{ mod+ } (b \text{ mod+ } c)$  (mod+ is associative),
- the neutral element is 0.
- each element  $a \in \mathbb{Z}_n$  has an inverse for this operation, namely  $n - a$  (because  $a \text{ mod+ } (n - a) \equiv a + (n - a) \pmod{n} \equiv n \equiv 0 \pmod{n}$ ).

Since the operation is commutative, i.e.  $(a \text{ mod+ } b) = (b \text{ mod+ } a)$ , this structure is actually a “commutative group”.

### 4.7.2 Multiplication in a group

If we define the operation  $\text{mod}^*$  on the set  $\mathbb{Z}_n$  where

$$a \text{ mod}^* b := (a * b) \pmod{n},$$

then  $\mathbb{Z}_n$  together with this operation is **usually not a group** because not all properties are fulfilled for each  $n$ .

**Example:**

- a) In  $\mathbb{Z}_{15}$ , for example, the element 5 does not have an inverse. That is to say, there is no  $a$  with  $5 * a \equiv 1 \pmod{15}$ . Each modulo product with 5 on this set gives 5, 10 or 0.
- b) In  $\mathbb{Z}_{55} \setminus \{0\}$ , for example, the elements 5 and 11 do not have multiplicative inverses. That is to say, there is no  $a \in \mathbb{Z}_{55}$  such that  $5 * a \equiv 1 \pmod{55}$  and no  $a$  such that  $11 * a \equiv 1 \pmod{55}$ . This is because 5 and 11 are not relatively prime to 55. Each modulo product with 5 on this set gives 5, 10, 15, ..., 50 or 0. Each modulo product with 11 on this set gives 11, 22, 33, 44 or 0.

On the other hand, there are subsets of  $\mathbb{Z}_n$  that form a group with the operation  $\text{mod}^*$ . If we choose all elements in  $\mathbb{Z}_n$  that are relatively prime to  $n$ , then this set forms a group with the operation  $\text{mod}^*$ . We call this set  $\mathbb{Z}_n^*$ .

**Definition 4.7.2.**  $\mathbb{Z}_n^*$  :

$$\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}.$$

$\mathbb{Z}_n^*$  is sometimes also called the reduced remainder set  $R'$  modulo  $n$ .

**Example:** For  $n = 10 = 2 * 5$  the following applies:

$$\text{full remainder set } R = \mathbb{Z}_n = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\text{reduced remainder set } R' = \mathbb{Z}_n^* = \{1, 3, 7, 9\} \longrightarrow \phi(n) = 4.$$

**Comment:**

$R'$  or  $\mathbb{Z}_n^*$  is always a genuine subset of  $R$  or  $\mathbb{Z}_n$  because 0 is always an element of  $R$  but never an element of  $R'$ . Since 1 and  $n - 1$  are always relatively prime to  $n$ , they are always elements of both sets.

If we select a random element in  $\mathbb{Z}_n^*$  and multiply it by every other element in  $\mathbb{Z}_n^*$ , then the products<sup>33</sup> are all in  $\mathbb{Z}_n^*$ , and the results are also a unique permutation of the elements in  $\mathbb{Z}_n^*$ . Since 1 is always an element of  $\mathbb{Z}_n^*$ , there is a unique “partner” in this set such that the product is 1. In other words:

**Theorem 4.7.1.** *Each element in  $\mathbb{Z}_n^*$  has a multiplicative inverse.*

**Example:**  $a = 3$  modulo 10 with  $\mathbb{Z}_n^* = \{1, 3, 7, 9\}$  it holds that  $a^{-1} = 7$ :

$$\begin{aligned} 3 &\equiv 3 * 1 \pmod{10}, \\ 9 &\equiv 3 * 3 \pmod{10}, \\ 1 &\equiv 3 * 7 \pmod{10}, \\ 7 &\equiv 3 * 9 \pmod{10}. \end{aligned}$$

The unique invertibility is an essential condition for cryptography (see section 4.10).

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<sup>33</sup>This is due to the fact that  $\mathbb{Z}_n^*$  is closed with respect to the multiplication and due to the gcd property:

$[a, b \in \mathbb{Z}_n^*] \Rightarrow [((a * b) \pmod{n}) \in \mathbb{Z}_n^*]$ , exactly:

$[a, b \in \mathbb{Z}_n^*] \Rightarrow [\gcd(a, n) = 1, \gcd(b, n) = 1] \Rightarrow [\gcd(a * b, n) = 1] \Rightarrow [((a * b) \pmod{n}) \in \mathbb{Z}_n^*]$ .

Mathematical game theory postulates players who respond rationally. Transactional game theory, on the other hand, deals with games that are not rational, perhaps even **irrational and thereby closer to reality**.

Quote 10: Eric Berne<sup>34</sup>

## 4.8 Euler function, Fermat's little theorem and Euler-Fermat

### 4.8.1 Patterns and structures

As mathematicians investigate the structure  $a * x \equiv b \pmod{m}$  (see chapter 4.5.2), so they are interested in the structure  $x^a \equiv b \pmod{m}$ .

Again here they are interested in the cases, if  $b = 1$  (value of the multiplicative inverse) and if  $b = x$  (the function  $f(x) = x^a \pmod{m}$  has a fixpoint). Concerning RSA fixed points: see 4.19.7.

### 4.8.2 The Euler phi function

Given  $n$ , the number of numbers from the set  $\{1, \dots, n-1\}$  that are relatively prime to  $n$  is equal to the value of the Euler<sup>35</sup> function  $\phi(n)$ .<sup>36</sup>

**Definition 4.8.1.** *The Euler phi function<sup>37</sup>  $\phi(n)$  specifies the number of elements in  $\mathbb{Z}_n^*$ .*

$\phi(n)$  also specifies how many whole numbers have multiplicative inverses in mod  $n$ .  $\phi(n)$  can be calculated very easily if we know the prime factors of  $n$ .

**Theorem 4.8.1.** *For a prime number, the following is true:  $\phi(p) = p - 1$ .*

**Theorem 4.8.2.** *If  $m$  is the product of two distinct primes, then:*

$$\phi(p * q) = (p - 1) * (q - 1) \quad \text{or} \quad \phi(p * q) = \phi(p) * \phi(q).$$

This case is important for the RSA procedure.

**Theorem 4.8.3.** *If  $n = p_1 * p_2 * \dots * p_k$  where  $p_1$  to  $p_k$  are distinct prime numbers (i.e. no factor occurs more than once), then the following is true (as a generalization of theorem 4.8.2):*

$$\phi(n) = (p_1 - 1) * (p_2 - 1) * \dots * (p_k - 1).$$

**Theorem 4.8.4.** *In general, the following is true for every prime number  $p$  and every  $n$  in  $\mathbb{N}$ :*

1.  $\phi(p^n) = p^{n-1} * (p - 1)$ .

2. *If  $n = p_1^{e_1} * p_2^{e_2} * \dots * p_k^{e_k}$ , where  $p_1$  to  $p_k$  are distinct prime numbers, then:*

$$\phi(n) = [(p_1^{e_1-1}) * (p_1 - 1)] * \dots * [(p_k^{e_k-1}) * (p_k - 1)] = n * ((p_1 - 1)/p_1) * \dots * ((p_k - 1)/p_k).$$

<sup>34</sup>Eric Berne, "Games People Play", rororo, (c) 1964, page 235.

<sup>35</sup>Leonhard Euler, Swiss mathematician, Apr 15, 1707 – Sep 18, 1783

<sup>36</sup>Also see the explanations about the Euler function  $\phi(n)$  in chapter 5.3.1 „The RSA procedure“.

<sup>37</sup>The Euler phi function is also often written as  $\Phi(n)$  or  $\phi(n)$ .

### Example:

- $n = 70 = 2 * 5 * 7 \Rightarrow$  using theorem 4.8.3:  $\phi(n) = 1 \cdot 4 \cdot 6 = 24$ .
- $n = 9 = 3^2 \Rightarrow$  using theorem 4.8.4:  $\phi(n) = 3^1 \cdot 2 = 6$ , because  $\mathbb{Z}_9^* = \{1, 2, 4, 5, 7, 8\}$ .
- $n = 2,701,125 = 3^2 * 5^3 * 7^4 \Rightarrow$  using theorem 4.8.4:

$$\phi(n) = [3^1 * 2] * [5^2 * 4] * [7^3 * 6] = 1,234,800.$$

### Comment: Number-theoretic functions in CT2

The Euler phi function is just one of several number-theoretic functions or statistics used. In CT2 you can get an overview and a quick comparison for different numbers. In the following figure as an example the phi function for the number 24 is highlighted.

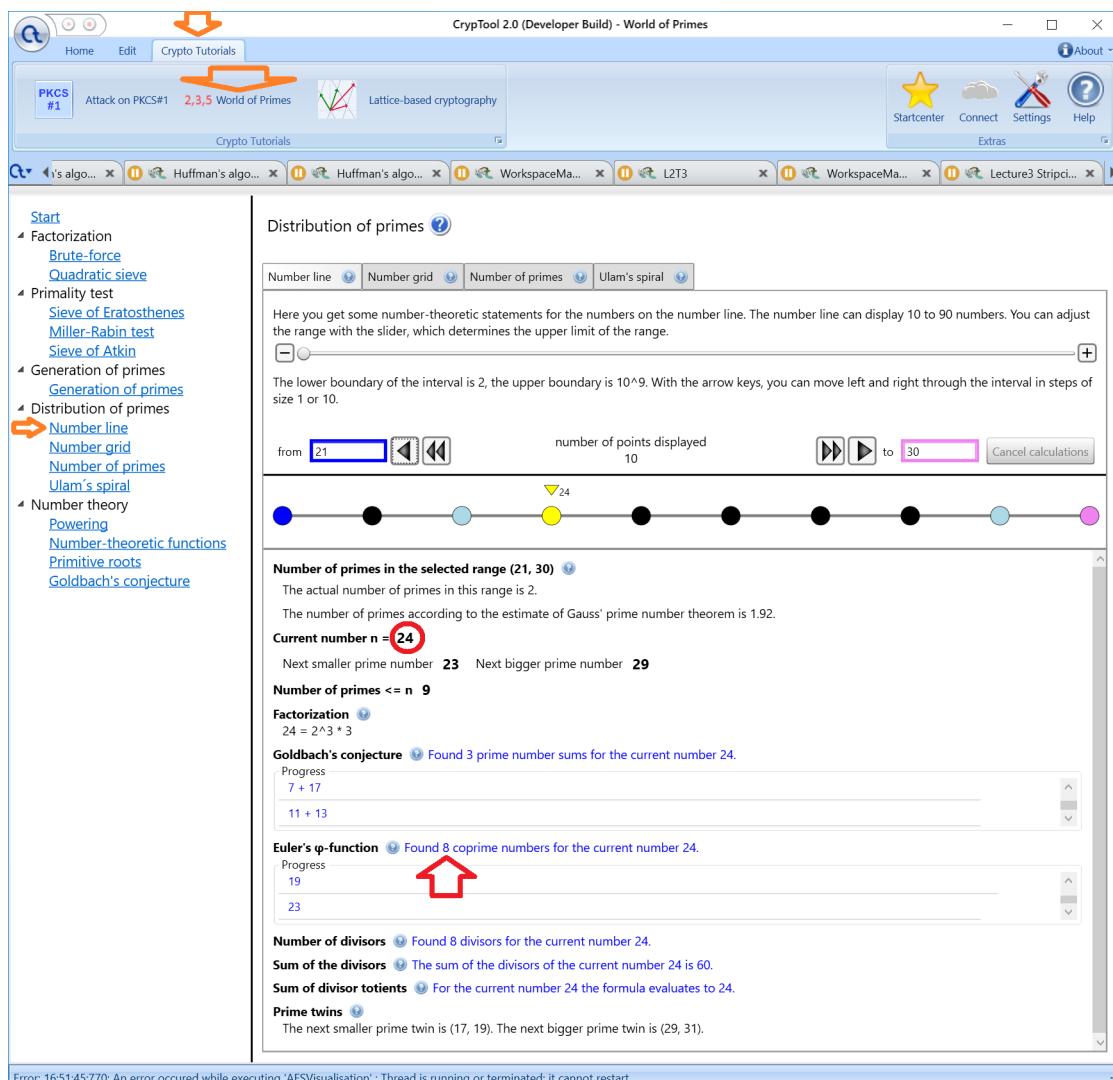


Figure 4.1: Number-theoretic functions in CT2<sup>38</sup>

<sup>38</sup>Graphics from CT2, menu Crypto Tutorials, World of Primes, Distribution of primes, Number line.

### 4.8.3 The theorem of Euler-Fermat

In order to prove the RSA procedure, we need Fermat's theorem and its generalisation (Euler-Fermat theorem) – [please see chapter 3.5](#).

**Theorem 4.8.5. Fermat's Little Theorem**<sup>39</sup> *Let  $p$  be a prime number and  $a$  be a random whole number, then:*

$$a^p \equiv a \pmod{p}.$$

An alternative formulation of Fermat's Little Theorem is as follows: Let  $p$  be a prime number and  $a$  be a random whole number that is relatively prime to  $p$ , then:

$$a^{p-1} \equiv 1 \pmod{p}.$$

**Theorem 4.8.6. Euler-Fermat theorem (generalization of Fermat's Little Theorem)**  
For all elements  $a$  in the group  $\mathbb{Z}_n^*$  (i.e.  $a$  and  $n$  are natural numbers that are co-prime):

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

This theorem states that if we raise a group element (here  $a$ ) to the power of the order of the group (here  $\phi(n)$ ), we always obtain the neutral element for multiplication (the number 1).

The 2nd formulation of Fermat's Little Theorem is derived directly from Euler's theorem if  $n$  is a prime number.

If  $n$  is the product of two prime numbers, we can - in certain cases - use Euler's theorem to calculate the result of a modular power very quickly. We have:  $a^{(p-1)*(q-1)} \equiv 1 \pmod{pq}$ .

**Examples for calculating a modular power:**

- What is  $5^2 \pmod{6}$  ?

With  $2 = 1 * 2$  and  $6 = 2 * 3$  where 2 and 3 are both prime;  $\phi(6) = 2$  because only 1 and 5 are relatively prime to 6, we obtain the equation  $5^2 \equiv 5^{\phi(6)} \equiv 1 \pmod{6}$ , without having to calculate the power.

- What is  $31^{792} \pmod{851}$  ?

With  $792 = 22 * 36$  and  $23 * 37 = 851$  where 23 and 37 are both prime, it follows for  $31 \in \mathbb{Z}_{851}^*$  that  $31^{792} \equiv 31^{\phi(23*37)} \equiv 31^{\phi(851)} \equiv 1 \pmod{851}$ .

### 4.8.4 Calculation of the multiplicative inverse

Another interesting application is a special case of determining the multiplicative inverses using the Euler-Fermat theorem (multiplicative inverses are otherwise determined using the extended Euclidean algorithm).

**Example:**

Find the multiplicative inverse of 1579 modulo 7351.

According to Euler-Fermat:  $a^{\phi(n)} = 1 \pmod{n}$  for all  $a$  in  $\mathbb{Z}_n^*$ . If we divide both sides by  $a$ , we get:  $a^{\phi(n)-1} \equiv a^{-1} \pmod{n}$ . For the special case that the modulo is prime, we have  $\phi(n) = p - 1$ . Therefore, the modular inverse is

$$a^{-1} = a^{\phi(n)-1} \equiv a^{(p-1)-1} \equiv a^{p-2} \pmod{p}.$$

For our example, this means:

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<sup>39</sup>Pierre de Fermat, French mathematician, Aug 17, 1601 – Jan 12, 1665.

Since the modulus 7351 is prime,  $p - 2 = 7349$ .

$$1579^{-1} \equiv 1579^{7349} \pmod{p}.$$

By cleverly breaking down the exponent, we can calculate this power relatively easily<sup>40</sup>:

$$7349 = 4096 + 2048 + 1024 + 128 + 32 + 16 + 4 + 1$$

$$1579^{-1} \equiv 4716 \pmod{7351}$$

#### 4.8.5 How many private RSA keys $d$ are there modulo 26

According to theorem 4.6.3, the arithmetic operations of modular expressions are performed in the exponents modulo  $\phi(n)$  rather than modulo  $n$ .<sup>41</sup>

In  $a^{e*d} \equiv a^1 \pmod{n}$ , if we wish to determine the inverses for the factor  $e$  in the exponent, we need to calculate modulo  $\phi(n)$ .

**Example:** (with reference to the RSA algorithm)

If we calculate modulo 26, which set can  $e$  and  $d$  come from?

Solution: We have  $e * d \equiv 1 \pmod{\phi(26)}$ .

The reduced remainder set  $R' = \mathbb{Z}_{26}^* = \{1, 3, 5, 7, 9, 11, 15, 17, 19, 21, 23, 25\}$  are the elements in  $\mathbb{Z}_{26}$ , which have a multiplicative inverse, that is which are relatively prime to 26 (see 4.7.2).

The reduced remainder set  $R''$  contains only the elements of  $R'$  that are relatively prime to  $\phi(n) = 12 : R'' = \{1, 5, 7, 11\}$ .

For every  $e$  in  $R''$  there exists a  $d$  in  $R''$  such that  $a \equiv (a^e)^d \pmod{n}$ .

For every  $e$  in  $R''$ , there exists therefore precisely one element (not necessarily different from  $e$ ) such that  $e * d \equiv 1 \pmod{\phi(26)}$ .

The general case, where  $n$  can be any integer (the sample here had  $n$  fixed to 26), is considered in chapter 4.19.6. There is a SageMath program, calculating the number of all  $d$ . For all  $e$  that are relatively prime to  $\phi(n)$  we can calculate  $d$  as follows using the Euler-Fermat theorem:

For  $a^{\phi(n)} \equiv 1 \pmod{n}$  is the same as saying  $a^{\phi(n)-1} \equiv a^{-1} \pmod{n}$ . Therefore

$$d \equiv e^{-1} \pmod{\phi(n)} \equiv e^{\phi(\phi(n))-1} \pmod{\phi(n)}.$$

The problems of factorizing  $n = pq$  with  $q \neq p$  and of finding  $\phi(n)$  have a similar degree of difficulty, and if we find a solution for one of the two problems, we also have a solution for the other<sup>42</sup> (please compare requisition 3 in section 4.10.1).

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<sup>40</sup>See section 4.6.4, “Fast calculation of high powers”.

<sup>41</sup>For the following example, we will adopt the usual practice for the RSA procedure of using “ $n$ ” rather than “ $m$ ” to denote the modulus.

<sup>42</sup>If we know the factors of  $n = p * q$  with  $p \neq q$ , then  $\phi(n) = (p - 1) * (q - 1) = n - (p + q) + 1$ . Additionally the factors  $p$  and  $q$  are solutions of the quadratic equation:  $x^2 - (p + q)x + pq = 0$ .

If only  $n$  and  $\phi(n)$  are known, then it is:  $pq = n$  and  $p + q = n - \phi(n) + 1$ . So you get  $p$  and  $q$  by solving the equation

$$x^2 + (\phi(n) - n - 1)x + n = 0$$

## 4.9 Multiplicative order and primitive roots<sup>43</sup>

The multiplicative order and the primitive root are two useful constructs (concepts) in elementary number theory.

Mathematicians often ask, in which conditions the repeated application of an operation results in the neutral element (compare [Patterns and Structures](#), chapter [4.8.1](#)).

For the  $i$ -times successive modular multiplication of a number  $a$  with  $i = 1, \dots, m - 1$  the product is the neutral element of the multiplication if and only if  $a$  and  $m$  are relatively prime.

**Definition 4.9.1.** *The multiplicative order  $\text{ord}_m(a)$  of a whole number  $a$  (mod  $m$ ) (where  $a$  and  $m$  are co-prime) is the smallest whole number  $i$  for which  $a^i \equiv 1 \pmod{m}$ .*

The following table shows that in a multiplicative group (here  $\mathbb{Z}_{11}^*$ ) not all numbers necessarily have the same order. The orders in this case are 1, 2, 5 and 10 and we notice that:

1. The orders are all factors of 10.
2. The numbers  $a = 2, 6, 7$  and 8 have the order 10 - we say that these numbers have the **maximum order** in  $\mathbb{Z}_{11}^*$ .

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<sup>43</sup>With the educational tool for number theory **NT** you can have a playful experience with primitive roots (see learning unit 2.2, pages 10-14/40 and 24-40/40).

NT can be called in CT1 via the menu path **Indiv. Procedures \ Number Theory Interactive \ Learning Tool for Number Theory**. See appendix [A.6](#).

Also see the SageMath samples in [4.19.3](#).

**Example 1:**

The following table 4.7<sup>44</sup> shows the values  $a^i \bmod 11$  for the exponents  $i = 1, 2, \dots, 10$ , and for the bases  $a = 1, 2, \dots, 10$  as well as the resulting value  $\text{ord}_{11}(a)$  for each  $a$ .

	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9	i=10	$\text{ord}_{11}(a)$
a=1	<b>1</b>	1	1	1	1	1	1	1	1	1	1
a=2	2	4	8	5	10	9	7	3	6	<b>1</b>	10
a=3	3	9	5	4	<b>1</b>	3	9	5	4	1	5
a=4	4	5	9	3	<b>1</b>	4	5	9	3	1	5
a=5	5	3	4	9	<b>1</b>	5	3	4	9	1	5
a=6	6	3	7	9	10	5	8	4	2	<b>1</b>	10
a=7	7	5	2	3	10	4	6	9	8	<b>1</b>	10
a=8	8	9	6	4	10	3	2	5	7	<b>1</b>	10
a=9	9	4	3	5	<b>1</b>	9	4	3	5	1	5
a=10	10	<b>1</b>	10	1	10	1	10	1	10	1	2

Table 4.7: Values of  $a^i \bmod 11, 1 \leq a, i < 11$  and according order of  $a \bmod 11$

Table 4.7 shows, for example, that the order of 3 modulo 11 has the value 5.

**Definition 4.9.2.** If  $a$  and  $m$  are co-prime and if  $\text{ord}_m(a) = \phi(m)$  (i.e.  $a$  has maximum order), then we say that  $a$  is a **primitive root** of  $m$ .<sup>45</sup>

Not for every modulo  $m$  there is a number  $a$ , which is a primitive root. In the table 4.7, only  $a = 2, 6, 7$  and  $8$  are a primitive root with respect to mod 11 ( $\text{ord}_{11}(a) = \phi(11) = 10$ ).

Using the primitive roots, we can clearly establish the conditions for which powers modulo  $m$  there is a unique inverse, and where the calculations in the exponents is manageable.<sup>46</sup>

The following two tables 4.8 and 4.9 show the multiplicative orders and primitive roots modulo 45 and modulo 46.

<sup>44</sup>The SageMath sample 4.5 contains the source code to generate table 4.7. See chapter 4.19.3 “Multiplicative order”.

<sup>45</sup>In chapter 4.19.4 “Primitive roots” there are SageMath programs to calculate primitive roots.

<sup>46</sup>There is also a very good overview about primitive roots in Wikipedia: [https://en.wikipedia.org/wiki/Primitive\\_root\\_modulo\\_n](https://en.wikipedia.org/wiki/Primitive_root_modulo_n).

**Example 2:**

The following table 4.8<sup>47</sup> shows the values  $a^i \bmod 45$  for the exponents  $i = 1, 2, \dots, 12$  and for the bases  $a = 1, 2, \dots, 12$  as well as the resulting value  $\text{ord}_{45}(a)$  for each  $a$ .

$a \setminus i$	1	2	3	4	5	6	7	8	9	10	11	12	$\text{ord}_{45}(a)$	$\phi(45)$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	24
2	2	4	8	16	32	19	38	31	17	34	23	1	12	24
3	3	9	27	36	18	9	27	36	18	9	27	36	—	24
4	4	16	19	31	34	1	4	16	19	31	34	1	6	24
5	5	25	35	40	20	10	5	25	35	40	20	10	—	24
6	6	36	36	36	36	36	36	36	36	36	36	36	—	24
7	7	4	28	16	22	19	43	31	37	34	13	1	12	24
8	8	19	17	1	8	19	17	1	8	19	17	1	4	24
9	9	36	9	36	9	36	9	36	9	36	9	36	—	24
10	10	10	10	10	10	10	10	10	10	10	10	10	—	24
11	11	31	26	16	41	1	11	31	26	16	41	1	6	24
12	12	9	18	36	27	9	18	36	27	9	18	36	—	24

Table 4.8: Values of  $a^i \bmod 45$ ,  $1 \leq a, i < 13$  and according order of  $a \bmod 45$

$\phi(45)$  is calculated using theorem 4.8.4:  $\phi(45) = \phi(3^2 * 5) = 3^1 * 2 * 4 = 24$ .

Since 45 is not a prime, there is no “multiplicative order” for all values of  $a$  (for all numbers that are not relatively prime to 45 : 3, 5, 6, 9, 10, 12, …, because  $45 = 3^2 * 5$ ).

**Example 3:**

Is 7 a primitive root modulo 45?

The necessary, but not sufficient requirement/condition  $\text{gcd}(7, 45) = 1$  is fulfilled. Table 4.8 shows that the number  $a = 7$  is not a primitive root of 45, because  $\text{ord}_{45}(7) = 12 \neq 24 = \phi(45)$ .

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<sup>47</sup>The SageMath sample 4.6 contains the source code to generate table 4.8. See chapter 4.19.3 “Multiplicative order”.

**Example 4:**

The following table 4.9<sup>48</sup> answers the question as to whether the number  $a = 7$  is a primitive root of 46.

The necessary, but not sufficient requirement/condition  $\gcd(7, 46) = 1$  is fulfilled.

$\phi(46)$  is calculated using theorem 4.8.2:  $\phi(46) = \phi(2 * 23) = 1 * 22 = 22$ . The number 7 is a primitive root of 46, because  $\text{ord}_{46}(7) = 2 = \phi(46)$ .

$a \setminus i$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	ord
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2	2	4	8	16	32	18	36	26	6	12	24	2	4	8	16	32	18	36	26	6	12	24	2	
3	3	9	27	35	13	39	25	29	41	31	1	3	9	27	35	13	39	25	29	41	31	1	3	
4	4	16	18	26	12	2	8	32	36	6	24	4	16	18	26	12	2	8	32	36	6	24	4	
5	5	25	33	27	43	31	17	39	11	9	45	41	21	13	19	3	15	29	7	35	37	1	5	
6	6	36	32	8	2	12	26	18	16	4	24	6	36	32	8	2	12	26	18	16	4	24	6	
7	7	3	21	9	17	27	5	35	15	13	45	39	43	25	37	29	19	41	11	31	33	1	7	
8	8	18	6	2	16	36	12	4	32	26	24	8	18	6	2	16	36	12	4	32	26	24	8	
9	9	35	39	29	31	3	27	13	25	41	1	9	35	39	29	31	3	27	13	25	41	1	9	
10	10	8	34	18	42	6	14	2	20	16	22	36	38	12	28	4	40	32	44	26	30	24	10	
11	11	29	43	13	5	9	7	31	19	25	45	35	17	3	33	41	37	39	15	27	21	1	11	
12	12	6	26	36	18	32	16	8	4	2	24	12	6	26	36	18	32	16	8	4	2	24	12	
13	13	31	35	41	27	29	9	25	3	39	1	13	31	35	41	27	29	9	25	3	39	1	13	
14	14	12	30	6	38	26	42	36	44	18	22	32	34	16	40	8	20	4	10	2	28	24	14	
15	15	41	17	25	7	13	11	27	37	3	45	31	5	29	21	39	33	35	19	9	43	1	15	
16	16	26	2	32	6	4	18	12	8	36	24	16	26	2	32	6	4	18	12	8	36	24	16	
17	17	13	37	31	21	35	43	41	7	27	45	29	33	9	15	25	11	3	5	39	19	1	17	
18	18	2	36	4	26	8	6	16	12	32	24	18	2	36	4	26	8	6	16	12	32	24	18	
19	19	39	5	3	11	25	15	9	33	29	45	27	7	41	43	35	21	31	37	13	17	1	19	
20	20	32	42	12	10	16	44	6	28	8	22	26	14	4	34	36	30	2	40	18	38	24	20	
21	21	27	15	39	37	41	33	3	17	35	45	25	19	31	7	9	5	13	43	29	11	1	21	
22	22	24	22	24	22	24	22	24	22	24	24	22	24	22	24	22	24	22	24	22	24	22	—	
23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	23	—	

Table 4.9: Values of  $a^i \pmod{46}$ ,  $1 \leq a, i < 24$  and according order of  $a \pmod{46}$

**Theorem 4.9.1.** Given a modulus  $n$  and a number  $a$ , relative prime to  $n$ , the following holds: The set  $\{a^i \pmod{n} \mid i = 1, \dots, \phi(n)\}$  equals the multiplicative group  $Z_n^*$  if and only if  $\text{ord}_n(a) = \phi(n)$ .<sup>49,50</sup>

The multiplicative group  $Z_n^*$  only contains all values from 1 to  $n - 1$ , if  $n$  is prime (see 4.7.2).

<sup>48</sup>The SageMath sample 4.7 contains the source code to generate table 4.9. See chapter 4.19.3 “Multiplicative order”.

<sup>49</sup>For prime moduli  $p$  all  $a$  with  $0 < a < p$  are of order  $\phi(p) = p - 1$ . Compare table 4.8 for an example. In this case  $a^i \pmod{n}$  goes through all the values  $1, \dots, p - 1$ . Exhausting all possible values of the set is an important cryptographic proposition (compare theorem 4.6.2). This determines a permutation  $\pi(p - 1)$ .

<sup>50</sup>Table 4.9 demonstrates that for composite moduli  $n$  not all  $a$  are of maximal order  $\phi(n)$ . In this example only 5, 7, 11, 15, 17, 19 and 21 are of order 22.

### Example 5: Length of cycles

The following tables 4.10 and 4.11<sup>51</sup> serve as samples to introduce cycle lengths – this is a topic which goes beyond the multiplicative order.

Cycle here means a sequence of numbers  $a^i \bmod n$  with  $1 \leq i < n$  for a given  $a$ , and a repeating sequence. According to the generation method as modular power, here each number is unique within a cycle. The cycles here don't have to contain the 1 – unless this cycles belongs to a multiplicative order  $\geq 1$  (they have the 1 always at the end of the cycle and at the position  $a^{n-1} \bmod n$ ).

With  $l$  we now mean the cycle length.

The maximum cycle length  $l_{max}$  is  $\phi(n)$ .

For the following tables 4.10 and 4.11  $\phi(n)$  is (according to theorem 4.8.4):

- $\phi(14) = \phi(2 * 7) = 1 * 6 = 6$ .
- $\phi(22) = \phi(2 * 11) = 1 * 10 = 10$ .

a) If the multiplicative order exists for  $a$ , (indendently whether  $a$  is prim) it is:  $ord_n(a) = l$ .

Samples: The maximum length  $l_{max}$ <sup>52</sup> is achieved e.g. for:

- $a = 3$  with  $l_{max} = ord_{14}(a) = 6$  in table 4.10, or
- $a = 10$  with  $l_{max} = ord_{22}(a) = 10$  in table 4.11.

b) Also, if no multiplicative order exists for  $a$ , the maximum cycle length can be achieved.<sup>53</sup>

Samples:

- In table 4.10:  $l_{max} = \phi(14) = 6$  for  $a = 10, 12$ .
- In table 4.11:  $l_{max} = \phi(22) = 10$  for  $a = 2, 6, 8, 18$ .

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<sup>51</sup>See chapter 4.19.3, “Multiplicative order” for the source code to generate the tables 4.10 und 4.11 using SageMath.

<sup>52</sup>We don't know of a formular telling for which a the length has a maximum.

<sup>53</sup>Here the sequences are built via  $a^i \bmod n$  with  $1 \leq i < n$ . For composite numbers  $n$  the sequences never contain all numbers  $1, \dots, n - 1$ .

This should not be mixed up with RSA, where the “sequence” is built differently,  $m^e \bmod n$  with  $0 \leq m < n$ , and this sequence then takes all numbers  $0, \dots, n - 1$  (permutation).

$a \setminus i$	1	2	3	4	5	6	7	8	9	10	11	12	13	$ord_{14}(a)$	$\phi(14)$	$l$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	6	1
2	2	4	8	2	4	8	2	4	8	2	4	8	2	0	6	3
3	3	9	13	11	5	1	3	9	13	11	5	1	3	6	6	6
4	4	2	8	4	2	8	4	2	8	4	2	8	4	0	6	3
5	5	11	13	9	3	1	5	11	13	9	3	1	5	6	6	6
6	6	8	6	8	6	8	6	8	6	8	6	8	6	0	6	2
7	7	7	7	7	7	7	7	7	7	7	7	7	7	0	6	1
8	8	8	8	8	8	8	8	8	8	8	8	8	8	0	6	1
9	9	11	1	9	11	1	9	11	1	9	11	1	9	3	6	3
10	10	2	6	4	12	8	10	2	6	4	12	8	10	0	6	6
11	11	9	1	11	9	1	11	9	1	11	9	1	11	3	6	3
12	12	4	6	2	10	8	12	4	6	2	10	8	12	0	6	6
13	13	1	13	1	13	1	13	1	13	1	13	1	13	2	6	2
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	1
15	1	1	1	1	1	1	1	1	1	1	1	1	1	1	6	1
16	2	4	8	2	4	8	2	4	8	2	4	8	2	0	6	3

Table 4.10: Values of  $a^i \pmod{14}$ ,  $1 \leq a < 17$ ,  $i < 14$

$a \setminus i$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	$ord_{22}(a)$	$l$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	8	16	10	20	18	14	6	12	2	4	8	16	10	20	18	14	6	12	2	0	10
3	3	9	5	15	1	3	9	5	15	1	3	9	5	15	1	3	9	5	15	1	3	5	5
4	4	16	20	14	12	4	16	20	14	12	4	16	20	14	12	4	16	20	14	12	4	0	5
5	5	3	15	9	1	5	3	15	9	1	5	3	15	9	1	5	3	15	9	1	5	5	5
6	6	14	18	20	10	16	8	4	2	12	6	14	18	20	10	16	8	4	2	12	6	0	10
7	7	5	13	3	21	15	17	9	19	1	7	5	13	3	21	15	17	9	19	1	7	10	10
8	8	20	6	4	10	14	2	16	18	12	8	20	6	4	10	14	2	16	18	12	8	0	10
9	9	15	3	5	1	9	15	3	5	1	9	15	3	5	1	9	15	3	5	1	9	5	5
10	10	12	10	12	10	12	10	12	10	12	10	12	10	12	10	12	10	12	10	12	10	0	2
11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	0	1
12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	12	0	1
13	13	15	19	5	21	9	7	3	17	1	13	15	19	5	21	9	7	3	17	1	13	10	10
14	14	20	16	4	12	14	20	16	4	12	14	20	16	4	12	14	20	16	4	12	14	0	5
15	15	5	9	3	1	15	5	9	3	1	15	5	9	3	1	15	5	9	3	1	15	5	5
16	16	14	4	20	12	16	14	4	20	12	16	14	4	20	12	16	14	4	20	12	16	0	5
17	17	3	7	9	21	5	19	15	13	1	17	3	7	9	21	5	19	15	13	1	17	10	10
18	18	16	2	14	10	4	6	20	8	12	18	16	2	14	10	4	6	20	8	12	18	0	10
19	19	9	17	15	21	3	13	5	7	1	19	9	17	15	21	3	13	5	7	1	19	10	10
20	20	4	14	16	12	20	4	14	16	12	20	4	14	16	12	20	4	14	16	12	20	0	5
21	21	1	21	1	21	1	21	1	21	1	21	1	21	1	21	1	21	1	21	1	21	2	2
22	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
23	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
24	2	4	8	16	10	20	18	14	6	12	2	4	8	16	10	20	18	14	6	12	2	0	10
25	3	9	5	15	1	3	9	5	15	1	3	9	5	15	1	3	9	5	15	1	3	5	5

Table 4.11: Values of  $a^i \bmod 22$ ,  $1 \leq a < 26$ ,  $i < 22$

## 4.10 Proof of the RSA procedure with Euler-Fermat

Using the Euler-Fermat theorem, we can “prove” the RSA<sup>54</sup> procedure in the group  $\mathbb{Z}_n^*$ .

### 4.10.1 Basic idea of public key cryptography

The basic idea behind public key cryptography is that all participants possess a different pair of keys ( $P$  and  $S$ ) and the public keys for all recipients are published. You can retrieve the public key  $P$  for a recipient from a directory just as you would look up some one’s phone number in the phone book. Furthermore, each recipient has a secret key  $S$  that is needed in order to decrypt the message and that is not known to anyone else. If the sender wishes to send a message  $M$ , he encrypts it using the public key  $P$  of the recipient before sending it:

The ciphertext  $C$  is determined as  $C = E(P; M)$ , where  $E$  (encryption) is the encryption rule. The recipient uses his private key  $S$  to decrypt the message with the decryption rule  $D : M = D(S; C)$ .

In order to ensure that this system works for every message  $M$ , the following four **requirements** must be met:

1.  $D(S; E(P; M)) = M$  for every  $M$  (invertibility) and  $M$  takes “very many” of its possible values.
2. All  $(S, P)$  pairs are different for all participants.
3. The time required to derive  $S$  from  $P$  is at least as high as the time required to decrypt  $M$  with no knowledge of  $S$ .
4. Both  $C$  and  $M$  can be calculated relatively easily.

The 1st requirement is a general condition for all cryptographic encryption algorithms.

The prerequisite of the 2nd requirement can easily be met because there is a “very” large number of prime numbers<sup>55</sup>. In addition, that this can be ensured by a central office that issues certificates (see chapter [4.11.5.4](#), S. 163).

It is this last requirement that makes the procedure actually usable. This is because it is possible to calculate the powers in a linear amount of time (because there is a restriction on the length of the numbers).

Although Whitfield Diffie and Martin Hellman formulated the general method as early as 1976, the actual procedure that met all four requirements was only discovered later by Rivest, Shamir and Adleman.

<sup>54</sup>The RSA procedure is the most common asymmetric cryptography procedure. Developed in 1978 by Ronald Rivest, Adi Shamir and Leonard Adleman, it can be used both for signatures and for encryption. Cryptographers always associate this procedure with the abbreviation “RSA” – the following remark is meant with humor to show that each letter combination can be used with several meanings: As country code “RSA” means the Republic of South Africa. In Britain the “Royal Society for the encouragement of Arts, Manufactures & Commerce” is commonly known as the “RSA”.

<sup>55</sup>According to the **prime number theorem** (chapter [3.7.2](#), p. 86) of Legendre and Gauss there are approximately  $n / \ln(n)$  prime numbers up to the number  $n$ . This means, for example, that there are  $6.5 * 10^{74}$  prime numbers under  $n = 2^{256}$  ( $= 1.1 * 10^{77}$ ) and  $3.2 * 10^{74}$  prime numbers under  $n = 2^{255}$ . Between  $2^{255}$  and  $2^{256}$  there are therefore  $3.3 * 10^{74}$  prime numbers with precisely 256 bits. Because of this large number of primes we cannot simply store them all – just by reasons from physics: see the number of atoms in the universe in the overview under [3.11](#).

#### 4.10.2 How the RSA procedure works

The individual steps for implementing the RSA procedure can be described as follows (see [Eck14, p. 213 ff] and [Sed90, p. 338 ff]). Steps 1 to 3 constitute key generation, steps 4 and 5 are the encryption, and steps 6 and 7 are the decryption:

1. Select two distinct random prime numbers<sup>56,57</sup>  $p$  and  $q$  and calculate  $n = p * q$ .<sup>58</sup>  
The value  $n$  is called the RSA modulus.<sup>59</sup>
2. Select an arbitrary  $e \in \{2, \dots, n - 1\}$  such that<sup>60</sup>:  
 $e$  is relatively prime to  $\phi(n) = (p - 1) * (q - 1)$ .  
We can then “throw away”  $p$  and  $q$ .<sup>61</sup>
3. Select  $d \in \{1, \dots, n - 1\}$  with  $e * d \equiv 1 \pmod{\phi(n)}$ , i.e.  $d$  is the multiplicative inverse of  $e$  modulo  $\phi(n)$ .<sup>62,63</sup> We can then “throw away”  $\phi(n)$ .
  - $(n, e)$  is the public key  $P$ .
  - $(n, d)$  is the private key  $S$  (only  $d$  must be kept secret).
4. For encryption, the message represented as a (binary) number is divided into parts such that each part of the number is less than  $n$ .
5. Encryption of the plaintext (or the parts of it)  $M \in \{1, \dots, n - 1\}$ :
 
$$C = E((n, e); M) := M^e \pmod{n}.$$
6. For decryption, the ciphertext represented as a binary number is divided into parts such that each part of the number is less than  $n$ .

---

<sup>56</sup>Compaq introduced the so-called multi-prime method with high marketing effort in 2000.  $n$  was the product of two big and one relative small prime:  $n = o * p * q$ . With theorem 4.8.3 we get:  $\phi(n) = (o - 1) * (p - 1) * (q - 1)$ . This method did not assert itself.

One reason probably is, that Compaq claimed a patent on it. Generally there is less understanding in Europe and with the Open Source Initiative, that one can claim patents on algorithms. But there is really no understanding outside the U.S., that one can get a patent for a special case (3 factors) of an algorithm (RSA), although the patent for the general case was almost expired.

JCT contains the multi-prime RSA method both within the **Visuals** menu of the Default Perspective as well as within the Algorithm Perspective.

<sup>57</sup>If the two primes  $p$  and  $q$  are equal then  $(m^e)^d \equiv m \pmod{n}$  is not true for all  $m < n$  (although  $e * d \equiv 1 \pmod{\phi(n)}$  is fulfilled). **Example:**

If  $n = 5^2$  then according to theorem 4.8.4 it is  $\phi(n) = 5 * 4 = 20$ ,  $e = 3$ ,  $d = 7$ ,  $e * d = 21 \equiv 1 \pmod{\phi(n)}$ . But it is  $(5^3)^7 \equiv 0 \pmod{25}$ .

<sup>58</sup>The GISA (German Information Security Agency) recommends, to choose the prime factors  $p$  and  $q$  almost the same, but not too close:

$$0.5 < |\log_2(p) - \log_2(q)| < 30.$$

They recommend to generate the primes independently and check that the restriction is fulfilled (see [BSI16]).

<sup>59</sup>In CT1 und often in the literature the RSA modulo is denoted with a capital “ $N$ ” .

<sup>60</sup>It is recommended by cryptanalytic reasons, but not necessary to make RSA work, to select  $e$  such that:  
 $\max(p, q) < e < \phi(n) - 1$ .

<sup>61</sup>The procedure also allows us to select  $d$  freely and then calculate  $e$ . However, this has practical disadvantages. We usually want to be able to encrypt messages “quickly”, which is why we choose a public exponent  $e$  such that it has a short bit length compared to the modulus  $n$  and as few binary ones as possible (e.g.  $2^{16} + 1$ ). So a fast exponentiation is possible when encrypting. We want to select the publicly known  $e$  to be an advantageous value that allows the exponential calculation to be performed quickly during encryption. The prime numbers 3, 17 and 65537 have proved to be particularly practical for this purpose. The most often used number is  $65537 = 2^{16} + 1$ , or in binary:  $10 \dots 0 \dots 01$  (this number is prime and therefore relatively prime to many other numbers).

<sup>62</sup>For reasons of security,  $d$  should not be too small.

<sup>63</sup>We start by determining either  $d$  or  $e$  depending on the implementation.

7. Decryption of the ciphertext (or the parts of it)  $C \in \{1, \dots, n - 1\}$ :

$$M = D((n, d); C) := C^d \pmod{n}.$$

The numbers  $d, e$  and  $n$  are usually extremely large (e. g.  $d$  and  $e$  300 bits,  $n$  600 bits).

**Comment:**

The security of the RSA algorithm depends as with all public key methods on the difficulty to calculate the private key  $d$  from the public key  $(n, e)$ .

Concretely for the RSA method does this mean:

1. It is hard to calculate  $\phi(n)$  for big compounds  $n$  and
2. It is hard to calculate the prime factors of big compounds  $n$  (factorization problem).<sup>64</sup> .

#### 4.10.3 Proof of requirement 1 (invertibility)

For pairs of keys  $(n, e)$  and  $(n, d)$  that possess fixed properties in steps 1 to 3 of the RSA procedure, the following must be true for all  $M < n$ :

$$M \equiv (M^e)^d \pmod{n} \quad \text{with} \quad (M^e)^d = M^{e*d}.$$

This means that the deciphering algorithm above works correctly.

We therefore need to show that:

$$M^{e*d} \equiv M \pmod{n}$$

We will show this in 3 steps using theorem 4.8.5 (Fermat's Little Theorem) (according to [Beu96, p. 131ff]).

**Step 1:**

In the first step we show that:  $M^{e*d} \equiv M \pmod{p}$

Since  $n = p * q$  and  $\phi(p * q) = (p - 1) * (q - 1)$  and since  $e$  and  $d$  are selected in such a way that  $e * d \equiv 1 \pmod{\phi(n)}$ , there is a whole number  $k$  such that:  $e * d = 1 + k * (p - 1) * (q - 1)$ .

$$\begin{aligned} M^{e*d} &\equiv M^{1+k*\phi(n)} \equiv M * M^{k*\phi(n)} \equiv M * M^{k*(p-1)*(q-1)} \pmod{p} \\ &\equiv M * (M^{p-1})^{k*(q-1)} \pmod{p} \quad \text{based on little Fermat : } M^{p-1} \equiv 1 \pmod{p} \\ &\equiv M * (1)^{k*(q-1)} \pmod{p} \\ &\equiv M \pmod{p} \end{aligned}$$

The requirement for using the simplified Euler-Fermat theorem (theorem 4.8.5) was that  $M$  and  $p$  are relatively prime.

Since this is not true in general, we need to consider the case when  $M$  and  $p$  are not relatively prime. Since  $p$  is a prime number, this implies that  $p$  is a factor of  $M$ . But this means:

$$M \equiv 0 \pmod{p}.$$

---

<sup>64</sup>There is no reason for the concern sometimes mentioned that there are not enough primes: Raising the dimension (exponent) of the modul always offers enough primes to consider – this is visualized in chapter 3.13 “Appendix: Visualization of the quantity of primes in higher ranges”.

If  $p$  is a factor of  $M$ , then  $p$  is also a factor of  $M^{e*d}$ . Therefore:

$$M^{e*d} \equiv 0 \pmod{p}.$$

Since  $p$  is a factor of both  $M$  and  $M^{e*d}$ , it is also a factor of their difference:

$$(M^{e*d} - M) \equiv 0 \pmod{p}.$$

And therefore our conjecture is also true in this special case.

### Step 2:

In exactly the same way we prove that:  $M^{e*d} \equiv M \pmod{q}$ .

### Step 3:

We now combine the conjectures from step 1 and 2 for  $n = p * q$  to show that:

$$M^{e*d} \equiv M \pmod{n} \text{ for all } M < n.$$

From step 1 and 2 we have  $(M^{e*d} - M) \equiv 0 \pmod{p}$  and  $(M^{e*d} - M) \equiv 0 \pmod{q}$ . Therefore,  $p$  and  $q$  are both factors of the same number  $z = (M^{e*d} - M)$ . Since  $p$  and  $q$  are **distinct** prime numbers, their product must also be a factor of this number  $z$ . Thus:

$$(M^{e*d} - M) \equiv 0 \pmod{p * q} \quad \text{or} \quad M^{e*d} \equiv M \pmod{p * q} \quad \text{or} \quad M^{e*d} \equiv M \pmod{n}.$$

□

### Comment 1:

We can also condense the three steps if we use the theorem 4.8.6 (Euler-Fermat) – i.e. not the simplified theorem where  $n = p$  and which corresponds to Fermat's Little Theorem:

$$(M^e)^d \equiv M^{e*d} \equiv M^{(p-1)(q-1)*k+1} \equiv \left( \underbrace{M^{(p-1)(q-1)}}_{\equiv M^{\phi(n)} \equiv 1 \pmod{n}} \right)^k * M \equiv 1^k * M \equiv M \pmod{n}.$$

### Comment 2:

When it comes to signing messages, we perform the same operations but first use the secret key  $d$ , followed by the public key  $e$ . The RSA procedure can also be used to create digital signatures, because:

$$M \equiv (M^d)^e \pmod{n}.$$

## 4.11 Regarding the security of the RSA algorithm<sup>65</sup>

There have always been discussions about the suitability of the RSA algorithm for digital signatures and encryption, e. g. after publications of breakthroughs in factorization. Nevertheless the RSA algorithm has become a de-facto standard since it was published more than 20 years ago (compare 7.1).

The security of the RSA algorithm rests — as with all cryptographic methods — on the following 4 central pillars:

- the complexity of the number theoretical problem on which the algorithm is based (here factorization of big numbers),
- the election of fitting parameters (here the length of the module  $N$ ),
- the adequate usage of the algorithm and key generation and
- the correct implementation of the algorithm.

Usage and key generation are well understood today. Implementation based on long integer arithmetic is very easy.

The following sections examine the RSA algorithm with respect to the first two points.

### 4.11.1 Complexity

Successful decryption or forgery of a signature — without knowing the private key — requires calculating the  $e$ -th root mod  $n$ . The private key, this is the multiplicative inverse of  $e$  mod  $\phi(n)$ , can be easily determined if  $\phi(n)$  is known.  $\phi(n)$  again can be calculated from the prime factors of  $n$ . Breaking of RSA therefore cannot be more difficult than factorization of the module  $n$ .

The best factorization method known today is a further development of the General Number Field Sieve (GNFS) , which was originally devised to factor only numbers of a special form (like Fermat numbers). The complexity of solving the factorization problem with the GNFS is asymptotically

$$O(l) = e^{c \cdot (l \cdot \ln 2)^{1/3} \cdot (\ln(l \cdot \ln 2))^{2/3} + o(l)}$$

Please refer to: [LL93a] and [Sil00]

This formula shows, that the factorization problem belongs to the class of problems with sub-exponential time complexity (i. e. time complexity grows asymptotically not as fast as exponential functions like  $e^l$  or  $2^l$ , but strictly slower, e. g. like  $e^{\sqrt{l}}$ ). This classification is all that is currently known; it does not preclude the possibility that the factorization problem can be solved in polynomial time (see 4.11.5.1).

$O(l)$  is the average number of processor steps depending on the bit length  $l$  of the number  $n$  to be factorized. For the best currently known factorization algorithm the constant  $c = (64/9)^{1/173} = 1923$ .

The inverse proposition, that the RSA algorithm can be broken only by factorization of  $n$ , is still not proven. Most number theorists consider the “RSA problem” and the factorization problem equivalent in terms of time complexity.

Please refer to: *Handbook of Applied Cryptography* [MvOV01].

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<sup>65</sup>Major parts of the first part of chapter 4.11 follow the article “Vorzüge und Grenzen des RSA-Verfahrens” written by F. Bourseau, D. Fox and C. Thiel [BFT02].

#### 4.11.2 Security parameters because of new algorithms

##### Factorization algorithms<sup>66</sup>

The complexity is basically determined by the length  $l$  of the modulus  $n$ . Higher values for this major parameter are oriented at the possibilities of the current algorithms for factorization:

- In 1994 a 129-digit RSA modulus (428 bit), published in 1977, was factorized by a distributed implementation of the Quadratic Sieve algorithm (QS), developed 1982 by Pomerance. This effort took 8 months.

Please refer to:

C. Pomerance: *The quadratic sieve factoring algorithm* [Pom84].

- In 1999 a 155-digit modulus (512 bit) was factored with an implementation of the general number field sieve algorithm (GNFS), developed by Buhler, Lenstra and Pomerance. The GNFS is more efficient than QS if  $n$  is longer than about 116 decimal digits. This effort took 5 months.

Please refer to:

J.P. Buhler, H.W. Lenstra, C. Pomerance: *Factoring integers with the number field sieve* [BLP93].

- Ten years later, end of 2009, a 232-digit modulus (768 bit) was factored by Kleinjung etc. after 2 1/2 years.

Please refer to:

T. Kleinjung, et. al.: *Factorization of a 768-bit RSA modulus* [Kle10].

This made practically evident that a module length of 768 bit no longer prevents from attackers.

Details about factorization progress since 1999 see chapter 4.11.4.

##### Lattice base reduction algorithms

The module length  $l$  is not the only parameter relevant for security. Beneath requirements from implementation and engineering the sizes and the proportions of the parameters  $e$ ,  $d$  and  $n$  are relevant.

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<sup>65</sup>With the educational tool for number theory NT you can gather more experience with current factorization algorithms (see learning unit 5.1-5.5, pages 1-15/15).

NT can be called in CT1 via the menu path **Indiv. Procedures \ Number Theory Interactive \ Learning Tool for Number Theory**. See appendix A.6.

The quadratic sieve (QS) can be found in CT1 and CT2; GNFS, the most modern factorisation method for moduli bigger than 130 decimal digits, is only part of CT2 (via YAFU and msieve).

CT2 has a GeneralFactorizer component based on YAFU. This is faster than the implemented functions in CT1. So CT2 offers the following factoring methods:

- brute-force with small primes
- Fermat
- Shanks square forms factorization (squofof)
- Pollard rho
- Pollard p-1
- Williams p+1
- Lenstra elliptic curve method (ECM)
- self-initializing quadratic sieve (SIQS)
- multiple polynomial quadratic sieve (MPQS)
- special number field sieve (SNFS)
- general number field sieve (GNFS).

According attacks based on lattice reductions are a real threat for (too) simple implementations of RSA. These attacks can be structured into the following four categories:

- Attacks against very small public keys  $e$  (e.g.  $e = 3$ ).
- Attacks against relatively small private exponents  $d$  (e.g.  $d < n^{0.5}$ ).
- Factorization of the modulus  $n$ , if one of the factors  $p$  or  $q$  is *partly* known.
- Attacks requiring, that a *part* of the private key  $d$  is known.

A good overview concerning these attacks can be found in the diploma thesis of Matthias Schneider [Sch04].

#### 4.11.3 Forecasts about factorization of large integers

Since 1980 a lot of progress has been made. Estimations about the future development of the ability to factor RSA modules vary and depend on some assumptions:

- progression in computing performance (Moore's law: every 18 month the computing power will double) and in grid computing.
- development of new algorithms.

Within the last years the module bit length feasible for factorization increased — even without new algorithms — by 10 bit per year. Larger numbers require not only more time to be factored, but also huge RAM storage for the solutions matrix being used by the best algorithms known today. This need for storage grows like the square root of the computation time, i. e. also sub-exponentially. Because RAM availability increased exponentially in the recent decades, it seems that this should not be the limiting factor.

An estimation of the evolution of secure key lengths was done by Lenstra/Verheul in 1999 [LV01] (compare figure 7.1 in chapter 7.1).

Within the article [BFT02] Dirk Fox<sup>67</sup> published his prognosis of an almost linear factorization progression, if all influencing factors are included: Each year the module length feasible for factorization increases by 20 bit on average. So his forecast was below the more optimistic estimations of GISA and NIST.

This forecast by Dirk Fox from the year 2001 seems to prove true by the latest factorization records of RSA-200 and RSA-768 (see chapter 4.11.4). His estimation for the year 2005, to achieve a bit length of 660 bit, was almost a precision landing (compare figure 4.2).

If the forecast withstands in the future then the factorization of an RSA modulus of 1024 bit can be expected in the year 2020.

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<sup>67</sup>His company Secorvo Ltd delivered a statement on the recommendation for key length selection published by the GISA (German Information Security Agency). Chapter 2.3.1 of this statement contains a competent and understandable discussion of RSA security (this document exists – to my knowledge – only in German): <https://www.secervo.de/publikationen/stellungnahme-algorithmenempfehlung-020307.pdf>

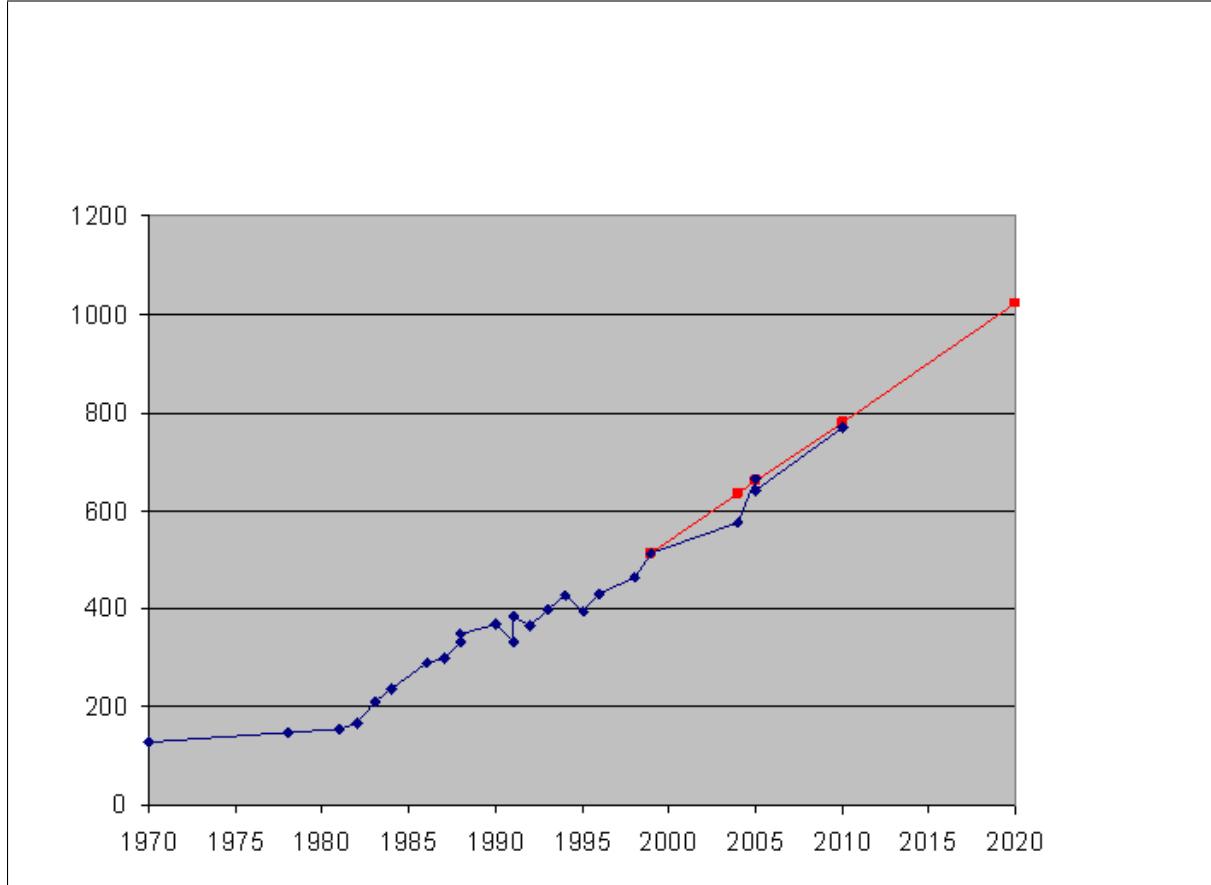


Figure 4.2: Comparison between the published factorization records (blue) and the predicted development (red) [Source Fox 2001; last addition 2011]

To let the possible happen, you again and again have to try the impossible.

Quote 11: Hermann Hesse<sup>68</sup>

#### 4.11.4 Status regarding factorization of concrete large numbers

An exhaustive overview about the factoring records of composed integers using different methods can be found on the following web pages:

<http://www.crypto-world.com>

[http://en.wikipedia.org/wiki/Integer\\_factorization\\_records](http://en.wikipedia.org/wiki/Integer_factorization_records)

[http://en.wikipedia.org/wiki/RSA\\_Factoring\\_Challenge](http://en.wikipedia.org/wiki/RSA_Factoring_Challenge)

The current record (as of Nov. 2012) obtained using the GNFS method (general number field sieve) factorized a general 232 decimal digit into its both prime factors.

The last records<sup>69</sup> with factorization algorithms for composed numbers are listed in the table 4.12.

	Decimal digits	Binary digits	Factored on	Factored by
RSA-768	232	768	Dec 2010	Thorsten Kleinjung et al.
RSA-200	200	663	May 2005	Jens Franke et al.
RSA-640 <sup>70</sup>	193	640	Nov 2005	Jens Franke et al.
RSA-576	174	576	Dec 2003	Jens Franke et al.
RSA-160	160	530	Apr 2003	Jens Franke et al.
RSA-155	155	512	Aug 1999	Herman te Riele et al.
...				
C307	307	1017	May 2007	Jens Franke et al.
C176	176	583	May 2005	Kazumaro Aoki et al.
C158	158	523	Jan 2002	Jens Franke et al.

Table 4.12: The current factoring records (as of Nov. 2012)

<sup>68</sup>Hermann Hesse, German/Swiss writer and Nobel Prize winner, July 2, 1877 – August 9, 1962.

<sup>69</sup>The 'RSA numbers' are certain large semiprime numbers (i.e., numbers with exactly two prime factors). They were generated and published by the company RSA Security: In the RSA Factoring Challenge the prime factors for these numbers are sought.

See <http://www.emc.com/emc-plus/rsa-labs/historical/the-rsa-factoring-challenge.htm>.

RSA Labs offers its challenges since the beginning of the 1990th. The first RSA Factoring Challenge labeled the numbers, from RSA-100 to RSA-500, according to their number of decimal digits; the second RSA Factoring Challenge labeled the numbers according to their number of binary digits. Within the second challenge cash prizes were offered for successful factorizations of RSA-576 to RSA-2048 (RSA-576, RSA-640 etc. using 64 bit steps upwards — An exception to this is RSA-617, which was created prior to the change in the numbering scheme). But the RSA challenges ended ahead of time in 2007, RSA Inc. retracted the prize. All till now unsolved RSA challenges of RSA Labs can also be found at the website of the cipher challenge "MysteryTwister C3" (<http://www.mysterytwisterc3.org>).

The 'C numbers' originate from the Cunningham project: <http://homes.cerias.purdue.edu/~ssw/cun/>. These are factors of Mersenne numbers, which have a very special form. This makes it an order of magnitude easier to factor them as moduli of the same length build for RSA.

<sup>70</sup>A research group of the GISA solved this challenge which was awarded with 20,000 US dollar using the GNFS method. The researchers needed about five months to divide this number into its both 320 bit long prime factors.

Experiences about the ellipsed time of factorization with the open source software Pari-GP, SageMath, CrypTool 1 and CrypTool 2) can be found in “Zeitexperimente zur Faktorisierung” (time experiments about factorization) (see [SW10]).

Below these last records listed in table 4.12 are explained in more detail<sup>71</sup>:

### RSA-155

On August 22, 1999 researchers from the Netherlands found the solution of this RSA challenge. They factorized a 155-digit number into its both 78-digit primes (see chapter 4.11.2).

This 512 bit RSA-155 meant to reach a kind of *magic* border.

### C158

On January 18, 2002 researchers at the German University of Bonn<sup>72</sup> factorized a 158-digit decimal number into its both prime factors (these are build with 73 and 86 decimal digits) using the GNFS method (general number field sieve).

This record got much less attention within the press than the solution of RSA-155.

The task of the researchers from Bonn was not initiated by a challenge, but they wanted to find the last prime factors of the integer  $2^{953} - 1$  (see “Wanted List” of the Cunningham Project<sup>73</sup>).

The 6 smaller prime factors, already found before have been:

3, 1907, 425796183929,  
1624700279478894385598779655842584377,  
3802306738549441324432139091271828121 and  
128064886830166671444802576129115872060027.

The first 3 factors can be easily computed<sup>74</sup>. The next three prime factors were found by P. Zimmerman<sup>75</sup>, T. Grandlund and R. Harley during the years 1999 and 2000 using the elliptic curve factorization method.

The last remaining factor, called “C158”, was known to be composite by then, but its factors were not known (the following 3 lines contain one number):

39505874583265144526419767800614481996020776460304936  
45413937605157935562652945068360972784246821953509354  
4305870490251995655335710209799226484977949442955603

---

The researchers around Professor Jens Franke (from the University of Bonn, the GISA, and the CWI) do not aim on getting cash prizes but in extending the research limits. So statements about the necessary length of a secure RSA modulus are more well-founded.

<sup>71</sup>The two methods, GNFS and SNFS, used to do so are shortly illustrated at the following web pages:

[http://en.wikipedia.org/wiki/Special\\_number\\_field\\_sieve](http://en.wikipedia.org/wiki/Special_number_field_sieve)  
[http://en.wikipedia.org/wiki/General\\_number\\_field\\_sieve](http://en.wikipedia.org/wiki/General_number_field_sieve)

<sup>72</sup><https://members.loria.fr/PZimmermann/records/gnfs158>

<sup>73</sup>Cunningham project: <http://homes.cerias.purdue.edu/~ssw/cun/>

<sup>74</sup>E.g. using CT1 via menu **Indiv. Procedures \ RSA Cryptosystem \ Factorization of a Number**.

CT1 can factorize in a reasonable time numbers no longer than 250 bit (Numbers bigger than 1024 bits are not accepted by CT1). CT2 is able to factorize numbers bigger than 250 bit length.

<sup>75</sup><http://homepages.loria.fr/PZimmermann/ecmnet/>

The factorization of C158 resulted in the following two 73- and 86-digit prime factors:

3388495837466721394368393204672181522815830368604993048084925840555281177

and

1165882340667125990314837655838327081813101  
2258146392600439520994131344334162924536139.

So now all 8 prime factors of  $2^{953} - 1$  have been found.

Links:

<https://members.loria.fr/PZimmermann/records/gnfs158>  
<http://www.crypto-world.com/announcements/c158.txt>

## RSA-160

On January 18, 2002 researchers at the German University of Bonn<sup>76</sup> factorized a 160-digit number into its both prime factors (these are build with each 80 decimal digits) using the GNFS method (general number field sieve).

The computations for the factorization of RSA-160 also took place at the German Information Security Agency (GISA) in Bonn.<sup>77</sup>

The 160-digit decimal number originates from the old challenge list of RSADSI. This number was retracted after RSA-155 (RSA512) had been factorized successfully. The prime factors of RSA-160 were still unknown. So this record of the team of Prof. Franke provides the solution of the old challenge, for which RSADSI didn't award a price anymore.

The composite number called "RSA-160" is (the following 3 lines contain one number):

215274110271888970189601520131282542925777358884567598017049  
767677813314521885913567301105977349105960249790711158521430  
2079314665202840140619946994927570407753

The factorization of RSA-160 resulted in the following two prime factors:

$p = 45427892858481394071686190649738831$   
656137145778469793250959984709250004157335359

and

$q = 47388090603832016196633832303788951$   
973268922921040957944741354648812028493909367

The calculations took place between December 2002 and April 2003.

<sup>76</sup><https://members.loria.fr/PZimmermann/records/rs160>  
<https://members.loria.fr/PZimmermann/records/factor.html>  
<http://www.crypto-world.com/FactorWorld.html>

<sup>77</sup>Every year the GISA creates a paper to describe which crypto algorithms are feasible to generate digital signatures according to the German signature law – under participation of experts from economy and science. To review signature methods based on the factorization problem the GISA also co-operates with researchers from the University of Bonn.

## RSA-200

On May 9, 2005 the research group of Prof. Jens Franke at the German University of Bonn<sup>78</sup> announced, that they achieved to factorize a 200-digit number into its both prime factors (these are build with each 100 decimal digits) using the GNFS method (general number field sieve).

The composite number called “RSA-200” is (the following 3 lines contain one number):

2799783391122132787082946763872260162107044678695542853756000992932  
6128400107609345671052955360856061822351910951365788637105954482006  
576775098580557613579098734950144178863178946295187237869221823983

The factorization of RSA-200 resulted in the following two prime factors:

$$p = 35324619344027701212726049781984643686711974001976  
25023649303468776121253679423200058547956528088349$$

and

$$q = 79258699544783330333470858414800596877379758573642  
19960734330341455767872818152135381409304740185467$$

The calculations took place between December 2003 and May 2005. The factorization done by the group around Bahr, Böhm, Franke, Kleinjung, Montgomery and te Riele lasted almost 17 months. The operating expense of the calculations was about 120,000 MIPS-years<sup>79</sup>.

## RSA-768

On December 12, 2009 the research group around Prof. Thorsten Kleinjung<sup>80</sup> announced, that they achieved to factorize a 232-digit number into its both prime factors (both factors have 116 decimal digits). They used the GNFS method (general number field sieve) in a way where they did “oversieving” on several hundred computers before starting the matrix step.

The composite number called “RSA-768” is (the following 3 lines contain one number):

123018668453011775513049495838496272077285356959533479219732245215172640050726  
365751874520219978646938995647494277406384592519255732630345373154826850791702  
6122142913461670429214311602221240479274737794080665351419597459856902143413

The factorization of RSA-768 resulted in the following two prime factors (each with 384 bit):

$$p = 3347807169895689878604416984821269081770479498371376856891  
2431388982883793878002287614711652531743087737814467999489$$

and

$$q = 3674604366679959042824463379962795263227915816434308764267  
6032283815739666511279233373417143396810270092798736308917$$

The calculations took about 2 1/2 years.<sup>81</sup>

<sup>78</sup><https://members.loria.fr/PZimmermann/records/rsa200>

<sup>79</sup>A MIPS-year (MY) is the quantity of operations a machine can perform in one year, if the machine constantly achieves one million integer operations per second (MIPS). For illustration: a INTEL Pentium 100 processor achieves about 50 MIPS. To factorize a 2048 bit module it is estimated to need about  $8.5 \cdot 10^{40}$  MY.

<sup>80</sup><http://eprint.iacr.org/2010/006.pdf> [Kle10]

<sup>81</sup>This was an “academic effort” – organisations with bigger resources could do it much faster.

## C307 / M1039

In May 2007 Prof. Franke, Prof. Kleinjung (University of Bonn), the Japanese telecommunication company NTT and Prof. Arjen Lenstra (Polytechnical University of Lausanne) announced, that they managed to factorize a 307 digit decimal number into its both prime factors with the SNFS method (special number field sieve) within 11 months (the two factors have 80 and 227 decimal digits).

The task of the researchers was not initiated by a challenge, but they wanted to find the last prime factors of the Mersenne number  $2^{1039} + 1$  (see “Wanted List” of the Cunningham Project<sup>82</sup>).

The number  $2^{1039} - 1$  consists of 3 prime factors: The smallest one,  $p7 = 5080711$  was already known.<sup>83</sup>

To complete this the second factor (co-divider) “C307” had to be factorized: Till then it was only known, that the last remaining factor was composite, but it was unknown, how many prime factors it had and what are the prime factors. The following 5 lines contain one number:

$C307 = 11594205740725730643698071488768946407538997917020177249868635353$   
8224838599667566080006095408005179472053993261230204874402860435302  
861914101440934535123347127396798850226307575280937916602855510550  
042581077117617761009413797078797380618700843777186828680889844712  
822002935201806074755451541370711023817

The factorization of C307 resulted in the following two 80- and 227-digit prime factors:

$$p80 = 558536666199362912607492046583159449686465270184  
88637648010052346319853288374753$$

and

$$p227 = 207581819464423827645704813703594695162939708007395209881208  
387037927290903246793823431438841448348825340533447691122230  
281583276965253760914101891052419938993341097116243589620659  
72167481161749004803659735573409253205425523689.$$

So now the number  $2^{1039} - 1$  is completely factorized in its 3 prime factors.

Links:

- [ <http://www.loria.fr/~zimmerma/records/21039->
- <http://www.crypto-world.com/announcements/m1039.txt>
- <http://www.crypto-world.com/FactorAnnouncements.html>

<sup>82</sup>Cunningham project: <http://homes.cerias.purdue.edu/~ssw/cun/>

Cunningham table: <http://homes.cerias.purdue.edu/~ssw/cun/pmain1215>

The numbers in the Cunningham table have the following syntax:

“(2,n)-” means  $2^n - 1$ ; “(2,n)+” means  $2^n + 1$ .

To describe the magnitude one writes  $p < n >$  or  $c < n >$ : “n” is the number of decimal digits and “p” and “c” tell, whether the number is prime or composite.

$2^{1039} - 1 = p7 * c307 = p7 * p80 * p227$

It is explained more precisely at the page of the Cunningham project:

“2651+ means  $2^{651} + 1$  and the size (c209 means 209 decimal digits) of the number which was factored. Then come the new factor(s), the discoverer and the method used. Recently, only the multiple polynomial quadratic sieve (ppmpqs), the elliptic curve method (ecm) and the number field sieve (nfs) have been used. ‘hmpqs’ stands for hypercube multiple polynomial quadratic sieve. Under ‘new factors’, ‘p90’ means a 90-digit prime and ‘c201’ is a 201-digit composite number.”.

<sup>83</sup>This one can also be found using CT1 via menu **Indiv. Procedures \ RSA Cryptosystem \ Factorization of a Number** — with the algorithms of Brent, Williams or Lenstra, which are “relatively” good to separate small factors. Analogously, you can do it in CT2 with the GeneralFactorizer component.

## Size of factorized numbers compared to primality proven numbers

As you notice the factorized compound numbers built of 2 prime factors are much smaller than the especially structured numbers, for which primality tests are able to decide whether these numbers are prime or not (see chapters 3.4, 3.5 and 3.6).

Length of the current world records in decimal notation:

$$\begin{aligned} [\text{RSA-768 number}] &\longleftrightarrow [\text{49th known Mersenne prime}] \\ 232 &\longleftrightarrow 22,338,618 \\ [\text{see table 4.12}] &\longleftrightarrow [\text{see table 3.1}] \end{aligned}$$

### 4.11.5 Further research results about primes and factorization

Prime numbers are part of very many topical research areas in number theory and computer science. Progress made with factorization is bigger than was estimated in 2005 – this is not only due to faster computers but also new mathematical knowledge. The current status of the according research is discussed in chapter 10.

The security of the RSA algorithm is based on the empirical observation that factoring large numbers is a hard problem. A module  $n$  (typically, 1024 bit) can be easily constructed as the product of two large primes  $p, q$  (typically, 500–600 bit each), by calculating  $n = pq$ . However, it is a hard problem to (reversely) extract  $p, q$  from  $n$ . In order to calculate the private key from the public key, you either need to know  $p$  and  $q$ , or the value of the Euler phi function  $\phi(n)$ .

Thus, any progress in efficiency of factorizing large integers will effect the security of the RSA. As a consequence, the underlying primes  $p, q$  and, thus, the module  $n$  (1024 bit as of today) have to be increased. In case of a quantum leap in factorization, the RSA algorithm might be compromised.

#### 4.11.5.1 Bernstein's paper and its implication on the security of the RSA algorithm

In his paper “Circuits for integer factorization: a proposal” (<http://cr.yp.to/djb.html>, published November 2001, D. J. Bernstein [Ber01] addresses the problem of factorizing large integers. Therefore, his results are of relevance from a RSA point of view. As a main result Bernstein claims that the implementation of the general number field sieve algorithm (GNFS) can be improved to factor, with the same effort as before, integers with three times more digits.

We note that the definition of *effort* is a crucial point: Bernstein claims that effort is the product of time and costs of the machine (including the memory used). The gist of the paper lies in the fact that he can reduce a big part of factorizing to sorting. Using Schimmler's scheme, sorting can be optimized by massive parallel computing. At the end of section 3 Bernstein explains this effect: The costs of  $m^2$  parallel computers with a constant amount of memory is a constant times  $m^2$ . The costs of a computer with a single processor and memory of size  $m^2$  is also of the order of  $m^2$ , but with a different constant factor. With  $m^2$  processors in parallel, sorting of  $m^2$  numbers (with Schimmler's scheme) can be achieved in time  $m$ , while a

$m^2$ -memory computer needs time of the order of  $m^2$ . Decreasing memory and increasing the number of processors, the computing time can be reduced by a factor  $1/m$  without additional effort in terms of total costs. In section 5 it is said that massive parallel computing can also increase efficiency of factorizing using Lenstra's elliptic-curve-method (a search algorithm has costs that increase in a quadratic square manner instead of cubically).

We note that all results achieved so far are asymptotic results. This means that they only hold in the limit  $n$  to infinity. Unfortunately, there is no upper limit for the residual error (i.e. the difference between the real and the asymptotic value) for finite  $n$  — a problem which has already been addressed by the author. As a consequence, one cannot conclude whether the costs (in the sense of Bernstein) for factorizing 1024–2048-bit RSA modules can be significantly reduced.

There is no doubt that Bernstein's approach is innovative. However, the reduction of computing time under constant costs comes along with a massive use of parallel computing — a scenario which seems not to be realistic yet. For example, formally 1 sec computing time on one machine and  $1/1,000,000$  sec time parallel computing time on 1,000,000 machines might have same costs. In reality, it is much harder to realize the second situation, and Bernstein does not take into account the fixed costs, in particular for building a network between all these computers.

Although distributed computing over a large network might help to overcome this problem, realistic costs for data transfer have to be taken into account — a point which was not addressed in Bernstein's proposal.

As long as there is neither (low cost) hardware nor a distributed computing approach (based on Bernstein's ideas), there should not be a problem for RSA. It has to be clarified from which magnitude of  $n$  on Bernstein's method could lead to a significant improvement (in the sense of the asymptotic result).

Arjen Lenstra, Adi Shamir et. al. analyzed the paper of Bernstein [LSTT]. In summary they expect a factorization improvement on how much longer the bit length of the keys could be with a factor of 1.17 (instead of factor 3 as proposed by Bernstein).

The abstract of their paper "Analysis of Bernstein's Factorization Circuit" says:

"... Bernstein proposed a circuit-based implementation of the matrix step of the number field sieve factorization algorithm. We show that under the non-standard cost function used in [1], these circuits indeed offer an asymptotic improvement over other methods but to a lesser degree than previously claimed: for a given cost, the new method can factor integers that are 1.17 times larger (rather than 3.01). We also propose an improved circuit design based on a new mesh routing algorithm, and show that for factorization of 1024-bit integers the matrix step can, under an optimistic assumption about the matrix size, be completed within a day by a device that costs a few thousand dollars. We conclude that from a practical standpoint, the security of RSA relies exclusively on the hardness of the relation collection step of the number field sieve."

RSA Security concludes in its analysis of the Bernstein paper [RSA02] from April, 8 2002 also – as expected – that RSA is still not compromised.

This is still an ongoing discussion.

When this section was written (June 2002) nothing was publicly known about, how far there exist implementations of his theoretical onsets and how much financing there was for his research project.

#### 4.11.5.2 The TWIRL device

In January 2003 Adi Shamir and Eran Tromer from the Weizmann Institute of Science published a preliminary draft called “*Factoring Large Numbers with the TWIRL Device*” raising concerns about the security of key sizes till 1024 bits [ST03a].

Their abstract summarizes their results very well: “The security of the RSA cryptosystem depends on the difficulty in factoring large integers. The best current factoring algorithm is the Number Field Sieve (NFS), and its most difficult part is the sieving step. In 1999 a large distributed computation involving thousands of workstations working for many months managed to factor a 512-bit RSA key, but 1024-bit keys were believed to be safe for the next 15-20 years. In this paper we describe a new hardware implementation of the NFS sieving step ... which is 3-4 orders of magnitude more cost effective than the best previously published designs ... . Based on a detailed analysis of all the critical components (but without an actual implementation), we believe that the NFS sieving step for 1024-bit RSA keys can be completed in less than a year with a \$10M device, and that the NFS sieving step for 512-bit RSA keys can be completed in less than ten minutes with a \$10K device. Coupled with recent results about the difficulty of the NFS matrix step ... this raises some concerns about the security of these key sizes.”

A detailed explanation from these two authors also can be found in the RSA Laboratories CryptoBytes [ST03b].

The 3-page article in the DuD issue of June 2003 [WLB03] contains a very good explanation, how the attack using the generalized number field sieve (GNFS) works and which progress is made, to factorize numbers. At GNFS we can distinguish 2 general steps: The sieve step (relation collecting) and the matrix reduction. Besides the sieve step is highly parallelizable, it dominates the overall calculation burden. Shamir and Tromer haven’t built a TWIRL device yet, but the estimated costs of 10 till 50 million Euro (in order to factorize a 1024-bit number) is not prohibitive for secret agencies or big criminal organizations, because the “costs for a single espionage satellite is estimated e.g. to be several billion USD”. The authors therefore recommend, to get as soon as possible rid of today used sensible RSA, Diffie-Hellman or ElGamal keys up to 1024 bit and to use then keys of at least 2048 bit length. The planned TCPA/Palladium hardware will use 2048-bit RSA keys!

So recommendations like the ones from the GISA (German Information Security Agency) to use higher key lengths are very valid.

#### 4.11.5.3 “Primes in P”: Primality testing is polynomial

In August 2002 the three Indian researchers M. Agrawal, N. Kayal and N. Saxena published the paper “*PRIMES in P*” about a new primality testing algorithm called AKS [AKS02]. They discovered a polynomial time deterministic algorithm for determining if a number is prime or not.

The importance of this discovery is that it provides number theorists with new insights and opportunities for further research. Lots of people over centuries have been looking for a polynomial time test for primality, and this result is a major theoretic breakthrough. It shows that new results can be generated from already known facts.

But even its authors note that other known algorithms may be faster (for example ECPP). The new algorithm works on any integer. For example the GIMPS project uses the Lucas-Lehmer primality test which takes advantage of the special properties of Mersenne numbers. This makes the Lucas-Lehmer test much faster, allowing to test numbers with millions of digits while general

purpose algorithms are limited to numbers with a few thousand digits.

Current research results on this topic can be found at:

<http://www.mersenne.org/>

<http://fatphil.org/math/AKS/> Original paper in English

<http://ls2-www.cs.uni-dortmund.de/lehre/winter200203/kt/material/primes.ps>

Good explanation in German by Thomas Hofmeister.

#### 4.11.5.4 Shared Primes: Modules with common prime factors

The RSA algorithm is based on the presumed difficulty of factoring large bi-prime integers (moduli), the factoring problem. However, as pointed out in Lenstra et al [LHA<sup>+</sup>12b] it is possible, given a set of moduli, to factor some of them if they share primes. In this case, the factoring problem is bypassed using the – relatively easy greatest common divisor (gcd) operation. On the other hand, it is no trivial task to extract common shared primes and to factor the according moduli efficiently for a very big number of given moduli (several millions).

Using the gcd only works if the RSA keys were not generated randomly. Taking into consideration the significance of strong cryptographic keys it is important to verify that all keys were generated following the principle of true randomness [ESS12].

When Lenstra et al published their paper [LHA<sup>+</sup>12b] in Feb 2012, they did not publish the source code. However, soon afterwards the source code of a similar program was published at the CrypTool website<sup>84</sup> in Python and C++, and – again a bit later – at the page used by [HDWH12]<sup>85</sup>. The fastest code known to me comes with [HDWH12].

These applications find all shared factors that may exist, given a finite set of moduli – even if this set includes millions of moduli. Such an application enables system administrators to test their own RSA keys.

The quite naive way to find all shared factors would be to compare each modul with all other moduli which has a complexity growing quadratically with the number of moduli.

A very efficient method using trees for comparing all gcd pairs is based on a publication of Dan Bernstein in 2005 [Ber05]. Bernstein uses a precalculation which leads to the product of all moduli. It's another example showing how helpful precalculations can be to break cryptographic systems (another famous example are rainbow tables used to find the origin of a hash value [Oec03]).

The following SageMath sample shows the very different run times when calculating a gcd and a factorization. The section after this sample will explain the essential part of the method used in [HDWH12]: Using two trees accelerates the calculation of the gcd pairs a lot.

The SageMath sample 4.1 shows that multiplication of factors, dividing a modul with a known factor, or calculating the gcd is very fast. However, factoring moduli steeply increases with longer moduli. Even the relatively small moduli used in this example show this: The smaller modul (69 decimal digits, 228 bit) took 76 seconds, while the bigger one (72 decimal digits, 239 bit) took almost 217 seconds.

In addition, the operations multiplication, division and gcd show big differences in runtime when the used operands are very different in size.

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<sup>84</sup><http://www.cryptool.org/en/ctp-dokumentation-en/361-ctp-paper-rsa-moduli>

<sup>85</sup><https://www.factorable.net/>

---

**SageMath sample 4.1** Comparing the runtime of calculating a gcd and performing a factorization

---

```
# Multiplication
sage: 3593875704495823757388199894268773153439 * 84115747449047881488635567801
302301541122639745170382530168903859625492057067780948293331060817639

sage: 3593875704495823757388199894268773153439 * 162259276829213363391578010288127
583139672825572068433667900695808357466165186436234672858047078770918753

# Division
sage: time 302301541122639745170382530168903859625492057067780948293331060817639 /
      3593875704495823757388199894268773153439
Wall time: 0.00 s
84115747449047881488635567801

sage: time 583139672825572068433667900695808357466165186436234672858047078770918753 /
      3593875704495823757388199894268773153439
Wall time: 0.00 s
162259276829213363391578010288127

# Calculate gcd
sage: time gcd (583139672825572068433667900695808357466165186436234672858047078770918753,
      302301541122639745170382530168903859625492057067780948293331060817639)
Wall time: 0.00 s
3593875704495823757388199894268773153439

# Factorize
sage: time factor (583139672825572068433667900695808357466165186436234672858047078770918753)
Wall time: 217.08 s
162259276829213363391578010288127 * 3593875704495823757388199894268773153439

sage: time factor (302301541122639745170382530168903859625492057067780948293331060817639)
Wall time: 76.85 s
84115747449047881488635567801 * 3593875704495823757388199894268773153439
```

---

## Efficient computing of all gcd pairs and explanation of the formula used to determine the shared primes

The excellent paper “Mining Your Ps and Qs: Detection of Widespread Weak Keys in Network Devices” [HWH12] explains the algorithm how the gcd’s of every pair of RSA moduli are calculated efficiently.

First the product  $P$  of all moduli  $m_i$  is calculated using a product tree. Then a remainder tree is build modulo the squares of the moduli. Then the gcd’s of a defined modul  $m_i$  and of the remainders  $z_i$  divided by this defined modul are calculated.

This is visualized in Figure 4.3 which is a copy from [HWH12] (where the moduli are called  $N_i$  instead of  $m_i$ ):

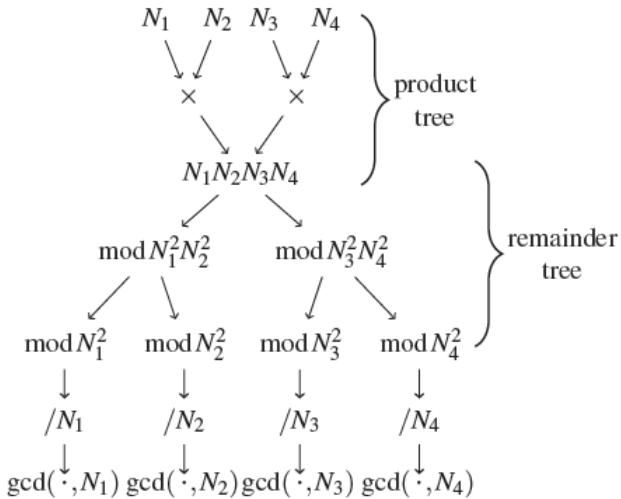


Figure 1: Computing all-pairs GCDs efficiently

---

### Algorithm 1 Quasilinear GCD finding

---

**Input:**  $N_1, \dots, N_m$  RSA moduli

- 1: Compute  $P = \prod N_i$  using a product tree.
- 2: Compute  $z_i = (P \bmod N_i^2)$  for all  $i$  using a remainder tree.

**Output:**  $\gcd(N_i, z_i/N_i)$  for all  $i$ .

---

Figure 4.3: Algorithm and figure to compute all gcd pairs efficiently

The paper [HWH12] explains well *how* the algorithm works, but not as well *why*. The product  $P$  of all moduli is a very big number, even compared to a single modul. Without the simplifications from the remainder tree you would go the following way: Calculate  $\gcd_i = \gcd(P/m_i, m_i)$  for all  $i$ . Compare each  $\gcd_i \neq 1$  with all other  $\gcd_j \neq 1$  with  $j \neq i$ . If two gcd’s are the same, then their moduli share a factor.<sup>86</sup>

As it’s very slow to calculate this for numbers with such a big difference in size, the remainder

---

<sup>86</sup>A prerequisite for getting only prime factors, is that duplicate moduli are removed before setting up the trees.

tree is used. Despite it seems to consist of more steps it's a huge simplification.

Within the remainder tree you get – at the end –  $(P \bmod m_i^2)/m_i$  for all  $i$ .<sup>87</sup>

The main remaining question now is: Why does  $\gcd((P \bmod m_i^2)/m_i, m_i)$  deliver the same result as  $\gcd(P/m_i, m_i)$ ? We prove that this identity is correct.<sup>88</sup>

$$\begin{aligned} \gcd((P \bmod m_i^2)/m_i, m_i) &\stackrel{!}{=} \gcd(P/m_i, m_i) \\ \iff 89 \end{aligned}$$

$$\begin{aligned} \gcd(((P \bmod m_i^2)/m_i) \bmod m_i, m_i) &\stackrel{!}{=} \gcd((P/m_i) \bmod m_i, m_i) \\ \iff 90 \end{aligned}$$

$$\begin{aligned} ((P \bmod m_i^2)/m_i) \bmod m_i &\stackrel{!}{=} (P/m_i) \bmod m_i \\ \iff 91 \end{aligned}$$

$$\begin{aligned} (P \bmod m_i^2)/m_i - P/m_i \equiv 0 \bmod m_i &\Leftrightarrow m_i \mid ((P \bmod m_i^2)/m_i - P/m_i) \\ 92 \end{aligned}$$

$$\begin{aligned} m_i \mid ((P - m_i^2 * \lfloor P/m_i^2 \rfloor - P))/m_i \\ 93 \end{aligned}$$

$$m_i \mid (m_i * \lfloor P/m_i^2 \rfloor)$$

As this is true, we can conclude that the two gcds are equivalent.

---

<sup>87</sup>It would not make sense to calculate modulo  $m_i$  instead of  $m_i^2$  on the left gcd, which would mean to use  $(P \bmod m_i)/m_i$ , as  $m_i|P$ , so  $P/m_i$  is always a whole number, which means  $(P \bmod m_i)$  is always = 0.

Sample with very small moduli:

$$m_1 = 2 * 3 = 6; \quad m_2 = 2 * 7 = 14; \quad P = 6 * 14 = 84$$

$$P \bmod m_1 = 84 \bmod 6 = 0; \quad P \bmod m_1^2 = 84 \bmod 36 = 12$$

$$P \bmod m_2 = 84 \bmod 14 = 0; \quad P \bmod m_2^2 = 84 \bmod 196 = 84$$

$$\gcd_1 = \gcd(12/6, 6) = \gcd(2, 6) = 2$$

$$\gcd_2 = \gcd(84/14, 14) = \gcd(6, 14) = 2$$

The way the tree is structured it also would not make sense to first divide and then do the modulo calculation, as making the division first would lead just to the given moduli but in reversed order.

It also would not make sense to calculate  $(P \bmod (m_i^3))/m_i^2$  as this is only additional effort with no improvement.

<sup>88</sup>P represents here the product of all moduli, and  $m_i$  represents any arbitrary modulus.

<sup>89</sup>According to Euklid's algorithm (first iteration) the following identity is true:

$$\gcd(a, b) = \gcd(b, a \bmod b) \text{ if } b \neq 0$$

This holds as per definition it is:  $\gcd(a, 0) = a$

Applied to our problem this means:

$$\gcd((P \bmod m_i^2)/m_i, m_i) = \gcd((P \bmod m_i^2)/m_i \bmod m_i, m_i)$$

$$\gcd(P/m_i, m_i) = \gcd(P/m_i \bmod m_i, m_i)$$

<sup>90</sup>The gcd's are equal if both their first arguments are equal.

<sup>91</sup>The following transformations are all equalities.

<sup>92</sup>Using the modulus operation (definition 4.4.2 at page 121) and division it is:  $a \bmod b = a - b * \lfloor a/b \rfloor$

So  $P \bmod m_i^2$  can be written as  $P - m_i^2 \lfloor P/m_i^2 \rfloor$ .

<sup>93</sup>P reduces itself, the exponent in the  $m_i$  enumerator is simplified with the  $m_i$  denominator.

It is our choices, that show what we truly are, far more than our abilities.

Quote 12: Joanne K. Rowling<sup>94</sup>

## 4.12 Applications of asymmetric cryptography using numerical examples

The results of modular arithmetic are used extensively in modern cryptography. Here we will provide a few examples from cryptography using small<sup>95</sup> numbers.<sup>96</sup>

Enciphering a text entails applying a function (mathematical operation) to a character string (number) to generate a different number. Deciphering entails reversing this function, in other words using the distorted image that the function has created from the plaintext in order to restore the original image. For example, the sender could take the plaintext  $M$  of a confidential message and add a secret number, the key  $S$ , to obtain the ciphertext  $C$ :

$$C = M + S.$$

The recipient can reconstruct the plaintext by reversing this operation, in other words by subtracting  $S$ :

$$M = C - S.$$

Adding  $S$  reliably makes the plaintext impossible to read. However, this encryption is rather weak, because all an interceptor needs to do to calculate the key is to obtain a plaintext and the associated ciphertext

$$S = C - M,$$

and can then read any subsequent messages encrypted using  $S$ .

The essential reason for this is that subtraction is just as simple an operation as addition.

### 4.12.1 One-way functions

If the key is to be impossible to determine even with knowledge of both the plaintext and the ciphertext, we need a function that is, on the one hand, relatively easy to calculate – we don't want to have problems encrypting messages. On the other hand, the inverse function should exist (otherwise information would be lost during encryption), but should be de facto incalculable.

What are possible candidates for such a **one way function**? We could take multiplication rather than addition, but even primary school children know that the inverse function, division, is only slightly more difficult than multiplication itself. We need to go one step higher in the hierarchy of calculation methods. It is still relatively simple to calculate the power of a number, but the corresponding two reverse functions – *taking roots* (find  $b$  in the equation  $a = b^c$  when  $a$  and  $c$  are known) and *calculating logarithms* (find  $c$  in the above equation when  $a$  and  $b$  are known) are so complicated that pupils normally do not learn them at school.

<sup>94</sup> Joanne K. Rowling, "Harry Potter and the Chamber of Secrets", Bloomsbury, 1998, last chapter "Dobby's reward", p. 245, by Dumbledore.

<sup>95</sup> In the RSA procedure, we call numbers "small" if the bit lengths are much less than 1024 bits (i.e. 308 decimal points). In practice, 1024 bits is currently considered the minimum length for a secure RSA modul.

<sup>96</sup> Within the series *RSA & Co. at school: Modern cryptology, old mathematics, and subtle protocols* there are didactically very well prepared articles with code samples in Python and SageMath. Unfortunately these are currently only available in German. See for instance [WSE15].

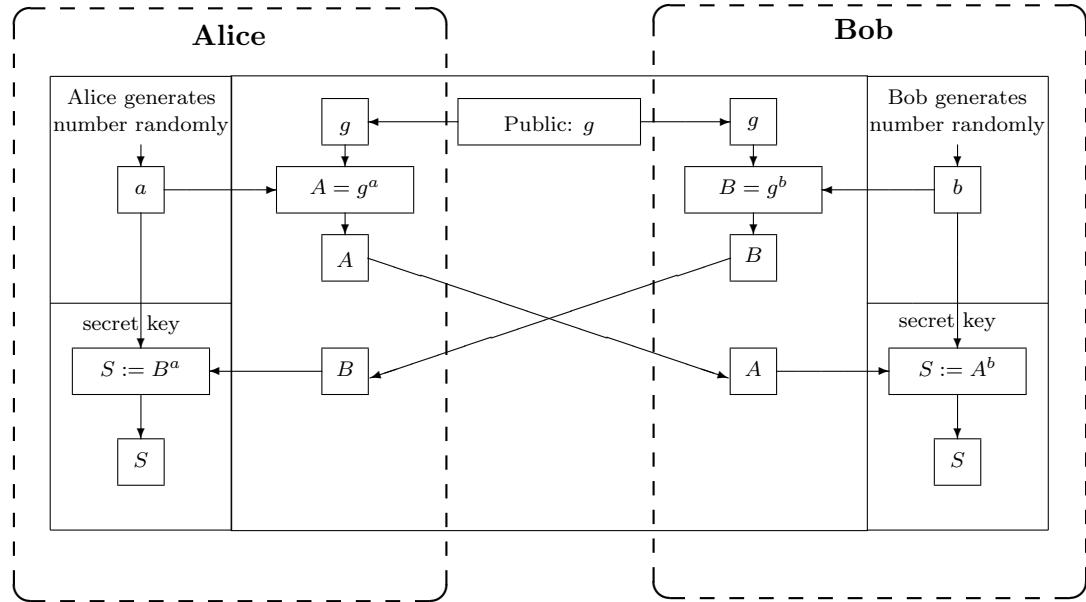
Although a certain structure can still be recognised for addition and multiplication, raising numbers to the power of another or calculating exponentials totally mixes up all the numbers. Knowing a few values of the function doesn't tell us much about the function as a whole (in contrast to addition and multiplication).

#### 4.12.2 The Diffie-Hellman key exchange protocol

Whitfield Diffie, Martin E. Hellman, and Ralph Merkle developed this key exchange protocol in Stanford in 1976.<sup>97</sup>

Alice and Bob<sup>98</sup> use a one way function to obtain a key  $S$ , the session key, for subsequent correspondence. This is then a secret that is only known to the two of them. Alice selects a random number  $a$  and keeps it secret. She applies a one way function to  $a$  to calculate the number  $A = g^a$  and sends it to Bob. He does the same, by selecting a secret random number  $b$ , calculating  $B = g^b$  and sending it to Alice. The number  $g$  is random and can be publicly known. Alice applies the one way function together with her secret number  $a$  to  $B$ , while Bob does the same with his secret number  $b$  and the received number  $A$ .

The result  $S$  is the same in each case because the one way function is commutative:  $(g^a)^b = (g^b)^a$ . But even Bob cannot reconstruct Alice's secret number  $a$  from the data available to him, while Alice cannot determine Bob's secret number  $b$ . And a perpetrator who knows  $g$  and has intercepted both  $A$  and  $B$  cannot use this knowledge to determine  $a, b$  or  $S$ .



#### Procedure:

Alice and Bob want to negotiate a secret session key  $S$  via a channel that may be intercepted.

<sup>97</sup>With CT1 this exchange protocol has been visualized: You can execute the single steps with concrete numbers using menu **Indiv. Procedures \ Protocols \ Diffie-Hellman Demonstration**.

In JCT you can find it in the default perspective via the menu item **Visuals \ Diffie-Hellman Key Exchange (EC)**.

<sup>98</sup>Bob and Alice are the default names used for the two authorized participants in a protocol (see [Sch96b, p. 23]).

1. They select a prime number  $p$  and a random number  $g$  and exchange this information openly.
2. Alice now selects  $a$ , a random number less than  $p$  and keeps it secret.  
Similarly, Bob selects  $b$ , a random number less than  $p$  and keeps it secret.
3. Alice now calculates  $A \equiv g^a \pmod{p}$ .  
Bob calculates  $B \equiv g^b \pmod{p}$ .
4. Alice sends the result  $A$  to Bob.  
Bob sends the result  $B$  to Alice.
5. In order to now determine the session key to be used by both, they both separately raise the respective results they have received to the power of their secret random number modulo  $p$ . This means:
  - Alice calculates  $S \equiv B^a \pmod{p}$  and
  - Bob calculates  $S \equiv A^b \pmod{p}$ .

Even if a spy intercepts  $g, p$ , and the interim results  $A$  and  $B$ , he cannot use these in order to determine the session key used – due to the difficulty of calculating the discrete logarithm<sup>99</sup>.

We will now use an example with (unrealistically) small numbers to illustrate this.

#### Example using small numbers:

1. Alice and Bob select  $g = 11, p = 347$ .
2. Alice selects  $a = 240$ , Bob selects  $b = 39$  and they keep  $a$  and  $b$  secret.
3. Alice calculates  $A \equiv g^a \equiv 11^{240} \equiv 49 \pmod{347}$ .  
Bob calculates  $B \equiv g^b \equiv 11^{39} \equiv 285 \pmod{347}$ .
4. Alice sends Bob:  $A \equiv 49$ ,  
Bob sends Alice:  $B \equiv 285$ .
5. Alice calculates  $B^a \equiv 285^{240} \equiv 268 \pmod{347}$ ,  
Bob calculates  $A^b \equiv 49^{39} \equiv 268 \pmod{347}$ .

Alice and Bob can now communicate securely using their shared session key  $S$ . Even if spies could intercept everything transferred via the connection  $g = 11, p = 347, A = 49$ , and  $B = 285$ ; they would not be able to calculate the secret key  $S$ .

However, this is only true for large numbers because then the discrete logarithm is extremely difficult to solve (see chapter 10).

---

<sup>99</sup>Further details about the [discrete logarithm problem](#) can be found in chapters 5.4 and 10.

### Comment:

For such small numbers as in the example above, the DH key exchange scheme can be attacked as the discrete logarithms can be easily calculated in order to reveal the exponents  $a$  or  $b$ .

After revealing  $a$  or  $b$ ,  $S$  can be calculated in the same way as Alice or Bob do it.

To get the discrete logarithms, here we need to calculate one of the following equations:

$a$  from Alice:  $11^x \equiv 49 \pmod{347}$ , that means  $\log_{11}(49) \pmod{347}$ .

$b$  from Bob:  $11^y \equiv 285 \pmod{347}$ , that means  $\log_{11}(285) \pmod{347}$ .

You can use SageMath to determine the discrete logarithm  $x$  that solves for instance the equation above (i.g. for Alice):

---

**SageMath sample 4.2** Sample with small numbers: calculating the discrete logs  $a$  and  $b$  in order to attack DH

---

```
# Get the secret key of Alice:  
### via numbers  
sage: discrete_log(mod(49,347),mod(11,347))  
67  
### via variables in the ring of integers  
sage: R=IntegerRing(347)  
sage: g=R(11)  
sage: A=R(49)  
sage: discrete_log(A,g)  
67  
  
# Get the secret key of Bob:  
sage: B=R(285)  
sage: discrete_log(B,g)  
39
```

---

As the SageMath function `discrete_log` expects as arguments only elements of a ring (integers between 0 and an upper limit), we could either enforce this type by entering the numbers directly with the according modulo operator: `discrete_log(mod(49, 347), mod(11, 347))`.

A much better alternative is to use the variables like in the formula above and let SageMath know that they are elements of a ring. After this “burdon” at the beginning for the initialization, you can write the formulars like you are used to: `discrete_log(A, g)`.<sup>100,101</sup>

---

<sup>100</sup>Such number theoretic tasks can also be solved using other tools like PariGP, LiDIA, BC, or Mathematica (see the list of web sites in the appendix at the end of this chapter). Here is the according syntax to get the discrete log for Alice:

- Pari-GP: `znlog(Mod(49,347),Mod(11,347))`
- LiDIA: `d1(11,49,347)`
- Mathematica: `MultiplicativeOrder[11, 347, 49]`

The general “Solve” function provides the “em tdep message”: The equations appear to involve the variables to be solved for in an essentially non-algebraic way.

All function calls deliver the result 67.

<sup>101</sup>Why have the functions delivered the value 67 for the discrete logarithm of Alice rather than 240 which Alice selected as exponent  $a$ ?

