Linear continuum mechanics

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Impressum

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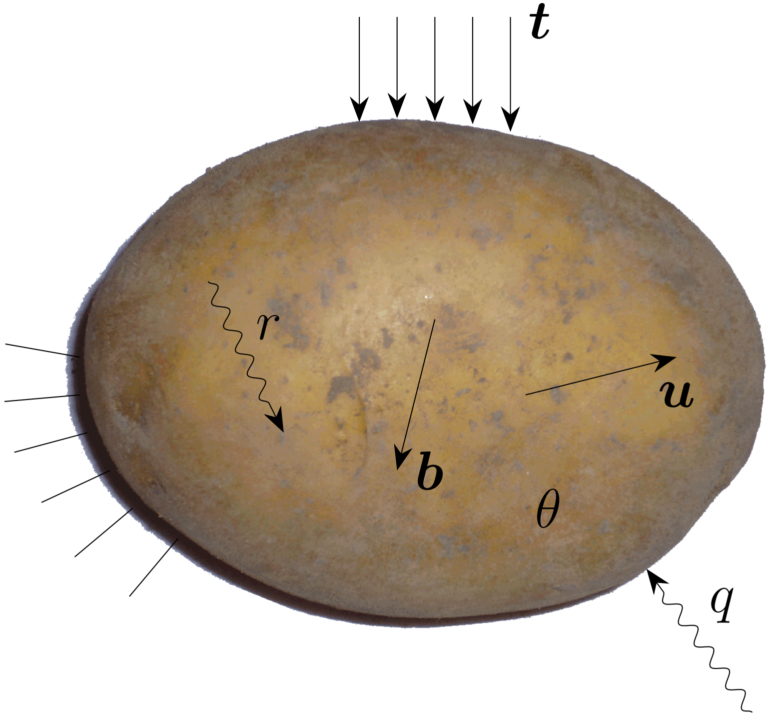
We will consider the **response** of a continuous body due to **external loading**.



- $\boldsymbol{u}$ displacement
- $\boldsymbol{b}$ body forces
- $\boldsymbol{t}$ tractions
- θ temperature
- r heat sources
- q heat fluxes

External loading

- Mechanical loading
  - Body forces $\boldsymbol{b}$ (far field forces) $$[\,\boldsymbol{b}\,] = \frac{\text{force}}{\text{length}^3}$$
  - Tractions $\boldsymbol{t}$ (contact forces) $$[\,\boldsymbol{t}\,] = \frac{\text{force}}{\text{length}^2}$$
- Non-mechanical loading
  - Heat sources r $$[\,r\,] = \frac{\text{power}}{\text{length}^3}$$
  - Heat fluxes q $$[\,q\,] = \frac{\text{power}}{\text{length}^2}$$

Response

- Mechanical response
  - Displacements $\boldsymbol{u}$, strain $\boldsymbol{\varepsilon}$ $\qquad [\,\boldsymbol{u}\,] = \text{length}, \quad [\,\boldsymbol{\varepsilon}\,] = 1$
  - Stress $\boldsymbol{\sigma}$ $\qquad [\,\boldsymbol{\sigma}\,] = \dfrac{\text{force}}{\text{length}^2}$
- Non-mechanical response
  - Temperature θ $\qquad [\,\theta\,] = \text{temperature}$
  - Heat flux vector $\boldsymbol{q}$ (Fourier Law) $\qquad [\,\boldsymbol{q}\,] = \dfrac{\text{power}}{\text{length}^2}$

Notation

We deal with scalars, vectors and tensors.

	Index notation	Symbolic notation	
Scalar	θ	θ	Tensor of 0^{th}-order
Vector	u_i	$\boldsymbol{u}$	Tensor of 1^{st}-order
Tensor	ε_{ij}	$\boldsymbol{\varepsilon}$	Tensor of 2^{nd}-order
$\vdots$	$\vdots$	$\vdots$	$\vdots$
Tensor	E_{ijkl}	$\boldsymbol{E}$	Tensor of 4^{th}-order

$$i, j, k, l = 1, 2, 3$$

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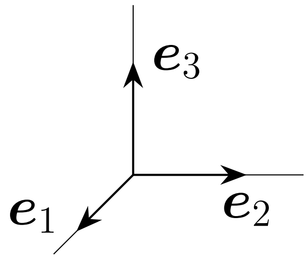
Representation of tensors

A tensor of any order consists of its **coordinates** and the **basis vectors** of the **coordinate system**.

Here cartesian coordinate system.



Orthonormal basis vectors $\boldsymbol{e}_1,\ \boldsymbol{e}_2,\ \boldsymbol{e}_3$

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$$

Example: Coordinate representation of a 1^{st}-order tensor (vector)

$$\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3 = \sum_i u_i\boldsymbol{e}_i = u_i\boldsymbol{e}_i$$

Example: Coordinate representation of a 2^{nd}-order tensor

$$\boldsymbol{\varepsilon} = \sum_{i,j} \varepsilon_{ij}\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \varepsilon_{ij}\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

We employ the Einstein summation convention by summing over repeated indices.

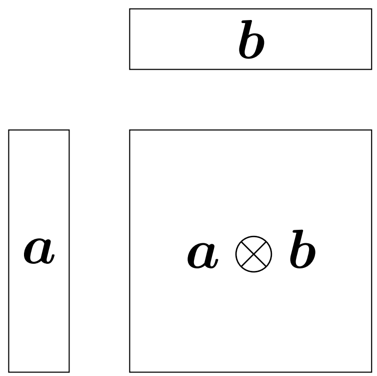
Example for basis dyads

$$[\boldsymbol{e}_1 \otimes \boldsymbol{e}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\boldsymbol{e}_1 \otimes \boldsymbol{e}_2] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Transpose of a tensor

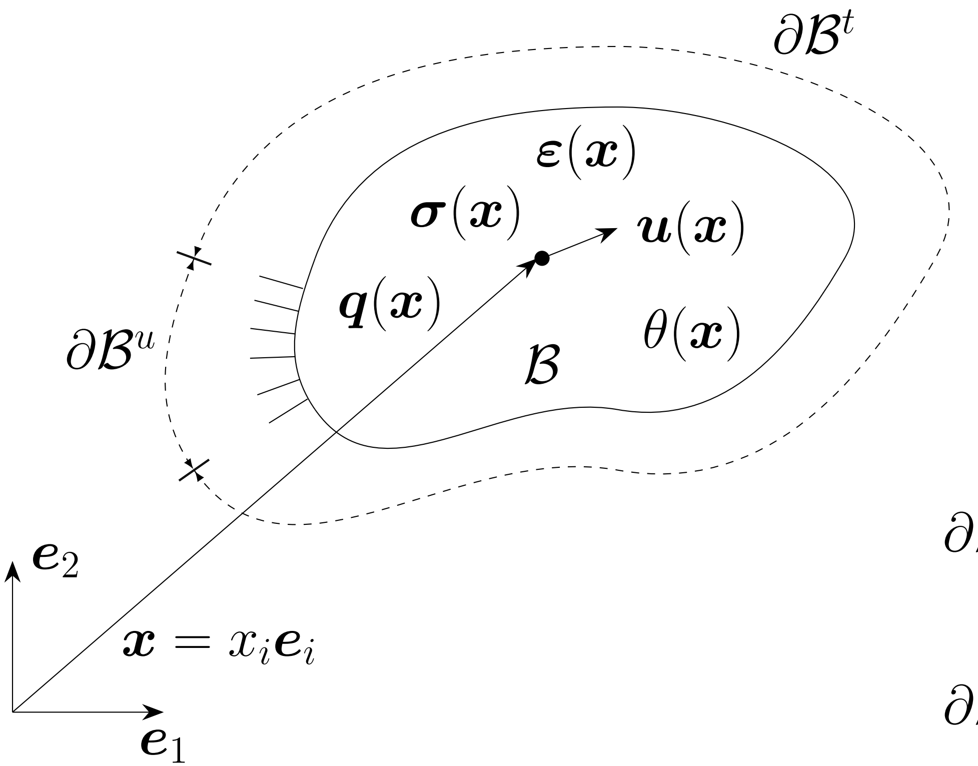
$$\boldsymbol{\sigma} = \sigma_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \boldsymbol{\sigma}^t = \sigma_{ji}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

Dyadic product



The big picture

In continuum mechanics all quantities are considered as fields parameterised in the coordinates of space.



All points $\boldsymbol{x}$ occupied by the continuous body define the configuration $\mathcal{B}$.

The boundary of $\mathcal{B}$ is denoted by $\partial\mathcal{B}$ with subdivision

$$\partial\mathcal{B} = \partial\mathcal{B}^u \cup \partial\mathcal{B}^t \ , \qquad \emptyset = \partial\mathcal{B}^u \cap \partial\mathcal{B}^t$$

and

$$\partial\mathcal{B} = \partial\mathcal{B}^\theta \cup \partial\mathcal{B}^q \ , \qquad \emptyset = \partial\mathcal{B}^\theta \cap \partial\mathcal{B}^q$$