The discrete logarithm is the smallest natural exponent that solves the equation  $11^x \equiv 49 \pmod{347}$ . Both  $x = 67$  and  $x = 240$  (the number selected in the example) satisfy the equation and can therefore be used to calculate the session key:  $285^{240} \equiv 285^{67} \equiv 268 \pmod{347}$ . If Alice and Bob had selected a primitive root modulo  $p$  as base  $g$ , then for every remainder from the set  $\{1, 2, \dots, p-1\}$  there is exactly one exponent from the set  $\{0, 1, \dots, p-2\}$ .

As an aside, there are 172 different primitive roots modulo 347, 32 of which are prime (not necessary). Since the number 11 selected for  $g$  in the example is not a primitive root of 347, the remainders do not take all values from the set  $\{1, 2, \dots, 346\}$ . Thus, for a particular remainder there may be more than one exponent or even no exponent at all in the set  $\{0, 1, \dots, 345\}$  that satisfies the equation.

## 4.13 The RSA procedure with actual numbers<sup>102</sup>

“Games are Nature’s way of preparing us to face difficult realities. Are you finally ready to face reality, Sergeant?”

Quote 13: Daniel Suarez<sup>103</sup>

Having described above [how the RSA procedure works](#), we will now work through the steps using actual, but small, numbers.

### 4.13.1 RSA with small prime numbers and with a number as message

Before applying the RSA procedure to a text, we will first demonstrate it directly using a single number<sup>104</sup> as message.<sup>105</sup>

1. Let the selected prime numbers be  $p = 5$  and  $q = 11$ .  
Thus,  $n = 55$  and  $\phi(n) = (p - 1) * (q - 1) = 40$ .
2.  $e = 7$  ( $e$  should<sup>106</sup> lie between 11 and 39, and must be relatively prime to 40).
3.  $d = 23$  (since  $23 * 7 \equiv 161 \equiv 1 \pmod{40}$ ),
  - Public key of the recipient: (55, 7),
  - Private key of the recipient: (55, 23).
4. Let the message be the number  $M = 2$  (so no division into blocks is required).
5. Encryption:  $C \equiv 2^7 \equiv 18 \pmod{55}$ .
6. The ciphertext is simply the number  $C = 18$  (we therefore do not need to divide it into blocks).

---

With the relevant SageMath commands you find:

`is_prime(347)=True, euler_phi(347)=346, gcd(11,347)=1, and multiplicative_order(mod(11, 347))=173.`

i	$11^i \pmod{347}$	
0	1	
1	11	
2	121	
3	290	
67	49	searched exponent
172	284	
173	1	= multiplicative order of $11^i \pmod{347}$
174	11	
175	121	
176	290	
240	49	searched exponent

Further information can be found in chapter [4.19.4 “Primitive roots”](#).

<sup>102</sup>Nguyen wrote a short, didactically very clear article about some basic number theory and SageMath usage [[Ngu09b](#)].

<sup>103</sup>Daniel Suarez, “Daemon”, Dutton Adult, 2010, Chapter 45, “Respawning”, p. 610, Sobol.

<sup>104</sup>In practice, RSA is not applied on texts, but only on big numbers.

<sup>105</sup>Using CT1 you can solve this with the menu **Indiv. Procedures \ RSA Cryptosystem \ RSA Demonstration**.

<sup>106</sup>See footnote [60](#) on page [147](#).

7. Decryption:  $M \equiv 18^{23} \equiv 18^{(1+2+4+16)} \equiv 18 * 49 * 36 * 26 \equiv 2 \pmod{55}$ .

We will now apply the RSA procedure to a text, first using the upper case alphabet (26 characters), then using the entire ASCII character set as the basis for the messages.

#### 4.13.2 RSA with slightly larger primes and a text of upper case letters

We have the text “ATTACK AT DAWN”, and the characters are coded according to table 4.13.<sup>107</sup>

Character	Numerical value	Character	Numerical value
Blank	0	M	13
A	1	N	14
B	2	O	15
C	3	P	16
D	4	Q	17
E	5	R	18
F	6	S	19
G	7	T	20
H	8	U	21
I	9	V	22
J	10	W	23
K	11	X	24
L	12	Y	25
		Z	26

Table 4.13: Capital letters alphabet

#### Key generation (steps 1 to 3):

1.  $p = 47, q = 79$  ( $n = 3713$ ;  $\phi(n) = (p - 1) * (q - 1) = 3588$ ).
2.  $e = 37$  ( $e$  should<sup>108</sup> lie between 79 and 3587, and must be relatively prime to 3588).
3.  $d = 97$  (since  $e * d = 1 \pmod{\phi(n)}$ ;  $37 * 97 \equiv 3589 \equiv 1 \pmod{3588}$ ).<sup>109</sup>

#### 4. Encryption:

Text:	A	T	T	A	C	K		A	T		D	A	W	N
Number:	01	20	20	01	03	11	00	01	20	00	04	01	23	14

This 28-digit number is divided into 4-digit parts (because 2626 is still smaller than  $n = 3713$ ):  
0120 2001 0311 0001 2000 0401 2314

All 7 parts are encrypted using:  $C \equiv M^{37} \pmod{3713}$ :<sup>110</sup>  
1404 2932 3536 0001 3284 2280 2235

#### 5. Decryption:

Ciphertext: 1404 2932 3536 0001 3284 2280 2235

<sup>107</sup>Using CT1 you can solve this with the menu **Indiv. Procedures \ RSA Cryptosystem \ RSA Demonstration**. This is also described in the tutorial/scenario in CT1's online help [Options, specify alphabet, number system, block length 2 and decimal representation].

<sup>108</sup>See footnote 60 on page 147.

<sup>109</sup>How to compute  $d = 97$  using the *extended gcd* algorithm is shown in appendix 4.14

<sup>110</sup>See chapter 4.19.5 “RSA examples with SageMath” for source code to do RSA encryption using SageMath.

This 28-digit number is divided into 4-digit parts.

All 7 parts are decrypted using:  $M \equiv C^{97} \pmod{3713}$ :

0120 2001 0311 0001 2000 0401 2314

The 2-digit numbers are transformed into capital letters and blanks.

Using the selected values it is easy for a cryptanalyst to derive the secret values from the public parameters  $n = 3713$  and  $e = 37$  by revealing that  $3713 = 47 * 79$ .

If  $n$  is a 1024-bit number, there is, according to present knowledge, little chance of this.

#### 4.13.3 RSA with even larger primes and a text made up of ASCII characters

In real life, the ASCII alphabet is used to code the individual characters of the message as 8-bit numbers.

The idea for this exercise<sup>111</sup> is taken from the example in [Eck14, p. 271].

Coded in decimal notation, the text “RSA works!” is as follows:

Text:	R	S	A	w	o	r	k	s	!	
Number:	82	83	65	32	119	111	114	107	115	33

We will work through the example in 2 variants. The steps 1 to 3 are common for both.

##### Key generation (steps 1 to 3):

1.  $p = 503$ ,  $q = 509$  ( $n = 256,027$ ;  $\phi(n) = (p-1)(q-1) = 255,016 = 2^3 * 127 * 251$ ).<sup>112</sup>

2.  $e = 65,537$

( $e$  should<sup>113</sup> lie between 509 and 255,015, and must<sup>114</sup> be relatively prime to 255,016).

3.  $d = 231,953$

(since  $e \equiv d^{-1} \pmod{\phi(n)}$ :  $65,537 * 231,953 \equiv 15,201,503,761 \equiv 1 \pmod{255,016}$ ).<sup>115</sup>

**Variant 1: All ASCII characters are en-/decrypted separately (no blocks are formed).**

##### 4. Encryption:

Text:	R	S	A	w	o	r	k	s	!	
Number:	82	83	65	32	119	111	114	107	115	33

The letters are not combined!<sup>116</sup>

<sup>111</sup>Using CT1 you can solve this exercise via the menu path **Indiv. Procedures \ RSA Cryptosystem \ RSA Demonstration**.

Using JCT you can solve this exercise via the menu path **Visuals \ RSA Cryptosystem** of the Default Perspective.

<sup>112</sup>See chapter [4.19.5 “RSA examples with SageMath”](#) for the source code to factorize the number  $\phi(n)$  using SageMath.

<sup>113</sup>See footnote [60](#) on page [147](#).

<sup>114</sup> $e$  cannot, therefore, be 2, 127 or 251 ( $65,537 = 2^{16} + 1$ ) ( $255,016 = 2^3 * 127 * 251$ ).

In real life,  $\phi(n)$  is not factorized but rather the Euclidean algorithm is used for the selected  $e$  to guarantee that  $\gcd(e, \phi(n)) = 1$ .

<sup>115</sup>Other possible combinations of  $(e, d)$  include: (3, 170, 011), (5, 204, 013), (7, 36, 431).

<sup>116</sup>For secure procedures we need large numbers that assume – as far as possible – all values up to  $n - 1$ . If the possible value set for the numbers in the message is too small, even large prime numbers cannot make the procedure secure. An ASCII character is represented by 8 bits. If we want larger values we must combine several numbers. Two characters need 16 bits, whereby the maximum value that can be represented is 65536. The modulus  $n$  must then be greater than  $2^{16} = 65536$ . This is applied in variant 2. When the numbers are combined, the leading zeros are kept in binary notation (just as if we were to write all numbers with 3 digits in decimal notation above and were then to obtain the sequence 082 083, 065 032, 119 111, 114 107, 115 033).

Each character is encrypted using:  $C = M^{65,537} \pmod{256,027}$ :<sup>117</sup>

212984 025546 104529 031692 248407  
100412 054196 100184 058179 227433

### 5. Decryption:

Ciphertext:

212984 025546 104529 031692 248407  
100412 054196 100184 058179 227433

Each character is decrypted using:  $M \equiv C^{231,953} \pmod{256,027}$ :

82 83 65 32 119 111 114 107 115 33

### Variant 2: The ASCII characters are en-/decrypted two at a time as blocks.

In variant 2 the block formation is done in two different sub-variants: (4./5. and 4'./5').

Text:	R	S	A	w	o	r	k	s	!	
Number:	82	83	65	32	119	111	114	107	115	33

### 4. Encryption:

Blocks are formed<sup>118</sup> (each ASCII character is encoded into a 8 digit binary number below):

21075 16672 30575 29291 29473<sup>119</sup>

Each block is encrypted using:  $C \equiv M^{65,537} \pmod{256,027}$ :<sup>120</sup>

158721 137346 37358 240130 112898

### 5. Decryption:

Ciphertext:

158721 137346 37358 240130 112898

Each block is decrypted using:  $M \equiv C^{231,953} \pmod{256,027}$ :

21075 16672 30575 29291 29473

### 4'. Encryption:

Blocks are formed: (each ASCII character is encoded into a 3 digit decimal number below):

82083 65032 119111 114107 115033<sup>121</sup>

Each block is encrypted using:  $C \equiv M^{65,537} \pmod{256,027}$ :<sup>122</sup>

---

<sup>117</sup>See chapter 4.19.5 “RSA examples with SageMath” for the source code for RSA exponentiation using SageMath.  
<sup>118</sup>

single character	binary representation	decimal representation
01010010, 82	01010010 01010011	= 21075
01010011, 83		
01000001, 65	01000001 00100000	= 16672
00100000, 32		
01110111, 119	01110111 01101111	= 30575
01101111, 111		
01110010, 114	01110010 01101011	= 29291
01101011, 107		
01110011, 115	01110011 00100001	= 29473
00100001, 33:		

<sup>119</sup>Using CT1 you can solve this with the menu **Indiv. Procedures \ RSA Cryptosystem \ RSA Demonstration** with the following options: all 256 ASCII characters, b-adic, block length 2 and decimal representation.

<sup>120</sup>See chapter 4.19.5 “RSA examples with SageMath” for the source code for RSA exponentiation using SageMath.

<sup>121</sup>The RSA encryption works correctly with the modulus  $n = 256,027$  because each ASCII block of two characters will be encoded into a number that is smaller or equal than the number 255, 255.

<sup>122</sup>See chapter 4.19.5 “RSA examples with SageMath” for the source code for RSA exponentiation using SageMath.

198967 051405 254571 115318 014251

**5'. Decryption:**

Ciphertext:

198967 051405 254571 115318 014251

Each block is decrypted using:  $M \equiv C^{2473} \pmod{67,519}$ :

82083 65032 119111 114107 115033

#### 4.13.4 A small RSA cipher challenge (1)

The task is taken from [Sti06, Exercise 4.6]: The pure solution has been published by Prof. Stinson.<sup>123</sup> However, it is not the result that is important here but rather the individual steps of the solution, that is, the explanation of the cryptanalysis.<sup>124</sup>

Two samples of RSA ciphertext are presented in Tables 4.14<sup>125</sup> and 4.15<sup>126</sup>. Your task is to decrypt them. The public parameters of the system are

$n = 18,923$  and  $e = 1261$  (for Table 4.14) and  
 $n = 31,313$  and  $e = 4913$  (for Table 4.15).

This can be accomplished as follows. First, factor  $n$  (which is easy because it is so small). Then compute the exponent  $d$  from  $\phi(n)$ , and, finally, decrypt the ciphertext. Use the square-and-multiply algorithm to exponentiate modulo  $n$ .

In order to translate the plaintext back into ordinary English text, you need to know how alphabetic characters are “encoded” as elements in  $\mathbb{Z}_n$ . Each element of  $\mathbb{Z}_n$  represents three alphabetic characters as in the following examples:

$$\begin{aligned} \text{DOG} &\mapsto 3 * 26^2 + 14 * 26 + 6 = 2398 \\ \text{CAT} &\mapsto 2 * 26^2 + 0 * 26 + 19 = 1371 \\ \text{ZZZ} &\mapsto 25 * 26^2 + 25 * 26 + 25 = 17,575. \end{aligned}$$

You will have to invert this process as the final step in your program.

The first plaintext was taken from “The Diary of Samuel Marchbanks”, by Robertson Davies, 1947, and the second was taken from “Lake Wobegon Days”, by Garrison Keillor, 1985.

---

<sup>123</sup><http://cacr.uwaterloo.ca/~dstinson/solns.html>

<sup>124</sup>The method of solving the problem is outlined in the scenario of the online help to CT1 and in the presentation on the CT website.

<sup>125</sup>The numbers of this table can be worked with via Copy and Paste.

<sup>126</sup>The numbers of this table are in the online help of CT1 in the chapter “Example illustrating the RSA demonstration”.

12423	11524	7243	7459	14303	6127	10964	16399
9792	13629	14407	18817	18830	13556	3159	16647
5300	13951	81	8986	8007	13167	10022	17213
2264	961	17459	4101	2999	14569	17183	15827
12693	9553	18194	3830	2664	13998	12501	18873
12161	13071	16900	7233	8270	17086	9792	14266
13236	5300	13951	8850	12129	6091	18110	3332
15061	12347	7817	7946	11675	13924	13892	18031
2620	6276	8500	201	8850	11178	16477	10161
3533	13842	7537	12259	18110	44	2364	15570
3460	9886	8687	4481	11231	7547	11383	17910
12867	13203	5102	4742	5053	15407	2976	9330
12192	56	2471	15334	841	13995	17592	13297
2430	9741	11675	424	6686	738	13874	8168
7913	6246	14301	1144	9056	15967	7328	13203
796	195	9872	16979	15404	14130	9105	2001
9792	14251	1498	11296	1105	4502	16979	1105
56	4118	11302	5988	3363	15827	6928	4191
4277	10617	874	13211	11821	3090	18110	44
2364	15570	3460	9886	9988	3798	1158	9872
16979	15404	6127	9872	3652	14838	7437	2540
1367	2512	14407	5053	1521	297	10935	17137
2186	9433	13293	7555	13618	13000	6490	5310
18676	4782	11374	446	4165	11634	3846	14611
2364	6789	11634	4493	4063	4576	17955	7965
11748	14616	11453	17666	925	56	4118	18031
9522	14838	7437	3880	11476	8305	5102	2999
18628	14326	9175	9061	650	18110	8720	15404
2951	722	15334	841	15610	2443	11056	2186

Table 4.14: RSA ciphertext A

6340	8309	14010	8936	27358	25023	16481	25809
23614	7135	24996	30590	27570	26486	30388	9395
27584	14999	4517	12146	29421	26439	1606	17881
25774	7647	23901	7372	25774	18436	12056	13547
7908	8635	2149	1908	22076	7372	8686	1304
4082	11803	5314	107	7359	22470	7372	22827
15698	30317	4685	14696	30388	8671	29956	15705
1417	26905	25809	28347	26277	7897	20240	21519
12437	1108	27106	18743	24144	10685	25234	30155
23005	8267	9917	7994	9694	2149	10042	27705
15930	29748	8635	23645	11738	24591	20240	27212
27486	9741	2149	29329	2149	5501	14015	30155
18154	22319	27705	20321	23254	13624	3249	5443
2149	16975	16087	14600	27705	19386	7325	26277
19554	23614	7553	4734	8091	23973	14015	107
3183	17347	25234	4595	21498	6360	19837	8463
6000	31280	29413	2066	369	23204	8425	7792
25973	4477	30989					

Table 4.15: RSA ciphertext B

#### 4.13.5 A small RSA cipher challenge (2)

The following task is a corrected version from the book written by Prof. Yan [Yan00, Example 3.3.7, p. 318]. However, it is not the result that is important here but rather the individual steps of the solution, that is, the explanation of the cryptanalysis.<sup>127</sup>

There are three tasks with completely different degrees of difficulty here. In each case we know the ciphertext and the public key  $(e, n)$ :

- (a) Known plaintext: find the secret key  $d$  using the additionally known original message.
- (b) Ciphertext-only: find  $d$  and the plaintext.
- (c) Calculate the RSA modulus, in other words factorization (with no knowledge of the message).

$$n = 63978486879527143858831415041, e = 17579$$

Message<sup>128</sup>:

1401202118011200,  
1421130205181900,  
0118050013010405,  
0002250007150400

Cipher:

45411667895024938209259253423,  
16597091621432020076311552201,  
46468979279750354732637631044,  
32870167545903741339819671379

##### Comment:

The original message consisted of a sentence containing 31 characters (coded with the capital letters alphabet from section 4.13.2). Each group of 16 decimal numbers is then combined to form one number (the last number is filled with zeros). These numbers are raised to the power of  $e$ .

When you decrypt the message you must fill the calculated numbers with leading zeros in order to obtain plaintext.

This needs to be stressed because the type of padding is extremely important during implementation and standardization for interoperable algorithms.

---

<sup>127</sup>The method of solving the problem is outlined in the scenario of the online help to CT1 and in the CrypTool presentation.

<sup>128</sup>The numbers of this table are in the online help of CT1 in the chapter “Example illustrating the RSA demonstration”.

## 4.14 Appendix: The greatest common divisor (gcd) of whole numbers and the two algorithms of Euclid<sup>129</sup>

The greatest common divisor of two natural numbers  $a$  and  $b$  is an important value that can be calculated very quickly. Here we make use of the fact that if a number  $c$  divides the numbers  $a$  and  $b$  (i.e. there exists an  $a'$  and a  $b'$  such that  $a = a' * c$  and  $b = b' * c$ ), then  $c$  also divides the remainder  $r$  of  $a/b$ . In short notion we can write: If  $c$  divides  $a$  and  $b$  it follows that  $c$  divides  $r = a - \lfloor a/b \rfloor * b$ .<sup>130</sup>

As the latter statement is valid for each common divisor  $c$  of  $a$  and  $b$  it follows that:

$$\gcd(a, b) = \gcd(a - \lfloor a/b \rfloor * b, b).$$

Using this information, the algorithm for calculating the gcd of two numbers can be written as follows (in pseudo code):

```

INPUT: a,b != 0
1. if ( a < b ) then x = a; a = b; b = x; // Swap a and b (a > b)
2. a = a - int(a/b) * b           // a is smaller than b, the
                                  // gcd(a, b) is unchanged
3. if ( a != 0 ) then goto 1.      // a falls after each step and
                                  // the algorithm ends when a==0.
OUTPUT "gcd(a,b) = " b    // b is the gcd of the original a and b

```

Also further relationships can be derived from the gcd: For this, we need the set of equations for  $a$  and  $b$ :

$$\begin{aligned} a &= 1 * a + 0 * b \\ b &= 0 * a + 1 * b, \end{aligned}$$

or, in matrix notation:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} a \\ b \end{pmatrix}.$$

We summarize this information in the extended matrix:

$$\left( \begin{array}{c|cc} a & 1 & 0 \\ b & 0 & 1 \end{array} \right)$$

If we apply the above gcd algorithm to this matrix, we obtain the *extended Euclid algorithm*<sup>131</sup> which can be used to calculate the multiplicative inverse:

---

<sup>129</sup>With the educational tool for number theory **NT** you can see

a) how Euklid's algorithm calculates the gcd (learning unit 1.3, pages 14-19/21) and  
b) how Euklid's enhanced algorithm finds the multiplicative inverse (learning unit 2.2, page 13/40).  
**NT** can be called in **CT1** via the menu path **Indiv. Procedures \ Number Theory Interactive \ Learning Tool for Number Theory**. See appendix A.6.

**CT2** contains within the crypto tutorial “**World of Primes** ⇒ Number Theory ⇒ Number-theoretic functions” the functions “Extended Euclidian algorithm” and “Modular multiplicative inverse”.

In **JCT** you can find it in the default perspective via the menu item **Visuals \ Extended Euclidian / Reciprocal Subtraction**.

<sup>130</sup>The Gauss bracket  $\lfloor x \rfloor$  of a real number  $x$  is defined via:  $\lfloor x \rfloor$  is the next integer less or equal  $x$ .

See footnote 134 on page 183.

<sup>131</sup>A more intuitive and generic way from the simple to the extended Euclidean algorithm can be found within the series *RSA & Co. at school: Modern cryptology, old mathematics, and subtle protocols*: see NF part 2 [WS06].

**INPUT:**  $a, b \neq 0$

$$0. \quad x_{1,1} := 1, x_{1,2} := 0, x_{2,1} := 0, x_{2,2} := 1$$

$$1. \quad \left( \begin{array}{c|cc} a & x_{1,1} & x_{1,2} \\ b & x_{2,1} & x_{2,2} \end{array} \right) := \left( \begin{array}{cc} 0 & 1 \\ 1 & -\lfloor a/b \rfloor * b \end{array} \right) * \left( \begin{array}{c|cc} a & x_{1,1} & x_{1,2} \\ b & x_{2,1} & x_{2,2} \end{array} \right).$$

2. **if** ( $b \neq 0$ ) **then goto** 1.

**OUTPUT:** “ $\gcd(a, b) = a * x + b * y$ :”, “ $\gcd(a, b) =$ ”  $b$ , “ $x =$ ”  $x_{2,1}$ , “ $y =$ ”  $x_{2,2}$

Since this algorithm only performs linear transformations, the same equations always apply

$$\begin{aligned} a &= x_{1,1} * a + x_{1,2} * b \\ b &= x_{2,1} * a + x_{2,2} * b \end{aligned}$$

We get the extended gcd equation at the end of the algorithm<sup>132</sup>:

$$\gcd(a, b) = a * x_{2,1} + b * x_{2,2}.$$

### Example:

Using the extended gcd we can determine for  $e = 37$  the multiplicative inverse number  $d$  to modulo 3588 (i.e.  $37 * d \equiv 1 \pmod{3588}$ ):

$$0. \quad \left( \begin{array}{c|cc} 3588 & 1 & 0 \\ 37 & 0 & 1 \end{array} \right)$$

$$1. \quad \left( \begin{array}{c|cc} 37 & 1 & 0 \\ 36 & 0 & -96 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & -(\lfloor 3588/36 \rfloor = 96) * 37 \end{array} \right) * \left( \begin{array}{c|cc} 3588 & 1 & 0 \\ 37 & 0 & 1 \end{array} \right).$$

$$2. \quad \left( \begin{array}{c|cc} 36 & 1 & -96 \\ 1 & -1 & 97 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & -(\lfloor 37/36 \rfloor = 1) * 36 \end{array} \right) * \left( \begin{array}{c|cc} 37 & 1 & 0 \\ 36 & 0 & -96 \end{array} \right).$$

$$3. \quad \left( \begin{array}{c|cc} 1 & -1 & 97 \\ 0 & 37 & -3588 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 1 & -(\lfloor 36/1 \rfloor = 36) * 1 \end{array} \right) * \left( \begin{array}{c|cc} 36 & 1 & -96 \\ 1 & -1 & 97 \end{array} \right).$$

**OUTPUT:**

$$\gcd(37, 3588) = a * x + b * y;$$

$$\gcd(37, 3588) = 1, x = -1, y = 97.$$

Thus

1. 37 and 3588 are relatively prime (37 has an inverse modulo 3588).
2.  $37 * 97 = (1 * 3588) + 1$  in other words  $37 * 97 \equiv 1 \pmod{3588}$ .  
and therefore the number 97 is the multiplicative inverse to 37 modulo 3588.

---

<sup>132</sup>By termination of the gcd algorithm, the program variables  $a$  and  $b$  contain the values  $a = 0$  and  $b = \gcd(a, b)$ . Please keep in mind, that the program variables are different to the numbers  $a$  and  $b$  and that they are only relevant for the scope of the algorithm.

## 4.15 Appendix: Forming closed sets

The property of closeness within a set is always defined in relation to an operation. The following shows how to construct the “closed set”  $G$  with respect to the operation  $+ \pmod{8}$  for a given initial set  $G_0$ :

$G_0 = \{2, 3\}$  ——addition of the numbers in  $G_0$  determines further numbers :

$$2 + 3 \equiv 5 \pmod{8} = 5$$

$$2 + 2 \equiv 4 \pmod{8} = 4$$

$$3 + 3 \equiv 6 \pmod{8} = 6$$

$G_1 = \{2, 3, 4, 5, 6\}$  ——addition of the numbers in  $G_1$  determines :

$$3 + 4 \equiv 7 \pmod{8} = 7$$

$$3 + 5 \equiv 8 \pmod{8} = 0$$

$$3 + 6 \equiv 9 \pmod{8} = 1$$

$G_2 = \{0, 1, 2, 3, 4, 5, 6, 7\}$  ——addition of the numbers in  $G_2$  does not extend the set!

$G_3 = G_2$  ——we say :  $G_2$  is closed for addition  $\pmod{8}$ .

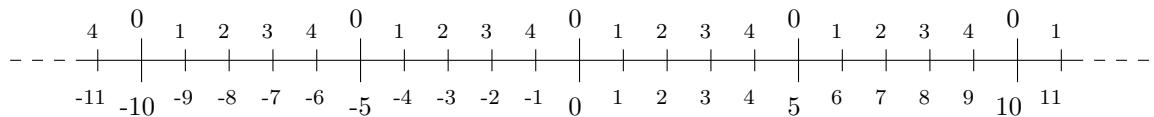
## 4.16 Appendix: Comments on modulo subtraction

Comment on subtraction modulo 5:  $2 - 4 = -2 \equiv 3 \pmod{5}$ .

It is therefore **not** true that  $-2 \equiv 2 \pmod{5}$  !

People often make the mistake of equating this. You can see this clearly if you place the permutation  $(0, 1, 2, 3, 4)$  in  $\mathbb{Z}_5$ , for example from  $-11$  to  $+11$ , over the range of numbers in  $\mathbb{Z}$ .

range of numbers modulo 5



## 4.17 Appendix: Base representation and base transformation of numbers, estimation of length of digits

For a given number  $z$  one may ask how to represent such a number. In general we use representations like  $z = 2374$  or  $z = \sqrt{2}$ . The second number consists of an infinite number of digits and therefore it can never be described precisely by the first representation. You can get around this problem by writing the number symbolically. But if you have to write it in digits, the number must be rounded.

We represent numbers usually in the decimal system (base 10). Computers are working with the binary representation of numbers — only for the display numbers are represented in decimal or sometimes hexadecimal (base 16) form.

This appendix describes how to generate arbitrary base representations of any positive integer and how to determine the number of required digits via the logarithm function.

### ***b*-adic sum representation of positive integers**

Given base  $b$ , each positive integer  $z$  can be represented as a  $b$ -adic sum

$$z = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0,$$

where  $a_i \in \{0, 1, \dots, b-1\}$ ,  $i = 0, 1, \dots, n$  are called *digits*.

For this sum, it follows that:

- 1) For arbitrary digits  $a_0, a_1, \dots, a_n$  it is:  $b^{n+1} > a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0$ .
- 2) There exist digits  $a_0, a_1, \dots, a_n$  (namely  $a_i = b-1$  for  $i = 0, \dots, n$ ), following that  $b^{n+1} - 1 \leq a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0$ .

(Using these inequalities it can be shown that each positive integer can be represented by a  $b$ -adic sum).

By writing the digits  $a_n a_{n-1} \cdots a_1 a_0$  in a row directly after each other (without the  $b^i$ ) the usual writing for numbers comes to hand.

#### **Example:**

Base  $b = 10$ :  $10278 = 1 \cdot 10^4 + 0 \cdot 10^3 + 2 \cdot 10^2 + 7 \cdot 10^1 + 8$

Base  $b = 16$ :  $FE70A = 15 \cdot 16^4 + 14 \cdot 16^3 + 7 \cdot 16^2 + 0 \cdot 16^1 + 10$ .

### **Number of digits to represent a positive integer**

For a positive integer  $z$  the length of the  $b$ -adic representation can be determined via the following steps. Starting from the inequality  $b^{n+1} > z \geq b^n$  we have — after applying the logarithm function on basis  $b$ <sup>133</sup>:  $n+1 > \log_b z \geq n$ . Therefore we have  $n = \lfloor \log_b z \rfloor$ .<sup>134</sup> We call  $l_b(z)$  the number of required digits to represent the number  $z$  on the base  $b$ . We have

$$l_b(z) := \lfloor \log_b z \rfloor + 1.$$

---

<sup>133</sup>Applying the logarithm formula on base  $b$  and  $b'$  we have  $\log_b z = \log_{b'} z / \log_{b'}(b)$ . It is therefore easy using e.g. logarithm tables for the base  $b' = 10$  to compute the logarithm of base  $b = 2$ .

<sup>134</sup>The function  $\lfloor x \rfloor$  determines the next integer smaller than  $x$  (in case  $x \geq 0$  the digits after the decimal point are truncated).

See footnote 130 on page 180.

**Example 1 (decimal→hex):**

We compute for the decimal number  $z = 234$  (EA in hex) the hexadecimal representation (number base  $b = 16$ )

$$l_{16}(z) = \lfloor \log_{16}(z) \rfloor + 1 = \lfloor \ln(z)/\ln(16) \rfloor + 1 = \lfloor 1.96... \rfloor + 1 = 1 + 1 = 2.$$

**Example 2 (decimal→binary):**

We compute for the decimal number  $z = 234$  (11101010 in binary) the binary representation (number base  $b = 2$ )

$$l_2(z) = \lfloor \log_2(z) \rfloor + 1 = \lfloor \ln(z)/\ln(2) \rfloor + 1 = \lfloor 7.87... \rfloor + 1 = 7 + 1 = 8.$$

**Example 3 (binary→decimal):**

We compute for the decimal number  $z = 11101010$  (234 decimal) the decimal representation (number base  $b = 10$ )

$$l_{10}(z) = \lfloor \log_{10}(z) \rfloor + 1 = \lfloor \ln(z)/\ln(10) \rfloor + 1 = \lfloor 2.36... \rfloor + 1 = 2 + 1 = 3.$$

**Algorithm to compute the base representation**

Given the number  $z$  one can compute the base  $b$  representation of  $z$  using the following algorithm:

```
input:  $z, b$ 
 $n := 0, z' := z$ 
while  $z' > 0$  do
     $a_n := z' \bmod b,$ 
     $z' := \lfloor z'/b \rfloor$ 
     $n := n + 1$ 
end do
output:  $a_n a_{n-1} \cdots a_1 a_0$  in base  $b$  representation.
```

**Example 1 (decimal→hex):**

The integer  $z = 234$  on the number base 10 will be transformed into the hex representation via  $a_0 = 234 \bmod 16 = 10 = A$ ,  $234/16 = 14 = E$ ,

$$a_1 = 14 \bmod 16 = E$$

and therefore we have EA.

**Example 2 (binary→decimal):**

The binary number  $z = 1000100101110101$  is transformed into the decimal representation via the following steps:

$$1000100101110101 = 1001 \pmod{1010} \implies a_0 = 9, \quad 1000100101110101 / 1010 = 110110111110$$

$$110110111110 = 1000 \pmod{1010} \implies a_1 = 8, \quad 110110111110 / 1010 = 10101111$$

$$10101111 = 1 \pmod{1010} \implies a_2 = 1, \quad 10101111 / 1010 = 100011$$

$$100011 = 101 \pmod{1010} \implies a_3 = 5, \quad 100011 / 1010 = 1$$

$$11 = 11 \pmod{1010} \implies a_4 = 3$$

therefore  $z = 35189$ .

## 4.18 Appendix: Interactive presentation about the RSA cipher

The following presentation (last update Nov. 2010) shows the basics of the RSA cipher in an interactive way.

There are three variants:

- Powerpoint 2007 (for download; dynamical, animated)<sup>135</sup>
- PDF (for download; static, no interaction)<sup>136</sup>
- Flash (can be started within the browser, requires JavaScript; time-controlled replay)<sup>137</sup>

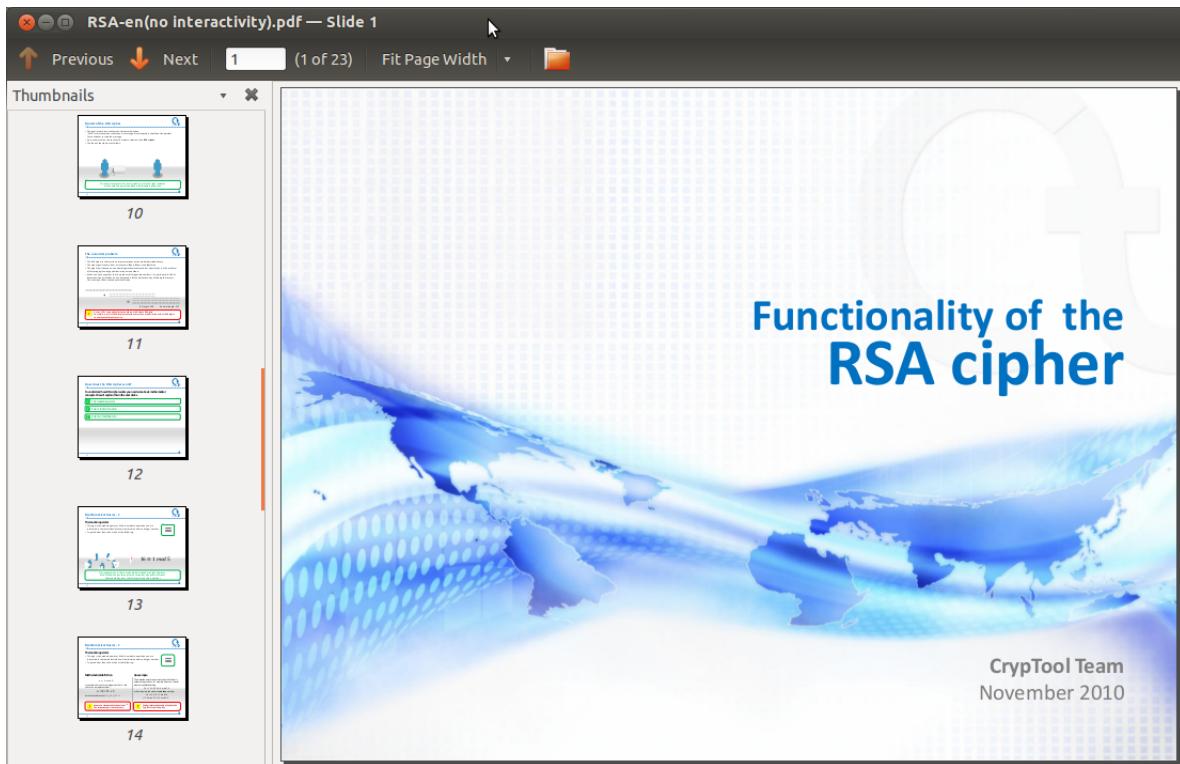


Figure 4.4: Screenshot RSA Presentation (PDF)

<sup>135</sup> <http://www.cryptool.org/images/ct1/presentations/RSA/RSA-de.pptx>

<sup>136</sup> [http://www.cryptool.org/images/ct1/presentations/RSA/RSA-de\(keine%20Interaktivitaet\).pdf](http://www.cryptool.org/images/ct1/presentations/RSA/RSA-de(keine%20Interaktivitaet).pdf)

<sup>137</sup> <http://www.cryptool.org/images/ct1/presentations/RSA/RSA-Flash-de/player.html>

## 4.19 Appendix: Examples using SageMath

“She would never be able to tell her parents ... about any of this. She couldn’t tell them about her code-breaking work. About her near death at the hands of the Daemon. About the shadowy entities pulling the strings of her government.”

Quote 14: Daniel Suarez<sup>138</sup>

Below you can find SageMath source code related to contents of the chapter 4 (“[Introduction to Elementary Number Theory with Examples](#)”).

### 4.19.1 Multiplication table modulo m

The multiplication table 4.4 (from page 129) for  $a \times i \pmod{m}$ , where  $m = 17$ ,  $a = 5$  and  $a = 6$ , and  $i$  ranges over all integers from 0 to 16 can be computed using SageMath as follows:

---

**SageMath sample 4.3** Multiplication tables for  $a \times i \pmod{m}$  with  $m = 17$ ,  $a = 5$  and  $a = 6$

```
sage: m = 17; a = 5; b = 6
sage: [mod(a * i, m).lift() for i in xrange(m)]
[0, 5, 10, 15, 3, 8, 13, 1, 6, 11, 16, 4, 9, 14, 2, 7, 12]
sage: [mod(b * i, m).lift() for i in xrange(m)]
[0, 6, 12, 1, 7, 13, 2, 8, 14, 3, 9, 15, 4, 10, 16, 5, 11]
```

---

The function `mod()` returns an object that represents integers modulo  $m$  (in our case  $m = 17$ ). From the `Mod` object you can get its single components either with the function `component` or with the function `lift`. We use the method `lift()` to convert that object to an integer representation.

The other multiplication table examples modulo 13 (table 4.5) and modulo 12 (table 4.6) on page 129 can similarly be computed by replacing `m = 17` with `m = 13` and `m = 12` respectively.

### 4.19.2 Fast exponentiation

The fast exponentiation modulo  $m$  can be computed using the SageMath function `power_mod()`. The result of this function is an integer. We can compute the exponentiation in the example in chapter “[Fast calculation of high powers](#)” on page 132 as follows:

---

**SageMath sample 4.4** Fast exponentiation mod  $m = 103$

```
sage: a = 87; m = 103
sage: exp = [2, 4, 8, 16, 32, 43]
sage: [power_mod(a, e, m) for e in exp]
[50, 28, 63, 55, 38, 85]
```

---

<sup>138</sup>Daniel Suarez, “Freedom”, Dutton Adult, 2010, Chapter 19, “Crossroad”, p. 229, Philips.

### 4.19.3 Multiplicative order

The order  $\text{ord}_m(a)$  of a number  $a$  in the multiplicative group  $\mathbf{Z}_m^*$  is the smallest number  $i \geq 1$  such that  $a^i \equiv 1 \pmod{m}$  holds (see chapter 4.9, “[Multiplicative order and primitive roots](#)”). To create table 4.7 on page 140 we can print all exponentiation  $a^i \pmod{11}$  as follows:

---

**SageMath sample 4.5** Table with all powers  $a^i \pmod{m}$  for  $m = 11$ ,  $a = 1, \dots, 10$

---

```
sage: m = 11
sage: for a in xrange(1, m):
....:     print [power_mod(a, i, m) for i in xrange(1, m)]
....:
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
[2, 4, 8, 5, 10, 9, 7, 3, 6, 1]
[3, 9, 5, 4, 1, 3, 9, 5, 4, 1]
[4, 5, 9, 3, 1, 4, 5, 9, 3, 1]
[5, 3, 4, 9, 1, 5, 3, 4, 9, 1]
[6, 3, 7, 9, 10, 5, 8, 4, 2, 1]
[7, 5, 2, 3, 10, 4, 6, 9, 8, 1]
[8, 9, 6, 4, 10, 3, 2, 5, 7, 1]
[9, 4, 3, 5, 1, 9, 4, 3, 5, 1]
[10, 1, 10, 1, 10, 1, 10, 1, 10]
```

and including the last column with the order of each  $a \pmod{11}$

```
sage: m = 11
sage: for a in xrange(1, m):
....:     lst= [power_mod(a, i, m) for i in xrange(1, m)]
....:     lst.append(multiplicative_order(mod(a,m)))
....:     print lst
....:
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
[2, 4, 8, 5, 10, 9, 7, 3, 6, 1, 10]
[3, 9, 5, 4, 1, 3, 9, 5, 4, 1, 5]
[4, 5, 9, 3, 1, 4, 5, 9, 3, 1, 5]
[5, 3, 4, 9, 1, 5, 3, 4, 9, 1, 5]
[6, 3, 7, 9, 10, 5, 8, 4, 2, 1, 10]
[7, 5, 2, 3, 10, 4, 6, 9, 8, 1, 10]
[8, 9, 6, 4, 10, 3, 2, 5, 7, 1, 10]
[9, 4, 3, 5, 1, 9, 4, 3, 5, 1, 5]
[10, 1, 10, 1, 10, 1, 10, 1, 10, 1, 2]
```

---

Table 4.8 on page 141 gives examples for the order modulo 45  $\text{ord}_{45}(a)$  and the Euler number  $\phi(45)$ .

The following SageMath code constructs a table similar to that on page 141.

---

**SageMath sample 4.6** Table with all powers  $a^i \pmod{45}$  for  $a = 1, \dots, 12$  plus the order of a

---

```
sage: m = 45
sage: for a in xrange(1, 13):
....:     lst = [power_mod(a, i, m) for i in xrange(1, 13)]
....:     try:
....:         lst.append(multiplicative_order(mod(a, m)))
....:     except:
....:         lst.append("None")
....:     lst.append(euler_phi(m))
....:     print lst
....:
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 24]
[2, 4, 8, 16, 32, 19, 38, 31, 17, 34, 23, 1, 12, 24]
[3, 9, 27, 36, 18, 9, 27, 36, 18, 9, 27, 36, 'None', 24]
[4, 16, 19, 31, 34, 1, 4, 16, 19, 31, 34, 1, 6, 24]
[5, 25, 35, 40, 20, 10, 5, 25, 35, 40, 20, 10, 'None', 24]
[6, 36, 36, 36, 36, 36, 36, 36, 36, 36, 36, 'None', 24]
[7, 4, 28, 16, 22, 19, 43, 31, 37, 34, 13, 1, 12, 24]
[8, 19, 17, 1, 8, 19, 17, 1, 8, 19, 17, 1, 4, 24]
[9, 36, 9, 36, 9, 36, 9, 36, 9, 36, 9, 36, 'None', 24]
[10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 'None', 24]
[11, 31, 26, 16, 41, 1, 11, 31, 26, 16, 41, 1, 6, 24]
[12, 9, 18, 36, 27, 9, 18, 36, 27, 9, 18, 36, 'None', 24]
```

---

The number  $\text{ord}_m(a)$  only exists if  $a$  is relatively prime to  $m$ , which can be checked with  $\text{gcd}(a, m)$ .

In the above code example, we put the calculation of the multiplicative order within a **try-except** block. This allows SageMath to catch any exceptions or errors raised by the function `multiplicative_order()`. If an exception or error is raised in the `try` block, then we know that  $\text{ord}_m(a)$  does not exist for that particular value of  $a$ , hence in the `except` block we append the string "None" to the row as represented by the object `lst`.

Table 4.9 on page 142 displays exponentiation  $a^i \pmod{46}$  as well as the order  $\text{ord}_{46}(a)$ .

SageMath can create that table as follows:

---

**SageMath sample 4.7** Table with all powers  $a^i \pmod{46}$  for  $a = 1, \dots, 23$  plus the order of a

---

```
sage: m = 46
sage: euler_phi(m)
22
sage: for a in xrange(1, 24):
....:     lst = [power_mod(a, i, m) for i in xrange(1, 24)]
....:     try:
....:         lst.append(multiplicative_order(mod(a, m)))
....:     except:
....:         lst.append("None")
....:     print lst
....:
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]
[2, 4, 8, 16, 32, 18, 36, 26, 6, 12, 24, 2, 4, 8, 16, 32, 18, 36, 26, 6, 12, 24, 2, 'None']
[3, 9, 27, 35, 13, 39, 25, 29, 41, 31, 1, 3, 9, 27, 35, 13, 39, 25, 29, 41, 31, 1, 3, 11]
[4, 16, 18, 26, 12, 2, 8, 32, 36, 6, 24, 4, 16, 18, 26, 12, 2, 8, 32, 36, 6, 24, 4, 'None']
[5, 25, 33, 27, 43, 31, 17, 39, 11, 9, 45, 41, 21, 13, 19, 3, 15, 29, 7, 35, 37, 1, 5, 22]
[6, 36, 32, 8, 2, 12, 26, 18, 16, 4, 24, 6, 36, 32, 8, 2, 12, 26, 18, 16, 4, 24, 6, 'None']
[7, 3, 21, 9, 17, 27, 5, 35, 15, 13, 45, 39, 43, 25, 37, 29, 19, 41, 11, 31, 33, 1, 7, 22]
[8, 18, 6, 2, 16, 36, 12, 4, 32, 26, 24, 8, 18, 6, 2, 16, 36, 12, 4, 32, 26, 24, 8, 'None']
[9, 35, 39, 29, 31, 3, 27, 13, 25, 41, 1, 9, 35, 39, 29, 31, 3, 27, 13, 25, 41, 1, 9, 11]
[10, 8, 34, 18, 42, 6, 14, 2, 20, 16, 22, 36, 38, 12, 28, 4, 40, 32, 44, 26, 30, 24, 10, 'None']
[11, 29, 43, 13, 5, 9, 7, 31, 19, 25, 45, 35, 17, 3, 33, 41, 37, 39, 15, 27, 21, 1, 11, 22]
[12, 6, 26, 36, 18, 32, 16, 8, 4, 2, 24, 12, 6, 26, 36, 18, 32, 16, 8, 4, 2, 24, 12, 'None']
[13, 31, 35, 41, 27, 29, 9, 25, 3, 39, 1, 13, 31, 35, 41, 27, 29, 9, 25, 3, 39, 1, 13, 11]
[14, 12, 30, 6, 38, 26, 42, 36, 44, 18, 22, 32, 34, 16, 40, 8, 20, 4, 10, 2, 28, 24, 14, 'None']
[15, 41, 17, 25, 7, 13, 11, 27, 37, 3, 45, 31, 5, 29, 21, 39, 33, 35, 19, 9, 43, 1, 15, 22]
[16, 26, 2, 32, 6, 4, 18, 12, 8, 36, 24, 16, 26, 2, 32, 6, 4, 18, 12, 8, 36, 24, 16, 'None']
[17, 13, 37, 31, 21, 35, 43, 41, 7, 27, 45, 29, 33, 9, 15, 25, 11, 3, 5, 39, 19, 1, 17, 22]
[18, 2, 36, 4, 26, 8, 6, 16, 12, 32, 24, 18, 2, 36, 4, 26, 8, 6, 16, 12, 32, 24, 18, 'None']
[19, 39, 5, 3, 11, 25, 15, 9, 33, 29, 45, 27, 7, 41, 43, 35, 21, 31, 37, 13, 17, 1, 19, 22]
[20, 32, 42, 12, 10, 16, 44, 6, 28, 8, 22, 26, 14, 4, 34, 36, 30, 2, 40, 18, 38, 24, 20, 'None']
[21, 27, 15, 39, 37, 41, 33, 3, 17, 35, 45, 25, 19, 31, 7, 9, 5, 13, 43, 29, 11, 1, 21, 22]
[22, 24, 22, 24, 22, 24, 22, 24, 22, 24, 22, 24, 22, 24, 22, 24, 22, 24, 22, 24, 22, 'None']
[23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 23, 'None']
```

---

The following code for generating the tables 4.10 and 4.11 at page 144 f. also delivers the result in a way, that it can be easily processed in LaTeX. The prerequisite is that all content is assigned to one SageMath object (here a matrix).<sup>139</sup>

---

**SageMath sample 4.8** Code for tables with all powers  $a^i \pmod{m}$  for variable  $a$  and  $i$  plus order of  $a$  and Eulerphi of  $m$

---

```
def power_mod_order_matrix(m, max_a, max_i):
    r = matrix(ZZ, max_a+1, max_i+3)
    for a in xrange(0, max_a+1):
        r[a, 0] = a
        for i in xrange(1, max_i+1):
            if a==0:
                r[a,i] = i
            else:
                r[a, i] = power_mod(a, i, m)
        try:
            r[a, max_i+1] = multiplicative_order(mod(a, m))
        except:
            r[a, max_i+1] = 0
            r[a, max_i+2] = euler_phi(m)
    return r

print "\n1: m=45;max_i=13;max_a=13";m=45;max_i=13;max_a=13
r = power_mod_order_matrix(m, max_a, max_i);print r;print latex(r)

print "\n2: m=46;max_i=25;max_a=25";m=46;max_i=25;max_a=25
r = power_mod_order_matrix(m, max_a, max_i);print r.str();print latex(r)

print "\n3: m=14;max_i=13;max_a=16";m=14;max_i=13;max_a=16
r = power_mod_order_matrix(m, max_a, max_i);print r;print latex(r)

print "\n4: m=22;max_i=21;max_a=25";m=22;max_i=21;max_a=25
r = power_mod_order_matrix(m, max_a, max_i);print r.str();print latex(r)
...
3: m=14;max_i=13;max_a=16
[ 0  1  2  3  4  5  6  7  8  9  10 11 12 13  0  6]
[ 1  1  1  1  1  1  1  1  1  1  1  1  1  1  1  6]
[ 2  2  4  8  2  4  8  2  4  8  2  4  8  2  0  6]
[ 3  3  9 13 11  5  1  3  9 13 11  5  1  3  6  6]
...
\left(\begin{array}{rrrrrrrrrrrrrr}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 0 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 6 \\
2 & 2 & 4 & 8 & 2 & 4 & 8 & 2 & 4 & 8 & 2 & 4 & 8 & 2 & 0 & 6 \\
3 & 3 & 9 & 13 & 11 & 5 & 1 & 3 & 9 & 13 & 11 & 5 & 1 & 3 & 6 & 6 \\
\end{array}\right)
```

---

<sup>139</sup>Remark about the SageMath program, especially the SageMath indices:

- for  $x$  in  $xrange(2, 5)$  delivers 2,3,4.
- $m = matrix(ZZ, 2, 5)$  has 2 rows and 5 columns.  
The cells are named  $m(0,0)$  to  $m(1,4)$ .
- All elements of the matrix have to be numerical, so “0” instead of “None” as in the tables before.
- The output of matrices can be controlled in SageMath with:

```
sage: from sage.matrix.matrix0 import set_max_cols, set_max_rows
sage: set_max_cols(100)
sage: set_max_rows(100)
```

- The length of the cycle in the last column of the tables 4.10 and 4.11 was added manually.

#### 4.19.4 Primitive roots

Computing a primitive root (see chapter 4.9, “[Multiplicative order and primitive roots](#)”) in SageMath is very straightforward. If  $n$  is an integer, the command `primitive_root(n)` computes *one* primitive root of the multiplicative group  $(\mathbf{Z}/n\mathbf{Z})^*$ , if one exists. If  $n$  is prime then this is the same as calculating a primitive root of  $\mathbf{Z}/n\mathbf{Z}$ .

Here, we calculate some primitive roots of a few integers.

---

**SageMath sample 4.9** Calculating one primitive root for a given prime

---

```
sage: primitive_root(4)
3
sage: primitive_root(22)
13
sage: for p in primes(1, 50):
....:     print p, primitive_root(p)
....:
2 1
3 2
5 2
7 3
11 2
13 2
17 3
19 2
23 5
29 2
31 3
37 2
41 6
43 3
47 5
```

---

If  $p$  is prime, then  $\mathbf{Z}/p\mathbf{Z}$  has at least one primitive root.

Sometimes we want to compute all the primitive roots of  $(\mathbf{Z}/n\mathbf{Z})^*$ , not just any primitive root of  $(\mathbf{Z}/n\mathbf{Z})^*$ . The following self-written function can do this.<sup>140</sup>

---

**SageMath sample 4.10** Function “enum\_PrimitiveRoots\_of\_an\_Integer” to calculate all primitive roots for a given number

---

```
def enum_PrimitiveRoots_of_an_Integer(M):
    """
    Return all the primitive roots of the integer M (if possible).
    """
    try:
        g = primitive_root(M)
    except:
        return None
    targetOrder = euler_phi(M)
    L=[]
    # Stepping through all odd integers from 1 up to M, not including
    # M. So this loop only considers values of i where 1 <= i < M.
    for i in xrange(1,M,2):
        testGen = Mod(g^i,M)
        if testGen.multiplicative_order() == targetOrder:
            L.append(testGen.lift())
    # removing duplicates
    return Set(L)

# AA_Start -- Testcases for enum_PrimitiveRoots_of_an_Integer(M)
print "AA_Start -- Testcases for enum_PrimitiveRoots_of_an_Integer(M)"
M=10; print "1-----Testcase: M = %s" % M
LL = enum_PrimitiveRoots_of_an_Integer(M)
if LL==None:
    print M
else:
    print LL
M=8; print "2-----Testcase: M = %s" % M
# M=8 hat keine primitive root mod m. Checke, ob per try - except abgefangen.
LL = enum_PrimitiveRoots_of_an_Integer(M)
if LL==None:
    print M
else:
    print LL
M=17; print "3-----Testcase: M = %s" % M
LL = enum_PrimitiveRoots_of_an_Integer(M)
if LL==None:
    print M
else:
    print LL
# AA_End -- Testcases

OUTPUT:
AA_Start -- Testcases for enum_PrimitiveRoots_of_an_Integer(M)
1-----Testcase: M = 10
{3, 7}
2-----Testcase: M = 8
8
3-----Testcase: M = 17
{3, 5, 6, 7, 10, 11, 12, 14}
```

---

<sup>140</sup>This code was developed in a SageMath script file and executed non-interactively. That is why you don't see “sage:” and “....:” at the beginning of the lines like in the SageMath samples before.

For example, here is a list of all primitive roots of the prime 541.

---

**SageMath sample 4.11** Table with all primitive roots for the given prime 541

---

```
sage: L=enum_PrimitiveRoots_of_an_Integer(541); L
{2, 517, 10, 523, 13, 14, 527, 528, 18, 531, 24, 539, 30, 37, 40, 51,
54, 55, 59, 62, 65, 67, 68, 72, 73, 77, 83, 86, 87, 91, 94, 96, 98,
99, 107, 113, 114, 116, 117, 126, 127, 128, 131, 132, 138, 150, 152,
153, 156, 158, 163, 176, 181, 183, 184, 195, 197, 199, 206, 208,
210, 213, 218, 220, 223, 224, 244, 248, 250, 257, 258, 259, 260,
261, 263, 267, 269, 270, 271, 272, 274, 278, 280, 281, 282, 283,
284, 291, 293, 297, 317, 318, 321, 323, 328, 331, 333, 335, 342,
344, 346, 357, 358, 360, 365, 378, 383, 385, 388, 389, 391, 403,
409, 410, 413, 414, 415, 424, 425, 427, 428, 434, 442, 443, 445,
447, 450, 454, 455, 458, 464, 468, 469, 473, 474, 476, 479, 482,
486, 487, 490, 501, 504, 511}
sage: len(L)
144
```

---

With a little bit of programming, we can count how many primitive roots are in a given range of integers. We can check this for all numbers or only for the primes within this range.

---

**SageMath sample 4.12** Function “count\_PrimitiveRoots\_of\_an\_IntegerRange” to calculate all primitive roots for a given range of integers

---

```
def count_PrimitiveRoots_of_an_IntegerRange(start, end, bPrimesOnly=True):
    """
        Compute all primitive roots of all numbers between start and end,
        inclusive, and count them.
        If the flag bPrimesOnly is True, it performs primality tests, so it
        allows us to count the number of primes from start to end, inclusive.
        If the flag bPrimesOnly is false, it additionally counts these even
        numbers which have no primitive root.
    """
    nCheckedNumb = 0
    nCheckedNumb_WithoutPrimitivRoots = 0
    nPrimitiveRoots = 0
    for n in xrange(start, end+1):
        if bPrimesOnly:
            if is_prime(n):
                nCheckedNumb += 1
                L = enum_PrimitiveRoots_of_an_Integer(n)
                nPrimitiveRoots += len(L)
            else:
                nCheckedNumb += 1
                L = enum_PrimitiveRoots_of_an_Integer(n)
                if L==None:
                    nCheckedNumb_WithoutPrimitivRoots += 1
                else:
                    nPrimitiveRoots += len(L)

        if bPrimesOnly:
            print "Found all %s" % nPrimitiveRoots + \
                  " primitive roots of %s primes." % nCheckedNumb
        else:
            if nCheckedNumb_WithoutPrimitivRoots == 0:
                print "Found all %s " % nPrimitiveRoots + \
                      "primitive roots of %s numbers." % nCheckedNumb
            else:
                print "Found all %s " % nPrimitiveRoots + \
                      "primitive roots of %s numbers." % \
                      (nCheckedNumb - nCheckedNumb_WithoutPrimitivRoots)
            print "(Total of numbers checked: %s, " % nCheckedNumb + \
                  "Amount of numbers without primitive roots: %s)" % \
                  nCheckedNumb_WithoutPrimitivRoots
```

---

Using the SageMath command `time`, we can also find out how long it takes on our computer.

---

**SageMath sample 4.13** Function “count\_PrimitiveRoots\_of\_an\_IntegerRange”: testcases and output

---

```
# BB_Start -- Testcases for count_PrimitiveRoots_of_an_IntegerRange(start, end, bPrimesOnly=True)
print "\n\nBB_Start -- Testcases for count_PrimitiveRoots_of_an_IntegerRange(start, end, True)"

print "\n1-----Testcase: (1, 500)"
time count_PrimitiveRoots_of_an_IntegerRange(1, 500)

print "\n2-----Testcase: (5, 6, False)"
time count_PrimitiveRoots_of_an_IntegerRange(5, 6, False)

print "\n3-----Testcase: (1, 500, False)"
time count_PrimitiveRoots_of_an_IntegerRange(1, 500, False)
# BB_End -- Testcases

OUTPUT:
BB_Start -- Testcases for count_PrimitiveRoots_of_an_IntegerRange(start, end, bPrimesOnly=True)

1-----Testcase: (1, 500)
Found all 8070 primitive roots of 95 primes.
Time: CPU 0.94 s, Wall: 0.97 s

2-----Testcase: (5, 6, False)
Found all 3 primitive roots of 2 numbers.
Time: CPU 0.00 s, Wall: 0.00 s

3-----Testcase: (1, 500, False)
Found all 11010 primitive roots of 170 numbers.
(Total of numbers checked: 500, Amount of numbers without primitive roots: 330)
Time: CPU 1.52 s, Wall: 1.59 s
```

---

Using our custom-defined function `enum_PrimitiveRoots_of_an_Integer`, we can find all primitive roots of one prime integer  $p$ .

The following function counts how many primes are in a given range and enumerates all their primitive roots.

From this list of primitive roots, we can determine the smallest and largest primitive root for  $\mathbf{Z}/p\mathbf{Z}$ , as well as count the number of primitive roots of  $\mathbf{Z}/p\mathbf{Z}$ .

---

**SageMath sample 4.14** Function “`count_PrimitiveRoots_of_a_PrimesRange`” to calculate the number of primitive roots for a given range of primes

---

```
def count_PrimitiveRoots_of_a_PrimesRange(start, end):
    """
    Compute all primitive roots of all primes between start and end,
    inclusive. This uses a primes iterator.
    """
    nPrimes = 0
    nPrimitiveRoots = 0
    for p in primes(start, end+1):
        L = enum_PrimitiveRoots_of_an_Integer(p)
        print p, len(L)
        nPrimes += 1
        nPrimitiveRoots += len(L)
    print "Found all %s" % nPrimitiveRoots + " primitive roots of %s primes." % nPrimes

# CC_Start -- Testcases for count_PrimitiveRoots_of_a_PrimesRange(start, end)
print "\n\nBB_Start -- Testcases for count_PrimitiveRoots_of_a_PrimesRange(start, end)"
print "-----Testcase: (1, 1500)"
time count_PrimitiveRoots_of_a_PrimesRange(1, 1500)
# CC_End -- Testcases

OUTPUT:
CC_Start -- Testcases for count_PrimitiveRoots_of_a_PrimesRange(start, end)
-----Testcase: (1, 1500)
2 1
3 1
5 2
7 2
11 4
13 4
17 8
19 6
23 10
29 12
31 8
37 12
...
1483 432
1487 742
1489 480
1493 744
1499 636
Found all 62044 primitive roots of 239 primes.
Time: CPU 7.55 s, Wall: 7.85 s
```

---

A variant of our function `count_PrimitiveRoots_of_a_PrimesRange` (slightly modified by Minh Van Nguyen) was used to generate a database of all primitive roots of all primes between 1 and 100,000.

---

**SageMath sample 4.15** Code to generate the database with all primitive roots for all primes between 1 and 100,000

---

```
start = 1
end = 10^5
fileName = "./primroots.dat"
file = open(fileName, "w")
for p in primes(start, end+1):
    L = enum_PrimitiveRoots_of_an_Integer(p)
    print p, len(L)
    # Output to a file. The format is:
    # (1) the prime number p under consideration
    # (2) the number of primitive roots of Z/pZ
    # (3) all the primitive roots of Z/pZ
    file.write(str(p) + " " + str(len(L)) + " " + str(L) + "\n")
    file.flush()
file.close()
```

---

This code and the function `enum_PrimitiveRoots_of_an_Integer` was executed in a Sage script file non-interactively. It took about 6 hours on a modern PC with SageMath 7.2.

Between 1 and 100,000 there are 9,592 primes. For them, more than 169 million primitive roots have been calculated. For each prime  $p > 3$  it holds that between 20 % and almost 50 % of all integers between 1 and  $p$  are an according primitive root.

The resulting file “primroots\_1-100000.dat” is a database of all primitive roots of all primes between 1 and 100,000 inclusive. It is a large file (about 1.1 GB uncompressed, and 156 MB compressed with 7Zip). You can find the compressed file at

[https://www.cryptool.org/images/ctp/documents/primroots\\_1-100000.7z](https://www.cryptool.org/images/ctp/documents/primroots_1-100000.7z).

Its content looks like this:

```
2      1      {1}
3      1      {2}
5      2      {2, 3}
7      2      {3, 5}
11     4      {8, 2, 6, 7}
...
99989  42840  {2, 3, 8, 10, 11, 13, 14, ..., 99978, 99979, 99981, 99986, 99987}
99991  24000  {65539, 6, 65546, 11, 12, ..., 65518, 65520, 87379, 65526, 65528}
```

The Sage script 4.16 calculates all primitive roots for all primes up to one million, and outputs for each prime the number of different primitive roots and the smallest primitive root.

---

**SageMath sample 4.16** Code to generate the database with the smallest primitive root for all primes between 1 and 1,000,000

---

```
start = 1
end = 10^6
fileName = "./primroot-smallest_up-to-one-million.dat"
file = open(fileName, "w")
file.write(timestring + "\n")
file.flush()
for p in primes(start, end+1):
    L = enum_PrimitiveRoots_of_an_Integer(p)
    # Output to commandline only p and number of prim roots of Z_p
    print p, len(L)
    # Output more to a file. The format is:
    # (1) the prime number p under consideration
    # (2) the number of primitive roots of Z_p
    # (3) the smallest primitive roots of Z_p
    LL = sorted(L) # necessary as the smallest primroot is not always
                    # found first by the enum fct (see L of p=43)
    file.write(str(p) + " " + str(len(L)) + " " + str(LL[0]) + "\n")
    file.flush()
file.flush()
file.close()
```

---

The Sage script 4.16 was stopped after several weeks (running on a modern PC with SageMath 7.2) after investigating all primes up to half a million. The result was stored in the file “primroot\_number-of-and-smallest\_up-to-prime-500107.dat”, which is 617 kB uncompressed, and 178 kB compressed with 7Zip.

You can find the compressed file at [https://www.cryptool.org/images/ctp/documents/primroot\\_number-of-and-smallest\\_up-to-prime-500107.7z](https://www.cryptool.org/images/ctp/documents/primroot_number-of-and-smallest_up-to-prime-500107.7z).

This database contains all primes  $p$  between 1 and 500,107 together with the according number of primitive roots and the according smallest prim root mod  $p$ . It holds that the number of primitive roots (for  $p > 3$ ) is always an odd number. We don't know a formula which delivers the number of primitive roots for a given number. So this database may be interesting to some number theorists.<sup>141</sup> Its content looks like this:

```
2      1      1
3      1      2
5      2      2
7      2      3
11     4      2
13     4      2
17     8      3
...
99989  42840  2
99991  24000  6
100003 28560  2
...
500069  250032 2
500083  151520 2
500107  156864 2
```

---

<sup>141</sup>See Re: [sage-devel] What can we do with a database of primitive roots?  
<https://groups.google.com/forum/m/#topic/sage-devel/TA5Nk2Gdh0I>

If you are looking only for the smallest primitive root, then this script could be accelerated dramatically by applying mathematical theory and searching more directly for possible candidates (instead of first generating all primitive roots with `enum_PrimitiveRoots_of_an_Integer`).

The database file “primroots\_1-100000.dat” then was used as input to create three graphics using the following code (Sage script 4.17).

---

**SageMath sample 4.17** Code to generate the graphics about the primitive roots

---

```
sage: # open a database file on primitive roots from 1 to 100,000
sage: file = open("/scratch/mvngu/primroots.dat", "r")
sage: plist = []      # list of all primes from 1 to 100,000
sage: nlist = []      # number of primitive roots modulo prime p
sage: minlist = []    # smallest primitive root modulo prime p
sage: maxlist = []    # largest primitive root modulo prime p
sage: for line in file:
....:     # get a line from the database file and tokenize it for processing
....:     line = line.strip().split(" ", 2)
....:     # extract the prime number p in question
....:     plist.append(Integer(line[0]))
....:     # extract the number of primitive roots modulo p
....:     nlist.append(Integer(line[1]))
....:     # extract the list of all primitive roots modulo p
....:     line = line[-1]
....:     line = line.replace("{", "")
....:     line = line.replace("}", "")
....:     line = line.split(", ")
....:     # sort the list in non-decreasing order
....:     line = [Integer(s) for s in line]
....:     line.sort()
....:     # get the smallest primitive root modulo p
....:     minlist.append(line[0])
....:     # get the largest primitive root modulo p
....:     maxlist.append(line[-1])
....:
sage: file.close()  # close the database file
sage: # plot of number of primitive roots modulo p
sage: nplot = point2d(zip(plist, nlist), pointsize=1)
sage: nplot.axes_labels(["x", "y"])
sage: nplot
sage: # plot of smallest primitive root modulo prime p
sage: minplot = point2d(zip(plist, minlist), pointsize=1)
sage: minplot.axes_labels(["x", "y"])
sage: minplot
sage: # plot of largest primitive root modulo prime p
sage: maxplot = point2d(zip(plist, maxlist), pointsize=1)
sage: maxplot.axes_labels(["x", "y"])
sage: maxplot
```

---

Figure 4.5 graphs the number of primitive roots for each prime between 1 and 100,000. The  $x$ -axis represents primes between 1 and 100,000, while the  $y$ -axis counts the number of primitive roots for each prime within that interval.

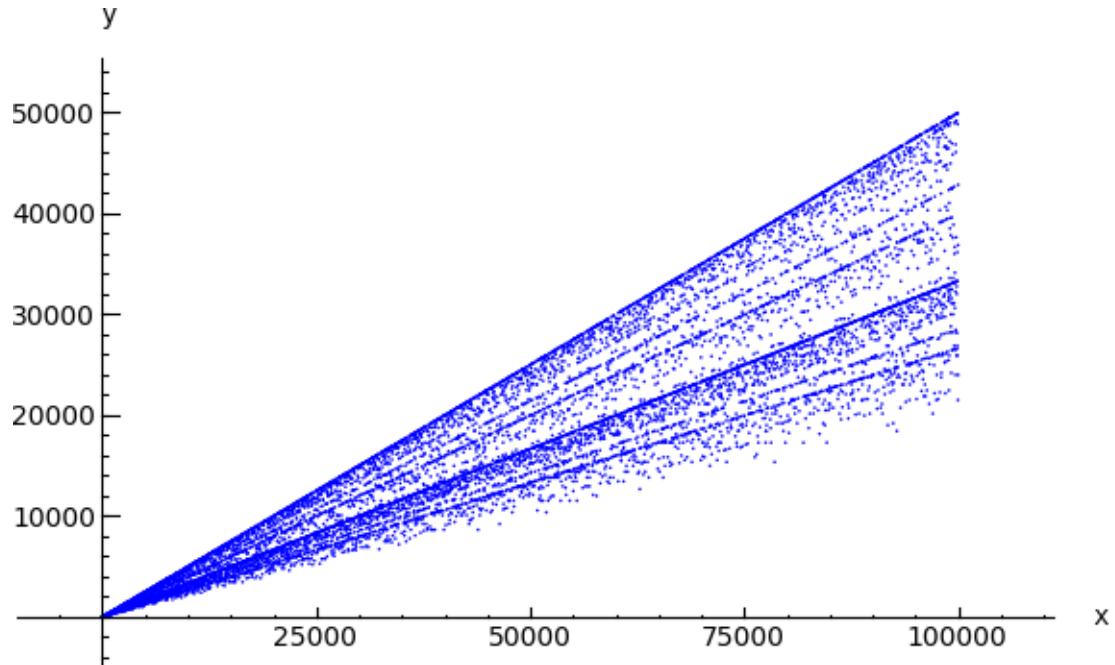


Figure 4.5: The number of primitive roots of all primes between 1 and 100,000.

Figure 4.6 graphs the smallest primitive roots of all primes between 1 and 100,000. The  $x$ -axis represents primes between 1 and 100,000. The  $y$ -axis represents the smallest primitive root of each prime within that interval.

Figure 4.7 shows a corresponding graph for the largest primitive root of each prime within the above interval.

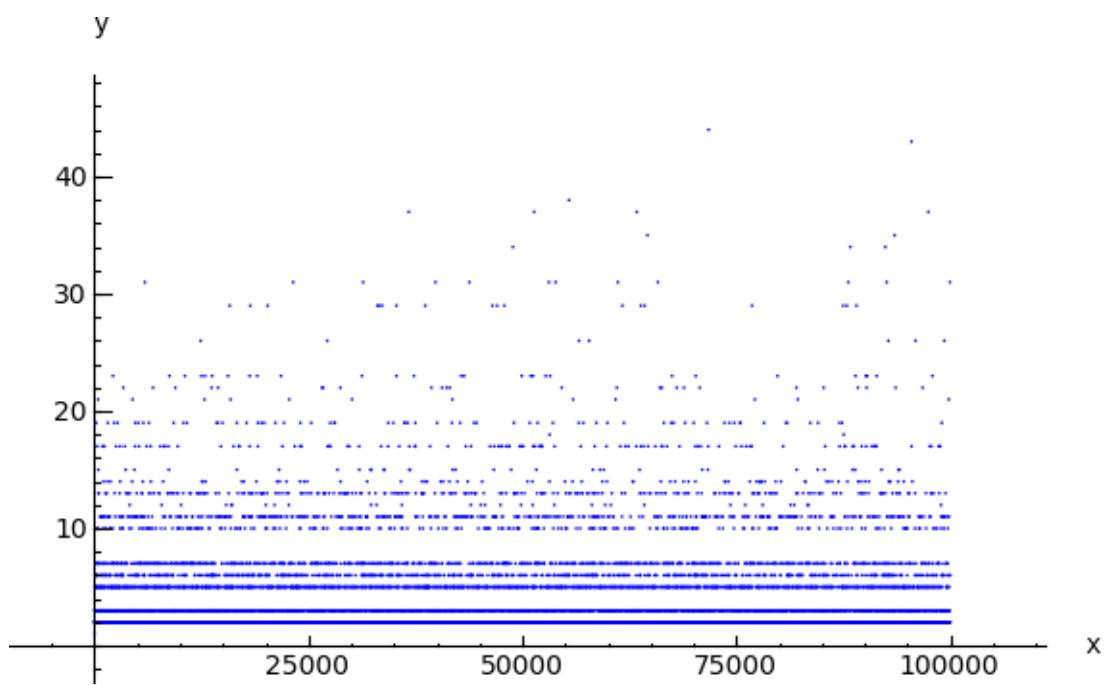


Figure 4.6: The smallest primitive roots of all primes between 1 and 100,000.

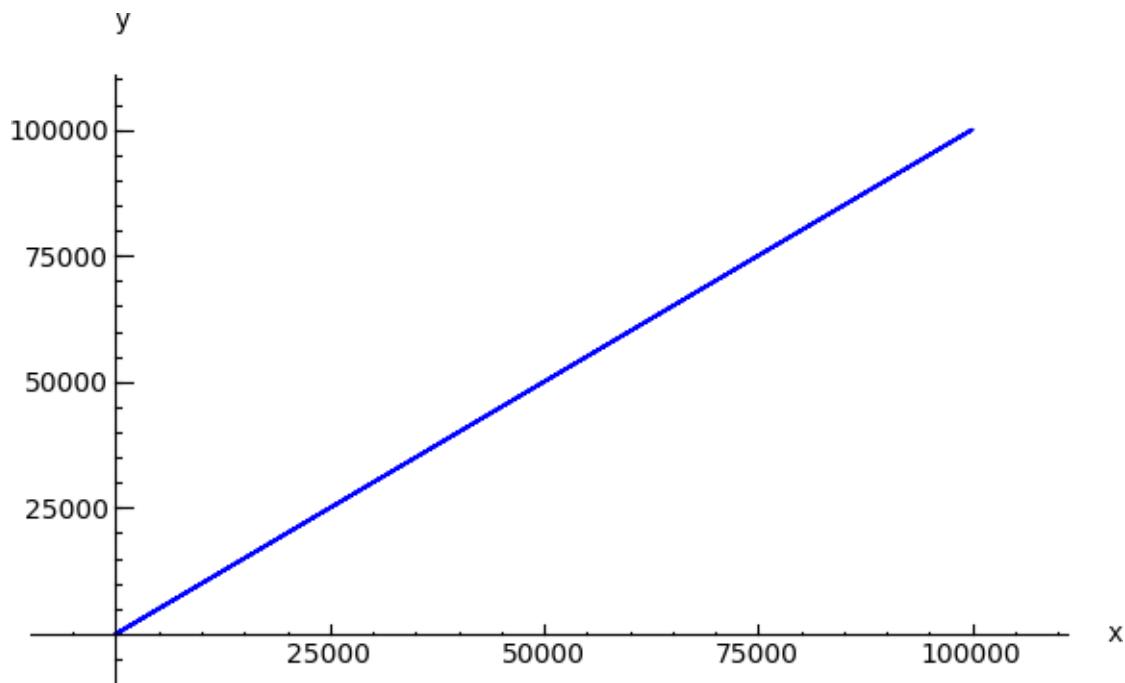


Figure 4.7: The largest primitive roots of all primes between 1 and 100,000.

#### 4.19.5 RSA examples with SageMath

Below is SageMath source code for the simple RSA examples in section 4.13 (“The RSA procedure with actual numbers”).

##### Example on page 172:

The RSA exponentiation  $M^{37} \pmod{3713}$  on message  $M = 120$  can be calculated in SageMath as follows:

```
sage: power_mod(120, 37, 3713)
1404
```

##### Example on page 173:

The factorization of  $\phi(256027) = 255016 = 2^3 * 127 * 251$  can be calculated using SageMath as follows:

---

##### SageMath sample 4.18 Factoring a number

---

```
sage: factor(255016)
2^3 * 127 * 251
```

---

##### Example on page 173:

SageMath can do RSA encryption as follows:

---

##### SageMath sample 4.19 RSA encryption by modular exponentiation of a number (used as message)

---

```
sage: A = [82, 83, 65, 32, 119, 111, 114, 107, 115, 33]
sage: e = 65537; m = 256027
sage: [power_mod(a, e, m) for a in A]
[212984, 25546, 104529, 31692, 248407, 100412, 54196, 100184, 58179, 227433]
```

---

##### Example on page 174:

RSA encryption using SageMath:

```
sage: A = [21075, 16672, 30575, 29291, 29473]
sage: e = 65537; m = 256027
sage: [power_mod(a, e, m) for a in A]
[158721, 137346, 37358, 240130, 112898]
```

##### Example on page 174:

RSA encryption using SageMath:

```
sage: A = [82083, 65032, 119111, 114107, 115033]
sage: e = 65537; m = 256027
sage: [power_mod(a, e, m) for a in A]
[198967, 51405, 254571, 115318, 14251]
```

#### 4.19.6 How many private RSA keys $d$ exist within a given modulo range?

The RSA encryption procedure was described in section 4.10.2 (“How the RSA procedure works”). Steps 1 to 3 constitute key generation, steps 4 and 5 are the encryption:

1. Select two distinct random prime numbers  $p$  and  $q$  and calculate  $n = p * q$ .  
The value  $n$  is called the RSA modulus.
2. Select an arbitrary  $e \in \{2, \dots, n - 1\}$  such that:  
 $e$  is relatively prime to  $\phi(n) = (p - 1) * (q - 1)$ .  
We can then “throw away”  $p$  and  $q$ .
3. Select  $d \in \{1, \dots, n - 1\}$  with  $e * d \equiv 1 \pmod{\phi(n)}$ ,  
i.e.  $d$  is the multiplicative inverse of  $e$  modulo  $\phi(n)$ . We can then “throw away”  $\phi(n)$ .  
 $\rightarrow (n, e)$  is the public key  $P$ .  
 $\rightarrow (n, d)$  is the private key  $S$  (only  $d$  must be kept secret).
4. For encryption, the message represented as a (binary) number is divided into parts such that each part of the number represents a number less than  $n$ .
5. Encryption of the plaintext (or the parts of it)  $M \in \{1, \dots, n - 1\}$ :

$$C = E((n, e); M) := M^e \pmod{n}.$$

The default way to crack a given RSA ciphertext  $C$  would be to use the public key of the recipient and to try to factorize  $n$ . Then you can go through the steps 2 and 3 and generate the private key  $e$ , which is normally used to decrypt a ciphertext.

According to the “prime number theorem”<sup>142</sup> the number of prime numbers  $PI(x)$  is asymptotic to  $x/\ln(x)$ . Between 1 and a given  $n$  there are about  $n/\ln(n)$  different primes.

If you don't want to use factorization but ask the question like in classic encryption, you may want to find out: How many possible private keys  $(n, d)$  are there for a given key size range  $n \in [a, b]$ ?<sup>143</sup>

SageMath source code 4.20 below defining the function `count_Number_of_RSA_Keys` can answer this question concretely (if the modulus is not too big).<sup>144</sup>

As there are many more private keys  $(n, d)$  within a bigger range of values for  $n$ , even brute-force factoring is much more efficient as brute-force trying all the keys.

---

<sup>142</sup>See section 3.7.2 (“Density and distribution of the primes”).

<sup>143</sup>Chapter 4.8.5 (“How many private RSA keys  $d$  are there modulo 26”), p. 138 deals with the special case  $n = 26$ .

<sup>144</sup>a) Calling `sage: count_Number_of_RSA_Keys(100, 1000)` means to consider the interval  $[100, 1000]$  for  $n$ .

$n$  is defined by the two primes  $p, q : n = p * q$ .

So here one prime can have the maximal value 500 because  $2 * 500 = 1000$  (while then the other prime will have the smallest possible prime value 2).

The number of possible combinations of primes is  $comb = 258$ .

The number of primes in the given range is 143.

The number of private keys is 34,816.

b) Calling `sage: count_Number_of_RSA_Keys(100, 100, True)` has the following output:

- Number of private keys for modulus in a given range: 0

- Number of primes in a given range: 0

The reason for that is, that with this call only  $n = 100$  is considered, and the function investigates only semiprime  $n$ : 100 is not semi prime, this means 100 is not the product of only two primes.

---

**SageMath sample 4.20** How many private RSA keys d are there if you know a range for the public key n?

---

```
def count_Number_of_RSA_Keys(start, end, Verbose=False):
    """
    How many private RSA keys (n,d) exist, if only modulus N is given, and start <= N <= end?
    (prime_range(u,o) delivers all primes >=u und < o).
    """
    a = start
    b = end
    s = 0
    comb = 0
    for p in prime_range(1, b/2+1):
        for q in prime_range(p + 1, b/2+1):
            if a <= p * q and p * q <= b:
                comb = comb +1
                s = s + (euler_phi(euler_phi(p * q))-1)
            if Verbose:
                print "p=%s, " % p + "q=%s, " % q + "s=%s" % s
    print "Number of private keys d for modulus in a given range: %s" % s + " (comb=%s), " % comb

    # Just for comparison: How many primes are in this range?
    s = 0
    for p in prime_range(a, b+1):
        if Verbose:
            print "a=%s, " % a + "b=%s, " % b + "p=%s" % p
        s = s + 1
    print "Number of primes in a given range: %s" % s

print "\n\nDD_Start -- Testcases for count_Number_of_RSA_Keys(start, end)"
print "\n-----Testcase: (100, 1000) [Should deliver 34.816]"
time count_Number_of_RSA_Keys(100, 1000)
print "\n-----Testcase: (100, 107, True) [Should deliver 23]"
time count_Number_of_RSA_Keys(100, 107, True)
u = 10^3; o = 10^4;
print "\n-----Testcase: (%s, " % u + "%s) [Should deliver 3.260.044]" % o
time count_Number_of_RSA_Keys(u, o)

OUTPUT:
DD_Start -- Testcases for count_Number_of_RSA_Keys(start, end)

-----Testcase: (100, 1000) [Should deliver 34.816]
Number of private keys d for modulus in a given range: 34816 (comb=258),
Number of primes in a given range: 143
Time: CPU 0.03 s, Wall: 0.04 s

-----Testcase: (100, 107, True) [Should deliver 23]
p=2, q=53, s=23
Number of private keys d for modulus in a given range: 23 (comb=1),
a=100, b=107, p=101
a=100, b=107, p=103
a=100, b=107, p=107
Number of primes in a given range: 3
Time: CPU 0.00 s, Wall: 0.00 s

-----Testcase: (1000, 10000) [Should deliver 3,260,044]
Number of private keys d for modulus in a given range: 3260044 (comb=2312),
Number of primes in a given range: 1061
Time: CPU 0.63 s, Wall: 0.66 s
```

---

#### 4.19.7 RSA fixed points $m^e = m \bmod n$ mit $m \in \{1, \dots, n - 1\}$

Also encryption methods can have fixed – cleartext messages where the according ciphertext matches the original. In mathematics, variables mapped by the algorithm (function) onto themselves are called fixed points. In cryptography the according messages are called “unconcealed messages”.

Generally speaking: The more fixed points an encryption algorithm contains, the easier it is to break it.

With the RSA procedure:  $n = pq$  is the product of two different prime numbers, and there exists  $e$  where  $\gcd(e, (p-1)(q-1)) = 1$ . The encryption is then  $c = m^e \bmod n$ . A fixed point in the RSA procedure is a message  $m$ , where:  $m = m^e \bmod n$ . The result of the encryption is the given message.

When the size of  $n$  is sufficiently big, the probability of the occurrence of fixed points in RSA is very small – as illustrated in Figure 4.8: In average, we found not more than 40 fixed points.

Students often presume the occurrence of fixed points high, because they counter a “relatively” large number of examples when experimenting with **small** prime numbers, as  $m = 0, 1$  and  $n-1$  are also always fixed points.

In practice, where large prime numbers are chosen, fixed points have no significance for the security of RSA. Therefore, this paragraph refers more to the mathematical questions.<sup>145</sup>

##### 4.19.7.1 The number of RSA fixed points

In this section we show how many RSA fixed points there are for  $m \in \{1, \dots, n - 1\}$ .

**Theorem 4.19.1.** *The number of the fixed points  $m^e = m \bmod n$  with  $m \in \{1, \dots, n - 1\}$  is  $\gcd(p - 1, e - 1) \cdot \gcd(q - 1, e - 1)$ .*

##### Proof

Given  $m^e = m \bmod n$ . According to the CRT<sup>146</sup>, the following statements are equivalent:

$$[m^e = m \bmod n] \Leftrightarrow [m^e = m \bmod p \text{ and } m^e = m \bmod q]$$

Furthermore, the decomposition on the right side is equivalent to:

$$m^{e-1} = 1 \bmod p \text{ and } m^{e-1} = 1 \bmod q.$$

We consider  $m^{e-1} = 1 \bmod p$  and search all  $(e-1)$  roots of unity<sup>147</sup> in  $\mathbb{Z}_p^*$ .

It holds:  $\mathbb{Z}_p^*$  for  $p$  prime is cyclic.  $\Rightarrow$  A generator  $g$  exists which produces  $\mathbb{Z}_p^*$ :  $\mathbb{Z}_p^* = \langle g \rangle$ .

---

<sup>145</sup>Thanks to Taras Shevchenko for gathering parts of the content of this section and to Volker Simon for writing the SageMath program 4.21 “Getfixpoints”.

<sup>146</sup>CRT = Chinese Remainder Theorem. [http://en.wikipedia.org/wiki/Chinese\\_Remainder\\_Theorem](http://en.wikipedia.org/wiki/Chinese_Remainder_Theorem)

<sup>147</sup>- In mathematics, a **root of unity** is a number  $x$  that equals 1 when raised to some integer power  $n$ .  
- An  $n$ -th root of unity  $x$  is **primitive** if it is not a  $k$ -th root of unity for all integers  $k$  smaller than  $n$ :

$$x^n = 1 \text{ and } x^k \neq 1 \quad (k = 1, 2, 3, \dots, n - 1)$$

- If  $F$  is a finite field and  $n$  is a positive integer, then a  $n$ th-root of unity in  $F$  is a solution of the equation

$$x^n - 1 = 0 \text{ in } F$$

The following theorem from [Kat01, Pg. 69] characterizes all  $(e - 1)$ -th roots of unity in  $\mathbb{Z}_p^*$ :

**Theorem 4.19.2.**  $g^\alpha$  is exactly then  $(e - 1)$ -th root of unity in  $\mathbb{Z}_p^*$ , when  $(e - 1)\alpha = 0 \pmod{p - 1}$ . There are  $\gcd(p - 1, e - 1)$  of these.

### Proof

The first theorem results directly from the small theorem from Fermat:

$$g^{\alpha(e-1)} = 1 \pmod{p} \Rightarrow \alpha(e-1) = 0 \pmod{p-1}$$

Let  $\delta = \gcd(p - 1, e - 1)$ .  $\alpha(e - 1) = 0 \pmod{p - 1}$  implies  $\frac{\alpha(e-1)}{\delta} = 0 \pmod{\frac{p-1}{\delta}}$ .

Since  $\frac{e-1}{\delta}$  and  $\frac{p-1}{\delta}$  are coprime (each was reduced by the gcd of their corresponding numerator),  $\alpha$  must be a multiple of  $\frac{p-1}{\delta}$ .

$$\alpha \frac{p-1}{\delta} \text{ with } \alpha = 1, \dots, \delta$$

These  $\delta$  different powers then correspond to the  $(e - 1)$ -th roots of unity  $g^{\alpha \frac{p-1}{\delta}} \pmod{p}$  in  $\mathbb{Z}_p^*$ .  $\square$

Analog for  $q$ : For  $m^{e-1} = 1 \pmod{q}$  we then have  $\gcd(q - 1, e - 1)$  many of  $(e - 1)$ -th roots of unity.

The number of combinations of the  $(e - 1)$ -th root of unity in  $\mathbb{Z}_p^*$  and  $\mathbb{Z}_q^*$  gives the total quantity of RSA fixed points:  $m^e = m \pmod{n}$  with  $m \in \{1, \dots, n - 1\}$ :

$$\gcd(p - 1, e - 1) \cdot \gcd(q - 1, e - 1)$$

Adding  $m = 0$  to the above, results in the theorem 4.19.3:

**Theorem 4.19.3.** If  $m \in \{0, \dots, n - 1\}$ , then the quantity of the RSA fixed points is:

$$(\gcd(p - 1, e - 1) + 1) \cdot (\gcd(q - 1, e - 1) + 1)$$

$\square$

### 4.19.7.2 Lower bound for the quantity of RSA fixed points

In the following section, we show that there is a lower bound for the quantity of RSA fixed points. This lower bound 6 exists when the two different RSA prime numbers are the smallest possible values (2 and 3).

**Theorem 1:** Given:  $p = 2, q = 3$

The quantity of RSA fixed points for  $p = 2$  and  $q = 3$  is

$$\underbrace{(\gcd(p - 1, e - 1) + 1)}_{=1} \cdot \underbrace{(\gcd(q - 1, e - 1) + 1)}_{=2} = 2 \cdot 3 = 6$$

**Theorem 2:** Given:  $p \neq q; p > 2, q > 2$

The quantity of RSA fixed points for  $p \neq q; p, q > 2$  is  $\geq 9$ .

#### Proof (of the 2nd theorem)

Since  $p$  and  $q$  are prime,  $(p - 1)$  and  $(q - 1)$  for  $p, q > 2$  are even.

The RSA algorithm requires to choose  $e$  so that  $1 < e < \phi(n) = (p - 1)(q - 1)$  and

$$\gcd(e, (p-1)(q-1)) = 1$$

Since  $(p-1)$  and  $(q-1)$  are even,  $e$  is odd  $\Rightarrow e-1$  is even.

Since  $(p-1)$  and  $(e-1)$  are even, then:

$$\gcd(p-1, e-1) \geq 2$$

$$\Rightarrow (\gcd(p-1, e-1) + 1) \geq 3 \text{ and } (\gcd(q-1, e-1) + 1) \geq 3$$

$$\Rightarrow (\gcd(p-1, e-1) + 1) \cdot (\gcd(q-1, e-1) + 1) \geq 9$$

□

### Samples:

For  $(e, n) = (17, 6)$ , all six possible messages  $\{0,1,2,3,4,5\}$  are fixed points (for  $n = 6$ , it is independent from the value of  $e$ ).

For  $(e, n) = (17, 10)$ , all 10 possible messages are fixed points.

For  $(e, n) = (19, 10)$ , only 6 of the 10 possible messages are fixed points.

#### 4.19.7.3 Unfortunate choice of $e$

In this section, we show that with  $e = 1 + \text{lcm}(p-1, q-1)$  each encryption results in a fixed point (independently of the size of  $p, q$ , or  $n$ , each  $m$  is mapped on itself); and then we generalize this to all unfortunate choices of  $e$ .

If  $e = 1$ , then for all  $m$ :  $c = m^e = m$ . This is the trivial case.

**Theorem:** Given:  $p, q > 2$

If  $e = 1 + \text{lcm}(p-1, q-1)$ , then for all  $m \in \{1, \dots, n-1\}$ :  $m^e = m \pmod{n}$ .

### Proof

Given:

- $e \cdot d = 1 \pmod{\phi(n)}$  or  $e \cdot d = 1 \pmod{\text{lcm}(p-1, q-1)}$
- $m^e \pmod{n} = m^{e \pmod{\phi(n)}} \pmod{n}$

Encryption of messages:

$c = m^e \pmod{n}$ , where  $c$  is the ciphertext and  $m$  is the plaintext.

Decryption of messages:

$m' = c^d \pmod{n}$ , where  $d$  is the multiplicative inverse of  $e$ .

We will show:  $c = m \pmod{n}$  for the chosen  $e$ .

$$c = m^e \pmod{n}$$

$$c = m^{1+\text{lcm}(p-1, q-1)} \pmod{n}$$

$$c = m^1 \cdot m^{k \cdot (p-1) \cdot (q-1)} \pmod{n}$$

$$c = m^1 \cdot m^{[k \cdot \phi(n)]} \pmod{\phi(n)} \pmod{n}$$

$$c = m^1 \cdot m^0 = m \pmod{n}$$

□

**Example: Fixed point property for all m:**

Given  $n = p \cdot q = 13 \cdot 37 = 481$

$$\Rightarrow \phi(n) = (p-1)(q-1) = 12 \cdot 36 = 432$$

$$\Rightarrow e = \text{lcm}(p-1, q-1) + 1 = \text{lcm}(12, 36) + 1 = 36 + 1 = 37.$$

With  $m \in \{4, 6, 7, 480\}$  we get in  $m^e \bmod n$  as:

$$4^{37} \bmod 481 = 4$$

$$6^{37} \bmod 481 = 6$$

$$7^{37} \bmod 481 = 7$$

$$480^{37} \bmod 481 = 480$$

There is not just the one single  $e$  (see above), where all  $m \in \{1, \dots, n-1\}$  have the fixed point property  $m^e = m \bmod n$ .<sup>148</sup>

**Theorem 4.19.4.** *The complete fixed point property of all  $m$  is valid for every  $e = j \cdot \text{lcm}(p-1, q-1) + 1$ , where  $j = 0, 1, 2, 3, 4, \dots$  to  $e \leq \phi(n)$ .*

**Example: Further values for  $e$  with fixed point properties:**

Given  $n = p \cdot q = 13 \cdot 37 = 481$  with  $\text{lcm}(p-1, q-1) = \text{lcm}(12, 36) = 36$ .

Then,  $e$  can have the following values:  $e = j \cdot \text{lcm}(p-1, q-1) + 1$  for  $j = 0, 1, 2, \dots, 11$ :

$$\Rightarrow e \in \{1, 37, 73, 109, 145, 181, 217, 253, 289, 325, 361, 397\}.$$

Starting  $j = 12$ , the following is valid:  $e = 12 \cdot \text{lcm}(12, 36) + 1 = 432 + 1 = 433 > 432 = \phi(n)$ .

Checking the four values above for  $m$  with  $e = 217$ , the results are:

$$4^{217} \bmod 481 = 4$$

$$6^{217} \bmod 481 = 6$$

$$7^{217} \bmod 481 = 7$$

$$480^{217} \bmod 481 = 480$$

**Theorem 4.19.5.** *The number of possible values for  $e$  with  $m^e = m \bmod n$  may be computed with the following:*

$$[\text{Quantity } e] = \left\lfloor \frac{\phi(n)}{\text{lcm}(p-1, q-1) + 1} \right\rfloor + 1 = \frac{\phi(n)}{\text{lcm}(p-1, q-1)}$$

In our example, this results in  $\frac{432}{\text{lcm}(12, 36)} = 12$  different values for  $e$ , where  $m^e = m \bmod n$  for all  $m$  in  $\mathbb{Z}_{481}$ .

---

<sup>148</sup>Sometimes these  $e$ , which make any message to a fixed point, are called “weak keys” ( $e, n$ ) of the RSA algorithm. This notation is different to the “weak keys”  $k$  in DES, where **every** message  $m$  relates to itself if the encryption is done twice. To my knowledge, for larger  $n$  the RSA procedure does not have weaks in this meaning:  $(m^e)^e = m$ . In JCT you can find weak DES keys in the default perspective via the menu item **Visuals \ Inner States of the Data Encryption Standard (DES)**.

#### 4.19.7.4 An empirical estimate of the quantity of fixed points for growing moduli

In this section, we make an empirical estimate of the quantity of fixed points for growing moduli (and  $e$  not weak).

For this, we randomly choose  $p$  and  $q$  from the six following ranges each characterized by its lower and upper bound:  $(2^2, 2^{10})$ ,  $(2^{10}, 2^{20})$ ,  $(2^{20}, 2^{40})$ ,  $(2^{40}, 2^{80})$ ,  $(2^{80}, 2^{160})$ ,  $(2^{160}, 2^{320})$ .

10 attempts were made for each range. For the exponent  $e$ , the standard value  $e = 2^{16} + 1$  was always chosen. The quantity of fixed points for all 60 attempts was computed with the program [4.21 “Getfixpoints.sage”](#).

The following five sets contain the randomly chosen value pairs  $(p,q)$  of the first five ranges.

$$From(2^2, 2^{10}) : (p, q) \in \{(127, 947), (349, 809), (47, 461), (587, 151), (19, 23), (709, 509), (653, 11), (859, 523), (823, 811), (83, 331)\}$$

$$From(2^{10}, 2^{20}) : (p, q) \in \{(447401, 526283), (474223, 973757), (100829, 126757), (35803, 116933), (577751, 598783), (558121, 607337), (950233, 248167), (451103, 73009), (235787, 164429), (433267, 287939)\}$$

$$From(2^{20}, 2^{40}) : (p, q) \in \{(58569604997, 321367332149), (286573447351, 636576727223), (134703821971, 134220414529), (161234614601, 711682765579), (19367840881, 804790726361), (932891507377, 521129503333), (337186437739, 426034644493), (986529569219, 604515928397), (276825557171, 654134442649), (639276602353, 1069979301731)\}$$

$$From(2^{40}, 2^{80}) : (p, q) \in \{(667530919106151273090539, 287940270633610590682889), (437090557112369481760661, 590040807609821698387141), (1131921188937480863054851, 813935599673320990215139), (874130181777177966406673, 632270193935624953596331), (599303355925474677078809, 717005631177936134003029), (752829320004631398659063, 714134510643836818718761), (1046313315092743492917349, 835721729660755006973833), (877161707568112212806617, 42831503328261105793649), (575464819450637793425803, 5425832051159043433027), (321404337099945148592363, 992663778486687980443879)\}$$

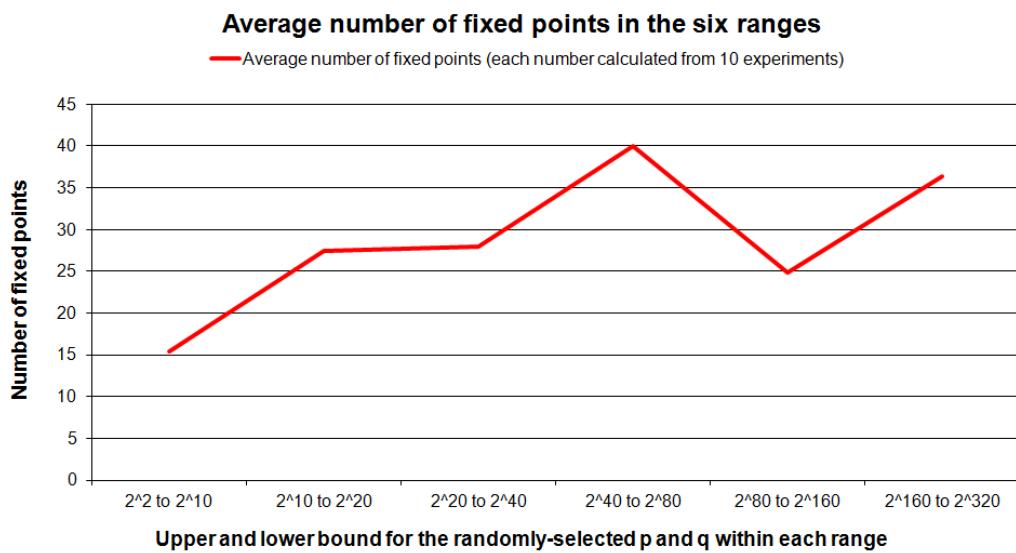
$$From(2^{80}, 2^{160}) : (p, q) \in \{(838952969674957834783403492645269831354775774659, 694309130163549038783972189350416942879771871411), (981985107290629501374187748859961786804311564643, 178616495258601001174141825667078950281544628693), (614446632627716919862227545890890553330513965359, 761232454374959264696945191327265643178491649141), (1421756952722008095585945863962560425554707936337, 986781711714138924140285492105143175328486228197), (862346475785474165539441761205023498091366178341, 438589995804600940885415547506719456975478582911), (1034081318899669345416602574034081247538053001533, 1207032778571434704618111297072774884748706223447), (308083812465705343620096534684980088954958466893, 350597371862294596793629011464584694618569736021), (830376326124356299120963861338027196931951857769, 924874232653136669722297184352059466357375363191), (85600581120154590810189237569820706006659829231, 297064381842806596646150718828138629443319259829), (1358984492013516052055790129324581847590275909129, 609402294805414245544586792657989060761523960427)\}$$


Figure 4.8: An empirical estimate of the quantity of fixed points for growing moduli

Figure 4.8 shows that within the six ranges of size, the average number of fixed points was not higher than 40.

#### 4.19.7.5 Example: Determining all fixed points for a specific public RSA key

The exercise is to determine all fixed points for  $(n, e) = (866959, 17)$ .

##### Solution:

We start by factoring  $n$ :  $866959 = 811 \cdot 1069$ .

The quantity of RSA fixed points results from the theorem 4.19.3:

$$(gcd(p-1, e-1)+1) \cdot (gcd(q-1, e-1)+1) = (gcd(811-1, 17-1)+1) \cdot (gcd(1069-1, 17-1)+1) = (2+1) \cdot (4+1) = 15$$

The SageMath program 4.21 (Getfixpoints) returns 15 fixed points for  $(n, e) = (866959, 17)$ :

0	1	23518	23519	47037
188964	212482	236000	654477	843440
843441	630959	677995	819922	866958

##### Example:

Using 843441 as a sample for validation:  $843441^{17} \bmod 866959 = 843441$

So  $m = 843441$  is actually a fixed point for the given  $(n, e)$ .

#### Meaning of the Variables in the SageMath Code 4.21:

- `gen_f_p = r.multiplicative_generator()`  
`r` is a residue class ring modulo  $p$  and `multiplicative_generator()` returns a generator element that was created by the ring modulo  $p$ .
- `power_mod(gen_f_p, Integer(i*(p-1)/gcd_p), p)`  
The `power_mod` function raises a number  $m$  to the power of  $e$  and returns the results modulo  $n$ .  
E.g.: `power_mod(m, e, n) := m^e \bmod n`
- `numpy.append(fp, power_mod(gen_f_p, Integer(i*(p-1)/gcd_p), p))`  
The `append` function extends an array (`fp`) by an additional element.
- `crt(Integer(r), Integer(s), Integer(p), Integer(q))`  
`CRT` is the acronym for Chinese Remainder Theorem. `crt(r, s, p, q)` solves the congruences  $x = r \bmod p$  and  $x = s \bmod q$  with the help of the Chinese Remainder Theorem.

---

**SageMath sample 4.21** Determining all fixed points for a specific public RSA key

---

```
import numpy

print "--- Search for fixpoints in Textbook-RSA given p, q, e ---";
fp=numpy.array([0])
fq=numpy.array([0])

#Edit e,p,q here
###EDIT BEGIN###
e=17;
p=811;
q=1069;
###EDIT END###

n=p*q;
print "Prime p: ",p;
print "Prime q: ",q;
print "Modul n: ",n;
print "Public exponent e: ", e;

r=Integer(p)
gen_f_p = r.multiplicative_generator(); print "\nGenerator of f_p: ",gen_f_p;
s=Integer(q)
gen_f_q = s.multiplicative_generator(); print "Generator of f_q: ",gen_f_q;

gcd_p = gcd(e-1,p-1)
gcd_q = gcd(e-1,q-1)
print "\ngcd(e-1,p-1): ", gcd_p;
print "gcd(e-1,q-1): ", gcd_q;

print "\nNumber of fixpoints: ",(gcd_p+1)*(gcd_q+1);
#Calculating fixpoints modulo F_p
#run i from 0 until gcd(e-1,p-1):
#g^( i*(p-1) / (gcd(e-1,p-1)) ) mod p

print "\nFixpoints modulo p";
print "0 (trivial fixpoint added manually)";
i=0;
for i in range(gcd_p):
    fix_p = power_mod(gen_f_p,Integer(i*(p-1)/gcd_p),p); print fix_p;
    fp = numpy.append(fp,fix_p)

print "\nFixpoints modulo q";
print "0 (trivial fixpoint added manually)";
j=0;
for j in range(gcd_q):
    fix_q = power_mod(gen_f_q,Integer(j*(q-1)/gcd_q),q); print fix_q;
    fq = numpy.append(fq,fix_q);

print "\nFixpoints for the public RSA key (n,e) = (", n, ", ", e, ")"
for r in fp:
    for s in fq:
        print crt(Integer(r),Integer(s),Integer(p),Integer(q))

print "\nRemark: You can verify each fixpoint with power_mod(m,e,n).";
```

---

## 4.20 Appendix: List of the definitions and theorems formulated in this chapter

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[http://www.bundesnetzagentur.de/cln\\_1411/DE/Service-Funktionen/ElektronischeVertrauensdienste/QES/WelcheAufgabenhatdieBundesnetzagentur/GeeigneteAlgorithmenfestlegen/geeignetealgorithmenfestlegen\\_node.html](http://www.bundesnetzagentur.de/cln_1411/DE/Service-Funktionen/ElektronischeVertrauensdienste/QES/WelcheAufgabenhatdieBundesnetzagentur/GeeigneteAlgorithmenfestlegen/geeignetealgorithmenfestlegen_node.html)  
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All links have been confirmed at July 12, 2016.

## Web links

1. Ron Knott's Fibonacci page  
Here, everything revolves around Fibonacci numbers.  
<http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html>
2. CrypTool  
Open source e-learning software to illustrate cryptography and cryptanalysis  
<http://www.cryptool.org>
3. Mathematica  
Commercial mathematics package  
<http://www.wolfram.com>
4. LiDIA  
Library with number-theory functions and the LC interpreter. Maintenance stopped in 2000.  
[http://cs.nyu.edu/exact/core/download/core\\_v1.3/core\\_v1.3/lidia/](http://cs.nyu.edu/exact/core/download/core_v1.3/core_v1.3/lidia/)
5. BC  
Interpreter with number-theory functions. Development stopped since 2006. The existing BC versions don't work correctly under new Windows versions.  
Keith Matthews regularly publishes updates for UNIX.  
<http://www.gnu.org/software/bc/bc.html>  
<http://www.numbertheory.org/gnubc/gnubc.html>
6. Pari-GP  
Fast and free interpreter with number theoretical functions. Also within the browser.  
Maintained by Karim Belabas.  
<http://pari.math.u-bordeaux.fr/>  
<http://pari.math.u-bordeaux.fr/gp.html>  
<http://en.wikipedia.org/wiki/PARI/GP>
7. SageMath  
Excellent, open source computer algebra system with Python as script language, used to build the code samples in this chapter. See the introduction in chapter A.7.  
<http://www.sagemath.org/>  
[http://en.wikipedia.org/wiki/Sage\\_%28mathematics\\_software%29](http://en.wikipedia.org/wiki/Sage_%28mathematics_software%29)
8. Münchenbach  
Only after I had completed the first versions of this article, I came across the website of Mr. Münchenbach (1999), which interactively and didactically uses elementary number theory to provide a sophisticated description of the fundamental mathematical ideas. It was created for a teaching project in the 11th grade of the technical grammar school and uses MuPAD (unfortunately only available in German):  
<http://www.hydrargyrum.de/kryptographie>
9. Wagner  
Web site of Mr. Wagner from 2012, who is responsible for the development of the curriculum of computer science in one of the German federal states (Bundesländer). Here you can get hold of a collection of texts and (Java-)programs (available only in German):  
<http://www.saar.de/~awa/kryptologie.html>  
[http://www.saar.de/~awa/menu\\_kryptologie.html](http://www.saar.de/~awa/menu_kryptologie.html)

10. GISA

German Information Security Agency

[https://www.bsi.bund.de/EN/TheBSI/thebsi\\_node.html](https://www.bsi.bund.de/EN/TheBSI/thebsi_node.html)

11. Factorization records and factoring challenges

<http://www.crypto-world.com/>, last update 2013

<http://www.crypto-world.com/FactorWorld.html>, Webseite von Scott Contini

<http://www.loria.fr/~zimmerma/records/factor.html>

<http://www.emc.com/emc-plus/rsa-labs/historical/the-rsa-factoring-challenge.htm>

<https://www.uni-bonn.de/Pressemitteilungen/004-2010>, about RSA-768

<https://members.loria.fr/PZimmermann/records/gnfs158>, about C158

[http://www.loria.fr/~zimmerma/records/rsa160](https://www.loria.fr/~zimmerma/records/rsa160)

12. The Cunningham Project

<http://www.cerias.purdue.edu/homes/ssw/cun/>, last modified May 11, 2016

13. Daniel J. Bernstein pages

<http://cr.yp.to>

14. Post-quantum cryptography

<http://pqcrypto.org/>

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# Chapter 5

## The Mathematical Ideas behind Modern Cryptography<sup>1</sup>

(Roger Oyono / Bernhard Esslinger / Joerg-Cornelius Schneider, Sep 2000; Updates: Nov 2000, Feb 2003, Apr 2007, Mar 2010, Jan 2013)

I don't know if its getting better, if we change it,  
but I know, that we have to change it, if it should become better.

Quote 15: Georg Christoph Lichtenberg<sup>2</sup>

### 5.1 One way functions with trapdoor and complexity classes

A **one way function** is a function that can be calculated efficiently, but whose inverse is extremely complicated and practically impossible to calculate.

To put it more precisely: A one way function is a mapping  $f$  from a set  $X$  to a set  $Y$ , such that  $f(x)$  can be calculated easily for each element  $x$  of  $X$ , whereas for (almost) every  $y$  from  $Y$  it is practically impossible to find an inverse image  $x$  (i.e. an  $x$  where  $f(x) = y$ ).

An everyday example of a one way function is a telephone book: the function to be performed is to assign a name to the corresponding telephone number. This can be done easily due to the fact that the names are sorted alphabetically. However, the inverse function - assigning a name to a given number - is obviously difficult if you only have a telephone book available.

One way functions play a decisive role in cryptography. Almost all cryptographic terms can be rephrased using the term one way function. Let's take for example public key encryption (asymmetric cryptography):

Each subscriber  $T$  to the system is assigned a private key  $d_T$  and what is known as a public key  $e_T$ . These keys must have the following property (public key property):

<sup>1</sup>With the educational tool for number theory NT you can apply some of the methods introduced here (RSA, Rabin, DH, ElGamal) (see learning unit 4.2 and 4.3, pages 9-17/17).

NT can be called in CT1 via the menu path **Indiv. Procedures \ Number Theory Interactive \ Learning Tool for Number Theory**. See appendix A.6.

You can find according functions within the programs CT1, CT2 and JCT: see the list of the included functions within appendix A.1, A.2 and A.3.

<sup>2</sup>Georg Christoph Lichtenberg, German writer and physicist (1742-1799),  
(also see: [http://en.wikipedia.org/wiki/Georg\\_Christoph\\_Lichtenberg](http://en.wikipedia.org/wiki/Georg_Christoph_Lichtenberg))

For an opponent who knows the public key  $e_T$ , it is practically impossible to determine the private key  $d_T$ .

In order to construct useful public key procedures, therefore, we look for a one way function that is “easy” to calculate in one direction , but is “difficult” (practically impossible) to calculate in the other direction, provided that a particular piece of additional information (trapdoor) is not available. This additional piece of information allows the inverse to be found efficiently. Such functions are called **trapdoor one way functions**. In the above case, the one-way function is the encryption via exponentiation with the public key  $e_T$  as exponent. The private key  $d_T$  is the trapdoor information.

In this process, we describe a problem as “easy” if it can be solved in polynomial time as a function of the length of the input. If the length of the input is  $n$  bits, then the time for calculating the function is proportional to  $n^a$ , where  $a$  is a constant. We say that the complexity of such problems is  $O(n^a)$  [Landau- or Big-O notation].

If you compare two functions  $2^n$  and  $n^a$ , where  $a$  is a constant, then there always exists a value for  $n$ , from which for all further  $n$  applies:  $n^a < 2^n$ . The function  $n^a$  has a lower complexity. Sample: for  $a = 5$  the following applies: from the length  $n = 23$ ,  $2^n$  is greater than  $n^5$ ; for further  $n$   $2^n$  clearly increases more quickly [( $2^{22} = 4,194,304$ ,  $2^{25} = 5,153,632$ ), ( $2^{23} = 8,388,608$ ,  $2^{25} = 6,436,343$ ), ( $2^{24} = 16,777,216$ ,  $2^{25} = 7,962,624$ )].

The term “practically impossible” is slightly less precise. In general, we can say that a problem cannot be solved efficiently, if the time required to solve it increases more quickly than the polynomial time as a function of the size of the input. If, for example, the length of the input is  $n$  bits and the time required for calculating the function is proportional to  $2^n$ , then the following currently applies: the function practically cannot be calculated for  $n > 80$ .

In order to develop a public key procedure that can be implemented in practice, it is therefore necessary to discover a suitable trapdoor one way function.

In order to tidy things up among this confusing multitude of possible problems and their complexities, we group problems with similar complexities into classes.

The most important complexity classes are the classes **P** and **NP**:

- The class **P**: This class contains those problems that can be solved in a polynomial amount of time.
- The class **NP**: The definition of this class doesn't look at the time required to solve a problem, but rather at the time required to verify a given solution. The class **NP** consists of those problems for which a given solution can be verified in a polynomial amount of time. Hereby, the term **NP** “non-deterministic” means polynomial and is based on a calculation model, i.e. on a computer that only exists in theory and can “guess” correct solutions non-deterministically then verify them in polynomial time.

The class **P** is contained in the class **NP**. A well-known unsolved problem is the question whether or not  $\mathbf{P} \neq \mathbf{NP}$  is true, i.e. whether or not **P** is a true subset. An important property of the class **NP** is that it also contains what are known as **NP**-complete problems. These are problems that represent the class **NP** as follows: If a “good” algorithm for such a problem exists, then “good” algorithms exist for all problems from **NP**. In particular: If **P** only contained one complete problem, i.e. if a polynomial solution algorithm existed for this problem, then **P** would be equal to **NP**. In this sense, the **NP**-complete problems are the most difficult problems in **NP**.

Many cryptographic protocols are formed in such a way that the “good” subscribers only have to solve problems from **P**, whereas a perpetrator is faced with problems from **NP**.

Unfortunately, we do not yet know whether one way functions actually exist. However, we can prove that one way functions exist if and only if **P** ≠ **NP** [BDG, S.63].

Some mathematicians have again and again claimed to have proven this equivalence, but so far the claims have always turned out to be false [Hes01].

A number of algorithms have been suggested for public key procedures. In many cases - although they at first appeared promising - it was discovered that they could be solved polynomially. The most famous failed applicant is the knapsack with trapdoor, suggested by Ralph Merkle [MH78].

## 5.2 Knapsack problem as a basis for public key procedures

### 5.2.1 Knapsack problem

You are given  $n$  objects  $G_1, \dots, G_n$  with the weights  $g_1, \dots, g_n$  and the values  $w_1, \dots, w_n$ . The aim is to carry away as much as possible in terms of value while restricted to an upper weight limit  $g$ . You therefore need to find a subset of  $\{G_1, \dots, G_n\}$ , i.e.  $\{G_{i_1}, \dots, G_{i_k}\}$ , so that  $w_{i_1} + \dots + w_{i_k}$  is maximised under the condition  $g_{i_1} + \dots + g_{i_k} \leq g$ .

Such questions belong to the **NP**-complete problems (not deterministically polynomial) that are difficult to calculate.

A special case of the knapsack problem is:

Given the natural numbers  $a_1, \dots, a_n$  and  $g$ , find  $x_1, \dots, x_n \in \{0, 1\}$  where  $g = \sum_{i=1}^n x_i a_i$  (i.e. where  $g_i = a_i = w_i$  is selected). This problem is also called a **0-1 knapsack problem** and is identified with  $K(a_1, \dots, a_n; g)$ .

Two 0-1 knapsack problems  $K(a_1, \dots, a_n; g)$  and  $K(a'_1, \dots, a'_n; g')$  are called congruent if two co-prime numbers  $w$  and  $m$  exist in such a way that

1.  $m > \max\{\sum_{i=1}^n a_i, \sum_{i=1}^n a'_i\}$ ,
2.  $g \equiv wg' \pmod{m}$ ,
3.  $a_i \equiv wa'_i \pmod{m}$  for all  $i = 1, \dots, n$ .

**Comment:**

Congruent 0-1 knapsack problems have the same solutions. No quick algorithm is known for clarifying the question as to whether two 0-1 knapsack problems are congruent.

A 0-1 knapsack problem can be solved by testing the  $2^n$  possibilities for  $x_1, \dots, x_n$ . The best method requires  $O(2^{n/2})$  operations, which for  $n = 100$  with  $2^{100} \approx 1.27 \cdot 10^{30}$  and  $2^{n/2} \approx 1.13 \cdot 10^{15}$  represents an insurmountable hurdle for computers. However, for special  $a_1, \dots, a_n$  the solution is quite easy to find, e.g. for  $a_i = 2^{i-1}$ . The binary representation of  $g$  immediately delivers  $x_1, \dots, x_n$ . In general, the a 0-1 knapsack problem can be solved easily if a permutation<sup>3</sup>  $\pi$  of  $1, \dots, n$  exists with  $a_{\pi(j)} > \sum_{i=1}^{j-1} a_{\pi(i)}$  with  $j = 1, \dots, n$ . If, in addition,  $\pi$  is the identity, i.e.

---

<sup>3</sup>A permutation  $\pi$  of the numbers  $1, \dots, n$  is a change in the order in which these numbers are listed. For example, a permutation  $\pi$  of  $(1, 2, 3)$  is  $(3, 1, 2)$ , i.e.  $\pi(1) = 3$ ,  $\pi(2) = 1$  and  $\pi(3) = 2$ .

$\pi(i) = i$  for  $i = 1, 2, \dots, n$ , then the sequence  $a_1, \dots, a_n$  is said to be super-increasing. Crypto procedure 5.1 solves the knapsack problem with a super-increasing sequence in the time of  $O(n)$ .

---

**Crypto Procedure 5.1** Solving knapsack problems with super-increasing weights

---

```

for  $i = n$  to 1 do
    if  $T \geq a_i$  then
         $T := T - s_i$ 
         $x_i := 1$ 
    else
         $x_i := 0$ 
    if  $T = 0$  then
         $X := (x_1, \dots, x_n)$  is the solution.
    else
        No solution exists.

```

---

### 5.2.2 Merkle-Hellman knapsack encryption

In 1978, Merkle and Hellman [MH78] specified a public key encryption procedure that is based on “defamiliarizing” the easy 0-1 knapsack problem with a super-increasing sequence into a congruent one with a super-increasing sequence. It is a block ciphering that ciphers an  $n$ -bit plaintext each time it runs, see crypto procedure 5.2 for the details.

In 1982, Shamir [Sha82] specified an algorithm for breaking the system in polynomial time without solving the general knapsack problem. Len Adleman [Adl83] and Jeff Lagarias [Lag83] specified an algorithm for breaking the twice iterated Merkle-Hellman knapsack encryption procedure in polynomial time. Ernst Brickell [Bri85] then specified an algorithm for breaking multiply iterated Merkle-Hellman knapsack encryption procedures in polynomial time. This made this procedure unsuitable as an encryption procedure. It therefore delivers a one way function whose trapdoor information (defamiliarization of the 0-1 knapsack problem) could be discovered by an eavesdropper.

## 5.3 Decomposition into prime factors as a basis for public key procedures

Primes form the basis for a large number of algorithms for public-key procedures.

### 5.3.1 The RSA procedure<sup>4,5</sup>

As early as 1978, R. Rivest, A. Shamir, L. Adleman [RSA78] introduced the most important asymmetric cryptography procedure to date.

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<sup>4</sup>Please compare chapters 4.10, ff.

<sup>5</sup>Using CT1 you can gain practical experience with the RSA procedure via the menu **Indiv. Procedures \ RSA Cryptosystem \ RSA Demonstration**. You can find RSA also in CT2 and JCT.

---

**Crypto Procedure 5.2** Merkle-Hellman (based on knapsack problems)

Let  $(a_1, \dots, a_n)$  be super-increasing. Let  $m$  and  $w$  be two co-prime numbers with  $m > \sum_{i=1}^n a_i$  and  $1 \leq w \leq m - 1$ . Select  $\bar{w}$  with  $w\bar{w} \equiv 1 \pmod{m}$  the modular inverse of  $w$  and set  $b_i := wa_i \pmod{m}$ ,  $0 \leq b_i < m$  for  $i = 1, \dots, n$ , and verify whether the sequence  $b_1, \dots, b_n$  is not super-increasing. A permutation  $b_{\pi(1)}, \dots, b_{\pi(n)}$  of  $b_1, \dots, b_n$  is then published and the inverse permutation  $\mu$  to  $\pi$  is defined secretly. A sender writes his/her message in blocks  $(x_1^{(j)}, \dots, x_n^{(j)})$  of binary numbers  $n$  in length, calculates

$$g^{(j)} := \sum_{i=1}^n x_i^{(j)} b_{\pi(i)}$$

and sends  $g^{(j)}$ , ( $j = 1, 2, \dots$ ).

The owner of the key calculates

$$G^{(j)} := \bar{w}g^{(j)} \pmod{m}, \quad 0 \leq G^{(j)} < m$$

and obtains the  $x_{\mu(i)}^{(j)} \in \{0, 1\}$  (and thus also the  $x_i^{(j)}$ ) from

$$\begin{aligned} G^{(j)} &\equiv \bar{w}g^{(j)} = \sum_{i=1}^n x_i^{(j)} b_{\pi(i)} \bar{w} \equiv \sum_{i=1}^n x_i^{(j)} a_{\pi(i)} \pmod{m} \\ &= \sum_{i=1}^n x_{\mu(i)}^{(j)} a_{\pi(\mu(i))} = \sum_{i=1}^n x_{\mu(i)}^{(j)} a_i \pmod{m}, \end{aligned}$$

by solving the easier 0-1 knapsack problems  $K(a_1, \dots, a_n; G^{(j)})$  with super-increasing sequence  $a_1, \dots, a_n$ .

---

**Crypto Procedure 5.3** RSA (based on the factorization problem)**Key generation:**

Let  $p$  and  $q$  be two different prime numbers and  $N = pq$ . Let  $e$  be any number relative prim to  $\phi(N)$ , i.e.  $\gcd(e, \phi(N)) = 1$ . Using the Euclidean algorithm, we calculate the natural number  $d < \phi(N)$ , such that

$$ed \equiv 1 \pmod{\phi(N)}.$$

whereby  $\phi$  is the **Euler phi Function**.

The output text is divided into blocks and encrypted, whereby each block has a binary value  $x^{(j)} \leq N$ .

**Public key:**

$$N, e.$$

**Private key:**

$$d.$$

**Encryption:**

$$y = e_T(x) = x^e \pmod{N}.$$

**Decryption:**

$$d_T(y) = y^d \pmod{N}.$$

---

### Comment: Euler phi function

The Euler phi function is defined as:

$\phi(n)$  is the number of natural numbers  $x < n$   
that do not have a common factor with  $n$ .

No common factor means: Two natural numbers  $a$  and  $b$  are co-prime if  $\gcd(a, b) = 1$ .

For the Euler phi function it holds that:

$$\phi(1) = 1, \quad \phi(2) = 1, \quad \phi(3) = 2, \quad \phi(4) = 2, \quad \phi(6) = 2, \quad \phi(10) = 4, \quad \phi(15) = 8.$$

For example,  $\phi(24) = 8$ , because

$$|\{x < 24 : \gcd(x, 24) = 1\}| = |\{1, 5, 7, 11, 13, 17, 19, 23\}|.$$

Table 5.1 shows values of  $\phi(n)$  up to  $n = 25$ .

$n$	$\phi(n)$	The natural numbers that are co-prime to $n$ and less than $n$ .
1	1	1
2	1	1
3	2	1, 2
4	2	1, 3
5	4	1, 2, 3, 4
6	2	1, 5
7	6	1, 2, 3, 4, 5, 6
8	4	1, 3, 5, 7
9	6	1, 2, 4, 5, 7, 8
10	4	1, 3, 7, 9
15	8	1, 2, 4, 7, 8, 11, 13, 14
20	8	1, 3, 7, 9, 11, 13, 17, 19
25	20	1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24

Table 5.1: Euler phi function

If  $p$  is a prime number, then  $\phi(p) = p - 1$ .

In the case of  $N = pq$ :

$$\phi(N) = pq(1 - 1/p)(1 - 1/q) = p(1 - 1/p)q(1 - 1/q) = (p - 1)(q - 1).$$

If we know the various prime factors  $p_1, \dots, p_k$  of  $n$ , then

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right).^6$$

---

<sup>6</sup>Further formulas for the Euler phi function are in chapter 4.8.2 “The Euler phi function”.

The function  $e_T$  is a one way function whose trapdoor information is the decomposition into primes of  $N$ .

At the moment, no algorithm is known that can factorize two prime numbers sufficiently quickly for extremely large values (e.g. for several hundred decimal places). The quickest algorithms known today [Sti06] factorize a compound whole number  $N$  in a time period proportional to  $L(N) = e^{\sqrt{\ln(N) \ln(\ln(N))}}$ . Some example values can be found in table 5.2.

$N$	$10^{50}$	$10^{100}$	$10^{150}$	$10^{200}$	$10^{250}$	$10^{300}$
$L(N)$	$1.42 \cdot 10^{10}$	$2.34 \cdot 10^{15}$	$3.26 \cdot 10^{19}$	$1.20 \cdot 10^{23}$	$1.86 \cdot 10^{26}$	$1.53 \cdot 10^{29}$

Table 5.2:  $L(N)$  value table [factorization effort related to the modul length]

To this date, it has not been proved that the problem of breaking RSA is equivalent to the factorization problem. Nevertheless, it is clear that the RSA procedure will no longer be safe if the factorization problem is “solved”.<sup>7</sup>

### 5.3.2 Rabin public key procedure (1979)

The Rabin public key procedure (crypto procedure 5.4) has been shown to be equivalent to breaking the factorization problem. Unfortunately, this procedure is susceptible to chosen-ciphertext attacks.

---

**Crypto Procedure 5.4** Rabin (based on the factorization problem)

Let  $p$  and  $q$  be two different prime numbers with  $p, q \equiv 3 \pmod{4}$  and  $n = pq$ . Let  $0 \leq B \leq n - 1$ .

**Public key:**

$$e = (n, B).$$

**Private key:**

$$d = (p, q).$$

**Encryption:**

$$y = e_T(x) = x(x + B) \pmod{n}.$$

**Decryption:**

$$d_T(y) = \sqrt{y + B^2/4} - B/2 \pmod{n}.$$


---

Caution: Because  $p, q \equiv 3 \pmod{4}$  the encryption is easy to calculate (if the key is known). This is not the case for  $p \equiv 1 \pmod{4}$ . In addition, the encryption function is not injective: There are precisely four different source codes that have  $e_T(x)$  as inverse image:  $x, -x - B, \omega(x + B/2) - B/2, -\omega(x + B/2) - B/2$ , where  $\omega$  is one of the four roots of unity. The source codes therefore must be redundant for the encryption to remain unique!

Backdoor information is the decomposition into prime numbers of  $n = pq$ .

---

<sup>7</sup>In 2000 the authors assumed that values of the order magnitude 100 to 200 decimal places are currently safe. They estimates that the current computer technology indicates that a number with 100 decimal places could be factorized in approximately two weeks at justifiable costs, and using an expensive configuration (e.g. of around 10 million US dollars), a number with 150 decimal places could be factorized in about a year, and a 200-digit number should remain impossible to factorize for a long time to come, unless there is a mathematical breakthrough. However, you can never be sure that there won't be a mathematical breakthrough tomorrow.

How easy it is to guess the future wrong is shown by the [factorization of RSA-200](#) (see chapter 4.11.4) – completely without a “mathematical breakthrough”.

## 5.4 The discrete logarithm as basis for public key procedures<sup>8</sup>

Discrete logarithms form the basis for a large number of algorithms for public-key procedures.

### 5.4.1 The discrete logarithm in $\mathbb{Z}_p$

Let  $p$  be a prime number and let  $g \in \mathbb{Z}_p^* = \{0, 1, \dots, p - 1\}$ . Then the discrete exponential function base  $g$  is defined as

$$e_g : k \longrightarrow y := g^k \pmod{p}, \quad 1 \leq k \leq p - 1.$$

The inverse function is called a discrete logarithm function  $\log_g$ ; the following holds:

$$\log_g(g^k) = k.$$

The problem of the discrete logarithm (in  $\mathbb{Z}_p^*$ ) is understood to be as follows:

Given  $p, g$  and  $y$ , determine  $k$  such that  $y = g^k \pmod{p}$ .

It is much more difficult to calculate the discrete logarithm than to evaluate the discrete exponential function (see chapter 4.9). Table 5.3 lists several procedures for calculating the discrete logarithm and their complexity [Sti06].

Name	Complexity
Baby-step-giant-step	$O(\sqrt{p})$
Silver-Pohlig-Hellman	polynomial in $q$ , the greatest prime factor of $p - 1$ .
Index-Calculus	$O(e^{(1+o(1))\sqrt{\ln(p)\ln(\ln(p))}})$

Table 5.3: Procedures for calculating the discrete logarithm over  $\mathbb{Z}_p^*$

The current record (as of April 2007) for calculating discrete logarithms was established in February 2007 by the group Kleinjung, Franke and Bahr at University of Bonn.<sup>9</sup> Kleinjung calculated the discrete logarithm modulo a 160 digit prime number  $p$  and generator  $g$ :

$$\begin{aligned} p &= \lfloor 10^{159}\pi \rfloor + 119849 \\ &= 314159265358979323846264338327950288419716939937510582097494 \\ &\quad 459230781640628620899862803482534211706798214808651328230664 \\ &\quad 7093844609550582231725359408128481237299 \\ g &= 2 \end{aligned}$$

---

<sup>8</sup>With the educational tool for number theory **NT** you can play with the distribution of the discrete logarithm values and apply Shank's baby-step-giant-step method: See learning units 6.1-6.3, pages 1-6/6.

NT can be called in CT1 via the menu path **Indiv. Procedures \ Number Theory Interactive \ Learning Tool for Number Theory**. See appendix A.6.

<sup>9</sup>[http://www.nabble.com/Discrete-logarithms-in-GF\(p\)----160-digits-t3175622.html](http://www.nabble.com/Discrete-logarithms-in-GF(p)----160-digits-t3175622.html)

The discrete logarithms  $k$  of the following integer  $y$  was determined:<sup>10</sup>

$$\begin{aligned} y &= \lfloor 10^{159} e \rfloor \\ &= 271828182845904523536028747135266249775724709369995957496696 \\ &\quad 762772407663035354759457138217852516642742746639193200305992 \\ &\quad 1817413596629043572900334295260595630738 \\ k &= \log_g(y) \mod p \\ &= 829897164650348970518646802640757844024961469323126472198531 \\ &\quad 845186895984026448342666252850466126881437617381653942624307 \\ &\quad 537679319636711561053526082423513665596 \end{aligned}$$

The search was performed with GNFS method (general number field sieve, Index-Calculus) and took about 17 CPU years on 3.2 GHz Xeon machines.

#### 5.4.2 Diffie-Hellman key agreement<sup>11</sup>

The mechanisms and algorithms of classical cryptography only take effect when the subscribers have already exchanged the secret key. In classical cryptography you cannot avoid exchanging secrets without encrypting them. Transmission safety here must be achieved using non-cryptographic methods. We say that we need a secret channel for exchanging secrets. This channel can be realised either physically or organisationally.

What is revolutionary about modern cryptography is, amongst other things, that you no longer need secret channels: You can agree secret keys using non-secret, i.e. public channels.

One protocol that solves this problem is that of Diffie and Hellman (crypto procedure 5.5).

---

##### Crypto Procedure 5.5 Diffie-Hellman key agreement

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Two subscribers  $A$  and  $B$  want to agree on a joint secret key.

Let  $p$  be a prime number and  $g$  a natural number. These two numbers do not need to be secret. The two subscribers then select a secret number  $a$  and  $b$  from which they calculate the values  $\alpha = g^a \mod p$  and  $\beta = g^b \mod p$ . They then exchange the numbers  $\alpha$  and  $\beta$ . To end with, the two subscribers calculate the received value to the power of their secret value to get  $\beta^a \mod p$  and  $\alpha^b \mod p$ .

Thus

$$\beta^a \equiv (g^b)^a \equiv g^{ba} \equiv g^{ab} \equiv (g^a)^b \equiv \alpha^b \mod p$$


---

The safety of the **Diffie-Hellman protocol** is closely connected to calculating the discrete logarithm mod  $p$ . It is even thought that these problems are equivalent.

#### 5.4.3 ElGamal public key encryption procedure in $\mathbb{Z}_p^*$

By varying the Diffie-Hellman key agreement protocol slightly, you can obtain an asymmetric encryption algorithm, crypto procedure 5.6. This observation was made by Taher ElGamal.

---

<sup>10</sup>The integer  $y$  was chosen as the first 159 digits of the Euler number  $e$ .

<sup>11</sup>With CT1 this exchange protocol has been visualized: you can execute the single steps with concrete numbers using menu **Indiv. Procedures \ Protocols \ Diffie-Hellman Demonstration**.

In JCT you can find it in the default perspective via the menu item **Visuals \ Diffie-Hellman Key Exchange (EC)**.

---

**Crypto Procedure 5.6** ElGamal (based on the discrete logarithm problem)

Let  $p$  be a prime number such that the discrete logarithm in  $\mathbb{Z}_p$  is difficult to compute. Let  $\alpha \in \mathbb{Z}_p^*$  be a primitive element. Let  $a \in \mathbb{N}$  and  $\beta = \alpha^a \pmod{p}$ .

**Public key:**

$$p, \alpha, \beta.$$

**Private key:**

$$a.$$

Let  $k \in \mathbb{Z}_{p-1}$  be a random number and  $x \in \mathbb{Z}_p^*$  the plaintext.

**Encryption:**

$$e_T(x, k) = (y_1, y_2),$$

where

$$y_1 = \alpha^k \pmod{p}$$

and

$$y_2 = x\beta^k \pmod{p}.$$

**Decryption:**

$$d_T(y_1, y_2) = y_2(y_1^a)^{-1} \pmod{p}$$

---

#### 5.4.4 Generalized ElGamal public key encryption procedure

The discrete logarithm can be generalized in any number of finite groups  $(G, \circ)$ . The following provides several properties of  $G$ , that make the discrete logarithm problem difficult. Instead of  $g \circ h$  we often write only  $gh$ .

**Calculating the discrete exponential function** Let  $G$  be a group with the operation  $\circ$  and  $g \in G$ . The (discrete) exponential function base  $g$  is defined as

$$e_g : k \mapsto g^k, \quad \text{for all } k \in \mathbb{N}.$$

where

$$g^k := \underbrace{g \circ \dots \circ g}_{k \text{ times}}.$$

The exponential function is easy to calculate:

**Lemma.** *The power  $g^k$  can be calculated in at most  $2\log_2 k$  group operations.*

**Proof**

Let  $k = 2^n + k_{n-1}2^{n-1} + \dots + k_12 + k_0$  be the binary representation of  $k$ . Then  $n \leq \log_2(k)$ , because  $2^n \leq k < 2^{n+1}$ .  $k$  can be written in the form  $k = 2k' + k_0$  with  $k' = 2^{n-1} + k_{n-1}2^{n-2} + \dots + k_1$ . Thus

$$g^k = g^{2k'+k_0} = (g^{k'})^2 g^{k_0}.$$

We therefore obtain  $g^k$  from  $g^{k'}$  by squaring and then multiplying by  $g$ . The claim is thus proved by induction to  $n$ .  $\square$

#### Problem of the discrete logarithm

Let  $G$  by a finite group with the operation  $\circ$ . Let  $\alpha \in G$  and  $\beta \in H = \{\alpha^i : i \geq 0\}$ . We need to find a unique  $a \in \mathbb{N}$  with  $0 \leq a \leq |H| - 1$  and  $\beta = \alpha^a$ . We define  $a$  as  $\log_\alpha(\beta)$ .

**Calculating the discrete logarithm** A simple procedure for calculating the discrete logarithm of a group element, that is considerably more efficient than simply trying all possible values for  $k$ , is the baby-step-giant-step algorithm.

**Theorem 5.4.1.** [baby-step-giant-step algorithm] Let  $G$  be a group and  $g \in G$ . Let  $n$  be the smallest natural number with  $|G| \leq n^2$ . Then the discrete logarithm of an element  $h \in G$  can be calculated base  $g$  by generating the following two lists each containing  $n$  elements and comparing these lists:

$$\begin{aligned} \text{giant-step list: } & \{1, g^n, g^{2n}, \dots, g^{n \cdot n}\}, \\ \text{baby-step list: } & \{hg^{-1}, hg^{-2}, \dots, hg^{-n}\}. \end{aligned}$$

After detecting a common element the calculation can be stopped. In order to calculate these lists, we need  $2n$  group operations.

### Proof

If  $g^{jn} = hg^{-i}$ , i.e.  $h = g^{i+jn}$ , then the problem is solved. If the lists are disjoint, then  $h$  cannot be represented as  $g^{i+jn}$ ,  $i, j \leq n$ . As all powers of  $g$  are thus recorded, the logarithm problem does not have a solution.  $\square$

You can use the baby-step-giant-step algorithm to demonstrate that it is much more difficult to calculate the discrete logarithm than to calculate the discrete exponential function. If the numbers that occur have approximately 1000 bits in length, then you only need around 2000 multiplications (see theorem 5.4.1) to calculate all  $g^k$ , but around  $2^{500} \approx 10^{150}$  operations to calculate the discrete logarithm using the baby-step-giant-step algorithm.

In addition to the baby-step-giant-step algorithm, there are also numerous other procedures for calculating the discrete logarithm [Sti06].

**The theorem from Silver-Pohlig-Hellman** In finite Abelian groups, the discrete logarithm problem can be reduced to groups of a lower order.

**Theorem 5.4.2.** [Silver-Pohlig-Hellman] Let  $G$  be a finite Abelian group with  $|G| = p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s}$ . The discrete logarithm in  $G$  can then be reduced to solving logarithm problems in groups of the order  $p_1, \dots, p_s$ .

If  $|G|$  contains a “dominant” prime factor  $p$ , then the complexity of the logarithm problem is approximately

$$O(\sqrt{p}).$$

Therefore, if the logarithm problem is to be made difficult, the order of the group used  $G$  should have a large prime factor. In particular, if the discrete exponential function in the group  $\mathbb{Z}_p^*$  is to be a one way function, then  $p - 1$  must be a large prime factor. In this case a generalized ElGamal procedure can be defined (crypto procedure 5.7).

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**Crypto Procedure 5.7** Generalized ElGamal (based on the factorization problem)

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Let  $G$  be a finite group with operation  $\circ$ , and let  $\alpha \in G$ , so that the discrete logarithm in  $H = \{\alpha^i : i \geq 0\}$  is difficult to calculate. Let  $a$  with  $0 \leq a \leq |H| - 1$  and let  $\beta = \alpha^a$ .

**Public key:**

$$\alpha, \beta$$

**Private key:**

$$a$$

Let  $k \in \mathbb{Z}_{|H|}$  be a random number,  $k \neq 0$ , and  $x \in G$  be a plaintext.

**Encryption:**

$$e_T(x, k) = (y_1, y_2),$$

where

$$y_1 = \alpha^k$$

and

$$y_2 = x \circ \beta^k.$$

---

**Decryption:**

$$d_T(y_1, y_2) = y_2 \circ (y_1^a)^{-1}$$

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[Elliptic curves](#) (see chapter 7) provide useful groups for public key encryption procedures.

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All links have been confirmed at July 13, 2016.

# Chapter 6

## Hash Functions and Digital Signatures

([Joerg-Cornelius Schneider](#) / [Bernhard Esslinger](#) / [Henrik Koy](#), Jun 2002; Updates: Feb 2003, Jun 2005, Jul 2009, Nov 2012)

We can make everything out of this world, but we cannot create a world, where humans in some ten thousand years can think: 'Ok, now it is enough. Everything should stay like it is. Let's do no changes any more, don't do inventions any more, because it cannot become better, and if, then we don't want this.'

Quote 16: Stanislaw Lem<sup>1</sup>

The aim of digital signatures is to guarantee the following two points:

- User authenticity:  
It can be checked whether a message really does come from a particular person.
- Message integrity:  
It can be checked whether the message has been changed (on route).

An asymmetric technique is used again (see encryption procedures). Participants who wish to generate a digital signature for a document must possess a pair of keys. They use their secret key to generate signatures and the recipient uses the sender's public key to verify whether the signature is correct. As before, it must be impossible to use the public key to derive the secret key.

In detail, a *Signature procedure*<sup>2</sup> looks like this:

Senders use their message and secret key to calculate the digital signature for the message. Compared to hand-written signatures, digital signatures therefore have the advantage that they also depend on the document to be signed. Signatures from one and the same participant are different unless the signed documents are completely identical. Even inserting a blank in the

<sup>1</sup>This was the answer of Stanislaw Lem to heavy critics at his philosophical main book "Summa Technologiae", 1964, where he thought about the possibility of an evolution creating artificial intelligence.

<sup>2</sup>With CT1 you can also generate and check digital signatures: Using the submenus of the main menu **Digital Signatures / PKI** or using menu **Indiv. Procedures \ RSA Cryptosystem \ Signature Demonstration (Signature Generation)**. Also with JCT (in the default and the algorithm perspektive) its possible to create different kinds of electronic signatures.

text would lead to a different signature. The recipient of the message would therefore detect any injury to the message integrity as this would mean that the signature no longer matches the document and is shown to be incorrect when verified.

The document is sent to the recipient together with the signature. The recipient can then use the sender's public key, the document and the signature to establish whether or not the signature is correct. The procedure we just described has in practice, however, a decisive disadvantage. The signature would be approximately as long as the document itself. To prevent an unnecessary increase in data traffic, and also for reasons of performance, we apply a cryptographic hash function<sup>3</sup> to the document – before signing. The output of the hash function will then be signed.

## 6.1 Hash functions

A *hash function*<sup>4</sup> maps a message of any length to a string of characters with a constant size, the hash value.

### 6.1.1 Requirements for hash functions

Cryptographically secure hash functions fulfill the following three requirements (the order is in a way that the requirements increase):

- Resistance against 1st pre-image attacks:

It should be practically impossible, for a given number, to find a message that has precisely this number as hash value.

Given (fix): hash value  $H'$ ,

Searched: message  $m$ , so that:  $H(m) = H'$ .

- Resistance against 2nd pre-image attacks:

It should be practically impossible, for a given message, to find another message, which has precisely the same hash value.

Given (fix): message  $m_1$  [and so the hash value  $H_1 = H(m_1)$ ],

Searched: message  $m_2$ , so that:  $H(m_2) = H_1$ .

- Collision resistance:

It should be practically impossible to find any two messages with the same hash value (it

---

<sup>3</sup>Hash functions are implemented within CT1 at several places.

Using menus **Individual Procedures \ Hash and Analysis \ Hash** you have the possibilities

- to apply one of 6 hash functions to the content of the current window,
- to calculate the hash value of a file,
- to test, how changes to a text change the according hash value,
- to calculate a key from a password according to the PKCS#5 standard,
- to calculate HMACs from a text and a secret key, and
- to perform a simulation, how digital signatures could be attacked by a targeted search for hash value collisions.

CT2 and JCT also contain different hash methods: See the functions' lists within the appendix A.2 and A.3.

<sup>4</sup>Hash algorithms compute a condensed representation of electronic data (message). When a message is input to a hash algorithm, the result is an output called a message digest. The message digests typically range in length from 128 to 512 bits, depending on the algorithm. Secure hash algorithms are typically used with other cryptographic algorithms, such as digital signature algorithms and keyed-hash message authentication codes, or in the generation of random numbers (bits).

doesn't matter what hash value).

Searched: 2 messages m1 and m2, so that:  $H(m1) = H(m2)$ .

### 6.1.2 Current attacks against hash functions // SHA-3

So far, no formal proof has been found that perfectly secure cryptographic hash functions exist.

During the past several years no new attacks against hash algorithms came up, and so the candidates that had not yet shown any weaknesses in their structure in practice (e.g. SHA-1<sup>5</sup> or RIPEMD-160<sup>6</sup>) were trusted.

At Crypto 2004 (August 2004)<sup>7</sup> this safety-feeling was disputed: Chinese researchers published collision attacks against MD4, SHA-0 and parts of SHA-1. This globally caused new motivation to engage in new hash attack methods.

The initially published result reduced the expected complexity for one SHA-1 collision search from  $2^{80}$  (brute-force) to  $2^{69}$  [WYY05b]. More recent announcements claim to further reduce the required effort to  $2^{63}$  [WYY05a] and  $2^{52}$  [MHP12]. This would bring collision attacks into the practical realm, as similar efforts have been mastered in the past (s. 1.3.3).

According to our current knowledge there is no need to run scared. But in the future digital signatures should use longer hash values and/or other hash algorithms.

Already before Crypto 2004 the U.S. National Institute of Standards and Technology (NIST) announced, to discontinue SHA-1 in the next few years. So it is recommended not to use SHA-1 for new products generating digital signatures. The SHA-2 family [NIS15] provides stronger algorithms.

To address new findings in cryptanalysis, in 2008 NIST opened a competition to develop a new cryptographic hash algorithm beyond the SHA-2 family: In October 2012 Keccak was announced as "SHA-3".<sup>8</sup>

---

<sup>5</sup>SHA-1 is a 160 bit hash function specified in FIPS 180-1 (by NIST), ANSI X9.30 Part 2 and [NIS13].

SHA means Secure Hash Algorithm, and is widely used, e.g. with DSA, RSA or ECDSA.

The current standard [NIS15] defines four secure hash algorithms – SHA-1, SHA-256, SHA-384, and SHA-512. For these hash algorithms there are also validation tests defined in the test suite FIPS 140-2.

The output length of the SHA algorithms was enhanced because of the possibility of birthday attacks: these make n-bit AES and a 2n-bit hash roughly equivalent:

- 128-bit AES – SHA-256
- 192-bit AES – SHA-384
- 256-bit AES – SHA-512.

With CT1 you can comprehend the birthday attack on digital signatures:

using the menu **Analysis \ Hash \ Attack on the Hash Value of the Digital Signature**.

CT2 contains an MD5 collider.

<sup>6</sup>RIPEMD-160, RIPEMD-128 and the optional extension RIPEMD-256 have object identifiers defined by the ISO-identified organization TeleTrusT, both as hash algorithm and in combination with RSA. RIPEMD-160 is also part of the ISO/IEC international standard ISO/IEC 10118-3:1998 on dedicated hash functions, together with RIPEMD-128 and SHA-1. Further details:

- <http://www.esat.kuleuven.ac.be/~bosseleae/ripemd160.html>
- <http://www.ietf.org/rfc/rfc2857.txt> ("The Use of HMAC-RIPEMD-160-96 within ESP and AH").

<sup>7</sup><http://www.iacr.org/conferences/crypto2004/>

<sup>8</sup><http://csrc.nist.gov/groups/ST/hash/sha-3/>

With CT2 you can execute and visualize the Keccak hash function – using in the Startcenter **Templates \ Hash Functions \ Keccak Hash (SHA-3)**.

Keccak also can be used as pseudo random number generator and as stream cipher: This can be found within the Startcenter templates via **Tools \ Keccak PRNG**, and **Cryptography \ Modern Ciphers \ Symmetric \ Keccak Streamcipher**.

Further information about this topic can be found in the article “Hash cracked – The consequences of the successful attacks on SHA-1” by Reinhard Wobst and Jürgen Schmidt<sup>9</sup> by Heise Security.

### 6.1.3 Signing with hash functions

“Manipulation was Sobol’s speciality ... the main investigation should branch off and try to discover Sobol’s master plan.”

Quote 17: Daniel Suarez<sup>10</sup>

The signature procedure with hash functions<sup>11</sup> is as follows: Rather than signing the actual document, the sender now first calculates the hash value of the message and signs this. The recipient also calculates the hash value of the message (the algorithm used must be known), then verifies whether the signature sent with the message is a correct signature of the hash value. If this is the case, the signature is verified to be correct. This means that the message is authentic, because we have assumed that knowledge of the public key does not enable you to derive the secret key. However, you would need this secret key to sign messages in another name.

Some digital signature schemes are based on asymmetric *encryption* procedures, the most prominent example being the RSA system, which can be used for signing by performing the private key operation on the hash value of the document to be signed.

Other digital signature schemes were developed exclusively for this purpose, as the DSA (Digital Signature Algorithm), and are not directly connected with a corresponding encryption scheme.

Both, RSA and DSA signature are discussed in more detail in the following two sections. After that we go one step further and show how digital signatures can be used to create the digital equivalent of ID cards. This is called Public Key Certification.

## 6.2 RSA signatures

As mentioned in the comment at the end of section 4.10.3 it is possible to perform the RSA private and public key operation in reverse order, i. e. raising  $M$  to the power of  $d$  and then to the power of  $e$  ( $\text{mod } N$ ) yields  $M$  again. Based on this simple fact, RSA can be used as a signature scheme.

The RSA signature  $S$  for a message  $M$  is created by performing the private key operation:

$$S \equiv M^d \pmod{N}$$

In order to verify, the corresponding public key operation is performed on the signature  $S$  and the result is compared with message  $M$ :

$$S^e \equiv (M^d)^e \equiv (M^e)^d \equiv M \pmod{N}$$

<sup>9</sup><http://www.heise.de/security/artikel/56634>.

Further references are e.g.:

<http://csrc.nist.gov/groups/ST/toolkit/index.html>.

<sup>10</sup>Daniel Suarez, “Daemon”, Dutton Adult, 2010, Chapter 14, “Meme Payload”, p. 142, Ross.

<sup>11</sup>Compare: [http://en.wikipedia.org/wiki/Digital\\_signature](http://en.wikipedia.org/wiki/Digital_signature).

If the result matches the message  $M$ , then the signature is accepted by the verifier, otherwise the message has been tampered with, or was never signed by the holder of  $d$ .

As explained above, signatures are not performed on the message itself, but on a cryptographic hash value of the message. To prevent certain attacks on the signature procedure (alone or in combination with encryption) it is necessary to format the hash value before doing the exponentiation, as described in the PKCS#1 (Public Key Cryptography Standard #1 [Lab02]). The fact that this standard had to be revised recently, after being in use for several years, can serve as an example of how difficult it is to get the details of cryptography right.

### 6.3 DSA signatures

In August of 1991, the U.S. National Institute of Standards and Technology (NIST) proposed a digital signature algorithm (DSA), which was subsequently adopted as a U.S. Federal Information Processing Standard (FIPS 186 [NIS13]).

The algorithm is a variant of the ElGamal scheme. Its security is based on the Discrete Logarithm Problem. The DSA public and private key and its procedures for signature and verification are summarised in crypto procedure 6.1.

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#### Crypto Procedure 6.1 DSA signature

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##### Public Key

$p$  prime

$q$  160-bit prime factor of  $p - 1$

$g = h^{(p-1)/q} \pmod{p}$ , where  $h < p - 1$  and  $h^{(p-1)/q} > 1 \pmod{p}$

$y \equiv g^x \pmod{p}$

*Remark:* Parameters  $p, q$  and  $g$  can be shared among a group of users.

##### Private Key

$x < q$  (a 160-bit number)

##### Signing

$m$  the message to be signed

$k$  choose at random, less than  $q$

$r = (g^k \pmod{p}) \pmod{q}$

$s = (k^{-1}(\text{SHA-1}(m) + xr)) \pmod{q}$

*Remark:*

- $(s, r)$  is the signature.
- The security of the signature depends not only on the mathematical properties, but also on using a good random source for  $k$ .
- SHA-1 is a 160-bit hash function.

##### Verifying

$w = s^{-1} \pmod{q}$

$u_1 = (\text{SHA-1}(m)w) \pmod{q}$

$u_2 = (rw) \pmod{q}$

$v = (g^{u_1}y^{u_2}) \pmod{p}$

*Remark:* If  $v = r$ , then the signature is verified.

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While DSA was specifically designed, so that it can be exported from countries regulating

export of encryption soft and hardware (like the U.S. at the time when it was specified), it has been noted [Sch96b, p. 490], that the operations involved in DSA can be used to emulate RSA and ElGamal encryption.

## 6.4 Public key certification

The aim of public key certification is to guarantee the connection between a public key and a user and to make it traceable for external parties. In cases in which it is impossible to ensure that a public key really belongs to a particular person, many protocols are no longer secure, even if the individual cryptographic modules cannot be broken.

### 6.4.1 Impersonation attacks

Assume Charlie has two pairs of keys ( $\text{PK}_1, \text{SK}_1$ ) and ( $\text{PK}_2, \text{SK}_2$ ), where  $\text{SK}$  denotes the secret key and  $\text{PK}$  the public key. Further assume that he manages to palm off  $\text{PK}_1$  on Alice as Bob's public key and  $\text{PK}_2$  on Bob as Alice's public key (by falsifying a public key directory).

Then he can attack as follows:

- Alice wants to send a message to Bob. She encrypts it using  $\text{PK}_1$  because she thinks that this is Bob's public key. She then signs the message using her secret key and sends it.
- Charlie intercepts the message, removes the signature and decrypts the message using  $\text{SK}_1$ . If he wants to, he can then change the message in any way he likes. He then encrypts the message again, but this time using Bob's genuine public key, which he has taken from a public key directory, signs the message using  $\text{SK}_2$  and forwards it to Bob.
- Bob verifies the signature using  $\text{PK}_2$  and will reach the conclusion that the signature is correct. He then decrypts the message using his secret key.

In this way Charlie can listen in on communication between Alice and Bob and change the exchanged messages without them noticing. The attack will also work if Charlie only has one pair of keys.

Another name for this type of attack is “man-in-the-middle attack”. Users are promised protection against this type of attack by public-key certification, which is intended to guarantee the authenticity of public keys. The most common certification method is the X.509 standard.

### 6.4.2 X.509 certificate

Each participant who wants to have an X.509 certificate ([IT97]) verifying that his public key belongs to a real person consults what is known as a certification authority (CA)<sup>12</sup>. He proves his identity to this CA (for example by showing his ID). The CA then issues him an electronic document (certificate) which essentially contains the name of the certificate-holder and the name of the CA, the certificate-holder's public key and the validity period of the certificate. The CA then signs the certificate using its secret key.

Anyone can now use the CA's public key to verify whether a certificate is falsified. The CA therefore guarantees that a public key belongs to a particular user.

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<sup>12</sup>Often called trust center, if the certificates are not only offered to a closed user group.

This procedure is only secure as long as it can be guaranteed that the CA's public key is correct. For this reason, each CA has its public key certified by another CA that is superior in the hierarchy. In the upper hierarchy level there is usually only one CA, which can of course then have its key certified by another CA. It must therefore transfer its key securely in another way. In the case of many software products that work with certificates (such as the Microsoft and Netscape Web browsers), the certificates of these root CAs are permanently embedded in the program right from the start and cannot be changed by users at a later stage. However, (public) CA keys, in particular those of the root entity, can also be secured by means of making them available publicly.

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All links have been confirmed at July 14, 2016.

# Chapter 7

## Elliptic Curves

(Bartol Filipovic / Matthias Bueger / Bernhard Esslinger / Roger Oyono, Apr 2000, Updates: Dec 2001, Jun 2002, Mar 2003, Nov 2009, Aug 2013, Aug 2016)

### 7.1 Elliptic-curve cryptography – a high-performance substitute for RSA?

In many business sectors secure and efficient data transfer is essential. In particular, the RSA algorithm is used in many applications. Although the security of RSA is beyond doubt, the evolution in computing power has caused a growth in the necessary key length. Today, 1024-bit RSA keys are standard, but the GISA (German Information Security Agency) recommends the usage of 2048-bit keys from 2006 on (compare section 4.11). The fact that most chips on smart cards cannot process keys extending ca. 2000 bit shows that there is a need for alternatives in the area of asymmetric cryptography. Elliptic-curve cryptography (ECC) is such an alternative. They are used widely on smartcards.

The efficiency of a cryptographic algorithm depends on the key length and the calculation effort that is necessary to provide a prescribed level of security. The major advantage of ECC compared to RSA is that it requires much shorter key lengths.

If we assume that the computing power increases by Moore's law (i. e. it doubles every 18 months)<sup>1</sup>, then the evolution of the key lengths for secure communication will be as in figure 7.1, which was generated from table 1 (on page 32 in [LV01]).<sup>2</sup>

In addition, a digital signature can be processed 10-times faster with ECC than with RSA. However, verification of a given signature is still more efficient with RSA than with ECC. Refer to figure 7.2 (source: J. Merkle, Elliptic-Curve Cryptography Workshop, 2001) for a comparison. The reason is that RSA public keys can be chosen relatively small as long as the secret key is long enough.

Nevertheless, thin clients like smart cards usually have to store the (long) secret key and

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<sup>1</sup>Moore's law formulates the empirical observation and the according forecast that the number of components or transistors on an integrated circuit doubles every two years. It referred originally only to the transistors density in an integrated circuit, however not, for example, to the increase in the storage density. This realization for transistors density in 1965 (with a correction 1975) was expressed by Gordon Moore, a co-founder of Intel. In recent years, the growth of computing power was even higher than the forecasted doubling every 2 years. Future limits are set by the transistors size of a few atoms, which could be achieved by 2020.

<sup>2</sup>Further information about key length comparison by Arjen Lenstra und Eric Verheul, plus more modern evaluations till 2015 can be found in the interactive website <http://www.keylength.com>.

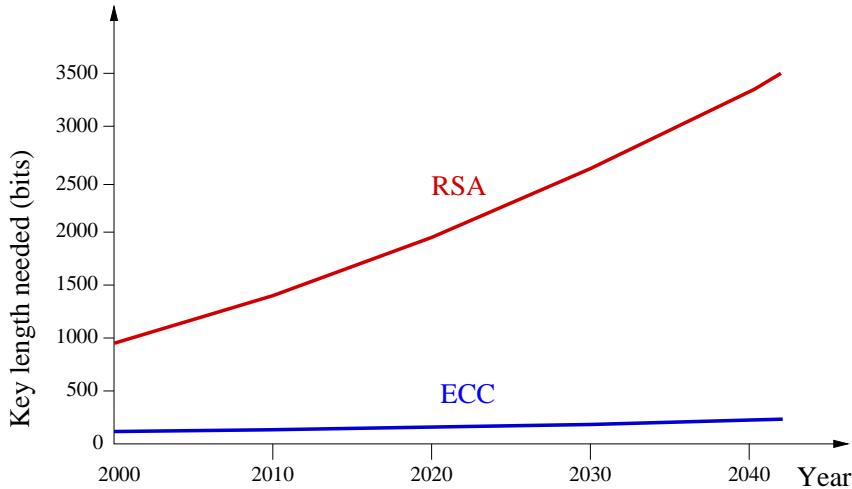


Figure 7.1: Prognosis of the key lengths regarded to be safe for RSA and for elliptic curves

have to process a digital signature rather than verify one. Therefore, there is a clear advantage in using ECC in terms of efficiency.

Nowadays, the major problem with ECC implementations is the lack of standardization. There is only one way to implement RSA, but there are many ways for ECC: One can work with different sets of numbers, different (elliptic) curves — described by parameters<sup>3</sup> — , and a variety of representations of the elements on the curve. Each choice has its advantages and disadvantages, and one can certainly construct the most efficient for each application. However, this causes problems in interoperability. But if all ECC-tools should be able to communicate with each other, they will have to support all different algorithms, which might put the advantage of efficient computation and the need of less storage capacity to the contrary.

Therefore, international standardization organizations like IEEE (P1363), ASC (ANSI X9.62, X9.63), ISO/IEC as well as major players like RSA labs or Certicom have recently started standardization initiatives. While the IEEE only describes the different implementations, the ASC has explicitly stated 10 elliptic curves and recommends their usage. The advantage of the ASC approach is that one needs only a single byte to indicate which curve is meant. However, it is not yet clear whether the ASC curves will become a de facto standard.

Although there is no need to replace RSA in any application today<sup>4</sup>, one seriously should take the usage of ECC into consideration<sup>5</sup>. GISA More current discussions about the security of ECC can be found in chapter 10.

## 7.2 Elliptic curves – history

Mathematicians have been researching elliptic curves for over 100 years. Over the course of time, many lengthy and mathematically complex results have been found and published which are connected to elliptic curves. A mathematician would say that elliptic curves (or the mathematics

<sup>3</sup>See chapter 7.4

<sup>4</sup>Current information about the security of the RSA algorithm can be found in chapters 4.11 and 10.

<sup>5</sup>Compare the technical guideline “Cryptographic Methods: Recommendations and Keylengths” of GISA from February 15th, 2016.

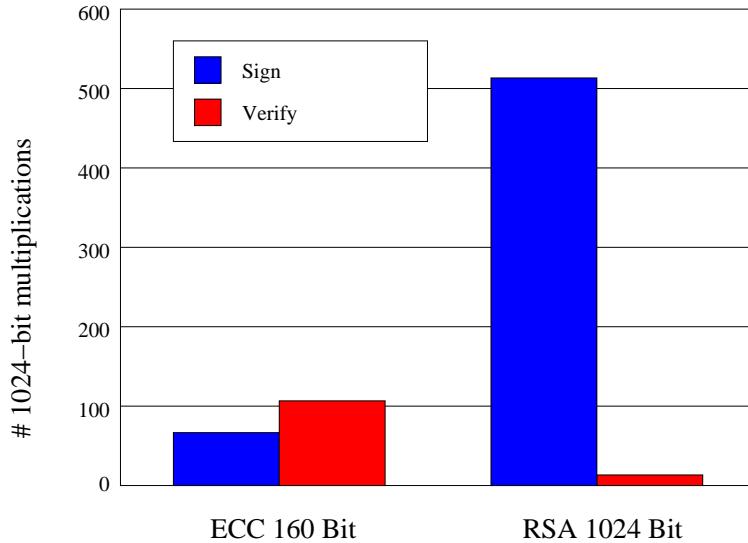


Figure 7.2: Comparison between signing and verification time for RSA and elliptic curves

behind them) are widely understood. This research was originally purely mathematical. That is to say, elliptic curves were investigated, for example, in the mathematical areas of number theory and algebraic geometry, which are generally highly abstract. Even in the recent past, elliptic curves played an important role in pure mathematics. In 1993 and 1994<sup>6</sup>, Andrew Wiles published mathematical works that triggered enthusiasm far beyond the specialist audience. In these works, he proved a conjecture put forward in the 1960's. To put it short, this conjecture was concerned with the connection between elliptic curves and what are called module forms. What is particularly interesting for most people is that the works of Wiles also proved the famous second theorem of Fermat. Mathematicians had spent centuries (Fermat lived from 1601 to 1665) trying to find a strict proof of this theorem. Understandably, therefore, Wiles' proof got a good response. Fermat formulated his theorem as follows (written in the borders of a book from Diophantus):

*Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos ejusdem nominis fas est dividere: cuius rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.*

With a free translation, using the denotation of modern mathematics, this means:  
No positive whole numbers  $x, y$  and  $z$  greater than zero exist such that  $x^n + y^n = z^n$  for  $n > 2$ . I have found an amazing proof of this fact, but there is too little space within the confines of this book to include it.

This is truly amazing: A statement that is relatively simple to understand (we are referring to Fermat's second theorem here) could only be proved after such a long period of time, although

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<sup>6</sup>1994 the gaps in the first proof have been closed by Wiles and Richard Taylor.

Fermat himself claimed to have found a proof. What's more, the proof found by Wiles is extremely extensive (all of Wiles publications connected with the proof made up a book in themselves). This should therefore make it obvious that elliptic curves are generally based on highly complex mathematics.

Anyway, that's enough about the role of elliptic curves in pure mathematics. In 1985 Neal Koblitz and Victor Miller independently suggested using elliptic curves in cryptography. Elliptic curves have thus also found a concrete practical application. Another interesting area of application for elliptic curves is for factorizing whole numbers (the RSA cryptographic system is based on the difficulty/complexity of finding prime factors of an extremely large number; compare section 4.11.). In this area, procedures based on elliptic curves have been investigated and used since 1987 (compare section 7.8).

There are also prime number tests based on elliptic curves.

Elliptic curves are used differently in the various areas: Encryption procedures based on elliptic curves are based on the difficulty of the problem known as elliptic curve discrete logarithm. The factorization of natural composite numbers  $n$  uses the fact that a large number of elliptic curves can be generated for  $n$ .

## 7.3 Elliptic curves – mathematical basics

This section provides information about *groups* and *fields*.<sup>7</sup>

### 7.3.1 Groups

Because the term *group* is used differently in everyday language than in mathematics, we will, for reasons of completeness, begin by introducing the essential statement of the formal definition of a group:

- A group is a non-empty set  $G$  on which an operation “.”. The set  $G$  is closed under this operation, which means that for any two elements  $a, b$  taken from  $G$ , performing the operation on them gives an element in  $G$ , i.e.  $ab = a \cdot b$  lies in  $G$ .
- For all elements  $a, b$  and  $c$  in  $G$ :  $(ab)c = a(bc)$  (associative law).
- There exists an element  $e$  in  $G$  that behaves neutrally with respect to the operation .. That means that for all  $a$  in the set  $G$ :  $ae = ea = a$ .
- For each element  $a$  in  $G$  there exists a so-called inverse<sup>8</sup> element  $a^{-1}$  in  $G$  such that:  $aa^{-1} = a^{-1}a = e$ .

If in addition it applies  $ab = ba$  (commutative law) for all  $a, b$  in  $G$ , then we call the group an *Abelian* group.

Since we may define different operations on the same set, we distinguish them by giving them different names (e.g. + addition or · multiplication).

The simplest example of an (Abelian) group is the group of whole numbers under the standard operation of addition. The set of whole numbers is denoted as  $\mathbb{Z}$ .  $\mathbb{Z}$  has an infinite number of

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<sup>7</sup>A didactically well prepared introduction into Elliptic Curves can be found in [SWE15].

<sup>8</sup>The inverse is uniquely determined because if  $x, y \in G$  are each inverse to  $a$ , i.e.  $ax = xa = e$  and  $ay = ya = e$ , then  $x = xe = x(ay) = (xa)y = ey = y$ .

elements:  $\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$ . For example, the operation of  $1 + 2$  lies in  $\mathbb{Z}$ , for  $1 + 2 = 3$  and 3 lies in  $\mathbb{Z}$ . The neutral element in the group  $\mathbb{Z}$  is 0. The inverse element of 3 is  $-3$ , for  $3 + (-3) = 0$ .

For our purpose, so-called *finite* groups play an important role. This means that there exists a set  $\mathcal{M}$  with a fixed number of elements and an operation  $+$  such that the above conditions are fulfilled. One example is the group  $\mathbb{Z}_n = \{0, 1, 2, 3, \dots, n-1\}$  of the remainders of the division by  $n \in \mathbb{N}$ , and the operation is an addition mod  $n$ . So, for example,  $a$  and  $b$  in  $\mathbb{Z}_n$  are subject to the operation  $a + b \bmod n$ .

**Cyclic groups** Cyclic groups<sup>9</sup> are those groups  $G'$  that possess an element  $g$  from which the group operation can be used to generate all other elements in the group. This means that for each element  $a$  in  $G'$  there exists a positive whole number  $i$  such that if  $g$  is subject to the operation  $i$  times (i.e. “ $g \cdot i$ ”),  $g + g + \dots + g = a$  (additive group) or  $g^i = g \cdot g \cdots g = a$  (multiplicative group). The element  $g$  is the *generator* of the cyclic group — each element in  $G'$  can be generated using  $g$  and the operation.

**Group order** Now to the order of an element of the group: Let  $a$  be in  $G$ . The smallest positive whole number  $r$  for which  $a$  subject to the operation with itself  $r$  times is the neutral element of the group  $G'$  (i.e.:  $r \cdot a = a + a + \dots + a = e$  respectively  $a^r = e$ ), is called the *order* of  $a$ .

The order of the group is the number of elements in the set  $G$ .

### 7.3.2 Fields

In mathematics, one is often interested in sets on which at least two (group) operations are defined — frequently called addition and multiplication. Most prominent are so called fields.

A field is understood to be a set  $K$  with two operations (denoted as  $+$  and  $\cdot$ ) which fulfills the following conditions:

- The set  $K$  forms an Abelian group together with the operation  $+$  (addition), where 0 is the neutral element of the operation  $+$ .
- The set  $K \setminus \{0\}$  also forms an Abelian group together with the operation  $\cdot$  (multiplication).
- For all elements  $a, b$  and  $c$  in  $K$ , we have  $c \cdot (a + b) = c \cdot a + c \cdot b$  and  $(a + b) \cdot c = a \cdot c + b \cdot c$  (distributive law).

Fields may contain an infinite number of elements (e.g. the field of real numbers). They are called *infinite* fields. In contrast we call a field *finite*, if it contains only a finite number of elements (e.g.  $\mathbb{Z}_p = \{0, 1, 2, 3, \dots, p-1\}$ , where  $p$  is a prime.  $\mathbb{Z}_p$  with addition mod  $p$  and multiplication mod  $p$ ).

**Characteristic of a field** Let  $K$  be a field and 1 be the neutral element of  $K$  with respect to the multiplicative operation “ $\cdot$ ”. Then the characteristic of  $K$  is said to be the order of 1

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<sup>9</sup>Cyclic groups can be in general also endless like the additive group of the integer numbers. We consider here only finite cyclic groups.

with respect to the additive operation. This means that the characteristic of  $K$  is the smallest positive integer  $n$  such that

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0.$$

If there is no such  $n$ , i.e. if  $1 + 1 + \cdots + 1 \neq 0$  no matter how many 1s we add, then we call  $K$  a field with characteristic 0.

Thus, fields with characteristic 0 are infinite since they contain the (pairwise distinct) elements  $1, 1 + 1, 1 + 1 + 1, \dots$ . On the other hand, fields with finite characteristic may be finite or infinite.

If the characteristic is finite, it has to be prime. This fact can easily be proved: Assume  $n = pq$ ,  $p, q < n$ , is the characteristic of a field  $K$ . By definition of  $n$ , the elements  $\bar{p} = \underbrace{1 + 1 + \cdots + 1}_{p \text{ times}}$

$\bar{q} = \underbrace{1 + 1 + \cdots + 1}_{q \text{ times}}$  of  $K$  are not equal to 0. Thus, there exist inverse elements  $\bar{p}^{-1}, \bar{q}^{-1}$  with respect to multiplication. It follows that  $(\bar{p}\bar{q})(\bar{p}^{-1}\bar{q}^{-1}) = 1$ , which contradicts the fact that  $\bar{p}\bar{q} = \bar{n} = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0$  and, hence,  $\underbrace{(\bar{p}\bar{q})(\bar{p}^{-1}\bar{q}^{-1})}_{=0} = 0$ .

#### Comment:

The field of real numbers has the characteristic 0; the field  $\mathbb{Z}_p$  has the characteristic  $p$ . If  $p$  is not prime,  $\mathbb{Z}_p$  is not a field at all.

The most simple field is  $\mathbb{Z}_2 = \{0, 1\}$ . It contains only two elements, the neutral elements with respect to addition and multiplication. In particular, we have  $0 + 0 = 0, 0 + 1 = 1 + 0 = 1, 1 + 1 = 0, 1 \cdot 1 = 1, 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0$ .

**Finite Fields** As mentioned above, each finite field has a characteristic  $p \neq 0$ , where  $p$  is a prime. On the other hand, given a prime  $p$  there is a field which has exactly  $p$  elements, that is  $\mathbb{Z}_p$ .

However, the number of elements of a field need not be prime in general. For example, it is not hard to construct a field with 4 elements<sup>10</sup>.

One can show that the order of any field is a prime power (i.e. the power of a prime number). On the other hand, we can construct a field with  $p^n$  elements for any given prime  $p$  and positive integer  $n$ . Since two fields that have the same number of elements can not be distinguished<sup>11</sup>, we say that there is **the field with  $p^n$  elements** and denote it by  $GF(p^n)$  or by  $\mathbb{F}_p^n$  (used mostly in the English-speaking world). Here  $GF$  stands for *Galois Field* to commemorate the French Mathematician Galois.

<sup>10</sup>The set  $K = \{0, 1, a, b\}$  fitted with the operation defined in the tabular below is a field:

+	0	1	a	b
0	0	1	a	b
1	1	0	b	a
a	a	b	0	1
b	b	a	1	0

und

.	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	b	1
b	0	b	1	a

<sup>11</sup>If  $K, K'$  are fields with  $k = p^n$  elements, then there is a one-to-one map  $\varphi : K \rightarrow K'$ , that respects the arithmetic of the field. Such a map is called an isomorphy. Isomorphic fields mathematically behave in the same way so that it makes no sense to distinguish between them. For example,  $\mathbb{Z}_2$  und  $K' = \{\text{ZERO}, \text{ONE}\}$  with zero-element *ZERO* and one-element *ONE* are isomorphic. We note that mathematical objects are only defined by their mathematical properties.

The fields  $GF(p)$  of prime order play a prominent role. They are called prime fields and often also denoted by  $\mathbb{Z}_p$ .<sup>12</sup>

## 7.4 Elliptic curves in cryptography

In cryptography elliptic curves are a useful tool.<sup>13</sup>

Such curves are described by some equation. A detailed analysis has shown that curves of the form<sup>14</sup>

$$F(x_1, x_2, x_3) = -x_1^3 + x_2^2 x_3 + a_1 x_1 x_2 x_3 - a_2 x_1^2 x_3 + a_3 x_2 x_3^2 - a_4 x_1 x_3^2 - a_6 x_3^3 = 0, \quad (7.1)$$

are especially useful. The variables  $x_1, x_2, x_3$  and parameters  $a_1, \dots, a_6$  are elements of a given field  $K$ , which has certain properties that make it useful from the cryptographic point of view. The underlying field  $K$  might be the well known field of real numbers or some finite field (see last section). In order to obtain a curve that is useful for cryptography, the parameters have to be chosen in a way that the following conditions hold

$$\frac{\partial F}{\partial x_1} \neq 0, \quad \frac{\partial F}{\partial x_2} \neq 0, \quad \frac{\partial F}{\partial x_3} \neq 0.$$

We identify points on the curve that can be derived from each other by multiplying each component with some scalar. This makes sense since  $(x_1, x_2, x_3)$  solves (7.1) if and only if  $\alpha(x_1, x_2, x_3)$  ( $\alpha \neq 0$ ) does. Formally, this means that we consider classes of equivalent points instead of single points, where points are called equivalent if one is the scalar multiple of the other one.

If we put  $x_3 = 0$  in the basic equation (7.1), then this equation collapses to  $-x_1^3 = 0$ , leading to  $x_1 = 0$ . Thus, the equivalence class which includes the element  $(0, 1, 0)$  is the only one that contains a point with  $x_3 = 0$ . For all points on the curve that are not equivalent to  $(0, 1, 0)$ , we may apply the following transformation

$$K \times K \times (K \setminus \{0\}) \ni (x_1, x_2, x_3) \mapsto (x, y) := \left( \frac{x_1}{x_3}, \frac{x_2}{x_3} \right) \in K \times K,$$

which reduces the number of variables to two instead of three. We note that the basic equation (7.1)  $F(x_1, x_2, x_3) = 0$  was chosen in a way that this transformation leads to the famous so-called Weierstrass-Equation<sup>15</sup> holds

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \quad (7.2)$$

Since all but one point (i.e. equivalence class) of the elliptic curve can be described using equation (7.2), this equation is often called the elliptic equation, and its solutions written as

$$\mathbf{E} = \{(x, y) \in K \times K \mid y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6\} \cup \{\mathcal{O}\}.$$

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<sup>12</sup>For prime fields additive as well as multiplicative group are cyclic. Furthermore, each field  $GF(p^n)$  contains a subfield that is isomorphic to the prime field  $\mathbb{Z}_p$ .

<sup>13</sup>We write “elliptic-curve cryptography” with a hyphen, according to “public-key cryptography”. Sadly, this isn’t used consistently in the literature.

<sup>14</sup>This curve is given by the zeros of a polynomial  $F$  of degree three in three variables. In general, expressions of the form  $P = \sum_{i_1, \dots, i_n} a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$  with  $i_1, \dots, i_n \in \mathbb{N}$  with coefficients  $a_{i_1 \dots i_n} \in K$  are called polynomials in  $n$  variables  $x_1, \dots, x_n$  with underlying field  $K$ , if  $\deg P := \max\{i_1 + \dots + i_n : a_{i_1 \dots i_n} \neq 0\}$  is finite, i.e. the sum has only finitely many non-zero terms (monomials). The sum of the exponents of the variables of each term of the sum is at most 3, at least one term of the sum has a single variable with 3 as value of the according exponent.

<sup>15</sup>Karl Weierstrass, 31.10.1815–19.12.1897, German mathematician, famous for his rigorous formal approach to mathematics.

Here,  $\mathcal{O}$  represents the point  $(0, 1, 0)$  that is loosely speaking mapped to infinity by the transformation (division by  $x_3$ ) that reduces the three variables to two.

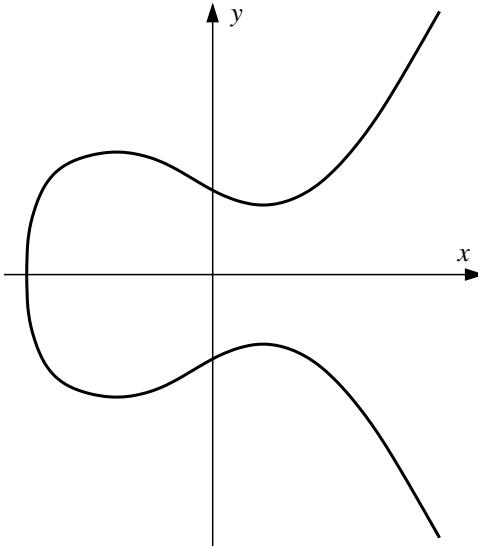


Figure 7.3: Example of an elliptic curve with the real numbers as underlying field

In contrast to figure 7.3 only finite fields  $K = GF(p^n)$  are used in elliptic-curve cryptography. The reason is loosely speaking that in modern communication engineering data processing is always based on discrete data (simply because computers accept only discrete data).

For practical reasons, it turned out to be useful to take either  $GF(p)$  with a large prime  $p$  or  $GF(2^n)$  with a (large) positive integer  $n$ . Using  $GF(p)$  has the advantage of providing a relatively simple arithmetic; on the other hand  $GF(2^n)$  allows a binary representation of each element that supports the way computers work. Other fields like, for example,  $GF(7^n)$  do not have any of these advantages and are, thus, not considered, although there is no mathematical reason why they should not.

A coordinate transformation can result in a simpler version<sup>16</sup> of the Weierstrass equation. Depending whether  $p > 3$ , different transformations are used, and we obtain

- in case of  $GF(p)$ ,  $p > 3$ , the elliptic curve equation of the form

$$y^2 = x^3 + ax + b \quad (7.3)$$

with  $4a^3 + 27b^2 \neq 0$

- in case of  $GF(2^n)$  the elliptic curve equation of the form

$$y^2 + xy = x^3 + ax^2 + b \quad (7.4)$$

with  $b \neq 0$ <sup>17</sup>.

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<sup>16</sup>Such a coordinate transformation is combination of a rotation and a dilatation of the coordinate system without changing the elliptic curve itself.

<sup>17</sup>The form (7.3) is called the standard form of the Weierstrass-equation. If the characteristic of the field is 2 or 3, we obtain  $4 = 0$  respectively  $27 = 0$ , which means that the condition on parameters  $a, b$  collapse. Loosely speaking, this is the reason why the transformation to the standard form does not work in these cases.

This conditions on the parameters  $a, b$  ensure that the elliptic equation can be used in the context of cryptography<sup>18</sup>.

Let  $|E|$  denote the number of elements of an elliptic curve  $E$  given an underlying field  $GF(k)$  (for practical reasons either  $k = p$  with  $p$  prim or  $k = 2^n$ ). Then Hasse's theorem [Sil09] yields  $||E| - k - 1| \leq 2 \cdot \sqrt{k}$ . This inequality is equivalent to  $k + 1 - 2\sqrt{k} < |E| < k + 1 + 2\sqrt{k}$ . In particular, this means that the number of elements of an elliptic curve is approximately  $k$  (for large  $k$ ).

## 7.5 Operating on the elliptic curve

In order to work with elliptic curves in practice, we define an operation (often written in an additive way  $+$ ) on the set of points on the curve. If we have a curve over the field  $GF(p)$ , we define the commutative operation  $+$  by

1.  $P + \mathcal{O} = \mathcal{O} + P = P$  for all  $P \in E$ ,
2. for  $P = (x, y)$  and  $Q = (x, -y)$  we set  $P + Q = \mathcal{O}$ ,
3. for  $P_1 = (x_1, x_2), P_2 = (x_2, y_2) \in E$  with  $P_1, P_2 \neq \mathcal{O}$  and  $(x_2, y_2) \neq (x_1, -y_1)$  we set  $P_3 := P_1 + P_2$ ,  $P_3 = (x_3, y_3)$  defined by

$$x_3 := -x_1 - x_2 + \lambda^2, \quad y_3 := -y_1 + \lambda(x_1 - x_3)$$

with the auxiliary quotient

$$\lambda := \begin{cases} \frac{y_1 - y_2}{x_1 - x_2} & \text{if } P_1 \neq P_2, \\ \frac{3x_1^2 + a}{2y_1} & \text{if } P_1 = P_2. \end{cases}$$

In particular, we obtain  $-P = (x, -y)$  for  $P = (x, y) \in E$ .

If we deal with a curve over the field  $GF(2^n)$ , we define the operation  $+$  in an analogous way by

1.  $P + \mathcal{O} = \mathcal{O} + P = P$  for all  $P \in E$ ,
2. for  $P = (x, y)$  and  $Q = (x, x + y)$  we set  $P + Q = \mathcal{O}$ ,
3. for  $P_1 = (x_1, x_2), P_2 = (x_2, y_2) \in E$  with  $P_1, P_2 \neq \mathcal{O}$  and  $(x_2, y_2) \neq (x_1, x_1 + y_1)$  we set  $P_3 := P_1 + P_2$ ,  $P_3 = (x_3, y_3)$  defined by

$$x_3 := -x_1 + x_2 + \lambda + \lambda^2 + a, \quad y_3 := y_1 + x_3 + \lambda(x_1 + x_3)$$

with auxiliary quotient

$$\lambda := \begin{cases} \frac{y_1 + y_2}{x_1 + x_2} & \text{if } P_1 \neq P_2, \\ x_1 + \frac{y_1}{x_1} & \text{if } P_1 = P_2. \end{cases}$$

In particular, we obtain  $-P = (x, -y)$  for  $P = (x, y) \in E$ .

(Note that  $-(-P) = (x, x + (x + y)) = (x, 2x + y) = (x, y)$ , since the underlying field has characteristic 2.)<sup>19</sup>

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<sup>18</sup>Formally we call such curves non singular.

<sup>19</sup>An animation of the addition of points on elliptic curves can be found on the Certicom homepage <https://www.certicom.com/ecc-tutorial> (Date of last update is unknown). See also the web link about the [Java-Tutorial](#) at the end of this chapter.

One can verify that  $+$  defines a group operation on the set  $E \cap \{\mathcal{O}\}$ . In particular this means that the sum of two points is again a point on the elliptic curve. How his operation works is geometrically visualized in the following section.

### How to add points on an elliptic curve

The following figures show how points on an elliptic curve over the field of real numbers are summed up using affine coordinates. We note that the point infinity  $\mathcal{O}$  cannot be shown in the affine plane.

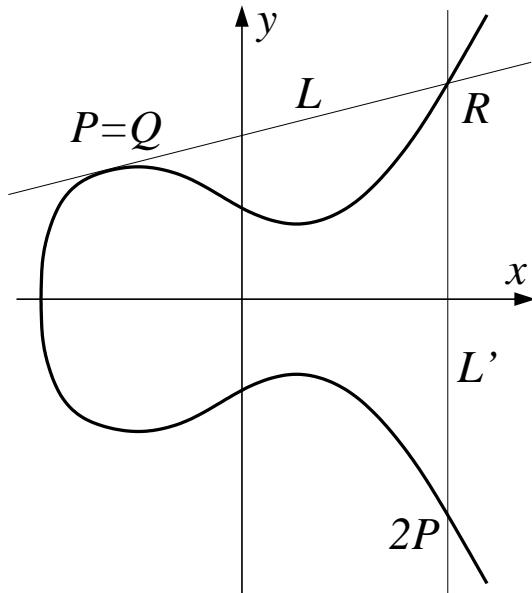


Figure 7.4: Doubling of a point

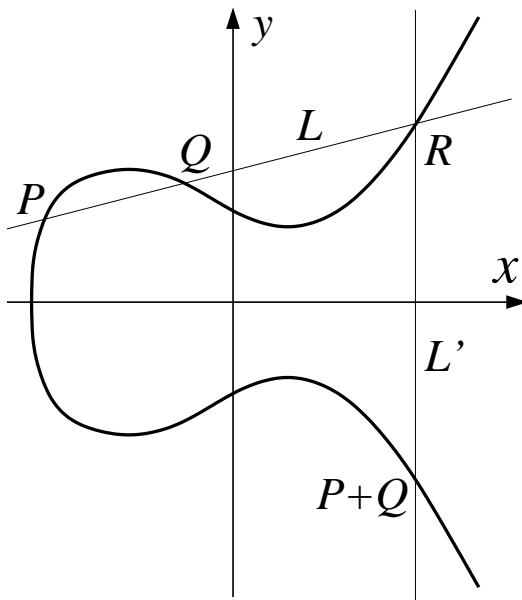


Figure 7.5: Summing up two different points over the real number field

## 7.6 Security of elliptic-curve cryptography: The ECDLP

As mentioned above in section 7.4, we only consider elliptic curves over the finite<sup>20</sup> fields  $GF(2^n)$  or  $GF(p)$  (for a large prime  $p$ ). This means that all parameters that describe the curve are taken from this underlying field. If  $E$  is an elliptic curve over such a field and  $P$  is a point on the curve  $E$ , then we can derive for all positive integers  $m$

$$mP := \underbrace{P + P + \cdots + P}_{m \text{ times}}.$$

Looking on this operation from the cryptographic point of view, it turns out to be very interesting by the following reason: On the one hand one needs only  $\log m$  operations to calculate  $mP$  — one simply has to calculate  $P, 2P, 2^2P, 2^3P, \dots$ , write  $m$  in a binary form and finally add all these multiples  $2^kP$  of  $P$  with respect to the binary representation of  $m$  — on the other hand it seems to be very hard to find  $m$  given  $P$  and  $Q = mP$  on  $E$ . Of course, we may simply calculate  $P, 2P, 3P, 4P, 5P, \dots$  and compare each of them with  $Q$ . But this will take as much as  $m$  operations.

Yet there is no algorithm known that efficiently derives  $m$  given  $P$  and  $Q$ . The best algorithms known so far need about  $\sqrt{q}$  operations where  $q$  is the (largest) prime factor of  $p - 1$ , in case the underlying field is  $GF(p)$ ; here  $m$  should be between 1 and  $q$  liegen so that one needs at most  $\log q$  operations to calculate  $mP$ . However, the quotient  $\frac{\sqrt{q}}{\log q}$  tends to  $+\infty$  very fast for large  $q$ .

If we choose the parameters sufficiently large (for example, let  $p$  be prime and at least 160 bits long), an computer will easily be able to calculate  $mP$  (in less than a second). The *inverse problem* however, to derive  $m$  from  $mP$  and  $P$ , can (still) not be solved in reasonable time.

This problem is known as the “Elliptic Curve Discrete Logarithm Problem” (for short ECDLP).

In elliptic-curve cryptography we formally look at points on the elliptic curve as elements of a group with point addition  $+$  as operation. Furthermore, we use only elliptic curves that have a sufficiently large number of points. However, in special cases curves may be weak and not useful due to other reasons. For such special cases the ECDLP can be much easier to solve than in the general case. This means that one has to look carefully at the parameters when choosing an elliptic curve for cryptographic applications.

Not useful for cryptography are *a-normal* (that are curves over  $\mathbb{Z}_p$ , for which the set  $\mathbf{E}$  consists of exactly  $p$  elements) and *supersingular* curves (that are curves, for which the ECDLP can be reduced to the “normal” discrete logarithms in another, smaller finite field). This means that there are cryptographically useful and non-useful elliptic curves. Given the parameters  $a$  and  $b$ , it is possible to determine whether a curve is useful or not. In many publications one can find parameters that turned out to be useful for cryptography. The open (scientific) discussion guarantees that these results take into account latest research.

Given a secure curve, the time that is needed to solve the ECDLP is strongly correlated with parameter  $p$  in case  $GF(p)$  respectively  $n$  in case of  $GF(2^n)$ . The larger these parameters become, the more time an attacker needs to solve the ECDLP — at least with the best algorithms known so far. Experts recommend bit-lengths of 200 for  $p$  for secure curves. A comparison with RSA modulus length shows why elliptic curves are so interesting for applications. We note that the computation effort for signing and encryption is closely related to the bit-length of the parameters. In addition the initiation process, i.e. the generation of the private-public-key-pair,

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<sup>20</sup>Discrete in contrast to continuous.

becomes more complicated the larger  $p$  is. Thus, one looks for the smallest parameters that still come along with the security required. It is remarkable that a length of 200 bits for  $p$  is sufficient to construct a *good* elliptic curve that is as secure as RSA with a 1024 bit RSA modulus (as far as we know today). For short, the reason for this advantage of ECC lies in the fact that the best algorithms known for solving the ECDLP need exponential time while the best algorithms for factorizing are sub-exponential (number field sieve, quadratic sieve or factorizing with elliptic curves). Hence, the parameters for a cryptosystem that is based on the problem of *factorizing large integers* have to be larger than the parameters for a system based on ECDLP.

## 7.7 Encryption and signing with elliptic curves

The *elliptic curve discrete logarithm problem* (ECDLP) is the basis for elliptic-curve cryptography. Based on this problem, there are different signature schemes. In order to apply one of these, we need:

- An elliptic curve  $\mathbf{E}$  with an underlying field  $GF(p^n)$ .
- A prime  $q \neq p$  and a point  $G$  on the elliptic curve  $\mathbf{E}$  with order  $q$ . This means that  $qG = \mathcal{O}$  and  $rG \neq \mathcal{O}$  for all  $r \in \{1, 2, \dots, q-1\}$ . Thus  $q$  is a factor of the group order (i.e. the number of elements)  $\#\mathbf{E}$  of  $E$ . Since  $q$  is prime,  $G$  generates a cyclic sub-group of  $\mathbf{E}$  of order  $q$ .

The parameters mentioned are often called *Domain* parameter. They describe the elliptic curve  $\mathbf{E}$  and the cyclic sub-group of  $\mathbf{E}$  on which the signature scheme is based.

### 7.7.1 Encryption

Using elliptic curves one can construct a key exchange protocol based on the Diffie-Hellman protocol (see chapter 5.4.2). The key exchanged can be used for a subsequent symmetric encryption. We note that in contrast to RSA there is no pair of private and public key that can be used for encryption and decryption!

In the notation of elliptic curves, the Diffie-Hellman protocol reads as follows: First both partners (A und B) agree on a group  $G$  and an integer  $q$ . Then they choose  $r_A, r_B \in \{1, 2, \dots, q-1\}$  at random, derive the points  $R_A = r_A G$ ,  $R_B = r_B G$  on the elliptic curve and exchange them (using an insecure channel). After that A easily obtains  $R = r_A R_B$ ; B gets the same point ( $R = r_A r_B G$ ) by calculating  $r_B R_A = r_B r_A G = r_A r_B G = R$ . We note that  $R_A, R_B$  are easy to derive as long as  $r_A$  respectively  $r_B$  are known  $G$ . However, the inverse operation, to get  $R_A$  respectively  $R_B$  from  $r_A$  respectively  $r_B$  is hard.

Using the best algorithms known so far, it is impossible for any attacker to obtain  $R$  without knowing either  $r_A$  or  $r_B$  — otherwise he would have to solve the ECDLP.

In order to prohibit a “Man-in-the-middle” attack, one may sign the values  $G, q, R_A, R_B$  as described in chapter 6.4.1.

### 7.7.2 Signing

Using the DSA signature scheme, one can proceed as follows: The signing party chooses a (non-trivial) number  $s \in \mathbb{Z}_q$ , which will be the private key, and publishes  $q, G$  and  $R = sG$ . We

note that  $s$  cannot be obtained from  $G$  and  $R$  are not sufficient — a fact on which the security of the signature scheme is based.

Given the message  $m$ , which should be signed, one first constructs a digital finger print using a hash-algorithm  $h$  such that  $h(m)$  has its values in  $\{0, 1, 2, \dots, q - 1\}$ . Thus,  $h(m)$  can be considered as an Element of  $\mathbb{Z}_q$ . Then the signing party chooses a random number  $r \in \mathbb{Z}_q$  and derives  $R = (r_1, r_2) = rG$ . We note that the first component  $r_1$  of  $R$  is an element of  $GF(p^n)$ . This component will then be projected onto  $\mathbb{Z}_q$ , i.e. in case of  $n = 1$  it is interpreted as the remainder of an element of  $\{0, 1, \dots, p - 1\}$  divided by  $q$ . This projection of  $r_1$  onto  $\mathbb{Z}_q$  is denoted by  $\bar{r}_1$ . Then one determines  $x \in \mathbb{Z}_q$  such that

$$rx - s\bar{r}_1 - h(m) = 0.$$

The triple  $(m, r_1, x)$  is then published as the digital signature of message  $m$ .

### 7.7.3 Signature verification

In order to verify a signature, one has to build  $u_1 = h(m)/x$ ,  $u_2 = \bar{r}_1/x$  (in  $\mathbb{Z}_q$  and derive

$$V = u_1G + u_2Q.$$

Since we have  $Q = sG$ , the point  $V = (v_1, v_2)$  satisfies  $v_1 = u_1 + u_2s$ . We note that this operations take place in the field  $GF(p^n)$ . The projection of  $GF(p^n)$  on  $\mathbb{Z}_q$  mentioned above should be chosen in such a way that  $\bar{v}_1 = u_1 + u_2s$  is an element of  $\mathbb{Z}_q$ . Then it follows that

$$\bar{v}_1 = u_1 + u_2s = h(m)/x + \bar{r}_1s/x = (h(m) + \bar{r}_1s)/x = rx/x = r.$$

Since  $R = rG$ , we obtain  $\bar{v}_1 = \bar{r}_1$ , i.e.  $R$  and  $V$  coincide modulo the projection onto  $\mathbb{Z}_q$ .

## 7.8 Factorization using elliptic curves

There are factorization<sup>21</sup> algorithms based on elliptic curves<sup>22</sup>. More precisely, these procedures exploit the fact that elliptic curves can be defined over  $\mathbb{Z}_n$  ( $n$  composite number). Elliptic curves over  $\mathbb{Z}_n$  do not form a group, because not every point on such an elliptic curve has an inverse point. This is connected with the fact that - if  $n$  is a composite number - there exist elements in  $\mathbb{Z}_n$  that do not have an inverse with respect to multiplication mod  $n$ . In order to add two points on an elliptic curve over  $\mathbb{Z}_n$ , we can calculate in the same way as on elliptic curves over  $\mathbb{Z}_p$ . Addition of two points (on an elliptic curve over  $\mathbb{Z}_n$ ), however, fails if and only if a factor of  $n$  has been found. The reason for this is that the procedure for adding points on elliptic curves gives elements in  $\mathbb{Z}_n$  and calculates the inverse elements for these (with respect to multiplication mod  $n$ ) in  $\mathbb{Z}_n$ . The extended Euclidean algorithm is used here. If the addition of two points (that lie of an elliptic curve over  $\mathbb{Z}_n$ ) gives an element in  $\mathbb{Z}_n$  that does not have an inverse element in  $\mathbb{Z}_n$ , then the extended Euclidean algorithm delivers a genuine factor of  $n$ .

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<sup>21</sup>John M. Pollard was involved in the development of many different factorization algorithms; also at factorization with ECC he was one of the leading heads. As an employee of British Telekom he never published much. At the RSA data Security Conference in 1999 he was awarded for his “outstanding contributions in mathematics”.

<sup>22</sup>In 1987 H.W. Lenstra published an often used factorization algorithm, based on elliptic curves (see [Len87]).

The biggest compound numbers currently factorized with elliptic curves have about 80 decimal digits:

<https://members.loria.fr/PZimmermann/records/top50.html>.

See also the web link about the [ECMNET project](#) at the end of this chapter.

Factorization using elliptic curves thus principally works as follows: Random curves over  $\mathbb{Z}_n$  are selected, as well as random points (that lie on this curve) and add them; you thus obtain points that also lie on the curve or find a factor of  $n$ . Factorization algorithms based on elliptic curves therefore work probabilistically. The opportunity of defining large number of elliptic curves over  $\mathbb{Z}_n$  allows you to increase the probability of finding two points which you can add to obtain a factor of  $n$ . These procedures are therefore highly suitable for parallelization.

## 7.9 Implementing elliptic curves for educational purposes

There are not many free programms offering ECC under a graphical user interface. The following subsections explain which according functionality is available in CrypTool and in SageMath.

### 7.9.1 CrypTool

CT1 offers elliptic curves for the digital signature function<sup>23</sup> and for ECC-AES hybrid encryption<sup>24</sup>.

It implements the basic algorithms for group operations, for generating elliptic curves, for importing and exporting parameters for elliptic curves over finite fields with  $p$  elements ( $p$  prime). The algorithms have been implemented in ANSI C and comply with draft no. 8 of the IEEE P1363 work group *Standard Specifications for Public Key Cryptography*

<http://grouper.ieee.org/groups/1363>.

The procedure implements the cryptographic primitives for generating and verifying signatures for the variations of Nyberg-Rueppel signatures and DSA signatures based on elliptic curves.

Step-by-step point addition on elliptic curves is visualized in CT1 and in JCT.<sup>25</sup>

### 7.9.2 SageMath

In SageMath elliptic curves are very well described at:<sup>26</sup>

- [http://doc.sagemath.org/html/en/constructions/elliptic\\_curves.html](http://doc.sagemath.org/html/en/constructions/elliptic_curves.html)
- [http://doc.sagemath.org/html/en/reference/plane\\_curves/index.html#elliptic-curves](http://doc.sagemath.org/html/en/reference/plane_curves/index.html#elliptic-curves)

Additionally there is an exhaustive, interactive [ECC tutorial](#) by Maike Massierer. This interactive introduction to Elliptic-Curve Cryptography is built up as a SageMath notebook.

SageMath notebooks are running after a logon within a browser<sup>27,28</sup>.

The [ECC notebook](#) created in 2008 by Massierer<sup>29</sup> consists of 8 parts (title page with contents plus 7 chapters) and aims to let even beginners understand what elliptic curves are:

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<sup>23</sup>The dialog box, which appears in CT1 after clicking the menu **Digital Signatures/PKI \ Sign Message**, offers the EC methods ECSP-DSA and ECSP-NR.

<sup>24</sup>Within CT1 you can find this technique using the menu path **Crypt \ Hybrid**.

<sup>25</sup>CT1: menu **Digital Signatures/PKI \ Signature Demonstration (Signature Generation)**, JCT (default perspective): menu **Visuals \ Elliptic Curve Calculations**.

<sup>26</sup>According SageMath samples can be found e.g. also in the "Elliptic Curve Cryptography (ECC) Tutorial"  
<http://www.williamstein.org/simuw06/notes/notes/node12.html>

<sup>27</sup>If you installed SageMath on your own (Unix) server, you first have to enter at the command line the command **notebook()**.

<sup>28</sup>The [ECC notebook](#) of Massierer needs the KASH3 library: Therefore e.g. with SageMath 4.2.1the package "kash3-2008-07-31.spkg" has to be installed (command **sage -i**).

<sup>29</sup>Instructions to use an interactive SageMath notebook: Update for the new SageMathCloud xxxxxxxxxxxxxxxx

- Some SageMath servers are publicly available and offer running samples as "Published Worksheets", which you

0. ECC Notebook (title page and contents)
1. Introduction and Overview
2. Motivation for the use of Elliptic Curves in Cryptography
3. Elliptic Curves in Cryptography
4. Cryptographic Protocols in ECC
5. Domain Parameter Generation for ECC Systems
6. Conclusion and Further Topics
7. References

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can run and download without login on. These worksheets are listed if you click on “Published” in the above right corner.

- Worksheets using the `interact` command currently need some additional todos for a user to work correctly: sign-in, make a copy and execute all commands again.

- Some of the ECC tutorial’s content uses a special math fonts that are not installed by default with most browsers. When you notice that formulas are not displayed correctly or even get an error message about missing fonts from your browser, you need to install the jsMath fonts for a better layout.

See <http://www.math.union.edu/~dpvc/jsMath/>.

After installing these fonts you can see the jsMath symbol at the lower border of your browser. If you click this symbol you can find the download page for the TIFF fonts. This font installation has to be done at every PC.

## 7.10 Patent aspects

If the field  $GF(2^n)$  is used instead of the prime field  $GF(p)$ , one has to make substantial changes in the implementation. The advantage of  $GF(2^n)$  lies in the fact that calculations in  $GF(2^n)$  can be implemented very efficiently using the binary representation. In particular, divisions are much easier to process compared to  $GF(p)$  (this is particularly important in the signature scheme mentioned above where a division is needed for processing a signature as well as for the verification).

In order to achieve maximal gain in efficiency, one may choose a field that allows special basis like polynomial basis (useful for software implementations) or normal basis (best for hardware implementations). For special  $n$  (like, for example,  $n = 163, 179, 181$ ) one may even combine both advantages. However, they are still non-standard.

Sometimes only the first component and one additional bit is used as representation of a point on the elliptic curve instead of the full two components. Since the first component together with the additional bit is sufficient to derive the full point, this representation minimizes the memory capacity needed. In particular, for normal basis this point compression can be implemented efficiently. In addition, the cryptographic protocols themselves become more effective. A disadvantage is, however, that *point compression* can be used for about half of all elliptic curves only and is protected under US patent (US Patent 6141420, Certicon), causing additional costs. In the general case  $GF(p^n)$  (and also in case  $n = 1$ ) often so called affine or projective co-ordinates are used. Depending on the application, these co-ordinates may result in a gain in efficiency as well.

A comprehensive description of all implementations and their advantages and disadvantages would go far beyond the scope of this paper. We only want to state that there is a variety of possible implementations for elliptic-curve cryptography, much more than for RSA. Therefore, there are serious efforts to reduce this large to a small number of standard implementations. Some standardization committees even try to reduce the complexity by focusing on a small number of (prescribed) curves (ASC approach).

It is still not clear whether these standardization initiatives will be successful or not. However, without agreed standards, ECC is not likely to become a real alternative for RSA. The committees might be forced to act fast if there was a break-through in factorization.

Current informationen about the patents situation can be found here<sup>30</sup>.

## 7.11 Elliptic curves in use

Today elliptic-curve cryptography is already broadly in use. A prominent example is the information network Bonn-Berlin<sup>31</sup>, used for the exchange of strictly confidential documents between different German federal governmental institutions in Berlin and Bonn. With the help of ECC a high security solution could be realized. Interoperability, however, played only a minor role.

In Austria ECC has been massively launched: A bank card with digital signature function.

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<sup>30</sup>[https://en.wikipedia.org/wiki/Elliptic\\_curve\\_cryptography](https://en.wikipedia.org/wiki/Elliptic_curve_cryptography)  
[https://en.wikipedia.org/wiki/ECC\\_patents](https://en.wikipedia.org/wiki/ECC_patents)

<https://cr.yp.to/ecdh/patents.html> Bernstein, Daniel J. (2006-05-23): “Irrelevant patents on elliptic-curve cryptography”. Retrieved 2016-07-14.

<sup>31</sup>The Informationsverbund Bonn-Berlin (IVBB) connects governmental institutions in the old and new German capital.

Both examples show typical applications for elliptic-curve cryptography: For high security solutions and for implementations on smartcards in which the key length is crucial (because of lack of physical memory available).

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All links have been confirmed at July 14, 2016.

# Chapter 8

## Introduction to Bitblock and Bitstream Ciphers

([Klaus Pommerening](#), 2015, Updates: Jan 2016, Apr 2016)

While number theoretic methods prevail for the construction and analysis of asymmetric encryption algorithms, modern symmetric encryption algorithms almost always rely on Boolean algebra, that is on the manipulation of bits. This involves a quite different kind of Mathematics and might be unfamiliar to beginners. Therefore in this chapter we attempt a smooth introduction into this mathematical subject. As previous knowledge we assume elementary mathematical notions such as “variable” and “function”, also for other domains than just the real numbers, and knowledge of elementary algebra, calculus, and number theory.

Let us start with the description how to interpret and process bits, and how to apply functions to them. Such functions are called Boolean functions, named after George Boole<sup>1</sup> who formalized logic by introducing the elementary logical operations, and thereby made logic a part of mathematics (“logical calculus”). Most modern symmetric ciphers, as well as hash functions, are expressed in terms of systems of Boolean functions.

The focus of this chapter is on introducing the mathematical foundations of ciphers that operate on bits. We won’t define concrete ciphers in detail but instead recommend the books by Menezes/Orschot/Vanstone [[MvOV01](#)], Oppliger [[Opp11](#)], Paar und Pelzl [[PP09](#)], Schmeh [[Sch03](#), [Sch16a](#)], and Stamp [[SL07](#)].

A word on nomenclature: In the existing literature these ciphers usually are called “block ciphers” or “stream ciphers” without the prefix “bit-”. Sometimes this usage might cause a misunderstanding since—in particular for stream ciphers—ciphers could operate on other character sets (alphabets, letters) as their basic units. For clarity in case of doubt it’s better to make the “bits” explicit parts of the notations.

This being said we could express the subject of this chapter—bitblock ciphers as well as bitstream ciphers—in other words as

**Symmetric encryption of information given by bits.**

The mathematical foundations and methods belong to the domains of

**Boolean algebra and finite fields.**

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<sup>1</sup>George Boole, English mathematician, logician, and philosopher, November 2, 1815 – December 8, 1864

## 8.1 Boolean Functions

### 8.1.1 Bits and their Composition

On the lowest level computers operate on bits, or small groups of bits, for example bytes that usually consist of 8 bits, or “words” consisting of 32 or 64 bits depending on the processor architecture. This text assumes some familiarity with the bits 0 and 1 and with elementary logical operations such as AND, OR, NOT, and “exclusive or” (XOR). Nevertheless we give a short description to get the terminology clear.