The big picture

Consequence of field description: We may compute derivatives of fields

Index notation	Symbolic notation
$\dfrac{\partial u_i}{\partial x_j} = u_{i,j}$	$\nabla \boldsymbol{u} = u_{i,j}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$

with $\nabla \boldsymbol{u}$ - gradient of $\boldsymbol{u}$

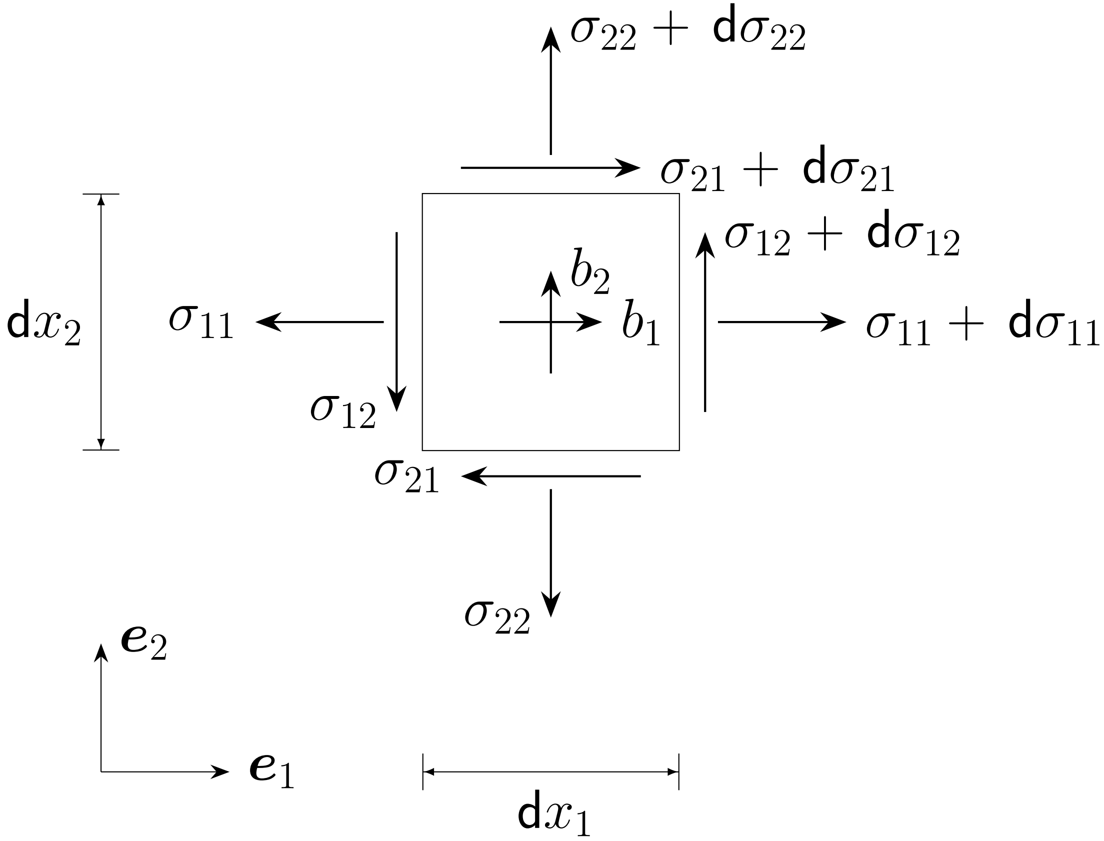
$\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ - basis dyad

$\otimes$ - dyadic product

Problems of continuum mechanics are described in terms of

$$\left.\begin{array}{l} \bullet\ \text{Kinematic equations} \\ \bullet\ \text{Balance laws} \\ \bullet\ \text{Constitutive laws} \end{array}\right\} \text{Solution methods}$$

Mechano-statics



$$\mathsf{d}\sigma_{11} = \frac{\partial \sigma_{11}}{\partial x_1}\,\mathsf{d}x_1$$
$$\mathsf{d}\sigma_{22} = \frac{\partial \sigma_{22}}{\partial x_2}\,\mathsf{d}x_2$$
$$\mathsf{d}\sigma_{12} = \frac{\partial \sigma_{12}}{\partial x_1}\,\mathsf{d}x_1$$
$$\mathsf{d}\sigma_{21} = \frac{\partial \sigma_{21}}{\partial x_2}\,\mathsf{d}x_2$$

Equlibrium of mechanical forces

$$\rightarrow: \quad \mathsf{d}\sigma_{11}\ \mathsf{d}x_2 + \mathsf{d}\sigma_{21}\ \mathsf{d}x_1 + b_1\ \mathsf{d}x_1\ \mathsf{d}x_2 = 0$$

$$\uparrow: \quad \mathsf{d}\sigma_{12}\ \mathsf{d}x_2 + \mathsf{d}\sigma_{22}\ \mathsf{d}x_1 + b_2\ \mathsf{d}x_1\ \mathsf{d}x_2 = 0$$

$$\rightarrow: \quad \frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + b_1 = 0$$

$$\uparrow: \quad \frac{\partial\sigma_{12}}{\partial x_1} + \frac{\partial\sigma_{22}}{\partial x_2} + b_2 = 0$$

Index notation	Symbolic notation
$\sigma_{ij,i} + b_j = 0 \quad \forall\ j = 1,2$	$\mathsf{div}\,\boldsymbol{\sigma}^t + \boldsymbol{b} = \boldsymbol{0}$

The big picture

Example for **constitutive law**: Hooke's law

Index notation

$$\sigma_{ij} = E_{ijkl}\,\varepsilon_{kl}$$

Symbolic notation

$$\boldsymbol{\sigma} = \boldsymbol{E} : \boldsymbol{\varepsilon}$$

with $\boldsymbol{E} = E_{ijkl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$ - elasticity tensor

Kinematics: Strain-displacement relation

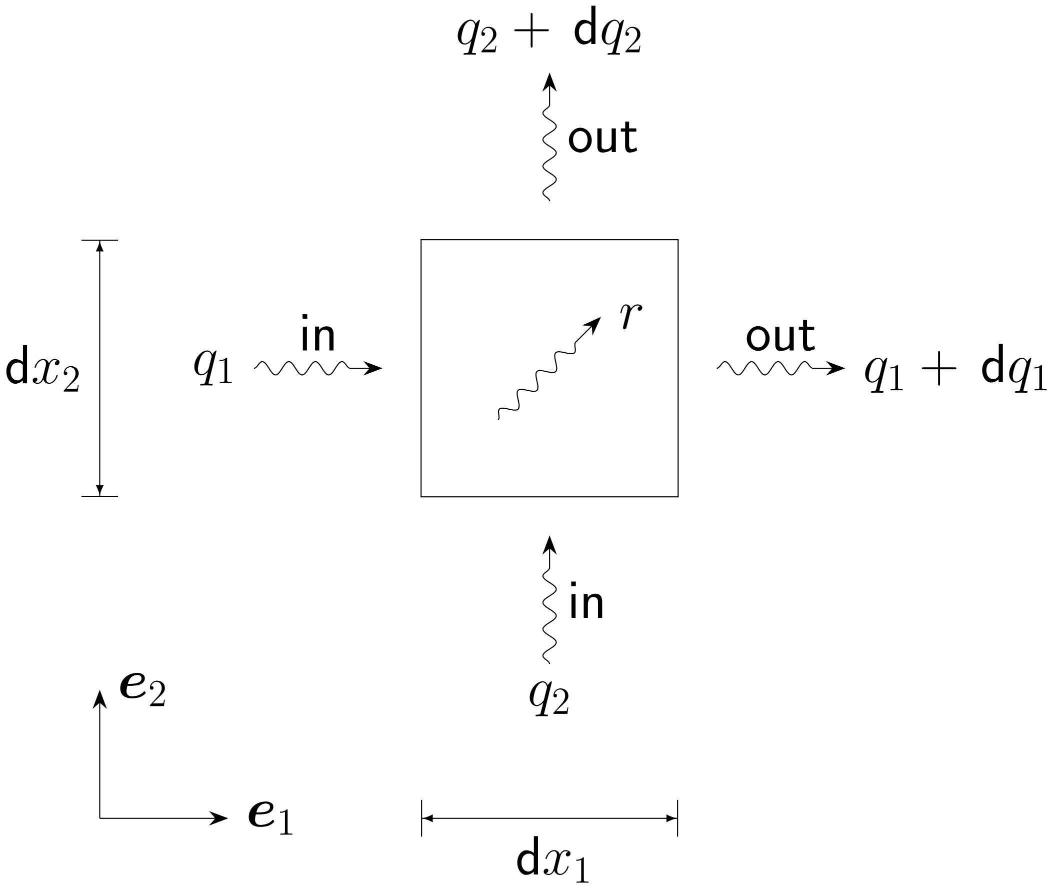
Index notation

$$\varepsilon_{kl} = \frac{1}{2}\left[u_{k,l} + u_{l,k}\right] = u_{(k,l)}$$

Symbolic notation

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left[\nabla\boldsymbol{u} + \nabla^t\boldsymbol{u}\right] = \nabla^{\mathsf{sym}}\,\boldsymbol{u}$$

Thermo-statics



$$\mathrm{d}q_1 = \frac{\partial q_1}{\partial x_1}\,\mathrm{d}x_1$$
$$\mathrm{d}q_2 = \frac{\partial q_2}{\partial x_2}\,\mathrm{d}x_2$$

Equilibrium of thermal power supply

$$-\,\mathrm{d}q_1\,\mathrm{d}x_2 - \,\mathrm{d}q_2\,\mathrm{d}x_1 + r\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0$$

$$-\frac{\partial q_1}{\partial x_1} - \frac{\partial q_2}{\partial x_2} + r = 0$$

Index notation

$$-q_{i,i} + r = 0$$

Symbolic notation

$$-\,\mathrm{div}\,\boldsymbol{q} + r = 0$$

Example for **constitutive law**: Fourier law

Index notation	Symbolic notation
$q_i = -\kappa\,\theta_{,i}$	$\boldsymbol{q} = -\kappa\,\nabla\theta$

with κ heat conduction coefficient

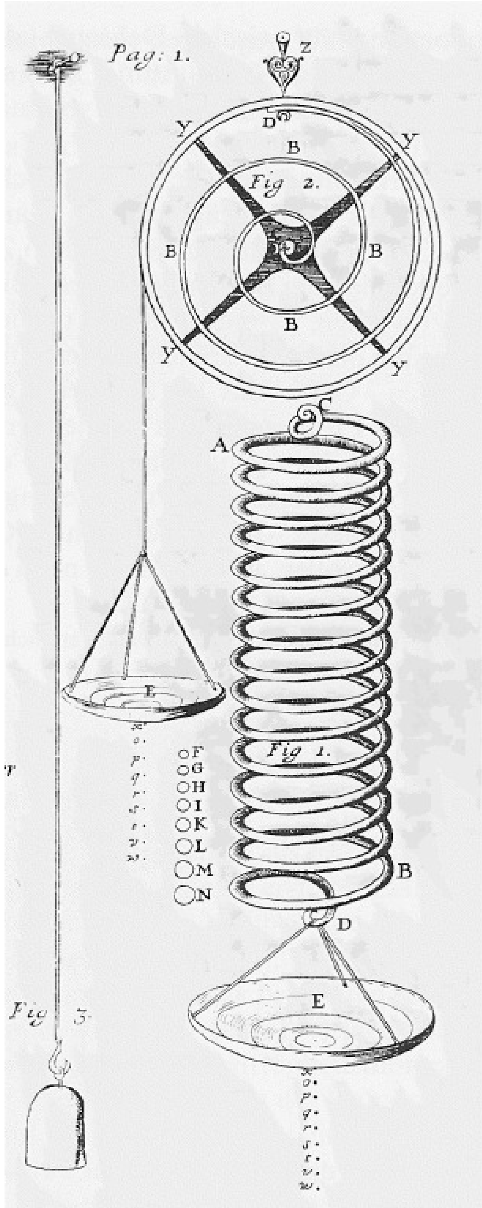
$$\kappa\,\theta_{,ii} + r = 0 \quad \text{with} \quad \theta_{,ii} = \Delta\theta$$