Bits have several distinct interpretations: logically as truth values “True” (T) and “False” (F), algebraically as objects 0 (corresponding to F) and 1 (corresponding to T). Mathematically they are the elements of the two element set  $\{0, 1\}$  that in this chapter is denoted by  $\mathbb{F}_2$ . Here is why:

Consider the residue class ring of  $\mathbb{Z}$  modulo 2. This ring has two elements and is a field since 2 is a prime number. Addition in this field exactly corresponds to the logical composition XOR, multiplication to the logical composition AND, as is seen in Table 8.1. Table 8.2 lists the transformation formulas between the elementary logical and algebraical operations auf.

logical				algebraic				
bits	composition			bits	composition			
$x$	$y$	OR	AND	XOR	$x$	$y$	+	.
F	F	F	F	F	0	0	0	0
F	T	T	F	T	0	1	1	0
T	F	T	F	T	1	0	1	0
T	T	T	T	F	1	1	0	1

Table 8.1: The most important compositions of bits. The logical XOR is identical with the algebraic +, the logical AND with the algebraic · (multiplication).

algebraic to logic
$x + y = (x \vee y) \wedge (\neg x \vee \neg y)$
$x \cdot y = x \wedge y$
logic to algebraic
$x \vee y = x + y + x \cdot y$
$x \wedge y = x \cdot y$
$\neg x = 1 + x$

Table 8.2: Transformation of algebraic operations to logical ones and vice versa

Because this algebraic structure as a field plays a predominant role in cryptography, we use the common notation  $\mathbb{F}_q$  for finite fields from algebra (often also noted as  $\text{GF}(q)$  for “Galois<sup>2</sup> Field” where  $q$  is the number of elements<sup>3</sup>). In this context it also makes sense to use the algebraic symbols + (for XOR) and · (for AND), and, as is common in mathematics, we often omit the multiplication dot. Cryptographers instead tend to use the symbols  $\oplus$  and  $\otimes$ , that however in mathematics are loaded with quite different meanings<sup>4</sup>. Therefore in this text we avoid them

<sup>2</sup>Évariste Galois, French mathematician, October 25, 1811 – May 31, 1832

<sup>3</sup>SageMath uses the notation  $\text{GF}(q)$ .

<sup>4</sup>direct sum and tensor product of vector spaces

except in diagrams.

For clarification we explicitly hint at some special aspects of algebraic calculations in the binary case (or in characteristic 2):

- Two equal summands in a sum cancel out, that is, together give 0. General rule:  $x + x = 0$ , or  $2x = 0$ .
- More generally an even number of equal summands always gives 0, an odd number of equal summands gives exactly this summand. General rule:

$$m x := \underbrace{x + \cdots + x}_m = \begin{cases} 0 & \text{for even } m \\ x & \text{for odd } m. \end{cases}$$

- For algebraic manipulations a subtraction means exactly the same operation as an addition—plus and minus signs are arbitrarily interchangeable. General rule:  $x + y = x - y$ .
- All three binomial formulas, for  $(x + y)^2$ ,  $(x - y)^2$ ,  $(x + y)(x - y)$ , collapse to a single one:

$$(x + y)^2 = x^2 + y^2.$$

Since mixed term occurs twice, resulting in a 0.

### 8.1.2 Description of Boolean Functions

First, we act the naive and define: A **Boolean function** is a rule (or an algorithm) that takes a certain number of bits and produces a new bit from them. Before rephrasing this naive definition more precisely in mathematical language (see Definition 8.1.1) we illuminate its meaning by some concrete illustrations.

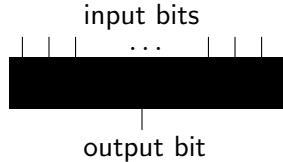
For a comprehensive treatment see [CS09] or [Pom08, Pom14] or the two articles by Claude Carlet<sup>5</sup> in [CH10]<sup>6</sup>.

As a first simple example, consider AND as a Boolean function: It takes two bits and produces one new bit by the well-known rules shown in Table 8.1. For a slightly more complex example take the function  $f_0$  that produces the value

$$f_0(x_1, x_2, x_3) = x_1 \text{ AND } (x_2 \text{ OR } x_3) \quad (8.1)$$

from three bits  $x_1, x_2, x_3$ .

A Boolean function may be depicted by a “black box”:



The mechanism inside this black box may be specified from several different points of view:

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<sup>5</sup>also see his list of publications at <http://www.math.univ-paris13.fr/~carlet/pubs.html>

<sup>6</sup>Carlet's articles are online at <http://www.math.univ-paris13.fr/~carlet/chap-fcts-Bool-corr.pdf> and <http://www.math.univ-paris13.fr/~carlet/chap-vectorial-fcts-corr.pdf>

- **mathematically** by a formula
- **informatically** by an algorithm
- **technically** by a circuit (or plugging diagram)
- **pragmatically** by a truth table (that is the complete lookup table of its values)

Our sample function  $f_0$  is mathematically defined by the Formula (8.1). The corresponding algorithm is also adequately specified by this formula since it has no branching points or conditional statements. As a circuit we visualize  $f_0$  as in Figure 8.1. The truth table is in Table 8.3.

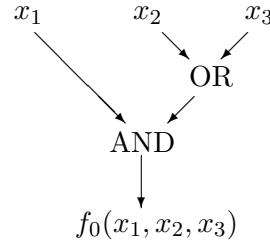


Figure 8.1: Example of a circuit

$x_1$	$x_2$	$x_3$	$f_0(x_1, x_2, x_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Table 8.3: Example of a truth table

The term “truth table” is motivated by the interpretation of the bits in logical calculus: 0 (= F) means “false”, 1 (= T) means “true”. The value  $f(x_1, \dots, x_n)$  of a Boolean function  $f$  indicates whether the complete expression is true or false whenever the single input bits  $x_1, \dots, x_n$  have the respective truth values.

The connection with electrical engineering—that is the connection between logical calculus and electric circuits—was essentially developed by Shannon<sup>7</sup>.

### 8.1.3 The Number of Boolean Functions

The truth table of  $f_0$ , Table 8.3, suggests an easy way of enumerating all Boolean functions: Three variables combine to  $8 = 2^3$  different input triples, since each input bit may assume the values 0 or 1 independently of the other ones. Furthermore a Boolean function  $f$  may assume the

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<sup>7</sup>Claude Elwood Shannon, American mathematician and electrical engineer, April 30, 1916 – February 24, 2001.

values 0 or 1 at each triple independently of the seven other triples. This makes 8 independent choices of 0 or 1, a total of  $2^8$ . Therefore the number of Boolean functions of three variables is  $256 = 2^8$ .

In the general case we have  $N = 2^n$  different allocations of the  $n$  input variables, and for each of these  $N$  input tuples the function may assume the values 0 or 1. This makes a total of  $2^N$  different choices. Thus the general formula is:

**Theorem 8.1.1.** *The number of different Boolean functions of  $n$  variables is  $2^{2^n}$ .*

For four variables we have  $2^{16} = 65536$  different functions. By the formula the number grows superexponentially with  $n$ , even the exponent grows exponentially.

All the 16 Boolean functions of two variables are listed in Section 8.1.7, Table 8.4.

#### 8.1.4 Bitblocks and Boolean Functions

Collections of bits are denoted by several different names<sup>8</sup>, depending on the context: vectors, lists, ( $n$ -) tuples, ... For certain sizes we often use special denotations such as bytes or octets (for 8 bits), words (for 32 or 64 bytes depending on the processor architecture) ... In the present chapter we usually use the denomination “bitblocks” that is common in cryptography. Thus a **bitblock** of length  $n$  is a list  $(x_1, \dots, x_n)$  of bits where the order matters. There are eight different bitblocks of length 3:

$$(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1).$$

Sometimes, if the danger of misunderstanding is negligible, we write them as bitstrings<sup>9</sup> without parentheses or commas:

$$000, 001, 010, 011, 100, 101, 110, 111. \quad (8.2)$$

We often use the abbreviation  $x$  for  $(x_1, \dots, x_n)$ . This short form highlights the fact that we consider bitblocks as objects of their own.

The  $2^n$  different bitblocks of length  $n$  are the elements of the cartesian product  $\mathbb{F}_2^n = \mathbb{F}_2 \times \dots \times \mathbb{F}_2$ . This cartesian product has a “natural” structure as a vector space over the field  $\mathbb{F}_2$ —bitblocks  $x$  and  $y \in \mathbb{F}_2^n$  may be added or multiplied by scalars  $a \in \mathbb{F}_2$ :

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n),$$

$$a \cdot (x_1, \dots, x_n) = (a \cdot x_1, \dots, a \cdot x_n).$$

Now we can write down the mathematically exact definition:

**Definition 8.1.1.** *A Boolean function of  $n$  variables is a map*

$$f: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2.$$

It takes a bitblock of length  $n$  as argument, and produces a single bit.

In this text we sometimes denote the set of all Boolean functions on  $\mathbb{F}_2^n$  by  $\mathcal{F}_n$ . By Theorem 8.1.1 its cardinality is  $\#\mathcal{F}_n = 2^{2^n}$ .

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<sup>8</sup>that all refer the same conceptual entities. However in Python or SageMath the different names sometimes denote different types.

<sup>9</sup>sometimes also as columns, that is  $n \times 1$  matrices when the focus is on the interpretation as vectors

**Convention** If we describe a Boolean function by its truth table, we usually order the truth table lexicographically<sup>10</sup> with respect to  $x \in \mathbb{F}_2^n$ , as in the example above. This order corresponds to the natural order of the integers  $a = 0, \dots, 2^n - 1$ , if these are expanded in base 2

$$a = x_1 \cdot 2^{n-1} + \dots + x_{n-1} \cdot 2 + x_n$$

and assigned to the corresponding bitblocks  $(x_1, \dots, x_n) \in \mathbb{F}_2^n$ .

### 8.1.5 Logical Expressions and Conjunctive Normal Form

For describing Boolean functions in mathematical terms, that is by formulas, there are two approaches (beyond truth tables):

- In the logical approach Boolean functions are expressed by disjunctions (the operation OR, also written as  $\vee$ ), conjunctions (the operation AND, also written as  $\wedge$ ), and negations (the operation NOT, also written as  $\neg$ ). Compositions of these operations are called **logical expressions**.
- In the algebraic approach Boolean functions are expressed by additions  $+$  and multiplications  $\cdot$  in the field  $\mathbb{F}_2$ . Compositions of these operations are called **(binary) polynomial expressions**<sup>11</sup>.

We'll see soon that both approaches describe all Boolean functions, and that we even can require a certain structure as so called normal forms. Of course there are algorithms to switch between the three representations—truth tables, logical expressions, and binary polynomial expressions—for all Boolean functions. But we cannot hope that these algorithms are efficient for large numbers  $n$  of variables—even writing down the truth table involves  $2^n$  bits. For the algorithmic treatment of Boolean functions in SageMath see also Appendix 8.4.

It seems that the algebraic approach allows a smoother handling of Boolean functions for cryptologic purposes due to its (yet to explore) more rigid structure. In contrast the logical approach more easily leads to a realization in hardware by circuits since the elementary Boolean operations have direct analogues as circuit elements (“gates”).

Since in the following the logical approach plays a minor role we state the result on normal forms without further reasoning. The possibility of a logical representation (without normalization) will follow as a corollary in Section 8.1.7, see Theorem 8.1.5.

**Theorem 8.1.2.** *Each Boolean function of  $n$  variables  $x_1, \dots, x_n$  has a representation of the form (conjunction)*

$$f(x) = s_1(x) \wedge \dots \wedge s_r(x)$$

*with some index  $r$  where the  $s_j(x)$  for  $j = 1, \dots, r$  each have the form (disjunctions)*

$$s_j(x) = t_{j1}(x) \vee \dots \vee t_{jn_j}(x)$$

---

<sup>10</sup>The lexicographic order orders strings (like a dictionary) by the value of their first characters—here 0 or 1 with  $0 < 1$ . If the first characters are equal, the order looks at the second character and so on. The sequence 011, 100, 101 is in lexicographic order. A counterexample is the sequence 100, 101, 011: here the third string begins with 0 that is smaller than the first character of the string preceding it. The sequence of the eight bitblocks of length 3 in Formula (8.2) is in lexicographic order.

<sup>11</sup>An expression that contains other kinds of operations is non-polynomial. For numbers we could think of using input variables as exponents. This however makes little sense for the Boolean variables 0 and 1. But see the definition of the Fourier transform in Section 8.2.8.

with a certain number  $n_j$  of terms  $t_{jk}(x)$  ( $j = 1, \dots, r$  and  $k = 1, \dots, n_j$ ), each of which in turn has the form  $x_i$  (an input bit) or  $\neg x_i$  (a negated input bit) for some index  $i$ <sup>12</sup>.

In other words: We can build each Boolean function by first forming a handful of expressions (the  $s_j(x)$ ) as OR of some of the input bits or their negations, and then join these expressions by AND (“conjunction of disjunctions”). This “normal form” cleanly separates AND- and OR-compositions into two layers—there is no further intermixture. The example function  $f_0$  from Section 8.1.2 was defined by the formula

$$f_0(x_1, x_2, x_3) = \underbrace{x_1}_{s_1(x)} \wedge \underbrace{(x_2 \vee x_3)}_{s_2(x)}$$

that already has the “conjunctive” form from Theorem 8.1.2 with

$$n_1 = 1, \quad s_1(x) = t_{11}(x) = x_1, \quad n_2 = 2, \quad t_{21}(x) = x_2, \quad t_{22}(x) = x_3.$$

This is no longer true if we expand it:

$$f_0(x) = (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$$

This example doesn’t display negated input bits. However in Table 8.4 we see some of them.

The form of a Boolean function according to Theorem 8.1.2 is called **conjunctive normal form (CNF)**. It is not unique<sup>13,14</sup>. Without further explanation we remark that there is a further simplification as a “canonical CNF” that guarantees a certain uniqueness. There is also an analogous disjunctive normal form (**DNF**) (a “disjunction of conjunctions”).

### 8.1.6 Polynomial Expressions and Algebraic Normal Form

We consider (binary) polynomial expressions in the variables  $x_1, \dots, x_n$ , such as  $x_1^2 x_2 + x_2 x_3 + x_3^2$ . Since we work over the field  $\mathbb{F}_2$  only the constants 0 and 1 occur as coefficients and these don’t show up explicitly. The observation<sup>15</sup> that  $a^2 = a$  for all elements  $a \in \mathbb{F}_2$ , and even  $a^e = a$  for all exponents  $e \geq 1$ , leads to another simplification of the expressions. As a consequence for binary polynomial expressions we need consider the variables  $x_1, \dots, x_n$  with exponents 0 and 1 only. Therefore our sample expression may be written as  $x_1 x_2 + x_2 x_3 + x_3$ . Another example:  $x_1^3 x_2 + x_1 x_2^2 = x_1 x_2 + x_1 x_2 = 0$ .

In general a **monomial expression** (or simply “monomial”) has the form

$$x^I := \prod_{i \in I} x_i \quad \text{with a subset } I \subseteq \{1, \dots, n\},$$

in other words it is a product of some of the variables where the subset  $I$  specifies the choice of “some”. Here is an illustrative example with  $n = 3$ :

$$I = \{2, 3\} \implies x^I = x_2 x_3$$

---

<sup>12</sup>In particular  $n_j \leq n$  for  $j = 1, \dots, r$ . Each individual input bit  $x_i$  occurs in each of the  $t_{jk}(x)$  either directly, or negated or not at all.

<sup>13</sup>For example we could add the terms  $\wedge (x_1 \vee x_2) \wedge (x_1 \vee x_3)$  to the normal form of  $f_0$ .

<sup>14</sup>The SageMath class `sage.logic.boolformula.BooleanFormula` provides transformations of a logical expression to the CNF by the function `convert_cnf()`, and to the corresponding truth table by the function `truthtable()`.

<sup>15</sup>Note  $0^2 = 0$  and  $1^2 = 1$ .

The total number of such monomial expressions is exactly  $2^n$ , corresponding to the number of choices of products of  $n$  potential factors. Here the empty set corresponds to an “empty” product of 0 factors whose usual interpretation is 1<sup>16</sup>. Thus

$$I = \emptyset \implies x^I = 1$$

A monomial expression has an immediate interpretation as a Boolean function. At first sight we don’t know whether all of these functions are distinct, but we’ll see this in a few moments.

A **polynomial expression** is a sum of monomial expressions—remember that we are in the binary case where coefficients take the values 0 or 1 only. Thus the most general (binary) polynomial expression has the form

$$\sum_{I \subseteq \{1, \dots, n\}} a_I x^I,$$

where all coefficients  $a_I$  are 0 or 1. In other words we add a subset of the  $2^n$  potential monomial expressions, and for this we have  $2^{2^n}$  choices. All these expressions give different Boolean functions, but we yet have to prove this. At first we prove that each Boolean function has a polynomial expression.

**Theorem 8.1.3 (ANF).** *For each Boolean function  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  there are coefficients  $a_I \in \mathbb{F}_2$  (that is = 0 or 1), where  $I$  runs through all subsets of  $\{1, \dots, n\}$ , such that  $f$  may be written as a polynomial expression in  $n$  variables of the form:*

$$f(x_1, \dots, x_n) = \sum_{I \subseteq \{1, \dots, n\}} a_I x^I. \quad (8.3)$$

### Proof

(Induction on  $n$ ) Start with  $n = 1$ <sup>17</sup>. The four Boolean functions of one variable  $x$  are the constants 0 and 1 and the functions given by  $x$  and  $1 + x$  (= the negation of  $x$ ). They all have the claimed form.

Now let  $n \geq 1$ . For  $x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$  we abbreviate  $(x_2, \dots, x_n) \in \mathbb{F}_2^{n-1}$  as  $x'$ . Then we can also write  $x = (x_1, x')$  instead of  $x = (x_1, \dots, x_n)$ .

Now take a function  $f \in \mathcal{F}_n$ . For each fixed value  $b$  of the first variable  $x_1$ , the choices being  $b = 0$  or  $b = 1$ , we consider the function  $x' \mapsto f(b, x')$  of the  $n - 1$  variables that  $x'$  consists of. By induction (for  $b = 0$  as well as for  $b = 1$ ) we know

$$f(b, x') = p_b(x') \quad \text{for all } x' \in \mathbb{F}_2^{n-1}$$

where  $p_0, p_1$  are polynomial expressions in  $x'$  of the desired form:

$$p_0(x') = \sum_{J \subseteq \{2, \dots, n\}} b_J x^J, \quad p_1(x') = \sum_{J \subseteq \{2, \dots, n\}} c_J x^J.$$

Therefore

$$f(x_1, x') = \begin{cases} p_0(x'), & \text{if } x_1 = 0, \\ p_1(x'), & \text{if } x_1 = 1, \end{cases} \quad \text{for all } x = (x_1, x') \in \mathbb{F}_2^n$$

since  $x_1$  assumes the values 0 or 1 only. We combine this conditional formula into

$$f(x_1, x') = (1 + x_1) p_0(x') + x_1 p_1(x') \quad \text{for all } x \in \mathbb{F}_2^n \quad (8.4)$$

---

<sup>16</sup>whereas “empty” sums usually are interpreted as 0.

<sup>17</sup>As a “typical mathematical sophistry” we could start with  $n = 0$ —the constant polynomial expressions 0 and 1 correspond to the two possible constant functions of 0 variables.

(To check this formula substitute  $x_1 = 0$  or  $x_1 = 1$ ). By expanding the right hand side and eliminating repeated monomials we get a polynomial expression in  $x$  of the claimed form:

$$\begin{aligned} f(x_1, x') &= p_0(x') + x_1(p_0(x') + p_1(x')) \\ &= \underbrace{\sum_{J \subseteq \{2, \dots, n\}} b_J x^J}_{\text{all monomials without } x_1} + \underbrace{\sum_{J \subseteq \{2, \dots, n\}} (b_J + c_J) x_1 x^J}_{\text{all monomials with } x_1}. \end{aligned}$$

□

The wording of this theorem is mathematically compact. As an illustration look at the second column of table 8.4, where the variables are  $x$  and  $y$  instead of  $x_1$  and  $x_2$ , and the coefficients are  $a, b, c, d$  instead of  $a_\emptyset, a_{\{1\}}, a_{\{2\}}, a_{\{1,2\}}$ . Each row of the table describes a Boolean function of two variables. The corresponding polynomial expression is the sum of the terms  $1, x, y, xy$  that have coefficient 1 in the representation by Equation (8.3), whereas terms with coefficients 0 don't show up explicitly.

Theorem 8.1.3 provides a representation of a Boolean function as a polynomial expression. This expression is called the **algebraic normal form (ANF)**<sup>18</sup>. The ANF is unique: Since the total number of polynomial expressions is  $2^{2^n}$ , and since they represent all  $2^{2^n}$  different Boolean functions, all these polynomial expressions must differ as functions, and furthermore this representation of a Boolean function as a polynomial expression must be unique. We have shown:

**Theorem 8.1.4.** *The representation of a Boolean function in algebraic normal form is unique.*

**Definition 8.1.2.** *The (algebraic) degree of a Boolean function  $f \in \mathcal{F}_n$  is the degree of its polynomial expression in algebraic normal form,*

$$\deg f = \max\{\#I \mid a_I \neq 0\}.$$

*It is always  $\leq n$ .*

The degree indicates how many different variables maximally occur in a monomial of the ANF.

**Example** Independently of the number of variables there are exactly two Boolean functions of degree 0: the two Boolean constants 0 and 1.

Functions of degree  $\leq 1$  are called affine functions. They are a sum of a constant and a Boolean linear form, see Section 8.1.9. If the degree is  $> 1$  the function is called nonlinear, even though the denomination “non-affine” would be more correct.

**Example** The Boolean function given by  $x \mapsto x_1 x_2 + x_2 x_3 + x_3$  has degree 2.

**Remark** Boolean functions have a high degree not by high powers of some variables but “only” by large products of different variables. Each single variable occurs with exponent at most 1 in each monomial of the ANF. Another way to express this fact is to say that all partial degrees—the degrees in the single variables  $x_i$  without regard for the other variables—are  $\leq 1$ .

---

<sup>18</sup>The transformation of ANF to truth table and vice versa is provided by the (internal) function `__convert()` of the class `BoolF()`, see SageMath sample 8.42. SageMath's own module `sage.crypto.boolean_function` also provides initialization by a truth table or by a Boolean polynomial, and functions `algebraic_normal_form()` and `truth_table()` for the transformations.

### 8.1.7 Boolean Functions of Two Variables

All the  $2^4 = 16$  Boolean functions of two variables  $x$  and  $y$  are enumerated in Table 8.4, as polynomial expressions in algebraic normal form  $a + bx + cy + dxy$ , and as logical expressions. The parameters  $a_I$  from Theorem 8.1.3 translate as follows:  $a = a_\emptyset$ ,  $b = a_{\{1\}}$ ,  $c = a_{\{2\}}$ ,  $d = a_{\{1,2\}}$ , the input variables as  $x = x_1$ ,  $y = x_2$ .

$a$	$b$	$c$	$d$	ANF	logical operation	CNF
0	0	0	0	0	False constant	$x \wedge \neg x$
1	0	0	0	1	True constant	$x \vee \neg x$
0	1	0	0	$x$	projection	$x$
1	1	0	0	$1 + x$	negation	$\neg x$
0	0	1	0	$y$	projection	$y$
1	0	1	0	$1 + y$	negation	$\neg y$
0	1	1	0	$x + y$	$x \text{ XOR } y$	$(x \vee y) \wedge (\neg x \vee \neg y)$
1	1	1	0	$1 + x + y$	$x \iff y$	$(\neg x \vee y) \wedge (x \vee \neg y)$
0	0	0	1	$xy$	$x \wedge y$	AND
1	0	0	1	$1 + xy$	$\neg(x \wedge y)$	NAND
0	1	0	1	$x + xy$	$x \wedge (\neg y)$	$x \wedge (\neg y)$
1	1	0	1	$1 + x + xy$	$x \implies y$	implication
0	0	1	1	$y + xy$	$(\neg x) \wedge y$	$(\neg x) \wedge y$
1	0	1	1	$1 + y + xy$	$x \iff y$	implication
0	1	1	1	$x + y + xy$	$x \vee y$	OR
1	1	1	1	$1 + x + y + xy$	$\neg(x \vee y)$	NOR

Table 8.4: The 16 operations on two bits (= Boolean functions of 2 variables), using Table 8.2  
(The order of the first column is lexicographic if  $a, b, c, d$  are considered in reverse order.)

We have already seen that each Boolean function admits a polynomial expression. To show that each Boolean function also admits a logical expression we only have to make sure that the algebraic operations  $+$  and  $\cdot$  have expressions by the logical operations  $\vee$ ,  $\wedge$ , and  $\neg$ . To see this look at the corresponding rows of Table 8.4. Thus we have shown (as a weak form of the here unproven Theorem 8.1.2):

**Theorem 8.1.5.** *Each Boolean function admits a logical expression, that is a representation by a composition of the logical operations  $\vee$ ,  $\wedge$ , and  $\neg$ .*

**Hint** In the algebraic interpretation the logical negation  $\neg$  corresponds to the addition of 1.

**Remark** The analogous form of the ANF for a Boolean function of three variables  $x, y, z$  is<sup>19</sup>

$$(x, y, z) \mapsto a + bx + cy + dz + exy + fxz + gyz + hxyz.$$

Here we see 8 coefficients  $a, \dots, h$ . This fits the observations that

<sup>19</sup>In this formula the letter  $f$ —in contrast with the common use in this text—denotes a coefficient, not a function. Mathematicians almost always use letters as symbols relative to the context, and only in exceptional cases with an absolute meaning. Such exceptions are the numbers  $e$ ,  $i$ , and  $\pi$ . But even  $i$  often denotes—in contexts without complex numbers—some other object, for example an index in a sum. Or sometimes  $e$  is used as exponent, or coefficient.

- a Boolean function of three variables has up to  $8 = 2^3$  monomials,
- and the number of such functions is  $2^{2^3} = 2^8 = 256$ .

**Example** What is the ANF of the function  $f_0$  from Section 8.1.2, written as  $f_0(x, y, z) = x \wedge (y \vee z)$  using the variables  $x, y, z$ ? By Table 8.4 we have  $(y \vee z) = y + z + yz$ , whereas  $\wedge$  simply is the multiplication in the field  $\mathbb{F}_2$ . Hence

$$f_0(x, y, z) = x \cdot (y + z + yz) = xy + xz + xyz,$$

and by the way we see that the degree of  $f_0$  is 3.

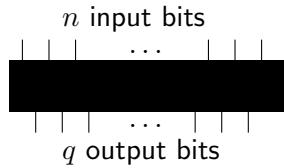
**Remark** From Table 8.4 we might directly read off a naive algorithm for translating logical expressions into (binary) polynomial expressions, and vice versa.

### 8.1.8 Boolean Maps

Cryptographic algorithms usually produce several bits at once, not only single bits. An abstract model for this is a **Boolean map**, that is a map<sup>20</sup>

$$f: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^q$$

with natural numbers  $n$  and  $q$ , illustrated by this picture



The images of  $f$  are bitblocks of length  $q$ . Decomposing them into their components,

$$f(x) = (f_1(x), \dots, f_q(x)) \in \mathbb{F}_2^q,$$

we see that we may interpret a Boolean map to  $\mathbb{F}_2^q$  as a  $q$ -tuple (or system) of Boolean functions

$$f_1, \dots, f_q: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2.$$

**Definition 8.1.3.** *The (algebraic) degree of a Boolean map  $f: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^q$  is the maximum of the algebraic degrees of its components,*

$$\deg f = \max\{\deg f_i \mid i = 1, \dots, q\}.$$

**Theorem 8.1.6.** *Each Boolean map  $f: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^q$  has a unique representation as*

$$f(x_1, \dots, x_n) = \sum_{I \subseteq \{1, \dots, n\}} x^I a_I$$

with  $a_I \in \mathbb{F}_2^q$ , and monomials  $x^I$  as in Theorem 8.1.3.

This representation of a Boolean map is also called **algebraic normal form**. It results from combining the algebraic normal forms of its component functions  $f_1, \dots, f_q$ . Compared with Theorem 8.1.3 the  $x^I$  and  $a_I$  occur in reversed order. This follows the convention that usually “scalars” (here the  $x^I \in \mathbb{F}_2$ ) precede “vectors” (here the  $a_I \in \mathbb{F}_2^q$ ). The  $a_I$  are the  $q$ -tuples of the respective coefficients of the component functions.

---

<sup>20</sup>The distinction between the concepts of “function” and “map” is somewhat arbitrary. Mathematicians often use it to indicate whether the values belong to a one-dimensional or multidimensional domain. Boolean maps—as systems of Boolean functions—often are denoted as “vector valued Boolean functions”, or “vectorial Boolean functions” (VBF).

### Example

Define a Boolean map  $g: \mathbb{F}_2^3 \rightarrow \mathbb{F}_2^2$  by a pair of logical expressions in three variables  $x, y, z$ :

$$g(x, y, z) := \begin{pmatrix} x \wedge (y \vee z) \\ x \wedge z \end{pmatrix}$$

where the components are written below each other, in column form, for clarity. We recognize the function  $f_0$  as the first component. The second component is the product  $x \cdot z$ . Hence the ANF of  $g$  is

$$g(x, y, z) = \begin{pmatrix} xy + xz + xyz \\ xz \end{pmatrix} = xy \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + xz \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + xyz \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The algebraic degree is 3, and the value table is in Table 8.5. Here the values  $g(x, y, z) \in \mathbb{F}_2^2$  of  $g$  are written as bitstrings<sup>21</sup> of length 2.

$x$	$y$	$z$	$g(x, y, z)$
0	0	0	00
0	0	1	00
0	1	0	00
0	1	1	00
1	0	0	00
1	0	1	11
1	1	0	10
1	1	1	11

Table 8.5: The value table of a sample Boolean map

#### 8.1.9 Linear Forms and Linear Maps

A Boolean function  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  is called **linear form** if it has degree 1 and absolute term 0. This means that its algebraic normal form has linear terms only:

$$f(x) = \sum_{i=1}^n s_i x_i \quad \text{for all } x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$$

with  $s_i \in \mathbb{F}_2$  for  $i = 1, \dots, n$ . Because the  $s_i$  are 0 or 1 a linear form is a partial sum

$$f(x) = \alpha_I(x) = \sum_{i \in I} x_i \quad \text{for all } x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$$

over a subset  $I \subseteq \{1, \dots, n\}$  of all indices, namely

$$I = \{i \mid s_i = 1\}.$$

In particular there are exactly  $2^n$  Boolean linear forms in  $n$  variables, and they correspond to the power set  $\mathfrak{P}(\{1, \dots, n\})$  in a natural way.

---

<sup>21</sup>The varying notation of bitblocks, sometimes in column form, sometimes as strings, doesn't aim at maximizing the confusion but suggests that in different contexts different notations are convenient. After all, two bits are two bits, no matter whether written in a line or in a column, with or without a separating comma, with or without parentheses or brackets.

Other common notations are (for  $I = \{i_1, \dots, i_r\}$ ):

$$f(x) = \alpha_I(x) = x[I] = x[i_1, \dots, i_r] = x_{i_1} + \dots + x_{i_r}.$$

The following theorem relates the definition with the notion of linear forms from linear algebra:

**Theorem 8.1.7.** *A Boolean function  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  is a linear form if and only if the following two conditions hold:*

- (i)  $f(x+y) = f(x) + f(y)$  for all  $x, y \in \mathbb{F}_2^n$ .
- (ii)  $f(ax) = af(x)$  for all  $a \in \mathbb{F}_2$  and all  $x \in \mathbb{F}_2^n$ .

### Proof

The representation by partial sums shows that each linear form meets the two conditions.

For the reverse direction let  $f$  be a Boolean function with (i) and (ii). Let  $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 1)$  be the “canonical unit vectors”. Then each  $x = (x_1, \dots, x_n) \in \mathbb{F}_2^n$  is a sum

$$x = x_1e_1 + \dots + x_ne_n.$$

Hence

$$f(x) = f(x_1e_1) + \dots + f(x_ne_n) = x_1f(e_1) + \dots + x_nf(e_n)$$

is the partial sum of the  $x_i$  over the index set consisting of the  $i$  for which the constant value  $f(e_i)$  is 1. Therefore  $f$  is a linear form in the sense of the definition above.  $\square$

A Boolean map  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^q$  is called linear if all of its component functions  $f_1, \dots, f_q$  are linear forms. As in the case  $q = 1$  we can show:

**Theorem 8.1.8.** *A Boolean map  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^q$  is linear if and only if the following two conditions hold:*

- (i)  $f(x+y) = f(x) + f(y)$  for all  $x, y \in \mathbb{F}_2^n$ .
- (ii)  $f(ax) = af(x)$  for all  $a \in \mathbb{F}_2$  and all  $x \in \mathbb{F}_2^n$ .

**Theorem 8.1.9.** *A Boolean map  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^q$  is linear if and only if it has the form*

$$f(x) = \sum_{i=1}^n x_i s_i$$

with  $s_i \in \mathbb{F}_2^q$ .

(Here again the  $x_i$  and  $s_i$  are written in reverse order.)

**Affine (Boolean) maps** are maps of algebraic degree  $\leq 1$ . They result from adding linear maps and constants.

In the case  $q = 1$ , that is for functions, the only possible constants are 0 and 1. Adding the constant 1 effects a logical negation, that is a “flipping” of all bits. Therefore we can say: *The affine Boolean functions are the linear forms and their negations.*

### 8.1.10 Systems of Boolean Linear Equations

Linear algebra over the field  $\mathbb{F}_2$  is quite simple, many complications known from other mathematical areas boil down to trivialities. Such is the case for the solution of systems of linear equations, explicitly written as

$$\begin{array}{lclcl} a_{11}x_1 & + & \cdots & + & a_{1n}x_n = b_1 \\ \vdots & & & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n = b_m \end{array}$$

with given  $a_{ij}$  and  $b_i \in \mathbb{F}_2$ , and unknown  $x_j$  for which we search solutions. In matrix terms this system has an elegant expression as

$$Ax = b$$

where  $A$  is an  $m \times n$  matrix, and  $x$  and  $b$  are column vectors, that is  $n \times 1$  or  $m \times 1$  matrices.

### Systems of Linear Equations in SageMath

To clarify the relation with “common” linear algebra we consider an example of a system of linear equations over the rational numbers:

$$\begin{array}{lclcl} x_1 & + & 2x_2 & + & 3x_3 = 0 \\ 3x_1 & + & 2x_2 & + & x_3 = -4 \\ x_1 & + & x_2 & + & x_3 = -1 \end{array}$$

and study how to handle this in SageMath. The complete solution is in SageMath sample 8.1. Here are the single steps:

1. Define the “coefficient matrix”  $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ .
2. Define the “image vector”  $b = (0, -4, 1)$ .
3. Let SageMath calculate a “solution vector”  $x$ . Since we wrote the left hand side of the system as matrix product  $Ax$  we have to use the method `solve_right()`.
4. Our system of linear equations could admit several solutions. We find them all by solving the corresponding “homogeneous”<sup>22</sup> system  $Az = 0$ . If  $z$  is a solution of the homogeneous system, then  $A \cdot (x + z) = Ax + Az = b + 0 = b$ , so  $x + z$  is a solution of the original (“inhomogeneous”) system. In this way we get all solutions. For if  $Ax = b$  and  $Ay = b$ , then  $A \cdot (y - x) = 0$ , hence the difference  $y - x$  solves the homogeneous system. For the solution of the homogeneous system we use the SageMath method `right_kernel()`.
5. The output appears somewhat cryptic. It says<sup>23</sup> that all solutions of the homogeneous system are multiples of the vector  $z = (1, -2, 1)$ .
6. We verify the solution  $y = x - 4z$  by checking that  $Ay = b$ .

---

<sup>22</sup>replacing the right hand side  $b$  by 0

<sup>23</sup>Since all coefficients were integers SageMath worked over  $\mathbb{Z}$  (= `Integer Ring`).

---

**SageMath sample 8.1** Solution of a system of linear equations over  $\mathbb{Q}$ 

---

```
sage: A = Matrix([[1,2,3],[3,2,1],[1,1,1]])
sage: b = vector([0,-4,-1])
sage: x = A.solve_right(b); x
(-2, 1, 0)
sage: K = A.right_kernel(); K
Free module of degree 3 and rank 1 over Integer Ring
Echelon basis matrix:
[ 1 -2  1]
sage: y = x - 4*vector([1,-2,1]); y
(-6, 9, -4)
sage: A*y
(0, -4, -1)
```

---

### Systems of Linear Equations in the Boolean Case

In the general case (over an arbitrary field) the underlying algorithm for solving a system of linear equations is Gaussian<sup>24</sup> elimination. This algorithm of course also hides in the SageMath method `solve_right()`.

In the Boolean case (over the field  $\mathbb{F}_2$ ) the solution of a system of linear equations by Gaussian elimination is extremely simple since all coefficients are 0 or 1, and multiplication and division are completely trivial. We don't need to deal with complicated coefficients (such as fractions over  $\mathbb{Q}$ ), or inexact coefficients (such as floating point numbers over  $\mathbb{R}$ ). So simple is the method that even for six unknowns calculating by “paper and pencil” almost outperforms the feeding of the corresponding small SageMath program with the correct input values. An example will illustrate this effect.

The idea of elimination is: reduce the system to a system with only  $n - 1$  unknowns, or “eliminate” one unknown.

**Case 1**  $x_n$  only occurs with coefficients  $a_{in} = 0$  for  $i = 1, \dots, m$ . In other words,  $x_n$  doesn't occur at all. Then the system is already reduced.

**Case 2**  $x_n$  has coefficient 1 in one of the equations. Then solve this<sup>25</sup> equation for  $x_n$ , and substitute the resulting expression for  $x_n$ ,

$$x_n = a_{i1}x_1 + \cdots + a_{in-1}x_{n-1} + b_i,$$

in the other  $m - 1$  equations. Thereafter the remaining equations contain only the unknowns  $x_1, \dots, x_{n-1}$ .

Continue recursively until there remains only one unknown or one equation. Now for the example that illustrates this simple procedure.

---

<sup>24</sup>Johann Carl Friedrich Gauß, German mathematician, astronomer, geodesist, and physicist, April 30, 1777 – February 23, 1855

<sup>25</sup>If we have more than one choice it doesn't matter which one we choose—in contrast to the situation over other fields where the search for an optimal “pivot element” constitutes an essential part of the algorithm.

## Example

$$\begin{array}{rccccc}
 x_1 & +x_3 & & +x_6 & = & 1 \\
 x_1 & +x_2 & +x_4 & +x_6 & = & 0 \\
 & x_2 & +x_3 & +x_5 & +x_6 & = & 0 \\
 x_1 & & +x_4 & +x_5 & & = & 1 \\
 & x_2 & +x_4 & +x_5 & & = & 1
 \end{array}$$

From the first equation we get  $x_6 = x_1 + x_3 + 1$  (using the rule that plus and minus are the same). Elimination results in a reduced system consisting of the equations 2 to 5 (note  $x_1 + x_1 = 0$  etc.):

$$\begin{array}{rccccc}
 & x_2 & +x_3 & +x_4 & = & 1 \\
 x_1 & +x_2 & & +x_5 & = & 1 \\
 x_1 & & +x_4 & +x_5 & = & 1 \\
 & x_2 & +x_4 & +x_5 & = & 1
 \end{array}$$

Solving the second equation of the reduced system for  $x_5$  and substituting  $x_5 = x_1 + x_2 + 1$  in the other ones gives

$$\begin{array}{rccccc}
 x_2 & +x_3 & +x_4 & = & 1 \\
 x_2 & & +x_4 & = & 0 \\
 x_1 & & +x_4 & = & 0
 \end{array}$$

Now the last two equations yield  $x_4 = x_2 = x_1$ , and then the first one yields  $x_3 = 1$ . Thus the complete solution is

$$x_1 = x_2 = x_4 = x_6 = a \quad \text{with } a \in \mathbb{F}_2 \text{ arbitrary}, \quad x_3 = 1, \quad x_5 = 1.$$

Since  $a$  may assume the values 0 and 1 our result consists of exactly two solutions:  $(0, 0, 1, 0, 1, 0)$  and  $(1, 1, 1, 1, 1, 1)$ .

## The Example in SageMath

SageMath sample 8.2 shows the solution in SageMath code. The SageMath method `solve_right()` gives the solution  $(0, 0, 1, 0, 1, 0)$  only. To get all solutions we have to solve the homogeneous system. Its solutions are the multiples of the vector  $v = (1, 1, 0, 1, 0, 1)$ , that is, the two vectors  $(0, 0, 0, 0, 0, 0) = 0 \cdot v$  and  $(1, 1, 0, 1, 0, 1) = 1 \cdot v$ . Thus the second solution of the inhomogeneous system is  $(0, 0, 1, 0, 1, 0) + (1, 1, 0, 1, 0, 1) = (1, 1, 1, 1, 1, 1)$ .

## Estimate of the Cost

What about the cost of solving a system of Boolean linear equations in general? Consider  $m$  equations with  $n$  unknowns. Then the matrix  $A$  of coefficients has size  $m \times n$ . The expanded matrix  $(A, b)$  has size  $m \times (n + 1)$ .

We only aim at a coarse estimate and neglect possible optimizations of the procedure. For simplicity we assume  $m = n$ . In the case  $m > n$  we would ignore additional equations<sup>26</sup>. In the case  $m < n$  we would append “null equations” (of the kind  $0 \cdot x_1 + \dots + 0 \cdot x_n = 0$ ).

The elimination step, that is the reduction of the problem size from  $n$  to  $n - 1$ , amounts to exactly one pass through all  $n$  rows of the *expanded* matrix:

---

<sup>26</sup>Of course we must check if the solutions we found also satisfy the additional equations.

---

**SageMath sample 8.2** Solution of a system of Boolean linear equations

---

```
sage: M = MatrixSpace(GF(2), 5, 6) # GF(2) = field with two elements
sage: A = M([[1,0,1,0,0,1],[1,1,0,1,0,1],[0,1,1,0,1,1],[1,0,0,1,1,0],\
[0,1,0,1,1,0]]); A
[1 0 1 0 0 1]
[1 1 0 1 0 1]
[0 1 1 0 1 1]
[1 0 0 1 1 0]
[0 1 0 1 1 0]
sage: b = vector(GF(2), [1,0,0,1,1])
sage: x = A.solve_right(b); x
(0, 0, 1, 0, 1, 0)
sage: K = A.right_kernel(); K
Vector space of degree 6 and dimension 1 over Finite Field of size 2
Basis matrix:
[1 1 0 1 0 1]
```

---

- At first we search the first entry 1 in column  $n$ , consisting of the coefficients of  $x_n$ . This costs at most  $n$  single bit comparisions.
- Then we add the chosen row (containing the first entry 1 in column  $n$ ) to all those rows below it that also contain a 1 in column  $n$ . This amounts (per row) to a single bit comparision and up to  $n$  bit additions—we ignore the  $n$ -th entry since we know already that it becomes 0.

All in all this makes  $n$  bit comparisions and at most  $n \cdot (n - 1)$  bit additions, a total of at most  $n^2$  bit operations. Let  $N(n)$  be the number of bit operations for the complete solution of the system. Then we have the following inequality:

$$N(n) \leq n^2 + N(n - 1) \quad \text{for all } n \geq 2.$$

Now  $N(1) = 1$ : We only have to check the one coefficient of the one unknown whether it is 0 or 1. From this we decide whether the equation has a unique solution (for coefficient 1), or whether it is never true (coefficient 0, right hand side  $b = 1$ ), or whether it is true for arbitrary values of the unknown (coefficient 0, right hand side  $b = 0$ ).

Then we conclude  $N(2) \leq 2^2 + 1$ ,  $N(3) \leq 3^2 + 2^2 + 1$  etc. By induction we immediately get

$$N(n) \leq \sum_{i=1}^n i^2.$$

The explicit value of this sum is well-known, and we have shown:

**Theorem 8.1.10.** *The number  $N(n)$  of needed bit comparisions and bit additions for solving a system of  $n$  Boolean linear equations with  $n$  unknowns is upper bounded by*

$$N(n) \leq \frac{1}{6} \cdot n \cdot (n + 1) \cdot (2n + 1).$$

A somewhat more sloppy wording of this result expresses the cost as  $O(n^3)$ . In any case it is “polynomial of small degree” in terms of the problem size  $n$ .

**Remark** The notation by “O” obscures the difference with the cost over arbitrary fields that is generally bounded by  $O(n^3)$ . The “felt” much better performance in the Boolean case is partly founded by the exact estimate in Theorem 8.1.10 that even in the worst case is about  $\frac{1}{3} \cdot n^3$ . Moreover in the Boolean case we count simple bit operations only, and not arithmetic operations or floating point instructions that are significantly more expensive.

### 8.1.11 The Representation of Boolean Functions and Maps

#### Various Interpretations of Bitblocks

We used the term bitblock for a variety of slightly different objects. A bitblock  $b = (b_1, \dots, b_n) \in \mathbb{F}_2^n$  describes:

- a vector  $b \in \mathbb{F}_2^n$  written as a row or a column. This is the primary meaning of the term bitblock<sup>27</sup>.
- an argument of a Boolean function or map of  $n$  variables, also used as row index of a value table (or truth table)
- a bitstring of length  $n$
- a subset  $I \subseteq \{1, \dots, n\}$  defined by  $b$  as indicator:  $i \in I \Leftrightarrow b_i = 1$
- a linear form  $\alpha$  on  $\mathbb{F}_2^n$  expressed as sum of the variables  $x_i$  with  $b_i = 1$ . The evaluation of  $\alpha$  comes down to the scalar product of vectors:  $\alpha(x) = b \cdot x$ .
- a monomial in  $n$  variables  $x_1, \dots, x_n$  with all partial degrees  $\leq 1$ . In this interpretation  $b_i$  specifies the exponent 0 or 1 of the variable  $x_i$ .
- an integer between 0 and  $2^n - 1$  in binary representation (that is in the base-2 system). The sequence of binary “digits” (= bits) is identical with the corresponding bitstring<sup>28</sup>. Conversely the integer is the index (beginning with 0) of the bitstring when the bitstrings are lexicographically ordered in a list.

Of course there are further interpretations—after all each piece of information has a binary coding. The bitblocks for  $n = 3$  are listed in Table 8.6. SageMath sample 8.4.3 has some transformation functions.

#### Representation of the Truth Table of a Boolean Function

The previous subsection described (and Table 8.6 illustrated) how to interpret the bitblocks  $x = (x_1, \dots, x_n)$  of length  $n$  as integers  $i(x) = 0, 1, \dots, 2^n - 1$  in base-2 representation. The example in Table 8.7 suggests how to describe the truth table of a Boolean function  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  in a parsimonious way by a bitblock  $b = (b_0, \dots, b_{2^n-1})$  of length  $2^n$ : simply take the last column in the order given by the indices  $i(x)$ . The general procedure for arbitrary  $n$  runs as follows:

$$b_{i(x)} = f(x) \quad \text{where } i(x) = x_1 \cdot 2^{n-1} + \cdots + x_{n-1} \cdot 2 + x_n \\ \text{for } x = (x_1, \dots, x_n) \in \mathbb{F}_2^n.$$

---

<sup>27</sup>in Python/SageMath implemented as list, or tuple, or vector

<sup>28</sup>The SageMath method `binary()` transforms an integer to a bitstring, suppressing leading zeros. Example: `10.binary()` yields '1010'.

integer	bitstring	subset	linear form	monomial
0	000	$\emptyset$	0	1
1	001	{3}	$x_3$	$x_3$
2	010	{2}	$x_2$	$x_2$
3	011	{2, 3}	$x_2 + x_3$	$x_2 x_3$
4	100	{1}	$x_1$	$x_1$
5	101	{1, 3}	$x_1 + x_3$	$x_1 x_3$
6	110	{1, 2}	$x_1 + x_2$	$x_1 x_2$
7	111	{1, 2, 3}	$x_1 + x_2 + x_3$	$x_1 x_2 x_3$

Table 8.6: Interpretations of bitblocks of length 3

This might look entangled, but it simply means: “Interpret  $x$  as the base-2 representation of an integer  $i(x)$ , and set  $f(x)$  as the bit at position  $i(x)$  from the bitblock  $b$ ”. An additional column  $i(x)$  in the truth table of the function  $f_0$  ( $f_0$  was defined in Formula 8.1) illustrates this procedure – see Table 8.7. The last column of this table, written in row form, is the bitblock  $b$ .

In this way, the bitblock  $(0, 0, 0, 0, 0, 1, 1, 1)$  or, even more parsimoniously, the bitstring

00000111

of length  $2^3 = 8$  completely specifies the truth table of  $f_0$ .<sup>29</sup>

$x_1$	$x_2$	$x_3$	$i(x)$	$f_0(x_1, x_2, x_3)$
0	0	0	0	0
0	0	1	1	0
0	1	0	2	0
0	1	1	3	0
1	0	0	4	0
1	0	1	5	1
1	1	0	6	1
1	1	1	7	1

Table 8.7: An extended truth table [for  $f_0(x_1, x_2, x_3) = x_1 \wedge (x_2 \vee x_3)$ ] with  $n = 3$  and  $2^n = 8$

### Representation of the Algebraic Normal Form

The algebraic normal form (ANF) also has a characterization by  $2^n$  bits: the coefficients of the  $2^n$  different monomials<sup>30</sup> (see Theorem 8.1.3, page 270). The monomials also have an interpretation as bitblocks, see the list above. Therefore we may view a bitblock  $a = (a_0, \dots, a_{2^n-1})$  as representation of the ANF of a Boolean function  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  in the following way:

$$f(x) = \sum_{i=0}^{2^n-1} a_i x_1^{e_1(i)} \cdots x_n^{e_n(i)} \quad \text{where } i = e_1(i) \cdot 2^{n-1} + \cdots + e_n(i) \\ \text{with } e_1(i), \dots, e_n(i) = 0 \text{ or } 1.$$

<sup>29</sup>In Python/SageMath bitblocks are implemented as lists.

<sup>30</sup>Remember that the ANF is a sum of monomials. Each monomial is a product of a subset of  $\{x_1, \dots, x_n\}$ , and hence has a representation as an integer between 0 and  $2^n - 1$ .

This formula means: “Interpret the  $n$ -tuple  $e$  of exponents of a monomial as the base-2 representation of an integer  $i$ . The  $i$ -th element of the bitblock  $a$  indicates whether this monomial occurs in the ANF of  $f$  or not.”

For the sample function  $f_0$  we saw already (or easily check<sup>31</sup>) that the ANF is

$$f_0(x) = x_1x_3 + x_1x_2 + x_1x_2x_3.$$

It involves the monomials with exponent triples  $101, 110, 111$  that correspond to the integers  $5, 6, 7$ . Therefore we set the bits at the positions  $5, 6, 7$  to 1, and the remaining bits to 0, and get the parsimonious representation of the ANF by a bitstring:

00000111.

**Warning** This is the same bitstring as for the truth table *by pure chance*—a special property of the function  $f_0$ ! The function  $f(x_1, x_2) = x_1$  has truth table 0011 (it takes the value 1 if and only if  $x_1 = 1$ , or if the argument has the form  $x = (1, \text{any bit})$ ) and ANF 0010 (since it contains the single monomial  $x_1$ ).

The SageMath class `BoolF()` has a method for calculating the ANF, see the following subsection and Appendix 8.4<sup>32</sup>. SageMath sample 8.3 demonstrates its application to  $f_0$ .

---

### SageMath sample 8.3 A Boolean function with truth table and ANF

---

```
sage: bits = "00000111"
sage: x = str2bbl(bits); x
[0, 0, 0, 0, 0, 1, 1, 1]
sage: f = BoolF(x)
sage: y = f.getTT(); y
[0, 0, 0, 0, 0, 1, 1, 1]
sage: z = f.getANF(); z
[0, 0, 0, 0, 0, 1, 1, 1]
```

---

**Remark** Evaluating a Boolean function  $f$  at all arguments  $x \in \mathbb{F}_2^n$  the naive way costs  $2^n$  evaluations  $f(x)$ , each with at most  $2^n$  summands, each of which needing at most  $n - 1$  multiplications. Thus the costs have an order of magnitude of about  $n \cdot 2^n \cdot 2^n$ . If we relate the costs to the input size  $N = 2^n$  they are essentially quadratic:  $N^2 \cdot \log_2(N)$ . A common method, binary recursion, or “divide-and-conquer”, divides a problem into two subproblems of half the input size, and leads to a significantly more efficient algorithm. Starting from Equation (8.4) in the end effect we achieve a reduction to almost linear costs  $3N \cdot \log_2 N$ . This algorithm<sup>33</sup> is implemented in the class `BoolF()`, see Section 8.4.6.

## Object-Oriented Implementation

For an implementation of Boolean functions in SageMath (or Python) see Appendix 8.4.6 (class `BoolF()`). SageMath itself has a class `sage.crypto.boolean_function` that contains many

<sup>31</sup>Remember:  $f(1, 1, 1) = 1 + 1 + 1 = 1$  since we add mod 2.

<sup>32</sup>This transformation that converts a bitstring of length  $2^n$ —the truth table—into another bitstring of length  $2^n$ —the coefficient list of the ANF—is sometimes called Reed-Muller transformation or binary Moebius transformation.

<sup>33</sup>also denoted as fast binary Moebius transformation, an analogue of the fast Fourier transformation (FFT)

of the needed methods, also the conversion of a truth table into ANF. Here we describe an independent implementation.

In general in an object oriented programming language we may define a class that abstracts the structure of an object “Boolean function”:

**Class BoolF:**

**Attributes:**

- **blist**: truth table as a list of bits (= bitblock in “natural” order as described in Section 8.1.4). This list is also used as internal representation of the Boolean function.
- **dim**: the dimension of the definition domain

**Methods:**

- **setTT**: Fill the truth table with a given bitblock (“TT” for Truth Table).
- **setANF**: Input the ANF (as a list) and internally transform it into the truth table.
- **setDim**: Set the dimension.
- **getTT**: Output the truth table as a bitblock.
- **valueAt**: Get the value of the Boolean function at a given argument.
- **getDim**: Output the dimension.
- **getANF**: Output the algebraic normal form (ANF) as bitblock (in the “natural” order as specified above).
- **deg**: Output the algebraic degree.

The first three, the “**set** methods”, are used only implicitly for the initialization of an object. To have an easily readable output we add methods **printTT** and **printANF**.

The needed functions for transforming bitlists into integers or bitstrings, and vice versa, are in Appendix 8.4.3.

The implementation of Boolean maps derives from this: Define a class **BoolMap** as a list of objects of the class **BoolF** with (at least) the analogous methods. Some of them may also be found in the SageMath module `sage.crypto.mq.sbox`<sup>34</sup>.

---

<sup>34</sup>Cryptographers often use the term “S-boxes” for Boolean maps in small dimensions.

## 8.2 Bitblock Ciphers

In classical cryptography the weakness of simple monoalphabetic substitutions is remedied in two different ways: first by polygraphic substitutions that encrypt groups of letters at once, second by polyalphabetic substitutions that change the substitution alphabet depending on the position in the plaintext.

If we consider bits instead of letters, then monoalphabetic substitutions are cryptographically useless since we have only two choices: either leave all bits unchanged or invert all bits. Thus the plaintext either remains unchanged, or changes in a trivial way. However the two principles of hardening monoalphabetic substitutions yield two classes of useful encryption methods for binary encoded informations:

- Bitblock ciphers split bitstrings into blocks of a fixed length and encrypt one complete block per step.
- Bitstream ciphers encrypt bit by bit, each one by another substitution (so each single bit is unchanged or flipped by a position-dependent rule).

No mathematically complete proof exists for the security of any bitblock or bitstream cipher. Thus the situation is even worse than for asymmetric ciphers where the proof of security often reduces to a well-studied, if not solved, mathematical problem. The best we can do is to consider a symmetric cipher as secure if none of the known attacks is significantly faster<sup>35</sup> than a complete exhaustion of the key space (also known as “brute force attack”).

### 8.2.1 General Description

Bitblock ciphers transform blocks of a fixed length<sup>36</sup>  $n$  to bitblocks of the same length controlled by a key that itself is a bitblock of a certain length  $l$ .

An adequate model of a bitblock cipher is a Boolean map

$$F: \mathbb{F}_2^n \times \mathbb{F}_2^l \longrightarrow \mathbb{F}_2^n$$

often interpreted as a family  $(F_k)_{k \in K}$  of Boolean maps

$$F_k: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n \quad \text{for all } k \in K = \mathbb{F}_2^l$$

where  $F_k(a) = F(a, k)$ .

### Choosing the Key Length

For the key length  $l$  we have an obvious criterion:  $l$  must be large enough to prevent an exhaustion of the key space, a “brute force attack”. The key space is the set  $\mathbb{F}_2^l$ , so it contains  $2^l$  different keys. We assume that the probabilities for all keys are the same, that is  $1/2^l$ . In other words we assume that keys are chosen uniformly at random.

With these assumption we have a lower bound of about 80 bits for a secure key length according to the state of the art [LV00]. Popular ciphers use keys of lengths 128 or more, so have a sufficient security margin. The outdated standard cipher DES used 56 bit keys. The technology of today breaks it quite quickly.

---

<sup>35</sup>In this context a factor of less than 10 would not be deemed as “significantly faster”.

<sup>36</sup>The extension of the cipher to bitstrings of arbitrary lengths is the subject of Section 8.2.4. For the moment we neglect this aspect.

## Choosing the Block Length

The block length  $n$  should be large enough to prevent analyses of patterns or frequencies. Even more it should prevent leaking any information about the plaintext into the ciphertext, for example the presence of repetitions.

If the attacker observes about  $2^{n/2}$  ciphertexts corresponding to random plaintexts encrypted with the same key the probability of a “collision”<sup>37</sup> is about  $\frac{1}{2}$ . Therefore this number  $2^{n/2}$  should exceed the number of available memory cells. And the key should change frequently—long before this number of blocks is reached.

From this point of view the frequently used block length 64 is risky. Only frequent key change could justify it, and only if the plaintext contains few repetitions<sup>38</sup>. A better cipher, as the current standard AES, uses blocks of 128 bits.

These considerations about key or block lengths are typical for the discussion of security in modern cryptography: We use large security margins and avoid any weaknesses, how small they might be, even if there is no known practical attack that uses them. But since we have a broad choice of good and fast ciphers that provide large security margins there is no need to rely on a weaker cipher, even if this precaution seems paranoid.

### 8.2.2 Algebraic Cryptanalysis

#### Attacks with Known Plaintext<sup>39</sup>

Let a bitblock cipher be given by a Boolean map

$$F: \mathbb{F}_2^n \times \mathbb{F}_2^l \longrightarrow \mathbb{F}_2^n.$$

By Theorem 8.1.6  $F$  is an  $n$ -tuple  $F = (F_1, \dots, F_n)$  of polynomial expressions in  $n + l$  variables all of whose partial degrees are  $\leq 1$ .

A known plaintext block  $a \in \mathbb{F}_2^n$  with corresponding ciphertext block  $c \in \mathbb{F}_2^n$  yields a system

$$F(a, x) = c$$

of  $n$  polynomial equations for the unknown key  $x \in \mathbb{F}_2^l$ .

Systems of equations of this type (over arbitrary fields) are subjects of algebraic geometry. The general theory is quite deep, in particular if we search for concrete solution procedures. However—couldn’t the fact that our polynomials have all their partial degrees  $\leq 1$  simplify the problem?

**Example 1** Let  $n = l = 2$ ,

$$F(a_1, a_2, x_1, x_2) = (a_1 + a_2x_1, a_2 + a_1x_2 + x_1x_2),$$

$a = (0, 1)$ ,  $c = (1, 1) \in \mathbb{F}_2^2$ . The equations for the key  $(x_1, x_2) \in \mathbb{F}_2^2$  are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 + x_1 \\ 1 + 0 + x_1x_2 \end{pmatrix}.$$

The immediate solution is  $x_1 = 1$ ,  $x_2 = 0$ .

---

<sup>37</sup>by the “birthday paradox”

<sup>38</sup>Using a good “mode of operation” avoids the danger of repetitions, see Section 8.2.4.

<sup>39</sup>An attack with known plaintext assumes that the attacker knows or guesses a small piece of plaintext, and then tries to deduce the key or some more plaintext that is unknown to her. For the present section we assume that the known plaintext is a complete bitblock.

**Example 2,** linear maps: If  $F$  is a *linear* map, then the system of equations is accessible by the efficient solution algorithms of linear algebra, see 8.1.10. We have  $n$  linear equations for  $l$  unknowns. If  $l < n$  the attacker needs some additional blocks of known plaintext, or she executes an exhaustion of the remaining  $n - l$  key bits. For this method to work  $F$  needs to be linear only in  $x$ .

**Example 3,** substitution: Often polynomial equations look complex at first sight but aren't so. Here is an example (over  $\mathbb{F}_2$ )

$$x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 + x_2 + x_3 = 0.$$

By the substitutions  $x_i = z_i + 1$  it is transformed to

$$z_1z_2z_3 + z_1 = 0$$

(for an easy proof look in the reverse direction). This has the solutions

$$z_1 = 0, z_2, z_3 \text{ arbitrary or } z_1 = z_2 = z_3 = 1.$$

Therefore the complete solution of the original equation is

$$x_1 = 1, x_2, x_3 \text{ arbitrary or } x_1 = x_2 = x_3 = 0.$$

There are two powerful general approaches for solving systems of (polynomial) equations over  $\mathbb{F}_2$ :

- SAT solvers [GJ79]<sup>40</sup>,
- elimination using Groebner bases [Bri10]<sup>41</sup>.

Both methods work well for a small number of unknowns. With a growing number of unknowns their complexity becomes unmanageable<sup>42</sup>. Of course we always find a solution by searching through the complete value table. But this naive method is inefficient (exponential in the number of unknowns, and so hopeless for 80 or more unknowns). But also the costs of SAT solvers and Groebner-basis methods grow exponentially with the number of unknowns. Not even the fact that all partial degrees are  $\leq 1$  is of vital help.

## The Complexity of the Algebraic Attack

The theoretical analysis of the cost for finding a solution leads to one of the central notions of complexity theory, **NP**-completeness.

<sup>40</sup>SAT denotes the satisfiability problem of propositional logic. Consider a logical expression in Boolean variables  $x_1, \dots, x_n$  and ask if there exist values of the variables that make the expression “True”. In other words consider a Boolean function  $f$  and ask if it assumes the value 1. A **SAT solver** is an algorithm that takes a logical expression in CNF and decides the satisfiability by finding a solution  $x$ , or showing there's no solution. The naive algorithm uses the truth table and exhausts the  $2^n$  possible arguments. However there are much faster algorithms, the most popular being the DPLL algorithm (after Davis, Putnam, Logemann, and Loveland) and BDD based algorithms (Binary Decision Diagram). The SageMath modules `sage.sat.solvers` and `sage.sat.boolean_polynomials` contain some of these algorithms.

<sup>41</sup>For an introduction to this theory see the textbooks [Bar09, CLO07, vZGG99], or the script [Seg04], or the paper [Laz83].

<sup>42</sup>In fact SAT was the first problem in history shown to be **NP**-complete.

**Theorem 8.2.1** (Garey/Johnson). *The problem of finding a solution for a system of polynomial equations over  $\mathbb{F}_2$  is **NP** complete.*

For a proof see the book by Garey/Johnson [GJ79].<sup>43</sup>

We won't explain the notion “**NP**-complete” but only mention that the (up to now unproven) “**P** ≠ **NP** conjecture” implies that an **NP**-complete problem admits no efficient algorithmic solution, or that there is no solution algorithm whose execution time grows at most polynomially with the number of input variables.

A common interpretation of this theorem is: For an appropriately chosen block cipher  $F: \mathbb{F}_2^n \times \mathbb{F}_2^l \rightarrow \mathbb{F}_2^n$  the attack with known plaintext (against the key  $k \in \mathbb{F}_2^l$ ) is not efficient. However from a strict mathematical point of view the theorem *doesn't prove anything* of practical relevance:

1. It relates to an algorithm for *arbitrary* polynomial equations (over  $\mathbb{F}_2$ ). It doesn't contain any assertion for special classes of polynomials, or for a concrete system of equations.
2. It gives a pure proof of (non-) existence, and provides no hint as how to construct a concrete example of a “difficult” system of equations. Note that we know that some concrete systems admit easy solutions.
3. Even if we could find concrete examples concrete examples of “difficult” systems the theorem would not make any assertion whether only some rare instances (the “worst cases”) are difficult, or almost all (the “generic cases”)—and this is what the cryptologist wants to know. Maybe there is an algorithm that solves polynomial systems for almost all tuples of unknowns in an efficient way, and only fails for a few exceptional tuples.

Despite these critical comments the theorem raises hope that there are “secure” bitblock ciphers, and the designers of bitblock ciphers follow the

**Rule of thumb** *Systems of linear equations for bits admit very efficient solutions. Systems of nonlinear equations for bits in almost all cases admit no efficient solution.*

### 8.2.3 The Structure of Bitblock Ciphers

In an ideal world we would know how to reliably measure the security of a bitblock cipher

$$F: \mathbb{F}_2^n \times \mathbb{F}_2^l \rightarrow \mathbb{F}_2^n$$

for realistic values of the block length  $n$  and the key length  $l$ , say of an order of magnitude of 128 bits or more.

In fact we know explicit measures of security, for example the linear potential, or the differential potential, that quantify the deviation from linearity, or the algebraic immunity, or others. Unfortunately all of these only give necessary, not sufficient, conditions for security, and moreover the efficient computability of these measures is limited to small block lengths  $n$ , about 8 or slightly larger.

Lacking a general efficient approach to security the design of bitblock ciphers usually relies on a structure that, although not obligatory, in practice seems to provide plausible security

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<sup>43</sup>A recent article on the difficulty of systems of polynom equations is [CGH<sup>+</sup>03].

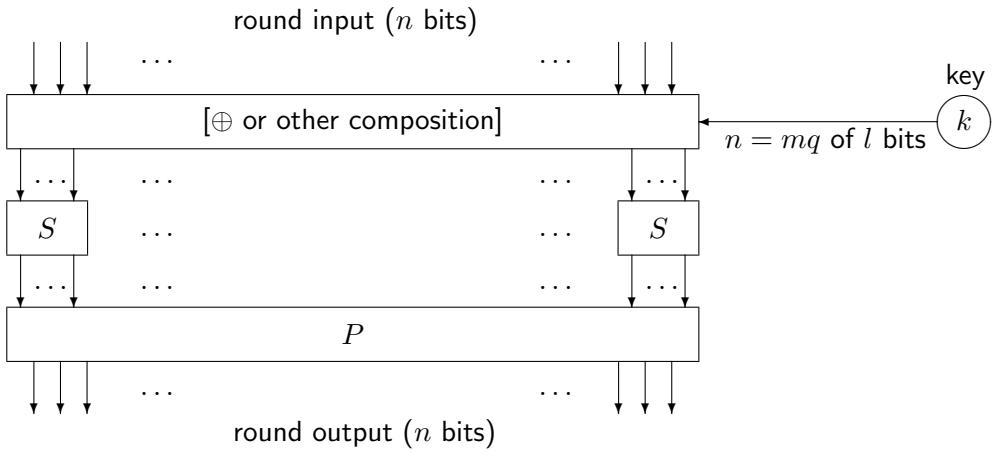


Figure 8.2: A single round of a bitblock cipher ( $S$  is a, maybe varying, S-box,  $P$ , a permutation,  $k$ , the key)

according to verifiable criteria. Most of the generally approved standard ciphers, such as DES and AES, follow this approach.

This common design scheme starts by constructing Boolean maps of small dimensions and then extending them to the desired block length in several steps:

1. Define one or more Boolean maps of small dimension  $q$  (= block length of the definition domain), say  $q = 4, 6$ , or  $8$ , that are good for several security criteria. These maps are called **S-boxes**<sup>44</sup>, and are the elementary building blocks of the cipher.
2. Mix the round input with some of the key bits and then apply  $m$  S-boxes in parallel (or apply the one S-box  $m$  times in parallel) to get a map with the desired input width  $n = mq$ .
3. Then permute the complete resulting bitblock over its total width.
4. These steps together are a “**round**” of the complete scheme. Assess the weaknesses of the round map, that mainly result from using S-boxes of small dimension. Then reduce these weaknesses in a reasonably controlled way by iterating the scheme over several rounds of the same structure but with a changing choice of key bits.
5. Don’t stop as soon as the security measures give satisfying values but add some surplus rounds to get a wide security margin.

Figure 8.2 outlines the scheme for a single round.

The complete scheme is a special case of a somewhat more general proposal that goes back to Shannon who required two basic features of block ciphers:

**Diffusion** The bits of the plaintext block “smear” over all parts of the block. This is done by applying permutations (a. k. a. as transpositions).

**Confusion** (complex dependencies) The interrelation between plaintext block and key on the one hand, as well as ciphertext block on the other hand should be as complex as possible (in particular as nonlinear as possible). Basic building blocks for this are substitutions.

---

<sup>44</sup>“S” stands for Substitution.

The overall effect of both requirements, taken together, should result in an unforeseeable change of ciphertext bits for a slight change of the key.

*The attacker should have no means to recognize whether a guessed key is “nearly correct”.*

For the construction of strong block ciphers Shannon proposed an alternating sequence of Substitutions and transpositions (= Permutations), so-called **SP-networks**:

$$\begin{aligned} \mathbb{F}_2^n &\xrightarrow{S_1(\bullet,k)} \mathbb{F}_2^n \xrightarrow{P_1(\bullet,k)} \mathbb{F}_2^n \longrightarrow \dots \\ &\dots \longrightarrow \mathbb{F}_2^n \xrightarrow{S_r(\bullet,k)} \mathbb{F}_2^n \xrightarrow{P_r(\bullet,k)} \mathbb{F}_2^n \end{aligned}$$

depending on a key  $k \in \mathbb{F}_2^l$ . In this scheme

$S_i = i\text{-th substitution}$

$P_i = i\text{-th permutation}$

$P_i \circ S_i = i\text{-th round}$

Alltogether the encryption function consists of  $r$  rounds.

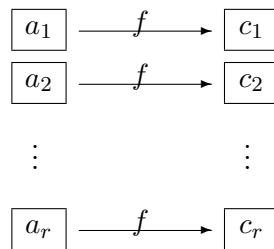
Note that the permutations are special linear maps  $P: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n$ . Some recent bitblock ciphers, the most prominent being AES, replace permutations by more general linear maps that provide an even better diffusion. However the proper term “**LP-network**” is not yet in use.

#### 8.2.4 Modes of Operation

Consider a block cipher<sup>45</sup>  $f: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n$ . If we want to apply it to longer or shorter bit sequences we must

1. split a bit sequence  $a$  into  $n$ -bit blocks  $a_1, \dots, a_r$ ,
2. fill (“pad”) the last block  $a_r$ , if necessary, up to length  $n$  with
  - zeroes
  - or random values
  - or context information.

Then the most obvious encryption algorithm is: Encipher the blocks one by one. This is called ECB mode (for “Electronic Code Book”). Schematically:



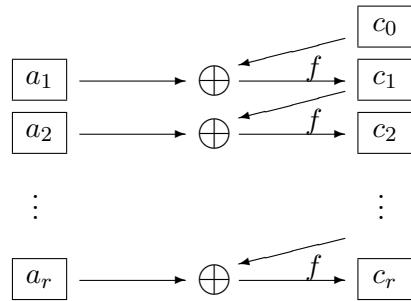

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<sup>45</sup>In this subsection the key plays no role. Therefore we omit “ $k$ ” in the notation.

ECB mode simply realizes a monoalphabetic substitution where the blocks in  $\mathbb{F}_2^n$  are interpreted as “letters”. For a sufficiently large  $n$  this is secure from a ciphertext-only attack. However the cipher leaks information on repeated blocks. For some plaintexts this is a real danger:

- For example MS-Word files contain long sequences consisting of the bytes 00000000 and 00000001.
- An even more alarming case is provided by image files with large single-color areas. They contain many identical blocks such that structures of the image may appear in the ciphertext file<sup>46</sup>.

In view of this weakness generating some additional diffusion between the plaintext blocks seems a good idea. A simple but effective approach is CBC (= Cipher Block Chaining). Choose a random start value  $c_0$  (also called IV = “Initialization Vector”). Then the procedure looks like this:



The formula for encryption in CBC mode is

$$\begin{aligned} c_i &:= f(a_i + c_{i-1}) \quad \text{for } i = 1, \dots, r \\ &= f(a_i + f(a_{i-1} + \dots + f(a_1 + c_0) \dots)). \end{aligned}$$

Each ciphertext block depends on *all previous* plaintext blocks (diffusion), and identical plaintext blocks in general encrypt to different ciphertext blocks.

The formula for decryption is

$$a_i = f^{-1}(c_i) + c_{i-1} \quad \text{for } i = 1, \dots, r.$$

**Question** Does it make sense to keep the initialization vector  $c_0$  secret and use it as an additional key component? (Then for the example of DES we had 56 proper key bits plus a 64 bit initialization vector, making a total of 120 key bits.)

**Answer** No!

**Reason** In the decryption process only  $a_1$  depends on  $c_0$ . This means that keeping  $c_0$  secret conceals known plaintext only for the first block. If the attacker knows the second or any later plaintext block, then she may attack the key as in ECB mode (by an attack with known plaintext).

There are several other modes of operation, see the Wikipedia entry “Block cipher mode of operation”. Worth mentioning is that the modes OFB (= Output Feedback) and CTR (= Counter) convert a bitblock cipher into a bitstream cipher.

---

<sup>46</sup>For a convincing example look at the Wikipedia entry “Block cipher mode of operation”.

### 8.2.5 Statistical Analyses

For cryptanalyzing bitblock ciphers we know some basic approaches:

1. exhaustion = brute-force searching the complete key space
2. algebraic attack, see Section 8.2.2
3. statistical attacks against hidden linearity:
  - (a) linear cryptanalysis (Matsui/Yamagishi 1992), the subject of Sections 8.2.6 ff.
  - (b) differential cryptanalysis (Murphy, Shamir, Biham 1990<sup>47</sup>)
  - (c) generalizations and mixtures of (a) and (b)

All these statistical attacks hardly break a cipher in the sense of classical cryptanalysis. They usually assume lots of known plaintexts, much more than an attacker could gather in a realistic scenario. Therefore a more adequate term is “analysis” instead of “attack”. The analyses make sense for finding measures for some partial aspects of security of bitblock ciphers. They measure security for example by the number of known plaintext blocks needed for the attack. If a cipher resists an attacker even with exaggerated assumptions on her capabilities, then we feel safe to trust it in real life.

Given an SP-network the analysis starts with the nonlinear components of the single rounds, in particular with the S-boxes. The next step is extending the potential attack over several rounds. This shows how the cost of the attack grows with the number of rounds. In this way we find criteria for the number of rounds for which the cipher is “secure”—at least from this special attack.

## Security Criteria for Bitblock Ciphers

To escape attacks bitblock ciphers, or their round maps, or their S-boxes, should fulfill some requirements.

- **Balance:** All preimages have the same number of elements, or in other words, the values of the map are uniformly distributed. Irregularities of the distribution would provide hooks for statistical cryptanalysis.
- **Diffusion/avalanche effect:** If a single plaintext bit changes, about 50% of the ciphertext bits change. This effect conceals similarity of plaintexts.
- **Algebraic complexity:** The determination of preimages or parts thereof should lead to equations whose solution is as difficult as possible. This requirement is related to the algebraic degree of the map, but only in an indirect way. A suitable measure is “algebraic immunity”.
- **Nonlinearity:** We know several criteria that measure linearity, also “hidden” linearity, and are relatively easy to describe and to handle. For example they quantify how susceptible Boolean maps are for linear or differential cryptanalysis [Pom14].

---

<sup>47</sup>known at IBM and NSA as early as in 1974. In contrast with differential cryptanalysis apparently linear cryptanalysis—though conceptually simpler—was unknown to the designers of DES. Accordingly the resistance of DES against linear cryptanalysis is suboptimal.

- The linear potential should be as low as possible, the linear profile as balanced as possible.
- The differential potential should be as low as possible, the differential profile as balanced as possible.

Some of these criteria are compatible with each other, some criteria contradict other ones. Therefore the design of a bitblock cipher requires a balance between different criteria. Instead of optimizing a map for a single criterion the designer should aim at a uniformly high level for all criteria.

### 8.2.6 The Idea of Linear Cryptanalysis

A comprehensive treatment of the statistical attacks would require some voluminous extra chapters. The least difficult one is linear cryptanalysis. Here we introduce its basic principles.

Consider a bitblock cipher  $F$  of block length  $n$  and key length  $l$ ,

$$F: \mathbb{F}_2^n \times \mathbb{F}_2^l \longrightarrow \mathbb{F}_2^n.$$

Imagine the arguments of  $F$  as plain texts  $a \in \mathbb{F}_2^n$  and keys  $k \in \mathbb{F}_2^l$ , the values of  $F$  as cipher texts  $c \in \mathbb{F}_2^n$ . A **linear relation** between a plaintext  $a \in \mathbb{F}_2^n$ , a key  $k \in \mathbb{F}_2^l$ , and a ciphertext  $c = F(a, k) \in \mathbb{F}_2^n$  is described by three linear forms

$$\alpha: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2, \quad \beta: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2, \quad \text{and} \quad \kappa: \mathbb{F}_2^l \longrightarrow \mathbb{F}_2$$

as an equation

$$\kappa(k) = \alpha(a) + \beta(c) \tag{8.5}$$

In the simplest case  $\alpha$ ,  $\beta$ , and  $\kappa$  each would pick a single bit from a bitblock, and equation (8.5) would express this one bit of the key as sum of one bit of plaintext with one bit of ciphertext. In the general case the linear forms are sums of several bits. If  $I = (i_1, \dots, i_r)$  is the index set that corresponds to the linear form  $\kappa$ —that is  $\kappa(k) = k_{i_1} + \dots + k_{i_r}$ —, then writing (8.5) more explicitly we get an equation for the sum of the involved key bits  $k_{i_1}, \dots, k_{i_r}$ :

$$k_{i_1} + \dots + k_{i_r} = \alpha(a) + \beta(c),$$

For an attack with known plaintext  $a$  this reduces the number of unknown key bits to  $l - 1$  by elimination of one of these bits.

In general the odds of the relation (8.5) for concrete random values of  $k$ ,  $a$ , and  $c$  are about fifty-fifty: both sides evaluate to 0 or 1 with probability  $\frac{1}{2}$ . Best for security is a frequency of 50% plaintexts  $a$  that make the relation true for a fixed key  $k$ , where  $c = F(a, k)$  is the corresponding ciphertext. This would make the relation indistinguishable from a pure accidental one. If the probability of the relation,

$$p_{F,\alpha,\beta,\kappa}(k) := \frac{1}{2^n} \cdot \#\{a \in \mathbb{F}_2^n \mid \kappa(k) = \alpha(a) + \beta(F(a, k))\},$$

is conspicuously larger than  $\frac{1}{2}$ , this reveals a biased probability for the values of the bits of  $k$ , and would result in a small advantage for the cryptanalyst. If on the other hand the probability is noticeably smaller than  $\frac{1}{2}$ , then the complementary relation  $\kappa(k) = \alpha(a) + \beta(c) + 1$  is true more often than by pure chance. This also is a weakness. Because the situation concerning the

deviation of the probabilities from the ideal value  $\frac{1}{2}$  is symmetric<sup>48</sup>, it makes sense to consider symmetric quantities<sup>49</sup>, the **input-output correlation**<sup>50</sup>:

$$\tau_{F,\alpha,\beta,\kappa}(k) := 2p_{F,\alpha,\beta,\kappa}(k) - 1$$

(in short: I/O-correlation) and the **potential** of a linear relation<sup>51</sup>:

$$\lambda_{F,\alpha,\beta,\kappa}(k) := \tau_{F,\alpha,\beta,\kappa}(k)^2.$$

The I/O-correlation takes values between  $-1$  and  $1$ . The potential takes values between  $0$  and  $1$ , and measures the deviation of the probability from  $\frac{1}{2}$ . In the best case it is  $0$ , in the worst,  $1$ . This “bad” extreme case would provide an exact and directly useable relation for the key bits. Figure 8.3 illustrates the connection. It is generated by SageMath sample 8.4.

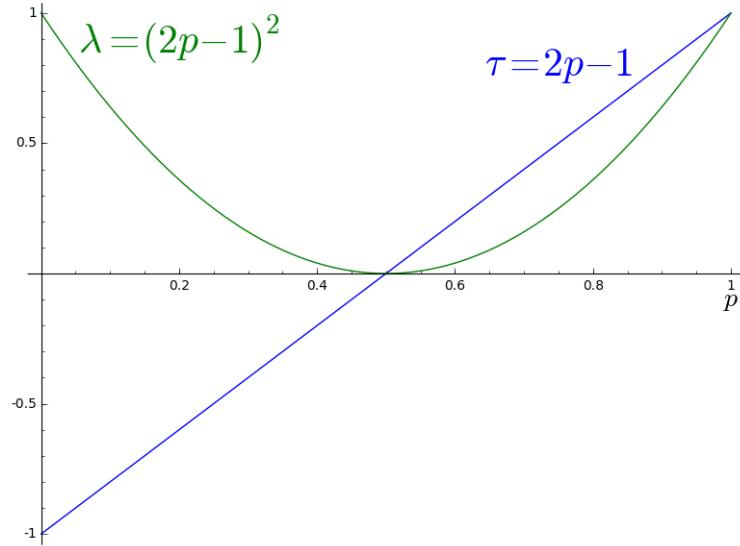


Figure 8.3: The relation between the probability  $p$ , the I/O-correlation  $\tau$ , and the potential  $\lambda$

Note that the key  $k$  is the target of the attack. As long as it is unknown, the value of  $p_{F,\alpha,\beta,\kappa}(k)$  is also unknown. Thus for cryptanalysis it makes sense to average the probabilities of a linear relation over all keys:

$$p_{F,\alpha,\beta,\kappa} := \frac{1}{2^{n+l}} \# \{(a, k) \in \mathbb{F}_2^n \times \mathbb{F}_2^l \mid \kappa(k) = \alpha(a) + \beta(F(a, k))\}. \quad (8.6)$$

This average probability is determined, at least theoretically, neglecting efficiency, by the definition of the cipher  $F$  alone. Calculating it however amounts to an exhaustion of all plaintexts and keys,

<sup>48</sup>and because the I/O-correlation and the potential are multiplicative, see Theorem 8.2.6

<sup>49</sup>Often in the literature these are used without giving them explicit names, as for example in Matsui’s original papers.

<sup>50</sup>This is the correlation of two Boolean functions on  $\mathbb{F}_2^n$ , namely  $\alpha + \kappa(k)$  and  $\beta \circ F_k$ . ( $\kappa(k)$  is a constant, i. e.  $0$  or  $1$ , for fixed  $k$ ). The first of these functions picks input bits, the second one output bits. In general the correlation of Boolean functions  $f, g: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  is the difference

$$c(f, g) := \frac{1}{2^n} \cdot [\#\{x \in \mathbb{F}_2^n \mid f(x) = g(x)\} - \#\{x \in \mathbb{F}_2^n \mid f(x) \neq g(x)\}]$$

<sup>51</sup>The historically first known record for the denomination “potential” is the contribution by Kaisa Nyberg at EUROCRYPT 1994. Mathematically less elegant, but frequently used, is the “bias”  $|p - \frac{1}{2}| = \sqrt{\lambda}/2$ .

---

**SageMath sample 8.4** Plot of I/O-correlation and potential

---

```
sage: plot1 = plot(2*x-1, (x,0,1))
sage: plot2 = plot((2*x - 1)**2, (x,0,1), color = 'green')
sage: xlabel = text('$p$', (1.0, -0.1), fontsize = 20, color = 'black')
sage: legend1 = text('$\tau = 2p - 1$', (0.75,0.8), fontsize = 30)
sage: legend2 = text('$\lambda = (2p - 1)^2$', (0.2,0.9), fontsize = 30,\\
    color = 'green')
sage: show(plot1 + plot2 + xlabel + legend1 + legend2)
```

---

and thus is unrealistic for a realistic cipher with large block lengths. We extend the definition for the “average case” also to I/O-correlation and potential<sup>52</sup>:

$$\begin{aligned}\tau_{F,\alpha,\beta,\kappa} &:= 2p_{F,\alpha,\beta,\kappa} - 1, \\ \lambda_{F,\alpha,\beta,\kappa} &:= \tau_{F,\alpha,\beta,\kappa}^2.\end{aligned}$$

Shamir<sup>53</sup> already in 1985 noticed that the S-boxes of DES admit linear relations with conspicuous probabilities. However it took another seven years until Matsui<sup>54</sup> (after first attempts by Gilbert and Chassé 1990 with the cipher FEAL) succeeded in making systematic use of this observation. For estimating<sup>55</sup>  $\kappa(k)$  he proceeded as follows (in the case  $p_{F,\alpha,\beta,\kappa} > \frac{1}{2}$ , else complementary<sup>56</sup>):

1. **Collect**  $N$  pairs of plaintexts and corresponding ciphertexts  $(a_1, c_1), \dots, (a_N, c_N)$ .

2. **Count** the number

$$t := \#\{i = 1, \dots, N \mid \alpha(a_i) + \beta(c_i) = 0\}.$$

3. **Decide** by majority depending on  $t$ :

- If  $t > \frac{N}{2}$ , estimate  $\kappa(k) = 0$ .
- If  $t < \frac{N}{2}$ , estimate  $\kappa(k) = 1$ .

The case  $t = \frac{N}{2}$  is worthless, however scarce—we might randomize the decision between 0 and 1, or output a suitable error code<sup>57</sup>. SageMath sample 8.5 contains the program code. A concrete application follows as example in the next subsection.

If we detect a linear relation whose probability differs from  $\frac{1}{2}$  in a sufficient way, then this procedure will have a good success probability for sufficiently large  $N$ . This allows to reduce the number of unknown key bits by 1, applying elimination.

As a theoretical result from these considerations we’ll get a connection between the number  $N$  of needed plaintext blocks and the success probability, see Table 8.11.

The more linear relations with sufficiently high certainty the attacker finds, the more she can reduce the size of the remaining key space until finally an exhaustion becomes feasible. A concrete example in Section 8.2.12 will illustrate this.

---

<sup>52</sup>Note that the I/O-correlation also is an average, but the potential is not!

<sup>53</sup>Adi Shamir, Israeli cryptologist, co-inventor of the RSA cipher, \*July 6, 1952

<sup>54</sup>Mitsuru Matsui, Japanese cryptologist, \*September 16, 1961

<sup>55</sup>This is a maximum likelihood estimation. One decides between several hypotheses (two in our case), and prefers the one hypothesis that attributes the highest probability to the observation.

<sup>56</sup>In the case  $p_{F,\alpha,\beta,\kappa} = \frac{1}{2}$  the method is useless.

<sup>57</sup>or both as in SageMath sample 8.5

---

**SageMath sample 8.5** Matsui’s test. The linear forms are  $\alpha$  for  $\alpha$ , and  $\beta$  for  $\beta$ . The list `pc` consists of  $N$  pairs of plaintexts and corresponding ciphertexts. The Boolean value `compl` indicates if the resulting bit must be inverted. The output is a triple consisting of the count  $t$  of zeros, the guessed bit, and the Boolean value that indicates whether the bit is deterministic (`True`) or (in the limit case) randomized (`False`). We use the function `binScPr` (“binary scalar product”) from SageMath sample 8.39 in Appendix 8.4.3.

---

```
def Matsui_Test(a, b, pc, compl = False):
    """Matsui's test for linear cryptanalysis"""
    N = len(pc)
    results = []
    for pair in pc:
        ax = binScPr(a,pair[0])
        by = binScPr(b,pair[1])
        result = (ax + by) % 2
        results.append(result)
    t = 0
    for bb in results:
        if bb == 0:
            t = t + 1
    if 2*t > N:
        if compl:
            return [t,1,True]
        else:
            return [t,0,True]
    elif 2*t_0 < N:
        if compl:
            return [t,0,True]
        else:
            return [t,1,True]
    else:
        return [t,randint(0,1),False]
```

---

## Example

For a concrete example with  $n = l = 4$  we consider the Boolean map<sup>58</sup>  $f$  that is given by the values in Table 8.8, and define the bitblock cipher

$$F: \mathbb{F}_2^4 \times \mathbb{F}_2^4 \longrightarrow \mathbb{F}_2^4 \quad \text{by } F(a, k) := f(a + k).$$

SageMath sample 8.6 defines this Boolean map  $f = \text{S0}$ , using the classes `BoolF` and `BoolMap` from Appendix 8.4.6. The columns of the defining matrix (implicit in the SageMath code) just give the values of the map as they are also found in the column  $y = f(x)$  of Table 8.8. (In other words, SageMath sample 8.6 and Table 8.8 give equivalent definitions of the map  $f$ .) A sample evaluation illustrates this (for the third column, representing the argument 0010).

We encrypt using the key  $k = 1000$  (that we’ll attack later as a test case). For a linear relation we consider the linear forms

$$\alpha(a) = a_4, \quad \beta(c) = c_1 + c_2 + c_4, \quad \kappa(k) = k_4.$$

---

<sup>58</sup>By the way  $f$  is the S-box  $S_0$  of LUCIFER, a precursor of DES developed around 1970.

---

**SageMath sample 8.6** A Boolean map (the S-box  $S_0$  of LUCIFER)

---

```
f1 = BoolF([1,1,0,1,1,1,0,0,0,0,0,1,0,0,1])
f2 = BoolF([1,1,1,0,1,1,0,0,0,1,0,0,0,1,1,0])
f3 = BoolF([0,1,1,1,1,0,1,0,1,1,1,0,0,0,0,0])
f4 = BoolF([0,1,1,0,0,1,1,0,0,0,1,1,1,0,1,0])
S0 = BoolMap([f1,f2,f3,f4])
# Sample evaluation
sage: S0.valueAt([0,0,1,0])
[0, 1, 1, 1]
```

---

$x$	$y = f(x)$	$\alpha(x) = x_4$	$\beta(y) = y_1 + y_2 + y_4$
0 0 0 0	1 1 0 0	0	0
0 0 0 1	1 1 1 1	1	1
0 0 1 0	0 1 1 1	0	0
0 0 1 1	1 0 1 0	1	1
0 1 0 0	1 1 1 0	0	0
0 1 0 1	1 1 0 1	1	1
0 1 1 0	1 0 1 1	0	0
0 1 1 1	0 0 0 0	1	0
1 0 0 0	0 0 1 0	0	0
1 0 0 1	0 1 1 0	1	1
1 0 1 0	0 0 1 1	0	1
1 0 1 1	0 0 0 1	1	1
1 1 0 0	1 0 0 1	0	0
1 1 0 1	0 1 0 0	1	1
1 1 1 0	0 1 0 1	0	0
1 1 1 1	1 0 0 0	1	1

Table 8.8: Value table of a Boolean map  $f: \mathbb{F}_2^4 \rightarrow \mathbb{F}_2^4$ , and two linear forms

In Section 8.2.7 we'll see that with these linear forms the relation  $\kappa(k) = \alpha(a) + \beta(c)$  for  $F$  has a quite large probability. Table 8.9 shows the ciphertexts belonging to three plaintexts  $a$  (that later we'll assume as known plaintexts). The values of  $c$  are taken from Table 8.8. The number  $t$  of observed values 0 of  $\alpha(a) + \beta(c)$  is  $t = 2$ . Hence the majority decision gives the estimate  $k_4 = 0$  (being in cheat mode we know it's correct).

This was easily done with pencil and paper. However it might be instructive to retrace the count in SageMath for a better understanding of more complex examples. SageMath sample 8.7 provides this. It uses the function `xor` from SageMath sample 8.39 in Appendix 8.4.3, as well `S0` from SageMath sample 8.6. The result `[2, 0, True]` says that we found 2 zeroes among the counted values, yielding the majority decision 0, and the output parameter `True` tells that the decision was deterministic, not randomized.

How successful will this procedure be in general? We have to analyse the problems:

1. How to find linear relations of sufficiently high probabilities?
2. Since in general bitblock ciphers consist of several rounds we ask:

$a$	$a + k$	$c$	$\alpha(a)$	$\beta(c)$	$\alpha(a) + \beta(c)$
0010	1010	0011	0	1	1
0101	1101	0100	1	1	0
1010	0010	0111	0	0	0

Table 8.9: Estimating a key bit after Matsui using three known plaintexts

---

**SageMath sample 8.7** An example of Matsui's test

---

```
sage: k = [1,0,0,0]
sage: alpha = [0,0,0,1]
sage: beta = [1,1,0,1]
sage: plist = [[0,0,1,0],[0,1,0,1],[1,0,1,0]]
sage: xlist = []
sage: xclist = []
sage: pclist = []
sage: for i in range(0,len(plist)):
....:     x = xor(plist[i],k)
....:     xlist.append(x)
....:
sage: xlist
[[1, 0, 1, 0], [1, 1, 0, 1], [0, 0, 1, 0]]
sage: for i in range(0,len(plist)):
....:     val = S0.valueAt(xlist[i])
....:     xclist.append([xlist[i],val])
....:     pclist.append([plist[i],val])
....:
sage: Matsui_Test(alpha,beta,pclist,False)
[2, 0, True]
```

---

- (a) How to find useful linear relations for the round function of an iterated bitblock cipher?
- (b) How to combine these over the rounds as a linear relation for the complete cipher?
- (c) How to calculate the probability of a combined linear relation for the complete cipher from the probabilities for the single rounds?

The answer to the first question and part (a) of the second one is: from the linear profile, see Section 8.2.8. The following partial questions lead to the analysis of linear paths, see Section 8.2.10, and the cumulation of probabilities, see Theorem 8.2.7. For (c) finally we'll find a useful rule of thumb.

### 8.2.7 Example A: A one-round cipher

We consider examples that are much too simple for real world applications but illustrate the principles of linear cryptanalysis in an easily intelligible way. We always assume round functions

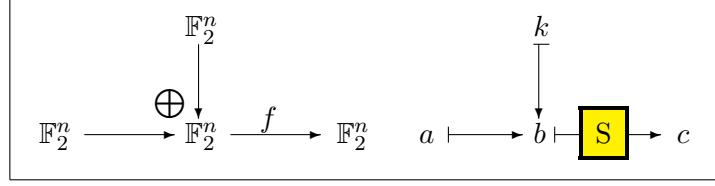


Figure 8.4: A (much too) simple example

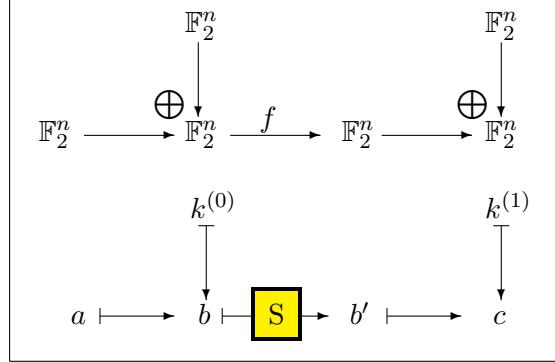


Figure 8.5: Example A: A One-Round Cipher

of the type  $f(a + k)$ , that is we add the key—or an  $n$ -bit part of it—to the plaintext<sup>59</sup> before applying a bijective S-box  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ . The simplest model is encryption by the formula

$$c = f(a + k),$$

see<sup>60</sup> Figure 8.4. This example is pointless because one block of known plaintext gives a solution<sup>61</sup> for  $k$ :

$$k = f^{-1}(c) + a.$$

The somewhat more involved example A stops this attack:

$$c = f(a + k^{(0)}) + k^{(1)}$$

<sup>59</sup>This is a quite special method of bringing the key into play but nevertheless realistic. The paradigmatic sample ciphers LUCIFER, DES, and AES do so. The term used with AES [DR02] is “key-alternating cipher structure”.

<sup>60</sup>The graphics here and later represent the map  $f$  sometimes by the S-box  $S$  in the elementwise assignments.

<sup>61</sup>We assume that the attacker knows the inverse map  $f^{-1}$  that is part of the decryption algorithm. One-way encryption methods that assume that  $f^{-1}$  is not efficiently deducible from  $f$  are the subject of another part of cryptography.

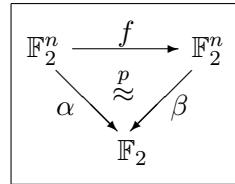


Figure 8.6: Diagram for an “approximative” linear relation

(see Figure 8.5). This is the simplest example for which the method of linear cryptanalysis makes sense: Let  $(\alpha, \beta)$  be a pair of linear forms with

$$\beta \circ f(x) \xrightarrow{p} \alpha(x), \quad (8.7)$$

where the symbol  $\xrightarrow{p}$  reads as “equal with probability  $p$ ”, or in other words

$$p = p_{f,\alpha,\beta} := \frac{1}{2^n} \cdot \#\{x \in \mathbb{F}_2^n \mid \beta \circ f(x) = \alpha(x)\}.$$

The diagram in Figure 8.6 illustrates Formula (8.7). Note that the linear form  $\kappa$  of the general theory is implicit in the present context: Since the key bits are simply added to plaintext and (“intermediary”) ciphertext we have  $\kappa = \alpha$  for  $k^{(0)}$ , and  $\kappa = \beta$  for  $k^{(1)}$ , hence  $\kappa(k^{(0)}, k^{(1)}) = \alpha(k^{(0)}) + \beta(k^{(1)})$ .

How does this scenario fit the general situation from Section 8.2.6? In example A we have

- key length  $l = 2n$ , key space  $\mathbb{F}_2^{2n}$ , and keys of the form  $k = (k^{(0)}, k^{(1)})$  with  $k^{(0)}, k^{(1)} \in \mathbb{F}_2^n$ .
- The cipher is defined by the map

$$F: \mathbb{F}_2^n \times \mathbb{F}_2^n \times \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n, \quad (a, k^{(0)}, k^{(1)}) \mapsto f(a + k^{(0)}) + k^{(1)}.$$

- The linear form  $\kappa: \mathbb{F}_2^n \times \mathbb{F}_2^n \longrightarrow \mathbb{F}_2$  is  $\kappa(k^{(0)}, k^{(1)}) = \alpha(k^{(0)}) + \beta(k^{(1)})$ .

Hence the probability of a linear relation for a fixed key  $k = (k^{(0)}, k^{(1)})$  is

$$\begin{aligned} p_{F,\alpha,\beta,\kappa}(k) &= \frac{1}{2^n} \cdot \#\{a \in \mathbb{F}_2^n \mid \kappa(k) = \alpha(a) + \beta(F(a, k))\} \\ &= \frac{1}{2^n} \cdot \#\{a \in \mathbb{F}_2^n \mid \alpha(k^{(0)}) + \beta(k^{(1)}) = \alpha(a) + \beta(f(a + k^{(0)}) + k^{(1)})\} \\ &= \frac{1}{2^n} \cdot \#\{a \in \mathbb{F}_2^n \mid \alpha(k^{(0)}) = \alpha(a) + \beta(f(a + k^{(0)}))\}, \end{aligned}$$

where we omitted  $\beta(k^{(1)})$  that occurs on both sides of the equation inside the curly set brackets.

This expression is independent of  $k^{(1)}$ , and the slightly rewritten equation

$$p_{F,\alpha,\beta,\kappa}(k) = \frac{1}{2^n} \cdot \#\{a \in \mathbb{F}_2^n \mid \alpha(a + k^{(0)}) = \beta(f(a + k^{(0)}))\}$$

shows that it assumes the same value for all  $k^{(0)}$ : With  $a$  also  $a + k^{(0)}$  runs through all of  $\mathbb{F}_2^n$  for a fixed  $k^{(0)}$ . Therefore this value must agree with the mean value over all  $k$ :

$$p_{F,\alpha,\beta,\kappa}(k) = p_{F,\alpha,\beta,\kappa} = \frac{1}{2^n} \cdot \#\{x \in \mathbb{F}_2^n \mid \alpha(x) = \beta(f(x))\} = p.$$

This consideration shows:

**Theorem 8.2.2.** *In the scenario of example A the probability  $p_{F,\alpha,\beta,\kappa}(k)$  assumes the same value*

$$p = \frac{1}{2^n} \cdot \#\{x \in \mathbb{F}_2^n \mid \alpha(x) = \beta(f(x))\}$$

for all keys  $k \in \mathbb{F}_2^{2n}$ . In particular  $p$  coincides with the mean value from Equation (8.6).

Using the notations from Figure 8.5 we have

$$\begin{aligned}\beta(c) &= \beta(b' + k^{(1)}) = \beta(b') + \beta(k^{(1)}) \\ &\stackrel{p}{\approx} \alpha(b) + \beta(k^{(1)}) = \alpha(a + k^{(0)}) + \beta(k^{(1)}) = \alpha(a) + \alpha(k^{(0)}) + \beta(k^{(1)}).\end{aligned}$$

This yields a linear relation for the bits of the key  $k = (k_1, k_2)$ :

$$\alpha(k^{(0)}) + \beta(k^{(1)}) \stackrel{p}{\approx} \alpha(a) + \beta(c).$$

Treating the complementary relation

$$\beta \circ f(x) \stackrel{1-p}{\approx} \alpha(x) + 1$$

in an analogous way we get:

**Theorem 8.2.3.** *In the scenario of example A let  $(\alpha, \beta)$  be a pair of linear forms for  $f$  with probability  $p$  as in Formula (8.7). Then  $\hat{p} = \max\{p, 1 - p\}$  is the success probability for determining a single key bit by this linear relation given one known plaintext block.*

### Example

Take  $n = 4$ , and for  $f$  take the S-box  $S_0$  of LUCIFER. As the two rightmost columns of Table 8.8 show the linear relation defined by  $(\alpha, \beta)$ , where  $\alpha(x) = x_4$  and  $\beta(y) = y_1 + y_2 + y_4$ , has probability<sup>62</sup>  $p_{f,\alpha,\beta} = \frac{14}{16} = \frac{7}{8}$ .

As concrete round keys take  $k_0 = 1000$  and  $k_1 = 0001$ . Table 8.10, running through all possible 16 plaintexts, shows that  $\alpha(a) + \beta(c)$  assumes the value  $1 = \alpha(k_0) + \beta(k_1)$  for this partial sum of key bits exactly 14 times—as expected.

How large is the success probability  $p_N$  of correctly estimating this partial sum, assuming  $N = 1, 2, \dots$  random known plaintexts from the set of  $2^n$  possible plaintexts? (For given linear forms  $\alpha$  and  $\beta$  with  $p = p_{f,\alpha,\beta}$ .) This is exactly the scenario of the hypergeometric distribution<sup>63</sup>. Therefore we have:

**Theorem 8.2.4.** *In example A let  $(\alpha, \beta)$  be a pair of linear forms that defines a linear relation for  $f$  with probability  $p$ . Then the success probability for determining a key bit by this linear relation from  $N$  known plaintexts is the cumulated probability  $p_N = p_N^{(s)}$  of the hypergeometric distribution with parameters  $2^n$ ,  $s = \hat{p} \cdot 2^n$ , and  $N$  where  $\hat{p} = \max\{p, 1 - p\}$ .*

If we neglect exact mathematical reasoning and work with asymptotic approximations (as is common in applied statistics), then we can replace the hypergeometric distribution by the normal distribution. The usual (quite vaguely stated) conditions for this approximation are “ $p$  not too different from  $\frac{1}{2}$ ,  $N \ll 2^n$ , but  $N$  not too small.” This gives the formula

$$p_N \approx \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\sqrt{N\lambda}} e^{-t^2/2} dt, \quad (8.8)$$

where  $\lambda = (2p - 1)^2$  is the potential of the linear relation. The values associated with the normal distribution<sup>64</sup> are well-known and yield Table 8.11. To get a success probability of about 95%

<sup>62</sup>providing strong evidence that the designers of LUCIFER weren't aware of linear cryptanalysis

<sup>63</sup>that we won't explain here

<sup>64</sup>Instead of the approximation by the normal distribution we could directly use the hypergeometric distribution. This would, in particular for small  $N$ , give a more precise value but not a closed formula as simple as (8.8).

$a$	$b$	$b'$	$c$	$\alpha(a) + \beta(c)$
0000	1000	0010	0011	1
0001	1001	0110	0111	1
0010	1010	0011	0010	0
0011	1011	0001	0000	1
0100	1100	1001	1000	1
0101	1101	0100	0101	1
0110	1110	0101	0100	1
0111	1111	1000	1001	1
1000	0000	1100	1101	1
1001	0001	1111	1110	1
1010	0010	0111	0110	1
1011	0011	1010	1011	1
1100	0100	1110	1111	1
1101	0101	1101	1100	1
1110	0110	1011	1010	1
1111	0111	0000	0001	0

Table 8.10: A linear relation for the key bits ( $b$  arises from  $a$  by adding  $k^{(0)}$ , resulting in “flipping” the first bit,  $b'$  from  $b$  by applying  $f$ , and  $c$  from  $b'$  by adding  $k^{(1)}$ ).

$N\lambda$	1	2	3	4	...	8	9
$p_N$	84,1%	92,1%	95,8%	97,7%	...	99,8%	99,9%

Table 8.11: Dependence of the success probability on the number of known plaintexts

we need  $N \approx \frac{3}{\lambda}$  known plaintexts according to the table. In the concrete example above we had  $p = \frac{7}{8}$ , hence  $\lambda = \frac{9}{16}$ , and the number of known plaintexts needed for a 95% success probability is  $N \approx 5$ . Using Table 8.9 we succeeded with only  $N = 3$  plaintexts. This is no great surprise because the a-priori probability of this success is about 90% (for  $N\lambda = \frac{27}{16} \approx 1,68\dots$ )<sup>65</sup>.

### 8.2.8 Approximation Table, Correlation Matrix, and Linear Profile

Linear relations for a Boolean map (or S-box)  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^q$  are true with certain frequencies (or probabilities). We collect these frequencies in a matrix of size  $2^n \times 2^q$ , called the **approximation table**<sup>66</sup> of  $f$ . This table gives, for each pair  $(\alpha, \beta)$  of linear forms, the number of arguments  $x$  where  $\beta \circ f(x) = \alpha(x)$ . Table 8.12 shows the approximation table of the S-box  $S_0$  of LUCIFER. The entry 16 in the upper left corner says that the relation  $0 = 0$  is true in all 16 possible cases. At the same time 16 is the common denominator by which we have to divide all other entries to get the probabilities. In the general case the upper left corner would be  $2^n$ . The remaining

<sup>65</sup>Here the condition “ $N$  not too small” for the approximation by the normal distribution is more than arguable. However determining the exact values for the hypergeometric distribution is easy: Consider an urn containing 16 balls, 14 black ones and 2 white ones, and draw 3 balls by random. Then the probability of all of them being black is  $\frac{26}{40}$ , the probability of two being black and one being white is  $\frac{13}{40}$ . Hence the probability of at least two balls being black is  $\frac{39}{40} = 97,5\%$ . This is clearly more than the 90% from the approximation (8.8). The remaining probabilities are  $\frac{1}{40}$  for exactly one black ball, and 0 for three white balls.

<sup>66</sup>When using references be aware that often all entries are diminished by  $2^{n-1}$ , for example by the SageMath function `linear_approximation_matrix()`.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	16	8	8	8	8	8	8	8	8	8	8	8	8	8	8	
1	8	6	6	8	8	6	6	8	8	6	6	8	8	14	6	
2	8	10	8	6	4	6	8	6	6	12	6	8	10	8	6	
3	8	12	10	6	12	8	10	6	6	6	8	8	10	10	8	
4	8	8	4	8	8	8	8	4	10	6	6	6	10	6	10	
5	8	10	10	12	8	10	6	8	10	8	4	10	10	8	6	
6	8	10	8	10	8	10	8	10	8	10	8	2	8	10	8	
7	8	8	10	6	8	8	2	6	8	8	10	6	8	8	10	
8	8	8	6	10	6	10	8	8	4	8	10	10	10	12	8	
9	8	10	8	10	6	4	10	8	8	6	8	6	6	8	10	
10	8	6	10	8	6	8	8	10	6	4	8	6	12	6	8	
11	8	12	8	8	6	6	6	10	10	6	10	10	8	8	12	
12	8	8	10	10	6	10	8	4	6	6	8	8	4	8	6	
13	8	6	12	6	6	8	10	8	10	8	6	8	8	10	12	
14	8	6	10	12	10	4	8	6	8	10	10	8	10	8	10	
15	8	8	8	8	10	6	6	10	4	8	4	8	6	6	10	

Table 8.12: Approximation table of the S-box  $S_0$  of LUCIFER. Row and column indices are linear forms represented by integers, see Section 8.1.11. To get the probabilities divide by 16.

entries of the first column (corresponding to  $\beta = 0$ ) are 8 because each non-zero linear form  $\alpha$  takes the value 0 in exactly half of all cases, that is 8 times<sup>67</sup>. For the first row an analogous argument is true—provided that  $f$  is bijective<sup>68</sup>.

The **correlation matrix** and the **linear profile**<sup>69</sup> are the analogous matrices whose entries are the I/O-correlations or the potentials of the corresponding linear relations. The correlation matrix arises from the approximation table by first dividing the entries by  $2^n$  (getting the probabilities  $p$ ) and then transforming the probabilities to I/O-correlations by the formula  $\tau = 2p - 1$ . To get the linear profile we have to square the single entries of the correlation matrix.

For  $S_0$  Table 8.13 shows the correlation matrix, and Table 8.14, the linear profile. Here again the first rows and columns hit the eye: The zeroes tell that a linear relation involving the linear form 0 has potential 0, hence is useless. The 1 in the upper left corner says that the relation  $0 = 0$  holds for any arguments, but is useless too. In the previous subsection we picked the pair  $(\alpha, \beta)$  where  $\alpha(x) = x_4$  (represented by  $0001 \doteq 1$ ) and  $\beta(y) = y_1 + y_2 + y_4$  (represented  $1101 \doteq 13$ ) in row 1, column 13. It assumes the maximum value<sup>70</sup>  $\frac{9}{16}$  for the potential that moreover also occurs at the coordinates (6, 11) and (7, 6).

### Efficient Calculation by Fourier Transformation

We can get the approximation table in the “naive” way by counting, and then derive the correlation matrix and the linear profile by a simple (elementwise) transformation. A more

<sup>67</sup>In the language of linear algebra we express this fact as: The kernel of a linear form  $\neq 0$  is a subspace of dimension  $n - 1$ .

<sup>68</sup>In the general case where  $q$  could be  $\neq n$  we would use the concept “balanced” that means that all preimages have the same size. Of course a map can be balanced only in the case  $q \leq n$ .

<sup>69</sup>also called linearity profile. Not to be confused with the linear complexity profile of a bit sequence that is defined by linear feedback shift registers (LFSRs) and sometimes also called linearity profile.

<sup>70</sup>We ignore the true, but useless, maximum value 1 in the upper left corner.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	0	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{3}{4}$	$-\frac{1}{4}$	0
2	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0
3	0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	0
4	0	0	$-\frac{1}{2}$	0	0	0	0	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
5	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$-\frac{1}{4}$
6	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{3}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$
7	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	$-\frac{3}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	$-\frac{1}{4}$
8	0	0	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0
9	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0	0	$-\frac{1}{4}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$-\frac{1}{2}$
10	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0
11	0	$\frac{1}{2}$	0	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{1}{2}$
12	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	0	$-\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{1}{4}$
13	0	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$	0
14	0	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{2}$	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	$\frac{1}{4}$
15	0	0	0	0	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Table 8.13: Correlation matrix of the S-box  $S_0$  of LUCIFER. Row and column indices are linear forms represented by integers.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{9}{16}$	$\frac{1}{16}$	0
2	0	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0
3	0	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0
4	0	0	$\frac{1}{4}$	0	0	0	0	$\frac{1}{4}$	$\frac{1}{16}$							
5	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
6	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$
7	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{9}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$
8	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{4}$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	0
9	0	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{4}$
10	0	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	0
11	0	$\frac{1}{4}$	0	0	$\frac{1}{16}$	0	0	$\frac{1}{4}$								
12	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{4}$	0	$\frac{1}{16}$	$\frac{1}{16}$
13	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{4}$	0
14	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	0	$\frac{1}{16}$
15	0	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

Table 8.14: Linear profile of the S-box  $S_0$  of LUCIFER. Row and column indices are linear forms represented by integers.

efficient algorithm uses the Fourier<sup>71</sup> transformation. In the binary case it is especially simple, and due to historically independent inventions is also called Hadamard<sup>72</sup> transformation, or Walsh<sup>73</sup> transformation. It transforms a *real valued* (!) function  $\varphi: \mathbb{F}_2^m \rightarrow \mathbb{R}$  into another real valued function  $\hat{\varphi}: \mathbb{F}_2^m \rightarrow \mathbb{R}$  defined by<sup>74</sup>

$$\hat{\varphi}(u) := \sum_{x \in \mathbb{F}_2^m} \varphi(x) \cdot (-1)^{u \cdot x}.$$

Here  $u \cdot x$  is the canonical scalar product in  $\mathbb{F}_2^m$ . The exponents are not integers but bits, however this is OK since over the basis  $-1$  for integer exponents only the residue classes modulo 2 matter.

Now consider a Boolean map  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^q$  and its **indicator function**  $\vartheta_f: \mathbb{F}_2^n \times \mathbb{F}_2^q \rightarrow \mathbb{R}$ ,

$$\vartheta_f(x, y) := \begin{cases} 1, & \text{if } y = f(x), \\ 0 & \text{else.} \end{cases}$$

Let us calculate its Fourier transform; set  $m = n + q$  and split the variables into blocks of lengths  $n$  and  $q$ :

$$\begin{aligned} \hat{\vartheta}_f(u, v) &= \sum_{x \in \mathbb{F}_2^n} \sum_{y \in \mathbb{F}_2^q} \vartheta_f(x, y) (-1)^{u \cdot x + v \cdot y} \\ &= \sum_{x \in \mathbb{F}_2^n} (-1)^{u \cdot x + v \cdot f(x)}. \end{aligned}$$

In the exponent we see the linear forms  $x \mapsto u \cdot x$  on  $\mathbb{F}_2^n$  that we denote by  $\alpha$ , and  $y \mapsto v \cdot y$  on  $\mathbb{F}_2^q$  that we denote by  $\beta$ . Then  $u$  is the bitblock representation of  $\alpha$ , and  $v$ , of  $\beta$ , and the exponent is

$$\alpha(x) + \beta \circ f(x).$$

an expression familiar from linear cryptanalysis. If  $\alpha(x) = \beta \circ f(x)$ , then the exponent is 0, hence the summand is 1. Otherwise the exponent is 1, the summand is  $-1$ . Thus we sum up  $2^n \cdot p_{f, \alpha, \beta}$  ones and  $2^n - 2^n \cdot p_{f, \alpha, \beta}$  “minus ones”, and the sum is

$$2^n \cdot [p_{f, \alpha, \beta} - (1 - p_{f, \alpha, \beta})] = 2^n \cdot \tau_{f, \alpha, \beta}.$$

Hence  $\hat{\vartheta}_f(u, v)$  is the I/O-correlation of  $(\alpha, \beta)$  up to a normalizing factor  $2^n$ .

The Fourier transform  $\hat{\vartheta}_f: \mathbb{F}_2^n \times \mathbb{F}_2^q \rightarrow \mathbb{R}$  of the indicator function of a Boolean map  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^q$  is called the (Walsh-) **spectrum** of  $f$ . We have shown:

**Theorem 8.2.5.** *The spectrum of a Boolean map  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^q$  coincides with  $2^n$  times the correlation matrix.*

This theorem is of eminent theoretical and practical importance:

- On the theoretical side it leads to very elegant and short proofs of statements about the correlation matrix and related objects [Pom14].

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<sup>71</sup>Joseph Fourier, French mathematician and physicist, March 21, 1768 – May 16, 1830

<sup>72</sup>Jacques Hadamard, French mathematician, December 8, 1865 – October 17, 1963

<sup>73</sup>Joseph L. Walsh, American mathematician, September 21, 1895 – December 6, 1973

<sup>74</sup>This is a special case of the discrete Fourier transformation. In the general case we would use the complex  $N$ -th root of unity  $\zeta = e^{2\pi i/N}$  instead of  $-1$ , and transform complex valued functions over the ring  $\mathbb{Z}/N\mathbb{Z}$ , instead over  $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ —or functions on  $\mathbb{Z}^m$  that have period  $N$  in each of the variables. In the binary case  $N = 2$ , and the 2nd root of unity  $-1$  is real, so we only need to consider real valued functions.

- On the practical side it allows the calculation of the correlation matrix (and consequently also of the approximation table and of the linear profile) by the *fast Fourier transformation*<sup>75</sup> that drops the cost by a factor of (essentially)  $2^n$  using binary recursion.

How large is the net effect of FFT? For simplicity we only consider the case  $n = q$ . The naive procedure counts  $2^n$  arguments in determining  $p_{f,\alpha,\beta}$  (and thereby also  $\tau_{f,\alpha,\beta}$ ) for fixed  $\alpha$  and  $\beta$ , the map being given by the value table. Hence the total cost is  $2^n \cdot 2^n \cdot 2^n$ .

We won't explain the FFT (see [Pom14]). It is at the heart of the function `wtr()` from Appendix 8.4.5. We remark without proof that the FFT of a real valued function  $\mathbb{F}_2^m \rightarrow \mathbb{R}$  takes  $3m \cdot 2^m$  simple operations with real numbers that we may count in the naive way for functions with values in  $\{-1, 1\}$  noting that the calculation involves only integers of moderate size. This makes a total of  $3 \cdot 2n \cdot 2^{2n}$  operations.

The fair way to describe the cost of an algorithm is as function of the size  $N$  of its input. Here the input is the value table of a Boolean map  $\mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ . Hence  $N = n \cdot 2^n$ —the value table describes  $n$  component functions each at  $2^n$  arguments. From this point of view the cost of the naive algorithm is (essentially) cubic, the cost of the fast algorithm (essentially) quadratic.

Anyway the cryptologist's preferred parameter is block length. From this point of view the cost grows exponentially in both cases although with a different rate. By the fast algorithm the calculation is feasible for “small” S-boxes, say up to a block length of 10.

SageMath sample 8.8 shows the calculation of the correlation matrix, the approximation table<sup>76</sup>, and the linear profile of  $S_0$ . (Remember to divide the entries of the resulting matrix `Spec` by 16, those of `linProf` by 256.)

---

**SageMath sample 8.8** Correlation matrix, approximation table, and linear profile of the S-box  $S_0$

---

```
sage: Spec = S0.wspec()
sage: ApprT = S0.linApprTable()
sage: linProf = S0.linProf()
```

---

If we call the method `linProf()` with the additional parameter `extended=True`, as in SageMath sample 8.9, then it outputs the maximal entry, as well as all index pairs where it occurs. A look at the approximation table or the correlation matrix then shows whether the corresponding linear relation has probability larger or smaller than  $\frac{1}{2}$ , specifying whether the resulting bit has to be complemented.

### 8.2.9 Example B: A two-round cipher

As a next step we iterate the round map

$$f: \mathbb{F}_2^n \times \mathbb{F}_2^q \rightarrow \mathbb{F}_2^n$$

of a bitblock cipher over two rounds using round keys  $k^{(i)} \in \mathbb{F}_2^q$  as illustrated in Figure 8.7<sup>77</sup>.

We consider linear relations

$$\kappa_1(k^{(1)}) \stackrel{p_1}{\approx} \alpha_1(c^{(0)}) + \beta_1(c^{(1)})$$

---

<sup>75</sup>abbreviated as FFT

<sup>76</sup>For calculating the approximation table the SageMath class `sage.crypto.mq.sbox.SBox` offers the function `S0.linear_approximation_matrix()` where the S-box has the definition `S0 =`

---

**SageMath sample 8.9** Linear profile of the S-box  $S_0$  with evaluation

---

```
sage: lProf = S0.linProf(extended=True)
sage: lProf[0]
[...]
sage: print("Maximum entry:", lProf[1], "| with denominator:", lProf[2])
('Maximum entry:', 144, '| with denominator:', 256)
sage: print("at indices:", lProf[3])
('at indices:', [[1, 13], [6, 11], [7, 6]])
sage: Spec = S0.wspec()
sage: for coord in lProf[3]:
....:     if (Spec[coord[0]][coord[1]] < 0):
....:         print ("For relation at", coord, "take complement.")
....:
('For relation at', [6, 11], 'take complement.')
('For relation at', [7, 6], 'take complement.')

```

---

with probability  $p_1$ , I/O-correlation  $\tau_1 = 2p_1 - 1$ , and potential  $\lambda_1 = \tau_1^2$ , and

$$\kappa_2(k^{(2)}) \stackrel{p_2}{\approx} \alpha_2(c^{(1)}) + \beta_2(c^{(2)})$$

with probability  $p_2$ , I/O-correlation  $\tau_2 = 2p_2 - 1$ , and potential  $\lambda_2 = \tau_2^2$ . We can combine these two linear relations if  $\alpha_2 = \beta_1$ , thereby getting a linear relation for some key bits expressed by the (known) plaintext  $c^{(0)} = a$  and the ciphertext  $c^{(2)} = c$ ,

$$\kappa_1(k^{(1)}) + \kappa_2(k^{(2)}) \stackrel{p}{\approx} \alpha_1(c^{(0)}) + \beta_2(c^{(2)}),$$

that holds with a certain probability  $p$ , and has I/O-correlation  $\tau$  and potential  $\lambda$ , that in general depend on  $k = (k^{(1)}, k^{(2)})$  and are difficult to determine. Therefore we again consider a simplified example B, see Figure 8.8. Encryption is done step by step by the formulas

$$b^{(0)} = a + k^{(0)}, \quad a^{(1)} = f_1(b^{(0)}), \quad b^{(1)} = a^{(1)} + k^{(1)}, \quad a^{(2)} = f_2(b^{(1)}), \quad c = a^{(2)} + k^{(2)}.$$

(Here  $f_1$  is given by the S-box  $S_0$ , and  $f_2$ , by the S-box  $S_1$  that could be identical with  $S_0$ <sup>78</sup>.) As for example A adding some key bits after the last round prevents the “stripping off” of  $f_2$ .

Comparing example B with the general settings in Section 8.2.6 we have:

- key length  $l = 3n$ , key space  $\mathbb{F}_2^{3n}$ , and keys of the form  $k = (k^{(0)}, k^{(1)}, k^{(2)})$  with  $k^{(0)}, k^{(1)}, k^{(2)} \in \mathbb{F}_2^n$ .
- Encryption is defined by the map

$$\begin{aligned} F: \mathbb{F}_2^n \times \mathbb{F}_2^n \times \mathbb{F}_2^n \times \mathbb{F}_2^n &\longrightarrow \mathbb{F}_2^n, \\ (a, k^{(0)}, k^{(1)}, k^{(2)}) &\mapsto f_2(f_1(a + k^{(0)}) + k^{(1)}) + k^{(2)}. \end{aligned}$$

---

<sup>78</sup>`mq.SBox(12,15,7,10,14,13,11,0,2,6,3,1,9,4,5,8)`. Warning: see the footnote on page 301.

<sup>77</sup>In a certain sense example A already was a two-round cipher since we used two partial keys. Adding one more S-box at the right side of Figure 8.5 would be cryptologically irrelevant, because this non-secret part of the algorithm would be known to the cryptanalyst who simply could “strip it off” (that is, apply its inverse to the cipher text) and be left with example A. In a similar way we could interpret example B as a three-round cipher. However this would be a not so common counting of rounds.

<sup>78</sup>We allow that the round functions of the different rounds differ. The reason is that usually the round function consists of several parallel S-boxes and the permutations direct an input bit through different S-boxes on its way through the rounds, see Section 8.2.12.

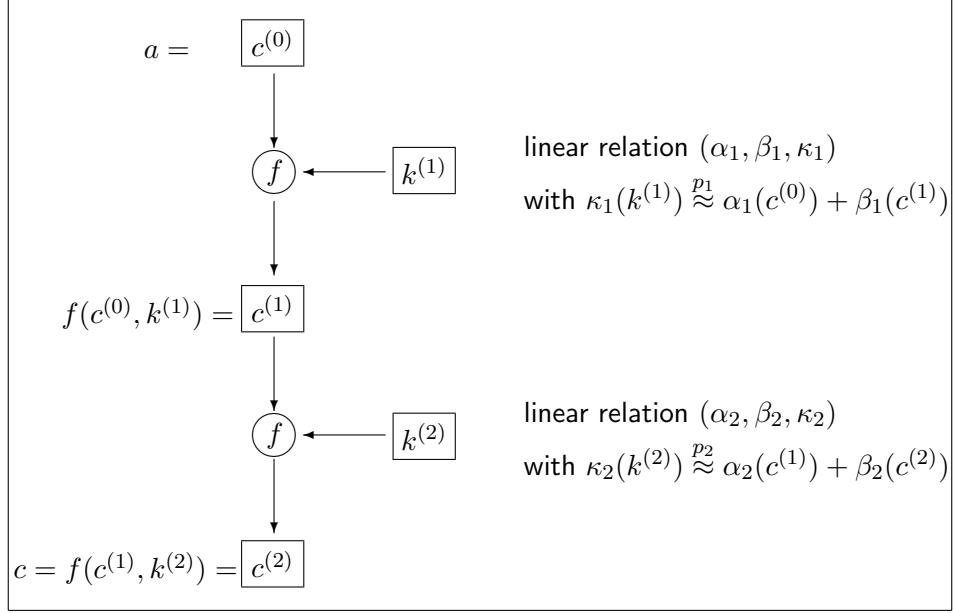


Figure 8.7: General two-round cipher

- The linear form  $\kappa: \mathbb{F}_2^n \times \mathbb{F}_2^n \times \mathbb{F}_2^n \longrightarrow \mathbb{F}_2$  is given by

$$\kappa(k^{(0)}, k^{(1)}, k^{(2)}) = \alpha(k^{(0)}) + \beta(k^{(1)}) + \gamma(k^{(2)}).$$

Here  $(\alpha, \beta)$  is a linear relation for  $f_1$  with probability  $p_1$ , I/O-correlation  $\tau_1$ , and potential  $\lambda_1$ , and  $(\beta, \gamma)$ , a linear relation for  $f_2$  with probability  $p_2$ , I/O-correlation  $\tau_2$ , and potential  $\lambda_2$  (the same  $\beta$  since we want to combine the linear relations), where

$$\begin{aligned} p_1 &= \frac{1}{2^n} \cdot \#\{x \in \mathbb{F}_2^n \mid \beta \circ f_1(x) = \alpha(x)\} \\ p_2 &= \frac{1}{2^n} \cdot \#\{y \in \mathbb{F}_2^n \mid \gamma \circ f_2(y) = \beta(y)\} \end{aligned}$$

With the notations of Figure 8.8 we have

$$\begin{aligned} \gamma(c) &= \gamma(a^{(2)}) + \gamma(k^{(2)}) \stackrel{p_2}{\approx} \beta(b^{(1)}) + \gamma(k^{(2)}) = \beta(a^{(1)}) + \beta(k^{(1)}) + \gamma(k^{(2)}) \\ &\stackrel{p_1}{\approx} \alpha(b^{(0)}) + \beta(k^{(1)}) + \gamma(k^{(2)}) = \alpha(a) + \alpha(k^{(0)}) + \beta(k^{(1)}) + \gamma(k^{(2)}) \end{aligned}$$

Hence we get a linear relation for the key bits as a special case of Equation (8.5) in the form

$$\alpha(k^{(0)}) + \beta(k^{(1)}) + \gamma(k^{(2)}) \stackrel{p}{\approx} \alpha(a) + \gamma(c)$$

with a certain probability  $p$  that is defined by the formula

$$\begin{aligned} p &= p_{F,\alpha,\beta,\gamma}(k) \\ &= \frac{1}{2^n} \cdot \#\{a \in \mathbb{F}_2^n \mid \alpha(k^{(0)}) + \beta(k^{(1)}) + \gamma(k^{(2)}) = \alpha(a) + \gamma(F(a, k))\}. \end{aligned}$$

We try to explicitly determine  $p$ . As for the one-round case we first ask how  $p$  depends on  $k$ . Insert the definition of  $F(a, k)$  into the defining equation inside the set brackets. Then  $\gamma(k^{(2)})$  cancels out and we are left with

$$p_{F,\alpha,\beta,\gamma}(k) = \frac{1}{2^n} \cdot \#\{a \in \mathbb{F}_2^n \mid \alpha(k^{(0)} + a) + \beta(k^{(1)}) = \gamma(f_2(k^{(1)} + f_1(k^{(0)} + a)))\}.$$

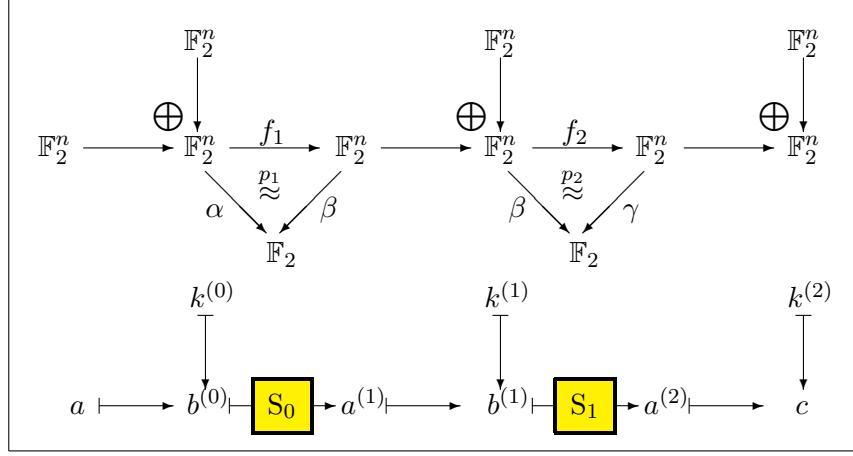


Figure 8.8: Example B: a two-round cipher

This is independent of  $k^{(2)}$ , and for all  $k^{(0)}$  assumes the same value

$$p_{F,\alpha,\beta,\gamma}(k) = \frac{1}{2^n} \cdot \#\{x \in \mathbb{F}_2^n \mid \alpha(x) = \beta(k^{(1)}) + \gamma(f_2(k^{(1)} + f_1(x)))\}$$

because  $x = k^{(0)} + a$  runs through  $\mathbb{F}_2^n$  along with  $a$ . This value indeed depends on  $k$ , but only on the middle component  $k^{(1)}$ . Now form the mean value  $\bar{p} := p_{F,\alpha,\beta,\gamma}$  over all keys:

$$\bar{p} = \frac{1}{2^{2n}} \cdot \#\{(x, k^{(1)}) \in \mathbb{F}_2^{2n} \mid \alpha(x) = \beta(k^{(1)}) + \gamma(f_2(k^{(1)} + f_1(x)))\}.$$

Inside the brackets we see the expression  $\gamma(f_2(k^{(1)} + f_1(x)))$ , and we know:

$$\gamma(f_2(k^{(1)} + f_1(x))) = \begin{cases} \beta(k^{(1)} + f_1(x)) & \text{with probability } p_2, \\ 1 + \beta(k^{(1)} + f_1(x)) & \text{with probability } 1 - p_2. \end{cases}$$

Here “probability  $p_2$ ” means that the statement is true for  $p_2 \cdot 2^{2n}$  of all possible  $(x, k^{(1)}) \in \mathbb{F}_2^{2n}$ . Substituting this we get

$$\begin{aligned} \bar{p} = \frac{1}{2^{2n}} \cdot & \left[ p_2 \cdot \#\{(x, k^{(1)}) \in \mathbb{F}_2^{2n} \mid \alpha(x) = \beta(f_1(x))\} \right. \\ & \left. + (1 - p_2) \cdot \#\{(x, k^{(1)}) \in \mathbb{F}_2^{2n} \mid \alpha(x) \neq \beta(f_1(x))\} \right] \end{aligned}$$

where now the defining equations of both sets are also independent of  $k^{(1)}$ . We recognize the definition of  $p_1$  and substitute it getting

$$\bar{p} = p_1 p_2 + (1 - p_1)(1 - p_2) = 2p_1 p_2 - p_1 - p_2 + 1.$$

This formula looks much more beautiful if expressed in terms of the I/O-correlations  $\bar{\tau} = 2\bar{p} - 1$  and  $\tau_i = 2p_i - 1$  for  $i = 1$  and 2:

$$\bar{\tau} = 2\bar{p} - 1 = 4p_1 p_2 - 2p_1 - 2p_2 + 1 = (2p_1 - 1)(2p_2 - 1) = \tau_1 \tau_2.$$

In summary we have proved:

$a$	$b^{(0)}$	$a^{(1)}$	$b^{(1)}$	$a^{(2)}$	$c$	$\beta(b^{(1)})$	$\gamma(a^{(2)})$	$\alpha(a) + \gamma(c)$
0000	1000	0010	0011	1001	1111	1	1	0
0001	1001	0110	0111	0100	0010	0	1	1
0010	1010	0011	0010	1110	1000	0	0	1
0011	1011	0001	0000	0111	0001	0	1	1
0100	1100	1001	1000	1100	1010	1	0	1
0101	1101	0100	0101	1011	1101	0	1	1
0110	1110	0101	0100	0011	0101	1	0	1
0111	1111	1000	1001	1101	1011	0	0	0
1000	0000	1100	1101	1111	1001	1	0	1
1001	0001	1111	1110	1000	1110	0	1	1
1010	0010	0111	0110	0000	0110	1	0	1
1011	0011	1010	1011	1010	1100	0	1	1
1100	0100	1110	1111	0101	0011	1	1	0
1101	0101	1101	1100	0110	0000	0	1	1
1110	0110	1011	1010	0001	0111	1	0	1
1111	0111	0000	0001	0010	0100	1	0	0

Table 8.15: The data flow in the concrete example for B, and some linear forms

**Theorem 8.2.6.** *For example B we have:*

- (i) *The probability  $p_{F,\alpha,\beta,\gamma}(k)$  depends only on the middle component  $k^{(1)}$  of the key  $k = (k^{(0)}, k^{(1)}, k^{(2)}) \in \mathbb{F}_2^n \times \mathbb{F}_2^n \times \mathbb{F}_2^n$ .*
- (ii) *The mean value of these probabilities over all keys  $k$  is  $p_{F,\alpha,\beta,\gamma} = \bar{p} = 2p_1p_2 - p_1 - p_2 + 1$ .*
- (iii) *The I/O-correlations and the potentials are multiplicative:*

$$\tau_{F,\alpha,\beta,\gamma} = \tau_1\tau_2 \quad \text{and} \quad \lambda_{F,\alpha,\beta,\gamma} = \lambda_1\lambda_2.$$

In Matsui's test we face the decision whether to use the linear relation or its negation for estimating a bit. We can't do better than use the mean value  $p_{F,\alpha,\beta,\gamma}$  since the key and the true probability  $p_{F,\alpha,\beta,\gamma}(k)$  are unknown. This could be an error since these two probabilities might lie on different sides of  $\frac{1}{2}$ .

---

#### SageMath sample 8.10 A Boolean map (S-box S<sub>1</sub> of LUCIFER)

---

```

g1 = BoolF([0,0,1,1,0,1,0,0,1,1,0,1,0,1,0])
g2 = BoolF([1,0,1,0,0,0,0,1,1,1,0,0,1,1,0,1])
g3 = BoolF([1,1,1,0,1,1,0,0,0,0,0,1,1,1,0,0])
g4 = BoolF([1,0,0,1,1,1,0,0,0,1,1,0,0,1,0,1])
S1 = BoolMap([g1,g2,g3,g4])

```

---

#### Example

Let  $n = 4$ ,  $S_0$  as in 8.2.7, and  $S_1$  as defined in SageMath sample 8.10<sup>79</sup> and as given in Table 8.15 (in different order) as transition from column  $b^{(1)}$  to column  $a^{(2)}$ . (This table is easily calculated

---

<sup>79</sup>By the way this is the second S-box of LUCIFER.

with pencil and paper, or by SageMath sample 8.11.) Consider the linear forms  $\alpha \doteq 0001$  and  $\beta \doteq 1101$  as in Section 8.2.7 with  $p_1 = \frac{7}{8}$ ,  $\tau_1 = \frac{3}{4}$ ,  $\lambda_1 = \frac{9}{16}$ . Furthermore let  $\gamma \doteq 1100$ . Then the linear relation for  $f_2$  defined by  $(\beta, \gamma)$  (see Table 8.16, row index 13, column index 12) has probability  $p_2 = \frac{1}{4}$ , I/O-correlation  $\tau_2 = -\frac{1}{2}$ , and potential  $\lambda_2 = \frac{1}{4}$ , the maximum possible value by Table 8.17<sup>80</sup>.

---

#### SageMath sample 8.11 Sample calculations for the example B (two-round cipher)

---

```
sage: n = 4
sage: alpha = [0,0,0,1]; beta = [1,1,0,1]; gamma = [1,1,0,0]
sage: k0 = [1,0,0,0]; k1 = [0,0,0,1]; k2 = [0,1,1,0]
sage: for i in range(0,2**n):
....:     a = int2bb1(i,n); b0 = xor(a,k0); a1 = S0.valueAt(b0)
....:     b1 = xor(k1,a1); a2 = S1.valueAt(b1); c = xor(a2,k2)
....:     bit1 = binScPr(beta,b1); bit2 = binScPr(gamma,a2)
....:     bit3 = (binScPr(alpha,a) + binScPr(gamma,c)) % 2
....:     print(a, b0, a1, b1, a2, c, bit1, bit2, bit3)
```

---

As concrete round keys take  $k^{(0)} = 1000$ ,  $k^{(1)} = 0001$ —as in Section 8.2.7—, and  $k^{(2)} = 0110$ . We want to gain the bit  $\alpha(k^{(0)}) + \beta(k^{(1)}) + \gamma(k^{(2)})$  (that in cheat mode we know is 0). Since  $\tau_1 \tau_2 < 0$  in the majority of cases  $\alpha(a) + \gamma(c)$  should give the complementary bit 1. Table 8.15 shows that in 12 of 16 cases this prediction is correct. Therefore  $1 - p = \frac{3}{4}$ ,  $p = \frac{1}{4}$ ,  $\tau = -\frac{1}{2}$ ,  $\lambda = \frac{1}{4}$ . Remember that this value depends on the key component  $k^{(1)}$ . In fact it slightly deviates from the mean value

$$\bar{p} = 2 \cdot \frac{7}{8} \cdot \frac{1}{4} - \frac{7}{8} - \frac{1}{4} + 1 = \frac{7}{16} - \frac{14}{16} - \frac{4}{16} + \frac{16}{16} = \frac{5}{16}.$$

SageMath sample 8.12 calculates the variation of the probability as function of the partial key  $k^{(1)}$ . The result shows the values  $\frac{1}{4}$  and  $\frac{3}{8}$  each 8 times, all lying on the “correct side” of  $\frac{1}{2}$  and having the correct mean value  $\frac{5}{16}$ .

There are other “paths” from  $\alpha$  to  $\gamma$ —we could insert any  $\beta$  in between. Calculating the mean probabilities by an additional loop in SageMath yields—besides the already known  $\frac{5}{16}$ —three times  $\frac{15}{32}$ , eleven times exactly  $\frac{1}{2}$ , and even a single  $\frac{17}{32}$  that lies on the “wrong” side of  $\frac{1}{2}$ . Thus only the one case we explicitly considered is really good.

As an alternative concrete example take  $\beta \doteq 0001$ . Here  $\lambda_1 = \frac{1}{16}$ ,  $p_1 = \frac{3}{8}$ ,  $\tau_1 = -\frac{1}{4}$ , and  $\lambda_2 = \frac{1}{16}$ ,  $p_2 = \frac{5}{8}$ ,  $\tau_2 = \frac{1}{4}$ . Hence  $\tau = -\frac{1}{16}$  and  $p = \frac{15}{32}$ . The target bit is  $\alpha(k^{(0)}) + \beta(k^{(1)}) + \gamma(k^{(2)}) + 1 = 1$ , and the success probability is  $1 - p = \frac{17}{32}$ . The mean value of  $p$  over all keys is  $\frac{15}{32}$  for this  $\beta$  in coincidence with the key-specific value.

### 8.2.10 Linear Paths

Consider the general case where the round map  $f: \mathbb{F}_2^n \times \mathbb{F}_2^q \rightarrow \mathbb{F}_2^n$  is iterated for  $r$  rounds with round keys  $k^{(i)} \in \mathbb{F}_2^q$ , in analogy with Figure 8.7. Let  $(\alpha_i, \beta_i, \kappa_i)$  be a linear relation for round  $i$ . Let  $\alpha_i = \beta_{i-1}$  for  $i = 2, \dots, r$ . Set  $\beta_0 := \alpha_1$ . Then the chain  $\beta = (\beta_0, \dots, \beta_r)$  is called a **linear path** for the cipher.

---

<sup>80</sup>The linear profile of  $S_1$  is more uniform than that of  $S_0$ .

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	16	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
1	8	10	8	10	8	6	12	10	10	4	6	8	10	8	10	8
2	8	6	4	10	6	8	6	8	8	10	4	6	10	8	10	8
3	8	8	8	8	6	6	6	6	10	6	6	10	4	8	8	12
4	8	8	8	4	8	8	8	4	6	6	6	10	10	10	10	6
5	8	6	8	10	4	6	8	6	8	6	12	6	8	10	8	6
6	8	10	12	10	6	12	6	8	10	8	6	8	8	10	8	6
7	8	8	8	12	10	10	10	6	4	8	8	8	6	10	10	10
8	8	8	6	10	10	6	8	8	10	10	8	12	8	12	6	6
9	8	6	6	8	6	12	8	10	8	6	10	12	10	8	8	10
10	8	6	6	8	12	10	6	8	10	4	8	6	6	8	8	6
11	8	4	10	10	8	8	10	6	8	8	6	10	8	4	6	6
12	8	8	6	6	6	10	12	8	8	8	6	6	6	10	4	8
13	8	10	6	8	6	8	8	10	6	8	8	10	4	6	10	4
14	8	10	6	8	8	10	10	4	12	10	10	8	8	6	10	8
15	8	4	10	6	8	8	10	10	10	8	8	6	10	12	8	

Table 8.16: Approximation table of the S-box  $S_1$  of LUCIFER. Row and column indices are linear forms represented by integers, see Section 8.1.11. For the probabilities divide by 16.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0
2	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0
3	0	0	0	0	$\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{4}$								
4	0	0	0	$\frac{1}{4}$	0	0	0	$\frac{1}{4}$	$\frac{1}{16}$							
5	0	$\frac{1}{16}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
6	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	0	0	$\frac{1}{16}$	0	$\frac{1}{16}$
7	0	0	0	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	0	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
8	0	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{4}$	0	$\frac{1}{16}$	$\frac{1}{16}$
9	0	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$
10	0	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{16}$	$\frac{1}{4}$	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$
11	0	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$
12	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	0	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	0
13	0	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$
14	0	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$
15	0	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{4}$	0

Table 8.17: Linear profile of the S-box  $S_1$  of LUCIFER. Row and column indices are linear forms represented by integers.

---

**SageMath sample 8.12** Dependence of the probability on the key

---

```
sage: n = 4; NN = 2**n
sage: alpha = [0,0,0,1]; beta = [1,1,0,1]; gamma = [1,1,0,0]
sage: reslist = []
sage: sum = 0
sage: for j in range(0,NN):
....:     k1 = int2bb1(j,n)
....:     ctr = 0
....:     for i in range(0,NN):
....:         x = int2bb1(i,n)
....:         u = S0.valueAt(x); y = xor(k1,u); z = S1.valueAt(y)
....:         bit1 = binScPr(alpha,x)
....:         bit2 = binScPr(beta,k1); bit3 = binScPr(gamma,z)
....:         if (bit1 == (bit2 + bit3) % 2):
....:             ctr += 1
....:     prob = ctr/NN
....:     reslist.append([k1, ctr, prob])
....:     sum += ctr
....:
sage: reslist
[[[0, 0, 0, 0], 4, 1/4],
 [[0, 0, 0, 1], 4, 1/4],
 [[0, 0, 1, 0], 4, 1/4],
 [[0, 0, 1, 1], 4, 1/4],
 [[0, 1, 0, 0], 6, 3/8],
 [[0, 1, 0, 1], 6, 3/8],
 [[0, 1, 1, 0], 6, 3/8],
 [[0, 1, 1, 1], 6, 3/8],
 [[1, 0, 0, 0], 6, 3/8],
 [[1, 0, 0, 1], 4, 1/4],
 [[1, 0, 1, 0], 4, 1/4],
 [[1, 0, 1, 1], 6, 3/8],
 [[1, 1, 0, 0], 4, 1/4],
 [[1, 1, 0, 1], 6, 3/8],
 [[1, 1, 1, 0], 6, 3/8],
 [[1, 1, 1, 1], 4, 1/4]]
sage: meanprob = sum/(NN*NN)
sage: print("Sum of counters:", sum, "| Mean probability:", meanprob)
('Sum of counters:', 80, '| Mean probability:', 5/16)
```

---

For a simplified scenario, let's call it example C as a generalization of example B, again we'll derive a useful result on the probabilities. So we consider the special but relevant case where the round keys enter the algorithm in an additive way, see Figure 8.9.

Given a key  $k = (k^{(0)}, \dots, k^{(r)}) \in \mathbb{F}_2^{n \cdot (r+1)}$  we compose the encryption function  $F$  successively with the intermediate results

$$a^{(0)} = a \mid b^{(0)} = a^{(0)} + k^{(0)} \mid a^{(1)} = f_1(b^{(0)}) \mid b^{(1)} = a^{(1)} + k^{(1)} \mid \dots$$

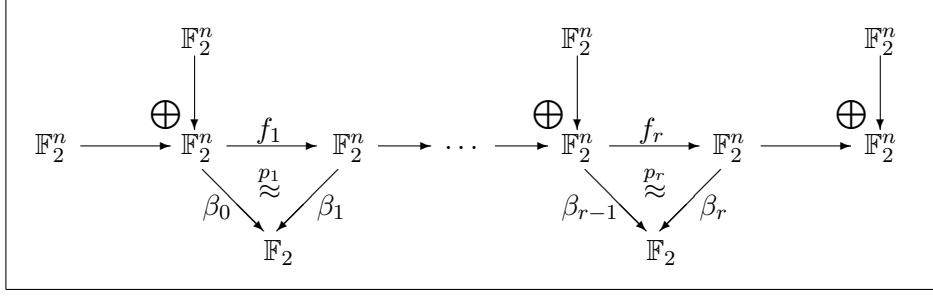


Figure 8.9: Example C: Multiple rounds, keys entered into the algorithm in an additive way

$$b^{(r-1)} = a^{(r-1)} + k^{(r-1)} \mid a^{(r)} = f_r(b^{(r-1)}) \mid b^{(r)} = a^{(r)} + k^{(r)} = c =: F(a, k)$$

The general formula is

$$\begin{aligned} b^{(i)} &= a^{(i)} + k^{(i)} \text{ for } i = 0, \dots, r, \\ a^{(0)} &= a \text{ and } a^{(i)} = f_i(b^{(i-1)}) \text{ for } i = 1, \dots, r. \end{aligned}$$

We consider a linear relation

$$\kappa(k) \xrightarrow{p} \beta_0(a) + \beta_r(c),$$

where

$$\kappa(k) = \beta_0(k^{(0)}) + \dots + \beta_r(k^{(r)}),$$

and  $p$  is the probability

$$p_{F,\beta}(k) = \frac{1}{2^n} \cdot \#\{a \in \mathbb{F}_2^n \mid \sum_{i=0}^r \beta_i(k^{(i)}) = \beta_0(a) + \beta_r(F(a, k))\}$$

that depends on the key  $k$ . Denote the mean value of these probabilities over all  $k$  by  $q_r$ . It depends on  $(f_1, \dots, f_r)$  and on the linear path  $\beta$ :

$$q_r := \frac{1}{2^{n \cdot (r+2)}} \cdot \#\{a, k^{(0)}, \dots, k^{(r)} \in \mathbb{F}_2^n \mid \sum_{i=0}^r \beta_i(k^{(i)}) = \beta_0(a) + \beta_r(F(a, k))\}.$$

Substitute  $F(a, k) = a^{(r)} + k^{(r)} = f_r(b^{(r-1)}) + k^{(r)}$  into the defining equation of this set:

$$\beta_0(k^{(0)}) + \dots + \beta_r(k^{(r)}) = \beta_0(a) + \beta_r(f_r(b^{(r-1)})) + \beta_r(k^{(r)}).$$

Then  $\beta_r(k^{(r)})$  cancels out, and we see that the count is independent of  $k^{(r)}$ . The remaining formula is

$$q_r = \frac{1}{2^{n \cdot (r+1)}} \cdot \#\{a, k^{(0)}, \dots, k^{(r-1)} \in \mathbb{F}_2^n \mid \sum_{i=0}^{r-1} \beta_i(k^{(i)}) = \beta_0(a) + \beta_r(f_r(b^{(r-1)}))\}.$$

In this formula the probability  $p_r$  is hidden: We have

$$\beta_r(f_r(b^{(r-1)})) = \begin{cases} \beta_{r-1}(b^{(r-1)}) & \text{with probability } p_r, \\ 1 + \beta_{r-1}(b^{(r-1)}) & \text{with probability } 1 - p_r, \end{cases}$$

where “with probability  $p_r$ ” means: in  $p_r \cdot 2^{n \cdot (r+1)}$  of the  $2^{n \cdot (r+1)}$  possible cases. Hence

$$\begin{aligned} q_r &= \frac{1}{2^{n \cdot (r+1)}} \cdot \left[ p_r \cdot \#\{a, k^{(0)}, \dots, k^{(r-1)} \mid \sum_{i=0}^{r-1} \beta_i(k^{(i)}) = \beta_0(a) + \beta_{r-1}(b^{(r-1)})\} \right. \\ &\quad \left. + (1 - p_r) \cdot \#\{a, k^{(0)}, \dots, k^{(r-1)} \mid \sum_{i=0}^{r-1} \beta_i(k^{(i)}) = 1 + \beta_0(a) + \beta_{r-1}(b^{(r-1)})\} \right] \\ &= p_r \cdot q_{r-1} + (1 - p_r) \cdot (1 - q_{r-1}), \end{aligned}$$

for the final counts exactly correspond to the probabilities for  $r - 1$  rounds.

This is the perfect entry to a proof by induction, showing:

**Theorem 8.2.7** (Matsuis Piling-Up Theorem). *In example C the mean value  $p_{F,\beta}$  of the probabilities  $p_{F,\beta}(k)$  over all keys  $k \in \mathbb{F}_2^{n(r+1)}$  fulfills*

$$2p_{F,\beta} - 1 = \prod_{i=1}^r (2p_i - 1).$$

*In particular the I/O-correlations and the potentials are multiplicative.*

### Proof

The induction starts with the trivial case<sup>81</sup>  $r = 1$ .

From the previous consideration we conclude

$$2q_r - 1 = 4p_r q_{r-1} - 2p_r - 2q_{r-1} + 1 = (2p_r - 1)(2q_{r-1} - 1),$$

and the assertion follows by induction on  $r$ .  $\square$

For real ciphers in general the round keys are *not* independent but derive from a “master key” by a specific key schedule. In practice however this effect is negligable. The method of linear cryptanalysis follows the rule of thumb:

*Along a linear path the potentials are multiplicative.*

Theorem 8.2.7, although valid only in a special situation and somewhat imprecise for real life ciphers, gives a good impression of how the cryptanalytic advantage (represented by the potential) of linear approximations decreases with an increasing number of rounds; note that the product of numbers smaller than 1 (and greater than 0) decreases with the number of factors. This means that the security of a cipher against linear cryptanalysis is the better, the more rounds it involves.

### 8.2.11 Parallel Arrangement of S-Boxes

The round map of an SP-network usually involves several “small” S-boxes in a parallel arrangement. On order to analyze the effect of this construction we again consider a simple example D, see Figure 8.10.

---

<sup>81</sup>Theorem 8.2.6 contains the case  $r = 2$  that we prove here once again. Nevertheless the separate treatment of this case was a good introduction and motivation.

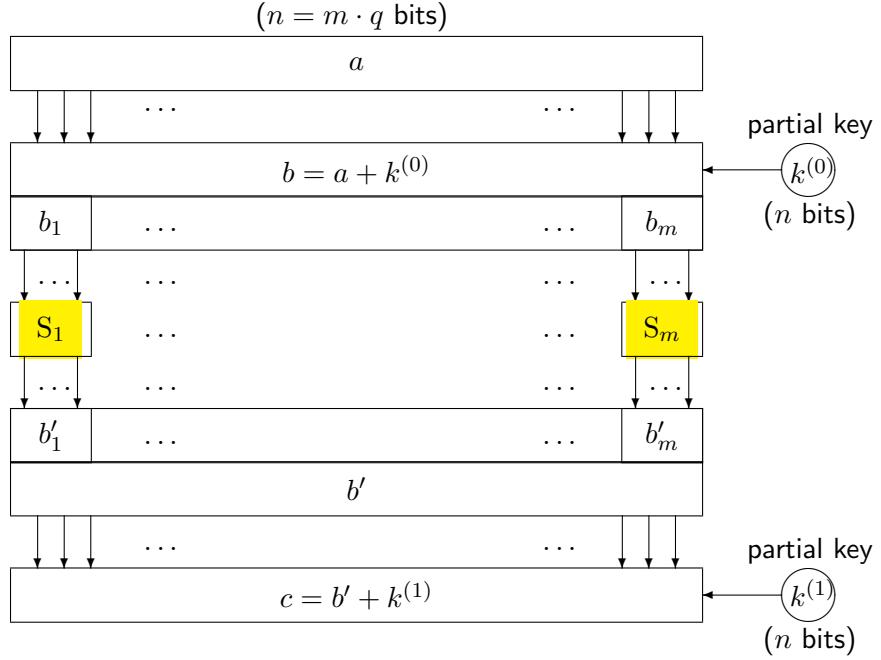


Figure 8.10: Example D: parallel arrangement of  $m$  S-boxes  $S_1, \dots, S_m$  of width  $q$

**Theorem 8.2.8.** Let  $S_1, \dots, S_m : \mathbb{F}_2^q \rightarrow \mathbb{F}_2^q$  be Boolean maps,  $n = m \cdot q$ , and  $f$ , the Boolean map

$$f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n, \quad f(x_1, \dots, x_m) = (S_1(x_1), \dots, S_m(x_m)) \text{ for } x_1, \dots, x_m \in \mathbb{F}_2^q.$$

Let  $(\alpha_i, \beta_i)$  for  $i = 1, \dots, m$  be linear relations for  $S_i$  with probabilities  $p_i$ . Let

$$\begin{aligned} \alpha(x_1, \dots, x_m) &= \alpha_1(x_1) + \dots + \alpha_m(x_m) \\ \beta(y_1, \dots, y_m) &= \beta_1(y_1) + \dots + \beta_m(y_m) \end{aligned}$$

Then  $(\alpha, \beta)$  is a linear relation for  $f$  with probability  $p$  given by

$$2p - 1 = (2p_1 - 1) \cdots (2p_m - 1).$$

### Proof

We consider the case  $m = 2$  only; the general case follows by a simple induction as for Theorem 8.2.7.

In the case  $m = 2$  we have  $\beta \circ f(x_1, x_2) = \alpha(x_1, x_2)$  if and only if

- either  $\beta_1 \circ S_1(x_1) = \alpha_1(x_1)$  and  $\beta_2 \circ S_2(x_2) = \alpha_2(x_2)$
- or  $\beta_1 \circ S_1(x_1) = 1 + \alpha_1(x_1)$  and  $\beta_2 \circ S_2(x_2) = 1 + \alpha_2(x_2)$ .

Hence  $p = p_1 p_2 + (1 - p_1)(1 - p_2)$ , and the assertion follows as for Theorem 8.2.6.  $\square$

As a consequence the I/O-correlations and the potentials are multiplicative also for a parallel arrangement. At first view this might seem a strengthening of the security, but this appearance is deceiving! We cannot detain the attacker from choosing all linear forms as zeroes except the “best” one. And the zero forms have probabilities  $p_i = 1$  and potentials 1. Hence the attacker picks a pair  $(\alpha_j, \beta_j)$  with maximum potential, and then sets  $\alpha(x_1, \dots, x_m) = \alpha_j(x_j)$  and  $\beta(y_1, \dots, y_m) = \beta_j(y_j)$ . In a certain sense she turns the other S-boxes, except  $S_j$ , “inactive”. Then the complete linear relation inherits exactly the probability and the potential of the “active” S-box  $S_j$ .

$a$	$a_3$	$c$	$c_0 + c_1 + c_3$	estimate
00011110	1	00000010	0	1
00101100	0	00111111	1	1
10110010	1	01011101	0	1
10110100	1	01010000	0	1
10110101	1	01010111	0	1

Table 8.18: Calculations for example D (parallel arrangement of  $m$  S-boxes)

### Example

Once again we consider a concrete example with  $m = 2$  and  $q = 4$ , hence  $n = 8$ . As S-boxes we take the ones from LUCIFER,  $S_0$  at the left, and  $S_1$  at the right, see Figure 8.10. For the left S-box  $S_0$  we take the linear relation with  $\alpha \doteq 0001$  and  $\beta \doteq 1101$ , that we know has probability  $p_1 = \frac{7}{8}$ . For the right S-Box  $S_1$  we take the relation  $(0, 0)$  (where both linear forms are 0). Since always  $0 = 0$  its probability is 1. The combined linear relation for  $f = (S_0, S_1)$  then also has probability  $p = \frac{7}{8}$  and potential  $\lambda = \frac{9}{16}$  by Theorem 8.2.8, and we know that linear cryptanalysis with  $N = 5$  pairs of plaintext and ciphertext has (more than) 95% success probability. We decompose all relevant bitblocks into bits:

**plaintext:**  $a = (a_0, \dots, a_7) \in \mathbb{F}_2^8$ ,

**ciphertext:**  $c = (c_0, \dots, c_7) \in \mathbb{F}_2^8$ ,

**key:**  $k = (k_0, \dots, k_{15}) \in \mathbb{F}_2^{16}$  where  $(k_0, \dots, k_7)$  serves as “initial key” (corresponding to  $k^{(0)}$  in Figure 8.10), and  $(k_8, \dots, k_{15})$  as “final key” (corresponding to  $k^{(1)}$ ).

Then  $\alpha(a) = a_3$ ,  $\beta(c) = c_0 + c_1 + c_3$ , and  $\kappa(k) = \alpha(k_0, \dots, k_7) + \beta(k_8, \dots, k_{15}) = k_3 + k_8 + k_9 + k_{11}$ . Hence the target relation is

$$k_3 + k_8 + k_9 + k_{11} = a_3 + c_0 + c_1 + c_3.$$

We use the key  $k = 1001011000101110$  whose relevant bit is  $k_3 + k_8 + k_9 + k_{11} = 1$ , and generate five random pairs of plaintext and ciphertext<sup>82</sup>, see Table 8.18. We see that for this example Matsui’s algorithm guesses the relevant key bit correctly with no dissentient.

### 8.2.12 Mini-Lucifer

As a slightly more complex example we define a toy cipher “Mini-Lucifer” that employs the S-boxes and a permutation of the true LUCIFER. Here is the construction, see Figure 8.11:

- Before and after each round map we add a partial key. We use two keys  $k^{(0)}$  and  $k^{(1)}$  in alternating order. They consist of the first or last 8 bits of the 16 bit master key. In particular for  $r \geq 3$  the round keys are not independent.
- The round function consists of a parallel arrangement of the two S-boxes, as in the example of Section 8.2.11, followed by the permutation P.

<sup>82</sup>We simulate an attacker who knows five pairs of plaintexts and corresponding ciphertexts by generating (arbitrarily at random) and using these five pairs. Our odds of correctly determining the wanted key bit are very high.

- The permutation  $P$  maps a single byte (octet) to itself as defined in SageMath sample 8.13.  
As usual for SP-networks we omit it in the last round.

SageMath sample 8.14 contains the SageMath/Python code.

---

**SageMath sample 8.13** The permutation  $P$  of LUCIFER

---

```
def P(b):
    """Lucifer's permutation"""
    pb = [b[2],b[5],b[4],b[0],b[3],b[1],b[7],b[6]]
    return pb
```

---



---

**SageMath sample 8.14** Mini-Lucifer over  $r$  rounds

---

```
def miniLuc(a,k,r):
    """Mini-Lucifer, encrypts 8-bit a with 16-bit key k over r rounds."""
    k0 = k[0:8]          # split into subkeys
    k1 = k[8:16]
    aa = a               # round input
    # --- begin round
    for i in range(0,r): # round number is i+1
        if (i % 2 == 0): # select round key
            rndkey = k0
        else:
            rndkey = k1
        b = xor(aa,rndkey)      # add round key
        bleft = b[0:4]           # begin substitution
        bright = b[4:8]
        bbleft = S0.valueAt(bleft)
        bbright = S1.valueAt(bright)
        bb = bbleft + bbright # end substitution
        if (i+1 == r):          # omit permutation in last round
            aa = bb
        else:
            aa = P(bb)
    # --- end round
    if (r % 2 == 0):          # add subkey after last round
        finkey = k0
    else:
        finkey = k1
    c = xor(aa,finkey)
    return c
```

---

Up to now we ignored permutations in linear cryptanalysis. How do they influence the analysis?

Well, let  $f$  be a Boolean map,  $(\alpha, \beta)$ , a linear relation for  $f$  with probability  $p$ , and  $P$ , a permutation of the range of  $f$ . Then we set  $\beta' = \beta \circ P^{-1}$ , a linear form, and immediately see

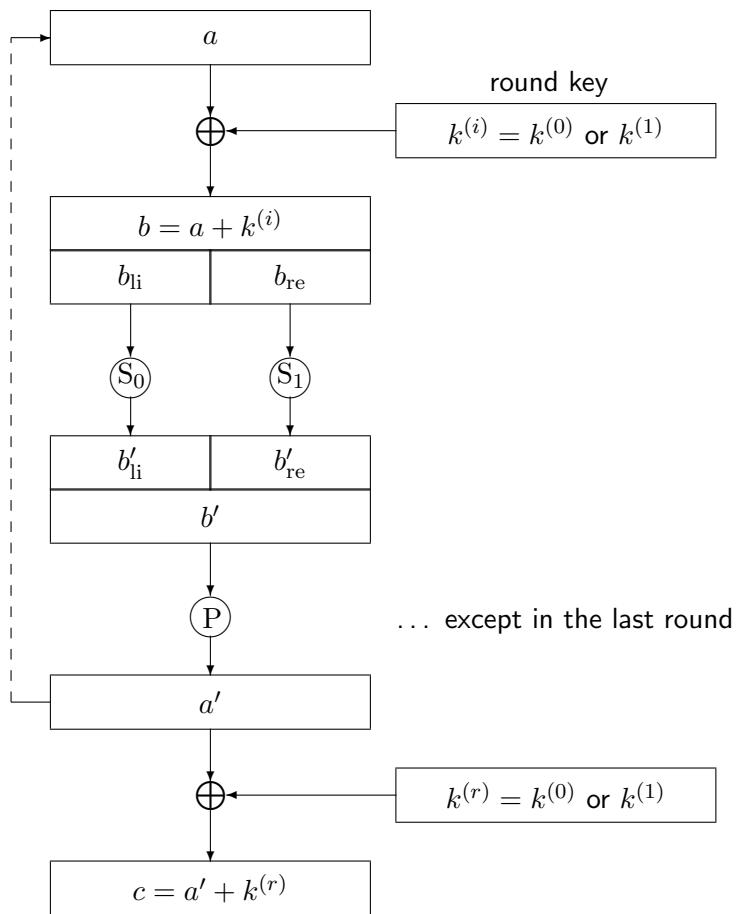


Figure 8.11: Mini-Lucifer over  $r$  rounds

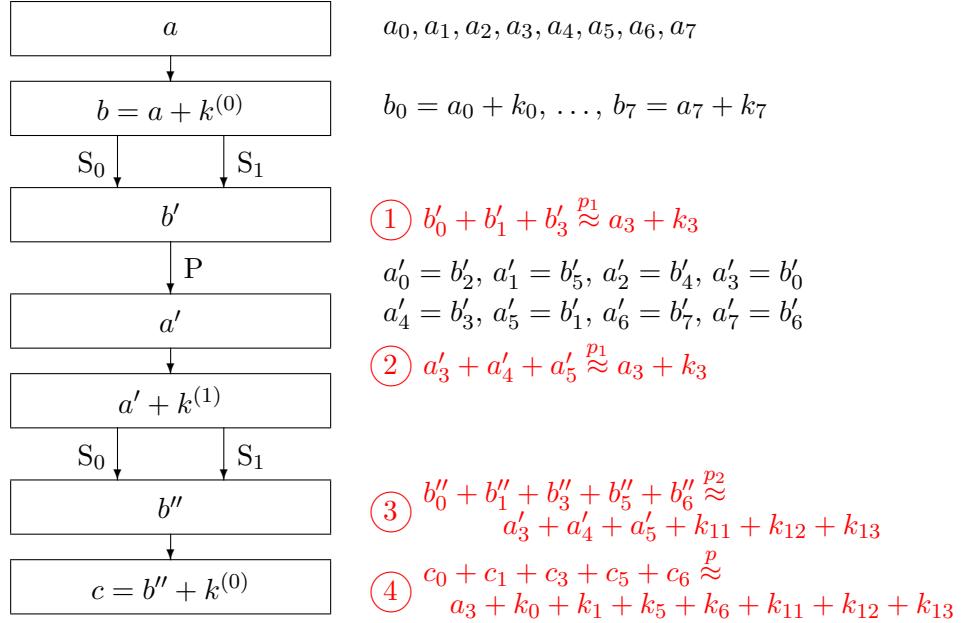


Figure 8.12: Mini-Lucifer with 2 rounds

that  $(\alpha, \beta')$  is a linear relation for  $P \circ f$  with the same probability  $p$ :

$$\begin{aligned} p &= \frac{1}{2^n} \cdot \#\{x \in \mathbb{F}_2^n \mid \beta(f(x)) = \alpha(x)\} \\ &= \frac{1}{2^n} \cdot \#\{x \in \mathbb{F}_2^n \mid (\beta \circ P^{-1})(P \circ f(x)) = \alpha(x)\}. \end{aligned}$$

The assignment  $\beta \mapsto \beta'$  simply permutes the linear forms  $\beta$ . In other words: appending a permutation to  $f$  permutes the columns of the approximation table, of the correlation matrix, and of the linear profile<sup>83</sup>.

*Inserting a permutation into the round function of an SP-network affects linear cryptanalysis in a marginal way only.*

We'll verify this assertion for a concrete example, and see how “marginal” the effect really is.

### Example

The concrete example is specified in Figure 8.12. The relation 1, namely

$$\beta(b') \stackrel{p_1}{\approx} \alpha(a + k^{(0)}), \quad \text{or explicitly } b'_0 + b'_1 + b'_3 \stackrel{p_1}{\approx} a_3 + k_3,$$

holds with probability  $p_1 = \frac{7}{8}$  between  $\alpha \hat{=} 0001$  and  $\beta \hat{=} 1101$ . The permutation  $P$  transforms it to the relation 2, namely

$$\beta \circ P^{-1}(a') \stackrel{p_1}{\approx} \alpha(a + k^{(0)}) = \alpha(a) + \alpha(k^{(0)}), \quad \text{or explicitly } a'_3 + a'_4 + a'_5 \stackrel{p_1}{\approx} a_3 + k_3.$$

But  $P$  also distributes the bits from the left-hand side of the relation over the two S-boxes of the next round. So the cryptanalytic trick of letting only one S-box per round become active works only for the first round.

---

<sup>83</sup>By the way the same holds if we replace the permutation by a more general bijective linear map.

*Inserting a permutation into the round function of an SP-network has the effect that linear cryptanalysis has to deal with more than one parallel S-box becoming active in later rounds.*

We'll soon see in the example that this effect reduces the potential. The relevant bits  $a'_3, a'_4, a'_5$ , or, after adding the key,  $a'_3 + k_{11}, a'_4 + k_{12}, a'_5 + k_{13}$ , split as input to the left S-box  $S_0$  of the second round (namely  $a'_3 + k_{11}$ ), and to the right one,  $S_1$  (namely  $a'_4 + k_{12}$  and  $a'_5 + k_{13}$ ).

On the left-hand side, for  $S_0$ , the linear form for the input is  $\beta'_1 \doteq 0001 \doteq 1$ , on the right-hand side, for  $S_1$ , we have  $\beta'_2 \doteq 1100 \doteq 12$ . From the linear profile of  $S_0$  we see that the maximum possible potential for  $\beta'_1$  is  $\lambda'_1 = \frac{9}{16}$  with  $p'_1 = \frac{7}{8}$ , assumed for  $\gamma_1 \doteq 13 \doteq 1101$ .

For  $\beta'_2$  the maximum potential is  $\lambda''_2 = \frac{1}{4}$ . Having two choices we choose  $\gamma_2 \doteq 6 \doteq 0110$  with probability  $p''_2 = \frac{3}{4}$ . The combined linear relation with  $\beta'(x) = \beta'_1(x_0, \dots, x_3) + \beta'_2(x_4, \dots, x_7)$  and, on the output side,  $\gamma(y) = \gamma_1(y_0, \dots, y_3) + \gamma_2(y_4, \dots, y_7)$  has I/O-correlation

$$2p_2 - 1 = (2p'_1 - 1)(2p''_2 - 1) = \frac{3}{8}$$

by Theorem 8.2.8, hence  $p_2 = \frac{11}{16}$ ,  $\lambda_2 = \frac{9}{64}$ .

The relation between  $\beta'(a' + k^{(1)})$  and  $\gamma(b'')$ , namely

$$\gamma(b'') \xrightarrow{p_2} \beta'(a' + k^{(1)}) = \beta'(a') + \beta'(k^{(1)}),$$

is labelled by 3 and written in explicit form in Figure 8.12. Combining 2 and 3 (and cancelling  $k_3$ ) yields the relation

$$\gamma(c) + \gamma(k^{(0)}) = \gamma(c + k^{(0)}) = \gamma(b'') \xrightarrow{p} \alpha(a) + \alpha(k^{(0)}) + \beta'(k^{(1)}),$$

given explicitly and labelled by 4 in the figure. Its probability  $p$  is given by Theorem 8.2.7 since the two round keys are independent. We get

$$2p - 1 = (2p_1 - 1)(2p_2 - 1) = \frac{3}{4} \cdot \frac{3}{8} = \frac{9}{32},$$

whence  $p = \frac{41}{64}$ . The corresponding potential is  $\lambda = \frac{81}{1024}$ .

The number  $N$  of needed plaintexts for a 95% success probability follows from the approximation in Table 8.11:

$$N = \frac{3}{\lambda} = \frac{1024}{27} \approx 38.$$

Note that there are only 256 possible plaintexts at all.

In the example the success probability derived from the product of the I/O-correlations (or of the potentials) of all active S-boxes. We had luck since in this example the involved partial keys were independent. In the general case this is not granted. Nevertheless the cryptanalyst relies on the empirical evidence and ignores the dependencies, trusting the rule of thumb:

*The success probability of linear cryptanalysis is (approximately) determined by the multiplicativity of the I/O-correlations (or of the potentials) of all the active S-boxes along the considered path (including all of its ramifications).*

The restriction in this rule of thumb concerns the *success probability* of linear cryptanalysis but not the *course of action*. The cryptanalyst is right if and only if she succeeds, no matter whether her method had an exact mathematical foundation for all details.

Now we obtained a single bit. So what?

Of course we may find more relations, and detect more key bits. However we have to deal with smaller and smaller potentials, and face an increasing danger of hitting a case where the probability for the concrete (target) key lies on the “wrong” side of  $\frac{1}{2}$ . Moreover we run into a multiple test situation reusing the same known plaintexts several times. This enforces an unpleasant adjustment of the success probabilities.

### Systematic Search for Linear Relations

The search for useful linear relations over several rounds has no general elegant solution. The published examples often use linear paths that more or less appear from nowhere, and it is not evident that they are the best ones.

Let  $n$  be the block length of the cipher, and  $r$ , the number of rounds. Then for each round the choice is between  $2^n$  linear forms, making a total of  $2^{n(r+1)}$  choices. This number also specifies the cost of determining the best relation by complete search. There are some simplifications that however don't reduce the order of magnitude of the cost:

- In the first round consider only linear forms that activate only one S-box.
- Then choose the next linear form such that it activates the least possible number of S-boxes of the next round (with high, but not necessarily maximum potential).
- If one of the relations in a linear path has probability  $\frac{1}{2}$ , or I/O-correlation 0, then the total I/O-correlation is 0 by multiplicativity, and this path may be neglected. The same is true componentwise if the linear forms split among the S-boxes of the respective round. However this negligence could introduce errors since we deal with average probabilities not knowing the key-dependent ones.

For our 2-round example with Mini-Lucifer the systematic search is feasible by pencil and paper or by a SageMath or Python script. Our example has the following characteristics:

- $\alpha = (\alpha_1, \alpha_2)$ <sup>84</sup> with  $\alpha_1 \hat{=} 1 \hat{=} 0001$  and  $\alpha_2 \hat{=} 0 \hat{=} 0000$
- $\beta = (\beta_1, \beta_2)$  with  $\beta_1 \hat{=} 13 \hat{=} 1101$  and  $\beta_2 \hat{=} 0 \hat{=} 0000$
- $\beta' = (\beta'_1, \beta'_2)$  with  $\beta'_1 \hat{=} 1 \hat{=} 0001$ ,  $\beta'_2 \hat{=} 12 \hat{=} 1100$
- $\gamma = (\gamma_1, \gamma_2)$  with  $\gamma_1 \hat{=} 13 \hat{=} 1101$ ,  $\gamma_2 \hat{=} 6 \hat{=} 0110$
- $\tau_1 = \frac{3}{4}$ ,  $\tau'_1 = \frac{3}{4}$ ,  $\tau''_1 = \frac{1}{2}$ ,  $\tau_2 = \frac{3}{8}$ ,  $\tau = \frac{9}{32}$ ,  $p = \frac{41}{64} = 0,640625$
- $c_0 + c_1 + c_3 + c_5 + c_6 \approx a_3 + k_0 + k_1 + k_5 + k_6 + k_{11} + k_{12} + k_{13}$

An alternative choice of  $\gamma_2$  is  $\gamma_2 \hat{=} 14 \hat{=} 1110$ ; this yields a linear path with the characteristics

- $\alpha \hat{=} (1, 0)$ ,  $\beta \hat{=} (13, 0)$ ,  $\beta' \hat{=} (1, 12)$ ,  $\gamma \hat{=} (13, 14)$ 
  - $\tau = -\frac{9}{32}$ ,  $p = \frac{23}{64} = 0,359375$
  - $c_0 + c_1 + c_3 + c_4 + c_5 + c_6 \approx a_3 + k_0 + k_1 + k_4 + k_5 + k_6 + k_{11} + k_{12} + k_{13}$

---

<sup>84</sup> $\alpha_1$  was formerly denoted  $\alpha$ . Now for uniformity we make both components of all linear forms explicit and index them by 1 and 2.

The systematic search finds two even “better” linear paths, characterized by

- $\alpha \hat{=} (8, 0), \beta \hat{=} (8, 0), \beta' \hat{=} (1, 0), \gamma \hat{=} (13, 0)$ 
  - $\tau = -\frac{3}{8}, p = \frac{5}{16} = 0, 3125$
  - $c_0 + c_1 + c_3 \overset{p}{\approx} a_0 + k_1 + k_3 + k_{11}$
- $\alpha \hat{=} (15, 0), \beta \hat{=} (8, 0), \beta' \hat{=} (1, 0), \gamma \hat{=} (13, 0)$ 
  - $\tau = -\frac{3}{8}, p = \frac{5}{16} = 0, 3125$
  - $c_0 + c_1 + c_3 \overset{p}{\approx} a_0 + a_1 + a_2 + a_3 + k_2 + k_{11}$

that do not completely exhaust the potential of the single S-boxes but on the other hand activate only one S-box of the second round, and thereby show the larger potential  $\lambda = \frac{9}{64}$ . Thus we get a 95% success probability with only

$$N = \frac{3}{\lambda} = \frac{64}{3} \approx 21$$

known plaintexts for determining one bit.

The designer of a cipher should take care that in each round the active bits fan out over as many S-boxes as possible. The inventors of AES, Daemen<sup>85</sup> and Rijmen<sup>86</sup>, call this design approach “wide-trail strategy”<sup>87</sup>. Figure 8.13 shows an example of a linear path with all its ramifications.

### Example (Continued)

For an illustration of the procedure we generate 25 pairs of known plaintexts and corresponding ciphertexts using the key  $k \hat{=} 1001011000101110$ . The first part of SageMath sample 8.16 does this job. It uses the function `randsel()` from SageMath sample 8.15. This function delivers  $NN$  different integers<sup>88</sup> in the interval  $[0, 255]$ . SageMath sample 8.17 shows the results of a random execution.

---

#### SageMath sample 8.15 Generation of *different* random integers

---

```
def randsel(max,NN):
    """Generates NN different random integers between 0 and max."""
    rndlist = []
    while (len(rndlist) < NN):
        new = randint(0,max)
        if (not(new in rndlist)):
            rndlist.append(new)
    rndlist.sort()
    return rndlist
```

---

<sup>85</sup>Joan Daemen, Belgian cryptologist, coinventor of the AES cipher, \*1965

<sup>86</sup>Vincent Rijmen, Belgian cryptologist, coinventor of the AES cipher, \*1970

<sup>87</sup>The design of AES strengthens this effect by involving a linear map instead of a mere permutation, thereby replacing the “P” of an SP-network by an “L”.

<sup>88</sup>in fact pseudo-random integers. More on this topic in Section 8.3

---

**SageMath sample 8.16** Linear cryptanalysis of Mini-Lucifer over 2 rounds

---

```
sage: key = str2bb1("1001011000101110")
sage: bit = [0,0,0,0]
sage: bit[0] = (key[0]+key[1]+key[5]+key[6]+key[11]+key[12]+key[13]) % 2
sage: bit[1] = (key[0]+key[1]+key[4]+key[5]+key[6]+key[11]+key[12]+key[13])%2
sage: bit[2] = (key[1]+key[3]+key[11]) % 2
sage: bit[3] = (key[2]+key[11]) % 2
sage: NN = 25
sage: plist = randsel(255,NN)
sage: klist = [[],[],[],[]]
sage: for i in range (0,NN):
....:     plain = int2bb1(plist[i],8)
....:     ciph = miniLuc(plain,key,2)
....:     print("pc pair nr", i+1, "is", plain, ciph)
....:     kbit = (plain[3]+ciph[0]+ciph[1]+ciph[3]+ciph[5]+ciph[6]) % 2
....:     klist[0].append(kbit)
....:     kbit = (1+plain[3]+ciph[0]+ciph[1]+ciph[3]+ciph[4]+ciph[5]+ciph[6])%2
....:     klist[1].append(kbit)
....:     kbit = (1+plain[0]+ciph[0]+ciph[1]+ciph[3]) % 2
....:     klist[2].append(kbit)
....:     kbit=(1+plain[0]+plain[1]+plain[2]+plain[3]+ciph[0]+ciph[1]+ciph[3])%2
....:     klist[3].append(kbit)
[...]
sage: for j in range(0,4):
....:     sum = 0
....:     for jj in range(0,NN):
....:         sum += klist[j][jj]
....:     if (bit[j] == 0):
....:         sum = NN - sum
....:     print("True bit:", bit[j], klist[j])
....:     print("    Relation", j+1, ":", sum, "of", NN , "guesses are correct.")
[...]
```

---

The target key bits (that we know in cheat mode) are `bit[j]` for  $j = 0, 1, 2, 3$ . We use all four good relations at the same time without fearing the possible reduction of the success probability. All of these relations assert the probable equality of the bits `bit[j]` and the sums of plain and cipher bits in the rows `kbit`. For the last three of these sums we have to take the complementary bits since the corresponding I/O-correlations are negative (the probabilities are  $< \frac{1}{2}$ ). This is done by adding the bit 1.

SageMath sample 8.18 shows the result of the analysis. As we see our guess is correct for all four bits<sup>89</sup>.

---

<sup>89</sup>We didn't use the function `Matsui_Test()` in order to gain some insight into the intermediate results.

---

**SageMath sample 8.17** 25 random pairs of plaintexts and ciphertexts from Mini-Lucifer over 2 rounds

---

```
pc pair nr 1 is [0,0,0,0,1,1,1,1] [0,0,0,0,1,0,1,0]
pc pair nr 2 is [0,0,0,1,0,0,0,1] [1,1,0,0,1,1,1,0]
pc pair nr 3 is [0,0,0,1,0,1,1,0] [1,1,0,0,1,0,0,1]
pc pair nr 4 is [0,0,1,1,1,1,0,1] [1,0,1,1,0,0,1,0]
pc pair nr 5 is [0,1,0,0,0,0,0,0] [1,1,1,0,0,1,1,1]
pc pair nr 6 is [0,1,0,0,1,0,0,0] [0,1,0,1,0,1,1,1]
pc pair nr 7 is [0,1,0,0,1,1,0,0] [1,1,1,0,1,0,1,0]
pc pair nr 8 is [0,1,0,0,1,1,0,1] [0,1,0,1,1,1,0,0]
pc pair nr 9 is [0,1,0,0,1,1,1,1] [0,1,1,1,1,0,1,0]
pc pair nr 10 is [0,1,1,0,0,1,1,1] [0,0,1,1,0,0,1,1]
pc pair nr 11 is [1,0,0,0,0,0,1,1] [1,1,1,1,0,1,0,0]
pc pair nr 12 is [1,0,0,1,0,0,1,1] [0,1,1,0,1,0,1,1]
pc pair nr 13 is [1,0,0,1,1,0,0,0] [0,1,1,0,0,1,1,1]
pc pair nr 14 is [1,0,1,0,1,0,1,1] [1,1,0,1,1,0,0,1]
pc pair nr 15 is [1,0,1,1,0,0,0,1] [1,1,0,0,1,0,0,0]
pc pair nr 16 is [1,0,1,1,0,0,1,0] [1,0,1,0,0,1,0,0]
pc pair nr 17 is [1,0,1,1,0,1,1,0] [1,1,0,0,0,1,0,0]
pc pair nr 18 is [1,0,1,1,1,0,0,1] [1,1,0,0,0,0,0,1]
pc pair nr 19 is [1,0,1,1,1,1,0,1] [1,0,1,1,1,1,1,1]
pc pair nr 20 is [1,1,0,0,0,1,0,0] [0,1,0,0,1,1,1,1]
pc pair nr 21 is [1,1,0,0,0,1,1,1] [0,0,1,1,1,1,1,1]
pc pair nr 22 is [1,1,0,1,1,1,1,1] [1,1,0,1,1,0,1,0]
pc pair nr 23 is [1,1,1,0,0,0,0,0] [1,1,1,0,1,1,1,0]
pc pair nr 24 is [1,1,1,0,0,1,0,0] [0,1,1,1,0,0,1,1]
pc pair nr 25 is [1,1,1,1,0,1,0,1] [1,1,1,1,0,1,0,1]
```

---

---

**SageMath sample 8.18** Linear cryptanalysis of Mini-Lucifer over 2 rounds

---

```
True bit: 1 [1,1,1,0,0,0,1,1,1,0,0,1,0,1,1,1,0,1,1,1,1,0,1,1]
      Relation 1 : 17 of 25 guesses are correct.
True bit: 1 [1,1,1,1,1,1,1,1,1,1,1,1,1,0,1,0,1,1,1,1,0,0,0]
      Relation 2 : 20 of 25 guesses are correct.
True bit: 1 [1,1,1,1,1,1,1,1,1,0,1,1,1,1,0,0,0,1,1,1,1,0,0,1]
      Relation 3 : 18 of 25 guesses are correct.
True bit: 0 [1,0,0,1,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0]
      Relation 4 : 20 of 25 guesses are correct.
```

---

As a consequence of our analysis we get a system of four linear equations for the 16 unknown key bits:

$$\begin{aligned} 1 &= k_0 + k_1 + k_5 + k_6 + k_{11} + k_{12} + k_{13} \\ 1 &= k_0 + k_1 + k_4 + k_5 + k_6 + k_{11} + k_{12} + k_{13} \\ 1 &= k_1 + k_3 + k_{11} \\ 0 &= k_2 + k_{11} \end{aligned}$$

that allow us to reduce the number of keys for an exhaustion from  $2^{16} = 65536$  to  $2^{12} = 4096$ . Note the immediate simplifications of the system:  $k_{11} = k_2$  from the last equation, and  $k_4 = 0$  from the first two.

As a cross-check we run some more simulations. The next four yield

- 15, 16, 19, 16
- 15, 16, 13, 17
- 15, 20, 19, 17
- 19, 19, 20, 18

correct guesses, and so on. Only run number 10 produced a wrong bit (the second one):

- 17, 12, 14, 17

then again run number 25. Thus empirical evidence suggests a success probability of at least 90% in this scenario.

## Analysis over Four Rounds

Now let's explore how an increasing number of rounds impedes linear cryptanalysis.

Consider the toy cipher Mini-Lucifer over four rounds. Searching an optimal linear path over four rounds is somewhat expensive, so we content ourselves with extending the best example from the two round case, the third one, over two additional rounds. Slightly adapting the notation we get:

- for the first round  $\beta_0 = \alpha \doteq (8, 0)$  and  $\beta_1 \doteq (8, 0)$  (the “old”  $\beta$ ) with  $\tau_1 = -\frac{1}{2}$ ,
- for the second round (applying the permutation  $P$  to  $\beta_1$ )  $\beta'_1 \doteq (1, 0)$  and  $\beta_2 \doteq (13, 0)$  (the “old”  $\gamma$ ) with  $\tau_2 = \frac{3}{4}$ ,
- for the third round  $\beta'_2 \doteq (1, 12)$  and  $\beta_3 \doteq (13, 6)$  with  $\tau_3 = \frac{3}{8}$ ,
- for the fourth round  $\beta'_3 \doteq (5, 13)$  and  $\beta = \beta_4 \doteq (3, 12)$  (the “new”  $\beta$ ) with  $\tau_4 = -\frac{1}{4}$ .

Figure 8.13 shows this linear path with its ramifications.

The repeated round keys we used are not independent. Therefore multiplicativity of I/O-correlations is justified by the rule of thumb only yielding an approximate value for the I/O-correlation of the linear relation  $(\alpha, \beta)$  over all of the four rounds:

$$\tau \approx \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{3}{8} \cdot \frac{1}{4} = \frac{9}{256} \approx 0,035.$$

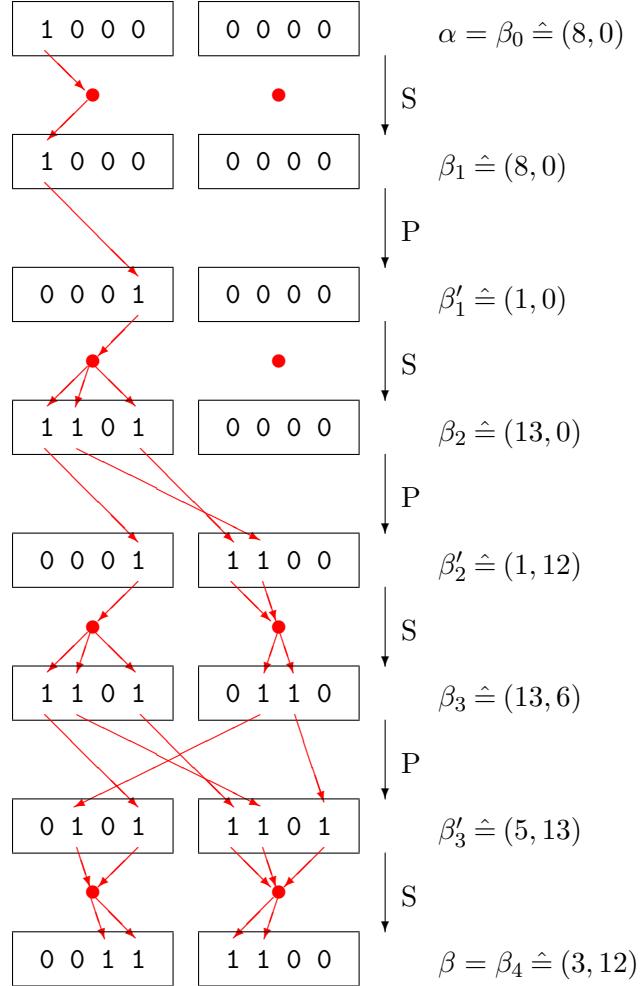


Figure 8.13: A linear path with ramifications (“trail”). For S the linear form in the range is chosen (for high potential), indicated by a red dot. For P the linear form in the range results by applying the permutation.

The other characteristics are

$$p \approx \frac{265}{512} \approx 0,518, \quad \lambda \approx \frac{81}{65536} \approx 0,0012, \quad N \approx \frac{65536}{27} \approx 2427,$$

the last one being the number of needed known plaintexts for a 95% success probability.

Comparing this with the cost of exhaustion over all 65536 possible keys we seem to have gained an advantage. However there are only 256 different possible plaintexts all together. So linear cryptanalysis completely lost its sense by the increased number of rounds.

### 8.2.13 Outlook

As we saw linear cryptanalysis provides some evidence for the security of a cipher, in particular for choosing the number of rounds. But only some parts of the theory have a mathematically satisfying basis. Most existing publications only give ad-hoc analyses of concrete ciphers. For example Matsui showed how for DES  $2^{43}$  known plaintexts reveal 14 key bits with high certainty,

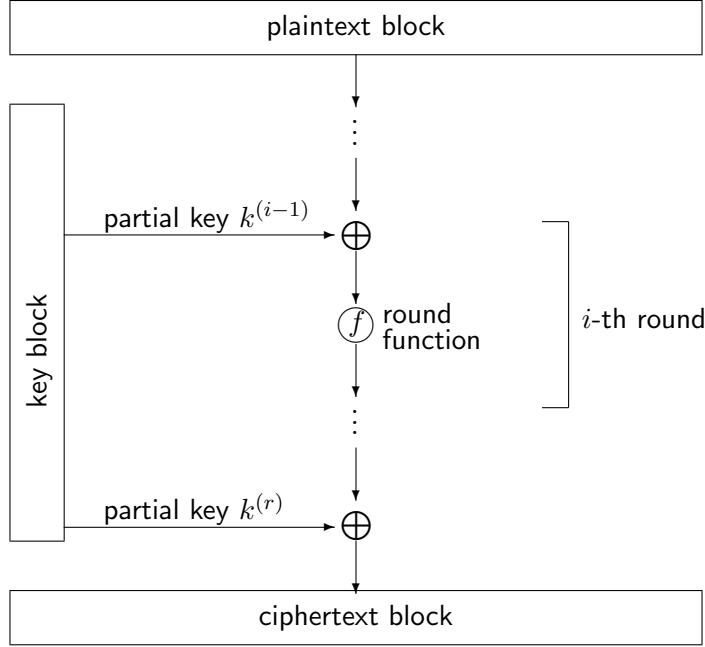


Figure 8.14: Structure of AES in the large

reducing the exhaustion to the remaining  $42 = 56 - 14$  key bits, a feasible task (at least if the analyst gets that many plaintexts).

The treatment of linear cryptanalysis serves as an example of similar analyses. Differential cryptanalysis as well as generalized and mixed variants follow similar lines of thought. For more information see the book [Sti06] that also explicitly specifies the most important ciphers DES and AES.

## AES

Figures 8.14 and 8.15 show the design of AES<sup>90</sup> in the large and show the realization of the design principles derived in the former subsections.

- The block length is  $n = 128$ , the key length,  $l = 128, 192$ , or  $256$ , the number of rounds,  $r = 10, 12$ , or  $14$ .
- At the beginning of each round and after the last round a partial key is added to the current bitblock, as in examples A, B, C, Figures 8.5, 8.8, 8.9. The complete algorithm involves  $r + 1$  partial keys.
- The 128-bit “partial keys”  $k^{(i)}$  are not partial keys in the proper sense but extracted from the master key  $k$  by a somewhat involved algorithm (“key schedule”). They are not independent.
- Each round starts by splitting the current 128-bit block into 16 parts each consisting of 8 bits. Each of these parts is fed into the same S-box  $S: \mathbb{F}_2^8 \rightarrow \mathbb{F}_2^8$ . This S-box has a mathematically quite elegant description that however assumes some advanced knowledge

---

<sup>90</sup>in CrypTool 2 found under “modern” / “symmetric”.

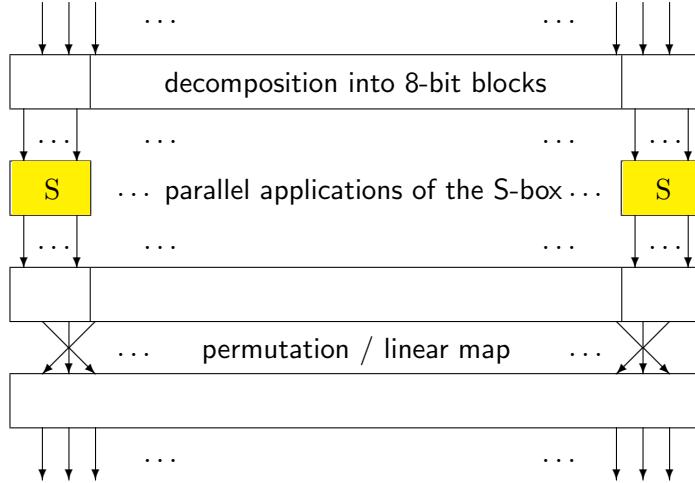


Figure 8.15: The round function  $f$  of AES

of abstract algebra, hence is omitted here. The linear potential of the S-box is  $\frac{1}{64}$ . The method `linProf()` of the class `boolMap`, SageMath sample 8.47, explicitly confirms this, but there is also a “deep” mathematical reason<sup>91</sup>.

- The “diffusion step” consists of a permutation followed by a linear map. This step is slightly more complex than for a pure SP-network as in Figure 8.2.

A final remark on the key schedule: If the round keys,  $k^{(i)}$  in our notation, are not simply partial keys but extracted by a more complex procedure, then the true master key is concealed. The cryptanalyst however attacks the “effective” key, consisting of the round keys  $k^{(i)}$ . This suffices to break the cipher. Nevertheless a complex key schedule makes a cipher more secure for it detains the attacker from explicitly utilizing the dependence of the various round keys, in particular if they consist of overlapping blocks of master key bits.

---

<sup>91</sup>that as a mathematical miracle involves counting the points of elliptic curves over finite fields of characteristic 2

## 8.3 Bitstream Ciphers

A bitstream cipher sequentially encrypts each single bit of a bitstring by an individual rule. The two possibilities are: leave the bit unchanged, or negate it. Leaving the bit unchanged is equivalent with (binary) adding 0, negating the bit is equivalent with adding 1. Thus every bitstream cipher may be interpreted as an XOR encryption<sup>92</sup> in the sense of the following subsection 8.3.1. We distinguish between

**synchronous bitstream ciphers** where the key stream is generated independently of the plaintext,

**asynchronous bitstream ciphers** where the key stream depends on the plaintext or other context parameters.

In this chapter we only treat synchronous bitstream ciphers. We also exclude stream ciphers over other character sets than the bits 0, 1.

### 8.3.1 XOR Encryption

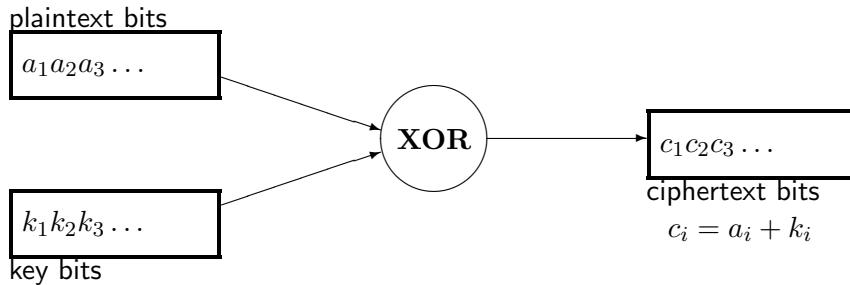


Figure 8.16: The principle of XOR encryption

The basic method of bitstream encryption is simply denoted by XOR. It interprets plaintexts<sup>93</sup> as sequences of bits. Also the key is a bit sequence, called **key stream**. The encryption algorithm adds the current bit of the plaintext and the current bit of the key stream by XOR. Figure 8.16 illustrates the algorithm<sup>94</sup>, Figure 8.17 shows an example.

$$\begin{array}{r}
 \text{a: } 01000100011101 \dots \\
 \text{k: } 10010110100101 \dots \\
 \hline
 \text{c: } 11010010111000 \dots
 \end{array}$$

Figure 8.17: Example of XOR encryption

<sup>92</sup>the key being the “difference” between ciphertext and plaintext as in Figure 8.19

<sup>93</sup>The SageMath method `ascii_to_bin()` from the module `sage.crypto.util` converts “ordinary” texts to bitstrings.

The inverse method is `bin_to_ascii()`. However these bitstrings belong to the class `StringMonoidElement`, so need a proper context. Therefore in SageMath sample 8.38 we define functions `txt2bb1` that transforms ASCII texts to bitblocks, and `bb12str` that transforms bitblocks to bitstrings.

<sup>94</sup>In CrypTool 2 under “classic”/ “XOR”, in Appendix 8.4.3 as `xor`, see SageMath sample 8.39.

In the twenties of the 20th century XOR ciphers were invented to encrypt teleprinter messages. These messages were written on five-hole punched tapes as in Figure 8.18. Another punched tape provided the key stream. Vernam<sup>95</sup> filed his U.S. patent in 1918. He used a key tape whose ends were glued together, resulting in a periodic key stream. Mauborgne<sup>96</sup> immediately recognized that a nonperiodic key is obligatory for security.

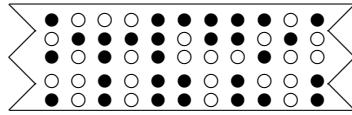


Figure 8.18: Punched tape—each column represents a five-bit character

In its strongest form, the one-time pad (OTP), XOR encryption is an example for perfect security in the sense of Shannon. As algorithm A5 or E<sub>0</sub> XOR helps to secure mobile phones or the Bluetooth protocol for wireless data transmission. As RC4 it is part of the SSL protocol that (sometimes) encrypts client-server communication in the World Wide Web, and of the PKZIP compression software. There are many other current applications, not all of them fulfilling the expected security requirements.

*The scope of XOR encryption ranges from simple ciphers that are trivially broken to unbreakable ciphers.*

### Advantages of XOR ciphers:

- Encryption and decryption are done by the same algorithm: Since  $c_i = a_i + k_i$  also  $a_i = c_i + k_i$ . Thus decryption also consists of adding key stream and ciphertext (elementwise binary).
- The method is extremely simple to understand and to implement
- ... and very fast—provided that the key stream is available. For high transfer rates one may precompute the key stream at both ends of the line.
- If the key stream is properly chosen the security is high.

### Disadvantages of XOR ciphers:

- XOR ciphers are vulnerable for known plaintext attacks: each correctly guessed plaintext bit reveals a key bit.
- If the attacker knows a piece of plaintext she also knows the corresponding piece of the key stream, and then is able to exchange this plaintext at will. For example she might replace “I love you” by “I hate you”, or replace an amount of \$1000 by \$9999. In other words the integrity of the message is poorly protected<sup>97</sup>.
- XOR ciphers provide no diffusion in the sense of Shannon’s criteria since each plaintext bit affects the one corresponding plaintext bit only<sup>98</sup>.

<sup>95</sup>Gilbert Vernam, U. S. American engineer, April 3, 1890 – February 7, 1960

<sup>96</sup>Joseph Mauborgne, Major General in the U. S. Army, February 26, 1881 – July 7, 1971

<sup>97</sup>To protect message integrity the sender has to implement an additional procedure.

<sup>98</sup>Block ciphers in the opposite were designed for using diffusion.

- Each reuse of a part of the key sequence (also in form of a periodic repetition) opens the door for an attack. The historical successes in breaking stream ciphers almost always used this effect, for example the attacks on encrypted teleprinters in World War II, or the project Venona during the Cold War.

A remark on the first item, the vulnerability for attacks with known plaintext: The common ISO character set for texts has a systematic weakness. The 8-bit codes<sup>99</sup> of the lower-case letters a..z all start with 011, of the upper-case letters A..Z, with 010. A supposed sequence of six lower-case letters (no matter which) reveals  $6 \cdot 3 = 18$  key bits.

In other words: We cannot prevent the attacker from getting or guessing a good portion of the plaintext. Thus the security against an attack with known plaintext is a fundamental requirement for an XOR cipher, even more than for any other cryptographic procedure.

### 8.3.2 Generating the Key Stream

The main naive methods for generating the key stream are:

- periodic bit sequence,
- running-text,
- “true” random sequence.

A better method uses a

- pseudo-random sequence

and leads to really useful procedures. The essential criterion is the quality of the pseudo-random generator.

#### Periodic Bit Sequence

**Example** We generate a key stream of period 8 by repeating  $k = 10010110$ . We represent letters by bytes in the ISO character set.

c		o		m		e				a	
a:	01000011 01101111 01101101 01100101 00100000 01100001										
k:	10010110 10010110 10010110 10010110 10010110 10010110										
	-----										
c:	11010101 11111001 11111011 11110011 10110110 11110111										
	-----										
	-----										
t				n		i		n		e	
01110100 00100000 01101110 01101001 01101110 01100101											
10010110 10010110 10010110 10010110 10010110 10010110											
-----											
11100010 10110110 11111000 11111111 11111000 11110011											
	-----										

---

<sup>99</sup>By the way the appearance of many zeroes in the leading bits of the bytes is an important identifying feature of texts in many european languages.

This encryption is easily done by hand, or by SageMath sample 8.19.

If we re-encode the ciphertext bytes in the ISO-8859-1 character set the cryptogram looks like this:

Ó ù ó ¶ ÷ â ¶ ø ý ø ó

This might bedazzle laypersons. An expert immediately notes that all characters are from the upper half of the possible 256 bytes. This observation suggests that the plaintext is in natural language, encrypted with a key whose leading bit is 1. If the attacker guesses that the conspicuous character ¶ = 10110110 corresponds to the space character 00100000, she derives the key as the difference 10010110. This breaks the cryptogram.

*Known or probable plaintext easily breaks periodic XOR encryption.*

---

#### SageMath sample 8.19 XOR encryption in Python/SageMath

---

```
sage: testtext = "Come at nine"
sage: bintext = txt2bb1(testtext)
sage: binstr = bb12str(bintext); binstr
'010000110110111101101101011001010010000001100001
011101000010000001101110011010010110111001100101'
sage: testkey = [1,0,0,1,0,1,1,0]
sage: keystr = bb12str(testkey); keystr
'10010110'
sage: cipher = xor(bintext,testkey)
sage: ciphstr = bb12str(cipher); ciphstr
'110101011111001111101111100111011011011110111
111000101011011110001111111111100011110011'
```

---

## MS Word and Periodic XOR

The following table (generated ad hoc by simple character counts) shows the frequencies of the most frequent bytes in MS Word files.

byte (hexadecimal)	bits	frequency
00	00000000	7–70%
01	00000001	0.8–17%
20 (space)	00100000	0.8–12%
65 (e)	01100101	1–10%
FF	11111111	1–10%

Note that these frequencies relate to the binary files, heavily depend on the type of the document, and may change with every software version. The variation is large, we often find unexpected peaks, and all bytes 00–FF occur. But all this doesn't matter here since we observe

*long chains of 00 bytes.*

For an MS Word file that is XOR encrypted with a periodically repeated key the ubiquity of zeroes suggests an efficient attack: Split the stream of ciphertext bits into blocks corresponding to

the length of the period<sup>100</sup> and add the blocks pairwise. If one of the plaintext blocks essentially consists of zeroes, then the sum is readable plaintext. Why? Consider the situation

	...	block 1	...	block 2	...
plaintext:	...	$a_1 \dots a_s$	...	0 $\dots$ 0	...
key:	...	$k_1 \dots k_s$	...	$k_1 \dots k_s$	...
ciphertext:	...	$c_1 \dots c_s$	...	$c'_1 \dots c'_s$	...

where  $c_i = a_i + k_i$  and  $c'_i = 0 + k_i = k_i$  for  $i = 1, \dots, s$ . Thus the key reveals itself in block 2, however the attacker doesn't recognize this yet. But tentatively pairwise adding all blocks she gets (amongst other things)

$$c_i + c'_i = a_i + k_i + k_i = a_i \quad \text{for } i = 1, \dots, s,$$

that is, a plaintext block. If she realizes this (for example recognizing typical structures), then she sees the key  $k_1, \dots, k_s$ .

Should it happen that the sum of two ciphertext blocks is zero then the ciphertext blocks are equal, and so are the corresponding plaintext blocks. The probability that both of them are zero is high. Thus the key could immediately show through. To summarize:

*XOR encryption with a periodic key stream is quite easily broken for messages with a known structure.*

This is true also for a large period, say 512 bytes = 4096 bits, in spite of the hyperastronomically huge key space of  $2^{4096}$  different possible keys.

## Running-Text Encryption

A classical approach to generating an aperiodic key is taking a data stream, or file, or text, that has at least the length of the plaintext. In classical cryptography this method was called running-text encryption, and the keys were taken from books<sup>101</sup> beginning at a certain position. The main method of breaking the cipher was finding or guessing the book. The same weakness affects the electronic analog that uses a file, or a CD, or DVD: As soon as the attacker knows the source of the key bits the key space is much too small—exhausting a file of several gigabytes is easily done, the costs are linear in the size of the file.

But even when the attacker is unable to guess the source of the key bits ciphertext-only cryptanalysis is possible: Plaintexts as well as keys contain structures that are not completely concealed by binary addition. We won't discuss this here<sup>102</sup> but summarize:

*XOR encryption with running-text keys is fairly easily broken.*

## True Random Sequence

The extreme choice for a key is a true random sequence of bits as key stream. Then the cipher is called **(binary) one-time pad (OTP)**. In particular no part of the key stream must be

---

<sup>100</sup>If the length of the period is unknown, determine it by the methods for periodic polyalphabetic substitutions from classical cryptanalysis named after Kasiski, Friedman, or Sinkov. Or simply try all possible lengths.