Laplace operator

$$\Delta\theta = \frac{\partial^2\theta}{\partial x_1 \partial x_1} + \frac{\partial^2\theta}{\partial x_2 \partial x_2} + \frac{\partial^2\theta}{\partial x_3 \partial x_3}$$

Stationary heat conduction equation

$$\kappa\Delta\theta + r = 0$$



– ut tensio sic vis –

Robert Hooke (1635–1703)

Lectures de potentia restitutiva, or of spring explaining the power of springing bodies [1678]

Mechano-statics

Weak form of balance equation: Index notation

$$\int_{\mathcal{B}} \delta u_j \left[\sigma_{ij,i} + b_j \right] \, \mathrm{d}v = 0$$

Product rule of differentiation

$$\delta u_j \, \sigma_{ij,i} = \left[\delta u_j \, \sigma_{ij} \right]_{,i} - \delta u_{j,i} \, \sigma_{ij}$$

Virtual work of stresses (internal virtual work)

$$\int_{\mathcal{B}} \delta u_{j,i} \, \sigma_{ij} \, \mathrm{d}v$$

Virtual work of 'tractions' and body forces (external virtual work)

$$\int_{\mathcal{B}} \left[\delta u_j \, \sigma_{ij} \right]_{,i} \, \mathrm{d}v + \int_{\mathcal{B}} \delta u_j b_j \, \mathrm{d}v$$

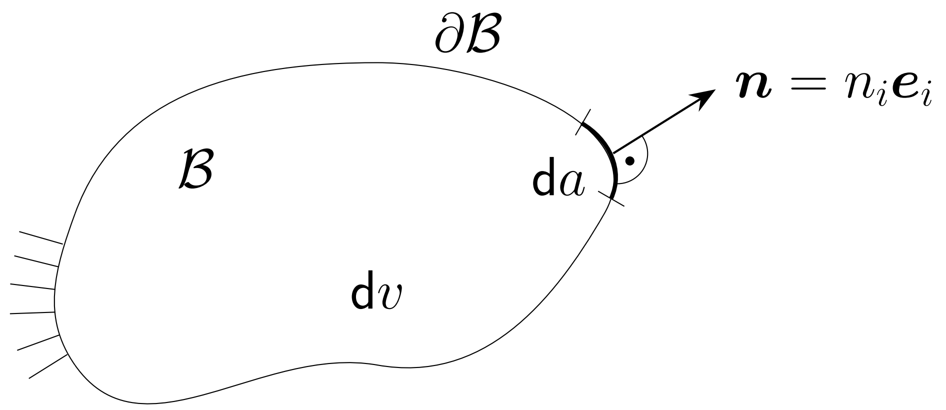
Gauss theorem

Index notation

$$\int_{\mathcal{B}} a_{i,i}\,\mathrm{d}v = \int_{\partial\mathcal{B}} a_i n_i\;\mathrm{d}a$$

Symbolic notation

$$\int_{\mathcal{B}} \mathrm{div}\,\boldsymbol{a}\,\mathrm{d}v = \int_{\partial\mathcal{B}} \boldsymbol{a}\cdot\boldsymbol{n}\,\mathrm{d}a$$



The big picture

Application of Gauss theorem to virtual work of 'tractions'

$$\int_{\mathcal{B}} [\delta u_j \, \sigma_{ij}]_{,i} \, \mathrm{d}v = \int_{\partial\mathcal{B}} \delta u_j \, \sigma_{ij} n_i \, \mathrm{d}a$$

Strain-displacement relation and Cauchy theorem

$$(\sigma_{ij} = \sigma_{ji}) \qquad \delta\varepsilon_{ij} = \delta u_{(j,i)} \qquad t_j = \sigma_{ij} \, n_i$$

The weak form of mechanical equilibrium corresponds to the principle of virtual work

$$\int_{\mathcal{B}} \delta\varepsilon_{ij} \, \sigma_{ij} \, \mathrm{d}v = \int_{\partial\mathcal{B}} \delta u_j \, t_j \, \mathrm{d}a + \int_{\mathcal{B}} \delta u_j \, b_j \, \mathrm{d}v \qquad \forall \, \delta u_j$$

Admissible virtual displacement, traction boundary condition

$$\int_{\partial\mathcal{B}} \delta u_j \, t_j \, \mathrm{d}a = \int_{\partial\mathcal{B}^t} \delta u_j \, t_j^p \, \mathrm{d}a$$

Requirement for virtual displacement

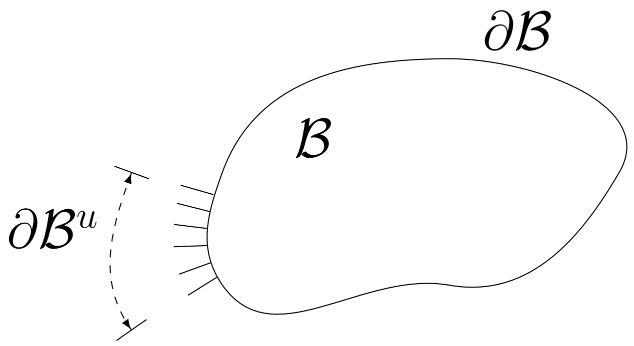
- virtual
- infinitesimal small
- admissible
- arbitrary

Index notation

$$\delta u_j = 0 \quad \text{on} \quad \partial \mathcal{B}^u$$

Symbolic notation

$$\delta \boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \partial \mathcal{B}^u$$



Boundary conditions

$$\partial\mathcal{B} = \partial\mathcal{B}^u \cup \partial\mathcal{B}^t \qquad \emptyset = \partial\mathcal{B}^u \cap \partial\mathcal{B}^t$$

$\boldsymbol{u} = \boldsymbol{u}^p$ on $\partial\mathcal{B}^u$ Dirichlet boundary condition

$(\boldsymbol{\sigma}^t \cdot \boldsymbol{n} =)\ \boldsymbol{t} = \boldsymbol{t}^p$ on $\partial\mathcal{B}^t$ Neumann boundary condition

Example: Beam bending

$$-EI_{yy}\, w'' = M_y$$
$$-EI_{yy}\, w''' = Q_z$$
$$EI_{yy}\, w'''' = q_z$$

Dirichlet BCs: $w,\ w'$
Neumann BCs: $M,\ Q$

In general:

Differential equation of order $2m$

$$F(u^{(2m)}, u^{(2m-1)}, \ldots, u', u) = 0$$

Dirichlet BCs: $u, \ldots, u^{(m-1)}$
Neumann BCs: $u^{(m)}, \ldots, u^{(2m-1)}$

Mechano-statics

Weak form of balance equation: Symbolic notation

$$\int_{\mathcal{B}} \delta \boldsymbol{u} \cdot \left[\operatorname{div} \boldsymbol{\sigma}^t + \boldsymbol{b} \right] \, \mathrm{d}v = 0$$

Product rule of differentiation

$$\delta \boldsymbol{u} \cdot \operatorname{div} \boldsymbol{\sigma}^t = \operatorname{div} \left(\delta \boldsymbol{u} \cdot \boldsymbol{\sigma}^t \right) - \nabla \delta \boldsymbol{u} : \boldsymbol{\sigma}^t$$

Virtual work of stresses (internal virtual work)

$$\int_{\mathcal{B}} \nabla \delta \boldsymbol{u} : \boldsymbol{\sigma}^t \, \mathrm{d}v$$

Virtual work of 'tractions' and body forces (external virtual work)

$$\int_{\mathcal{B}} \operatorname{div} \left(\delta \boldsymbol{u} \cdot \boldsymbol{\sigma}^t \right) \, \mathrm{d}v + \int_{\mathcal{B}} \delta \boldsymbol{u} \cdot \boldsymbol{b} \, \mathrm{d}v$$

The big picture

Application of Gauss theorem to virtual work of 'tractions'

$$\int_{\mathcal{B}} \text{div}\left(\delta \boldsymbol{u} \cdot \boldsymbol{\sigma}^t\right) \, \text{d}v = \int_{\partial\mathcal{B}} \delta \boldsymbol{u} \cdot \boldsymbol{\sigma}^t \cdot \boldsymbol{n} \, \text{d}a$$

Strain-displacement relation and Cauchy theorem

$$(\boldsymbol{\sigma} = \boldsymbol{\sigma}^t) \qquad \delta \boldsymbol{\varepsilon} = \nabla^{\text{sym}} \delta \boldsymbol{u} \qquad \boldsymbol{t} = \boldsymbol{\sigma}^t \cdot \boldsymbol{n}$$

The weak form of mechanical equilibrium corresponds to the principle of virtual work

$$\int_{\mathcal{B}} \delta \boldsymbol{\varepsilon} : \boldsymbol{\sigma} \, \text{d}v = \int_{\partial\mathcal{B}} \delta \boldsymbol{u} \cdot \boldsymbol{t} \, \text{d}a + \int_{\mathcal{B}} \delta \boldsymbol{u} \cdot \boldsymbol{b} \, \text{d}v \qquad \forall \, \delta \boldsymbol{u}$$