<sup>101</sup>one of several ways of using a book as cryptographic key that in classical cryptography are denoted as “book ciphers”

<sup>102</sup>JCrypTool offers an automatic recognition of plaintext and key that uses ciphertext only, see “Analysis”/“Viterbi analysis”.

repeated at any time. The notation “pad” comes from the idea of a tear-off calendar—each sheet is destroyed after use. This cipher is unbreakable, or “perfectly secure”. Shannon gave a formal proof of this, see [Sti06].

Without mathematical formalism the argument is as follows: The ciphertext divulges no information about the plaintext (except the length). It could result from *any* plaintext of the same length: simply take the (binary) difference of ciphertext and alleged plaintext as key. Consider the ciphertext  $c = a + k$  with plaintext  $a$  and key  $k$ , all represented by bitstreams and added bit by bit as in Figure 8.16. For an arbitrary different plaintext  $b$  the formula  $c = b + k'$  likewise shows a valid encryption using  $k' = b + c$  as key.

This property of the OTP could be used in a scenario of forced decryption<sup>103</sup> to produce an innocuous plaintext, as exemplified in Figure 8.19.

If the one-time pad is perfect—why not use it in any case?

- The key management is unwieldy: Key agreement becomes a severe problem since the key is as long as the plaintext and awkwardly to memorize. Thus the communication partners have to agree on the key stream prior to transmitting the message, and store it. Agreeing on a key only just in time needs a secure communication channel—but if there was one why not use it to transmit the plaintext in clear?
- The key management is inappropriate for mass application or multi-party communication because of its complexity that grows with each additional participant.
- The problem of message integrity requires an extended solution for OTP like for any XOR cipher.

There is another, practical, problem when encrypting on a computer: How to get random sequences? “True random” bits arise from physical events like radioactive decay, or thermal noise on an optical sensor. The apparently deterministic machine “computer” can also generate true random bits, for instance by special chips that produce usable noise. Moreover many events are unpredictable, such as the exact mouse movements of the user, or arriving network packets that, although not completely random, contain random ingredients that may be extracted. On Unix systems these are provided by `/dev/random`.

However these random bits, no matter how “true”, are not that useful for encryption by OTP. The problem is on the side of the receiver who cannot reproduce the key. Thus the key stream must be transmitted independently.

There are other, useful, cryptographic applications of “true” random bits: Generating keys for arbitrary encryption algorithms that are unpredictable for the attacker. Many cryptographic protocols rely on “nonces” that have no meaning except for being random, for example the initialization vectors of the block cipher modes of operation, or the “challenge” for strong authentication (“challenge-response protocol”).

For XOR encryption—as approximation to the OTP—algorithmically generated bit sequences are much more practicable. But the attacker should have no means to distinguish them from true random sequences. This is the essence of the concept “pseudo-randomness”, and generating pseudo-random sequences is of fundamental cryptologic relevance.

*XOR encryption with a pseudo-random key stream spoils the perfect security of the one-time pad. But if the pseudo-random sequence is cryptographically strong (Section 8.3.9) the attacker has no chance to exploit this fact.*

---

<sup>103</sup>also known as “rubber hose cryptanalysis”

Plain bits and text:

01010100 01101000 01101001 01110011 00100000 01101101	This m
01100101 01110011 01110011 01100001 01100111 01100101	essage
00100000 01101001 01110011 00100000 01101000 01100001	is ha
01111010 01100001 01110010 01100100 01101111 01110101	zardou
01110011 00101110	s.

Key bits:

11001000 11010110 00110011 11000000 00111011 10001110
00001000 11101111 01001001 11100101 10111100 10111001
00010010 11000110 01110011 11010111 11000100 01100000
11100110 00010111 01101010 10111011 00010101 11011000
11110000 01000010

Cipher bits:

10011100 10111110 01011010 10110011 00011011 11100011
01101101 10011100 00111010 10000100 11011011 11011100
00110010 10101111 00000000 11110111 10101100 00000001
10011100 01110110 00011000 11011111 01111010 10101101
10000011 01101100

Pseudokey bits:

11001000 11010110 00110011 11000000 00111011 10001110
00001000 11101111 01001001 11100101 10111100 10111001
00010010 11000110 01110011 11010111 11000101 01101111
11110010 00011001 01111011 10101010 00010101 11011000
11110000 01000010

Pseudodecrypted bits and text:

01010100 01101000 01101001 01110011 00100000 01101101	This m
01100101 01110011 01110011 01100001 01100111 01100101	essage
00100000 01101001 01110011 00100000 01101001 01101110	is in
01101110 01101111 01100011 01110101 01101111 01110101	nocuou
01110011 00101110	s.

Figure 8.19: XOR encryption of a hazardous message, and an alleged alternative plaintext

### 8.3.3 Pseudo-Random Generators

Pseudo-random generators mimic true random processes by deterministic algorithms. Usually such an algorithm is called random generator, omitting the prefix “pseudo” if there is no danger of confusion. The main difference between a pseudo-random sequence and a true random sequence is its reproducibility.

Cipher designers hope to approximate the ideal properties of the one-time pad by using a pseudo-random bit sequence as key stream treating the short start string as “effective key”. Even for random generators of modest quality the resulting ciphertexts are immune against statistical analyses. The all-dominant problem is the security against known plaintext attacks.

Thus the critical question for a pseudo-random sequence and for a random generator is:

*Given a known chunk (maybe fragmented) of the sequence, is there a way to determine some more bits of the sequence, be it forwards or backwards?*

For “classical” random generators that are popular in statistical applications and simulations the answer is YES, see Section 8.3.4. But we’ll learn about random generators that—presumably—are cryptographically secure. The cipher designer faces the problem of finding a good trade-off between efficiency and security.

The main serious methods of generating pseudo-random bit sequences, or key streams, are:

- (feedback) shift registers (FSR) and combinations thereof, with theoretical foundations in Boolean algebra,
- perfect random generators with theoretical foundations in number theory.

Figure 8.20 shows the schematic functionality of a random generator. It hides an inner state that changes with each step by a given algorithm. This algorithm is controlled by parameters some of which are “public”, but some of which are secret and serve as components of the key. The initial state (= start value) is a true random value and likewise secret. With each step the random generator outputs a value, depending on its current inner state, until an exterior intervention stops it.

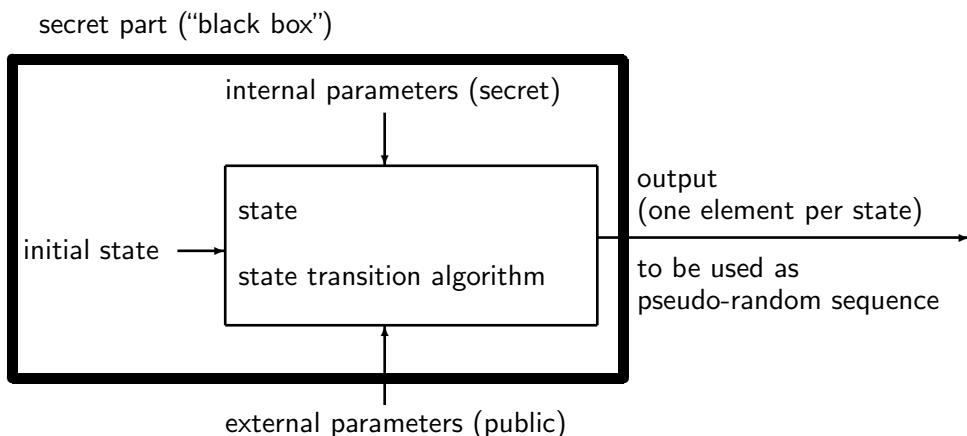


Figure 8.20: (Pseudo-)random generator

Thus the random generator transforms a short, truly random, bit sequence, the initial state, into a long pseudo-random sequence. Cryptologists call this effect “key expansion”.

## Feedback Shift Registers

Feedback shift registers (FSR) are a classical and popular method of generating pseudo-random sequences. The method goes back to Golomb<sup>104</sup> in 1955, but is often named after Tausworthe who picked up the idea in a 1965 paper. FSRs are especially convenient for hardware implementation.

An FSR of length  $l$  is specified by a Boolean function  $f: \mathbb{F}_2^l \rightarrow \mathbb{F}_2$ , the “feedback function”. Figure 8.21 shows the mode of operation. The output consists of the rightmost bit  $u_0$ , all the other bits are shifted to the right by one position, and the leftmost cell is filled by the bit  $u_l = f(u_{l-1}, \dots, u_0)$ . Thus the recursive formula

$$u_n = f(u_{n-1}, \dots, u_{l-1}) \quad \text{für } n \geq l \quad (8.9)$$

represents the complete sequence. SageMath sample 8.20 defines a general FSR with feedback function  $f$ , SageMath sample 8.21 uses it to generate a bit sequence by a concrete sample feedback function. Table 8.19 illustrates the stepping of the register. Note that the result doesn’t look convincingly random, and take this as warning that the choice of the parameters needs significantly more care.

The bits  $u_0, \dots, u_{l-1}$  form the start value. The key expansion transforms the short sequence  $u = (u_0, \dots, u_{l-1})$  (the effective key) of length  $l$  into a key stream  $u_0, u_1, \dots$  of arbitrary length<sup>105</sup>. Additionally in this context treating the internal parameters, that is the feedback function  $f$  or some of its coefficients, as components of the key makes sense. This makes the effective key length larger than  $l$ .

In this respect the realization in hardware differs from a software implementation: Hardware allows using an adjustable feedback function only by complex additional circuits. Thus in this case we usually assume an unchangeable feedback function, and (at least in the long run) we cannot prevent the attacker from figuring it out. In contrast a software implementation allows a comfortable change of the feedback function at any time such that it may serve as part of the key.

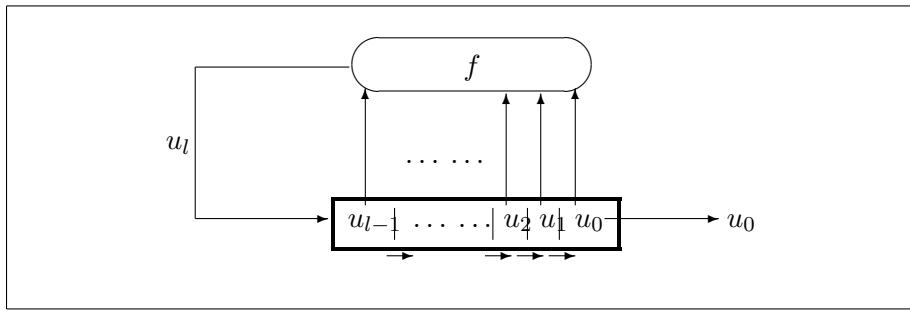


Figure 8.21: A feedback shift register (FSR) during the first iteration step. The Boolean function  $f$  calculates a new bit from the current state of the register. This new bit is slid in from the left.

## The Period of a Finite-State Machine

In computer science a feedback shift register is a special case of a finite-state machine. Therefore, the sequence of its states is periodic. Here is why.

<sup>104</sup>Solomon W. Golomb, U. S. American mathematician and engineer, \*May 30, 1932

<sup>105</sup>In a cryptographic context the bits  $u_0, u_1, \dots$  form the secret key stream. In other contexts there might be no need to conceal the output bits, but even then hiding the initial state might make sense, starting the output sequence at  $u_l$ .

---

**SageMath sample 8.20** A feedback shift register (FSR) in Python/SageMath

---

```
def fsr(f,x,n):
    """Generate a feedback shift register sequence.
    Parameters: Boolean function f, start vector x,
    number n of output bits."""
    u = x
    outlist = []
    for i in range (0,n):
        b = f.valueAt(u)
        c = u.pop()
        u.insert(0,b)
        outlist.append(c)
    return outlist
```

---

---

**SageMath sample 8.21** A (very poor) pseudo-random sequence in Python/SageMath

---

```
sage: bits = "1000010111001001"
sage: x = str2bb1(bits)
sage: f = BoolF(x)
sage: start = [0,1,1,1]
sage: bitlist = fsr(f, start, 32)
sage: print(bb12str(bitlist))
11101010101010101010101010101010
```

---

Let  $M$  be a finite set of  $m = \#M$  elements. Imagine  $M$  as the collection of all possible states of a machine. Consider a map (“transition”)

$$g: M \longrightarrow M.$$

For each element (“initial state”)  $x_0 \in M$  define a sequence  $(x_i)_{i \geq 0}$  in  $M$  by the recursive formula  $x_i = g(x_{i-1})$  for  $i \geq 1$ . After a preperiod of length  $\mu$  the sequence runs into a period of length  $\nu$ , see Figure 8.22, an explanation follows.

Since the set  $M$  is finite the states must eventually repeat. Thus there are smallest integers  $\mu \geq 0$  and  $\nu \geq 1$  such that  $x_{\mu+\nu} = x_\mu$ : To see this simply take  $\mu$  as the first index such that the element  $x_\mu$  reappears in the sequence at another position, and  $\mu + \nu$  as the first index where this repetition occurs. Then also (by induction)

$$x_{i+\nu} = g(x_{i+\nu-1}) = g(x_{i-1}) = x_i \quad \text{for } i > \mu.$$

feedback	state	output	
start	0111	→	1
$f(0111) = 1$	1011	→	1
$f(1011) = 0$	0101	→	1
$f(0101) = 1$	1010	→	0
$f(1010) = 0$	0101	from here on	periodic

Table 8.19: The stepping of a sample FSR

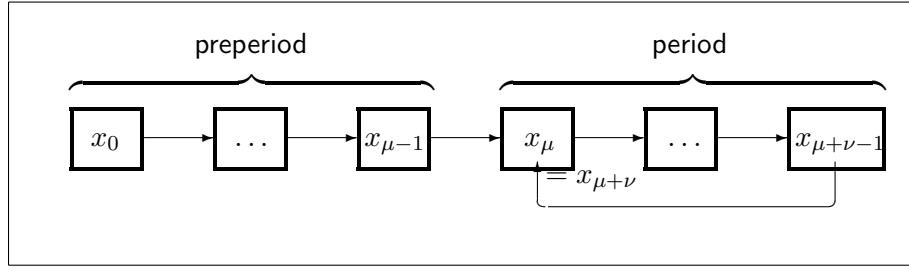


Figure 8.22: Period and preperiod

Here  $0 \leq \mu \leq m - 1$ ,  $1 \leq \nu \leq m$ ,  $\mu + \nu \leq m$ . The values  $x_0, \dots, x_{\mu+\nu-1}$  are all different, and the values  $x_0, \dots, x_{\mu-1}$  never reappear in the sequence.

**Definition:**  $\mu$  is called the (length of the) **preperiod**,  $\nu$ , the (length of the) **period**.

*A pseudo-random generator in the sense of Figure 8.20 inevitably generates periodic sequences. The best we can hope for is a period so huge that the practical application never exhausts its size.*

### Linear Shift Registers

The simplest and best understood instances of FSRs are the linear feedback shift registers (LFSR). Their feedback functions  $f$  are linear. From Section 8.1.9 we know that a linear function is simply a partial sum from an  $l$ -bit block:

$$f(u_{n-1}, \dots, u_{n-l}) = \sum_{j=1}^l s_j u_{n-j}, \quad (8.10)$$

where the coefficients  $s_j$  are 0 or 1. If  $I$  is the subset of indices  $j$  with  $s_j = 1$ , then the iteration (8.9) takes the form

$$u_n = \sum_{j \in I} u_{n-j}. \quad (8.11)$$

A simple graphical representation of an LFSR is shown in Figure 8.23. Here, the subset  $I$  defines the contacts (“taps”) that feed the respective cells into the feedback sum.

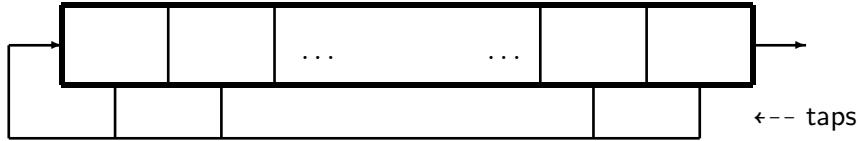


Figure 8.23: Simple graphical representation of an LFSR

For a good choice of the parameters—that we won’t discuss further—the sequence has a period of about  $2^l$ , the number of possible different states of the register, and statistical tests are hardly able to distinguish it from a uniformly distributed true random sequence [Gol82]. It is remarkable that such a simple approach generates pseudo-random sequences of fairly high quality! Of course the initial state  $u = (0, \dots, 0)$  is inappropriate. For an initial state  $\neq 0$  the

maximum possible period is  $2^l - 1$ . Without further explanation we remark that obtaining this period is easy<sup>106</sup>.

Using an LFSR for bitstream encryption the secret inner parameters—the coefficients  $s_1, \dots, s_l$ —as well as the initial state  $u_0, \dots, u_{l-1}$  together constitute the key. In contrast the length  $l$  of the register is assumed as known to the attacker.

SageMath sample 8.22 implements an LFSR<sup>107,108</sup> as function `lfsr()`<sup>109</sup>; the output is a pseudo-random bitstream. A sample call of this function for an LFSR of length 16, generating 1024 bits, is in SageMath sample 8.23, the output, in Table 8.20 (without parentheses or delimiters).

We could apply a series of statistical tests to this bitstream, for example tests of uniform distribution, and would always see good results. Instead we visualize the sequence in Figure 8.24 for optical inspection—of course an even more insufficient proof. However the superficial impression shows a quite random sequence. SageMath sample 8.23 generated this picture.

Don't take offense at the sequence of nine (black) ones in the third to last row: The probability of nine ones in nine random bits is  $(1/2)^9 = 1/512$ . Therefore in a random bitstream of length 1024 a “run” of this kind occurs with high probability.

*Neither the usual statistical tests nor the visual impression are valid testimonials of the quality of a pseudo-random sequence.*

As we'll see the random properties of LFSR sequences are poor. Cryptanalysis detects deficiencies that evade standard statistical tests.

---

#### SageMath sample 8.22 Defining an LFSR in Python/SageMath

---

```
def lfsr(s,x,n):
    """Generate a linear feedback shift register sequence.
    Parameters: Coefficient vector s, start vector x, number n of
    output bits."""
    l = len(s)
    assert l == len(x), "lfsr_Error: Bad length of start vector."
    u = x                               # in Python use u = x.copy()
    outlist = []
    for i in range (0,n):
        b = binScPr(s, u)
        c = u.pop()
        u.insert(0,b)
        outlist.append(c)
    return outlist
```

---

<sup>106</sup>The necessary and sufficient condition is that the “feedback polynomial”  $1 + s_1x + s_2x^2 + \dots + s_lx^l$  is primitive as polynomial over the field  $\mathbb{F}_2$ . Caution: Don't confuse the feedback polynomial and the feedback function, the former being a (formal) polynomial in a single variable, the latter a Boolean linear form in  $l$  variables.

<sup>107</sup>or the function `sage.crypto.lfsr.lfsr_sequence` of SageMath

<sup>108</sup>SageMath sample 8.51 provides a more systematic approach by defining a class `LFSR`.

<sup>109</sup>It uses `binScPr()` from SageMath sample 8.39, the “scalar product” of two binary vectors, or the evaluation of the linear form defined by `s` at the bitblock `u`.

```
11001000110101100011001111000000
00111011100011100000100011101111
01001001111001011011110010111001
00010010110001100111001111010111
11000100011000001110011000010111
01101010101110110001010111011000
11110000010000100010111100011110
10100111000001111000100001011000
0101010100010111110110011011101
11001001110111110001011000100010
1110010010111110011011001010011
00001100100001100110100011100100
11101000100101110110011011001010
11011100100110111001011100000011
00100010111101111000110000010001
0111010000110011111101000100101
00111010001111000100000000110110
10000101110101110001100000010001
1101101101110111001000110101001
1000111110110101010011111100001
11101110111101011001010110001010
00000100001001100110001110100110
0001010010110100000010101100100
1001011010101111110111111011101
11001010010001001011011111110
10100101001111110110100100010001
10111100011001111001011111010110
011101110100100010100101101111
01100111011000000111011111010000
11011101111111110000010001000100
10010111111110101011101110111111
01110010110000010001111001100111
```

Table 8.20: A pseudo-random bit sequence from an LFSR

---

**SageMath sample 8.23** A pseudo-random bit sequence in Python/SageMath

---

```
sage: coeff = [0,1,1,0,1,0,0,0,0,0,0,0,0,0,0,1]
sage: start = [0,1,1,0,1,0,1,1,0,0,0,1,0,0,1,1]
sage: bitlist = lfsr(coeff, start, 1024)
sage: print(bitlist)
### Visualization
sage: m = matrix(GF(2),32,bitlist)
sage: row = [1]*32
sage: n = matrix(GF(2),32,row*32) # All entries 1
sage: p = m+n                      # Toggle bits of m ---> 1 = black
sage: p.subdivide(range(0,33),range(0,33))
sage: matrix_plot(p, subdivisions=True)
```

---

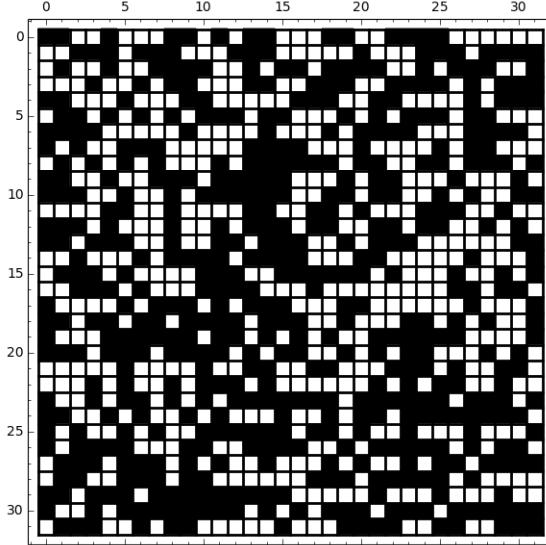


Figure 8.24: Visualization of the pseudo-random bit sequence from Figure 8.20, generated by SageMath sample 8.20 (1 = black, 0 = white)

### 8.3.4 Algebraic Attack on LFSRs

Even simple random generators such as LFSRs produce bit sequences that are virtually indistinguishable from true random sequences by statistical methods, and so provide no hooks for statistical methods of cryptanalysis. This is not true for attacks with known plaintext. The resulting equations for the key bits are accessible for algebraic cryptanalysis. If the key stream originates from a known source trying to solve these equations promises success. In particular this holds for LFSRs.

Consider a key bitstream  $u_0, u_1, \dots$  generated by an LFSR by formulas (8.10) or (8.11). Assume a plaintext  $a$  is XOR encrypted using this key stream, resulting in the ciphertext  $c$ , where  $c_i = a_i + u_i$  for  $i = 0, 1, \dots$ . What are the prospects of an attacker who knows a chunk of the plaintext?

Well, assume she knows the first  $l + 1$  bits<sup>110</sup> of the plaintext. She immediately derives the corresponding bits  $u_0, \dots, u_l$  of the key stream, in particular the initial state of the LFSR. For the yet unknown coefficients  $s_i$  she knows a linear relation:

$$s_1 u_{l-1} + \dots + s_l u_0 = u_l.$$

Each additional known plaintext bit yields one more relation, and having  $l$  relations, from  $2l$  bits of known plaintext, the easy linear algebra over the field  $\mathbb{F}_2$  (in non-degenerate cases) finds a unique solution. In the next subsections we'll prove, using some deeper mathematical methods:

**Theorem 8.3.1.** *An LFSR of length  $l$  is completely predictable from the first  $2l$  bits for the cost of about  $\frac{1}{3} \cdot l^3$  bit operations.*

### Prediction of LFSRs

Assume we know the first  $2l$  bits  $u_0, \dots, u_{2l-1}$  from an LFSR of length  $l$ . For an elegant formulation of the linear algebra methods we introduce the **state vectors**

$$u_{(i)} = (u_i, \dots, u_{i+l-1}) \quad \text{for } i = 0, 1, \dots$$

The vector  $u_{(i)}$  is the register content for step  $i$  (in reversed order compared with Figure 8.21). Thus the analysis focusses on the states, not directly on the output. The recursion (8.10) in matrix form (for  $n \geq l$ ) is

$$\begin{pmatrix} u_{n-l+1} \\ \vdots \\ u_{n-1} \\ u_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ s_l & s_{l-1} & \dots & s_1 \end{pmatrix} \begin{pmatrix} u_{n-l} \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{pmatrix}$$

or more parsimoniously (the indices being substituted by  $m = n - l + 1$ )

$$u_{(m)} = S \cdot u_{(m-1)} \quad \text{for } m \geq 1$$

where  $S$  is the coefficient matrix. As a further step we collect  $l$  consecutive state vectors  $u_{(i)}, \dots, u_{(i+l-1)}$  in a state matrix

$$U_{(i)} = \begin{pmatrix} u_i & u_{i+1} & \dots & u_{i+l-1} \\ u_{i+1} & u_{i+2} & \dots & u_{i+l} \\ \vdots & \vdots & \ddots & \vdots \\ u_{i+l-1} & u_{i+l} & \dots & u_{2l-2} \end{pmatrix}$$

and set  $U = U_{(0)}$ ,  $V = U_{(1)}$ . This gives the formula

$$V = S \cdot U$$

that expresses the unknown coefficients  $s_1, \dots, s_l$  by the known plaintext bits  $u_0, \dots, u_{2l-1}$ . Most notably it allows us to write down the solution immediately—provided that the matrix  $U$  is invertible:

$$S = V \cdot U^{-1}.$$

The matrix  $S$  explicitly displays the coefficients  $s_1, \dots, s_l$ . We'll discuss the invertibility later on.

---

<sup>110</sup>If she knows any  $l + 1$  bits, even non-contiguous, the idea of attack is the same, only the formalism is slightly more involved.

## Example

Assume we are given a ciphertext:

```
10011100 10100100 01010110 10100110 01011101 10101110
01100101 10000000 00111011 10000010 11011001 11010111
00110010 11111110 01010011 10000010 10101100 00010010
11000110 01010101 00001011 11010011 01111011 10110000
10011111 00100100 00001111 01010011 11111101
```

We suspect that the cipher is XOR with a key stream from an LFSR of length  $l = 16$ . The context suggest that the text is in German and begins with the word “Treppunkt” (meeting point). To solve the cryptogram we need 32 bits of plaintext, that is the first four letters only, presupposed that the theory applies. This gives 32 bits of the key stream:

$$\begin{array}{l} 01010100 \ 01110010 \ 01100101 \ 01100110 = T \ r \ e \ f \\ 10011100 \ 10100100 \ 01010110 \ 10100110 \quad \text{cipher bits} \\ \hline 11001000 \ 11010110 \ 00110011 \ 11000000 \quad \text{key bits} \end{array}$$

SageMath sample [8.24](#) determines the coefficient matrix. Its last row tells us that all  $s_i = 0$  except  $s_{16} = s_5 = s_3 = s_2 = 1$ .

Now we know the LFSR and the initial state, and can reconstruct the complete key stream—yes, it is the same as in Figure [8.20](#)—and write down the plaintext (that by the way begins a bit differently from our guess).

## Proof of the Theorem

We have shown that the coefficients are uniquely determined assuming the state matrix  $U = U_{(0)}$  is invertible. As a consequence in this case the LFSR is completely known, and all output bits are predictable. We have yet to discuss the case where the matrix  $U$  is singular.

If one of the first  $l$  state vectors (= rows of the matrix  $U$ ) is zero, then all following state vectors are zero too, and prediction is trivial.

Thus we may assume that none of these vectors are zero, but that they are linearly dependent. Then there is a smallest index  $k \geq 1$  such that  $u_{(k)}$  is contained in the subspace spanned by  $u_{(0)}, \dots, u_{(k-1)}$ , and we find coefficients  $t_1, \dots, t_k \in \mathbb{F}_2$  such that

$$u_{(k)} = t_1 u_{(k-1)} + \dots + t_k u_{(0)}.$$

Then also  $u_{(k+1)} = S \cdot u_{(k)} = t_1 S \cdot u_{(k-1)} + \dots + t_k S \cdot u_{(0)} = t_1 u_{(k)} + \dots + t_k u_{(1)}$ , and by induction we get

$$u_{(n)} = t_1 u_{(n-1)} + \dots + t_k u_{(n-k)} \quad \text{for all } n \geq k.$$

This formula predicts all the following bits.

The statement on the cost follows from Theorem [8.1.10](#).

## Discussion

- For a singular state matrix this consideration yields a shorter LFSR (of length  $k < l$ ) that generates exactly the same sequence. Then our method doesn't determine the coefficients of the original register but nevertheless correctly predicts the sequence.

---

**SageMath sample 8.24** Determining a coefficient matrix

---

```
sage: l = 16
sage: kbits =
[1,1,0,0,1,0,0,0,1,1,0,1,0,1,1,0,0,0,1,1,0,0,1,1,1,1,0,0,0,0,0]
sage: ulist = []
sage: for i in range(0,l):
    state = kbits[i:(l+i)]
    ulist.append(state)
sage: U = matrix(GF(2),ulist)
sage: det(U)
1
sage: W = U.inverse()
sage: vlist = []
sage: for i in range(1,l+1):
    state = kbits[i:(l+i)]
    vlist.append(state)
sage: V = matrix(GF(2),vlist)
sage: S = V*W
sage: S
[0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0]
[0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0]
[1 0 0 0 0 0 0 0 0 0 0 1 0 1 1 0]
```

---

- If the bits the attacker knows aren't just the first ones but  $2l$  contiguous ones at a later position, then the theorem yields only the prediction of the following bits. In the main case of an invertible state matrix  $U$  the LFSR is completely known and may be run backwards to get the previous bits. For a singular state matrix we achieve the same effect using the shorter LFSR constructed above.
- The situation where  $2l$  bits of the key stream are known but at non-contiguous positions is slightly more involved. We get linear relations that contain additional (unknown) intermediate bits. If  $m$  is the number of these then we get  $l + m$  linear equations for  $l + m$  unknown bits.
- What if the length  $l$  of the LFSR is unknown? Exhaustively trying all values  $l = 1, 2, 3, \dots$

is nasty but feasible. A better approach is provided by the Berlekamp-Massey<sup>111</sup> algorithm that is efficient also without knowledge of  $l$ . We won't treat it in this chapter.

## Summary

Given a random generator as in Figure 8.20 cryptanalytic targets are:

- the secret parameters,
- the initial state,
- additional parts of the output (“prediction problem”),

given some parts of the output. As we saw for LFSRs the prediction problem has a solution even when the internal parameters remain unknown. Thus:

*Cryptanalysis of a random generator first of all means solving the prediction problem. A random generator is cryptographically secure if its prediction problem admits no efficient solution.*

*Linear feedback shift registers are not cryptographically secure.*

### 8.3.5 Approaches to Nonlinearity for Feedback Shift Registers

LFSRs are popular—in particular among electrical engineers and military—for several reasons:

- very easy implementation,
- extreme efficiency in hardware,
- good qualification as random generators for statistical applications and simulations,
- unproblematic operation in parallel even in large quantities.

But unfortunately from a cryptological view they are completely insecure if used naively. To capitalize their positive properties while escaping their cryptological weakness there are several approaches.

#### Approach 1, Nonlinear Feedback

Nonlinear feedback follows the scheme from Figure 8.21 with a nonlinear Boolean function  $f$ . We won't pursue this approach here. We saw a very simple toy example in SageMath sample 8.21. There is a general proof that in realistic use cases NLFSRs<sup>112</sup> are cryptographically useless if used in the direct naive way [Pom16].

#### Approach 2, Nonlinear Output Filter

The nonlinear ouput filter (nonlinear feedforward) realizes the scheme from Figure 8.25. The shift register itself is linear, the Boolean function  $f$ , nonlinear.

The nonlinear ouput filter is a special case of a nonlinear combiner.

---

<sup>111</sup>in SageMath contained as `sage.crypto.lfsr.berlekamp_massey`, in CrypTool 2 under “cryptanalysis”/“generic”/“Berlekamp-Massey algorithm”

<sup>112</sup>for Non Linear Feedback Shift Register

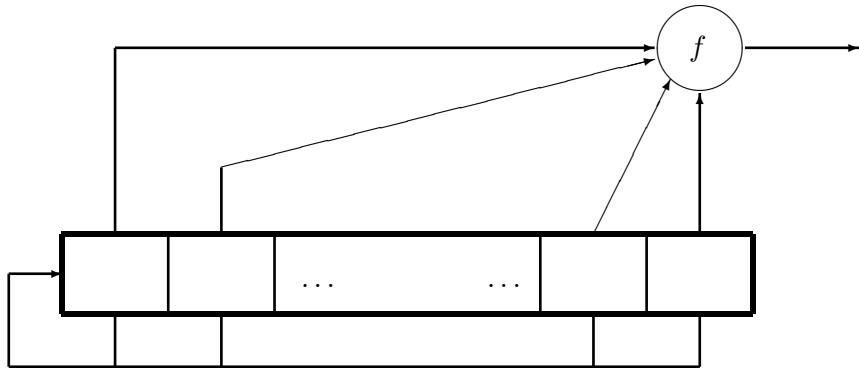


Figure 8.25: Nonlinear ouput filter for an LFSR

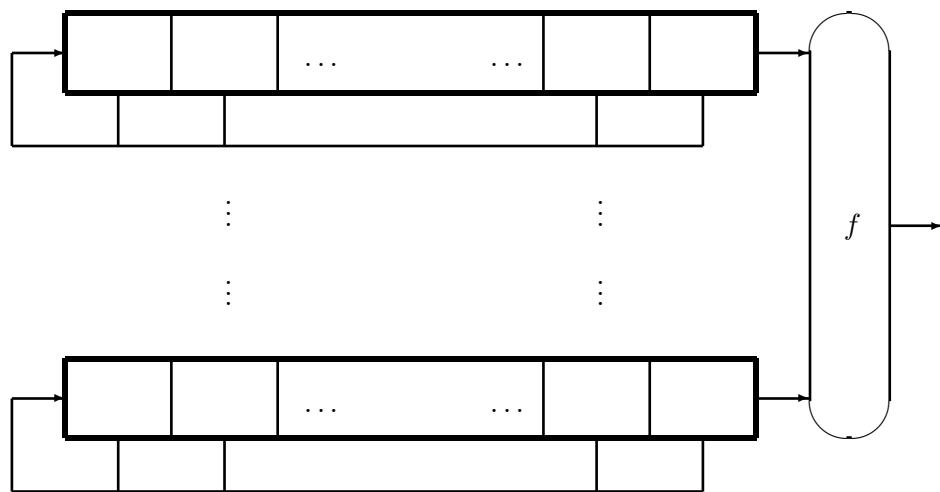


Figure 8.26: Nonlinear combiner

### Approach 3, Nonlinear Combiner

The nonlinear combiner uses a “battery” of  $n$  LFSRs—preferably of different lengths—operated in parallel. The output sequences of the LFSRs serve as input<sup>113</sup> of a Boolean function  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ , see Figure 8.26. We’ll see in Section 8.3.7 how to cryptanalyze this random generator.

### Approach 4, Output Selection/Decimation/Clocking

There are different ways of controlling a battery of  $n$  parallel LFSRs by another LFSR:

- **Output selection** takes the current output bit of exactly one of the LFSRs from the “battery”, depending on the state of the auxiliary register, and outputs it as the next pseudo-random bit. More generally we could choose “ $r$  from  $n$ ”.
- For **decimation** one usually takes  $n = 1$ , and outputs the current bit of the one battery

<sup>113</sup>hence the occasional denotation “nonlinear feedforward”

register only if the auxiliary register is in a certain state, for example its own current output is 1. Of course this kind of decimation applies to arbitrary bit sequences in an analogous way.

- For **clocking** we look at the state of the auxiliary register and depending on it decide which of the battery registers to step in the current cycle (and by how many positions), leaving the other registers in their current states<sup>114</sup>.

These methods turn out to be special cases of nonlinear combiners if properly rewritten. Thus approach 3 represents the most important method of making the best of LFSRs.

The encryption standard A5/1 for mobile communications uses three LFSRs of lengths 19, 22 und 23, each with maximum possible period, and slightly differently clocked. It linearly (by simple binary addition) combines the three output streams. The—even weaker—algorithm A5/2 controls the clocking by an auxiliary register. Both variants can be broken on a standard PC in real-time.

The Bluetooth encryption standard E<sub>0</sub> uses four LFSRs and combines them in a nonlinear way. This method is somewhat stronger than A5, but also too weak for real security [Sch03].

### Example: The Geffe Generator

The Geffe generator provides a simple example of output selection. Its description is in Figure 8.27. The output is  $x$ , if  $z = 0$ , and  $y$ , if  $z = 1$ . Expressed by a formula:

$$\begin{aligned} u &= \begin{cases} x, & \text{if } z = 0, \\ y, & \text{if } z = 1 \end{cases} \\ &= (1 - z)x + zy = x + zx + zy. \end{aligned}$$

This formula shows how to interpret the Geffe generator as a nonlinear combiner with a Boolean function  $f: \mathbb{F}_2^3 \rightarrow \mathbb{F}_2$  of degree 2. For later use we implement  $f$  in SageMath sample 8.25.

---

#### SageMath sample 8.25 The Geffe function

---

```
sage: geff = BoolF(str2bb1("00011100"),method="ANF")
sage: geff.printTT()
Value at 000 is 0
Value at 001 is 0
Value at 010 is 0
Value at 011 is 1
Value at 100 is 1
Value at 101 is 0
Value at 110 is 1
Value at 111 is 1
```

---

### 8.3.6 Implementation of a Nonlinear Combiner

A nonlinear combiner uses several LFSRs, operated in parallel. This suggests an implementation of LFSRs as objects of a class **LFSR**<sup>115</sup>.

---

<sup>114</sup>This reminds of the control logic of rotor machines in classical cryptography.

<sup>115</sup>see also CrypTool 2, “protocols”/“LFSR” or “NLFSR”



Figure 8.27: Geffe generator

### Class LSFR:

#### Attributes:

- `length`: the length of the register
- `taplist` (constant): the list of coefficients (or taps) that define the bits for feedback
- `state` (variable): the state of the register

#### Methods:

- `setLength`: define the length (used only implicitly for initialization)
- `setTaps`: define the list of taps (used only implicitly for initialization)
- `setState`: set the state of the register
- `getLength`: output the length
- `nextBits`: generate a given number of output bits, and set the next state

For observing the register a method (generically called `__str__` in Python) is convenient that outputs the attributes in human-readable form.

The complete implementation is in SageMath sample 8.51 in Section 8.4.9.

### Example: Geffe Generator

First we choose<sup>116</sup> three LFSRs of lengths 15, 16, 17, whose periods are  $2^{15} - 1 = 32767$ ,  $2^{16} - 1 = 65535$ , and  $2^{17} - 1 = 131071$ . These are pairwise coprime, see SageMath sample 8.26. Combining their outputs (in each step) as bitblocks of length 3 yields a sequence with a period that has an impressive length of 281459944554495, about  $300 \times 10^{12}$  (300 billions<sup>117</sup>). SageMath sample 8.27 defines the three LFSRs. The recursive formula for the third one, the control register `reg17`, is  $u_n = u_{n-3} + u_{n-17}$ , since exactly the taps 3 and 17 are “active”. We let each of the LFSRs generate a sequence of length 100, see SageMath sample 8.28. The Geffe function combines them in SageMath sample 8.29.

<sup>116</sup>using the lists of primitive polynomials from [MvOV01]

<sup>117</sup>European billions. For Americans this are 300 trillions.

---

**SageMath sample 8.26** Calculating a period

---

```
sage: n15 = 2**15 - 1; n15
32767
sage: n15.factor()
7 * 31 * 151
sage: n16 = 2**16 - 1; n16
65535
sage: n16.factor()
3 * 5 * 17 * 257
sage: n17 = 2**17 - 1; n17
131071
sage: n17.factor()
131071
sage: period = n15 * n16 * n17; period
281459944554495
```

---

---

**SageMath sample 8.27** Three LFSRs

---

```
sage: reg15 = LFSR([1,0,0,0,0,0,0,0,0,0,0,0,0,0,1])
sage: reg15.setState([0,1,1,0,1,0,1,1,0,0,0,1,0,0,1])
sage: print(reg15)
Length: 15 | Taps: 10000000000001 | State: 011010110001001
sage: reg16 = LFSR([0,1,1,0,1,0,0,0,0,0,0,0,0,0,1])
sage: reg16.setState([0,1,1,0,1,0,1,1,0,0,0,1,0,0,1,1])
sage: print(reg16)
Length: 16 | Taps: 0110100000000001 | State: 0110101100010011
sage: reg17 = LFSR([0,0,1,0,0,0,0,0,0,0,0,0,0,0,1])
sage: reg17.setState([0,1,1,0,1,0,1,1,0,0,0,1,0,0,1,1])
sage: print(reg17)
Length: 17 | Taps: 0010000000000001 | State: 01101011000100111
```

---

---

**SageMath sample 8.28** Three LFSR sequences

---

```
sage: nofBits = 100
sage: outlist15 = reg15.nextBits(nofBits)
sage: print(outlist15)
[1, 0, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 0,
 0, 0, 1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1,
 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0,
 1, 1, 1, 1, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0,
 0, 1, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1]
sage: outlist16 = reg16.nextBits(nofBits)
sage: print(outlist16)
[1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 1,
 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1,
 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0,
 1, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0, 1,
 1, 0, 1, 1, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1]
sage: outlist17 = reg17.nextBits(nofBits)
sage: print(outlist17)
[1, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1,
 0, 0, 0, 0, 0, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0,
 1, 1, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1,
 0, 1, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, 0, 1, 0, 1]
```

---

---

**SageMath sample 8.29** The combined sequence

---

```
sage: outlist = []
sage: for i in range(0,nofBits):
....:     x = [outlist15[i],outlist16[i],outlist17[i]]
....:     outlist.append(geff.valueAt(x))
....:
sage: print(outlist)
[1, 1, 0, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1,
 0, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 0, 0, 1, 1,
 1, 0, 1, 0, 1, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 0,
 1, 1, 1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0,
 0, 0, 1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0]
```

---

### 8.3.7 Correlation Attacks—the Achilles Heels of Combiners

Let  $f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  be the combining function of a nonlinear combiner. The number

$$K_f := \#\{x = (x_1, \dots, x_n) \in \mathbb{F}_2^n \mid f(x) = x_1\}$$

counts the coincidences of the value of the function with its first argument. If it is  $> 2^{n-1}$ , then the probability of a coincidence,

$$p = \frac{1}{2^n} \cdot K_f > \frac{1}{2},$$

is above average, and the combined output sequence “correlates” with the output of the first LFSR more than expected by random. If  $p < \frac{1}{2}$ , then the correlation deviates from the expected value in the other direction.

The cryptanalyst can exploit this effect in an attack with known plaintext. We suppose that she knows the “hardware”, that is the taps of the registers, and also the combining function  $f$ . She seeks the initial states of all the LFSRs. We assume she knows the bits  $k_0, \dots, k_{r-1}$  of the key stream<sup>118</sup>. For each of the  $2^{l_1}$  initial states of the first LFSR she generates the sequence  $u_0, \dots, u_{r-1}$ , and counts the coincidences. The expected values are

$$\frac{1}{r} \cdot \#\{i \mid u_i = k_i\} \approx \begin{cases} p & \text{for the correct initial state of LFSR 1,} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

If  $r$  is large enough, she can determine the true initial state of LFSR 1 (with high probability) for a cost of  $\sim 2^{l_1}$ . She continues with the other registers, and finally identifies the complete key with a cost of  $\sim 2^{l_1} + \dots + 2^{l_n}$ . Note that the cost is exponential, but significantly lower than the cost  $\sim 2^{l_1} \dots 2^{l_n}$  of the naive exhaustion of the key space.

In the language of linear cryptanalysis from 8.2.6 she made use of the linear relation

$$f(x_1, \dots, x_n) \stackrel{p}{\approx} x_1$$

for  $f$ . Clearly she could use any linear relation as well to reduce the complexity of key search<sup>119</sup>.

---

<sup>118</sup>for simplicity of exposition the first ones. The argument works in the same way for any  $r$  known key bits.

<sup>119</sup>A more in-depth analysis of the situation leads to the notion of correlation immunity that is related with the linear potential.

## Correlations from the Geffe Generator

From the truth table 8.21 we get the correlations produced by the Geffe generator. Thus the probabilities of coincidences are

$$p = \begin{cases} \frac{3}{4} & \text{for register 1 } (x), \\ \frac{3}{4} & \text{for register 2 } (y), \\ \frac{1}{2} & \text{for register 3 } (z = \text{control bit}). \end{cases}$$

A correlation attack easily detects the initial states of registers 1 and 2—the battery registers—given only a short piece of an output sequence. Afterwards exhaustion finds the initial state of register 3, the control register.

$x$	0	0	0	0	1	1	1	1
$y$	0	0	1	1	0	0	1	1
$z$	0	1	0	1	0	1	0	1
$f(x, y, z)$	0	0	0	1	1	1	0	1

Table 8.21: Truth table of the Geffe function (in horizontal order)

We exploit this weakness of the Geffe generator in SageMath sample 8.30 that continues SageMath sample 8.25. Since we defined the linear profile for objects of the class `BoolMap` only, we first of all have to interpret the function `geff` as a Boolean map, that is a one-element list of Boolean functions. Then the linear profile is represented by a matrix of 2 columns and 8 rows. The first column [64, 0, 0, 0, 0, 0, 0, 0] shows the coincidences with the linear form 0 in the range. So it contains no useful information, except the denominator 64 that applies to all entries. The second row [0, 0, 16, 16, 16, 16, 0, 0] yields the list of coincidence probabilities  $p$  (after dividing it by 64) in Table 8.22, using the formula

$$p = \frac{1}{2} \cdot (\pm\sqrt{\lambda} + 1).$$

If  $\lambda = 0$ , then  $p = 1/2$ . If  $\lambda = 1/4$ , then  $p = 1/4$  or  $3/4$ . For deciding between these two values for  $p$  we use Table 8.21.

linear form representation	0	$z$	$y$	$y+z$	$x$	$x+z$	$x+y$	$x+y+z$
potential	0	0	$1/4$	$1/4$	$1/4$	$1/4$	0	0
probability $p$	$1/2$	$1/2$	$3/4$	$1/4$	$3/4$	$3/4$	$1/2$	$1/2$

Table 8.22: Coincidence probabilities of the Geffe function

---

### SageMath sample 8.30 Linear profile of the Geffe function

---

```
sage: g = BoolMap([geff])
sage: linProf = g.linProf(); linProf
[[64,0], [0,0], [0,16], [0,16], [0,16], [0,16], [0,0], [0,0]]
```

---

In SageMath sample 8.31 we apply this finding to the 100 element sequence from SageMath sample 8.29. The function `coinc` from SageMath sample 8.39 (in the appendix) counts the

coincidences. For the first register we find 73 coincidences, for the second one 76, for the third one only 41. This confirms the values 75, 75, 50 predicted by our theory (taking into account statistical variability).

---

**SageMath sample 8.31** Coincidences for the Geffe generator

---

```
sage: coinc(outlist15,outlist)
73
sage: coinc(outlist16,outlist)
76
sage: coinc(outlist17,outlist)
41
```

---

### Cryptanalysis of the Geffe Generator

These results promise an effortless analysis of our sample sequence. For an assessment of the success probability we consider a bitblock  $b \in \mathbb{F}_2^r$  and first ask how large is the probability that a random bitblock  $u \in \mathbb{F}_2^r$  coincides with  $b$  at exactly  $t$  positions. For an answer we have to look at the symmetric binomial distribution (where  $p = \frac{1}{2}$  is the probability of coincidence at a single position): The probability of exactly  $t$  coincidences is

$$B_{r,\frac{1}{2}}(t) = \frac{\binom{r}{t}}{2^r}.$$

Hence the cumulated probability of up to  $T$  coincidences is

$$\sum_{t=0}^T B_{r,\frac{1}{2}}(t) = \frac{1}{2^r} \cdot \sum_{t=0}^T \binom{r}{t}.$$

If  $r$  is not too large, then we may explicitly calculate this value for a given bound  $T$ . If on the other hand  $r$  is not too small, then we approximate the value using the normal distribution. The mean value of the number of coincidences is  $r/2$ , the variance,  $r/4$ , and the standard deviation,  $\sqrt{r}/2$ .

In any case for  $r = 100$  the probability of finding at most (say) 65 coincidences is 0.999, the probability of surpassing this number is 1 %. For the initial state of register 1 we have to try  $2^{15} = 32786$  possibilities (generously including the zero state  $0 \in \mathbb{F}_2^{15}$  into the count). So we expect about 33 oversteppings with at least 66 coincidences. One of these should occur for the true initial state of register 1 that we expect to produce about 75 coincidences. Maybe it even produces the maximum number of coincidences.

SageMath sample 8.32 shows that this really happens. However the maximum number of coincidences, 73, occurs twice in the histogram. The first occurrence happens at index 13705, corresponding to the initial state 011010110001001, the correct solution. The second occurrence, at index 31115, see SageMath sample 8.33, yields the false solution 111100110001011 that eventually leads to a contradiction.

SageMath sample 8.34 shows the analogous analysis of register 2. Here the maximum of coincidences, 76, is unique, occurs at index 27411 corresponding to the initial state 0110101100010011, and provides the correct solution.

To complete the analysis we must yet determine the initial state of register 3, the control register. The obvious idea is to exhaust the  $2^{17}$  different possibilities. There is a shortcut since

---

**SageMath sample 8.32** Analysis of the Geffe generator—register 1

---

```
sage: clist = []
sage: histogr = [0] * (nofBits + 1)
sage: for i in range(0,2**15):
....:     start = int2bb1(i,15)
....:     reg15.setState(start)
....:     testlist = reg15.nextBits(nofBits)
....:     c = coinc(outlist,testlist)
....:     histogr[c] += 1
....:     clist.append(c)
....:
sage: print(histogr)
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 12, 12, 37, 78, 116, 216,
 329, 472, 722, 1003, 1369, 1746, 1976, 2266, 2472, 2531, 2600,
 2483, 2355, 2149, 1836, 1574, 1218, 928, 726, 521, 343, 228, 164,
 102, 60, 47, 36, 13, 8, 7, 4, 2, 1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
sage: mm = max(clist)
sage: ix = clist.index(mm)
sage: block = int2bb1(ix,15)
sage: print "Maximum =", mm, "at index", ix, ", start value", block
Maximum = 73 at index 13705 , start value\
[0, 1, 1, 0, 1, 0, 1, 1, 0, 0, 0, 1, 0, 0, 1]
```

---

**SageMath sample 8.33** Analysis of the Geffe generator—continued

---

```
sage: ix = clist.index(mm,13706); ix
31115
sage: print int2bb1(ix,15)
[1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 1, 0, 1, 1]
```

---

we already know 51 of the first 100 bits of the control register: At a position where the values of registers 1 and 2 differ, the control bit is necessarily 0 if the final output coincides with register 1, and 1 otherwise. Only at positions where registers 1 and 2 coincide the corresponding bit of register 3 is undetermined.

```
register 1: 1001000110101101100001001101101000001110110110000
register 2: 11001000110101100011001111000000001110111000111000
register 3: -1-00--0-1101-110001---00-1-00-1--1101--110---0---
bitsequence: 11010001110001101101001001001100001100111010110000

... 00101101101111110010010010101110001110011001011
... 00100011101111010010011110010110111100101110010001
... ---110-----1-1-11-0-100---01--01-1-001-1-00-1-
... 00100001101111010010001101010100110100110001001
```

In particular we already know 11 of the 17 initial bits, and are left with only  $2^6 = 64$  possibilities

---

**SageMath sample 8.34** Analysis of the Geffe generator—register 2

---

```
sage: clist = []
sage: histogr = [0] * (nofBits + 1)
sage: for i in range(0,2**16):
....:     start = int2bbi(i,16)
....:     reg16.setState(start)
....:     testlist = reg16.nextBits(nofBits)
....:     c = coinc(outlist,testlist)
....:     histogr[c] += 1
....:     clist.append(c)
....:
sage: print(histogr)
[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0, 1, 0, 2, 3, 4, 8, 17, 25, 51, 92, 171,
 309, 477, 750, 1014, 1423, 1977, 2578, 3174, 3721, 4452, 4821,
 5061, 5215, 5074, 4882, 4344, 3797, 3228, 2602, 1974, 1419,
 1054, 669, 434, 306, 174, 99, 62, 38, 19, 10, 3, 0, 1, 0, 0,
 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
 0, 0, 0, 0, 0, 0]
sage: mm = max(clist)
sage: ix = clist.index(mm)
sage: block = int2bbi(ix,16)
sage: print "Maximum =", mm, "at index", ix, ", start value", block
Maximum = 76 at index 27411 , start value\
[0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1]
```

---

to try.

But even this may be further simplified, since the known and the unknown bits obey linear relations of the type  $u_n = u_{n-3} + u_{n-17}$ . The unknown bits of the initial state are  $u_0, u_2, u_5, u_6, u_8, u_{13}$ . The solution follows the columns of Table 8.23, that immediately give

$$u_0 = 1, u_2 = 1, u_6 = 0.$$

The remaining solutions are

$$u_8 = u_{22} = u_{39} = 0, u_5 = u_{22} + 1 = u_8 + 1 = 1, u_{13} = u_{30} + 1 = 0.$$

Hence the initial state of the control register is 01101011000100111, and we know this is the correct solution. We don't need to bother with the second possible solution for register 1 since we already found a constellation that correctly reproduces the sequence.

### 8.3.8 Design Criteria for Nonlinear Combiners

From the forgoing discussion we derive design criteria for nonlinear combiners:

- The battery registers should be as long as possible.
- The combining function  $f$  should have a low linear potential.

$u_{17} = u_{14} + u_0$	$0 = 1 + u_0$	$u_0 = 1$	
$u_{19} = u_{16} + u_2$	$1 = 0 + u_2$	$u_2 = 1$	
$u_{20} = u_{17} + u_3$	$u_{20} = 0 + 0$	$u_{20} = 0$	
$u_{22} = u_{19} + u_5$	$u_{22} = u_5 + 1$	$u_5 = u_{22} + 1$	
$u_{23} = u_{20} + u_6$	$0 = u_{20} + u_6$	$u_6 = u_{20}$	$u_6 = 0$
$u_{25} = u_{22} + u_8$	$u_{25} = u_{22} + u_8$	$u_8 = u_{22} + u_{25}$	$u_8 = u_{22}$
$u_{27} = u_{24} + u_{10}$	$u_{27} = 0 + 1$	$u_{27} = 1$	
$u_{28} = u_{25} + u_{11}$	$0 = u_{25} + 0$	$u_{25} = 0$	
$u_{30} = u_{27} + u_{13}$	$u_{30} = u_{27} + u_{13}$	$u_{13} = u_{27} + u_{30}$	$u_{13} = u_{30} + 1$
$u_{33} = u_{30} + u_{16}$	$u_{33} = u_{30} + 0$	$u_{30} = u_{33}$	$u_{30} = 1$
$u_{36} = u_{33} + u_{19}$	$0 = u_{33} + 1$	$u_{33} = 1$	
$u_{39} = u_{36} + u_{22}$	$u_{39} = 0 + u_{22}$	$u_{22} = u_{39}$	
$u_{42} = u_{39} + u_{25}$	$0 = u_{39} + u_{25}$	$u_{39} = u_{25}$	$u_{39} = 0$

Table 8.23: Determination of the control register’s initial state

How long should the battery registers be? There are some algorithms for “fast” correlation attacks using the Walsh transformation, in particular against sparse linear feedback functions (that use only a small number of taps) [MS89]. These don’t reduce the complexity class of the attack (“exponential in the length of the shortest register”) but reduce the cost by a significant factor. So they are able to attack registers whose feedback functions have up to 100 monomials with coefficients in their ANF. As a consequence

- The single LFSRs should have a length of at least 200 bits, and use about 100 taps each.

To assess the number  $n$  of LFSRs we bear in mind that the combining function should be “correlation immune”, in particular have a low linear potential. A well-chosen Boolean function of 16 variables should suffice<sup>120</sup>.

Rueppel<sup>121</sup> found an elegant way out to make the correlation attack break down: Use a “time-dependent” combining function, that is a family  $(f_t)_{t \in \mathbb{N}}$ . The bit  $u_t$  of the key stream is calculated by the function  $f_t$ . We won’t analyze this approach here.

Observing that the correlation attack needs knowledge of the taps, the security could be somewhat better if the taps are secret. Then the attacker has to perform additional exhaustions that multiply the complexity by factors such as  $2^{l_1}$  for the first LFSR alone. This scenario allows choosing LFSRs of somewhat smaller lengths. But bear in mind that for a hardware implementation the taps are parts of the algorithm, not of the key, that is they are public parameters in the sense of Figure 8.20.

## Efficiency

LFSRs and nonlinear combiners allow efficient realizations by special hardware that produces one bit per clock cycle. This rate can be enlarged by parallelization. From this point of view estimating the cost of execution on a usual PC processor is somewhat inadequate. Splitting each of the  $\geq 200$  bit registers into 4 parts of about 64 bits shifting a single register requires at least 4 clock cycles, summing up to 64 clock cycles for 16 registers. Add some clock cycles for the

<sup>120</sup>There are no known recommendations in the literature.

<sup>121</sup>Rainer A. Rueppel, Swiss cryptographer

combining function. Thus one single bit would take about 100 clock cycles. A 2-GHz processor, even with optimized implementation, would produce at most  $2 \cdot 10^9 / 100 = 20$  million bits per second.

As a summary we note:

*Using LFSRs and nonlinear combining functions we can build useful and fast random generators, especially in hardware.*

Unfortunately there is no satisfying theory for the cryptologic security of this type of random generators, even less a mathematical proof. Security is assessed by plausible criteria that—as for bitblock ciphers—are related to the nonlinearity of Boolean functions.

### 8.3.9 Perfect (Pseudo-)Random Generators

As we saw the essential cryptologic criterion for random generators is unpredictability. In the 1980s cryptographers, guided by an analogy with asymmetric cryptography, found a way of modelling this property in terms of complexity theory: Prediction should boil down to a known “hard” algorithmic problem such as factoring or discrete logarithm. This idea established a new quality standard for random generators, much stronger than statistical tests, but eventually building on unproven mathematical hypotheses. Thus the situation with respect to the security of random generators is comparable to asymmetric encryption.

As an interesting twist it soon turned out that in a certain sense unpredictability is a universal property: For an unpredictable sequence there is *no efficient algorithm at all* that distinguishes it from a true random sequence, a seemingly much stronger requirement. See Theorem 8.3.2 (Yao’s theorem). This universality justifies the denomination “perfect” for the corresponding random generators. In particular there is no efficient statistical test that is able to distinguish the output of a perfect random generator from a true random sequence. Thus, on the theoretical side, we have a very appropriate model for random generators that are absolutely strong from a statistical viewpoint, and invulnerable from a cryptological viewpoint. In other words:

*Perfect random generators are cryptographically secure and statistically undistinguishable from true random sources.*

*Presumably perfect random generators exist, but there is no complete mathematical proof of their existence.*

The first concrete approaches to the construction of perfect random generators, the best known being the BBS generator (for Blum<sup>122</sup>, Blum<sup>123</sup>, Shub<sup>124</sup>), yielded algorithms that were too slow for most practical uses (given the then current CPUs). But modified approaches soon provided random generators that are passably fast and nevertheless (presumably) cryptographically secure.

### 8.3.10 The BBS Generator

As with the RSA cipher we consider an integer module  $m$  that is a product of two large prime numbers. For the BBS generator we choose<sup>125</sup> **Blum primes**  $p$ ; these are primes  $\equiv 3 \pmod{4}$ . A product of two Blum primes is called a **Blum integer**.

---

<sup>122</sup>Lenore Blum, U.S. American mathematician and computer scientist, \*December 18, 1942

<sup>123</sup>Manuel Blum, U.S. American mathematician and computer scientist, \*April 26, 1938

<sup>124</sup>Michael Shub, U.S. American mathematician, \*August 17, 1943

<sup>125</sup>for technical reasons not to be discussed here

The BBS generator works in the following way: As a first step choose two large random Blum primes  $p$  and  $q$ , and form their product  $m = pq$ . As a second step choose a random integer seed  $s$  with  $1 \leq s \leq m - 1$ , and coprime<sup>126,127</sup> with  $m$ .

Now we proceed with generating a pseudo-random sequence: Take  $x_0 = s^2 \bmod m$  as initial state<sup>128</sup>, and form the sequence of inner states of the random generator:  $x_i = x_{i-1}^2 \bmod m$  for  $i = 1, 2, 3, \dots$ . In each step output that last significant bit of the binary representation, that is  $u_i = x_i \bmod 2$  for  $i = 0, 1, 2, \dots$ , or in other words, the parity of  $x_i$ .

### Example

Of course an example with small numbers is practically irrelevant, but it illustrates the algorithm: Take  $p = 7$ ,  $q = 11$ ,  $m = 77$ ,  $s = 53$ . Then  $s^2 = 2809$ , hence  $x_0 = 37$ , and  $u_0 = 1$  since  $x_0$  is odd. The naive SageMath sample 8.35 shows the beginning of the sequence of states:

$i$	0	1	2	3	...
$x_i$	37	60	58	53	...
$u_i$	1	0	0	1	...

---

#### SageMath sample 8.35 A (much too) simple example for BBS

---

```
sage: p = 7
sage: q = 11
sage: m = p*q; m
77
sage: s = 53
sage: x0 = (s^2) % m; x0
37
sage: x1 = (x0^2) % m; x1
60
sage: x2 = (x1^2) % m; x2
58
sage: x3 = (x2^2) % m; x3
53
```

---

Treating the Blum primes  $p$  and  $q$  as secret is essential for the security of the BBS generator. They serve for forming  $m$  only, afterwards they may even be destroyed. In contrast with RSA there is no further use for them. Likewise all the non-output bits of the inner states  $x_i$  must be secret.

The standard distribution of SageMath contains the BBS generator. It consists of the procedures:

- `random_blum_prime()` in the module `sage.crypto.util`. To generate a random Blum prime  $p$  with a given number  $k$  of bits (= digits of the binary representation) call it

---

<sup>126</sup>If we catch an  $s$  not coprime with  $m$ , we have factorized  $m$  by hazard. This might happen, but is extremely unlikely, and can easily be captured at initialization time.

<sup>127</sup>If  $x_i < \sqrt{m}$ , then  $x_i^2 \bmod m = x_i^2$ , the integer square, so  $x_{i+1}^2$  has the same parity as  $x_i$ . In order to avoid a constant segment at the beginning of the output, often the boundary area  $s < \sqrt{m}$ , as well as  $s > m - \sqrt{m}$ , is excluded. However if we really choose  $s$  as a true random value, the probability for  $s$  falling into these boundary areas is extremely low. But to be on the safe side we may require  $\sqrt{m} \leq s \leq m - \sqrt{m}$ .

<sup>128</sup>We want  $x_0$  to be a quadratic residue.

as `p = random_blum_prime(2**k-1, 2**k)`. The correctness of this algorithm is only empirically founded: In fact there is always<sup>129</sup> a prime between  $2^{k-1}$  and  $2^k$  but this needn't be a Blum prime. Nevertheless empiricism tells us that there are lots of Blum primes in this interval, namely about  $2^k/(k \log(2))$ . Thus an attack by exhaustion will fail.

- `blum_blum_shub()` from `sage.crypto.stream`. To generate a sequence of  $r$  pseudo-random bits first generate two random Blum primes  $p$  and  $q$  and an initial value  $x_0 = s^2 \bmod pq$ , and then call the procedure as `blum_blum_shub(r, x_0, p, q)`.

SageMath sample 8.36 demonstrates the procedure. The intermediate results  $p$ ,  $q$ , and  $x_0$  are shown in Tables 8.24, 8.25, and 8.26, the result, in Table 8.27. By convention  $s$  as well as the factors  $p$  and  $q$  must be kept secret. Moreover there is no reason to reveal the product  $m = pq$ . However considering the progress of factorization algorithms we better should use Blum integers of at least 2048 bits<sup>130</sup>. And in any case  $s$  must be a true random value! We neglected this duty by choosing  $s$  as a pure power.

---

#### SageMath sample 8.36 Generating a sequence of BBS pseudo-random bits

---

```
sage: from sage.crypto.util import random_blum_prime
sage: from sage.crypto.stream import blum_blum_shub
sage: p = random_blum_prime(2^511, 2^512)
sage: q = random_blum_prime(2^511, 2^512)
sage: x0 = 11^248 % (p*q)           # s = 11^124 % (p*q)
sage: blum_blum_shub(1000,x0,p,q)
```

---

8	445	834	617	855	090	512	176	000	413	196	767	417	799	332
626	936	992	170	472	089	385	128	414	279	550	732	184	808	226
736	683	775	727	426	619	339	706	269	080	823	255	441	520	165
438	397	334	657	231	839	251								

Table 8.24: A Blum prime  $p$  with 512 bits (154 decimal places)

12	580	605	326	957	495	732	854	671	722	855	802	182	952	894
232	088	903	111	155	705	856	898	413	602	721	771	810	991	595
365	229	641	230	483	180	760	744	910	366	324	916	344	823	400
588	340	927	883	444	616	787								

Table 8.25: A Blum prime  $q$  with 512 bits (155 decimal places)

### 8.3.11 Perfectness and the Factorization Conjecture

Informally we define a **pseudo-random generator** (shortly: a random generator) as an efficient algorithm that takes a “short” bitstring  $s \in \mathbb{F}_2^n$  and converts it into a “long” bitstring  $s \in \mathbb{F}_2^r$ .

---

<sup>129</sup>This is a special case of Bertrand’s postulate, proved by Chebyshev in 1850: There is a prime between  $n$  and  $2n$  (for all  $n \geq 2$ ).

<sup>130</sup>more on this in Section 8.3.11

1	842	408	460	334	540	507	430	929	434	383	083	145	786	026
412	146	359	363	362	017	837	922	966	741	162	861	257	645	571
680	482	798	249	771	263	305	761	292	545	408	040	659	753	561
970	871	645	393	254	757	072	936	076	922	069	587	163	804	708
256	246	366	137	431	776	175	309	050	064	068	198	002	904	756
218	898	942	856	431	647	438	473	529	312	261	281			

Table 8.26: An initial value  $x_0$

1010	0110	0011	0100	0000	0111	1111	0100	1111	0111	0010	1001			
0000	0100	1111	0000	0010	1010	1011	1111	1000	0101	1110	0011			
1110	1000	1001	1100	1000	1000	0110	0111	0011	0011	1010	0011			
1100	1111	0011	1000	1011	0110	1011	1110	0110	1110	0111	1000			
1101	0011	1101	0010	1000	1101	0000	1100	0100	1011	1110	0011			
0110	0010	1011	0000	1010	1001	0110	0000	0011	1010	0011	1111			
1010	0110	0101	1000	1011	0100	0100	1111	1010	1011	0001	1100			
0000	0011	1101	1001	0001	0000	1111	1010	1001	0111	0111	0111			
0000	1010	0101	0111	0111	0001	0110	1001	0011	1011	0000	0011			
1000	0000	0111	0110	0110	1010	0110	0011	0111	1100	0010	0110			
0011	1001	1010	1111	0001	0010	1111	0010	1100	1111	0110	0100			
0001	1000	0101	0011	0000	0101	1111	1100	0101	0000	0100	0100			
0100	0101	0010	1110	1010	1011	1011	0110	0101	1011	1111	1110			
1100	1001	1011	0110	1001	0111	0111	1110	0101	0111	0011	0100			
1101	1110	0011	1111	1101	0100	1111	1011	1010	0010	0111	1111			
1010	1000	1100	1001	1010	1001	1010	0111	0100	0100	1010	0110			
0011	0010	1110	0111	0101	0111	1101	0000	0110	0000	1110	1100			
0101	1010	0111	1000	0101	1111	0010	1101	0110	0100	0010	1101			
0000	1101	0111	1011	0010	1010	1000	0110	0100	0111	1100	0000			
1101	0000	1011	1111	0101	1011	0011	1110	0010	1110	1101	0001			
1110	1111	1000	0111	1010	0000	1100	0101	0110	0001					

Table 8.27: 1000 BBS pseudo-random bits

The terminology of complexity theory allows us to give a mathematically exact<sup>131</sup> definition by considering parameter-dependent families of Boolean maps  $G_n: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^{r(n)}$ , and analyzing their behaviour when the parameter  $n$  grows to infinity. Such an algorithm—represented by the family  $(G_n)$  of Boolean maps—can be efficient only if the “expanding function”  $r: \mathbb{N} \rightarrow \mathbb{N}$  grows at most polynomially with the parameter  $n$ , otherwise even writing down the output sequence in an efficient way is impossible. Then we measure the cost somehow in a meaningful way, for example count the number of needed bit operations that likewise must be at most polynomial with respect to the asymptotic behaviour.

On the attacker’s side we consider algorithms that predict further bits, or aim at detecting some other weaknesses of our random generator. We analyze the costs of these algorithms also as functions of  $n$ . In case the cost grows faster than any polynomial, say exponentially, we rate the attack as inefficient.

Pursuing this approach would require a lot of additional formalisms including a model of probabilistic algorithms that are essential tools for the cryptanalyst. This would take us to far apart for the moment being. However we bear in mind that there is a mathematically correct theory formalizing the intuitive idea of efficiency. Relying on this knowledge we don’t hesitate to reasoning the naive way, and draft the following definition that in the given form is mathematically incorrect but might be made correct.

**Definition 8.3.1.** Consider a pseudo-random generator. A **next bit predictor** is an algorithm that takes a piece  $u_0, \dots, u_{r-1}$  from the beginning of the pseudo-random sequence and calculates the next bit  $u_r$ , without using the internal parameters of the pseudo-random generator.

The pseudo-random generator **passes the prediction test** if there is no efficient next bit predictor.

For example LFSRs don’t pass the prediction test: We constructed an efficient next bit predictor in Theorem 8.3.1.

**Definition 8.3.2.** Consider a pseudo-random generator. A **distinguisher** is an algorithm that decides whether a given sequence is purely random or is generated by the pseudo-random generator, without using the internal parameters of the pseudo-random generator.

The pseudo-random generator is **perfect** if there is no efficient distinguisher for it.

In particular no efficient statistical test is able to distinguish a perfect pseudo-random generator from a true random source. It is a bit of a surprise that the seemingly much weaker property of passing the prediction test already implies perfectness. In other words the prediction test is “universal”:

**Theorem 8.3.2.** (Yao’s<sup>132</sup> criterion) A pseudo-random generator is perfect if and only if it passes the prediction test.

Stated without proof.

Unfortunately this approach only gives qualitative results, and so it is somewhat dissatisfying. However, as often in complexity theory, this is the best we can achieve.

## The (Conjectured) Perfectness of the BBS Generator

The **factorization hypothesis** states that there is no efficient algorithm that decomposes large natural numbers into their prime factors. This hypothesis is the base of the security of RSA, as

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<sup>131</sup>but not completely satisfying from a practical point of view

<sup>132</sup>Andrew Yao (Yáo Qizhi), Chinese American computer scientist, \*December 24, 1946

well, as stated in Theorem 8.3.3, of the perfectness of the BBS generator:

**Theorem 8.3.3.** (Blum/Blum/Shub/Vazirani<sup>133</sup>/Vazirani<sup>134</sup>) *Assume the factorization hypothesis holds. Then the BBS generator is perfect.*

We omit the proof (that is quite involved). Sloppily expressed the theorem says:

*Whoever is able to predict a single bit of a BBS sequence given a partial sequence is also able to factor the module.*

This statement assumes that the attacker knows the module  $m$  of the BBS generator. However the module might also be secret, that is, considered as a part of the key. Assuming this the cryptographic security of BBS should even be better—but no proof of this stronger statement seems to be known, not even an informal one.

### 8.3.12 Examples and Practical Considerations

We saw that the BBS generator is perfect under a plausible but unproven assumption, the factorization hypothesis. However we don't know relevant concrete details, for example what parameters might be inappropriate. We know that certain initial states generate output sequences with short periods. Some examples of this effect are known, but we are far from a complete answer. However the security proof (depending on the factorization hypothesis) doesn't require additional assumptions. Therefore we may confidently use the BBS generator with a pragmatic attitude: randomly choosing the parameters (primes and initial state) the probability of hitting “bad” values is extremely low, much lower than finding a needle in a haystack, or even in the universe.

Nevertheless some questions are crucial for getting good pseudo-random sequences from the BBS generator in an efficient way:

- How large should we choose the module  $m$ ?
- How many bits can we use for a fixed module and initial state without compromising the security?

The provable results—relative to the factorization hypothesis—are qualitative only, not quantitative. The recommendation to choose a module that escapes the known factorization methods also rests on heuristic considerations only, and doesn't seem absolutely mandatory for a module that itself is kept secret. The real quality of the pseudo-random bit sequence, be it for statistical or for cryptographic applications, can only be assessed by empirical criteria for the time being. We are confident that the danger of generating a “bad” pseudo-random sequence is extremely small<sup>135</sup>, in any case negligible, for modules that escape the presently known factorization algorithms, say at least of a length of 2048 bits, and for a true random choice of the module and the initial state.

For the length of the useable output sequence we only know the qualitative criterion “at most polynomially many” that is useless in a concrete application. But even if we only use

---

<sup>133</sup>Umesh Vazirani, Indian-U.S. American computer scientist

<sup>134</sup>Vijay Vazirani, Indian-U.S. American computer scientist, \*April 20, 1957

<sup>135</sup>Émile Borel (Félix Édouard Justin Émile Borel, French mathematician and politician, January 7, 1871 – February 3, 1956) proposed an informal ranking of negligibility of extremely small probabilities:  $\leq 10^{-6}$  from a human view;  $\leq 10^{-15}$  from a terrestrial view;  $\leq 10^{-45}$  from a cosmic view. By choosing a sufficiently large module  $m$  for RSA or BBS we easily undercut Borel's bounds by far.

“quadratically many” bits we wouldn’t hesitate to take 4 millions bits from the generator with a  $\geq 2000$  bit module. Should we need substantially more bits we would restart the generator with new parameters after every few millions of bits.

An additional question suggests itself: Are we allowed to output more than a single bit of the inner state in each iteration step to enhance the practical benefit of the generator? At least 2 bits?

Vazirani and Vazirani, and independently Alexi, Chor, Goldreich<sup>136</sup>, and Schnorr<sup>137</sup> gave a partial answer to this question, unfortunately also a qualitative one only: at least  $O(\log_2 \log_2 m)$  of the least significant bits are “safe”. Depending on the constants that hide in the “O” we need to choose a sufficiently large module, and trust empirical experience. A common recommendation is using  $\lfloor \log_2 \log_2 m \rfloor$  bits per step. Then for a module  $m$  of 2048 bits, or roughly 600 decimal places, we can use 11 bits per step. Calculating  $x^2 \bmod m$  for a  $n$  bit number  $m$  takes  $(\frac{n}{64})^2$  multiplications of 64-bit integers and subsequently the same number of divisions of the type “128 bits by 64 bits”. For  $n = 2048$  this makes a total of  $2 \cdot (2^5)^2 = 2048$  multiplicative operations to generate 11 bits, or about 200 operations per bit. A well-established rule of thumb says that a modern CPU executes one multiplicative operation per clock cycle<sup>138</sup>. Thus on a 2-GHz CPU with 64-bit architecture we may expect roughly  $2 \cdot 10^9 / 200 \approx 10$  million bits per second, provided the algorithm is implemented in an optimized way. This consideration shows that the BBS generator is almost competitive with a software implementation of a sufficiently secure nonlinear combiner of LFSRs, and is fast enough for many purposes if executed on a present day CPU.

The cryptographic literature offers several pseudo-random generators that follow similar principles as BBS:

**The RSA generator (Shamir).** Choose a random module  $m$  of  $n$  bits as a product of two large primes  $p, q$ , and an exponent  $d$  that is coprime with  $(p-1)(q-1)$ , furthermore a random initial state  $x = x_0$ . The state transition is  $x \mapsto x^d \bmod m$ . Thus we calculate  $x_i = x_{i-1}^d \bmod m$ , and output the least significant bit, or the  $\lfloor \log_2 \log_2 m \rfloor$  least significant bits. If the RSA generator is not perfect, then there exists an efficient algorithm that breaks the RSA cipher. Since calculating  $d$ -th powers is more expensive by a factor  $n$  than squaring the cost is higher than for BBS: for a random  $d$  the algorithm needs  $O(n^3)$  cycles per bit.

**The index generator (Blum/Micali).** As module choose a random large prime  $p$  of  $n$  bits, and find a primitive root<sup>139</sup>  $a$  for  $p$ . Furthermore choose a random initial state  $x = x_0$ , coprime with  $p - 1$ . Then calculate  $x_i = a^{x_{i-1}} \bmod p$ , and output the most significant bit of  $x_i$ , or the  $\lfloor \log_2 \log_2 p \rfloor$  most significant bits. The perfectness of the index generator relies on the hypothesis that calculating discrete logarithms mod  $p$  is hard. The cost per bit also is  $O(n^3)$ .

**The elliptic index generator (Kaliski).** It works like the index generator, but replacing the group of invertible elements of the field  $\mathbb{F}_p$  by an elliptic curve over  $\mathbb{F}_p$  (such a curve is a finite group in a canonical way).

---

<sup>136</sup>Oded Goldreich, Israeli mathematician and computer scientist, \*February 4, 1957

<sup>137</sup>Claus-Peter Schnorr, German mathematician and computer scientist, \*August 4, 1943

<sup>138</sup>Special CPUs that use pipelines and parallelism are significantly faster.

<sup>139</sup>A primitive root for  $p$  is an integer whose powers run through all residue classes  $\neq 0 \bmod p$ , or in algebraic terms, a generating element of the multiplicative group mod  $p$ .

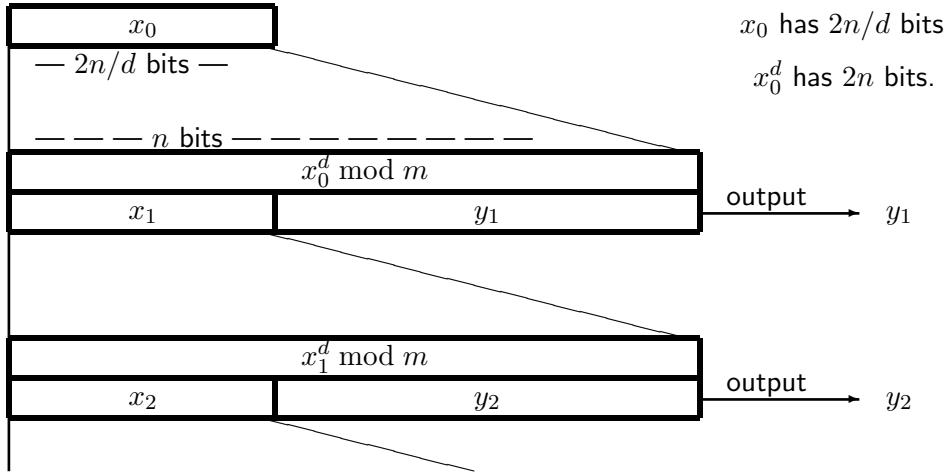


Figure 8.28: Micali-Schnorr generator

### 8.3.13 The Micali-Schnorr Generator

Micali<sup>140</sup> and Schnorr proposed a pseudo-random generator that is a descendent of the RSA generator. Fix an odd number  $d \geq 3$ . The parameter set is the set of all products  $m$  of two primes  $p$  and  $q$  whose bit length differs by at most 1, and such that  $d$  is coprime with  $(p-1)(q-1)$ . For an  $n$ -bit number  $m$  let  $h(n)$  be an integer  $\approx \frac{2n}{d}$ . Then the  $d$ -th power of an  $h(n)$ -bit number is (approximately) a  $2n$ -bit number.

In the  $i$ -th step calculate  $z_i = x_{i-1}^d \bmod m$ . Take the first  $h(n)$  bits as the new state  $x_i$ , that is  $x_i = \lfloor z_i / 2^{n-h(n)} \rfloor$ , and output the remaining bits, that is  $y_i = z_i \bmod 2^{n-h(n)}$ . Thus the bits of the result  $z_i$  are partitioned into two *disjoint* parts: the new state  $x_i$ , and the output  $y_i$ . Figure 8.28 illustrates this scheme.

But why may we hope that this random generator is perfect? This depends on the hypothesis: There is no efficient test that distinguishes the uniform distribution on  $\{1, \dots, m-1\}$  from the distribution of  $x^d \bmod m$  for uniformly distributed  $x \in \{1, \dots, 2^{h(n)}\}$ . If this hypothesis is true, then the Micali-Schnorr generator is perfect. This argument seems tautologic, but heuristic considerations show a relation with the security of RSA and with factorization. Anyway we have to concede that this “proof of security” seems considerably more airy than that for BBS.

How fast do the pseudo-random bits tumble out of the machine? As elementary operations we again count the multiplication of two 64-bit numbers, and the division of a 128-bit number by a 64-bit number with 64-bit quotient. We multiply and divide by the classical algorithms<sup>141</sup>. Thus the product of  $s$  (64-bit) words and  $t$  words costs  $st$  elementary operations. The cost of division is the same as the cost of the product of divisor and quotient.

The concrete recommendation<sup>142</sup> by the inventors is:  $d = 7$ ,  $n = 512$ . The output of each step consists of 384 bits, withholding 128 bits as the new state. The binary power algorithm for a 128-bit number  $x$  with exponent 7 costs several elementary operations:

- $x$  has 128 bits, hence 2 words.
- $x^2$  has 256 bits, hence 4 words, and costs  $2 \cdot 2 = 4$  elementary operations.

<sup>140</sup>Silvio Micali, U.S. American computer scientist, \*October 13, 1954

<sup>141</sup>Multiplication by fast Fourier transformation has an advantage only for much larger numbers.

<sup>142</sup>Today we would choose a larger  $n$ .

- $x^3$  has 384 bits, hence 6 words, and costs  $2 \cdot 4 = 8$  elementary operations.
- $x^4$  has 512 bits, hence 8 words, and costs  $4 \cdot 4 = 16$  elementary operations.
- $x^7$  has 896 bits, hence 14 words, and costs  $6 \cdot 8 = 48$  elementary operations.
- $x^7 \bmod m$  has  $\leq 512$  bits, and likewise costs  $6 \cdot 8 = 48$  elementary operations.

This makes a total of 124 elementary operations; among them only one reduction  $\bmod m$  (for  $x^7$ ). Our reward consists of 384 pseudo-random bits. Thus we get about 3 bits per elementary operation, or, by the assumptions in Section 8.3.12, about 6 milliards<sup>143</sup> bits per second. Compared with the BBS generator this amounts to a factor of about 1000.

Parallelization increases the speed virtually without limit: The Micali-Schnorr generator allows complete parallelization. Thus distributing the work among  $k$  CPUs brings a profit by the factor  $k$  since the CPUs can work independently of each other without need of communication.

### 8.3.14 Summary and Outlook

Bitstream ciphers need cryptographically secure random generators. These probably exist, however their security—*like the security of almost all ciphers*—is mathematically not completely proven.

But implementing a useful bitstream cipher takes more than just a good random generator:

- Message integrity requires additional means such as a combination with a cryptographic hash function.
- The operational conditions must prevent the reuse of (parts of) the key stream in a reliable way. This means that the key management requires utmost prudence. A possible approach<sup>144</sup> is using a longtime general key that consists of certain inner parameters of the random generator, and use the remaining parameters including the initial state as one-time message key.

In contrast with bitblock ciphers where we have the accepted standard AES (and the outdated standard DES) for bitstream ciphers there is no established standard. Closest to standardization is the eSTREAM portfolio developed in a European project from 2004 until 2008. It recommends a bunch of several ciphers [Sch03].

Unfortunately several “proprietary” ciphers, mostly bitstream ciphers developed in back rooms by cryptologic amateurs, found their way into security critical applications, relied on “security by obscurity”, but could easily be analyzed by reverse engineering, and teared to shreds by cryptologists. Therefore we finish this chapter with an advice that in an analogous form applies to all parts of cryptography:

*Never trust a random generator whose algorithm is kept secret, or for which no analysis results are publicly available. Statistical analyses are insufficient as security proofs, just as little as gargantuan periods, or a gigantic choice of initial states.*

---

<sup>143</sup>European milliards = American billions

<sup>144</sup>This was a usual approach with the cipher machines of World War II.

## 8.4 Appendix: Boolean Maps in SageMath

### 8.4.1 What's in SageMath?

SageMath has several functions for bitblocks and Boolean functions. These are scattered over different modules, use different data types, and have some limitations. We could live with them. Nevertheless in this text we mostly use our own independent (and free) implementation<sup>145</sup>.

#### Bitblocks

The SageMath function `binary()` converts integers into bitstrings. Sample usage: `123456789.binary()`, result: `'11101011011100110100010101'`.

The module `sage.crypto.util` has the functions `ascii_to_bin()`, `bin_to_ascii()`, and `ascii_integer()`. Somewhere else, in `sage.crypto.block_cipher.sdes`, we find a function `sdes.string_to_list()` that converts a bitstring into a list.

#### Logical Expressions

Functions for logical expressions are in the modules `sage.logic.boolformula`, `sage.sat.converters`, `sage.sat.solvers`, and `sage.sat.boolean_polynomials`. However these require a mode of thinking different from the approach in this text that is adapted to cryptographic considerations.

#### Boolean Functions and Maps

For dealing with Boolean functions we may look at the module `sage.crypto.boolean_function`, and use its functions `truth_table()`, `algebraic_normal_form()`, and `walsh_hadamard_transform()`, that are implemented as methods of the class `BooleanFunction`.

For dealing with Boolean maps we have the functions `cnf()`, `linear_approximation_matrix()`, and `difference_distribution_matrix()` as methods of the class `SBox` from the module `sage.crypto.mq.sbox`.

### 8.4.2 New SageMath Functions for this Chapter

The SageMath classes and functions defined in the following can be found in the module `bitciphers.sage`. They were developed with Python 3.4.1<sup>146</sup> under OS X, and tested with SageMath 6.2.

Their usage requires a SageMath installation. In command line mode attach the module via the SageMath commands

```
load_attach_path(path='/my/path', replace=False)  
attach('bitciphers.sage')
```

---

<sup>145</sup>This is known as “reinventing the wheel”. Forgive it in view of intended uniformity and consistency. Working through the functions in the SageMath samples should facilitate learning the algorithms. A further aspect: Our modules are written in pure Python and may be used without installing SageMath.

<sup>146</sup>The only relevant difference with Python 2 is the operator `//` for the division of integers.

If you use the worksheet frontend (for a SageMath server), you better copy the single functions each into an input cell<sup>147</sup>.

### 8.4.3 Conversion Routines for Bitblocks

---

**SageMath sample 8.37** Conversion routines for bitblocks

---

```
def int2bbl(number,dim):
    """Converts number to bitblock of length dim via base-2
    representation."""
    n = number                      # catch input
    b = []                           # initialize output
    for i in range(0,dim):
        bit = n % 2                  # next base-2 bit
        b = [bit] + b                # prepend
        n = (n - bit)//2
    return b

def bbl2int(bbl):
    """Converts bitblock to number via base-2 representation."""
    ll = len(bbl)
    nn = 0                           # initialize output
    for i in range(0,ll):
        nn = nn + bbl[i]*(2**((ll-1-i)))
    return nn
```

---

<sup>147</sup>A complete SageMath worksheet containing all samples of this text is available as `bitciphers.sws`, or in human-readable PDF format as `bitciphers_sws.pdf`.

---

**SageMath sample 8.38** Conversion routines for bitblocks (continued)

```
def str2bbl(bitstr):
    """Converts bitstring to bitblock."""
    ll = len(bitstr)
    xbl = []
    for k in range(0,ll):
        xbl.append(int(bitstr[k]))
    return xbl

def bbl2str(bbl):
    """Converts bitblock to bitstring."""
    bitstr = ""
    for i in range(0,len(bbl)):
        bitstr += str(bbl[i])
    return bitstr

def txt2bbl(text):
    """Converts ASCII-text to bitblock."""
    ll = len(text)
    xbl = []
    for k in range(0,ll):
        n = ord(text[k])
        by = int2bbl(n,8)
        xbl.extend(by)
    return xbl

def bbl2sub(bbl):
    """Converts bitblock to subset."""
    ll = len(bbl)
    set = []
    for i in range(0,ll):
        if (bbl[i] == 1):
            set.append(i+1)
    return set
```

---

---

**SageMath sample 8.39** Various compositions of bitblocks

---

```
def coinc(x,y):
    """Counts coincidences between 2 lists."""
    l1 = len(x)
    assert l1 <= len(y), "coinc_Error: Second bitblock too short."
    nn = 0
    for i in range(0,l1):
        if (x[i] == y[i]):
            nn += 1
    return nn

def binScPr(x,y):
    """Scalar product of two binary vectors (lists) mod 2."""
    l = len(x)
    assert l == len(y), "binScPr_Error: Blocks have different lengths."
    res = 0
    for i in range (0,l):
        res += x[i] * y[i]
    return res %2

def xor(plain,key):
    """Binary addition of bitblocks.
    Crops key if longer than plain.
    Repeats key if shorter than plain.
    """
    lk = len(key)
    lp = len(plain)
    ciph = []
    i = 0
    for k in range(0,lp):
        cbit = (plain[k] + key[i]) % 2
        ciph.append(cbit)
        i += 1
        if i >= lk:
            i = i-lk
    return ciph
```

---

#### 8.4.4 Matsui's Test

---

SageMath sample 8.40 Matsui's test

---

```
def Mats_tst(a, b, pc, compl = False):
    """Matsui's test for linear cryptanalysis"""
    NN = len(pc)
    results = []
    for pair in pc:
        ax = binScPr(a,pair[0])
        by = binScPr(b,pair[1])
        result = (ax + by) % 2
        results.append(result)
    t_0 = 0
    for bb in results:
        if bb == 0:
            t_0 = t_0 + 1
    if 2*t_0 > NN:
        if compl:
            return [t_0,1,True]
        else:
            return [t_0,0,True]
    elif 2*t_0 < NN:
        if compl:
            return [t_0,0,True]
        else:
            return [t_0,1,True]
    else:
        return [t_0,randint(0,1),False]
```

---

#### 8.4.5 Walsh Transformation

---

SageMath sample 8.41 Walsh transformation of bitblocks

---

```
def wtr(xx):
    """Fast Walsh transform of a list of numbers"""
    max = 4096                      # max dim = 12
    ll = len(xx)
    assert ll <= max, "wtr_Error: Bitblock too long."
    dim = 0                          # dimension
    m = 1                            # 2**dimension
    while m < ll:
        dim = dim+1
        m = 2*m
    assert ll == m, "wtr_Error: Block length not a power of 2."
    x = copy(xx)                     # initialize auxiliary bitblock
    y = copy(xx)                     # initialize auxiliary bitblock
    mi = 1                           # actual power of 2
    for i in range(0,dim):          # binary recursion
        for k in range(0,ll):
            if ((k//mi) % 2 == 1):   # picks bit nr i
                y[k] = x[k-mi] - x[k]
            else:
                y[k] = x[k+mi] + x[k]
        for k in range(0,ll):
            x[k] = y[k]
        mi = 2*mi                  # equals 2**i in the next step
    return x
```

---

#### 8.4.6 A Class for Boolean Functions

---

SageMath sample 8.42 A class for Boolean functions

---

```
class BoolF(object):
    """Boolean function
    Attribute: a list of bits describing the truth table of the function
    Attribute: the dimension of the domain"""

    __max = 4096                                # max dim = 12

    def __init__(self,blist,method="TT"):
        """Initializes a Boolean function with a truth table
        or by its algebraic normal form if method is ANF."""
        ll = len(blist)
        assert ll <= self.__max, "BoolF_Error: Bitblock too long."
        dim = 0                                     # dimension
        m = 1                                       # 2**dim
        while m < ll:
            dim = dim+1
            m = 2*m
        assert ll == m, "booltestError: Block length not a power of 2."
        self.__dim = dim
        if method=="TT":
            self.__tlist = blist
        else:
            self.__tlist=self.__convert(blist)

    def __convert(self,xx):
        """Converts a truth table to an ANF or vice versa."""
        x = copy(xx)                               # initialize auxiliary bitblock
        y = copy(xx)                               # initialize auxiliary bitblock
        mi = 1                                     # actual power of 2
        for i in range(0,self.__dim): # binary recursion
            for k in range(0,2**(self.__dim)):
                if ((k//mi) % 2 == 1): # picks bit nr i
                    y[k] = (x[k-mi] + x[k]) % 2 # XOR
                else:
                    y[k] = x[k]
        for k in range(0,2**(self.__dim)):
            x[k] = y[k]
        mi = 2*mi                                  # equals 2**i in the next step
        return x
```

---

---

**SageMath sample 8.43** Boolean functions (continued)

```
def getTT(self):
    """Returns truth table as bitlist."""
    return self.__tlist

def valueAt(self,xx):
    """Evaluates Boolean function."""
    l1 = len(xx)
    assert l1 == self.__dim, "booltestError: Block has false length."
    index = bbl2int(xx)
    return self.__tlist[index]

def getDim(self):
    """Returns dimension of definition domain."""
    return self.__dim

def getANF(self):
    """Returns algebraic normal form as bitlist."""
    y = self.__convert(self.__tlist)
    return y

def deg(self):
    """Algebraic degree of Boolean function"""
    y = self.__convert(self.__tlist)
    max = 0
    for i in range (0,len(y)):
        if y[i] != 0:
            b = int2bbl(i,self.__dim)
            wt = sum(b)
            if wt > max:
                max = wt
    return max
```

---

---

**SageMath sample 8.44** Boolean functions: Walsh spectrum and human-readable output

---

```
def wspec(self):
    """Calculate Walsh spectrum."""
    ff = copy(self.__tlist)
    ll = len(ff)
    gg = []
    for i in range(0,ll):
        bit = ff[i]
        if (bit):
            gg.append(-1)
        else:
            gg.append(1)
    ff = wtr(gg)
    return ff

def printTT(self):
    """Prints truth table to stdout."""
    for i in range(0,2**self.__dim):
        bb = int2bb1(i,self.__dim)
        print "Value at " + bbl2str(bb) + " is " + repr(self.__tlist[i])

def printANF(self):
    """Prints algebraic normal form to stdout."""
    y = self.__convert(self.__tlist)
    for i in range(0,2**self.__dim):
        monom = int2bb1(i,self.__dim)
        print "Coefficient at " + bbl2str(monom) + " is " + repr(y[i])
```

---

#### 8.4.7 A Class for Boolean Maps

---

**SageMath sample 8.45** A class for Boolean maps

---

```
class BoolMap(object):
    """Boolean map
    Attribute: a list of Boolean functions
    Attribute: the dimensions of domain and range"""

    __max = 8                                # max dim = 8

    def __init__(self,flist):
        """Initializes a Boolean map with a list of Boolean functions."""
        qq = len(flist)
        assert qq <= self.__max, "BoolMap_Error: Too many components."
        ll = len(flist[0].getTT())
        dim = 0                                     # dimension
        m = 1                                       # 2**dim
        while m < ll:
            dim = dim+1
            m = 2*m
        assert ll == m, "BoolMap_Error: Block length not a power of 2."
        assert dim <= self.__max, "BoolMap_Error: Block length exceeds maximum."
        self.__dimd = dim
        self.__dimr = qq
        for i in range(1,qq):
            li = len(flist[i].getTT())
            assert li == ll, "BoolMap_Error: Blocks of different lengths."
        self.__flist = flist

    def getFList(self):
        """Returns component list."""
        return self.__flist

    def getDim(self):
        """Returns dimension of preimage and image domain."""
        return [self.__dimd, self.__dimr]
```

---

---

**SageMath sample 8.46** Boolean maps (continued)

```
def getTT(self):
    """Returns truth table as list of bitlists."""
    nn = 2**self._dimd
    qq = self._dimr
    clist = []
    for j in range(0,qq):
        clist.append(self._flist[j].getTT())
    transp = []
    for j in range(0,nn):
        trrow = []
        for i in range(0,qq):
            trrow.append(clist[i][j])
        transp.append(trrow)
    return transp

def printTT(self):
    """Prints truth table to stdout."""
    nn = 2**self._dimd
    qq = self._dimr
    print("Dimensions of truth table:", nn, "by", qq)
    clist = []
    for j in range(0,qq):
        clist.append(self._flist[j].getTT())
    transp = []
    for j in range(0,nn):
        trrow = []
        for i in range(0,qq):
            trrow.append(clist[i][j])
        transp.append(trrow)
    for j in range(0,nn):
        bb = int2bb1(j,self._dimd)
        print("Value at", bb, "is", transp[j])

def valueAt(self,xx):
    """Evaluates Boolean map."""
    ll = len(xx)
    assert ll == self._dimd, "boolF_Error: Block has false length."
    index = bbl2int(xx)
    vlist = []
    for j in range(0,self._dimr):
        vlist.append(self._flist[j].getTT()[index])
    return vlist
```

---

---

**SageMath sample 8.47 Boolean maps (continued)**

---

```
def wspec(self):
    """Calculate Walsh spectrum."""
    dd = self.getDim()
    tt = self.getTT()
    m = 2**(dd[0])
    t = 2**(dd[1])
    nullv = [0] * t
    charF = []
    for k in range(0,m):
        charF.append(copy(nullv))
    for k in range(0,m):
        index = bbl2int(tt[k])
        charF[k][index] = 1
    blist = []
    for k in range(0,m):
        blist.extend(charF[k])
    speclist = wtr(blist)
    specmat = []
    for k in range(0,m):
        specmat.append(speclist[k*t:k*t+t])
    return specmat

def linApprTable(self):
    """Calculate the linear approximation table."""
    lpr = self.wspec()
    dd = self.getDim()
    m = 2**(dd[0])
    t = 2**(dd[1])
    for k in range(0,m):
        for i in range(0,t):
            lpr[k][i] = (lpr[k][i] + m)//2
    return lpr
```

---

---

**SageMath sample 8.48 Boolean maps: linear profile**

---

```
def linProf(self, extended=False):
    """Calculate linear profile. If extended is True, also
       calculate maximum potential and corresponding linear forms."""
    lpr = self.wspec()
    dd = self.getDim()
    m = 2**dd[0]
    t = 2**dd[1]
    for k in range(0,m):
        for i in range(0,t):
            lpr[k][i] = lpr[k][i] * lpr[k][i]
    if extended:
        flatlist = []
        for row in lpr:
            flatlist.extend(row)
        denominator = flatlist.pop(0)
        mm = max(flatlist)
        ixlist = []
        for k in range(0,m):
            for i in range(0,t):
                if lpr[k][i] == mm:
                    ixlist.append([k,i])
        return [lpr, mm, denominator, ixlist]
    else:
        return lpr
```

---

#### 8.4.8 Lucifer and Mini-Lucifer

---

SageMath sample 8.49 S-boxes and bit permutation of Lucifer

---

```
#----- Define S0 -----
f1 = BoolF([1,1,0,1,1,1,0,0,0,0,1,0,0,1])
f2 = BoolF([1,1,1,0,1,1,0,0,0,1,0,0,0,1,0])
f3 = BoolF([0,1,1,1,1,0,1,0,1,1,1,0,0,0,0])
f4 = BoolF([0,1,1,0,0,1,1,0,0,0,1,1,1,0,1])
S0 = BoolMap([f1,f2,f3,f4])

#----- Define S0 inverse -----
fi1 = BoolF([0,1,1,1,1,1,0,1,1,0,0,0,0,0])
fi2 = BoolF([1,0,0,0,1,1,0,0,1,1,0,1,0,1,1,0])
fi3 = BoolF([1,1,0,1,0,1,0,1,1,0,1,1,0,0,0,0])
fi4 = BoolF([1,1,0,0,1,0,1,0,1,0,1,0,0,1,0,1])
S0inv = BoolMap([fi1,fi2,fi3,fi4])

#----- Define S1 -----
g1 = BoolF([0,0,1,1,0,1,0,0,1,1,0,1,0,1,1,0])
g2 = BoolF([1,0,1,0,0,0,0,1,1,1,0,0,1,1,0,1])
g3 = BoolF([1,1,1,0,1,1,0,0,0,0,0,1,1,1,0,0])
g4 = BoolF([1,0,0,1,1,1,0,0,0,1,1,0,0,1,0,1])
S1 = BoolMap([g1,g2,g3,g4])

#----- Define S1 inverse -----
gi1 = BoolF([0,1,0,0,0,1,1,0,1,0,1,0,1,1,0,1])
gi2 = BoolF([1,0,0,1,1,1,1,0,1,0,0,1,0,0,0,1])
gi3 = BoolF([1,1,0,0,1,1,0,0,1,1,1,0,0,0,1,0])
gi4 = BoolF([0,0,1,0,1,1,0,0,0,1,1,1,0,1,0,1])
S1inv = BoolMap([gi1,gi2,gi3,gi4])

def P(b):
    """Lucifer's bit permutation"""
    pb = [b[2],b[5],b[4],b[0],b[3],b[1],b[7],b[6]]
    return pb
```

---

---

**SageMath sample 8.50** Mini-Lucifer over r rounds

---

```
def miniLuc(a,k,r):
    """Mini-Lucifer, encrypts 8-bit a with 16-bit key k over r rounds."""
    l1 = len(a)
    assert l1 == 8, "miniLuc_Error: Only blocks of length 8 allowed."
    lk = len(k)
    assert lk == 16, "miniLuc_Error: Only keys of length 16 allowed."
    k0 = k[0:8]           # split into subkeys
    k1 = k[8:16]
    aa = a                # round input
    # --- begin round
    for i in range(0,r): # round number is i+1
        if (i % 2 == 0): # select round key
            rndkey = k0
        else:
            rndkey = k1
        b = xor(aa,rndkey)      # add round key
        bleft = b[0:4]          # begin substitution
        bright = b[4:8]
        bbleft = S0.valueAt(bleft)
        bbright = S1.valueAt(bright)
        bb = bbleft + bbright # end substitution
        if (i+1 == r):         # omit permutation in last round
            aa = bb
        else:
            aa = P(bb)
    # --- end round
    if (r % 2 == 0):         # add subkey after last round
        finkey = k0
    else:
        finkey = k1
    c = xor(aa,finkey)
    return c
```

---

#### 8.4.9 A Class for Linear Feedback Shift Registers

---

**SageMath sample 8.51** A class for linear feedback shift registers

---

```
class LFSR(object):
    """Linear Feedback Shift Register
    Attributes: the length of the register
                a list of bits describing the taps of the register
                the state
    """
    __max = 1024 # max length

    def __init__(self,blist):
        """Initializes a LFSR with a list of taps
        and the all 0 state."""
        ll = len(blist)
        assert ll <= self.__max, "LFSR_Error: Bitblock too long."
        self.__length = ll
        self.__taplist = blist
        self.__state = [0] * ll

    def __str__(self):
        """Defines a printable string telling the internals of
        the register."""
        outstr = "Length: " + str(self.__length)
        outstr += " | Taps: " + bbl2str(self.__taplist)
        outstr += " | State: " + bbl2str(self.__state)
        return outstr

    def getLength(self):
        """Returns the length of the LFSR."""
        return self.__length

    def setState(self,slist):
        """Sets the state."""
        sl = len(slist)
        assert sl == self.__length, "LFSR_Error: Bitblock has wrong length."
        self.__state = slist
```

---

---

**SageMath sample 8.52** A class for linear feedback shift registers (continued)

---

```
def nextBits(self,n):
    """Returns the next n bits as a list and updates the state."""
    outlist = []
    a = self.__taplist
    u = self.__state
    for i in range (0,n):
        b = binScPr(a,u)
        c = u.pop()
        u.insert(0,b)
        outlist.append(c)
    self.__state = u
    return outlist
```

---

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# Chapter 9

## Homomorphic Ciphers

([Martin Franz](#), Jan 2013)

### 9.1 Introduction

Homomorphic ciphers are public-key cryptosystems with special properties. They allow performing certain arithmetic operations on encrypted ciphertexts, without knowing the corresponding plaintexts and without having to decrypt the ciphertexts first. These special properties have led to a huge amount of applications for homomorphic ciphers, e.g. in the domain of cloud computing. A very famous and relatively new cryptosystem with homomorphic properties is the Paillier cryptosystem. But also some of the older and well established cryptosystems, such as ElGamal or RSA, have homomorphic properties.

### 9.2 Origin of the term “homomorphic”

We first clarify the meaning and the origin of the term “homomorphic”. This term in cryptography is derived from its counterpart in mathematics: in mathematics, a homomorphism is a structure-preserving map between two algebraic structures. In the common sense this means, that a homomorphism  $f : X \rightarrow Y$  maps the structure of  $X$  to the structure of  $Y$ . Using an example, this can be easily illustrated: Let  $(X, +)$  and  $(Y, *)$  two algebraic groups with group operations  $+$  and  $*$ , respectively. A homomorphism  $f : X \rightarrow Y$  maps any given  $x \in X$  to a value  $y \in Y$ , in a way that it holds:

$$f(x_1 + x_2) = f(x_1) * f(x_2)$$

for any two  $x_1, x_2$  in  $X$ . This means, that for any two values  $x_1, x_2$  it does not matter whether we first compute their sum (group operation of  $X$ ) and then apply  $f$  (this is the left side of the above given equation); or, whether we first apply  $f$  to the values  $x_1, x_2$ , and then compute their product in  $Y$ , thus apply the group operation of  $Y$ . Please note that the operations  $+$  and  $*$  were chosen here only as an example, they always depend on the algebraic group they belong to. Naturally, the same relation holds for homomorphisms between groups with the same group operation.

**Example:** Let  $X = \mathbb{Z}$  be the set of integer values. The set  $\mathbb{Z}$  together with the addition operation forms an algebraic group  $G_1 = (\mathbb{Z}, +)$ . Similarly, the real values  $\mathbb{R}$  without the value zero together with the multiplication operation form a group  $G_2 = (\mathbb{R} \setminus \{0\}, *)$ . The function  $f : \mathbb{Z} \rightarrow \mathbb{R} \setminus \{0\}$ ,  $z \mapsto e^z$  is a homomorphism, since for all  $z_1, z_2 \in \mathbb{Z}$  it holds:  $f(z_1 + z_2) = e^{(z_1 + z_2)} =$

$f(z_1) * f(z_2)$ . On the contrary,  $f : \mathbb{Z} \rightarrow \mathbb{R} \setminus \{0\}, z \rightarrow z^2$  is an example for a function which is not a homomorphism.

### 9.3 Decryption function is a homomorphism

In the remainder of this chapter we will consider public-key cryptosystems with a special property, namely that its decryption function is a homomorphism. A public-key cryptosystem with this property will be called homomorphic.

Let us for now assume, the above described homomorphism  $f$  is the decryption function of a known cryptosystem. This means that we can perform certain algebraic operations in the ciphertext space, knowing which effects this will have on their plaintexts. Following the above given example:

$Y$  corresponds to the set of cipher texts,  $X$  is the set of plaintexts. For two plaintexts  $x_1, x_2$  with corresponding ciphertexts  $y_1, y_2$  it holds:

$$f(y_1 * y_2) = f(y_1) + f(y_2) = x_1 + x_2$$

This equation can be interpreted as follows: If we multiply two ciphertexts  $y_1, y_2$  with each other and subsequently decrypt their product, then we will obtain the sum of the originally encrypted values  $x_1$  and  $x_2$ . Everybody can – without knowledge of the plaintexts, without having to decrypt and even without knowing the private decryption key – compute a product of the two ciphertexts and knows that upon decryption the owner of the private key will obtain the sum of the two originally encrypted plaintexts.

## 9.4 Examples of homomorphic ciphers

### 9.4.1 Paillier cryptosystem

The most famous cryptosystem with homomorphic properties is the one by Paillier [Pai99]. First we will see how the Paillier key generation process works. After that, we will show that the Paillier cryptosystem indeed has homomorphic properties.

#### 9.4.1.1 Key generation

First, we generate two random prime numbers  $p, q$  in a way that their product  $n = pq$  forms a valid RSA modulus. As for common RSA, the value  $n$  should have a bit length of at least 1024 bits.

Using the prime values  $p$  and  $q$ , we can compute the value  $\lambda = lcm(p - 1, q - 1)$ .  $lcm$  here denotes the least common multiple. The RSA modulus  $n$  will now be the public key, while the private key is the value  $\lambda$ .

#### 9.4.1.2 Encryption

Let  $m$  be the message which will be encrypted, where  $m$  is taken from the plaintext space  $\mathbb{Z}_n$ . For each encryption, we first choose a random element  $r$  from the plaintext space  $\mathbb{Z}_n$ . Subsequently, using the public key, we compute the ciphertext  $n$  as:

$$c = E(m, r) = (n + 1)^m * r^n \bmod n^2$$

#### 9.4.1.3 Decryption

Given the private key  $\lambda$  and a ciphertext  $c \in \mathbb{Z}_{n^2}^*$ , we first compute  $S = c^\lambda \bmod n^2$  and subsequently  $T = \phi(n)^{(-1)} \bmod n^2$ , where  $\phi$  denotes the Euler function.

Finally, we compute the plaintext  $m = D(c) = (S - 1)/n * T \bmod n$ .

#### 9.4.1.4 Homomorphic property

We will now show that the Paillier cryptosystem has the homomorphic property as described above. For this, we will use  $E$  to denote the encryption and  $D$  to denote the decryption function of the Paillier cryptosystem. For simplicity, we set  $g := n + 1$ . For any two plaintexts  $m_1, m_2$  and random values  $r_1, r_2$  we obtain ciphertexts  $c_1, c_2$  as

$$c_1 = g^{m_1} * r_1^n \bmod n^2 \text{ and } c_2 = g^{m_2} * r_2^n \bmod n^2,$$

respectively. Now it is easy to see that for the product  $c_3 = c_1 * c_2$  it holds

$$c_3 = (g^{m_1} * r_1^n \bmod n^2) * (g^{m_2} * r_2^n \bmod n^2) = g^{m_1+m_2} * (r_1 * r_2)^n \bmod n^2 = E(m_1 + m_2, r_1 * r_2)$$

Thus, the product of two given ciphertexts is in fact a valid ciphertext, namely the encryption of the sum of the originally encrypted messages. Now it is straightforward to see that the decryption function is a homomorphism. Given two plaintexts  $m_1, m_2$  it holds

$$D(E(m_1, r_1) * E(m_2, r_2)) = D(E(m_1 + m_2, r_1 * r_2)) = m_1 + m_2 = D(E(m_1, r_1)) + D(E(m_2, r_2))$$

### 9.4.2 Other cryptosystems

Also older public-key cryptosystems can have homomorphic properties. Both the ElGamal cryptosystem and RSA constitute famous examples. We will show their homomorphic properties by means of some easy examples.

#### 9.4.2.1 RSA

Let  $(e, n)$  be the public RSA key ( $e$  the public encryption exponent,  $n$  the RSA modulus). For any two messages  $m_1, m_2$  we obtain encryptions  $c_1 = m_1^e \bmod n$  and  $c_2 = m_2^e \bmod n$ . Now for the product of these two encryptions it holds:  $c_1 * c_2 = m_1^e * m_2^e \bmod n = (m_1 * m_2)^e \bmod n$ . Thus, we obtain an encryption of the product of the two messages  $m_1$  and  $m_2$ . As it is straightforward to see, this property holds for any two plaintexts  $m_1, m_2$  and similar as for Paillier, the decryption function is a homomorphism. As we have seen here, RSA is an example for a homomorphism, where both groups have the same group operation.

#### 9.4.2.2 ElGamal

Similar to RSA we can also show the homomorphic properties of the ElGamal cryptosystem. Let  $(p, g, K)$  the public key while the private key is  $k$  (thus, it holds  $g^k \bmod p = K$ ). For any two messages  $m_1, m_2$  and random values  $r, s$  we obtain encryptions  $(R, c_1) = (K^r \bmod p, m_1 * g^r \bmod p)$  and  $(S, c_2) = (K^s \bmod p, m_2 * g^s \bmod p)$ . As for RSA, we verify that their product  $(R * S, c_1 * c_2)$  is an encryption of  $m_1 * m_2$ . Again it is straightforward to see that the decryption function is a homomorphism.

## 9.5 Applications

The homomorphic property of the Paillier cryptosystem can be used to add two encrypted values or to multiply any value under encryption with a known constant (note that the multiplication corresponds to the repeated application of the addition operation). This makes homomorphic ciphers to important and easy to use base primitives in cryptographic applications.

1. One of these applications is the so called “Electronic Voting”. Electronic voting allows a large number of voters to submit their ballots in an encrypted form. This is important in situations, where the voters cannot come together to the same location. This happens, for example, if the voters can only communicate over the internet via email. If the voting behavior of the single parties should remain secret, then the use of homomorphic ciphers is a good solution to this problem. The main principle of electronic voting using homomorphic ciphers is as follows.

All voters encrypt their ballots, using homomorphic encryption (see at the left site of the screenshot). The screenshot depicts the next steps (1 to 3):

- All voters (on the left in the figure 9.1) encrypt the value 1 if they opt positive and the value 0, if opposed to the decision.
- Using the homomorphic property, one can compute the sum of all encrypted ballots. Since this happens on encrypted values, the voting behavior of all participants remains secret.
- At the end, the result of the election is determined and published, this happens by decrypting the sum which was computed using the homomorphic property.

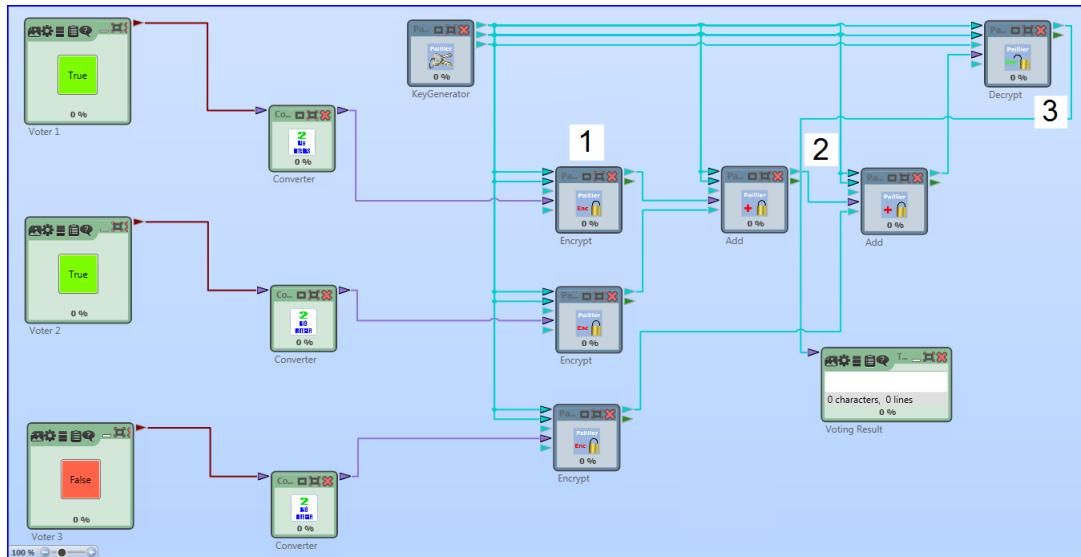


Figure 9.1: Voting example for Paillier

2. A second application of homomorphic ciphers is “Secure Multiparty Computation”. Here, two or more parties can compute any commonly known function. Each of the parties provides one or more of the inputs for the function to be computed. The goal of the secure computations is to keep all private inputs secret, while only the result of the function is

revealed. The use of homomorphic encryption helps to perform these computations on encrypted data. However, since the Paillier encryption only allows to compute additions of encrypted values (and, e.g. no multiplications can be performed), a number of additional methods and techniques have to be applied. The Wikipedia page [Wikb] offers a great start for reading more about this topic and more advanced techniques for Secure Multiparty Computation.

3. Furthermore it is expected, that homomorphic encryption will provide great advantages in the areas of “Cloud Computing”. Using so called “fully-homomorphic encryption” [Wikka] it will be possible to run large applications on external servers only on encrypted data. For this, necessarily one needs to be able to perform both arithmetic operations, the addition and the multiplication, on encrypted data (in contrast to Paillier encryption, which only allows performing additions). Such a crypto system was first presented in 2009 [Gen09].

## 9.6 Homomorphic ciphers in CrypTool

### 9.6.1 CrypTool 2

In CrypTool 2 you will find an implementation of the Paillier cryptosystem. Among the available components, there are components for key generation (Paillier Key Generator), an example for encryption and decryption with Paillier (called Paillier Text), as well as examples which illustrate the homomorphic properties of the cryptosystem (Paillier Addition, Paillier Blinding and Paillier Voting).

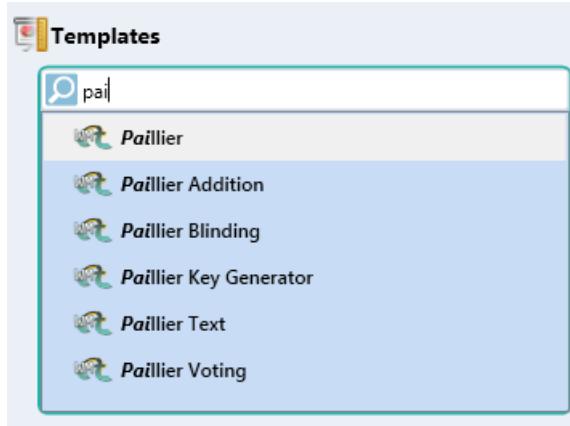


Figure 9.2: Paillier cryptosystem in CrypTool 2 (CT2)

### 9.6.2 JCrypTool

In JCrypTool there is an implementation (see Figure 9.3), which visualizes the homomorphic properties of various cryptosystem. For RSA and Paillier it shows, that multiplications, for RSA, and additions for Paillier, respectively, can be performed on encrypted values. For the fully-homomorphic cryptosystem by Gentry it is possible to perform both multiplications, as well as additions on encrypted values.

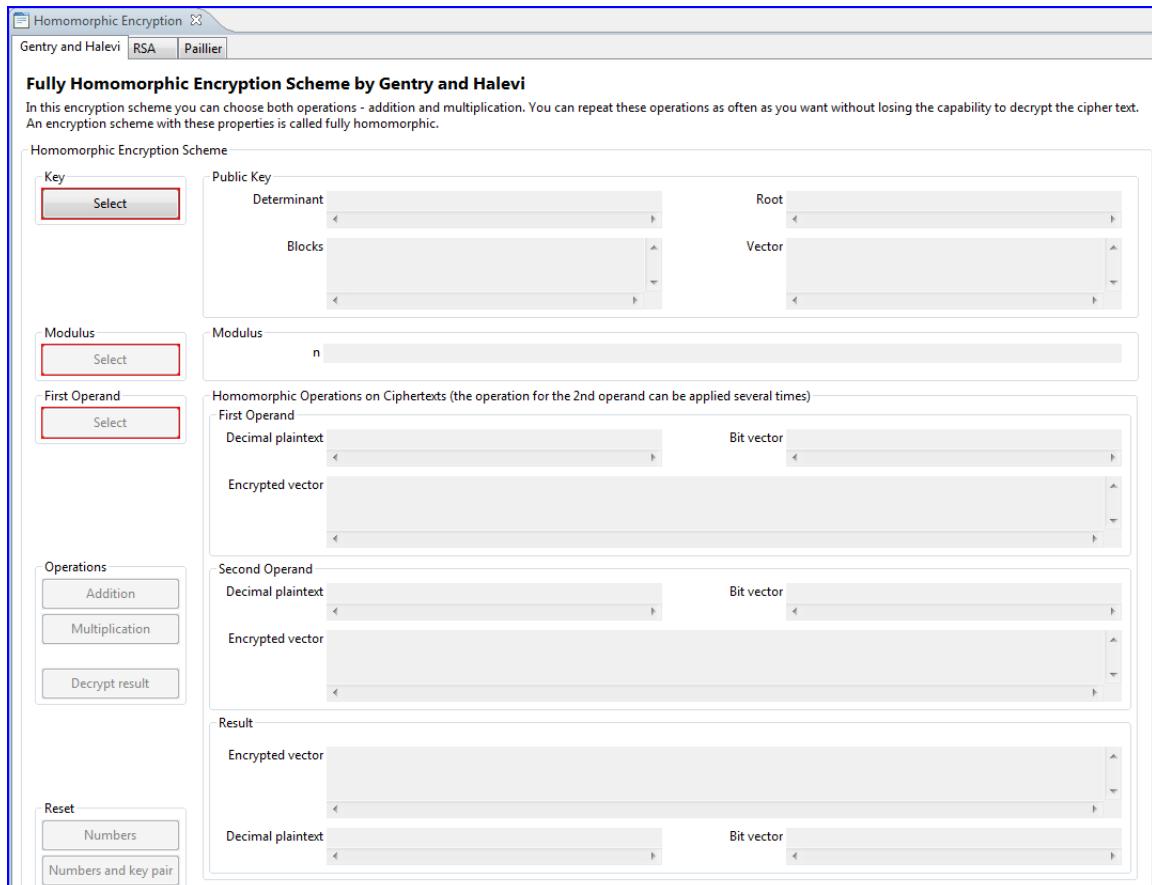


Figure 9.3: Visualization of homomorphic properties in JCrypTool (JCT)

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## Chapter 10

# Survey on Current Academic Results for Solving Discrete Logarithms and for Factoring – And How to React in Practice

([Antoine Joux](#), [Arjen Lenstra](#), & [Alexander May](#); Apr 2014)

**Abstract:** Recent algorithmic developments for solving discrete logarithms in finite fields of small characteristic led to some uncertainty among cryptographic users fueled by the media about the impact for the security of currently deployed cryptographic schemes (see for instance the discussion in [PRSS13] using the catchword “cryptocalypse”). This survey provides a broader picture about the currently best algorithms for computing discrete logarithms in various groups and about the status of the factorization problem. Our goal is to clarify what currently can be done algorithmically and what cannot be done without further major breakthroughs. In particular, we currently do not see a way how to extend the current algorithmic progress for finite fields of small characteristic to either the case of large characteristic finite fields or to the integer factorization problem.

## 10.1 Generic Algorithms for the Discrete Logarithm Problem in any Group

**Management Summary:** The hardness of the Discrete Logarithm Problem depends on the group over which it is defined. In this chapter we review cryptanalytical algorithms that work for any group. From a cryptographic point of view it is desirable to identify groups for which one is unable to find better algorithms. One candidate for these groups are elliptic curve groups.

In this chapter, we describe *general* cryptanalytical algorithms that apply for *any* finite abelian group. That means, any group used in cryptography – e.g. multiplicative groups of finite fields or of elliptic curves – are susceptible to this kind of algorithms. We will see that we can always compute a discrete logarithm in a group of order  $n$  in  $\mathcal{O}(\sqrt{n})$  steps by Pollard's Rho Method. This in turn means that for achieving a security level of  $2^k$  one has to choose a group of order at least  $2^{2k}$ . E.g. for achieving a security level of 80 bits, one has to choose a group of order at least 160 bits. This explains why in practice we usually take elliptic curve groups with at least 160 bit order.

Moreover, let  $G$  be a group of order  $n$  and let  $n = p_1^{e_1} \cdots p_\ell^{e_\ell}$  be the prime factorization of  $n$ . Then we will see that discrete logarithms in  $G$  can be computed in time  $\mathcal{O}(e_1\sqrt{p_1} + \dots + e_\ell\sqrt{p_\ell})$ . Notice that this bound is equal to Pollard's bound  $O(\sqrt{n})$  if and only if  $n$  is a prime. Otherwise, the complexity of computing the discrete logarithm is mainly determined by the size of the largest prime divisor of its group order. This explains why e.g. Schnorr/DSA signatures are implemented in groups which contain by construction a prime factor of size at least 160 bits. This also explains why usually elliptic curve groups have prime order or order containing only a very small **smooth co-factor**.

### 10.1.1 Pollard Rho Method

Let  $G$  be a finite abelian group. Let  $g$  be a generator of some large subgroup  $G' = \{g, g^2, \dots, g^n\} \subseteq G$  (e.g.  $g$  could generate  $G$  itself). Let  $y = g^x$ . Then the discrete logarithm problem is to find on input  $g$  and  $y$  the output  $x \bmod n$ . We write  $x = \text{dlog}_g(y)$ .

Pollard's Rho method tries to generate elements  $g^{a_i}y^{b_i} \in G'$  with  $a_i, b_i \in \mathbb{N}$  in a pseudo-random but deterministic fashion. Let us assume for simplicity that we generate random elements from the  $n$  elements in  $G'$ . Then by the birthday paradox, we expect to find after only  $\mathcal{O}(\sqrt{n})$  steps two elements which are identical. In our case, this means that

$$g^{a_i}y^{b_i} = g^{a_j}y^{b_j}.$$

This can be rewritten as  $g^{\frac{a_i-a_j}{b_j-b_i}} = y$ . This in turn implies that we can recover our discrete logarithm as  $x \equiv \frac{a_i-a_j}{b_j-b_i} \bmod n$ .

Hence, with Pollard's Rho method one can compute discrete logarithms in any finite abelian group of order  $n$  in  $\mathcal{O}(\sqrt{n})$  steps. By using so-called cycle-finding techniques, one can also show that Pollard's Rho Method can be implemented within constant space.

Moreover, it is also possible to improve the efficiency of square root algorithms when multiple discrete logarithms in the same group are desired: When computing  $L$  distinct logarithms, one can reduce the global cost from  $\mathcal{O}(L\sqrt{n})$  to  $\mathcal{O}(\sqrt{Ln})$  [FJM14].

### 10.1.2 Silver-Pohlig-Hellman Algorithm

As before let  $y = g^x$  for a generator  $g$  of order  $n$ . We have to compute the discrete logarithm  $x \bmod n$ . Moreover, let  $n = p_1^{e_1} \cdot \dots \cdot p_\ell^{e_\ell}$  be the prime factorization of  $n$ . Then by the Chinese Remainder Theorem  $x \bmod n$  is uniquely defined by the system of congruences

$$\begin{aligned} x &\equiv x_1 \bmod p_1^{e_1} \\ &\vdots \\ x &\equiv x_\ell \bmod p_\ell^{e_\ell}. \end{aligned} \tag{10.1}$$

The algorithm of Silver-Pohlig-Hellman computes all discrete logarithms  $x_i \bmod p_i$  in the subgroups of order  $p_i$  in  $\mathcal{O}(\sqrt{p_i})$  steps by using Pollard's Rho method. Then it is quite easy to find a logarithm modulo the prime power  $x_i \bmod p_i^{e_i}$  by a Hensel lifting process that performs  $e_i$  calls to the discrete logarithm procedure modulo  $p_i$ . In a Hensel lifting process, we start by a solution  $x_i \bmod p_i$ , and then consecutively compute  $x_i \bmod p_i^2$ ,  $x_i \bmod p_i^3$ , etc. until  $x_i \bmod p_i^{e_i}$  (see [May13] for Hensel's formula).

Finally, one computes the desired discrete logarithm  $x \bmod n$  from the above system of equations (10.1) by Chinese Remaindering. In total, the running time is mainly determined by computing  $x_i \bmod p_i$  for the largest prime factor  $p_i$ . That is, the running time is roughly  $\mathcal{O}(\max_i\{\sqrt{p_i}\})$ .

### 10.1.3 How to Measure Running Times

Throughout this survey, we want to measure the running time of analysis algorithms for discrete logarithms as a function of the bit-size of  $n$ . Note that any integer  $n$  can be written with (roughly)  $\log n$  bits, where  $\log$  is to base 2. Thus, the *bit-size* of  $n$  is  $\log n$ .

For expressing our running times we use the notation  $L_n[b, c] = \exp^{c \cdot (\ln n)^b (\ln \ln n)^{1-b}}$  for constants  $b \in [0, 1]$  and  $c > 0$ . Notice that  $L_n[1, c] = e^{c \cdot \ln n} = n^c$  is a function that is for constant  $c$  a polynomial in  $n$ . Therefore, we say that  $L_n[1, c]$  is *polynomial* in  $n$ . Also notice that  $L_n[1, c] = n^c = (2^c)^{\log_2 n}$  is a function that is exponential in  $\log n$ . Therefore, we say that  $L_n[1, c]$  is *exponential* in the bit-size  $\log n$  of  $n$ . So our Pollard Rho algorithm achieves exponential running time  $L[1, \frac{1}{2}]$ .

On the other end,  $L_n[0, c] = e^{c \cdot \ln \ln n} = (\ln n)^c$  is *polynomial* in the bit-size of  $n$ . Notice that the first parameter  $b$  is more important for the running time than the second parameter  $c$ , since  $b$  interpolates between polynomial and exponential running time. We shortly denote  $L_n[b]$  if we do not want to specify the constant  $c$ .

Some of the most important algorithms that we discuss in the subsequent sections achieve a running time of  $L_n[\frac{1}{2} + o(1)]$  or  $L_n[\frac{1}{3} + o(1)]$  (where the  $o(1)$ -part vanishes for  $n \rightarrow \infty$ ), which is a function that grows faster than any polynomial but slower than exponential. For cryptographic schemes, such attacks are completely acceptable, since the desired security level can be easily achieved by a moderate adjustment of the key sizes.

However, the recent algorithm of Joux et al. for computing discrete logarithms in finite fields of small characteristic achieves a running time of  $L_n[o(1)]$ , where  $o(1)$  converges to 0 for  $n \rightarrow \infty$ . This means that these algorithms are quasi polynomial time, and the underlying fields are no longer acceptable for cryptographic applications. A finite field  $\mathbb{F}_{p^n}$  has small characteristic if  $p$  is small, i.e. the base field  $\mathbb{F}_p$  is small and its extension degree  $n$  is usually large. In the recent algorithms we need a small  $p$ , since the algorithms enumerate over all  $p$  elements in the base field  $\mathbb{F}_p$ .

#### 10.1.4 Insecurity in the Presence of Quantum Computers

In 1995, Shor published an algorithm for computing discrete logarithms and factorizations on a quantum computer. He showed that computing discrete logarithms in *any* group of order  $n$  can be done in polynomial time which is almost  $\mathcal{O}(\log n^2)$ . The same running time holds for computing the factorization of an integer  $n$ . This running time is not only polynomial, but the attacks are even more efficient than the cryptographic schemes themselves! This in turn means that the problem cannot be fixed by just adjusting key sizes.

Thus, if we face the development of large-scale quantum computers in the next decades, then all classical dlog- and factoring-based cryptography has to be replaced. However, one should stress that the construction of large quantum computers with many qubits appears to be way more difficult than its classical counterpart, since most small quantum systems do not scale well and face decoherence problems.

**Recommendation:** It seems hard to predict the developments in constructing quantum computers. But experts in quantum physics currently do not see any mayor obstacle that would hinder the development of large quantum computers in the long term. It seems crucial to keep track of current progress in this area, and to have some alternative quantum-resistant cryptosystems ready to enroll within the next 15 years.

**References and further reading:** We recommend to read the books of Menezes, van Oorschot and Vanstone [MvOV01], Joux [Jou09] and Galbraith [Gal12] for a survey of cryptanalytic techniques. An introductory course in cryptanalysis is provided by May's lecture notes on cryptanalysis [May08, May12](German). An introduction to quantum algorithms can be found in the books of Homeister [Hom07](German) and Mermin [Mer08].

The algorithms of this section were originally presented in the superb works of Pollard [Pol75, Pol00] and Shor [Sho94]. Generic algorithms for multiple dlogs have recently been studied in [FJM14].

## 10.2 Best Algorithms for Prime Fields $\mathbb{F}_p$

**Management Summary:** Prime fields  $\mathbb{F}_p$  are – besides Elliptic Curves – the standard group for the discrete logarithm problem. There has been no significant algorithmic progress for this groups in the last 20 years. They are still a good choice for cryptography.

In Chapter 10.1, we learned that in any finite abelian group of order  $n$ , we can determine discrete logarithms in  $\mathcal{O}(\sqrt{n})$  steps. Notice that both the Pollard Rho Method as well as the Silver-Pohlig-Hellman algorithm from Chapter 10.1 used no other property of *representations* of group elements than their uniqueness. In these methods, one simply computes group elements by group operations and checks for equality of elements. Algorithms of this type are called *generic* in the literature.

It is known that generic algorithms cannot compute discrete logarithms in time better than the Silver-Pohlig-Hellman algorithm [Sho97b]. Thus, the algorithms of Chapter 10.1 can be considered optimal if no further information about the group elements is used.

However, when we specify our group  $G$  as the multiplicative group of the finite field  $\mathbb{F}_p$ , where  $p$  is a prime, we can actually exploit the representation of group elements. Natural representatives of  $\mathbb{F}_p$  are the integer  $0, \dots, p - 1$ . Thus, we can e.g. use the prime factorization of these integers. This is done in the so-called *Index Calculus type* discrete logarithm algorithms. This type of algorithm currently forms the class with the best running times for discrete logarithm over prime fields, prime extensions (Chapter 10.3) and for the factorization problem (Chapter 10.4).

We will now illustrate an Index Calculus algorithm with a very easy example.

### 10.2.1 An Introduction to Index Calculus Algorithms

An Index Calculus algorithm consists of three basic steps.

**Factor base:** Definition of a factor base  $F = \{f_1, \dots, f_k\}$ . We want to express group elements as powers of elements of the factor base.

**Relation finding:** Find elements  $z_i := g^{x_i} \in G$  for some integer  $x_i$  that can be written in the factor base, that is

$$g^{x_i} = \prod_{j=1}^k f_j^{e_{ij}}.$$

When we write this equality to the base  $g$ , we obtain a *relation*

$$x_i \equiv \sum_{j=1}^k e_{ij} \text{dlog}_g(f_j) \pmod{n},$$

where  $n$  is the order of  $g$ . A relation is a linear equation in the  $k$  unknowns

$$\text{dlog}_g(f_1), \dots, \text{dlog}_g(f_k)$$

. Once we have  $k$  linear independent relations of this type, we can compute these unknowns by linear algebra. This means we actually first compute all discrete logarithms of the factor base elements before we compute our desired individual logarithm of  $y$ .

**Dlog computation:** Express  $yg^r = g^{x+r} = \prod_{j=1}^k f_j^{e_j}$  in the factor base for some integer  $r$ . This gives us another relation

$$x + r \equiv \sum_{j=1}^k e_j \text{dlog}_g(f_j) \pmod{n},$$

which can be easily solved in the only unknown  $x = \text{dlog}_g y$ .

Let us provide an easy example for an Index Calculus algorithm that computes  $x = \text{dlog}_2(5)$  in  $\mathbb{F}_{11}^*$ . Since 2 generates the multiplicative group  $\mathbb{F}_{11}^*$ , the order of 2 is 10.

**Factor base:** Define  $F = \{-1, 2\}$ .

**Relation finding:**  $2^1 = (-1)^0 2^1$  gives us a first trivial relation

$$1 \equiv 0 \cdot \text{dlog}_2(-1) + 1 \cdot \text{dlog}_2(2) \pmod{10}.$$

If we compute  $2^6 = 64 \equiv -2 \pmod{11}$  we obtain a second relation

$$6 \equiv 1 \cdot \text{dlog}_2(-1) + 1 \cdot \text{dlog}_2(2) \pmod{10}.$$

Therefore, we can solve the system of linear equations

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \text{dlog}_2(-1) \\ \text{dlog}_2(2) \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 6 \end{pmatrix} \pmod{10}.$$

We obtain as the unique solution  $\text{dlog}_2(-1) \equiv 5$  and  $\text{dlog}_2(2) \equiv 1$ .

**Dlog computation:** Since  $5 \cdot 2^1 = 10 \equiv -1 \pmod{11}$  we obtain that

$$x + 1 \equiv 1 \cdot \text{dlog}_2(-1) + 0 \cdot \text{dlog}_2(2) \pmod{10}.$$

This leads to the solution  $x \equiv 4 \pmod{10}$ .

**Runtime:** Choosing a large factor base makes it easier to find relations, since it increases the likelihood that a certain number splits in the factor base. On the other hand, for a large factor base we have to find more relations in order to compute the dlogs of all factor base elements. An optimization of this tradeoff leads to a running time of  $L_p[\frac{1}{2}]$  for the relation finding step and also  $L_p[\frac{1}{2}]$  for performing the individual discrete logarithm computation in step 3.

Let us briefly discuss the advantages and disadvantages of the above simple Index Calculus algorithm from a cryptanalyst's point of view.

#### Advantages:

- For  $g^{x_i} = \prod_{j=1}^k f_j^{e_{ij}}$  it is trivial to compute the discrete logarithm on the left hand side.

#### Disadvantages:

- We need to factor relatively large numbers  $g^{x_i}$  over the integers. One can show that this intrinsically leads to a running time of  $L_p[\frac{1}{2}]$ , and there is no hope to get below the constant  $\frac{1}{2}$ .
- We need to compute all discrete logarithms of the factor base elements. This is inherent to all Index Calculus algorithms.

We will eliminate the first disadvantage by allowing factorizations over number fields. The second disadvantage is eliminated by choosing a factor base that allows for very efficient discrete logarithm computations of its elements.

### 10.2.2 The Number Field Sieve for Calculating the Dlog<sup>1</sup>

A number field  $\mathbb{Q}[\alpha]$  is a  $k$ -dimensional vector space over  $\mathbb{Q}$  and can be obtained by adjoining a root  $\alpha$  of some irreducible degree- $k$  polynomial  $f$  to  $\mathbb{Q}$ . This means we can write every element of  $\mathbb{Q}[\alpha]$  as  $a_0 + a_1\alpha + \dots + a_{k-1}\alpha^{k-1}$  with  $a_i \in \mathbb{Q}$ . If we restrict the  $a_i$  to integers we are in the ring  $\mathbb{Z}[\alpha]$ .

The number field sieve is also an index calculus algorithm. Compared to the previous approach it has the advantage to involve smaller numbers. This is done by choosing a specific representation of the prime field  $\mathbb{F}_p$ , which is implicitly defined as a finite field where two polynomials of small degree with small coefficients possess a common root. There are several methods that allow to construct such polynomials with a common root modulo  $p$ . In particular, for primes of a special form, *i.e.* with a sparse representation, it is possible to construct polynomials which are much better than in the general case. One typical construction that works well is to choose a number  $m$  and write  $p$  in basis  $m$  as  $\sum_{i=0}^t a_i m^i$ . We then find that  $f_1(X) = X - m$  and  $f_2(X) = \sum_{i=0}^t a_i m^i$  have  $m$  as a common root modulo  $p$ .

Equipped with two polynomials  $f_1$  and  $f_2$  of this form, with  $m$  as their common root modulo  $p$ , we obtain the following commutative diagram:

$$\begin{array}{ccc} & \mathbb{Z}[X] & \\ & \swarrow \quad \searrow & \\ \mathbb{Q}[X]/(f_1(X)) & & \mathbb{Q}[X]/(f_2(X)) \\ & \searrow \xrightarrow{X \mapsto m} \swarrow \xrightarrow{X \mapsto m} & \\ & \mathbb{F}_p & \end{array}$$

Let  $r_1, r_2$  be roots of  $f_1, f_2$ , respectively. Then we are working with the number fields  $\mathbb{Q}[r_1] \simeq \mathbb{Q}[X]/(f_1(X))$  and  $\mathbb{Q}[r_2] \simeq \mathbb{Q}[X]/(f_2(X))$ .

**Factor base:** Consists of small-norm prime elements in both number fields.

**Relation finding:** The basic principle of the number field sieve consists of sending elements of the form  $a + bX$  to both sides of the diagram and to write a relation when both sides factor into the factor base. Technically, this is quite challenging, because we need to introduce several tools to account for the fact that the left and right sides are not necessarily *unique factorization domains*. As a consequence, we need to factor elements into ideals and take care of the obstructions that arise from the class groups and unit groups. This procedure gives us the discrete logarithms of the factor base elements

**Discrete log computation:** Express the desired logarithm as a linear combination of the factor base elements.

**Runtime:** The Number Field Sieve is the most efficient currently known algorithm for the large characteristic discrete logarithm problem. In the general case – which means that  $p$  is not of a special form, e.g. close to a prime power – its complexity is  $L_p[\frac{1}{3}, (\frac{64}{9})^{1/3}]$ .

**References and further reading:** For an introduction to Index Calculus and the involved mathematical tools see May's lecture notes on number theory [May13](german) and the number

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<sup>1</sup>When calculating the dlog there is only the term **number field sieve** and no distinction between general vs. special. This is in the opposite to the number field sieve for factorization in section 10.4.1.

theory book by Müller-Stach, Piontkowski [MSP11]. For gaining a deep understanding of the Number Field Sieve, one has to study the book of Lenstra, Lenstra [LL93b] that contains all original works that led to the development of the Number Field Sieve algorithm in the late 80s and early 90s.

As a good start for understanding the Number Field Sieve, we recommend to first study its predecessors that are described in the original works of Adleman [Adl79], Coppersmith [COS86] and Pomerance [Pom84, Pom96].

## 10.3 Best Known Algorithms for Extension Fields $\mathbb{F}_{p^n}$ and Recent Advances

**Management Summary:** The groups over extension fields are attacked by the new algorithms of Joux et al. Before the invention of these attacks, the security of extension field groups appeared to be similar to the prime order groups from the last chapter. The new attacks render these groups completely insecure. However, the new attacks do not affect the security of prime order groups.

We will first discuss the formerly best algorithm from 2006 due to Joux and Lercier that achieves a running time of  $L_n[\frac{1}{3}]$ . We will then describe the recent developments that led to the dramatic improvement in the running time down to  $L_n[o(1)]$ , which is quasi polynomial time.

### 10.3.1 The Joux-Lercier Function Field Sieve (FFS)

Any finite field  $\mathbb{F}_{p^n}$  can be represented by a polynomial ring  $\mathbb{F}_p[x]/f(x)$ , where  $f(x)$  is an irreducible polynomial over  $\mathbb{F}_p$  with degree  $n$ . Thus, any element in  $\mathbb{F}_{p^n}$  can be represented by a univariate polynomial with coefficients in  $\mathbb{F}_p$  of degree less than  $n$ . Addition of two elements is the usual addition of polynomials, where the coefficients are reduced modulo  $p$ . Multiplication of two elements is the usual multiplication of polynomials, where the result is reduced modulo  $f(x)$  in order to again achieve a polynomial of degree less than  $n$ .

It is important to notice that the **description length** of an element is  $n\mathcal{O}(\log p)$ . Thus, a polynomial time algorithm achieves a running time which is polynomial in  $n$  and  $\log p$ . We will also consider fields of *small characteristic*  $p$ , where  $p$  is constant. Then polynomial running time means polynomial in  $n$ .

It is known that for any  $p$  there are always polynomials  $f(x)$  of degree  $n$  that are irreducible over  $\mathbb{F}_p$ . Usually, there are many of these polynomials, which in turn means that we obtain different representations of a finite field when choosing different polynomials  $f(x)$ . However, it is also known that all of these representations are isomorphic, and the isomorphisms are efficiently computable.

This fact is used in the algorithm of Joux and Lercier, who exploit different representations  $\mathbb{F}_p[x]/f(x)$  and  $\mathbb{F}_p[y]/g(y)$  of the same field. This is illustrated in the following commutative diagram.

$$\begin{array}{ccccc}
 & & \mathbb{F}_p[X, Y] & & \\
 & \swarrow & & \searrow & \\
 \mathbb{F}_p[X] & & & & \mathbb{F}_p[Y] \\
 & \searrow & & \swarrow & \\
 & & \mathbb{F}_{p^n} & &
 \end{array}$$

$\begin{matrix} & \nearrow Y \mapsto f(X) & \searrow X \mapsto g(Y) \\ \mathbb{F}_p[X] & & & \mathbb{F}_p[Y] \\ \searrow X \mapsto x & & \swarrow Y \mapsto y \\ & \mathbb{F}_{p^n} & & \end{matrix}$

**Factor base:** We choose all degree-1 polynomials  $x - a$  and  $y - b$  from  $\mathbb{F}_p[x] \cup \mathbb{F}_p[y]$ . Thus, the factor base has size  $2p$ .

**Relation finding:** On both sides, that is for polynomials  $h$  from  $\mathbb{F}_p[x]/f(x)$  and from  $\mathbb{F}_p[y]/g(y)$ , we try to factor into the linear factors from the factor base. This can be done by an easy

gcd computation  $\gcd(h, x^p - x)$  in time  $\mathcal{O}(p)$  for each polynomial. It can be shown that the number of polynomials that have to be tested is bounded by  $L_{p^n}[\frac{1}{3}]$ .

**Discrete log computation:** This step is done by writing a polynomial as a linear combination of polynomials of smaller degree and by repeating recursively, until degree-1 is found. This recursion is called a (degree) decent and requires running time  $L_{p^n}[\frac{1}{3}]$ , just like the relation finding step.

### 10.3.2 Recent Improvements for the Function Field Sieve

The first recent improvement upon the Joux-Lercier FFS was presented at Eurocrypt 2013 by Joux, who showed that it is possible to drastically lower the complexity of finding relations by replacing the classical sieving approach with a new technique based on a linear change of variables called *pinpointing*.

At the Crypto Conference 2013, Göloglu, Granger, McGuire, and Zumbrägel presented another approach, related to pinpointing that works very efficiently with a characteristic 2 subfield. Their paper was considered so important by the cryptographic community that they received the best paper award.

The new results hold for finite fields  $\mathbb{F}_{q^n}$  of characteristic two, i.e.  $q = 2^\ell$ . Notice that we use the standard convention that denotes primes by  $p$  and prime powers by  $q = p^\ell$ . For these fields  $\mathbb{F}_{q^n}$  the relation finding step in the Joux-Lercier algorithm simplifies, since one can construct polynomials that split with a higher probability than generic polynomials of the same degree.

Let us give a high-level description of the ideas of their improvement.

**Factor base:** All degree-1 polynomials as in the Joux-Lercier algorithm.

**Relation finding:** Göloglu, Granger, McGuire, and Zumbrägel show that one can construct a special type of polynomials over  $\mathbb{F}_q[x]$  – the so-called Bluher polynomials – that by construction split over  $\mathbb{F}_q[x]$ . So similar to our simple version of Index Calculus for integers in Section 10.2.1, we obtain one side of the equation for free. The cost for splitting the polynomials in  $\mathbb{F}_q[y]$  is roughly  $\mathcal{O}(q)$  and the cost for finding the discrete logarithms of the factor base elements is roughly  $\mathcal{O}(n \cdot q^2)$ . We will explain below why this gives us the discrete logarithms of the factor base in *polynomial time* for properly chosen parameters.

**Discrete log computation:** The individual discrete logarithm computation is similar to the Joux-Lercier algorithm.

**Runtime:** We are computing in a field  $\mathbb{F}_{q^n}$ , where  $q = 2^\ell$ . Hence, a polynomial time algorithm would require running time polynomial in the parameters  $n$  and  $\log q$ . However, the relation finding above takes time  $\mathcal{O}(n \cdot q^2)$ , which is polynomial in  $n$  but exponential in  $\log q$ . So actually the algorithm performs very poorly with respect to the size of the base field  $\mathbb{F}_q = \mathbb{F}_{2^\ell}$ .

The trick to work around this is to decrease the size of the base  $q$  to  $q'$  while slightly increasing the extension degree  $n$  to  $n'$ . Our goal is that the new base field size  $q'$  roughly equals the new extension degree  $n'$ , that is  $q' \approx n'$ . In this case, we again obtain a running time which is polynomial in  $n'$  and  $q'$ , but now  $q'$  is also polynomially bounded by  $n'$ . So in total, for step 2 our running time is in total polynomially bounded by  $n'$ .

Let us give a simple example of how this can be done for concrete parameters. Assume that we wish to compute a discrete logarithm in  $\mathbb{F}_{(2^{100})^{100}}$ . Then we would lower the base field

to  $q' = 2^{10}$  and at the same time increase the extension degree to  $n' = 1000$ , i.e. compute in  $\mathbb{F}_{(2^{10})^{1000}}$ . Notice that this can always be done by using the efficiently computable isomorphisms between finite fields of the same cardinality.

*Warning:* One might be tempted to bypass the above with the selection of exponents that do not split appropriately, i.e. by choosing  $\mathbb{F}_{2^p}$  with prime  $p$ . However, we can always embed our finite field in some larger field – as well as the respective discrete logarithms. Hence finite fields with small characteristic have to be considered insecure, independently of the special form of the extension degree  $n$ .

While the relation finding in step 2 of Göloglu, Granger, McGuire, and Zumbrägel can be done in *polynomial time*, the individual log computation is still time-consuming. If one does it naively, step 3 is even more time-consuming than in Joux-Lercier because of the increased extension degree  $n'$ . If one balances out the running times of step 2 and 3, one ends up with an improved overall running time of  $L_{q^n}[\frac{1}{3}, (\frac{4}{9})^{\frac{1}{3}}]$ .

### 10.3.3 Quasi-Polynomial Dlog Computation of Joux et al

In the previous section, it was shown that the discrete logarithms of all elements of a factor base can be computed in polynomial time. However, it remained a hard problem to use that fact for computing individual logarithms.

This problem has been recently solved by Joux [Jou13a] and Barbulescu, Gaudry, Joux and Thomé [BGJT13]. In the paper of Joux, it was shown that the individual logarithm step can be performed in  $L[\frac{1}{4}]$ . Shortly after, this was improved by Barbulescu, Gaudry, Joux and Thomé to  $L[o(1)]$ , which is a function that grows slower than  $L[\epsilon]$  for any  $\epsilon > 0$ . So they achieve quasi polynomial time.

Let us briefly describe the modifications of these two papers to the Function Field Sieve algorithm.

**Factor base:** Consists of degree-1 polynomials as before.

**Relation finding:** One starts with the trivial initial polynomial

$$h(x) = x^q - x = \prod_{\alpha \in \mathbb{F}_q} (x - \alpha)$$

that obviously factors in the factor base. Now, one applies linear and rational transformations (called homographies) to  $h(x)$ , which preserve its property to split over the factor base. One can show that there are sufficiently many independent homographies in order to construct sufficiently many relations. So out of one trivial polynomial  $h(x)$ , we obtain for free all  $\mathcal{O}(q)$  relations. This enables us to compute the discrete logarithms of the factor base elements in time  $\mathcal{O}(q)$ .

**Discrete log computation:** Barbulescu et al present an efficient *degree decent* algorithm which on input of a polynomial  $p(x)$  of degree  $n$  outputs a linear relation between the discrete log of  $p(x)$  and  $\mathcal{O}(nq^2)$  polynomials of degree  $\frac{n}{2}$  in time polynomial in  $q$  and  $D$ . This implies that we get a tree of polynomials, where the degree drops in every level by a factor of two, which in turns implies a tree depth of  $\log n$ . This results in a running time of  $\mathcal{O}(q^{\mathcal{O}(\log n)})$ .

**Runtime:** As in the previous Section 10.3.2 let us assume that the size  $q$  of the base field is of the same size as the extension degree  $n$ , i.e.,  $q = \mathcal{O}(n)$ . Then step 2 runs in time  $\mathcal{O}(q) = \mathcal{O}(n)$ , which is polynomial in  $n$ . Step 3 runs in time  $\mathcal{O}(q^{\mathcal{O}(\log n)}) = \mathcal{O}(n^{\mathcal{O}(\log n)}) = L_{q^n}[o(1)]$ . Notice that  $n^{\log n} = 2^{\log^2 n}$  grows faster than any polynomial function in  $n$  but slower than any sup-exponential function  $2^{nc}$  for some  $c > 0$ .

#### 10.3.4 Conclusions for Finite Fields of Small Characteristic

To give some examples what the theoretical quasi-polynomial run time of the previous results implies in practice, we illustrate in Table 10.1 what can currently be achieved in computing discrete logarithms.

Date	Field	Bitsize	Cost (CPU hours)	Algorithm
2012/06/17	$3^{6.97}$	923	895 000	[JL06]
2012/12/24	$p^{47}$	1175	32 000	[Jou13b]
2013/01/06	$p^{57}$	1425	32 000	[Jou13b]
2013/02/11	$2^{1778}$	1778	220	[Jou13a]
2013/02/19	$2^{1778}$	1991	2200	[GGMZ13]
2013/03/22	$2^{4080}$	4080	14 100	[Jou13a]
2013/04/11	$2^{6120}$	6120	750	[Jou13a]
2013/05/21	$2^{6168}$	6168	550	[Jou13a]

Table 10.1: Small characteristic records

**Recommendation:** The use of small characteristic fields for discrete log-based is **completely insecure**, no matter which key sizes are used. Fortunately, we are not aware of such a usage in actual applications in wide-spread/standardized cryptographic schemes.

#### 10.3.5 Do these Results Transfer to other Index Calculus Type Algorithms?

From a crypto user's point of view, one could worry that the current breakthrough results that drop the complexity for discrete log computations in small characteristic fields from  $L[\frac{1}{3}]$  to  $L[o(1)]$  apply to discrete logarithms in other groups as well. For instance, one might be concerned by the actual security level of discrete log based cryptography in finite fields  $\mathbb{F}_p$  of *large* characteristic.

**Conjecture:** We believe that the new techniques do not carry over to large-characteristic finite fields or elliptic curves that currently comprise the standard for cryptographic constructions.

Let us briefly collect some reasons, why the current techniques do not carry over to these groups, and which problems have to be solved before we see any significant progress in the running time for these groups.

- **Runtime:** Notice that all Index Calculus algorithms described in this section are polynomial in the base field size  $q$  and thus exponential in the bit-length  $\mathcal{O}(\log q)$ . So the hardness of the discrete logarithm problem seems to stem from the hardness in the base field, whereas the extension degree  $n$  does not contribute to make the problem significantly harder.

In particular, we note that each equation – constructed from the polynomial  $x^q - x$  as done in the new small characteristic algorithms – contains at least  $q$  terms. Thus, whenever  $q$  becomes bigger than  $L[1/3]$ , even writing a single equation of this type would cost more than the full complexity of the Number Field Sieve from Section 10.2.2.

Notice that there is a similar situation for discrete logarithms in elliptic curve groups. When we use an elliptic curve over  $\mathbb{F}_q$  in general the best known algorithm is the generic Pollard Rho algorithm from Chapter 10.1 with running time  $\mathcal{O}(\sqrt{q})$ . However, Gaudry's algorithm – that we discuss in Section 10.5.2 – requires for elliptic curves over  $\mathbb{F}_{q^n}$  only running time  $q^{2-\frac{2}{n}}$ , which is way better than the generic bound  $\mathcal{O}(q^{\frac{n}{2}})$ . Like the algorithms in this Chapter, Gaudry's algorithm is of Index Calculus type. And similar to the algorithms in this Chapter, the complexity of the discrete logarithm problem seems to be concentrated in the parameter  $q$  rather than the parameter  $n$ .

- **Polynomials vs Numbers:** Notice that the current results make heavy use of polynomial arithmetic and of subfields of  $\mathbb{F}_{q^n}$ . However, neither is polynomial arithmetic available for  $\mathbb{F}_p$  nor do there exist subfields for prime order groups. We would like to argue that many problems are efficiently solvable for polynomials, whereas they appear to be notoriously hard for integers. For instance, it is known that polynomials over finite fields and over the rationals can be efficiently factored by the algorithms of Berlekamp and Lenstra-Lenstra-Lovasz, whereas there is no equivalent algorithm for the integers. There is also an efficient algorithm for finding shortest vectors in polynomial rings due to von zur Gathen, where its integer lattice counterpart is known to be NP-hard.

What makes integers intrinsically harder than polynomials is the effect of carry bits. When we multiply two polynomials, we know by the convolution product exactly which coefficients contribute to which coefficients in the product, which is not true for integer multiplication due to the carry bits.

- **Complexity of Steps 2 & 3:** Any algorithmic breakthrough for index calculus type discrete logarithms would have to efficiently solve the discrete logarithms of a well-defined factor base *and* express the desired logarithm in terms of this factor base. But currently, we do not have an efficient method for either step in the case of large prime fields  $\mathbb{F}_p$ .

**References and further reading:** Coppersmith's algorithm [Cop84] from the mid 80s was for a long time the reference method for computing discrete logarithms in small characteristic fields. The Joux-Lercier Function Field Sieve was introduced 2006 in [JL06].

The recent advances started at Eurocrypt 2013 with Joux's pinpointing technique [Jou13b]. At Crypto 2013, Göloglu, Granger, McGuire and Jens Zumbrägel [GGMZ13] already improved the constant  $c$  in the  $L[\frac{1}{3}, c]$  running time. The improvement to running time  $L[\frac{1}{4}]$  was then presented in the work of Joux [Jou13a]. Eventually, Barbulescu, Gaudry, Joux and Thomé [BGJT13] proposed an algorithm for the decent that led to running time  $L[o(1)]$ .

## 10.4 Best Known Algorithms for Factoring Integers

**Management Summary:** The best algorithm for factoring shows close similarity to the best algorithm for computing discrete logarithms in prime order groups. It seems that the new attacks do not help to improve any of the two algorithms.

The best algorithm for computing the prime factorization of integers, the so-called Number Field Sieve, is very similar to the best algorithm for computing discrete logarithm in  $\mathbb{F}_p$  from Section 10.2.2, and much less similar to the algorithm for  $\mathbb{F}_{q^n}$  from Chapter 10.3.

In a nutshell, all known, sophisticated algorithms that factor RSA moduli  $n = pq$  for primes  $p, q$  of the same size, rely on the same basic simple idea. Our goal is to construct  $x, y \in \mathbb{Z}/n\mathbb{Z}$  such that

$$x^2 \equiv y^2 \pmod{n} \text{ and } x \not\equiv \pm y \pmod{n}.$$

This immediately yields the factorization of  $n$ , since  $n$  divides the product  $x^2 - y^2 = (x+y)(x-y)$  by the first property, but  $n$  does neither divide  $x+y$  nor  $x-y$  by the second property. Thus one prime factor of  $n$  has to divide  $x+y$ , whereas the other one has to divide  $x-y$ . This in turn means that  $\gcd(x \pm y, n) = \{p, q\}$ .

The factorization algorithms only differ in the way in which these  $x, y$  are computed. The intention is to compute  $x, y$  with  $x^2 \equiv y^2 \pmod{n}$  in an “independent” way. If this independence is given, it is easy to show that  $x \not\equiv \pm y \pmod{n}$  holds with probability  $\frac{1}{2}$ , since every square in  $\mathbb{Z}/n\mathbb{Z}$  has 4 square roots by the Chinese Remainder Theorem – two different roots modulo  $p$  and two different roots modulo  $q$ .

### 10.4.1 The Number Field Sieve for Factorization (GNFS)<sup>2</sup>

Let  $n \in \mathbb{N}$  be the integer that we want to factor. In the Number Field Sieve algorithm we start by constructing two polynomials  $f, g$  that share a common root  $m$  modulo  $N$ . Usually this is done by simply defining  $g(X) = X - m \pmod{n}$  and constructing some low degree polynomial  $f(X)$  with  $f(m) \equiv 0 \pmod{n}$  (e.g. by expanding  $n$  in base  $m$  as in Section 10.2.2).

Since  $f$  and  $g$  are different, they define different rings  $\mathbb{Z}[X]/f(X)$  and  $\mathbb{Z}[X]/g(X)$ . But since  $f$  and  $g$  share the same root  $m$  modulo  $n$ , both rings are isomorphic to  $\mathbb{Z}/n\mathbb{Z}$ ; and this isomorphism can be explicitly computed by the mapping  $X \mapsto m$ . This is illustrated in the following commutative diagram.

$$\begin{array}{ccc} & \mathbb{Z}[X] & \\ & \swarrow & \searrow \\ \mathbb{Q}[X]/(f(X)) & & \mathbb{Q}[X]/(g(X)) \\ & \searrow X \mapsto m & \swarrow X \mapsto m \\ & \mathbb{Z}/n\mathbb{Z} & \end{array}$$

**Factor base:** Consists of small-norm prime elements in both number fields.

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<sup>2</sup>The term number field sieve here always means the **general** number field sieve (GNFS). In the context of factorization there is a difference between a special and a general number field sieve – this is in the opposite to section 10.2.2.

CT2 contains an implementation of GNFS using msieve and YAFU.

**Relation finding:** We look for arguments  $\tilde{x}$  such that simultaneously  $\pi_f := f(\tilde{x})$  splits in  $\mathbb{Q}[X]/(f(X))$  and  $\pi_g := g(\tilde{x})$  splits in  $\mathbb{Q}[X]/(g(X))$  into the factor base. Such elements are called relations.

**Linear Algebra:** By linear algebra, we search for a product of the elements  $\pi_f$  which is a square and whose corresponding product of the  $\pi_g$  is also a square. If we send these elements via our homomorphism  $X \mapsto m$  to  $\mathbb{Z}/n\mathbb{Z}$ , we obtain elements  $x^2, y^2 \in \mathbb{Z}/n\mathbb{Z}$  such that  $x^2 \equiv y^2 \pmod{n}$ . If we first compute the square roots of  $\pi_f$  and  $\pi_g$  in their respective number fields before applying the homomorphism, we obtain  $x, y \in \mathbb{Z}/n\mathbb{Z}$  with  $x^2 \equiv y^2 \pmod{n}$ , as desired. The independence of  $x, y$  here stems from the different representations in both number fields.

**Runtime:** The above algorithm is up to some details – e.g. the square root computation in the number field – identical to the algorithm of Section 10.2.2 and shares the same running time  $L[\frac{1}{3}, (\frac{64}{9})^{1/3}]$ .

#### 10.4.2 Relation to the Index Calculus Algorithm for Dlogs in $\mathbb{F}_p$

Firstly, we know that computing discrete logarithms in composite order groups  $\mathbb{Z}/n\mathbb{Z}$  is at least as hard as factoring  $n = pq$ . This in turn means that any algorithm that computes discrete logarithms in  $\mathbb{Z}/n\mathbb{Z}$  computes the factorization of  $n$ :

$$\text{Dlogs in } \mathbb{Z}/n\mathbb{Z} \Rightarrow \text{Factoring } n.$$

Let us briefly give the idea of this relation. We compute the order  $k = \text{ord}(a)$  for an arbitrary  $a \in \mathbb{Z}/n\mathbb{Z}$  by our dlog-algorithm, i.e. we compute the smallest positive integer  $k$  such that  $a^k \equiv 1 \pmod{n}$ . If  $k$  is even, then  $a^{\frac{k}{2}} \not\equiv 1$  is a square root of 1. We have  $a^{\frac{k}{2}} \not\equiv -1$  with probability at least  $\frac{1}{2}$ , since 1 has 4 square roots modulo  $n$ . Set  $x \equiv a^{\frac{k}{2}} \pmod{n}$  and  $y = 1$ . Then we obtain  $x^2 \equiv 1 \equiv y^2 \pmod{n}$  and  $x \not\equiv \pm y \pmod{n}$ . By the discussion at the beginning of the Chapter this allows us to factor  $n$ .

Secondly, we also know that both problems factoring and computing discrete logarithms in  $\mathbb{F}_p$  are together at least as hard as computing discrete logarithms in  $\mathbb{Z}/n\mathbb{Z}$ . In short

$$\text{Factoring} + \text{Dlogs in } \mathbb{F}_p \Rightarrow \text{Dlogs in } \mathbb{Z}/n\mathbb{Z}.$$

This fact can be easily seen by noticing that factoring and dlogs in  $\mathbb{F}_p$  together immediately give an efficient version of the Silver-Pohlig-Hellman algorithm from Section 10.1. We first factor the group order  $n$  in prime powers  $p_i^{e_i}$ , and then compute the discrete logarithms in  $\mathbb{F}_{p_i}$  for each  $i$ . Just as in the Silver-Pohlig-Hellman algorithm we lift the solution modulo  $p_i^{e_i}$  and combine these lifted solutions via Chinese Remaindering.

We would like to stress that these two known relations do not tell much about whether there is a reduction

$$\text{Factoring} \Rightarrow \text{Dlog in } \mathbb{F}_p \quad \text{or} \quad \text{Dlog in } \mathbb{F}_p \Rightarrow \text{Factoring}.$$

Both directions are a long-standing open problem in cryptography. Notice however that the best algorithms for factoring and dlog in  $\mathbb{F}_p$  from Sections 10.2.2 and 10.4.1 are remarkably similar. Also historically always algorithmic progress for one problem immediately implied progress for the other problem as well. Although we have no formal proof, it seems to be fair to say that both problems seem to be closely linked from an algorithmic perspective.

### 10.4.3 Integer Factorization in Practice

Given the current state of the art of academic integer factorization research, even moderately sized – but properly chosen – RSA moduli offer a reasonable amount of protection against open community cryptanalytic efforts. The largest RSA challenge number factored by a public effort has just 768 bits [KAF<sup>+</sup>10] and required the equivalent of about 2000 years of computing on a single 2 GHz core. Attacking a 1024-bit RSA modulus is about a thousand times harder. Such an effort must be expected to be out of reach for academic efforts for several more years. Doubling the size to 2048-bit moduli increases the computational effort by another factor of  $10^9$ . Without substantial new mathematical or algorithmic insights 2048-bit RSA must be considered to be out of reach for at least two more decades.

### 10.4.4 Relation of Key Size vs. Security for Dlog in $\mathbb{F}_p$ and Factoring

The running time of the best algorithm for a problem defines the security level of a cryptosystem. E.g. for 80-bit security, we want that the best algorithm requires at least  $2^{80}$  steps.

As we already noted, the best running time for discrete logs in  $\mathbb{F}_p$  and for factoring is  $L[\frac{1}{3}, (\frac{64}{9})^{1/3}]$ . The most accurate way to use this formula is to actually measure the running time for a large real world factorization/dlog computation, and then extrapolate to large values. Assume that we know that it took time  $T$  to factor a number  $n_1$ , then we extrapolate the running time for some  $n_2 > n_1$  by the formula

$$T \cdot \frac{L_{n_1}[\frac{1}{3}, (\frac{64}{9})^{1/3}]}{L_{n_2}[\frac{1}{3}, (\frac{64}{9})^{1/3}]}.$$

So, we use the L-formula to estimate the relative factor that we have to spend in addition. Notice that this (slightly) overestimates the security, since the L-formula is asymptotic and thus becomes more accurate in the numerator than in the denominator – the denominator should include a larger error term. So in practice, one obtains (only slightly) less security than predicted by this formula.

We computed the formula for several choices of the bit-size of an RSA number  $n$ , respectively a dlog prime  $p$ , in Table 10.2. Recall from Section 10.4.1 that the running time of the Number Field Sieve algorithm for factoring is indeed a function of  $n$  and not of the prime factors of  $n$ .

We start with RSA-768 that has been successfully factored in 2009 [KAF<sup>+</sup>10]. In order to count the number of instructions for factoring RSA-768, one has to define what an *instruction unit* is. It is good practice in cryptography to define as a unit measure the time to evaluate DES in order to obtain comparability of security levels between secret and public key primitives. Then by definition of this unit measure, DES offers 56-bit security against brute-force key attacks.

In terms of this unit measure, the factorization of RSA-768 required  $T = 2^{67}$  instructions. From this starting point, we extrapolated the security level for larger bit-sizes in Table 10.2.

We successively increase the bit-size by 128 up to 2048 bits. We see that in the beginning, this leads to roughly an increase of security of 5 bits per 128-bit step, whereas in the end we only have an increase of roughly 3 bits per 128-bit step.

By Moore's law the speed of computers doubles every 1.5 years. Hence after  $5 \cdot 1.5 = 7.5$  years we have an increase of  $2^5$ , which means that currently we should roughly increase our bit-size by 128 bits every 7.5 years; and when we come closer to 2000 bits our increase of 128-bit steps should be in intervals of no later than 4.5 years. For more conservative choices that also

bit-size	security
768	67.0
896	72.4
1024	77.3
1152	81.8
1280	86.1
1408	90.1
1536	93.9
1664	97.5
1792	100.9
1920	104.2
2048	107.4

Table 10.2: Bitsize of  $n, p$  versus security level

anticipate some algorithmic progress rather than just an increase in computers' speed see the recommendations in Chapter 10.7.

**References and further reading:** An introduction to several factorization algorithms including the Quadratic Sieve – the predecessor of the Number Field Sieve – can be found in May's lecture notes on number theory [May13]. We recommend Blömer's lecture notes on algorithmic number theory [Blö99] as an introduction to the Number Field Sieve.

The development of the Number Field Sieve is described in the textbook of Lenstra and Lenstra [LL93b] that includes all original papers. The relation of discrete logarithms and factoring has been discussed by Bach [Bac84]. Details of the current factorization record for RSA-768 can be found in [KAF<sup>+</sup>10].

## 10.5 Best Known Algorithms for Elliptic Curves $E$

**Management Summary:** Elliptic curves are the second standard group for the discrete logarithm problem. The new attacks do not affect these groups, their security remains unchanged.

We would like to discuss elliptic curves  $E[p^n]$  over finite extension fields  $\mathbb{F}_{p^n}$  and elliptic curves  $E[p]$  over prime fields  $\mathbb{F}_p$ . The later are usually used for cryptographic purposes. The reason to discuss the former too is to illustrate – similar to the previous chapters – the vulnerabilities of extension fields  $\mathbb{F}_{p^n}$  as opposed to prime field  $\mathbb{F}_p$ . However, we would like to point out that we assume in the following – in contrast to the previous chapter – that  $n$  is fixed. This is because as opposed to the algorithm of Joux et al, the algorithms for  $E[p^n]$  have complexities which depend exponentially on  $n$ .

We present two different approaches for elliptic curves over extension fields: cover (or Weil descent) attacks introduced by Gaudry, Hess and Smart (GHS), and decomposition attacks proposed by Semaev and Gaudry. In some cases, it is possible to combine the two approaches into an even more efficient algorithm as shown by Joux and Vitse [JV11].

### 10.5.1 The GHS Approach for Elliptic Curves $E[p^n]$

This approach introduced by Gaudry, Hess and Smart aims at transporting the discrete logarithm problem from an elliptic curve  $E$  defined over an extension field  $\mathbb{F}_{p^n}$  to an higher genus curve defined over a smaller field, for example  $\mathbb{F}_p$ . This can be done by finding a curve  $H$  over  $\mathbb{F}_p$  together with a surjective morphism from  $H$  to  $E$ . In this context, we say that the curve  $H$  is a cover of  $E$ . Once such a curve  $H$  is obtained, it is possible using the so called coNorm technique to pull back a discrete logarithm problem on  $E$  to a discrete logarithm problem on the Jacobian of  $H$ . If the genus  $g$  of the target curve is not too large, this can lead to an efficient discrete logarithm algorithm. This uses the fact that there exists an index calculus algorithm on high genus curve of genus  $g$  over  $\mathbb{F}_p$  with complexity  $\max(g!p, p^2)$ . This was introduced by Enge, Gaudry and Thomé [EGT11].

Ideally, one would like the genus  $g$  to be equal to  $n$ . However, this is not possible in general. Classifying the possible covers for elliptic curve seems to be a difficult task.

### 10.5.2 Gaudry-Semaev Algorithm for Elliptic Curves $E[p^n]$

Let  $Q = \alpha P$  be a discrete logarithm on an elliptic curve  $E[p^n]$ . So the goal is to find the integer  $\alpha \in \mathbb{N}$  such that  $k$  times the point  $P \in E[p^n]$  added to itself is equal to the point  $Q \in E[p^n]$ .

Gaudry's discrete logarithm algorithm is of index calculus type. We briefly outline the basic steps.

**Factor base:** Consists of all points  $(x, y)$  on the elliptic curve  $E[p^n]$  such that  $x \in \mathbb{F}_p$ . That is  $x$  lies in the ground field  $\mathbb{F}_p$  rather than in the extension.

**Relation finding:** Given a random point  $R = aP$ , with  $a \in \mathbb{N}$ , we try to write  $R$  as a sum of exactly  $n$  points from the factor base, where  $n$  is the extension degree. This is achieved by using the  $n$ -th Semaev polynomial  $f_{n+1}$ . This polynomial is a symmetric polynomial of degree  $2^{n-2}$  in  $n+1$  unknowns  $x_1, \dots, x_{n+1}$  which encodes the fact that there exists points with respective abscissae  $x_1, \dots, x_{n+1}$  which sum to zero. Of course, the coefficients of  $f$

depend on the curve  $E$ . Replacing  $x_{n+1}$  by the abscissa of  $R$ , we can find a decomposition of  $R$  as a sum of points from the factor base by searching for a solution  $(x_1, \dots, x_n)$  in the basefield  $\mathbb{F}_p$ . In order to do this, one first rewrites  $f$  as a multivariate system of  $n$  equations by decomposing the constants that appear in the polynomial over some basis of  $\mathbb{F}_{p^n}$  over  $\mathbb{F}_p$ . This system of  $n$  equations in  $n$  unknowns can be solved using a Gröbner basis computation.

**Individual Discrete Log Computation:** To compute the discrete logarithm of  $Q$ , it suffices to find one additional relation that express a random multiple of  $Q$ , namely  $R = aQ$  in terms of the points in the factor base. This is done in the exact same way as the generation of relations in the previous step.

**Runtime:** The factor base can be computed in time  $\mathcal{O}(p)$ . Every  $R$  can be written as a sum of  $n$  factor base elements, i.e. yields a relation, with probability exponentially small in  $n$  (but independent of  $p$ ). If it yields a solution the running time of a Gröbner basis computation is also exponential in  $n$  (but polynomial in  $\log p$ ). In total, we need roughly  $p$  relations which can be computed in time linearly in  $p$  and exponentially in  $n$ . Since we assumed  $n$  to be fixed, we do not care about the bad behavior in  $n$ . The linear algebra step on a  $(p \times p)$ -matrix can then be performed in  $\mathcal{O}(p^2)$ , since the matrix is sparse – every row contains exactly  $n$  non-zero entries. With additional tricks one achieves a running time of  $\mathcal{O}(p^{2-\frac{2}{n}})$  for Gaudry's algorithm.

This should be compared to the generic bound of  $\mathcal{O}(p^{\frac{n}{2}})$  that we achieve when using Pollard's Rho algorithm from Chapter 10.1. Similar to Chapter 10.3, almost the whole complexity of the problem seems to be concentrated in the size of the base field  $p$ , and not in the extension degree  $n$ . Notice that as in Chapter 10.3, Gaudry's algorithm is exponential in  $\log p$ .

### 10.5.3 Best Known Algorithms for Elliptic Curves $E[p]$ over Prime Fields

**Generic discrete log solving:** In general, the best algorithm that we know for arbitrary elliptic curves  $E[p]$  is Pollard's Rho method with a running time of  $\mathcal{O}(\sqrt{p})$ . For the moment, it seems that nobody knows how to exploit the structure of an elliptic curve group or its elements in order to improve over the generic bound.

We would also like to point out that *random* elliptic curves, i.e. where the elliptic curve parameters  $a, b$  in the defining Weierstrass equation  $y^2 \equiv x^3 + ax + b \pmod{p}$  are chosen in a uniformly random manner, are among the hard instances. To further harden elliptic curves, one chooses for standardization only those curves that have (almost) prime order. This means that the co-factor of the largest prime in the group order is usually 1, which abandons the use of Silver-Pohlig-Hellman's algorithm.

**Embedding  $E[p]$  into  $\mathbb{F}_{p^k}$ :** It is known that in general elliptic curves  $E[p]$  can be embedded into a finite field  $\mathbb{F}_{p^k}$ , where  $k$  is the so-called *embedding degree*. In  $\mathbb{F}_{p^k}$  we could use the Number Field Sieve for discrete logarithm computations. Hence such an embedding would be attractive if  $L_{p^k}[\frac{1}{3}]$  is smaller than  $\sqrt{p}$ , which is the case only if the embedding degree  $k$  happens to be very small. However, for almost all elliptic curves the embedding degree is known to be huge, namely comparable to  $p$  itself.

Some constructions in cryptography, e.g. those that make use of bilinear pairings, exploit the advantages of a small embedding degree. Thus, in these schemes elliptic curves are explicitly chosen with a small embedding degree, e.g.  $k = 6$ , which balances out the hardness of the discrete logarithm problem on  $E[p]$  and in  $\mathbb{F}_p^k$ .

**The xedni calculus algorithm:** In 2000, Silverman published his *xedni calculus algorithm*

(read xedni backwards) that uses the group structure of  $E[p]$  for discrete logarithm computations, and thus is the only known non-generic algorithm that works directly on  $E[p]$ . However, it was soon after his publication discovered that the so-called lifting process in Silverman's algorithm has a negligible probability of succeeding in computing a discrete logarithm.

#### 10.5.4 Relation of Key Size vs. Security for Elliptic Curves $E[p]$

Similar to the discussion in Section 10.4.4 about key sizes for dlog in  $\mathbb{F}_p$  and for factoring, we want to evaluate how key sizes have to be adapted for elliptic curves  $E[p]$  in order to guard against an increase in computer speed. For elliptic curves, such an analysis is comparably simple. The best algorithm that we know for the dlog in  $E[p]$  is Pollard's Rho method with running time

$$L_p[1, \frac{1}{2}] = \sqrt{p} = 2^{\frac{\log p}{2}}.$$

This means that for achieving a security level of  $k$  bits, we have to choose a prime  $p$  with  $2k$  bits. In other words, increasing the bit-size of our group by 2 bits leads to increase of 1 bit in security. By Moore's law we loose 1 bit of security every 1.5 years just from an increase of computer's speed. In order to guard against this loss over 10 years, it thus suffices to increase the group-size by just  $7 \cdot 2 = 14$  bits. Notice that as opposed to the case of dlog in  $\mathbb{F}_p$  and factoring in Section 10.4.4 this increase is linear and independent of the starting point. That means to guard against technological speedups over 20 years, an increase of 28 bits is sufficient.

Of course, this analysis only holds if we do not have to face any major breakthrough in computer technology or algorithms. For a more conservative choice see the advice in Chapter 10.7.

#### 10.5.5 How to Securely Choose Elliptic Curve Parameters

A comprehensive description on how to choose elliptic curve domain parameters over finite fields can be found in RFC 5639 "ECC Brainpool Standard Curves and Curve Generation" by Manfred Lochter and Johannes Merkle [LM10, LM05]. This RFC defines a publicly verifiable way of choosing pseudo-random parameters for elliptic curve parameters, and thus it excludes the main source for embedding a trapdoor in the definition of a group. The authors discuss all *known* properties of a curve  $E[p]$  that might potentially weaken its security:

- **A small embedding degree** for the embedding into a finite field. This would allow for the use of more efficient finite field algorithms. Especially, the requirement excludes supersingular curves of order  $p + 1$ .
- **Trace one curves** which have order  $|E[p]| = p$ . These curves are known to be weak by the discrete logarithm algorithms of Satoh-Araki [SA98], Semaev [Sem98] and Smart [Sma99].
- **Large class number.** This excludes that  $E[p]$  can be efficiently lifted to a curve defined over some algebraic number field. This requirement is quite conservative, since even for small class numbers there is currently no efficient attack known.

Moreover the authors insist on the following useful properties.

- **Prime order.** This simply rules out subgroup attacks.
- **Verifiable pseudo random number generation.** The seeds for a pseudo random number generator are chosen in a systematic way by Lochter and Merkle, who use in

their construction the first 7 substrings of lenght 160 bit of the fundamental constant  $\pi = 3.141 \dots$

In addition, Lochter and Merkle specify a variety of curves for  $p$ 's of bit-lengths in the range 160 to 512. For TLS/SSL there is also a new set of proposed Brainpool curves available [LM13].

The work of Bos, Costello, Longa and Naehrig [BCLN14] gives a valuable introduction for practitioners on how to choose elliptic curve parameters that are secure and also allow for efficient implementation in various coordinate settings (Weierstrass, Edwards, Montgomery). Additionally, Bos et al focus on side-channel resistance against timing attacks by proposing constant-time scalar multiplications.

We highly recommend the SafeCurve project by Daniel Bernstein and Tanja Lange [BL14] that provides an excellent overview for several selection methods, their benefits and drawbacks. The goal of Bernstein and Lange is to provide security of Elliptic Curve Cryptography – rather than just strength of elliptic curves against discrete logarithm attacks. Therefore, they take into account various types of side-channels that may leak secrets in an implementation.

**References and further reading:** For an introduction to the mathematics of elliptic curves and their cryptographic applications we refer to the textbooks of Washington [Was08], Galbraith [Gal12], and Silverman [Sil99].

This section described the results of the original works of Gaudry, Hess, Smart [GHS02], Gaudry [Gau09], Semaev [Sem04], and the xedni algorithm of Silverman [Sil99].

## 10.6 Possibility of Embedded Backdoors in Cryptographic Keys

**Management Summary:** All cryptography seems to offer the possibility of embedding backdoors. Dlog schemes offer some advantage over factoring-based schemes in the sense that carefully chosen system-wide parameters protect *all* users.

The possibility of embedding trapdoors in cryptographic schemes to bypass cryptography and thus to decrypt/sign/authenticate without the use of a secret key is a long recognized problem that has been intensively discussed in the cryptographic community – e.g. at the panel discussion at Eurocrypt 1990. However, the wide-spread use of NSA’s backdoors as described by Edward Snowden has recently renewed the interest in this topic.

It appears that by construction some schemes are way more vulnerable than others. E.g. for discrete-log based schemes the definition of the group parameters is a system-wide parameter that is used by any user in the scheme. Thus, a party that is able to manipulate the definition of a group in such a way that enables this party to compute discrete logarithms in this group efficiently, can decrypt *all* communication. On the other hand, a carefully specified secure group also offers security for *all* users.

Currently, there is some speculation whether the NSA influenced NIST, the U.S. standardization agency, to standardize certain elliptic curves. But the definition of a group is not the only way to embed backdoors. All cryptographic schemes rely inherently on a good source of (pseudo-)random bits. It is well known that so-called semantic security of encryption schemes cannot be achieved without randomness, and every cryptographic secret key is assumed to be randomly chosen. Thus a weak pseudo-random generator opens the door for bypassing cryptography. Such a weak pseudo-random generator was standardized by NIST as Special Publication 800-90, although there have been warnings by the cryptographic community.

For factoring-based schemes the situation is slightly different than for discrete log-based schemes. As opposed to discrete log schemes, there are no system-wide parameters that define a group. Nevertheless, there are known ways to embed e.g. information about the factorization of the RSA modulus  $N$  in the RSA public exponent  $e$ . Moreover, recent attacks on RSA public key infrastructures [LHA<sup>+</sup>12a, HDWH12] show that it appears to be a difficult problem to generate RSA public keys with different primes in the public, mainly due to bad initializations of pseudo-random generators. This of course does only affect badly chosen keys of individuals as opposed to all users of cryptographic scheme.

**Recommendation:** Dlog-based schemes seem to be easier to control from a crypto designers perspective, since here all users have to take the same system-wide parameters.

We do not discuss the possibility of malware here – which may render obsolete any cryptographic protection method – or how to protect against it. But we would like to stress the follow (somewhat trivial) warning that addresses a crucial point in practice.

**Warning:** Cryptography can only protect data if it is properly implemented and does not leak its (immanent) secret. So in addition to the mathematical hardness of the underlying problems, we also have to trust in the implementor of a cryptographic scheme. This trust does not only include that the cryptographic scheme is implemented in the way it was originally designed – without embedding of any backdoors –, but also that the implementor does not reveal the

generated secret keys to a third party.

It seems that in the NSA affair, some companies were forced to reveal secret keys. Thus, one has to keep in mind that one has to buy cryptographic schemes from a completely reliable company that has not been compromised.

**References and further reading:** For a nice discussion of how to embed undetectable backdoors in various cryptographic schemes, see the original works of Young and Yung [[YY96](#), [YY97](#)]. See [[LHA<sup>+</sup>12a](#)] for a current attack on a significant portion of RSA keys in practice due to bad pseudo random number generation.

## 10.7 Advice for Cryptographic Infrastructure

**Management Summary:** Despite of recent discrete logarithm attacks, discrete logarithm-based schemes over *prime order groups* and *elliptic curve groups* remain secure. The same holds for factoring-based schemes. All discrete logarithm-based groups with small characteristic are completely insecure. Our suggestion is to choose elliptic curve groups.

### 10.7.1 Suggestions for Choice of Scheme

As we saw in the previous Chapters, discrete log-based schemes in  $\mathbb{F}_p$  and over  $E[p]$  remain secure, as well as factoring-based schemes. In the following, we suggest key sizes for these schemes that provide a sufficient security level **for the next two decades** under the assumption that no major algorithmic breakthrough occurs.

System	Key size in bits
Dlog in $\mathbb{F}_p$	2000 until 2019, then 3000
Factoring	2000 until 2019, then 3000
Dlog in $E[p]$	224 until 2015, then 250

Table 10.3: Security level 100 bit, source: BSI [BSI12], ANSSI [Age13]

Our preference is to use **elliptic curve groups**  $E[p]$  since they offer the following advantages:

- Algorithms for discrete logarithms in  $\mathbb{F}_p$  and factoring are closely linked. So any progress in one of these two might imply some progress for the other. But such a progress is unlikely to affect the security of elliptic curve groups.
- The best algorithms for  $E[p]$  are those of generic type from Chapter 10.1, which are inferior to the best algorithms for prime order discrete logarithm and factoring with  $L[\frac{1}{3}]$  running time. This in turn means that the key growth that compensates technological progress of faster computers is much smaller for  $E[p]$  – roughly 2 bits every 1.5 years according to Moore’s law.
- Getting algorithmic progress by using the group structure of  $E[p]$  seems to be harder than for  $\mathbb{F}_p$ , since as opposed to  $\mathbb{F}_p$  we do not even have an initial starting group-structure Index Calculus algorithm that we could improve.
- If an elliptic curve  $E[p]$  is properly chosen, i.e. the group is computationally hard and backdoor-free, then all users profit from the hardness of the discrete logarithm problem. Notice that this choice is crucial: If the group is not secure, then also all users suffer from its insecurity.

**Warning:** One should keep in mind that the suggestions above only hold in a world without large quantum computers. It seems crucial to keep track of current progress in this area, and to have some alternative quantum-resistant cryptosystems ready to enroll within the next 15 years.

**References and further reading:** For a good and conservative choice of key sizes we highly recommend to follow the suggestions of the Bundesamt für Sicherheit in der Informationstechnik (BSI) [BSI12] and the Agence nationale de la sécurité des systèmes d’information [Age13]. Both

sources also provide various valuable recommendations how to correctly implement and combine different cryptographic primitives.

**Remark from the editor in June 2016:** Since April 2014, quite a lot of things have changed (there have been new records in dlog finite fields and some marginal improvements of the L(1/3) algorithms in some contexts). However, this does not affect the overall conclusion that (only) small characteristic finite fields are no longer secure.

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All links have been confirmed at July 15, 2016.

# Chapter 11

# Crypto 2020 — Perspectives for Long-Term Cryptographic Security

([Johannes Buchmann](#), [Erik Dahmen](#), [Alexander May](#), and [Ulrich Vollmer](#), TU Darmstadt, May 2007)

Cryptography is a basic building block of all IT security solutions. Yet, for how long are the cryptographic tools we use today going to remain secure? Is this time long enough to ensure the confidentiality of medical data, to name just one example? Even in the short-term, the potential for havoc is great if certain keys are broken. Just think of the digital signatures that protect the authenticity of automatic updates for the Windows operating system.

## 11.1 Widely used schemes

In 1978, Rivest, Shamir and Adleman suggested the RSA public key encryption and signature schemes [[RSA78](#)]. RSA is still the most widely used public key scheme. The security of RSA depends on the difficulty of factoring so-called RSA moduli which are products of two large prime numbers. In their 1978 paper, the inventors of RSA suggested the use of RSA moduli with 200 decimal digits for long-term security. Later, the company RSA Security published a list of RSA moduli of increasing size, the RSA Challenge numbers. RSA Security offered prizes totaling \$ 635,000 for the factorization of these numbers, cf. <http://www.emc.com/emc-plus/rsa-labs/historical/the-rsa-challenge-numbers.htm>.

In 2005, that is 27 years after the invention of RSA, Bahr, Boehm, Franke, and Kleinjung from Bochum University managed to factor a 200 digit RSA challenge number ([www.mat.uniroma2.it/~eal/rsa640.txt](http://www.mat.uniroma2.it/~eal/rsa640.txt)). A key with size originally thought to be secure for a very long time was broken with a computation that took them just five months. This illustrates the tremendous progress factoring technology has made within the last 30 years. This progress is based on break-through mathematical ideas — e.g. the number field sieve proposed by John Pollard — as well as significant developments in computer hardware and software implementation technology.<sup>1</sup>

In 2000, Lenstra and Verheul developed an extrapolation formula that is supposed to help us forecast the security one can achieve with RSA and other important cryptographic schemes in the long term ([www.keylength.com](http://www.keylength.com)). The formula suggests the use of 850 digit RSA moduli if

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<sup>1</sup>Please compare chapter [4.11 Considerations regarding the security of the RSA algorithm](#), and especially chapters [4.11.4](#) and [4.11.5](#). For current cryptanalytical results against RSA and Dlog see chapter [10](#).

one wishes to protect data for the next 30 years. This corresponds to a 3072 bit RSA key.

Yet, even a well thought out extrapolation formula is no security guarantee! At any time, a brilliant mathematical idea can allow us to factor large numbers easily, and destroy the security of RSA. In 1996, Peter Shor showed that a quantum computer — a new type of computer that leverages the laws of quantum mechanics to speed up certain types of computations — can in principle be used for the fast factorization of large numbers [Sho97a]. Despite intensive research in the area, it is still too early to judge whether we are ever going to be able to build quantum computers of sufficient capacity to apply Shor's algorithm to numbers of relevant size.<sup>2</sup> Recent announcements of significant progress in this area made by the start-up company D-Wave ([www.dwavesys.com](http://www.dwavesys.com)) have been greeted with a lot of scepticism, even ridicule.

The development of attacks on another frequently used scheme called DSA (Digital Signature Algorithm) and the Elliptic Curve Cryptography (ECC) class of schemes moves in analogy to those on RSA. The security of these schemes depends on the difficulty of computing discrete logarithms. Even today, there is significant algorithmic progress. Quantum computers would render these schemes insecure.

What's the state of affairs with the so-called secret-key encryption schemes? In 1977, DES was introduced as the Data Encryption Standard [DES77]. Twenty-one years later, the Electronic Frontier Foundation (EFF) built the special purpose machine Deep Crack which needed just 56 hours to break a DES key. The problem with DES was that it used keys which were too short. It seems that the inventors of DES did not foresee the speed of hardware development. The Advanced Encryption Standard AES [AES02], successor to DES, is deemed secure at the moment even though there are interesting, if still inefficient, methods to attack AES with algebraic methods.

## 11.2 Preparation for tomorrow

Is the security of today's cryptography measuring up to its increasing importance? The experience shows: Carefully designed and implemented cryptographic schemes have a life time of five to twenty years. Whoever uses RSA, ECC or AES for short-term protection of data may feel safe. Moreover, it is also possible to achieve long-term authenticity, integrity and non-reputability of data, e.g., using the multiple signature scheme suggested by Sönke Maseberg [Mas02].

However, current schemes cannot guarantee long-term confidentiality. And what is to be done in twenty years from now? What should we do if, quasi over-night, unexpected mathematical progress renders an important cryptographic scheme insecure? Three things are necessary to prepare us for this event:

- a pool of secure alternative cryptographic schemes,

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<sup>2</sup>Required qubits for attacks on RSA, DSA and ECDSA using key with a bit length n:

RSA	$2n + 3$
DSA	$2n + 3$
ECDSA $2^n$	$\sim 2n + 8 \log n$
ECDSA p	$\sim 4n$

Please compare chapter 5.3 in "SicAri – Eine Sicherheitsplattform und deren Werkzeuge für die ubiquitäre Internetnutzung, KB2.1 – Abschlussbericht, Übersicht über Angriffe auf relevante kryptographische Verfahren", version 1.0, Mai 17, 2005, Prof. Dr. Johannes Buchmann et al., TUD-KryptC and cv cryptovision GmbH ([http://www.cdc.informatik.tu-darmstadt.de/~schepers/kb\\_21\\_angriffe.pdf](http://www.cdc.informatik.tu-darmstadt.de/~schepers/kb_21_angriffe.pdf)) and the dissertation of Axel Schmidt at the same faculty.

- infrastructures that enable us to exchange one cryptographic scheme for another, easily and quickly, and
- methods that ensure long-term confidentiality.

For many years, the cryptography group at the Technische Universität Darmstadt and its spin-off, the company FlexSecure ([www.flexsecure.de](http://www.flexsecure.de)), have worked to provide these tools. The trust center software FlexiTrust which is employed by the German National Root Certification Authority and the German Country Signing Authority offers an infrastructure within which cryptographic schemes can be easily exchanged. The open source library FlexiProvider implements a multitude of cryptographic schemes. Lately, we have intensified our research into “Post Quantum Cryptography” (PQC) seeking cryptographic schemes which remain secure even in the event that powerful quantum computers are built.

The security of public key cryptography traditionally rests on the difficulty of the solution of certain mathematical problems. Today, the following alternatives to the factorization and discrete logarithm problems are discussed in depth: the decoding problem, the shortest and closest vector problem in lattices, and the problem of solving large systems of multivariate quadratic equations. It is conjectured that quantum computers offer little advantage if we try to solve these problems efficiently.

### 11.3 New mathematical problems

Let us look at these alternatives a little more closely. The first encryption scheme based on the decoding problem was proposed by McEliece [[McE78](#)]. The background: Error-correcting codes are used to transmit or store electronic data in such a way that they remain undistorted even if a small number of bits are changed in transit or on the storage media. This property is used in, e.g., compact discs (CDs). The data on a CD can be reconstructed even if the disc has been slightly scratched.

In a code-based encryption scheme a message is encrypted by adding a fixed number of errors to (i.e. flipping a fixed numbers of bits of) the encoded message. Decoding requires knowledge of a suitable decoding procedure which eliminates these errors efficiently. This method is the secret key. Code-based encryption is in general very efficient. At the moment, research focus on the question which codes lead to secure encryption schemes with keys which are as small as possible.

Encryption on the basis of lattice problems is very similar to that on the basis of error-correcting codes. Lattices are regular structures of points in space. For instance, the points where the lines on squared paper cross form a two-dimensional lattice. For cryptographic usage, the dimension of the lattices is chosen to be much larger. Encryption works as follows: The plain-text is used to construct a lattice point which is then slightly distorted in such a way that it is no longer a lattice point, but close to one. Whoever knows a secret about the lattice is able to find this lattice point in the vicinity of the given point in space. The lattice point in turn yields the plain text. A particularly efficient lattice based encryption scheme is NTRU Encrypt (<https://www.securityinnovation.com/products/ntru-crypto>) . However, because NTRU was introduced fairly recently (in 1998), and its specification underwent several changes due to a variety of attacks, more cryptanalytic scrutiny is required to achieve confidence in its security.

## 11.4 New signatures

In 1979, Ralph Merkle proposed a remarkable framework for new signature schemes in his PhD thesis [Mer79]. Contrary to all other signature schemes, its security does not rest on the difficulty of a number-theoretic, algebraic or geometric problem. The only thing it requires is something which other signature schemes need anyway: a cryptographically secure hash function and a secure pseudo-random number generator. Each new hash function leads to a new signature algorithm. In consequence, the Merkle scheme has the potential to solve the problem of long-term availability of digital signature schemes.

Merkle uses in his construction so-called One-Time Signatures: Each new signature requires a new signing key and a new verification key. The idea Merkle had was to reduce the validity of many verification keys using a hash tree to the validity of a unique public hash value. When generating keys for the Merkle scheme one has to determine the number of signatures one can make with it in advance. For a long time this seemed a significant disadvantage. In [BCD<sup>+</sup>06], however, a variant of Merkle's scheme was proposed which allows to compute  $2^{40}$  signatures with a single key pair.<sup>3</sup>

## 11.5 Quantum cryptography – a way out of the impasse?

From the point of view of today's state of the art of cryptography, the problem of long-term confidentiality remains unsolved: There is *no* practical method to protect the confidentiality of an encrypted message over a very long period of time.

One way out of that dilemma may be to employ quantum cryptography: it allows for key agreement schemes (of very long keys for one-time pads) whose security is guaranteed by the laws of quantum mechanics, cf., e.g., [BB85]. At the moment, however, quantum cryptography is still rather inefficient, and it is unclear which cryptographic functionalities can be implemented on top of it.

## 11.6 Conclusion

What's on the balance sheet of today's cryptography? We have good tools to ensure short and medium term security. Software developers can employ these tools in their applications with good conscience as long as they make sure that components can quickly be exchanged when they become insecure.

In order to guarantee IT security for the future, too, we need to prepare a portfolio of secure cryptographic schemes. This portfolio needs to contain schemes which are suitable for the world of ubiquitous computing with many less powerful computers. It also needs to contain schemes which remain secure in the event that powerful quantum computers are built. Several promising candidates have been discussed in this article. They need to be studied carefully and prepared for use in everyday scenarios. The question how to ensure long-term confidentiality remains an important open research problem upon which cryptographic research should focus.

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<sup>3</sup>In JCT you can find in the default perspective below the main menu item **Visuals** several components and variants of this: the one-time signature WOTS+, the normal Merkle signature (MSS) and the extended Merkle signature scheme (XMSS).

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- [Sho97] Shor, Peter W.: *Polynomial time algorithms for prime factorization and discrete logarithms on a quantum computer*. SIAM Journal on Computing, 26(5):1484–1509, 1997.

All links in the article have been confirmed at July 15, 2016.

# Appendix A

## Appendix

- 1 CrypTool 1 Menu Tree
- 2 CrypTool 2 Templates
- 3 JCrypTool Functions
- 4 CrypTool-Online Functions
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- 6 Learning Tool for Elementary Number Theory
- 7 Short Introduction into the Computer Algebra System SageMath
- 8 Authors of the CrypTool Book

## A.1 CrypTool 1 Menus

This appendix contains at the following page the complete menu tree of CrypTool version 1.4.31<sup>1</sup>. The main menu of CT1 contains both, generic service functions in the six main menu items

- File
- Edit
- View
- Options
- Window
- Help,

and the actual crypto functions in the following four main menus

- Encrypt / Decrypt
- Digital Signature / PKI
- Individual Procedures
- Analysis.

Within **Individual Procedures** you find visualizations of single algorithms and of protocols. Some procedures are implemented both for a fast performance (mostly under the main menu **Encrypt/Decrypt**) and for a step-by-step visualization.

Which of the menu items in CrypTool 1 are active (that means not greyed), depends on the type of the currently active document window: The brute-force analysis for DES e. g. is only available, if the active window is opened in the hexadecimal view. On the other hand the menu item “Generate Random Numbers...” is always available (even if no document is opened).

---

<sup>1</sup>Since 2010, changes for the stable CrypTool 1 (**CT1**) focus mainly on bugfixes. Many new developments go into the two successor versions CrypTool 2 (**CT2**) and JCrypTool (**JCT**):

- Web site CT2: <http://www.cryptool.org/en/ct2-documentation-en>
- Web site JCT: <http://www.cryptool.org/en/jct-volunteer-en>

These successor versions are currently (Nov. 2012) still beta versions; but they are close to their first release and they are stable enough since quite a while to be used by end-users.

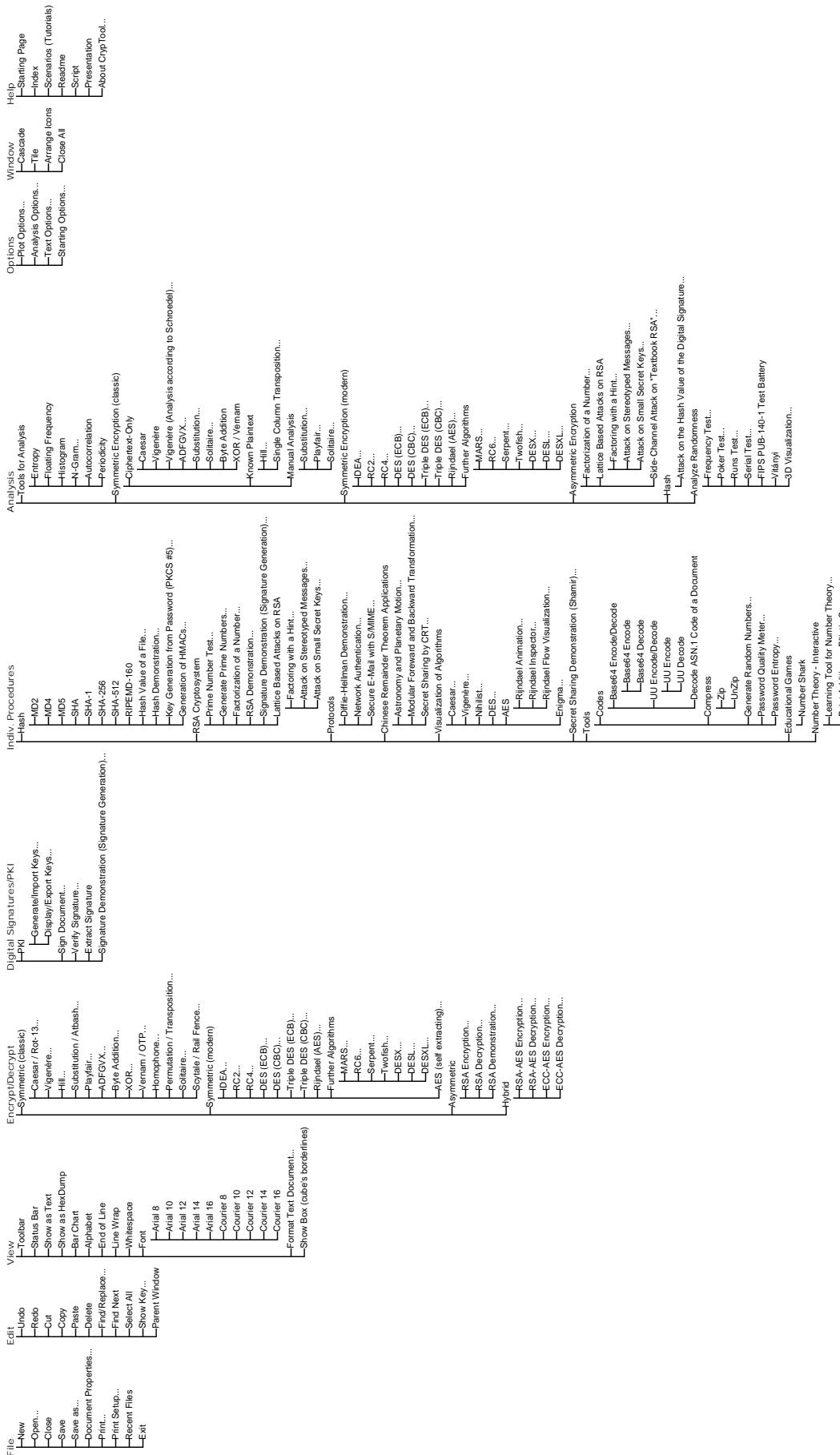


Figure A.1: Complete overview of the menu tree of CT1 (CrypTool 1.4.31)

## A.2 CrypTool 2 Templates

This appendix contains on the following pages the complete tree with all templates in CT2.<sup>2</sup> When you start CT2 it first shows the Startcenter.

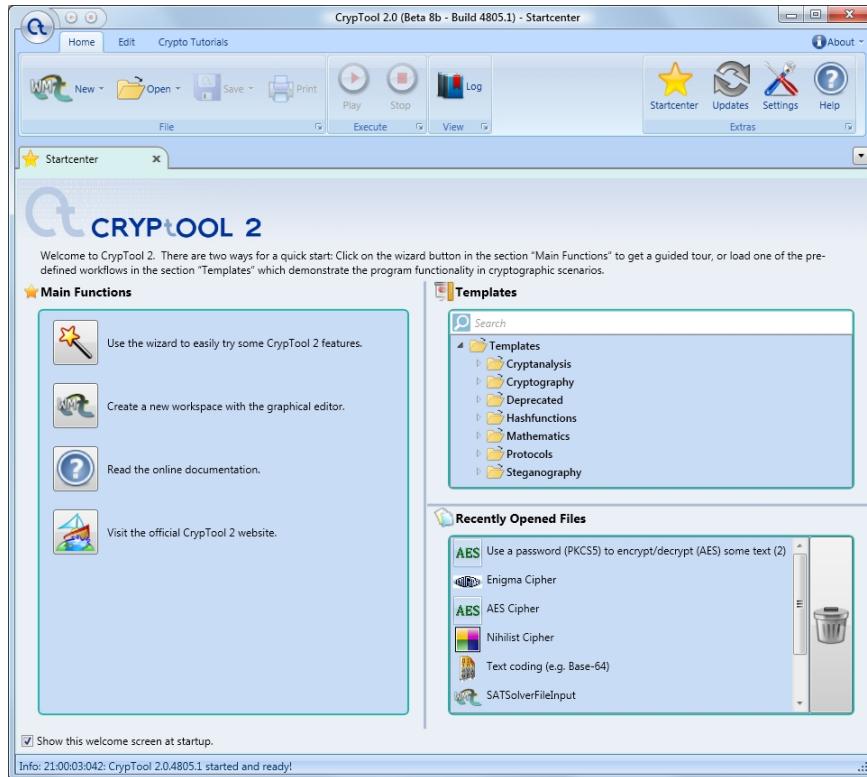


Figure A.2: Startcenter in CT2 (Beta 8b, May 2012)

Within the Startcenter you have three main ways to use the implemented functionality:

- via the Wizard, which leads you to the functions.
- via the Workspace, where you can compose the components (e.g. an encryption method, a text input function, ...) by yourself according to the visual programming concept.
- via the template tree, which offers ready-to-run workflows.

The Wizard asks questions so you can get to the desired scenarios (e.g. base64 coding) and then runs the according functions. You can afterwards save the chosen scenario as a normal template including the entries you used.

The empty workspace allows to drag on it any component from the navigation bar on the left and connect these components in the way you like. Most of the crypto functionality implemented in CT2 is offered using these components (e.g. Enigma, AES).

The template tree contains at least one template for each implemented component. The offered templates contain read-to-run workflows. If you change e.g. within the AES template your input, you can see at once, how the output is modified dynamically (e.g. adding another block via padding, influence of the chosen chaining, ...).