Admissible virtual displacement, traction boundary condition

$$\int_{\partial\mathcal{B}} \delta \boldsymbol{u} \cdot \boldsymbol{t} \, \text{d}a = \int_{\partial\mathcal{B}^t} \delta \boldsymbol{u} \cdot \boldsymbol{t}^p \, \text{d}a$$

Thermo-statics

Weak form of balance equation: Index notation

$$\int_{\mathcal{B}} \delta\theta \left[-q_{i,i} + r \right] \, \mathrm{d}v = 0$$

Product rule of differentiation

$$-\delta\theta q_{i,i} = -\left[\delta\theta \, q_i\right]_{,i} + \delta\theta_{,i} \, q_i$$

Virtual work of heat flux vector (internal virtual work)

$$-\int_{\mathcal{B}} \delta\theta_{,i} \, q_i \; \mathrm{d}v$$

Virtual work of 'heat flux' and heat sources (external virtual work)

$$-\int_{\mathcal{B}} \left[\delta\theta \, q_i\right]_{,i} \; \mathrm{d}v + \int_{\mathcal{B}} \delta\theta \, r \; \mathrm{d}v$$

Application of Gauss theorem to virtual work of 'heat flux'

$$-\int_{\mathcal{B}} [\delta\theta\, q_i]_{,i}\ \mathrm{d}v = -\int_{\partial\mathcal{B}} \delta\theta\, q_i\, n_i\ \mathrm{d}a$$

Cauchy theorem

$$q = -q_i\, n_i$$

The weak form of thermal equilibrium corresponds to the principle of virtual work

$$-\int_{\mathcal{B}} \delta\theta_{,i}\, q_i\ \mathrm{d}v = \int_{\partial\mathcal{B}} \delta\theta\, q\ \mathrm{d}a + \int_{\mathcal{B}} \delta\theta\, r\ \mathrm{d}v \qquad \forall\,\delta\theta$$

Admissible virtual temperature, heat flux boundary condition

$$\int_{\partial\mathcal{B}} \delta\theta\, q\ \mathrm{d}a = \int_{\partial\mathcal{B}^q} \delta\theta\, q^p\ \mathrm{d}a$$

Requirement for virtual temperature

- virtual
- infinitesimal small
- admissible
- arbitrary

$$\delta\theta = 0 \quad \text{on} \quad \partial\mathcal{B}^\theta$$

Boundary conditions

$$\partial\mathcal{B} \;=\; \partial\mathcal{B}^\theta \cup \partial\mathcal{B}^q \qquad\qquad \emptyset \;=\; \partial\mathcal{B}^\theta \cap \partial\mathcal{B}^q$$

$$\theta = \theta^p \quad \text{on} \quad \partial\mathcal{B}^\theta \qquad \text{Dirichlet boundary condition}$$

$$(-\boldsymbol{q} \cdot \boldsymbol{n} =)\; q = q^p \quad \text{on} \quad \partial\mathcal{B}^q \qquad \text{Neumann boundary condition}$$

Thermo-statics

Weak form of balance equation: Symbolic notation

$$\int_{\mathcal{B}} \delta\theta \left[-\,\mathsf{div}\,\boldsymbol{q} + r\right] \,\mathrm{d}v = 0$$

Product rule of differentiation

$$-\delta\theta \;\mathsf{div}\,\boldsymbol{q} = -\,\mathsf{div}\,(\delta\theta\,\boldsymbol{q}) + \nabla\delta\theta \cdot \boldsymbol{q}$$

Virtual work of heat flux vector (internal virtual work)

$$-\int_{\mathcal{B}} \nabla\delta\theta \cdot \boldsymbol{q} \;\mathrm{d}v$$

Virtual work of 'heat flux' and heat sources (external virtual work)

$$-\int_{\mathcal{B}} \mathsf{div}\,(\delta\theta\,\boldsymbol{q}) \;\mathrm{d}v + \int_{\mathcal{B}} \delta\theta \; r \,\mathrm{d}v$$

Application of Gauss theorem to virtual work of 'heat flux'

$$- \int_{\mathcal{B}} \operatorname{div}(\delta\theta\, \boldsymbol{q})\, \mathrm{d}v = - \int_{\partial\mathcal{B}} \delta\theta\, \boldsymbol{q} \cdot \boldsymbol{n}\, \mathrm{d}a$$

Cauchy theorem

$$q = -\boldsymbol{q} \cdot \boldsymbol{n}$$

The weak form of thermal equilibrium corresponds to the principle of virtual work

$$- \int_{\mathcal{B}} \nabla\delta\theta \cdot \boldsymbol{q}\, \mathrm{d}v = \int_{\partial\mathcal{B}} \delta\theta\, q\, \mathrm{d}a + \int_{\mathcal{B}} \delta\theta\, r\, \mathrm{d}v \qquad \forall\, \delta\theta$$

Admissible virtual temperature, heat flux boundary condition

$$\int_{\partial\mathcal{B}} \delta\theta\, q\, \mathrm{d}a = \int_{\partial\mathcal{B}^q} \delta\theta\, q^p\, \mathrm{d}a$$

Summary of boundary value problem (BVP)

Mechano-statics				Thermo-statics		
$\boldsymbol{\varepsilon} = \nabla^{\text{sym}}\boldsymbol{u}$	in	$\mathcal{B}$	'Kinematics'	$\boldsymbol{\gamma} = \nabla\theta$	in	$\mathcal{B}$
$\operatorname{div}\boldsymbol{\sigma}^t + \boldsymbol{b} = \boldsymbol{0}$	in	$\mathcal{B}$	'Equilibrium'	$-\operatorname{div}\boldsymbol{q} + r = 0$	in	$\mathcal{B}$
$\boldsymbol{\sigma} = \boldsymbol{E} : \boldsymbol{\varepsilon}$	in	$\mathcal{B}$	Constitutive law	$\boldsymbol{q} = -\kappa\boldsymbol{\gamma}$	in	$\mathcal{B}$
$\boldsymbol{u} = \boldsymbol{u}^p$	on	$\partial\mathcal{B}^u$	Dirichlet BC	$\theta = \theta^p$	on	$\partial\mathcal{B}^\theta$
$\boldsymbol{t} = \boldsymbol{t}^p$	on	$\partial\mathcal{B}^t$	Neumann BC	$q = q^p$	on	$\partial\mathcal{B}^q$
with $\boldsymbol{t} = \boldsymbol{\sigma}^t \cdot \boldsymbol{n}$				with $q = -\boldsymbol{q} \cdot \boldsymbol{n}$		

15 unknowns

$u_i,\ \varepsilon_{ij},\ \sigma_{ij}$

versus 15 equations

7 unknowns

$\theta,\ \gamma_i,\ q_i$

versus 7 equations

Geometrically linear strain tensor

$$\boldsymbol{\varepsilon} = \nabla^{\mathsf{sym}}\boldsymbol{u} = \frac{1}{2}[u_{i,j} + u_{j,i}]\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

with $\quad \varepsilon_{ij} = \frac{1}{2}[u_{i,j} + u_{j,i}] = \boldsymbol{e}_i \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{e}_j$

Matrix arrangement of coefficients

$$[\varepsilon_{ij}] = \begin{bmatrix} u_{1,1} & \frac{1}{2}[u_{1,2} + u_{2,1}] & \frac{1}{2}[u_{1,3} + u_{3,1}] \\ \frac{1}{2}[u_{2,1} + u_{1,2}] & u_{2,2} & \frac{1}{2}[u_{2,3} + u_{3,2}] \\ \frac{1}{2}[u_{3,1} + u_{1,3}] & \frac{1}{2}[u_{3,2} + u_{2,3}] & u_{3,3} \end{bmatrix}$$

Symmetric and skew-symmetric 2nd-order tensors

- Symmetric 2nd-order tensors satisfy:

$$\boldsymbol{A} = \boldsymbol{A}^t, \qquad A_{ij} = A_{ji}$$

- Skew-symmetric 2nd-order tensors satisfy:

$$\boldsymbol{A} = -\boldsymbol{A}^t, \qquad A_{ij} = -A_{ji}$$

Theorem: Every 2nd-order tensor can be uniquely decomposed into a symmetric and skew-symmetric contribution

$$\boldsymbol{A} = \boldsymbol{A}^{\mathsf{sym}} + \boldsymbol{A}^{\mathsf{skw}},$$

where

$$\boldsymbol{A}^{\mathsf{sym}} = \frac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^t], \qquad \boldsymbol{A}^{\mathsf{skw}} = \frac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^t].$$

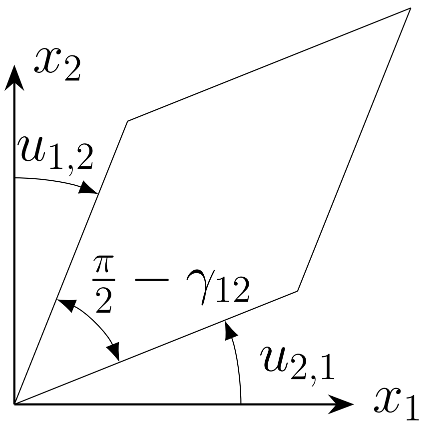
Observations

- Symmetry $\varepsilon_{ij} = \varepsilon_{ji}$
- $\varepsilon_{ij}, \quad i = j$ normal strains: measure length changes of fibres

$$\text{normal strain} = \frac{\text{new} - \text{old}}{\text{old}} \text{ length}$$

- $\varepsilon_{ij}, \quad i \neq j$ shear strains: measure angle changes between fibres
- $2\varepsilon_{ij} = \gamma_{ij}, \quad i \neq j$ engineering shear strains

The **tensorial** shear strains correspond to half of the **engineering** shear strains:

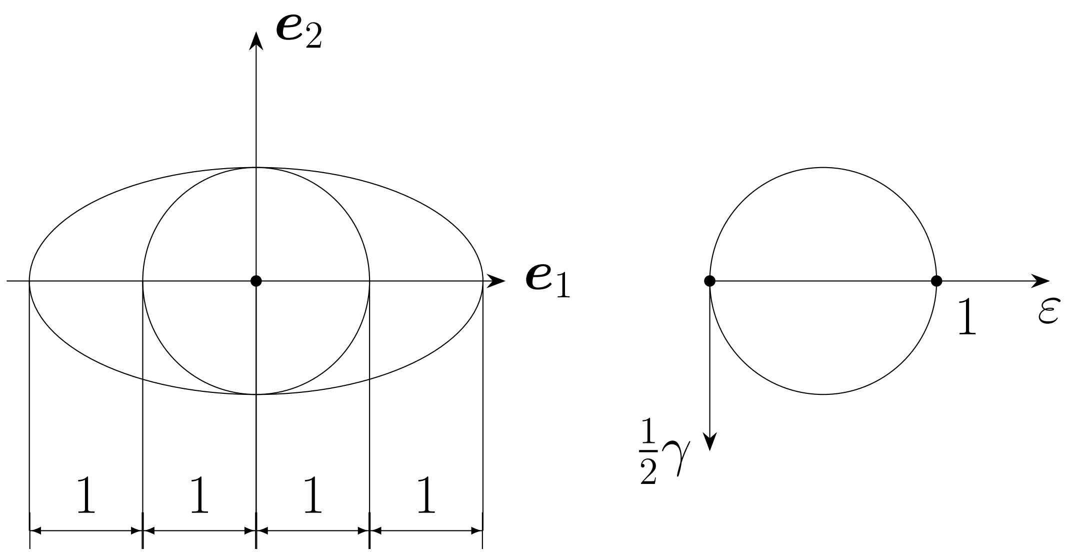


$$\gamma_{12} = u_{1,2} + u_{2,1} = 2\varepsilon_{12}$$

Mohr circle

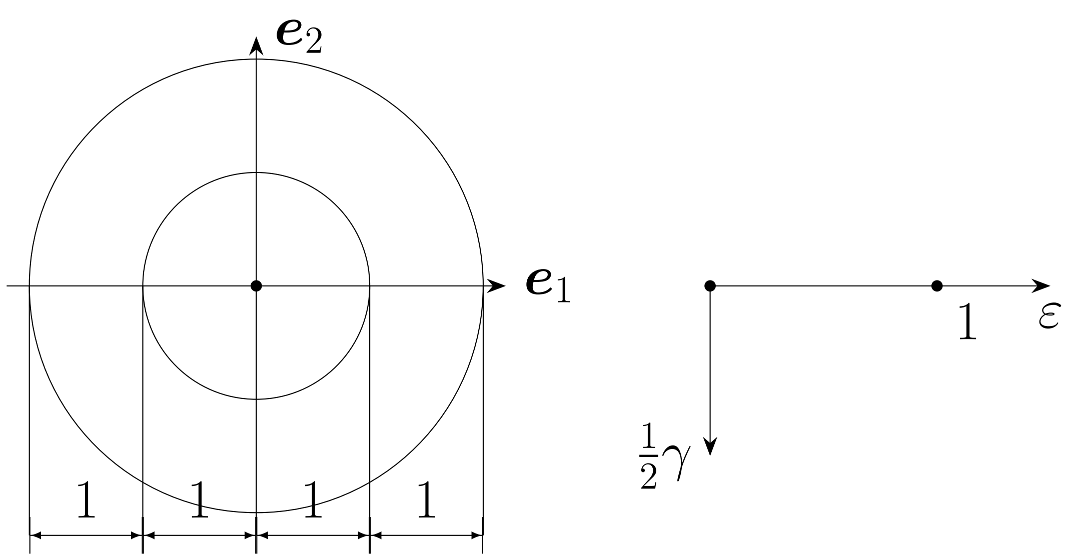
Uniaxial extension

$$[u_i] = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \qquad [\varepsilon_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



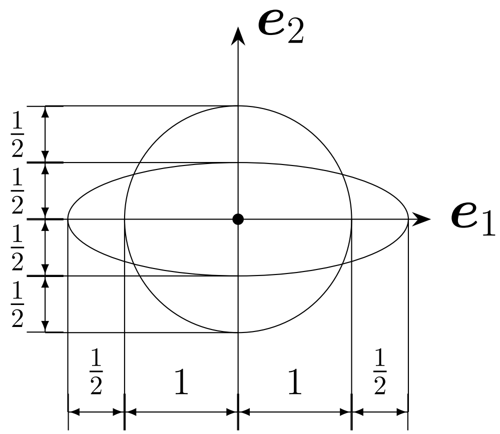
Dilatation

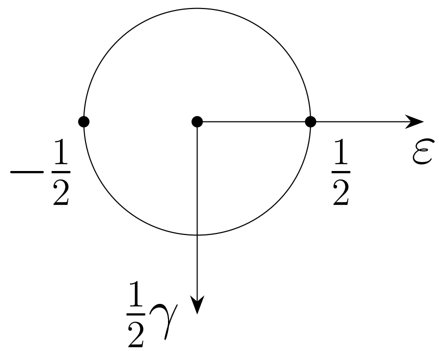
$$[u_i] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad [\varepsilon_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Pure shear

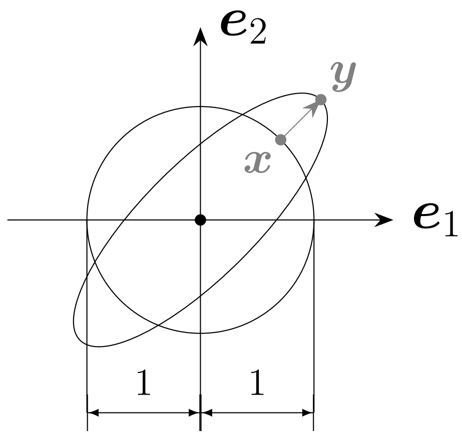
$$[u_i] = \frac{1}{2}\begin{bmatrix} x_1 \\ -x_2 \\ 0 \end{bmatrix} \qquad [\varepsilon_{ij}] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

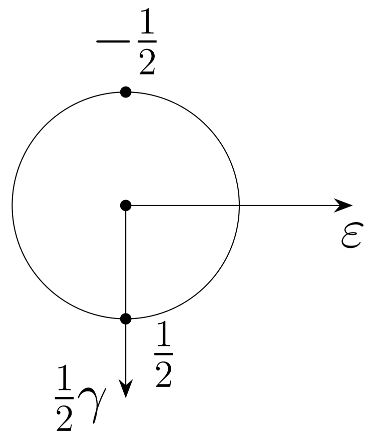




Simple shear

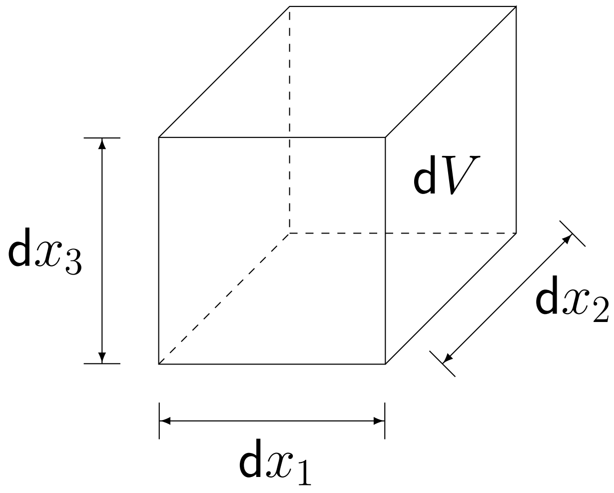
$$[u_i] = \frac{1}{2}\begin{bmatrix} x_2 \\ x_1 \\ 0 \end{bmatrix} \qquad [\varepsilon_{ij}] = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

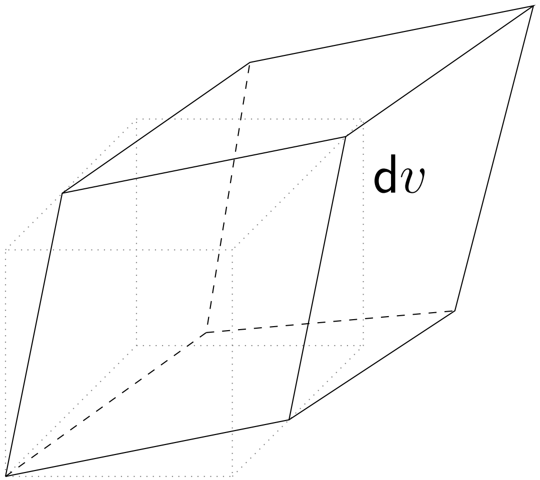




Example: $[x_i] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \implies [u_i] = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \implies [y_i] = [x_i] + [u_i] = \frac{3}{2\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 1.06 \\ 1.06 \\ 0 \end{bmatrix}$

Volume change





Volume before deformation

$$\mathrm{d}V = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

Volume after deformation

$$\mathrm{d}v = [1+\varepsilon_{11}]\,\mathrm{d}x_1\,[1+\varepsilon_{22}]\,\mathrm{d}x_2\,[1+\varepsilon_{33}]\,\mathrm{d}x_3$$

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}V} &= [1+\varepsilon_{11}]\,[1+\varepsilon_{22}]\,[1+\varepsilon_{33}]\\ &= 1+\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}+\text{higher order terms}\end{aligned}$$

We neglect higher order terms since we deal with small strains:

$$\varepsilon^2 \ll \varepsilon$$

Volume strain (dilatation)

$$d = \frac{\mathrm{d}v - \mathrm{d}V}{\mathrm{d}V}$$ volume strain $= \dfrac{\text{new} - \text{old}}{\text{old}}$ volume

$$= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$ trace of strain tensor

$$= \boldsymbol{\varepsilon} : \boldsymbol{I}$$ $\boldsymbol{I} = \delta_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ - unit tensor δ_{ij} - Kronecker delta

$$= [\varepsilon_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j] : [\delta_{kl}\,\boldsymbol{e}_k \otimes \boldsymbol{e}_l]$$

with $[\delta_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$

$$= \varepsilon_{ij}\,\delta_{kl}\,\delta_{ik}\,\delta_{jl}$$

$$= \varepsilon_{ii}$$ trace of $\boldsymbol{\varepsilon}$ renders d

$$= u_{i,i}$$

$$= \operatorname{div} \boldsymbol{u}$$

Volumetric strain tensor

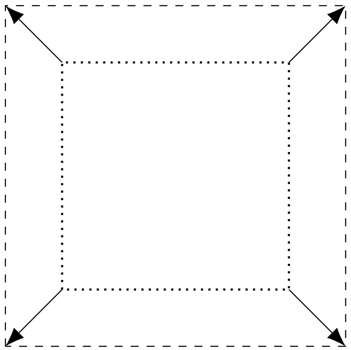
$$\boldsymbol{\varepsilon}^{\text{vol}} = \frac{1}{3} d\boldsymbol{I} \quad \text{with} \quad \boldsymbol{\varepsilon}^{\text{vol}} : \boldsymbol{I} = d$$

$\boldsymbol{\varepsilon}^{\text{vol}}$ contains volume changing but shape preserving contributions to $\boldsymbol{\varepsilon}$.