<sup>2</sup>You can find further information about CT2 at: <http://www.cryptool.org/en/ct2-documentation-en>

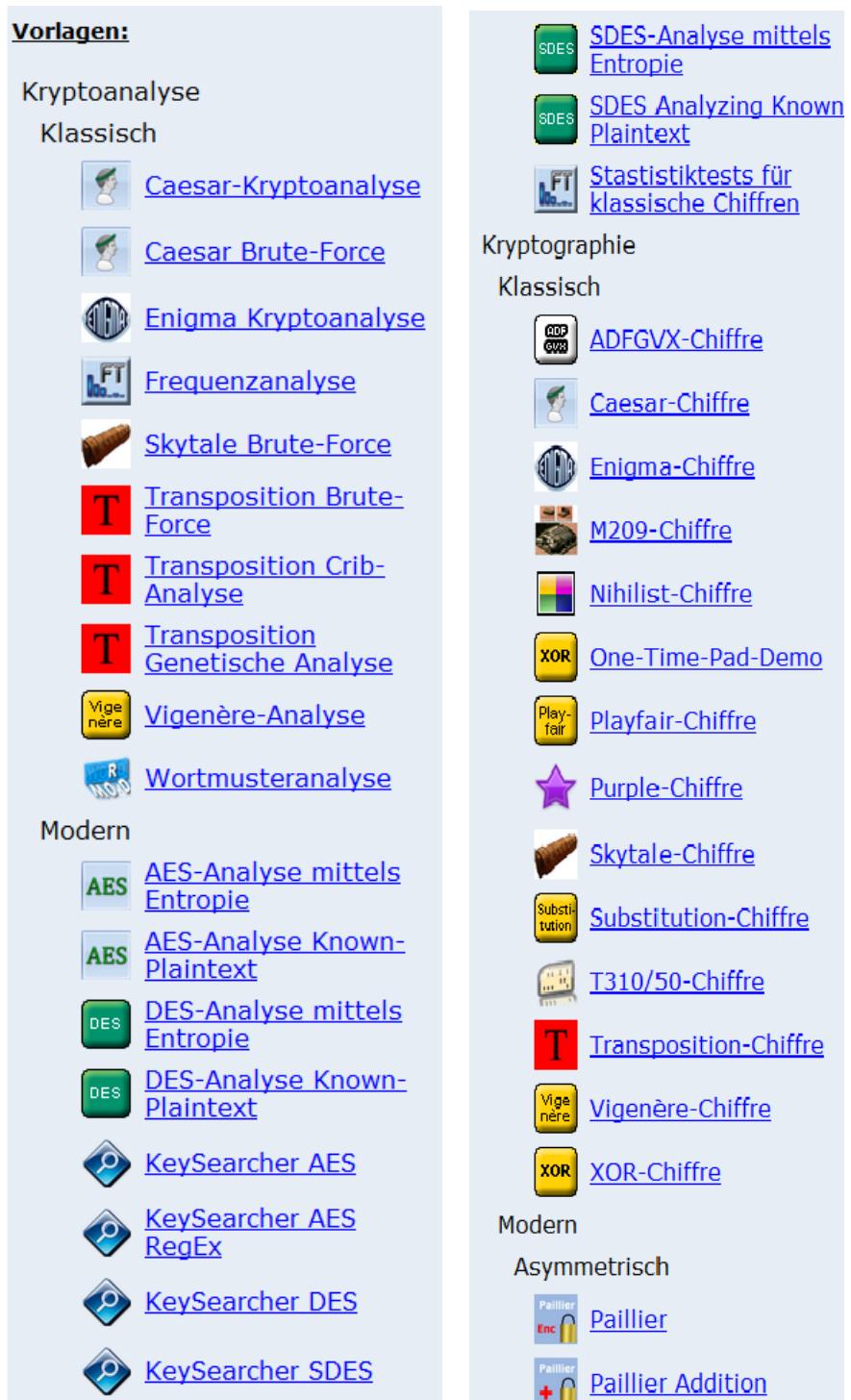


Figure A.3: Screenshot of the template tree of CT2 (NB4882.1, July 2012), Part 1

### A.3 JCrypTool Functions

On the following pages this appendix contains a list of all functions in JCrypTool.<sup>3</sup>

When you start JCT the first time it shows the Welcome window.

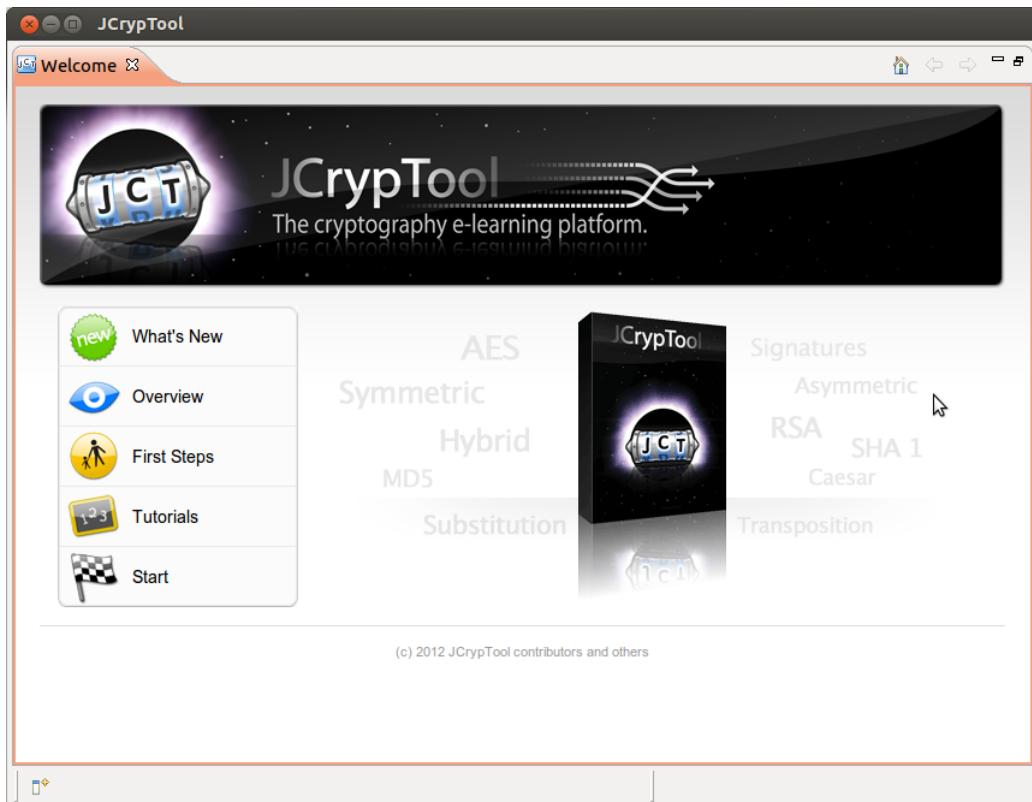


Figure A.4: Welcome screenshot in JCT (RC6, July 2012)

After pressing “Start” you can directly use the different functions. The functions implemented in JCT are presented in two different perspectives:

- Default Perspective
- Algorithm Perspective

All functions of the **Default Perspective** can be found both in the menus and in the navigation bar called “Crypto Explorer” (at the right side). The Default Perspective contains all important methods like classic transposition or modern AES, and many visualizations (e.g. Diffie-Hellman key exchange or calculations on elliptic curves).

All functions of the **Algorithm Perspective** can be found in the navigation bar called “Algorithms” (in this perspective also at the right side). The Algorithm Perspective contains all detail settings of the various algorithms, it especially offers post-quantum computing algorithms (PQC).

---

<sup>3</sup>You can find further information about JCT at: <http://www.cryptoool.org/en/jct-volunteer-en>  
This list was generated using the CT Portal website.

Function	JCryptTool*	Path in JCT
ADFGVX	D	Algorithms\ Classic\ ADFGVX
Autokey Vigenère	D	Algorithms\ Classic\ Autokey-Vigenère
Bifid	D	Algorithms\ Classic\ Bifid
Caesar	D	Algorithms\ Classic\ Caesar
Double Box	D	Algorithms\ Classic\ Double Box
Playfair	D	Algorithms\ Classic\ Playfair
Substitution	D	Algorithms\ Classic\ Substitution
Transposition	D	Algorithms\ Classic\ Transposition
Vigenère	D	Algorithms\ Classic\ Vigenère
XOR	D	Algorithms\ Classic\ XOR
3DES	F	Algorithms\ Block Ciphers\ DESede
AES	D\F	Algorithms\ Symmetric\ AES Algorithms\ Block Ciphers\ Rijndael
Camellia	F	Algorithms\ Block Ciphers\ Camellia
Dragon	D	Algorithms\ Symmetric\ Dragon
ECIES	F	Algorithms\ Hybrid Ciphers\ ECIES
ElGamal	D\F	Algorithms\ Asymmetric\ ElGamal Algorithms\ Asymmetric Block Ciphers\ ElGamal
IDEA	D\F	Algorithms\ Symmetric\ IDEA Algorithms\ Block Ciphers\ IDEA
LFSR	D	Algorithms\ Symmetric\ LFSR
MARS	F	Algorithms\ Block Ciphers\ MARS
McEliece Fujisaki	F	Algorithms\ Hybrid Ciphers\ McElieceFujisakiCipher
McEliece Kobara Imai	F	Algorithms\ Hybrid Ciphers\ McElieceKobaraImaiCipher
McEliece PKCS	F	Algorithms\ Asymmetric Block Ciphers\ McEliecePKCS
McEliece Pointcheval	F	Algorithms\ Hybrid Ciphers\ McEliecePointchevalCipher
MeRSA	F	Algorithms\ Asymmetric Block Ciphers\ MeRSA
Misty-1 (Kasumi)	F	Algorithms\ Block Ciphers\ Misty1
MpRSA	F	Algorithms\ Asymmetric Block Ciphers\ MpRSA
Niederreiter	F	Algorithms\ Asymmetric Block Ciphers\ Niederreiter
RC2	F	Algorithms\ Block Ciphers\ RC2
RC5	F	Algorithms\ Block Ciphers\ RC5
RC6	D\F	Algorithms\ Symmetric\ RC6 Algorithms\ Block Ciphers\ RC6
RSA	D\F	Algorithms\ Asymmetric\ RSA Algorithms\ Asymmetric Block Ciphers\ RSA_PKCS1_v1_5 Algorithms\ Asymmetric Block Ciphers\ RSA_PKCS1_v2_1
SAFER+	F	Algorithms\ Block Ciphers\ SAFER+
SAFER++	F	Algorithms\ Block Ciphers\ SAFER++
Serpent	F	Algorithms\ Block Ciphers\ Serpent
Shacal	F	Algorithms\ Block Ciphers\ Shacal
Shacal2	F	Algorithms\ Block Ciphers\ Shacal2
Twofish	F	Algorithms\ Block Ciphers\ Twofish
XML-Security	D	Algorithms\ XML Security\
CBC MACs	F	Algorithms\ Message Authentication Codes\ CBCMac
Cipher-based MACs (CMAC)	F	Algorithms\ Message Authentication Codes\ CMac
DHA-256	F	Algorithms\ Message Digests\ DHA256
FORK-256	F	Algorithms\ Message Digests\ FORK256
HMACs	D\F	Algorithms\ MAC\ HMacMD5 Algorithms\ Message Authentication Codes\ Hmac
MD4	F	Algorithms\ Message Digests\ MD4
MD5	D\F	Algorithms\ Hash\ MD5 Algorithms\ Message Digests\ MD5
PKCS#5	F	Algorithms\ Password-based ciphers\
RIPEMD	F	Algorithms\ Message Digests\ RIPEMD
SHA3 candidates	D	Algorithms\ Hash\ SHA3-Candidates

Figure A.5: Screenshot of the functions of JCT (RC6, July 2012), Part 1

Function	JCrypTool*	Path in JCT
SHA family	D\F	Algorithms\ Hash\ SHA Algorithms\ Message Digests\ SHA
Tiger	F	Algorithms\ Message Digests\ Tiger
Two-Track-MAC	F	Algorithms\ Message Authentication Codes\ TwoTrackMac\
VSH	F	Algorithms\ Message Digests\ VSH
CMSS	F	Algorithms\ Signatures\ CMSSSignature\
DSA	D\F	Algorithms\ Signature\ DSA Algorithms\ Signatures\ DSASignature\
ECDSA	F	Algorithms\ Signatures\ ECDSASignature\
GMSS	F	Algorithms\ Signatures\ GMSSSignature\
IQDSA	F	Algorithms\ Signatures\ IQDSASignature\
IQGQ	F	Algorithms\ Signatures\ IQGQSignature\
IQRDSA	F	Algorithms\ Signatures\ IQRDSSignature\
Merkle OTS	F	Algorithms\ Signatures\ MerkleOTSSignature\
MeRSA	F	Algorithms\ Signatures\ MeRSA
MpRSA	F	Algorithms\ Signatures\ MpRSA
Niederreiter CFS	F	Algorithms\ Signatures\ NiederreiterCFS
PKCS#1 (RSA Signatures)	F	Algorithms\ Signatures\ RSASignaturePKCS1v15\
RSASSA-PSS	F	Algorithms\ Signatures\ RSASSA-PSS
SHA1 ECNR	F	Algorithms\ Signatures\ SHA1\ECNR
SSL (MD5 and SHA1 with RSA)	F	Algorithms\ Signatures\ SHA1\SSL_MD5andSHA1withRSA
Blum Blum Shub (B.B.S.)	F	Algorithms\ Pseudo Random Number Generators\ BBSRandom
Elliptic Curve PRNG	F	Algorithms\ Pseudo Random Number Generators\ ECPRNG
SHA1	D	Algorithms\ Random Number Generator\ SHA1
SHA1 PRNG	F	Algorithms\ Pseudo Random Number Generators\ SHA1PRNG
Entropy	D	Analysis\ Entropy Analysis
Frequency Test	D	Analysis\ Frequency Analysis
Friedman Test	D	Analysis\ Friedman Test
Transposition Analysis	D	Analysis\ Transposition Analysis
Vigenère Analysis	D	Analysis\ Vigenère Breaker
Ant Colony Optimization	D	Visuals\ Ant Colony Optimization
Chinese Remainder Theorem	D	Visuals\ Chinese Remainder Theorem
Differential Power Analysis	D	Visuals\ Differential Power Analysis \ Double and Add
Diffie-Hellman Key Exchange	D	Visuals\ Diffie-Hellman Key Exchange (EC)
ElGamal	D	Visuals\ ElGamal Cryptosystem
Elliptic Curve Cryptography	D	Visuals\ ECC Demonstration
Extended Euclidian	D	Visuals\ Extended Euclidian
Feige Fiat Shamir	D	Visuals\ Feige Fiat Shamir
Fiat Shamir	D	Visuals\ Fiat Shamir
Graph Isomorphism	D	Visuals\ Graph Isomorphism
Grille	D	Visuals\ Grille
Homomorphic Encryption	D	Visuals\ Homomorphic Encryption
Kleptography	D	Visuals\ Kleptography
Magic Door	D	Visuals\ Magic Door
Multipartite Key Exchange	D	Visuals\ Multipartite Key Exchange
RSA	D	Visuals\ RSA Cryptosystem
Shamir's Secret Sharing	D	Visuals\ Shamir's Secret Sharing
Verifiable Secret Sharing (VSS)	D	Visuals\ Verifiable Secret Sharing
Simple Power Analysis	D	Visuals\ Simple Power Analysis \ Square and Multiply
Viterbi	D	Visuals\ Viterbi
XML-Security	D	Algorithms\ XML Security\
Number Shark	D	Games\ Number Shark
Sudoku Solver	D	Games\Sudoku
*	D = Default Perspective F = Functional Perspective	

Figure A.6: Screenshot of the functions of JCT (RC6, July 2012), Part 2

## A.4 CrypTool-Online Functions

This appendix contains a list of all functions in CrypTool-Online (CTO).<sup>4</sup>

The following screenshot shows the crypto functions implemented on CTO:

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<sup>4</sup>You can find further information about CTO at: [www.cryptool-online.org](http://www.cryptool-online.org)  
This list was generated using the functions list at the CT Portal website:  
<http://www.cryptool.org/ctp-documentation-en/ctp-functions-en>

Function	JCryptTool*	Path in JCT
ADFGVX	D	Algorithms\ Classic\ ADFGVX
Autokey Vigenère	D	Algorithms\ Classic\ Autokey-Vigenère
Bifid	D	Algorithms\ Classic\ Bifid
Caesar	D	Algorithms\ Classic\ Caesar
Double Box	D	Algorithms\ Classic\ Double Box
Playfair	D	Algorithms\ Classic\ Playfair
Substitution	D	Algorithms\ Classic\ Substitution
Transposition	D	Algorithms\ Classic\ Transposition
Vigenère	D	Algorithms\ Classic\ Vigenère
XOR	D	Algorithms\ Classic\ XOR
3DES	F	Algorithms\ Block Ciphers\ DESede
AES	D\F	Algorithms\ Symmetric\ AES Algorithms\ Block Ciphers\ Rijndael
Camellia	F	Algorithms\ Block Ciphers\ Camellia
Dragon	D	Algorithms\ Symmetric\ Dragon
ECIES	F	Algorithms\ Hybrid Ciphers\ ECIES
ElGamal	D\F	Algorithms\ Asymmetric\ ElGamal Algorithms\ Asymmetric Block Ciphers\ ElGamal
IDEA	D\F	Algorithms\ Symmetric\ IDEA Algorithms\ Block Ciphers\ IDEA
LFSR	D	Algorithms\ Symmetric\ LFSR
MARS	F	Algorithms\ Block Ciphers\ MARS
McEliece Fujisaki	F	Algorithms\ Hybrid Ciphers\ McElieceFujisakiCipher
McEliece Kobara Imai	F	Algorithms\ Hybrid Ciphers\ McElieceKobaraImaiCipher
McEliece PKCS	F	Algorithms\ Asymmetric Block Ciphers\ McEliecePKCS
McEliece Pointcheval	F	Algorithms\ Hybrid Ciphers\ McEliecePointchevalCipher
MeRSA	F	Algorithms\ Asymmetric Block Ciphers\ MeRSA
Misty-1 (Kasumi)	F	Algorithms\ Block Ciphers\ Misty1
MpRSA	F	Algorithms\ Asymmetric Block Ciphers\ MpRSA
Niederreiter	F	Algorithms\ Asymmetric Block Ciphers\ Niederreiter
RC2	F	Algorithms\ Block Ciphers\ RC2
RC5	F	Algorithms\ Block Ciphers\ RC5
RC6	D\F	Algorithms\ Symmetric\ RC6 Algorithms\ Block Ciphers\ RC6
RSA	D\F	Algorithms\ Asymmetric\ RSA Algorithms\ Asymmetric Block Ciphers\ RSA_PKCS1_v1_5 Algorithms\ Asymmetric Block Ciphers\ RSA_PKCS1_v2_1
SAFER+	F	Algorithms\ Block Ciphers\ SAFER+
SAFER++	F	Algorithms\ Block Ciphers\ SAFER++
Serpent	F	Algorithms\ Block Ciphers\ Serpent
Shacal	F	Algorithms\ Block Ciphers\ Shacal
Shacal2	F	Algorithms\ Block Ciphers\ Shacal2
Twofish	F	Algorithms\ Block Ciphers\ Twofish
XML-Security	D	Algorithms\ XML Security\
CBC MACs	F	Algorithms\ Message Authentication Codes\ CBCMac
Cipher-based MACs (CMAC)	F	Algorithms\ Message Authentication Codes\ CMac
DHA-256	F	Algorithms\ Message Digests\ DHA256
FORK-256	F	Algorithms\ Message Digests\ FORK256
HMACs	D\F	Algorithms\ MAC\ HMacMD5 Algorithms\ Message Authentication Codes\ Hmac
MD4	F	Algorithms\ Message Digests\ MD4
MD5	D\F	Algorithms\ Hash\ MD5 Algorithms\ Message Digests\ MD5
PKCS#5	F	Algorithms\ Password-based ciphers\
RIPEMD	F	Algorithms\ Message Digests\ RIPEMD
SHA3 candidates	D	Algorithms\ Hash\ SHA3-Candidates

Figure A.7: Screenshot of the functions of CTO (November 2012)

## A.5 Movies and Fictional Literature with Relation to Cryptography

Cryptographic applications – classical as well as modern ones – have been used in literature and movies. In some media they are only mentioned and are a pure admixture; in others they play a primary role and are explained in detail; and sometimes the purpose of the story, which forms the framework, is primarily to transport this knowledge and achieve better motivation. Here is the beginning of an overview.

### A.5.1 For Grownups and Teenagers

[Poe1843] Edgar Allan Poe,

*The Gold Bug*, 1843.<sup>5</sup>

In this short story Poe tells as first-person narrator about his acquaintanceship with the curious Mr. Legrand. They detect the fabulous treasure of captain Kidd via a gold bug and a vellum found at the coast of New England.

The cipher consists of 203 cryptic symbols and it proves to be a general monoalphabetic substitution cipher (see chapter 2.2.1). The story tells how they solve the riddle step by step using a combination of semantic and syntax analysis (frequency analysis of single letters in English texts).<sup>6</sup>

In this novel the code breaker Legrand says the famous statement: “Yet it may be roundly asserted that human ingenuity cannot concoct a cipher which human ingenuity cannot resolve – given the according dedication.”

[Verne1885] Jules Verne,

*Mathias Sandorf*, 1885.

This is one of the most famous novels of the French author Jules Verne (1828-1905), who was called “Father of Science fiction”.

In “Mathias Sandorf” he tells the story of the freedom fighter Earl Sandorf, who is betrayed to the police, but finally he can escape.

The whistle-blowing worked, because his enemies captured and decrypted a secret message sent to him. For decryption they needed a special grille, which they stole from him. This turning grille was a quadratic piece of jig with 6x6 squares, of which 1/4 (nine) were holes (see the [turning grille](#) in chapter 2.1.1).

[Kipling1901] Rudyard Kipling,

*Kim*, 1901.

Rob Slade’s review<sup>7</sup> of this novel says: “Kipling packed a great deal of information and

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<sup>5</sup>See [https://en.wikipedia.org/wiki/The\\_Gold-Bug](https://en.wikipedia.org/wiki/The_Gold-Bug).

Part 1 of the series *RSA & Co. at school: Modern cryptology, old mathematics, and subtle protocols* contains a didactical description for usage at school. Unfortunately this series is currently only available in German. See [WLS98], pp 52 ff (“Das Gold des Gehenkten”).

All material about a gold-bug lesson (double period) can be found at <http://www.informatik-im-kontext.de/> via “E-Mail (nur?) für Dich” => “Vertraulichkeit mit Verschlüsselungsverfahren”.

Poe not only was a well-known writer, but also a talented cryptographer. His story is also told in the book *Code Breaking* [Kip97].

<sup>6</sup>Part of the learning material above is a Python program which can be used to decrypt the ciphertext with Python and with SageMath. See the code example in A.5.3.

<sup>7</sup>See <http://catless.ncl.ac.uk/Risks/24.49.html#subj12>.

concept into his stories, and in “Kim” we find The Great Game: espionage and spying. Within the first twenty pages we have authentication by something you have, denial of service, impersonation, stealth, masquerade, role-based authorization (with ad hoc authentication by something you know), eavesdropping, and trust based on data integrity. Later on we get contingency planning against theft and cryptography with key changes.” The book is out of copyright<sup>8</sup>.

[Doyle1905] Arthur Conan Doyle,

*The Adventure of the Dancing Men*, 1905.

In this Sherlock Holmes short story (first published in 1903 in the “Strand Magazine”, and then in 1905 in the collection “The Return of Sherlock Holmes” the first time in book-form) Sherlock Holmes has to solve a cipher which at first glance looks like a harmless kid’s picture.

But it proves to be the monoalphabetic substitution cipher (see chapter 2.2.1) of the criminal Abe Slaney. Sherlock Holmes solves the riddle using frequency analysis.

[Sayers1932] Dorothy L. Sayers,

*Have his carcase*, Harper/Victor Gollancz Ltd., 1932.

In this novel the writer Harriet Vane finds a dead body at the beach. The police believe the death is suicide. Harriet Vane and the elegant amateur sleuth Lord Peter Wimsey together clear of the disgusting murder in this second of Sayers’s famous Harriet Vane mystery series.

This requires to solve a cryptogram. Surprisingly the novel not only describes the Playfair cipher in detail, but also the cryptanalysis of this cipher (see [Playfair](#) in chapter 2.2.3).

[Simmel1970] Johannes Mario Simmel,

*And Jimmy went to the Rainbow* (original title: *Und Jimmy ging zum Regenbogen*), Knaur Verlag, 1970.

The novel plays between 19938 and 1967 in Vienna. The main character Manual Aranda uncovers step by step the past of his murdered father. Important for the plot is an encrypted manuscript, which is decrypted in chapter 33. In the novel the cipher is called “25-fold Caesar cipher”. It is actually a Vigenère cipher with a 25 character key.

A movie of the novel appeared in 1971.

[Crichton1987] Michael Crichton,

*Sphere*, Pan Books, 1987.

A team of different scientists is send to the ground of the ocean in order to investigate a highly developed 900 m long space ship. The human peculiarities and psychological problems of the researchers surface more and more, because of life threatening events and isolation. There are many mysteries: While the space ship lies on the ground for 300 years, it has English markings and a life of its own, and materializing of the researcher’s imaginations appear. On a computer screen a cipher text appears, which is completely printed

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<sup>8</sup>You can read it at:

<http://whitewolf.newcastle.edu.au/words/authors/K/KiplingRudyard/prose/Kim/index.html>,  
<http://kipling.thefreelibrary.com/Kim> or  
<http://www.readprint.com/work-935/Rudyard-Kipling>.

in the book. The genius mathematician Harry deciphers the simple helical substitution code.

[**Seed1990**] Directed by Paul Seed,  
*House of Cards*, 1990.

In this movie Ruth tries to solve the secret, which made her daughter fall silent. Here two young people suffering from autism communicate via 5- and 6-digit primes (see chapter 3). After more than 1 hour the movie contains the following encrypted two series of primes:

$$\begin{aligned} 21,383; & \quad 176,081; \quad 18,199; \quad 113,933; \quad 150,377; \quad 304,523; \quad 113,933 \\ & \quad 193,877; \quad 737,683; \quad 117,881; \quad 193,877 \end{aligned}$$

[**Robinson1992**] Directed by Phil Alden Robinson,  
*Sneakers*, Universal Pictures Film, 1992.

In this movie the “sneakers”, computer experts under their boss Martin Bishop, try to get back the deciphering box SETEC from the “bad guys”. SETEC, invented by a genius mathematician before he was killed, allows to decrypt all codes from any nation. In the movie the code is not described in any way<sup>9</sup>.

[**Baldacci1997**] David Baldacci,  
*Total Control*, Mass Market Paperback, 1997.

Jason Archer, executive with a technology company suddenly disappears. Sidney Archer tries to find out about her husband’s surprising death. She gets a clue how the global financial system is abused and that the real control belongs to those with the most money. Here even good passwords don’t help ...

[**Natali1997**] Directed by Vincenzo Natali,  
*Cube*, Mehra Meh Film, 1997.

In this Canadian low-budget-movie 7 complete strangers of widely varying personality characteristics are involuntarily placed in an kafkaesque maze of cubical rooms containing deadly traps.

To get out the persons have to move through these rooms. To find out which rooms are dangerous, mathematics is crucial: Each cubic room has at its entrance a numerical marking consisting of three sets of three digits. First they deduce that all rooms marked at their entrance with at least one prime number are trapped. Later it comes out that a trapped room can also be marked by a number which is a power of a prime (so traps are  $p^n$ , e.g.  $128 = 2^7$  or  $101 = 101^1 = \text{prime}$ , but not  $517 = 11 * 47$ ).

[**Becker1998**] Directed by Harold Becker,  
*Mercury Rising*, Universal Pictures Film, 1998.

The NSA developed a new cipher, which is pretended to be uncrackable by humans and computers. To test its reliability some programmers hide a message encrypted with this cipher in a puzzle magazine.

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<sup>9</sup>Leonard Adleman (the “A” within RSA) worked as mathematical consultant for “Sneakers”. He describes the funny story about his contribution at his homepage <http://www.usc.edu/dept/molecular-science/fm-sneakers.htm>. It is assumed that the cipher used everywhere is RSA. According to that within the chip a fast, unknown factorization method is implemented.

Simon, a nine year old autistic boy, cracks the code. Instead of fixing the code, a government agent sends a killer. FBI agent Art Jeffries (Bruce Willis) protects the boy and sets a snare for the killers.

The code is not described in any way.

**[Brown1998]** Dan Brown,

*Digital Fortress*, E-Book, 1998.

Dan Brown's first novel was published in 1998 as e-book, but it was largely unsuccessful then.

The National Security Agency (NSA) uses a huge computer, which enables it to decrypt all messages (needless to say only of criminals and terrorists) within minutes even if they use the most modern encryption methods.

An apostate employee invents an unbreakable code and his computer program Diabolus forces the super computer to do self destructing operations. The plot, where also the beautiful computer expert Susan Fletcher has a role, is rather predictable.

The idea, that the NSA or another secret service is able to decrypt any code, is currently a popular topic. In "Digital Fortress" the super computer has 3 million processors – nevertheless from today's view this is by no means sufficient to hack modern ciphers.

**[Elsner1999]** Dr. C. Elsner,

*The Dialogue of the Sisters*, c't, Heise, 1999.

In this short story, which is included in the CrypTool package as PDF file, the sisters confidentially communicate using a variant of RSA (see chapter 4.10 and the following). They are residents of a madhouse being under permanent surveillance.

**[Stephenson1999]** Neal Stephenson,

*Cryptonomicon*, Harper, 1999.

This very thick novel deals with cryptography both in WW2 and today. The two heroes from the 40ies are the excellent mathematician and cryptanalyst Lawrence Waterhouse, and the overeager and morphine addicted US marine Bobby Shaftoe. They both are members of the special allied unit 2702, which tries to hack the enemy's communication codes and at the same time to hide the own existence.

This secretiveness also happens in the present plot, where the grandchildren of the war heroes – the dedicated programmer Randy Waterhouse and the beautiful Amy Shaftoe – team up.

Cryptonomicon is notably heavy for non-technical readers in parts. Several pages are spent explaining in detail some of the concepts behind cryptography. Stephenson added a detailed description of the Solitaire cipher (see chapter 2.4), a paper and pencil encryption algorithm developed by Bruce Schneier which is called "Pontifex" in the book. Another, modern algorithm called "Arethusa" is not explained in detail.

**[Elsner2001]** Dr. C. Elsner,

*The Chinese Labyrinth*, c't, Heise, 2001.

In this short story, which is included in the CrypTool package as PDF file, Marco Polo has to solve problems from number theory within a competition to become a major consultant of the Great Khan. All solutions are included and explained.

**[Colfer2001]** Eoin Colfer,  
*Artemis Fowl*, Viking, 2001.

In this book for young people the 12 year old Artemis, a genius thief, gets a copy of the top secret “Book of the Elfs”. After he decrypted it with his computer, he finds out things, men never should have known.

The used code is not described in detail or revealed.

**[Howard2001]** Directed Ron Howard,  
*A Beautiful Mind*, 2001.

This is the film version of Sylvia Nasar’s biography of the game theorist John Nash. After the brilliant but asocial mathematician accepts secret work in cryptography, his life takes a turn to the nightmarish. His irresistible urge to solve problems becomes a danger for himself and his family. Nash is – within his belief – a most important hacker working for the government.

Details of his way analysing code are not described in any way.

**[Apted2001]** Directed by Michael Apted,  
*Enigma*, 2001.

This is the film version of Robert Harris’ “historical fiction” *Enigma* (Hutchinson, London, 1995) about the World War II code-breaking work at Bletchley Park in early 1943, when the actual inventor of the analysis Alan Turing (after Polish pre-work) already was in the US. So the fictional mathematician Tom Jericho is the lead character in this spy-thriller. Details of his way analysing the code are not described.

**[Isau2003]** Ralf Isau,  
*The Museum of the stolen memories* (original title: *Das Museum der gestohlenen Erinnerungen*), Thienemann-Verlag, 1997/2003.

In this exciting novel the last part of the oracle can only be solved with the joined help of the computer community.

The book got several awards and exists in 8 different languages, but not in English yet.

**[Brown2003]** Dan Brown,  
*The Da Vinci Code*, Doubleday, 2003.

The director of the Louvre is found murdered in his museum in front of a picture of Leonardo da Vinci. And the symbol researcher Robert Langdon is involved in a conspiracy. The plot mentions different classic codes (substitution like Caesar or Vigenère, as well as transposition and number codes). Also there are hints about Schneier and the sunflower. The second part of the book contains a lot of theological considerations.

This book has become one of the most widely read books of all time.

**[Hill2003]** Tobias Hill,  
*The Cryptographer*, Faber & Faber, 2003.

London 2021: The company SoftMark developed and establish an electronic currency, which guarantees highest security standards by an unbreakable code. The inventor and company founder, called the cryptographer because of his mathematical talent, has become the

richest man in the world. But the code was hacked, and in a worldwide economic crisis his company goes bankrupt. Additionally the tax investigator Anna Moore is set on him.

**[McBain2004]** Scott McBain,

*Final Solution*, manuscript not published by Harper Collins, 2004 (German version has been published in 2005).

In a near future politicians, chiefs of military and secret services of many different countries take over all the power. With a giant computer network called "Mother" and complete surveillance they want to cement their power and commercialisation of life forever. Humans are only assessed according to their credit rating and globally acting companies elude of any democratic control. Within the thriller the obvious injustice, but also the realistic likelihood of this development are considered again and again.

With the help of a cryptographer a code to destroy was built into the super computer "Mother": In a race several people try to start the deactivation (Lars Pedersen, Oswald Plevy, the female American president, the British prime minister and an unknown Finish named Pia, who wants to take revenge for the death of her brother). On the opposite side a killing group acts under the special guidance of the British foreign minister and the boss of the CIA.

**[Preston2005]** Douglas Preston,

*Tyrannosaur Canyon*, Forge Books, 2005.

A very exciting thriller which also struggles with the question why the dinosaurs died off.

Archeologist Stem Weathers is shot in a canyon. Before his murderer appears he gives his notebook to Tom Broadbent, a local animal doctor, coming by accidentally.

The notebook contains on 60 pages only digits. Therefore Tom takes it to Wyman Ford an ex-CIA cryptanalyst, who now lives in a nearby abbey, after his wife was killed in action. Wyman first declines and says that self-invented code are "idiot ciphers" – devised by an idiot and easily crackable by each idiot. The notebook then proves to be not that easy. After intensive analysis he finds out that the digits are no code but the output of an earth radar device showing the picture of a well-preserved T.rex.

After around 250 pages of endless chases a surprising turn comes up: Masago, head of a so-called black-detachment unit of the CIA. He explains: New weapons invented once always have been used. Mankind will kill herself, but its his task to postpone that as far as possible. As head of the LS480 department he will prevent by any means possible that terrorists get any new dangerous biological weapon.

When scanning the dead body of Weathers the murder only found some rock cuttings he took. These rocks are investigated by a young researcher named Melody Crookshank although she doesn't know where the rock cuttings come from. She finds within them a very special kind of virus apparently coming from outer-space.

**[Burger2006]** Wolfgang Burger,

*Heidelberg Lies (original title: Heidelberger Lügen)*, Piper, 2006.

This detective story playing in the Rhein-Neckar area in Germany has several independent strands and local stories, but mainly it is about Kriminalrat Gerlach from Heidelberg. On page 207 f. the cryptographic reference for one strand is shortly explained: The soldier Hörrle had copied circuit diagrams of a new digital NATO decryption device and the

murdered man had tried to sell his perceptions to China.

[Twinig2006] James Twinig,

*The Black Sun*, HarperCollins, 2006.

A history-based thriller with some artificially constructed elements, dealing also with a treasure hunt to get the hidden uranium of the nazis, and naturally the future of the world depends on today's bad guys being stopped in time ...

Heros are Tom Kirk, a London-based ex-CIA agent and former professional art thief, and Dominique de Lecourt, who loves challenges including riddles and codes.

The only cryptographic parts are a “Sprungcode” (the criminals use this method to communicate via newsletter adverts), steganography (used to hide the Enigma key), and an Enigma message (containing the encrypted coordinates of the treasure).

At the beginning of the plot an Enigma device is stolen with high efforts which is necessary to let the story play in the constructed way. But in the reality today such a theft is completely needless, as there are great software emulators for the Enigma ...

[Vidal2006] Agustin Sanchez Vidal,

*Kryptum*, Dtv, 2006.

The first novel of the Spanish professor of art history has some similarities with Dan Brown’s “The Da Vinci Code” from 2003, but allegedly Vidal started his writing of the novel already in 1996. Vidal’s novel is a mixture between historic adventure and mystery thriller. It was a huge success in Spain and Germany. There is currently no English version available.

In the year 1582 Raimundo Randa is waiting to be condemned to death – he was all life long trying to solve a mystery. This mystery is about a parchment with cryptic characters, where a unique power is behind. Around 400 years later the American scientist Sara Toledano is fascinated by this power until she vanishes in Antigua. Her colleague, the cryptographer David Calderon, and her daughter Rachel are searching for her and simultaneously they try to solve the code. But also secret organizations like the NSA chase after the secret of the “last key”. They don’t hesitate to kill for it.

[Larsson2006] Stieg Larsson,

*Perdition* (original title: *Flickan som lekte med elden*), 2006.

The author was posthumously awarded in 2006 with the Scandinavian thriller award. The super hero Lisbeth Salander uses PGP and occupies herself with mathematical riddles like the Fermat theorem.

[Schroeder2008] Rainer M. Schröder,

*The Judas Documents* (original title: *Die Judas-Papiere*), Arena, 2008.

In the year 1899 Lord Pembroke has three men and one woman in his grip. So they have to follow his order to try to decipher the encrypted messages in the notebook of his dead brother Mortimer and to find the missing gospel according to Judas, which could shock the whole of Christendom. The four people therefor have to solve riddles at many places in the world. The story explains some classic ciphers like Polybius and Freemason.

**[Eschbach2009]** Andreas Eschbach,

*A King for Germany* (original title: *Ein König für Deutschland*), Lübbe, 2009.

The novel deals with manipulations of electronic voting machines.

Vincent Merrit, a young US-American programmer, is blackmailed, to write such a programme. Beside commercially oriented blackmailers also Massively Multiplayer Online Role-Playing Games (MMORPGs) and Live Action Role Playing (LARP) have a role. Because Merrit assumed that his programme will be misused, he installed a trapdoor: If a party with the name VWM participates at the election, it automatically gets 95 % of the votes ...

The fictional story line is based on many verifiable and well researched facts, which are referenced in footnotes.

While the cryptographic protocols itself could be made secure, their implementation and their organisational management stays susceptible against misuse.

Currently there is no English translation of the book.

**[Juels2009]** Ari Juels,

*Tetrakty*s, Emerald Bay Books, 2009.

The plot exposes the vulnerability of modern computer based identity, authenticity, and security interweaving modern cryptography with classical art and literature. Cryptographer and classicist Ambrose Jerusalem is a UC Berkeley graduate with a beautiful girlfriend and a comfortable future, until the NSA recruits him to track a strange pattern of computer break-ins. Many small pieces provide disturbing evidence that someone has broken RSA encryption. Even more bizarre, a secret cult of latter-day followers of Pythagoras, the great Greek mathematician and philosopher who believed reality could be understood only through a mystical system of numbers, appears to be behind the attacks.

**[Suarez2009]** Daniel Suarez,

*Daemon*, Penguin Books, 2009, 632 pages.

This is considered as one of the most exciting books during the last few years – its a near-science-fiction thriller combining developments in the real world and possibilities coming from current research like from the Google X Lab (augmented reality head-mounted displays (HMD) like Google glass, self-driving cars, 3-D printers, ...) to a plausible story.

After the computer genius and game developer Matthew Sobol died a daemon starts acting in the internet, which seemingly ruthless manipulates and trains more and more humans and companies.

By ruling the data everybody seems to be a helpless victim. All the communication of his mercenary soldiers is affected by high-tech and encryption – also the communication between the distributed instances of his incarnation. Core is an MMORPG game (massive multiplayer online role-playing game) which reminds many of WoW. Here also encryption is used, e.g. to advertise the best players: m0wFG3PRCoJVTs7JcgBwsOXb3U7yPxBB

The plot is without redundancy, complex, manifold, very fascinating and with it critics about the plutocrats it also contains concrete social elements. The end is open. And the ideas seem to be realizable in the very next future ...

**[Suarez2010]** Daniel Suarez,

*Freedom (TM)*, Penguin Books, 2010, 486 pages.

“The propulsive, shockingly plausible sequel to the bestseller Daemon” (see above). Freedom

(TM) (Daemon #2) patches a number of holes the writer left in the first book. The prose is tighter, the descriptions more direct, the characters are fleshed out, especially Loki. Having laid the groundwork in “Daemon”, Suarez uses this foundation in order to explore a new concept of social organization based on empowering information technology and the reasoning why and how the battle runs between the old potentates and the Daemon society, which also evolves further already during the story. Cryptography is a natural part of modern technology and modern warfare as described in this book. The new society emerging in “Freedom (TM)” is based on the darknet, an alternative to the internet using fast wireless meshes in order to increase the durability and availability of the network. Despite the story is shocking in some parts, it appears to be realistic and not far away from the parallel usage of modern technology integrated into our modern lives as a virtual world overlaying our real world.

[**Olsberg2011**] Karl Olsberg,

*Rafael 2.0* (original title: *Rafael 2.0*), Thienemann Verlag, 2011, 240 pages.

Michael and Rafael Ogilvy are talented twins who get along very well. Before the terminally ill Rafael dies, his father developed a virtual computer effigy of him, an artificial intelligence (AI). This is a good kept secret until Michael one day finds out what his father is hiding before him. However, his first horror soon turns into joy. So he still has something that reminds him of his brother.

But this computer system is also interesting for the military. One day Michael’s father is kidnapped and the company and thus also the computer program Rafael 2.0 fall into the wrong hands. Michael is banished by his uncle in a boarding school, from which he can flee. Henceforth, Michael and his friends try their best to find his father, of whom they assume that he was abducted by a competing company. From here the story gets really exciting ... Michael learns that there is another artificial intelligence, Metraton, which is not so well-disposed to the people. Nothing is too much engrossed, young teenagers are the target audience. Nevertheless, depth and substance are created when for instance the machinations in acquisitions are discussed.

From a crypto perspective, the section about factoring is thrilling: With a variant Michael can detect whether the computer is cheating ...

[**Burger2011**] Wolfgang Burger,

*The fifth murderer* (original title: *Der fünfte Mörder*), Piper, 2011.

Location & time of the story: Germany / Heidelberg, 1990 - 2009. Episode 7 of the Alexander-Gerlach series. Inspector Alexander Gerlach became almost a victim of a bomb blast when the sport utility vehicle (SUV) of a Bulgarian panderer exploded. Gerlach starts investigating because he wants to prevent a gang warfare, but then his bosses call him off. When the journalist Machatschek supports Gerlach, he communicates with him only via Skype using an add-on encryption program which he believes is the most secure in the world.

[**Eschbach2011**] Andreas Eschbach,

*Master of the universe – master of all staff* (original title: *Herr aller Dinge*), Lübbe, 2011.

This novel would have deserved a much broader audience: The idea in it of the “most terrific of all crimes”, which is the origin of the whole story, is new and almost revolutionary, but also infinitely sad. Along the failing partnership of Hiroshi (inventor genius) and Charlotte important topics like justice, human wealth and power are dealt with.

From a crypto perspective, Hiroshi uses distributed calculations and developed an encryption and backup system which misleads the government which bugged him.

**[Elsberg2012]** Marc Elsberg,

*Blackout – Tomorrow its too Late* (original title: *Blackout – Morgen ist es zu spät*), Blanvalet, 2012, 800 pages.

At a cold day in winter all power supply networks in Europe break down. Agencies, energy suppliers and security companies are in the dark and unable to solve the problem. The Italian computer scientist Piero Manzano believes that this caused by terrorists using hackers: All customers use since some years smart meters, electricity meters controlled by software which was manipulated. Despite the integrated security and encryption components they have been hacked, and are out of order by wrong control sequences. The terrifying consequences at various locations are described realistically and excitingly. And in the same way the reactions of the human beings ...

**[Olsberg2013]** Karl Olsberg,

*The Eigths Revelation* (original title: *Die achte Offenbarung*), Aufbau Taschenbuch, 2013, 460 pages.

Can a message from the past change our future? An ancient, encrypted manuscript fell into the hands of the historian Paul Brenner. The more he decodes the text, the more puzzling is the content: Because the book tells with remarkable precision events years ahead of the time of its presumed creation. While highly dangerous genetic material disappears from a US laboratory, someone tries to prevent at any price, that Paul deciphers the last, the eighth revelation. A gripping thriller about a shockingly realistic apocalypse with many human aspects ...

As a reader, you can participate in the deciphering of the manuscript.

The experiments of Paul to make the right persons aware of his discovery and to correct it later, are described very exciting – even chief editors have a dilemma with conspiracy.

The cipher on the last book page is offered as a challenge in the crypto competition MTC3: <https://www.mysterytwisterc3.org/en/challenges/level-i/the-last-note>

**[Takano2014]** Kazuaki Takano,

*Genocide of One*, 2014. (Original in Japanese: “Jenosaido”, 2011; as paperback in English again under the title “Extinction”, 2016)

The cover text says: He is a new kind of human. He may mean the end for the rest of us... One bright morning in Washington D.C., the US President learns of a terrifying new threat to national security. Soon afterwards, American mercenary Jonathan Yeager is asked to lead a team into the Congo to eliminate a mysterious enemy – a job which will help him pay for treatment for his dying son. But when they reach Africa, the threat turns out to be a three-year-old child named Akili: the next step in human evolution. The soldiers are under orders to kill the boy before his full potential can be realized. Yet Akili’s advanced knowledge might be the only hope Yeager has to save his son’s life... With time running out to choose a side, Yeager must decide whether to follow his orders or to save a creature who may not be as harmless or innocent as he appears. Because Akili is already the smartest being on the planet, with the power to either save humanity – or destroy it. This is a very exciting book. After having overcome the first 100-200 pages you’ll be awarded with surprising insights. According to the reviews it’s very well researched, but not for superficial readers.

From a crypto perspective, RSA and OTP are directly drivers of the story and are explained correctly. Breaking RSA by factorization is so important that the CIA wouldn't accept that this knowledge isn't in their ownership ...

[Elsberg2014] Marc Elsberg,

*ZERO – They Know what you are Doing* (original title: *ZERO – Sie wissen, was du tust*), Blanvalet Verlag, 2014, 480 pages.

London. In a pursuit a boy is shot. His death takes the journalist Cynthia Bonsant to the acclaimed internet platform Freemee. Freemee collects and analyzes data, and thus promises its millions of users – rightly – a better life and more success. There is only one who warns about Freemee and about the power that the online newcomer could give just a few: ZERO, the most searched online activist in the world. As Cynthia begins precisely to research, she's becoming the quarry. And in a world of cameras, headsets and smartphones there is no escape ...

Highly topical and menacing: the transparent person under control. The novel takes place in the near future (near fiction) and contains many contemporary references such as PRISM, predictive analytics, gamification. By the way, references to well-known science fiction media like "The Running Man", "Monkey Wrench Gang", "V as Vendetta" (V wears a Guy Fawkes mask, now the hallmark of Anonymous), "Network" and "Body Snatchers" are processed.

Technologically / cryptologically the protagonists move on the highest level, which is not further explained: Alice Kinkaid communicates with a Raspberry Pi. Cynthia's daughter Vi uses mesh networks. See

[https://de.wikipedia.org/wiki/Zero\\_%E2%80%93\\_Sie\\_wissen,\\_was\\_du\\_tust](https://de.wikipedia.org/wiki/Zero_%E2%80%93_Sie_wissen,_was_du_tust),

<http://www.zero-das-buch.de/actiontrailer.php>

[Lagercrantz2015] David Lagercrantz,

*The Girl in the Spider's Web*, Quercus, 2015.

This is the 4th novel in the Millennium series, and the first not written by Stieg Larsson. While Mikael Blomkvist's print medium is struggling to survive the reader gets more and more insight in the inner structures and the combinations of publishers, secret services, public agencies, organized crime, and industrial espionage. Here, no care is taken for single humans, and normal humans would have no chance against this mix of interests. However, the special skills of Lisbeth Salander makes a difference, and so the NSA is informed, that parts of it are led and misused by organized crime. The characters of the Millennium trilogy have been developed further in a credible way. Very exiting.

From a crypto perspective, Lisbeth and the autist August deal with elliptic curves to crack RSA.

See [https://en.wikipedia.org/wiki/The\\_Girl\\_in\\_the\\_Spider%27s\\_Web](https://en.wikipedia.org/wiki/The_Girl_in_the_Spider%27s_Web).

**Remark 1:** A long list of (partly commented) samples of cryptology in fictional literature can be found on the following German web page:

[http://www.staff.uni-mainz.de/pommeren/Kryptologie/Klassisch/0\\_Unterhaltung/](http://www.staff.uni-mainz.de/pommeren/Kryptologie/Klassisch/0_Unterhaltung/)

For some older authors (e.g. Jules Verne, Karl May, Arthur Conan Doyle, Edgar Allan Poe) there are links to the original and relevant text pieces.

**Remark 2:** You can find title pages of some of these books on the web site of Tobias Schrödel, who collects classic books about cryptography:

[http://tobiasschroedel.com/crypto\\_books.php](http://tobiasschroedel.com/crypto_books.php)

**Remark 3:** If you know of further books and movies, where cryptography has a major role then we would be very glad if you could send us the exact title and a short explanation about the movie/book's content. We will insinuate your possible enthusiasm for a title. Thanks a lot.

### A.5.2 For Kids and Teenagers

The following list contains movies and “kid books”. The kid books contain both stories, and collections of simple ciphers, prepared in a didactic and exciting manner (please send us similar English kid books and kid movies, because at the moment our list contains mostly German kid books):

[Mosesxxxx] [no named author],

*Top secret – The Book for Detectives and Spies* (original title: *Streng geheim – Das Buch für Detektive und Agenten*), Edition moses, [no year named].

This is a thin book for small kids with Inspector Fox and Dr. Chicken.

[Arthur196x] Robert Arthur,

*The Three Investigators: The Secret Key* (German version: *Die 3 ????: Der geheime Schlüssel nach Alfred Hitchcock* (volume 119), Kosmos-Verlag (from 1960)

The three detectives Justus, Peter and Bob have to decrypt covered and encrypted messages within this story to find out what is behind the toys of the Copperfield company.

[Para1988] Para,

*Ciphers* (original title: *Geheimschriften*), Ravensburger Taschenbuch Verlag, 1988 (1st edition 1977).

On 125 pages filled with a small font this mini format book explains many methods which young children can apply directly to encrypt or hide their messages. A little glossary and a short overview about the usage of encryption methods in history complete this little book.

Right at page 6 it summarizes for beginners in an old fashion style “The Important Things First” about paper&pencil encryption (compare chapter 2):

- “It must be possible to encrypt your messages at any place and at any location with the easiest measures and a small effort in a short time.
- Your cipher must be easy to remember and easy to read for your partners. But strangers should not be able to decrypt them.  
Remember: Fastness before finesse, security before carelessness.
- Your message must always be as short and precise as a telegram. Shortness outranks grammar and spelling. Get rid of all needless like salutations or punctuation marks. Preferably use only small or only capital letters.”

[Müller-Michaelis2002] Matthias Müller-Michaelis,

*The manual for detectives. Everything you need to know about ciphers, codes, reading tracks and the biggest detectives of the world* (original title: *Das Handbuch für Detektive. Alles über Geheimsprachen, Codes, Spurenlesen und die großen Detektive dieser Welt*), Südwest, 2002.

[Kippenhahn2002] Rudolf Kippenhahn,

*Top secret! – How to encrypt messages and to hack codes* (original title: *Streng geheim! – Wie man Botschaften verschlüsselt und Zahlencodes knackt*), rororo, 2002.

In this novel a grandpa, an expert for secret writings teaches his four grandchildren and

their friends, how to encrypt messages which nobody should read. Because there is someone who hacks their secrets, the grandpa has to teach them more and more complicated methods.

Within this story, which forms the framework, the most important classic encryption methods and its analysis are explained in a manner exciting and appropriate for children.

**[Harder2003]** Corinna Harder and Jens Schumacher,

*Top secret. The big book for detectives* (original title: *Streng geheim. Das große Buch der Detektive*), Moses, 2003.

**[Talke-Baisch2003]** Helga Talke and Milena Baisch,

*Your mission in the weird villa. Riddle thriller* (original title: *Dein Auftrag in der unheimlichen Villa. Kennwort Rätselkrimi*), Loewe, 2003.

From 4th form, <http://www.antolin.de>

Young detectives are faced simple ciphers and codes during their missions.

**[Flessner2004]** Bernd Flessner,

*The Three Investigators: Manual for Secret Messages* (original title: *Die 3 ????: Handbuch Geheimbotschaften*), Kosmos, 2004.

On 127 pages you learn in an easy and exciting manner, structured by the method types, which secret languages (like the one of the Navajo Indians or dialects) and which secret writings (real encryption or hiding via technical or linguistic steganography) existed and how simple methods can be decrypted.

The author tells where in history the methods were used and in which novel authors used encryption methods [like in Edgar Allan Poe's "The Gold Bug", like with Jules Verne's hero Mathias Sandorf or like with Astrid Lindgren's master detective Blomquist who used the ROR language (similar inserting ciphers are the spoon or the B language)].

This is a didactically excellent introduction for younger teens.

**[Zübert2005]** Directed by Christian Zübert,

*The Treasure of the White Hawks* (original title: *Der Schatz der weißen Falken*), 2005.

This exciting adventure movie for kids ties in with the tradition of classics like "Tom Sawyer and Huckleberry Finn" or Enid Blytons "Five Friends". The plot happens in summer 1981. In an old half tumbledown villa three young kids find the treasure map of the "White Hawks", which they decrypt with the help of a computer. Traced by another gang they aim to go to an old castle.

**[Dittert2011]** Christoph Dittert,

*The Three Investigators: Secret Messages* (German version: *Die 3 ????: Geheimnisvolle Botschaften*) (volume 160), Kosmos, 2011

In the house of Professor Mathewson an old hand-made book was stolen. The three detectives Justus, Peter and Bob are getting attacked by a ruthless opponent, who seems to be always a step ahead. A major part in this story is played by a palimpsest, an ancient manuscript page, written upon newly. Using X-rays they can make visible again the old text below. Its not only the story which is exciting, but also the way, how the instruction for a treasure hunt is encrypted. Despite using the simple railfence cipher it's not easy

to solve it, as the message is distributed onto two slips and as the printed symbols don't mean single letters.

**Remark 1:** You can find title pages of many of these kid books on the web site of Tobias Schrödel, who collects classic books about cryptography:

[http://tobiasschroedel.com/crypto\\_books.php](http://tobiasschroedel.com/crypto_books.php)

**Remark 2:** If you know of further books, which address cryptography in a didactic and for children adequate way, then we would be very glad if you could send us the exact book title and a short explanation about the book's content. Thanks a lot.

### A.5.3 Code for the light fiction books

Chapter A.5.1 lists as first book “The Gold Bug” by E.A. Poe.

Using the following Python code<sup>10</sup> you can decrypt the ciphertext of Captain Kidd (see the original text of “The Gold Bug” in <http://pinkmonkey.com/dl/library1/gold.pdf>, page 21) with Python or with SageMath.

The code already contains the ascii characters of the ciphertext and the correlated alphabets for the plaintext and the ciphertext of this monoalphabetic cipher.

The easiest way to perform the decryption is using the SageMathCell server (<http://sagecell.sagemath.org/>) in a browser: There you can switch between the programming languages Sage and Python. The code can be executed by inserting it with “copy and paste” and then pressing “Evaluate”.

---

**SageMath sample A.1** Decryption of the Gold-Bug ciphertext from the novel of E.A. Poe (with Python)

---

```
# Hier geht's los ... Start here
import string
PA = 'ETHSONAIRDGLBVPFYMUC'
print 'Plaintext alphabet PA: ', PA, ' Length of PA', len(PA)
CA = "8;4)+*56(!302'.1:9?-"
print 'Ciphertext alphabet CA: ', CA, ' Length of CA', len(CA)
codetableC2P = string.maketrans(CA,PA)
C = '''53++!305))6*;4826)4+.4+);806*;48!8'60))85;1+(;:+*8!83(88)5*!;46
(;88*96*?;8)*+;(485);5*!2:++(;4956*2(5*-4)8'8*;4069285);)6!8)4++;1(+9;4
8081;8:8+1;48!85;4)485!528806*81(+9;48;(88;4(+?34;48)4+;161;:188;+?;'''
P = string.translate(C, codetableC2P);
print 'Kidd decrypted:', P
# ... und hier sind wir schon fertig ... and here we are done!
```

---

**Remark 1:** When printing the ciphertext of Poe, Pinkmonkey “cheated” similarly like the code author by using only ascii characters.

In the archive of an original publication (e.g. at <https://archive.org/details/goldbug00poegoog> at page 95) you can see, that Poe used characters which were common in the letterpress printing (and most of them are also part of the ascii set). It is very unlikely that an untaught pirate would use just such characters for his ciphertext ...

**Remark 2:** The code uses the Python functions “maketrans” und “translate” from the String package (see e.g. the Python 2.7 documentation, section “functions”).

So both alphabets (for the plaintext and the ciphertext) are inserted as a simple string, and “maketrans” creates a mapping table. The actual encryption is done by “translate”. For the decryption you just have to switch the arguments of “maketrans” for the two alphabets. The otherwise necessary transformations between characters and their ascii numbers (using “str” and “ord”) can be avoided. This is ideal for monoalphabetic ciphers – especially for lessons at the junior high school.

---

<sup>10</sup>See [WLS98] and the file 2\_MonoKidd.zip at <http://bscw.schule.de/pub/bscw.cgi/159132>.

**Remark 3:** The following screenshot shows the execution of the Python code on the SageMathCell server (<http://sagecell.sagemath.org/>).

SageMathCell is the browser interface for Sage. Its functions are delivered for free on the Sage cell server (Sage is a comprehensive open-source math software system).

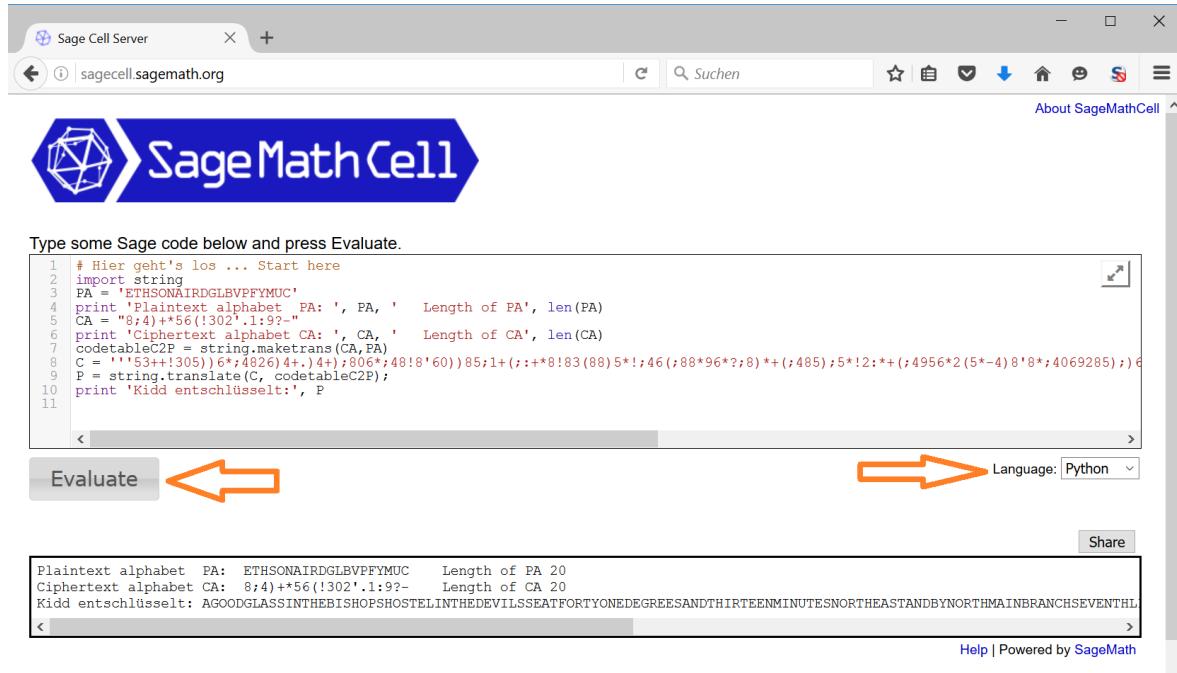


Figure A.8: Usage of SageMathCell to decipher Poe's Gold Bug (with Python)

As result we get:

```

Plaintext alphabet PA: ETHSONAIRDGLBVPFYMUC Length of PA 20
Ciphertext alphabet CA: 8;4)+*56(!302'.1:9?- Length of CA 20
Kidd decrypted: AGOODGLASSINTHEBISHOPSHOSTELINTHEDEVILSSEATFORTYONEDEGREES
                ANDTHIRTEENMINUTESNORTHEASTANDBYNORTHMAINBRANCHSEVENTHLIMBEASTSIDESHOOTFROMTHELEFTEYEOTHEDEATHSHEADABEELINEFROMTREETHROUGHTHESHOTFIFTYFEETOUT

```

It's evident how less code is needed with Python or Sage for such tasks. In the above sample there were 2 unnecessary comment lines, 3 lines for input, 3 lines of real code, and 3 lines for the output; really necessary were only 7 lines of code.

## A.6 Learning Tool for Elementary Number Theory

CT1 contains an interactive educational tool for elementary *number theory*, called “NT”.<sup>11</sup>

The educational tool “NT” (number theory) by Martin Ramberger introduces number theory and visualizes many of the methods and concepts. Where necessary it shows the according mathematical formulas. Often you can apply the mathematical methods dynamically with your own small numerical examples.

The content of this educational tool is mainly based on the books by J. Buchmann and H. Scheid [Buc16, Sch06].

This visualized educational tool was built with Authorware 4.<sup>12</sup>

**Request for enhancement/upgrade:** It would be desirable to update it to a new version Authorware or to use another development platform. If there are developers interested to do this, I’d be more than happy (please send an email at the author of this CrypTool script).

**Figures:** The figures A.9 till A.16 give you an impression of the educational tool “NT”:

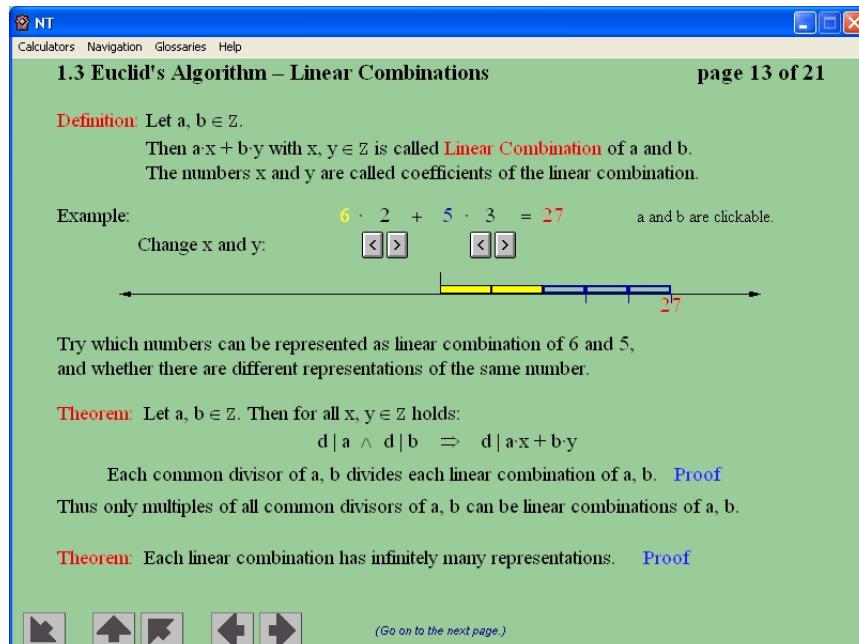


Figure A.9: Each common divisor of two integers also divides all its linear combinations

<sup>11</sup>NT can be called in CT1 via the menu path **Indiv. Procedures \ Number Theory Interactive \ Learning Tool for Number Theory**.

<sup>12</sup>As Authorware is outdated and as the vendor didn't make tools available to easily port it to his successor products, the program ZT will not be further developed.

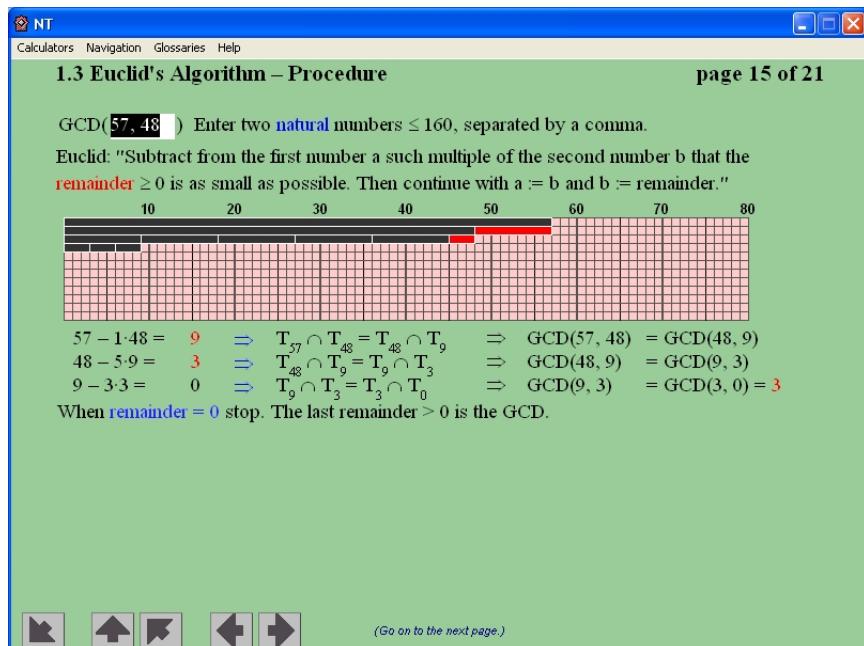


Figure A.10: Euklid's algorithm to determine gcd

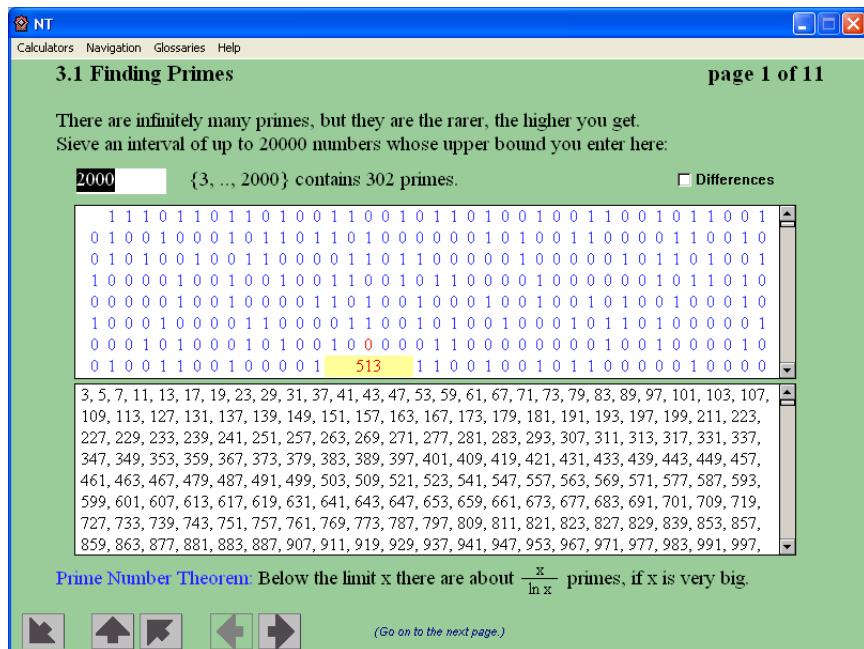


Figure A.11: Distribution of primes and its differences

**NT**

Calculators Navigation Glossaries Help

### 3.2 Fermat Test

page 4 of 11

Each prime  $p$  passes a test that results from Fermat's [Little Theorem](#):

Try for  $a, b \in \{2, \dots, p-1\}$ , if  $b^{p-1} \equiv 1 \pmod{p}$ .

This test is called **Fermat Test**. Unfortunately some composite numbers pass it as well.

Example:  $341 = 11 \cdot 31$ , even so is  $2^{340} \equiv 1 \pmod{341}$ .

A passed test gives no information, one repeats it with a different base  $b$ :

$n = 341$	$2^{n-1} \equiv 1 \pmod{n}$	Test passed
GCD(b, n) = 1	<input type="button" value="b &lt;"/> <input type="button" value="b &gt;"/>	

**Definition:** Let  $n$  be a composite number,  $b$  coprime to  $n$ .  
 If  $b^{n-1} \equiv 1 \pmod{n}$ , then one calls
 

- **n Pseudo Prime to Base b**,
- **b Liar for** (the primality of) **n**,

 otherwise one calls **b Witness against** (the primality of) **n**.

**Theorem:** If there are any witnesses against  $n$ ,  
 then they make up at least 50% of all  $b \in \{1, \dots, n\}$  coprime to  $n$ . [Proof](#)

(Go on to the next page.)

Figure A.12: Finding primes with the prime number test of Fermat

**NT**

Calculators Navigation Glossaries Help

### 4.1 Cryptography – Encryption ought to be Reversible

page 4 of 17

This method applies addition and multiplication. Its keys are pairs  $(a, b)$ . Try out which pairs make the encryption reversible.

`abcdefghijklmnopqrstuvwxyz Nicht alle a funktionieren.`

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 0
N I C H T A L L E A F U N K T I O N I E R E N	14 9 3 8 20 0 1 12 12 5 0 1 0 6 21 14 11 20 9 15 14 9 5 18 5 14

Encryption by the function:  $E(x) = a \cdot x + b \pmod{27}$        $a = 2$

number of keys:  $\varphi(27) \cdot 27$        $b = 0$

B D F H J L N P R T V X Z A C E G I K M O Q S U W Y	2 4 6 8 10 12 14 16 18 20 22 24 26 1 3 5 7 9 11 13 15 17 19 21 23 25 0
A R F P M B X X J B L O A V M R C A R J I J A	1 18 6 16 13 0 2 24 24 10 0 2 0 12 15 1 22 13 18 3 1 18 10 9 10 1

Decryption by the inverse function:  $D(x) = a' \cdot (x - b') \pmod{27}$        $a' = 14$

It must hold:  $a \cdot a' \equiv 1 \pmod{27}$ .       $b' = 0$

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z	1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 0
N I C H T A L L E A F U N K T I O N I E R E N	14 9 3 8 20 0 1 12 12 5 0 1 0 6 21 14 11 20 9 15 14 9 5 18 5 14

Figure A.13: Reversibility of encryption mechanisms exemplified with additive ciphers

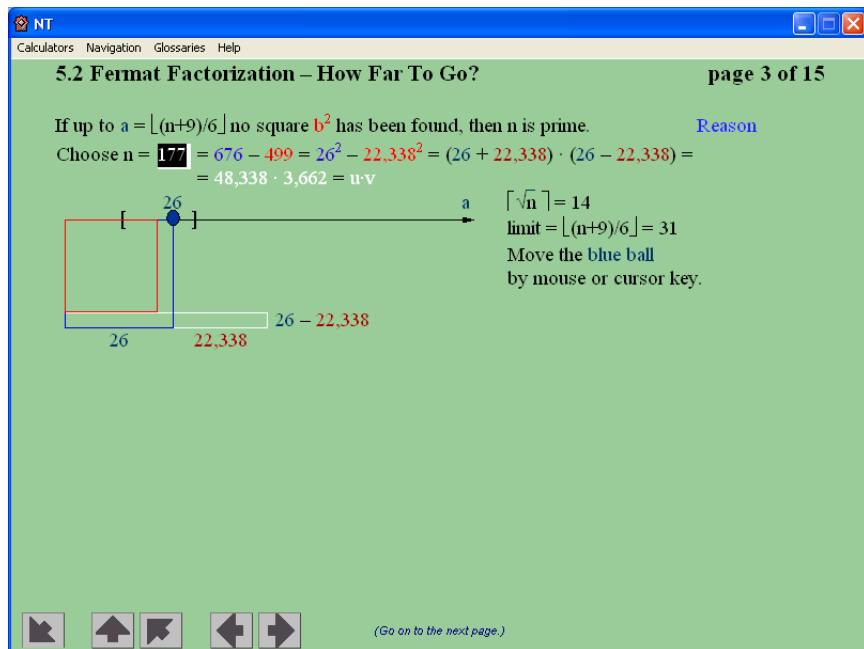


Figure A.14: Fermat factorization using the third binomial formula

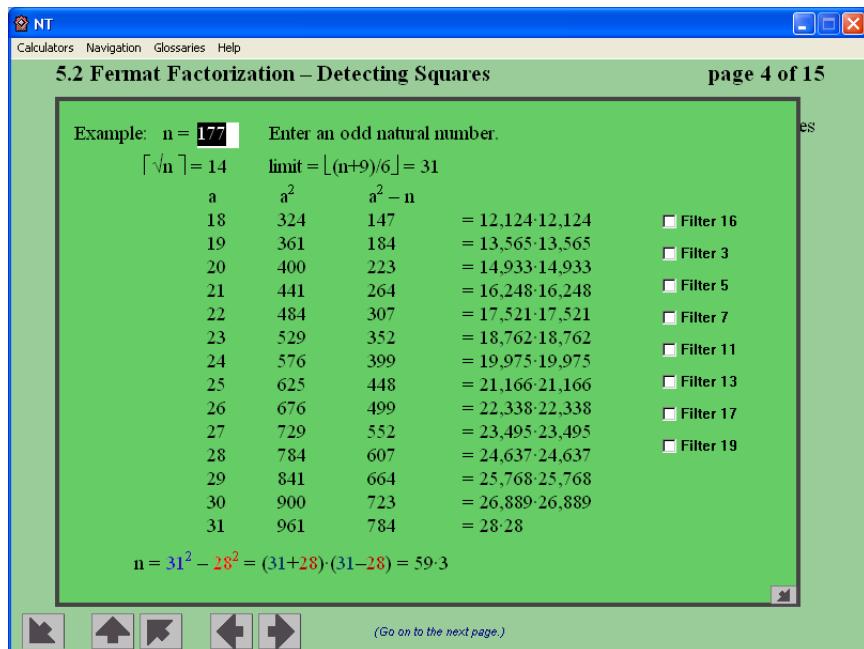


Figure A.15: Fermat factorization: detecting squares

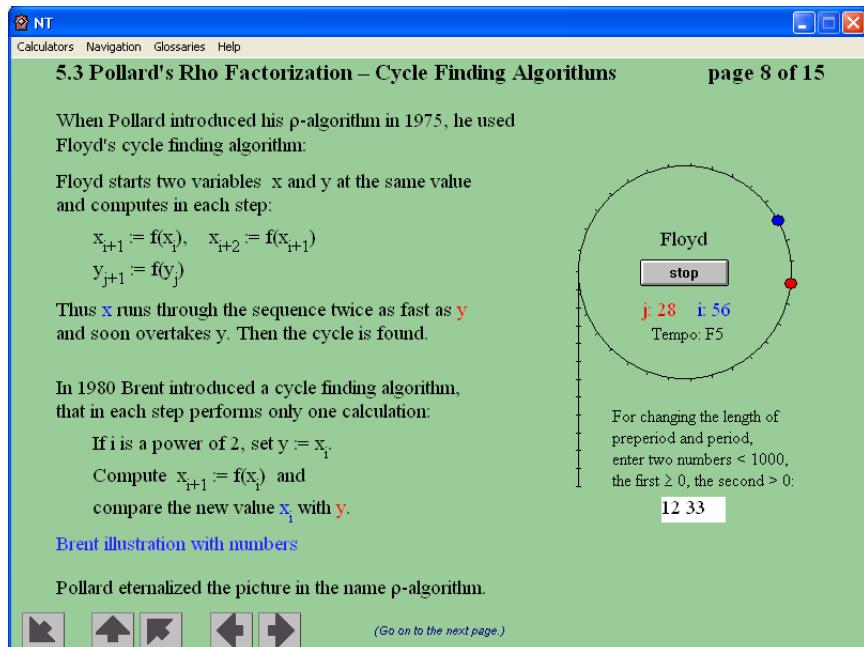


Figure A.16: Pollard's rho factorization: Floyd's cycle finding algorithm

# Bibliography

- [Buc16] Buchmann, Johannes: *Einführung in die Kryptographie*. Springer, 6th edition, 2016. Paperback.
- [Kip97] Kippenhahn, Rudolf: *Verschlüsselte Botschaften: Geheimschrift, Enigma und Chipkarte*. rowohlt, 1st edition, 1997. New edition 2012, Paperback, *Verschlüsselte Botschaften: Geheimschrift, Enigma und digitale Codes*.
- [Sch06] Scheid, Harald: *Zahlentheorie*. Spektrum Akademischer Verlag, 4th edition, 2006.
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[http://bscw.schule.de/pub/bscw.cgi/d637160/RSA\\_u\\_Co\\_T1.pdf](http://bscw.schule.de/pub/bscw.cgi/d637160/RSA_u_Co_T1.pdf).

## A.7 Short Introduction into the CAS SageMath

This book includes numerous code samples using SageMath. SageMath is an open source computer algebra system (CAS) that supports teaching, study and research in mathematics. It combines many high-quality open source packages<sup>13</sup> and provides access to their functionalities via a common interface, namely, a Python<sup>14</sup> based programming language.

SageMath can be used as a powerful desktop calculator, as a tool to help (undergraduate) students study mathematics, or as a programming environment for prototyping algorithms and research in algorithmic aspects of mathematics.

You can get a quick impression of SageMath e.g. with the references in this footnote<sup>15</sup>.

The official SageMath online documentation<sup>16</sup> is available at: <http://www.sagemath.org>.

In the meantime there are lots of PDF and HTML documents about using SageMath, so we name only a few of them as a good starting point<sup>17</sup>.

With respect to studying cryptography, SageMath modules can be used to complement a first course in cryptography<sup>18</sup>.

Comprehensive introductions into cryptography are in this footnote<sup>19</sup>.

### SageMath user interfaces

SageMath is available free of charge and can be downloaded from the following website:

<http://www.sagemath.org>

---

<sup>13</sup>To get an impression of how big SageMath is: After downloading the source of SageMath 4.1, it took around 5 hours on an average Linux PC to compile the whole system including all libraries. The compiled version occupied 1.8 GB disk space.

<sup>14</sup>There is also an easy interface to the C language, called Cython, which can be used to substantially speed up functions in SageMath.

See [http://openwetware.org/wiki/Open\\_writing\\_projects/Sage\\_and\\_cython\\_a\\_brief\\_introduction](http://openwetware.org/wiki/Open_writing_projects/Sage_and_cython_a_brief_introduction).

<sup>15</sup>- “Invitation to Sage” by David Joyner, last update 2009

<http://sage.math.washington.edu/home/wdj/teaching/calc1-sage/an-invitation-to-sage.pdf>

- “The SDSU Sage Tutorial”,

<http://www-rohan.sdsu.edu/~mosulliv/sagetutorial/>

<http://www-rohan.sdsu.edu/~mosulliv/sagetutorial/sagecalc.html>

- “SAGE For Newbies” by Ted Kosan, 2007,

[http://sage.math.washington.edu/home/tkosan/newbies\\_book/sage\\_for\\_newbies\\_v1.23.pdf](http://sage.math.washington.edu/home/tkosan/newbies_book/sage_for_newbies_v1.23.pdf)

<sup>16</sup>The corresponding official PDF documents can be downloaded at

<http://www.sagemath.org/help.html>, <http://www.sagemath.org/doc> and <http://planet.sagemath.org>.

<sup>17</sup>- “Library”: <http://www.sagemath.org/library/index.html>,

- “Documentation Project”: <http://wiki.sagemath.org/DocumentationProject>,

- “Teaching”: [http://wiki.sagemath.org/Teaching\\_with\\_SAGE](http://wiki.sagemath.org/Teaching_with_SAGE).

<sup>18</sup>- Module sources in the directory SAGE\_ROOT/devel/sage-main/sage/crypto.

- Overview, what crypto currently is in SageMath:

<http://www.sagemath.org/doc/reference/sage/crypto/>

- Discussions about teaching related aspects of development crypto in SageMath:

[http://groups.google.com/group/sage-devel/browse\\_thread/thread/c5572c4d8d42d081](http://groups.google.com/group/sage-devel/browse_thread/thread/c5572c4d8d42d081)

<sup>19</sup>- David Kohel’s notes from 2008 are a ready course

<http://www.sagemath.org/library/crypto.pdf> or the same eventually newer at

<http://sage.math.washington.edu/home/wdj/teaching/kohel-crypto.pdf> .

- “Introduction to Cryptography with Open-Source Software”, a very good book from Alasdair McAndrew, CRC, 2011

The default interface to SageMath is **command line** based, as shown in figure A.17. However, there is a graphical user interface to the software as well in the form of the SageMath notebook (see figure A.18). We can even use SageMath **notebooks**<sup>20</sup> online at different servers, without having to install SageMath locally, e.g.

```
http://www.sagenb.org or  
http://sage.mathematik.uni-siegen.de:8000
```

SageMath runs under many Linux distributions, Mac OS X, and Windows. For the Windows platform, a complete distribution of SageMath currently only runs as a VMware image.

```

-----
| Sage Version 4.0.2, Release Date: 2009-06-18
| Type notebook() for the GUI, and license() for information.
-----
sage: alph = AlphabeticStrings(); subcipher = SubstitutionCryptosystem(alph)
sage: key = alph([25 - i for i in xrange(26)]); key
ZYXWVUTSRQPONMLKJIHGFEDCBA
sage: encrypt = subcipher(key)
sage: msg = alph.encoding("Substitute this with something else nice."); msg
SUBSTITUTE THIS WITH SOMETHING ELSE NICE
sage: encrypt(msg)
HFYHGRGFGVGSRHDRGSHLNVGSRMTVOHVHRXY
sage:
sage: trancipher = TranspositionCryptosystem(alph, 7)
sage: key = trancipher.random_key(); key
(1,3)(2,7,4,6,5)
sage: trancipher.enciphering(key, msg); msg
BTSIUTSESUITHTMWOISHHEEGTNIEELCSIN
SUBSTITUTE THIS WITH SOMETHING ELSE NICE
sage: 
```

Figure A.17: SageMath command line interface

<sup>20</sup>Further details about SageMath notebooks can be found at chapter 7.9.2 (“[Implementing elliptic curves for educational purposes](#)” ⇒ “[SageMath](#)”).

<sup>21</sup>To start the SageMath gui locally: Enter notebook() at the SageMath prompt, and then your favorite browser (Iceweasel, Firefox, IE, ...) is started e.g. with the URL <http://localhost:8000>.

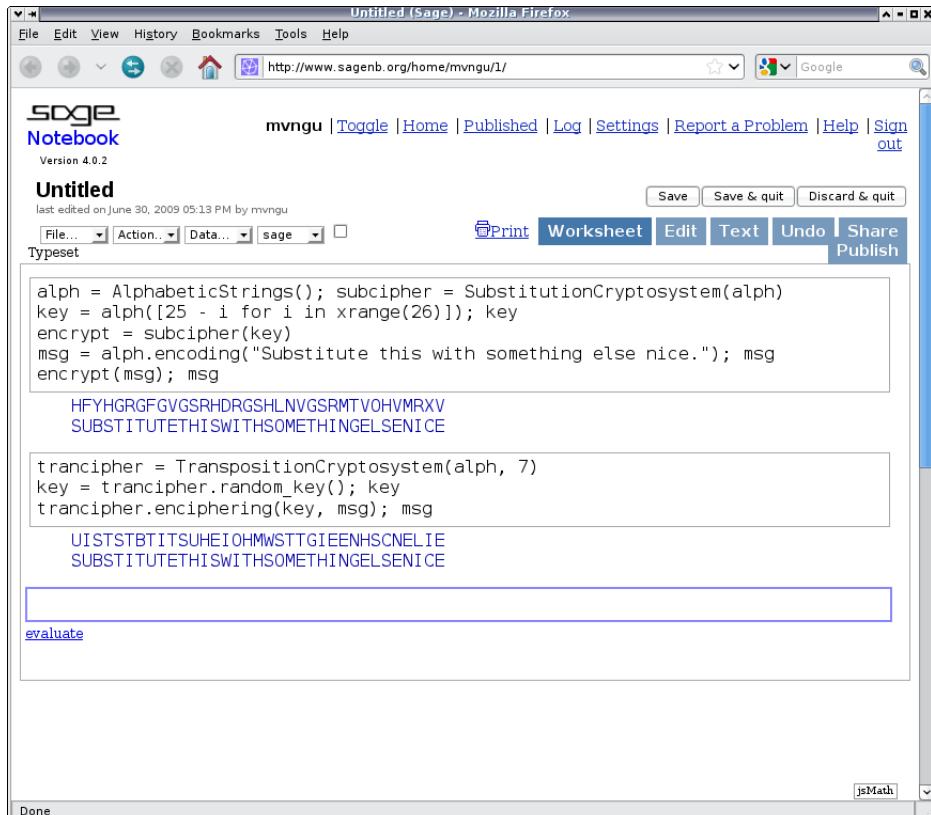


Figure A.18: SageMath notebook interface<sup>21</sup>

## Getting help with using SageMath

Upon loading SageMath from the command line, we are presented with something similar to the following:

```
mnemonic:~$ sage
-----
| Sage Version 4.1, Release Date: 2009-07-09 |
| Type notebook() for the GUI, and license() for information. |
-----

sage: help
Type help() for interactive help, or help(object) for help about object.
sage:
sage:
sage: help()

Welcome to Python 2.6! This is the online help utility.

If this is your first time using Python, you should definitely check out
the tutorial on the Internet at http://docs.python.org/tutorial/.

Enter the name of any module, keyword, or topic to get help on writing
Python programs and using Python modules. To quit this help utility and
return to the interpreter, just type "quit".

To get a list of available modules, keywords, or topics, type "modules",
"keywords", or "topics". Each module also comes with a one-line summary
```

of what it does; to list the modules whose summaries contain a given word such as "spam", type "modules spam".

Plenty of help is provided in the form of the official SageMath documentation that is distributed with every release of SageMath (see Figure A.19). The official SageMath standard documentation

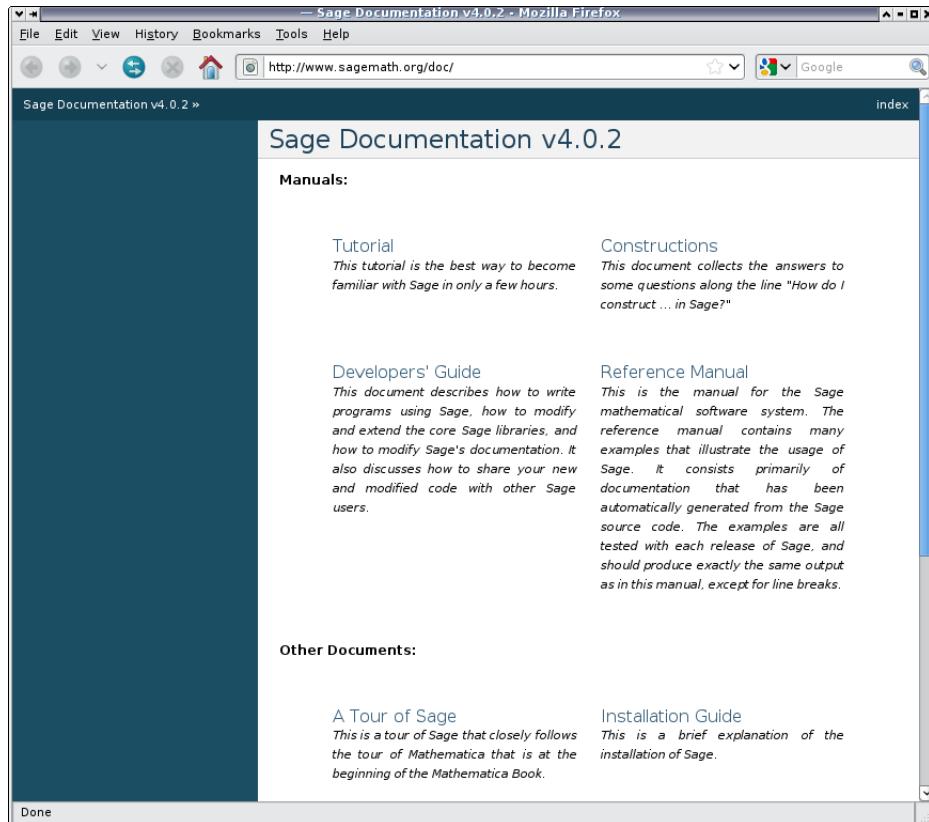


Figure A.19: The SageMath standard documentation

includes the following documents:

- Tutorial — This tutorial is designed to help SageMath beginners become familiar with SageMath. It covers many features that beginners should be familiar with, and takes one to three hours to go through.
- Constructions — This document is in the style of a SageMath “cookbook”. It is a collection of answers to questions about constructing various objects in SageMath.
- Developers’ Guide — This guide is for developers who want to contribute to the development of SageMath. Among other issues, it covers coding style and conventions, modifying the core SageMath libraries, modifying the SageMath standard documentation, and code review and distribution.
- Reference Manual — This manual provides complete documentation on the major features of SageMath. The description of a class is accompanied by numerous code samples. All code samples in the reference manual are tested before each SageMath release.
- Installation Guide — This guide explains how to install SageMath under various platforms.

- A Tour of Sage — This is a tour of SageMath that showcases various features of SageMath that are useful for beginners.
- Numerical Sage — This document introduces tools available under SageMath that are useful for numerical computation.
- Three Lectures about Explicit Methods in Number Theory Using Sage — This document is about using SageMath to perform computations in advanced number theory.

From within a SageMath session, we can obtain a list of commands matching some pattern. To do so, we type the first few characters and then press the “Tab” key:

```
sage: Su[TAB]
Subsets           Subwords           SuzukiGroup
SubstitutionCryptosystem  SupersingularModule
```

If we know the exact name of a command, we can use the `help` function to obtain further information on that command, or append the question mark “?” to the command name. For example, the command `help(SubstitutionCryptosystem)` provides documentation on the built-in class `SubstitutionCryptosystem`. We can get documentation on this class with the question mark as follows:

```
sage: SubstitutionCryptosystem?
Type:type
Base Class:<type 'type'>
String Form:<class 'sage.crypto.classical.SubstitutionCryptosystem'>
Namespace:Interactive
File:/home/mvngu/usr/bin/sage-3.4.1/local/lib/python2.5/site-packages/sage/crypto/classical.py
Docstring:

Create a substitution cryptosystem.

INPUT:
- ``S`` - a string monoid over some alphabet

OUTPUT:
- A substitution cryptosystem over the alphabet ``S``.

EXAMPLES::

sage: M = AlphabeticStrings()
sage: E = SubstitutionCryptosystem(M)
sage: E
Substitution cryptosystem on Free alphabetic string monoid
on A-Z
sage: K = M([ 25-i for i in range(26) ])
sage: K
ZYXWVUTSRQPONMLKJIHGfedcba
sage: e = E(K)
sage: m = M("THECATINTHEHAT")
sage: e(m)
GSVXZGRMGSVSZG

TESTS::

sage: M = AlphabeticStrings()
```

```
sage: E = SubstitutionCryptosystem(M)
sage: E == loads(dumps(E))
True
```

For further assistance on specific problems, we can also search the archive of the **sage-support** mailing list at

<http://groups.google.com/group/sage-support>

## Examples using the built-in mathematical functions in SageMath

Here are a few little examples<sup>22</sup> (all in console mode, for ease) to see what you can do with SageMath:

---

**SageMath sample A.2** Some small general samples in SageMath from different areas in mathematics

---

```
# * Calculus:  
sage: x=var('x')  
sage: p=diff(exp(x^2),x,10)*exp(-x^2)  
sage: p.simplify_exp()  
1024 x^10 + 23040 x^8 + 161280 x^6 + 403200 x^4 + 302400 x^2 + 30240  
  
# * Linear Algebra:  
sage: M=matrix([[1,2,3],[4,5,6],[7,8,10]])  
sage: c=random_matrix(ZZ,3,1);c  
[ 7 ]  
[-2 ]  
[-2 ]  
sage: b=M*c  
sage: M^-1*b  
[ 7 ]  
[-2 ]  
[-2 ]  
  
# * Number theory:  
sage: p=next_prime(randint(2^49,2^50));p  
1022095718672689  
sage: r=primitive_root(p);r  
7  
sage: pl=log(mod(10^15,p),r);pl  
1004868498084144  
sage: mod(r,p)^pl  
1000000000000000  
  
# * Finite Fields (\url{http://en.wikipedia.org/wiki/Finite_field}):  
sage: F.<x>=GF(2)[]  
sage: G.<a>=GF(2^4, name='a', modulus=x^4+x+1)  
sage: a^2/(a^2+1)  
a^3 + a  
sage: a^100  
a^2 + a + 1  
sage: log(a^2,a^3+1)  
13  
sage: (a^3+1)^13  
a^2
```

---

<sup>22</sup>The examples are from the blog of Dr. Alasdair McAndrew, Victoria University,  
<http://amca01.wordpress.com/2008/12/19/sage-an-open-source-mathematics-software-system>

## Writing code samples with SageMath

When you start using a CAS (computer algebra system) you normally type in the single commands on the command line as in the above example<sup>23</sup>.

But if you develop your own functions, modify them and call them, then it is much easier to do the development in your own editor, save it to a script file and execute the functions non-interactively on the command line manually. Both ways to develop code were applied in chapter 1.8 (“Appendix: Examples using SageMath”), chapter 2.5 (“Appendix: Examples using SageMath”), chapter 3.14 (“Appendix: Examples using SageMath”) and in chapter 4.19 (“Appendix: Examples using SageMath”).

To program and test SageMath code using an editor there are two useful commands: `load()` and `attach()`<sup>24</sup>.

Suppose you have a function definition like this:

```
def function(var1):
    r"""
    DocText.
    """
    ...
    return (L)
```

which has been saved to the file `primroots.sage`.

To load this function into SageMath (and do a syntax check at once), use `load()` as follows:

```
sage: load primroots.sage
```

and you can then proceed to use on the command line any variable or function defined in that SageMath script<sup>25</sup>.

Normally we also want to edit our own SageMath script and reload the content of the changed script into SageMath again. In that case, you can use the command `attach()` (you also can apply `attach()` directly after loading the script, even before having changed the script; and you can even omit `load()`, as this is contained in `attach()`):

---

<sup>23</sup>The standard way for presenting SageMath code starts the lines with “sage:” and “...”.

```
sage: m = 11
sage: for a in xrange(1, m):
....:     print [power_mod(a, i, m) for i in xrange(1, m)]
....:
```

This script usually uses the above convention for presenting SageMath code, if the code doesn’t come from a SageMath script. When people copy and paste the SageMath code from this script, in order to enter it at the command line, they should leave out “sage:” and “...” from the script (nevertheless in most cases the command prompt can deal with these prefixes correctly).

<sup>24</sup>See SageMath tutorial about Programming, chapter “Loading and Attaching Sage files”,

<http://www.sagemath.org/doc/tutorial/programming.html#loading-and-attaching-sage-files>.

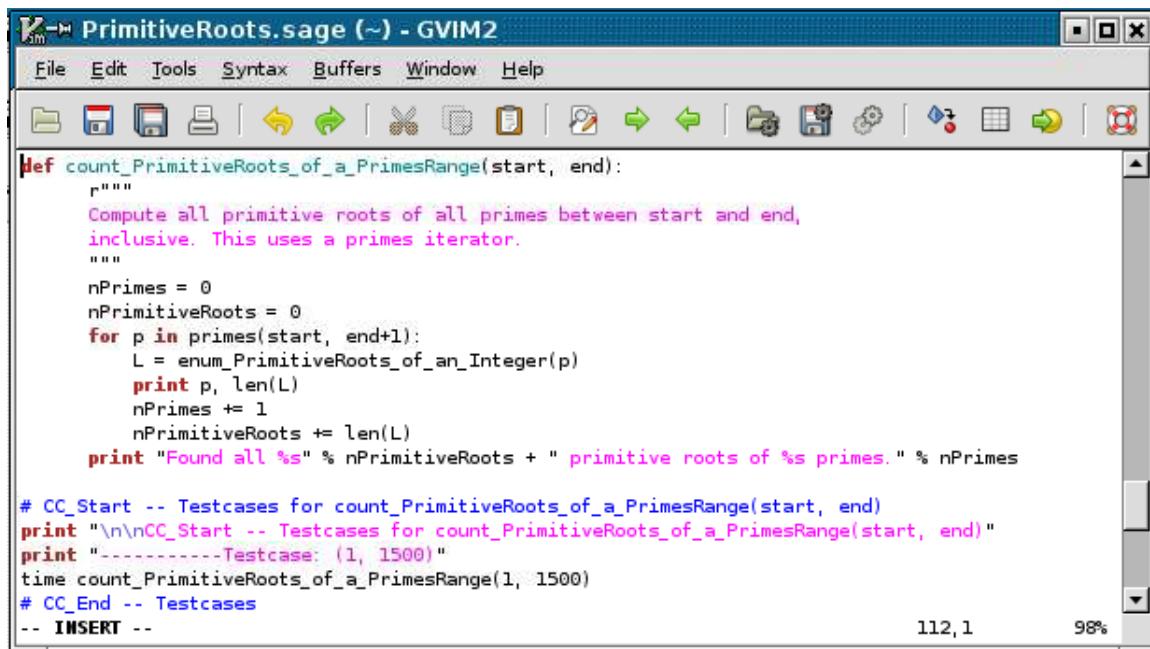
<sup>25</sup>Notes:

- Don’t use white spaces in your file name.
- It’s recommended that your SageMath script has the file extension “.sage”, instead of “.py”. With a SageMath script whose file name ends in “.sage”, when you load it into SageMath then the default SageMath environment is also loaded to make sure that it works as if you have defined your function from the SageMath command line. This also applies if you run the script from a bash shell using `$ sage primroots.sage`.
- If you run your script as above, then SageMath first parses your script, writes it to another file called “primroots.py” (note the “.py” extension), adds all necessary variables to “primroots.py” as well as writing any import statements to that file. That way, your SageMath script is executed as if you had typed the definitions in your script to the SageMath command line. An important difference is that all output needs a `print` statement.

```
sage: attach primroots.sage
```

Now edit the SageMath script in a text editor, but don't exit SageMath. After you saved it within your text editor, the changed function definition is reloaded into the running SageMath session after the next typing of Enter (and a syntax check is done at once). This reloading is done automatically for you, provided that all changes to your script have been saved. You can think of the command `attach()` as a way of telling SageMath to watch for all changes to a file, and reloading the file again once SageMath notices that there have been changes. With this command, you don't have to copy and paste between your text editor and the SageMath command line interface.

Here is a picture of SageMath code in the editor GVIM with activated code highlighting (see figure A.20).



The screenshot shows a GVIM window titled "PrimitiveRoots.sage (~) - GVIM2". The code in the buffer is:

```
def count_PrimitiveRoots_of_a_PrimesRange(start, end):
    """
    Compute all primitive roots of all primes between start and end,
    inclusive. This uses a primes iterator.
    """
    nPrimes = 0
    nPrimitiveRoots = 0
    for p in primes(start, end+1):
        L = enum_PrimitiveRoots_of_an_Integer(p)
        print p, len(L)
        nPrimes += 1
        nPrimitiveRoots += len(L)
    print "Found all %s" % nPrimitiveRoots + " primitive roots of %s primes." % nPrimes

# CC_Start -- Testcases for count_PrimitiveRoots_of_a_PrimesRange(start, end)
print "\n\nCC_Start -- Testcases for count_PrimitiveRoots_of_a_PrimesRange(start, end)"
print "-----Testcase: (1, 1500)"
time count_PrimitiveRoots_of_a_PrimesRange(1, 1500)
# CC_End -- Testcases
-- INSERT --
```

The status bar at the bottom right shows "112, 1" and "98%".

Figure A.20: SageMath sample shown in an editor with code highlighting

If you prefer to see the output of an attached file as if you would have typed in the commands on the commandline directly (not only what is shown via `print`) then you could use the command `iload()`: Each line is loaded one at a time. To load the next line, you have to press the `Enter` key. You have to repeatedly press the `Enter` key until all lines of the SageMath script are loaded into the SageMath session.

```
sage: iload primroots.sage
```

Some more hints:

- To get the version of your SageMath environment: `version()`
- To move quickly to the SageMath code examples in this script,
  - either look in the index at `SageMath -> Code examples`,
  - or have a look at the appendix “[List of SageMath Code Examples](#)”.
- The SageMath samples in this script are delivered with CrypTool.  
For further details see the end of the overview “[List of SageMath Code Examples](#)”.

## A.8 Authors of the CrypTool Book

This appendix lists the authors of this document.

Please refer to the top of each individual chapter for their contribution.

### **Bernhard Esslinger**

Initiator of the CrypTool project, editor and main author of this book. Professor for IT security and cryptography at the University of Siegen. Formerly: CISO of SAP AG, and head IT security and head Crypto Competence Center at Deutsche Bank.  
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---

### **Matthias Büger**

Contributor to chapter 7 (“[Elliptic Curves](#)”), research analyst at Deutsche Bank.

### **Bartol Filipovic**

Original author of the CT1 elliptic curve implementation and author of the corresponding chapter in this book.

### **Martin Franz**

Author of chapter 9 (“[Homomorphic Ciphers](#)”). Works and carries out research in the area of applied cryptography.

### **Henrik Koy**

Main developer and co-ordinator of CT1 development version 1.3 and 1.4; book reviewer and TeX guru; cryptographer and project leader IT at Deutsche Bank.

### **Roger Oyono**

Implementer of the CT1 factorization dialog and original author of chapter 5 (“[The Mathematical Ideas behind Modern Cryptography](#)”).

### **Klaus Pommerening**

Author of chapter 8 (“[Introduction to Bitblock and Bitstream Ciphers](#)”), Professor for mathematics and computer science at Johannes-Gutenberg-Universität. Retired.

### **Jörg Cornelius Schneider**

Design and long-term support of CrypTool; crypto enthusiast, IT architect and senior project leader IT at Deutsche Bank.

### **Christine Stötzel**

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- The source code of the SageMath samples in this script is delivered as SageMath program files within the CT1 setup program. All samples within one chapter are collected in one file. After installing CrypTool 1 you find the SageMath samples within the subdirectory **sagemath** in the following 4 files:
  - SageMath-Samples-in-Chap01.sage
  - SageMath-Samples-in-Chap02.sage
  - SageMath-Samples-in-Chap03.sage
  - SageMath-Samples-in-Chap04.sage
- All samples have been tested with SageMath Version 5.3 (Release Date 2012-09-08).

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