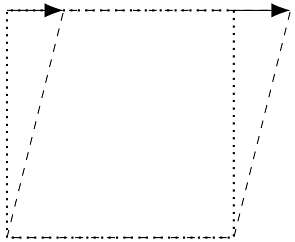
Deviatoric strain tensor

$$\boldsymbol{\varepsilon}^{\text{dev}} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\text{vol}} \quad \text{with} \quad \boldsymbol{\varepsilon}^{\text{dev}} : \boldsymbol{I} = 0$$

$\boldsymbol{\varepsilon}^{\text{dev}}$ contains volume preserving but shape changing contributions to $\boldsymbol{\varepsilon}$.



Dilatation



Simple Shear

Representation in terms of fourth-order projection tensors

Symbolic notation

$$\boldsymbol{A}^{\mathsf{sym}} = \boldsymbol{1}^{\mathsf{sym}} : \boldsymbol{A}$$

$$\boldsymbol{A}^{\mathsf{vol}} = \boldsymbol{1}^{\mathsf{vol}} : \boldsymbol{A}$$

$$\boldsymbol{A}^{\mathsf{dev}} = \boldsymbol{1}^{\mathsf{dev}} : \boldsymbol{A}$$

Index notation

$$A_{ij}^{\mathsf{sym}} = 1_{ijkl}^{\mathsf{sym}} \, A_{kl}$$

$$A_{ij}^{\mathsf{vol}} = 1_{ijkl}^{\mathsf{vol}} \, A_{kl}$$

$$A_{ij}^{\mathsf{dev}} = 1_{ijkl}^{\mathsf{dev}} \, A_{kl}$$

with fourth-order projection tensors

$$\boldsymbol{1}^{\mathsf{sym}}$$

$$\boldsymbol{1}^{\mathsf{vol}} = \frac{1}{3} \boldsymbol{I} \otimes \boldsymbol{I}$$

$$\boldsymbol{1}^{\mathsf{dev}} = \boldsymbol{1} - \boldsymbol{1}^{\mathsf{vol}}$$

$$1_{ijkl}^{\mathsf{sym}} = \frac{1}{2} \left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right]$$

$$1_{ijkl}^{\mathsf{vol}} = \frac{1}{3} \, \delta_{ij}\delta_{kl}$$

$$1_{ijkl}^{\mathsf{dev}} = \delta_{ik}\delta_{jl} - \frac{1}{3} \, \delta_{ij}\delta_{kl}$$

Application of fourth-order projection tensors to strain tensor

Symbolic notation	Index notation
$\boldsymbol{\varepsilon} = \mathbf{1}^{\mathsf{sym}} : \nabla \boldsymbol{u}$	$\varepsilon_{ij} = 1^{\mathsf{sym}}_{ijkl}\, u_{k,l}$
$\boldsymbol{\varepsilon}^{\mathsf{vol}} = \mathbf{1}^{\mathsf{vol}} : \boldsymbol{\varepsilon}$	$\varepsilon^{\mathsf{vol}}_{ij} = 1^{\mathsf{vol}}_{ijkl}\, \varepsilon_{kl}$
$\boldsymbol{\varepsilon}^{\mathsf{dev}} = \mathbf{1}^{\mathsf{dev}} : \boldsymbol{\varepsilon}$	$\varepsilon^{\mathsf{dev}}_{ij} = 1^{\mathsf{dev}}_{ijkl}\, \varepsilon_{kl}$

Proof of $\boldsymbol{\varepsilon} = \mathbf{1}^{\mathsf{sym}} : \nabla \boldsymbol{u}$

$$1^{\mathsf{sym}}_{ijkl}\, u_{k,l} = \frac{1}{2}\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]\, u_{k,l} = \frac{1}{2}\left[u_{i,j} + u_{j,i}\right] = \varepsilon_{ij}$$

Proof of $\boldsymbol{\varepsilon}^{\mathsf{vol}} = \mathbf{1}^{\mathsf{vol}} : \boldsymbol{\varepsilon}$

$$1^{\mathsf{vol}}_{ijkl}\, \varepsilon_{kl} = \frac{1}{3}\, \delta_{ij}\delta_{kl}\, \varepsilon_{kl} = \frac{1}{3}\, \varepsilon_{kk}\, \delta_{ij} = \frac{1}{3}\, d\, \delta_{ij} = \varepsilon^{\mathsf{vol}}_{ij}$$

Projection tensors are idempotent

- zeroth-order tensor

$$1\ ; \qquad 1^2 = 1$$

- second-order tensor

$$\boldsymbol{I}\ ; \qquad \boldsymbol{I}\cdot\boldsymbol{I} = \boldsymbol{I} \qquad \delta_{ij}\,\delta_{jk} = \delta_{ik}$$

- fourth-order tensor

$$\mathbf{1}^{\mathsf{sym}}\ ; \qquad \mathbf{1}^{\mathsf{sym}} : \mathbf{1}^{\mathsf{sym}} = \mathbf{1}^{\mathsf{sym}} \qquad 1^{\mathsf{sym}}_{ijkl}\,1^{\mathsf{sym}}_{klmn} = 1^{\mathsf{sym}}_{ijmn}$$

$$\mathbf{1}^{\mathsf{vol}}\ ; \qquad \mathbf{1}^{\mathsf{vol}} : \mathbf{1}^{\mathsf{vol}} = \mathbf{1}^{\mathsf{vol}} \qquad 1^{\mathsf{vol}}_{ijkl}\,1^{\mathsf{vol}}_{klmn} = 1^{\mathsf{vol}}_{ijmn}$$

$$\mathbf{1}^{\mathsf{dev}}\ ; \qquad \mathbf{1}^{\mathsf{dev}} : \mathbf{1}^{\mathsf{dev}} = \mathbf{1}^{\mathsf{dev}} \qquad 1^{\mathsf{dev}}_{ijkl}\,1^{\mathsf{dev}}_{klmn} = 1^{\mathsf{dev}}_{ijmn}$$

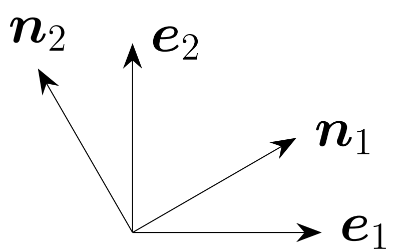
Orthogonality

$$\mathbf{1}^{\mathsf{vol}} : \mathbf{1}^{\mathsf{dev}} = \mathbf{0} \quad \text{(zero)} \qquad \mathbf{1}^{\mathsf{vol}} \perp \mathbf{1}^{\mathsf{dev}}$$

Eigenvalue problem

We seek a rotated orthonormal coordinate system with basic vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$ so that the corresponding shear strains vanish:

$$\boldsymbol{n}_a \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{n}_b = 0 \quad \text{for} \quad a \neq b$$



$$\begin{aligned} \boldsymbol{\varepsilon} &= \varepsilon_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= \lambda_1 \boldsymbol{n}_1 \otimes \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 \otimes \boldsymbol{n}_2 + \lambda_3 \boldsymbol{n}_3 \otimes \boldsymbol{n}_3 = \sum_{a=1}^{3} \lambda_a \boldsymbol{n}_a \otimes \boldsymbol{n}_a \end{aligned}$$

(no summation convention for a, b)

with λ_a — eigenvalues (principal strains),

$\boldsymbol{n}_a$ — eigenvectors (principal directions),

and $\boldsymbol{n}_a \cdot \boldsymbol{n}_b = \delta_{ab}$ — orthonormality

$$\boldsymbol{\varepsilon} \cdot \boldsymbol{n}_a = \lambda_a \, \boldsymbol{n}_a$$

Homogeneous linear 'system of equations'

$$[\boldsymbol{\varepsilon} - \lambda_a \boldsymbol{I}] \cdot \boldsymbol{n}_a = \boldsymbol{0}$$

Analysis of strain

Homogeneous linear 'system of equations'

$$[\boldsymbol{\varepsilon} - \lambda \boldsymbol{I}] \cdot \boldsymbol{n} = \boldsymbol{0}$$

Trivial solution

$$\boldsymbol{n} = \boldsymbol{0}$$

Condition for non-trivial solution

$$\det(\boldsymbol{\varepsilon} - \lambda \boldsymbol{I}) = \boldsymbol{0}$$

Characteristic equation

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

Principal invariants versus spectral invariants

$$\begin{aligned} I_1 &= \boldsymbol{\varepsilon} : \boldsymbol{I} &&= \lambda_1 + \lambda_2 + \lambda_3 \\ I_2 &= \frac{1}{2}[I_1^2 - \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon}] &&= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ I_3 &= \det(\boldsymbol{\varepsilon}) &&= \lambda_1\,\lambda_2\,\lambda_3 \end{aligned}$$

Basic invariants versus spectral invariants

$$\begin{aligned} \bar{I}_1 &= \boldsymbol{\varepsilon} : \boldsymbol{I} &&= \lambda_1 + \lambda_2 + \lambda_3 \\ \bar{I}_2 &= \boldsymbol{\varepsilon}^2 : \boldsymbol{I} &&= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \\ \bar{I}_3 &= \boldsymbol{\varepsilon}^3 : \boldsymbol{I} &&= \lambda_1^3 + \lambda_2^3 + \lambda_3^3 \end{aligned}$$

Cayley-Hamilton Theorem

Each symmetric second-order tensor satisfies its own characteristic equation.

$$\boldsymbol{\varepsilon}^3 - I_1\boldsymbol{\varepsilon}^2 + I_2\boldsymbol{\varepsilon} - I_3\boldsymbol{I} = \boldsymbol{0}$$

$$I_3 = \frac{1}{3}\left[\boldsymbol{\varepsilon}^3 : \boldsymbol{I} - I_1\boldsymbol{\varepsilon}^2 : \boldsymbol{I} + I_2\boldsymbol{\varepsilon} : \boldsymbol{I}\right]$$

$$\text{if exists } \boldsymbol{\varepsilon}^{-1} \quad \Longrightarrow \quad \boldsymbol{\varepsilon}^{-1} = \frac{1}{I_3}\left[\boldsymbol{\varepsilon}^2 - I_1\boldsymbol{\varepsilon} + I_2\boldsymbol{I}\right]$$

$$\text{Condition for } I_3 \neq 0 \quad \Longrightarrow \quad \lambda_a \neq 0 \quad \forall a$$

$K_a,\ \bar{K}_a$: Invariants of $\boldsymbol{\varepsilon}^{\mathsf{vol}}$

$$K_1 = d \qquad \bar{K}_1 = d$$

$$K_2 = \frac{1}{3}d^2 \qquad \bar{K}_2 = \frac{1}{3}d^2$$

$$K_3 = \frac{1}{27}d^3 \qquad \bar{K}_3 = \frac{1}{9}d^3$$

$J_a,\ \bar{J}_a$: Invariants of $\boldsymbol{\varepsilon}^{\mathsf{dev}}$

$$J_1 = 0 \qquad \bar{J}_1 = 0$$

$$-J_2 = \frac{1}{2}[\boldsymbol{\varepsilon}^{\mathsf{dev}}]^2 : \boldsymbol{I} \qquad \bar{J}_2 = [\boldsymbol{\varepsilon}^{\mathsf{dev}}]^2 : \boldsymbol{I}$$

$$J_3 = \frac{1}{3}[\boldsymbol{\varepsilon}^{\mathsf{dev}}]^3 : \boldsymbol{I} \qquad \bar{J}_3 = [\boldsymbol{\varepsilon}^{\mathsf{dev}}]^3 : \boldsymbol{I}$$

Integrability conditions

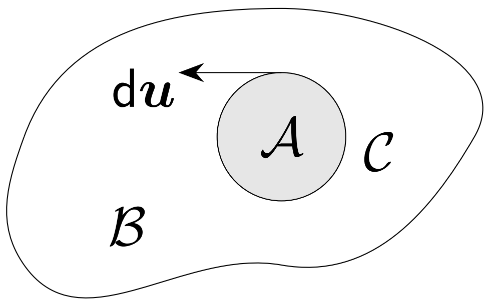
What are the conditions for a given distortion field $\boldsymbol{h}$ to be integrable to a single-valued displacement field such that $\boldsymbol{h} = \nabla \boldsymbol{u}$?

- Consider closure failure:

$$\oint_{\mathcal{C}} \mathrm{d}\boldsymbol{u} = \oint_{\mathcal{C}} \boldsymbol{h} \cdot \mathrm{d}\boldsymbol{x} \quad \text{with} \quad \boldsymbol{h} \doteq \nabla \boldsymbol{u}$$

- Use Stokes' theorem:

$$\oint_{\mathcal{C}} \boldsymbol{h} \cdot \mathrm{d}\boldsymbol{x} = \int_{\mathcal{A}} \mathrm{rot}^t\, \boldsymbol{h} \cdot \boldsymbol{n}\, \mathrm{d}a$$



- Closure gap has to vanish for all $\mathcal{C}$.
- Localization to pointwise statement:

$$\mathrm{rot}\, \boldsymbol{h} = \boldsymbol{0} \quad \text{with} \quad \mathrm{rot}\, \boldsymbol{h} = -[\nabla \boldsymbol{h} : \boldsymbol{e}]^t$$

Remark: $\mathcal{B}$ has to be simply connected ("no holes").

Dislocation density tensor

Non-symmetric dislocation density tensor has to vanish:

$$\boldsymbol{\alpha} = \text{rot}\,\boldsymbol{h} = h_{jk,l}e_{lki}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \stackrel{!}{=} \boldsymbol{0}$$

Proof:

- $\boldsymbol{\alpha} = \boldsymbol{0}$ is necessary:

$$\boldsymbol{h} = \nabla \boldsymbol{u} \implies \text{rot}\,\boldsymbol{h} = u_{j,kl}e_{lki}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = -u_{j,lk}e_{kli}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = -\,\text{rot}\,\boldsymbol{h}$$

$$\implies \boldsymbol{\alpha} = \boldsymbol{0}$$

- $\boldsymbol{\alpha} = \boldsymbol{0}$ is sufficient:

$$\text{rot}\,\boldsymbol{h} = \boldsymbol{0} \implies \oint_{\mathcal{C}} \boldsymbol{h} \cdot \mathrm{d}\boldsymbol{x} = \int_{\mathcal{A}} \text{rot}^t\,\boldsymbol{h} \cdot \boldsymbol{n}\,\mathrm{d}a = \boldsymbol{0} \quad \text{for all closed } \mathcal{C}$$

$$\implies \text{Set } \boldsymbol{u}(\boldsymbol{x}) = \int_{\boldsymbol{x}_0}^{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{\xi}) \cdot \mathrm{d}\boldsymbol{\xi} \text{ for any fixed } \boldsymbol{x}_0 \in \mathcal{B}$$

(integral is independent of the curve connecting $\boldsymbol{x}_0$ and $\boldsymbol{x}$)

$$\implies \boldsymbol{h} = \nabla \boldsymbol{u}$$

(also known as Poincaré lemma)

Integrability conditions

$$
\begin{aligned}
\alpha_{11} &= h_{13,2} - h_{12,3} = 0 \\
\alpha_{12} &= h_{23,2} - h_{22,3} = 0 \\
\alpha_{13} &= h_{33,2} - h_{32,3} = 0 \\
\alpha_{21} &= h_{11,3} - h_{13,1} = 0 \\
\alpha_{22} &= h_{21,3} - h_{23,1} = 0 \\
\alpha_{23} &= h_{31,3} - h_{33,1} = 0 \\
\alpha_{31} &= h_{12,1} - h_{11,2} = 0 \\
\alpha_{32} &= h_{22,1} - h_{21,2} = 0 \\
\alpha_{33} &= h_{32,1} - h_{31,2} = 0
\end{aligned}
$$

Remark:

$$
\left.
\begin{array}{l}
\text{3 displacement coordinates } \boldsymbol{u} = u_i \boldsymbol{e}_i \\
\text{9 kinematic equations } \boldsymbol{h} = \nabla \boldsymbol{u} \\
\text{9 integrability equations } \boldsymbol{\alpha} = \boldsymbol{0} \\
\text{3 constraint equations div } \boldsymbol{\alpha} = \boldsymbol{0}
\end{array}
\right\} \quad 3 = 9 \text{ - } [9 \text{ - } 3]
$$

Compatibility conditions

What are the conditions for a given symmetric strain field $\boldsymbol{\varepsilon}$ to be compatible with a single-valued displacement field $\boldsymbol{u}$ such that $\boldsymbol{\varepsilon} = \nabla^{\mathsf{sym}}\boldsymbol{u}$?

- Consider closure failure:

$$\oint_{\mathcal{C}} \mathrm{d}\boldsymbol{u} = \oint_{\mathcal{C}} [\boldsymbol{\varepsilon} + \boldsymbol{\omega}] \cdot \mathrm{d}\boldsymbol{x} \quad \text{with} \quad \boldsymbol{\varepsilon} \doteq \nabla^{\mathsf{sym}}\boldsymbol{u}, \ \boldsymbol{\omega} \doteq \nabla^{\mathsf{skw}}\boldsymbol{u}$$

- Use Stokes' theorem:

$$\oint_{\mathcal{C}} [\boldsymbol{\varepsilon} + \boldsymbol{\omega}] \cdot \mathrm{d}\boldsymbol{x} = \int_{\mathcal{A}} \mathrm{rot}^t(\boldsymbol{\varepsilon} + \boldsymbol{\omega}) \cdot \boldsymbol{n} \, \mathrm{d}a$$

- Closure gap has to vanish for all $\mathcal{C}$.
- Localization to pointwise statement:

$$\mathrm{rot}\, \boldsymbol{\varepsilon} = -\, \mathrm{rot}\, \boldsymbol{\omega} \neq \mathbf{0}$$

- But: $\mathrm{rot}\,\mathrm{rot}\, \boldsymbol{\varepsilon} = -\, \mathrm{rot}\,\mathrm{rot}\, \boldsymbol{\omega} = \mathbf{0}$

Remark: $\mathcal{B}$ has to be simply connected ("no holes").

Incompatibility tensor

Symmetric incompatibility tensor has to vanish:

$$\boldsymbol{\eta} = \operatorname{inc}\boldsymbol{\varepsilon} = \operatorname{rot}\operatorname{rot}\boldsymbol{\varepsilon} = e_{ikm}\varepsilon_{kl,mn}e_{jln}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \stackrel{!}{=} \mathbf{0}$$

Proof:

- $\boldsymbol{\eta} = \mathbf{0}$ is necessary:

$$\begin{aligned}\boldsymbol{\varepsilon} = \nabla^{\text{sym}}\boldsymbol{u} \implies \operatorname{rot}\operatorname{rot}\boldsymbol{\varepsilon} &= \varepsilon_{kl,mn}e_{ikm}e_{jln}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= \tfrac{1}{2}[u_{k,lmn}e_{ikm}e_{jln} + u_{l,kmn}e_{ikm}e_{jln}]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= \tfrac{1}{2}[-u_{k,nml}e_{ikm}e_{jnl} - u_{l,mkn}e_{imk}e_{jln}]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= -\operatorname{rot}\operatorname{rot}\boldsymbol{\varepsilon} \\ \implies \boldsymbol{\eta} = \mathbf{0} &\end{aligned}$$

- $\boldsymbol{\eta} = \mathbf{0}$ is sufficient:

(i) $\quad \text{rot rot}\, \boldsymbol{\varepsilon} = \mathbf{0} \implies \exists\, \boldsymbol{w}$ with $\text{rot}\, \boldsymbol{\varepsilon} = \nabla \boldsymbol{w}$ (by Poincaré lemma)

(ii) $\quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^t \implies \text{div}\, \boldsymbol{w} = \text{tr rot}\, \boldsymbol{\varepsilon} = e_{jkl}\varepsilon_{lk,j} = 0$

(iii) $\quad$ Set $\boldsymbol{\omega} = -\boldsymbol{e} \cdot \boldsymbol{w} \implies$

$$\begin{aligned} \text{rot}\, \boldsymbol{\omega} &= -[\nabla[-\boldsymbol{e} \cdot \boldsymbol{w}] : \boldsymbol{e}]^t \\ &= e_{jkl} w_{l,m} e_{kmi} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= [\delta_{ij}\delta_{lm} - \delta_{il}\delta_{jm}] w_{l,m} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ &= \text{div}\, \boldsymbol{w}\, \boldsymbol{I} - \nabla \boldsymbol{w} = -\,\text{rot}\, \boldsymbol{\varepsilon} \end{aligned}$$

(iv) $\quad \text{rot}(\boldsymbol{\varepsilon} + \boldsymbol{\omega}) = \mathbf{0} \implies \exists\, \boldsymbol{u}$ with $\boldsymbol{\varepsilon} + \boldsymbol{\omega} = \nabla \boldsymbol{u}$ (by Poincaré lemma)

and $\boldsymbol{\varepsilon} = \nabla^{\text{sym}} \boldsymbol{u}, \ \boldsymbol{\omega} = \nabla^{\text{skw}} \boldsymbol{u}$

St. Venant compatibility conditions

$$
\begin{aligned}
\eta_{11} &= \varepsilon_{22,33}+\varepsilon_{33,22}-2\varepsilon_{23,32}=0\\
\eta_{22} &= \varepsilon_{33,11}+\varepsilon_{11,33}-2\varepsilon_{31,13}=0\\
\eta_{33} &= \varepsilon_{11,22}+\varepsilon_{22,11}-2\varepsilon_{12,21}=0\\
\eta_{12} &= \varepsilon_{13,32}+\varepsilon_{23,31}-\varepsilon_{33,12}-\varepsilon_{12,33}=0\\
\eta_{23} &= \varepsilon_{21,13}+\varepsilon_{31,12}-\varepsilon_{11,23}-\varepsilon_{23,11}=0\\
\eta_{31} &= \varepsilon_{32,21}+\varepsilon_{12,23}-\varepsilon_{22,31}-\varepsilon_{31,22}=0
\end{aligned}
$$

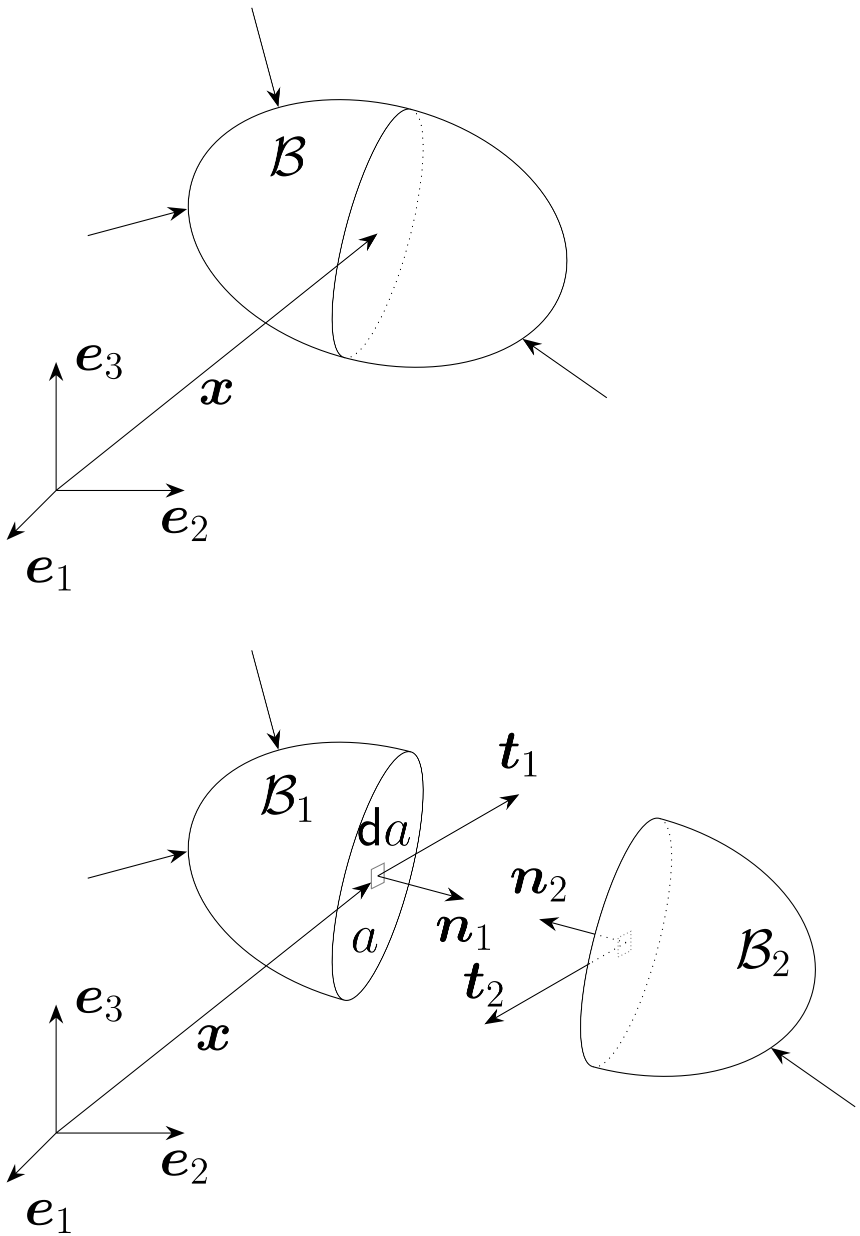
Remark:

$$
\left.\begin{array}{l}
\text{3 displacement coordinates } \boldsymbol{u}=u_i\boldsymbol{e}_i\\
\text{6 kinematic equations } \boldsymbol{\varepsilon}=\nabla^{\mathsf{sym}}\boldsymbol{u}\\
\text{6 compatibility equations } \boldsymbol{\eta}=\boldsymbol{\eta}^t=\boldsymbol{0}\\
\text{3 constraint equations div } \boldsymbol{\eta}=\boldsymbol{0}
\end{array}\right\}\quad 3 = 6 - [6 - 3]
$$

De plus, la pression ou tension exercée contre un plane quelconque se déduit très facilement, tant en grandeur qu'en direction, des pressions ou tensions exercées contre trois plans rectangulaires donnés

Augustin Louis Cauchy (1789–1857)

Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non elastiques [1823]



Area of cut-surface

$$a = \int_{\text{cut-surface}} \mathrm{d}a$$

Outward pointing surface normal

$$\boldsymbol{n}_\alpha, \quad \alpha = 1, 2$$

with $\quad \boldsymbol{n}_1 = -\boldsymbol{n}_2 = \boldsymbol{n} \quad$ and $\quad |\boldsymbol{n}| = 1$

Vectorial area element

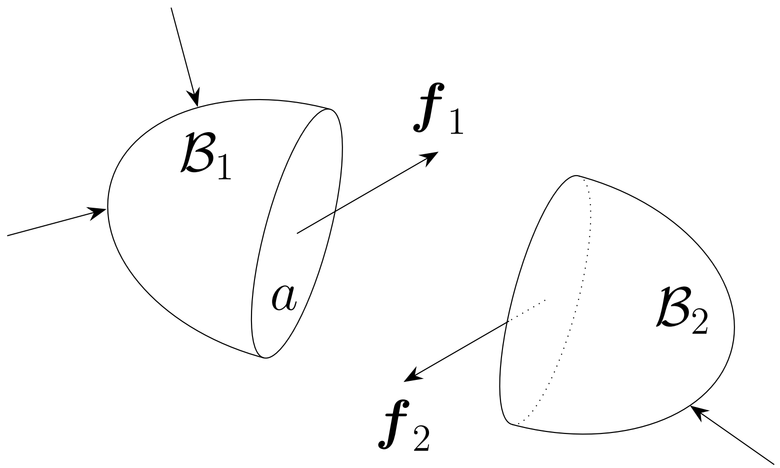
$$\mathrm{d}\boldsymbol{a} = \boldsymbol{n}\,\mathrm{d}a \quad \text{with} \quad |\,\mathrm{d}\boldsymbol{a}| = \mathrm{d}a$$

Traction vector

$$\boldsymbol{t}_\alpha = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{f}_\alpha}{\Delta a} = \frac{\mathrm{d}\boldsymbol{f}_\alpha}{\mathrm{d}a} \qquad \left[\frac{\text{force}}{\text{length}^2}\right]$$

with $\boldsymbol{t}_\alpha$ contribution to the resultant $\boldsymbol{f}_\alpha$

$$\boldsymbol{f}_\alpha = \int_{\text{cut-surface}} \boldsymbol{t}_\alpha \, \mathrm{d}a$$



actio et reactio

$$\boldsymbol{f}_1 = -\boldsymbol{f}_2$$

1. **Cauchy postulate**

   The traction vector at a point depends only on the normal vector to arbitrary cut-surfaces.

2. **Cauchy lemma**

   The traction vector is a homogeneous function of degree one in the normal vector to arbitrary cut-surfaces (homogeneous function of degree n: $\varepsilon^n f(x) = f(\varepsilon x)$).

3. **Cauchy theorem**

   The traction vector follows as a linear map of the normal vector to arbitrary cut-surfaces (by a second order tensor: Cauchy stress tensor).

Classification of terminology

1. **Postulate**

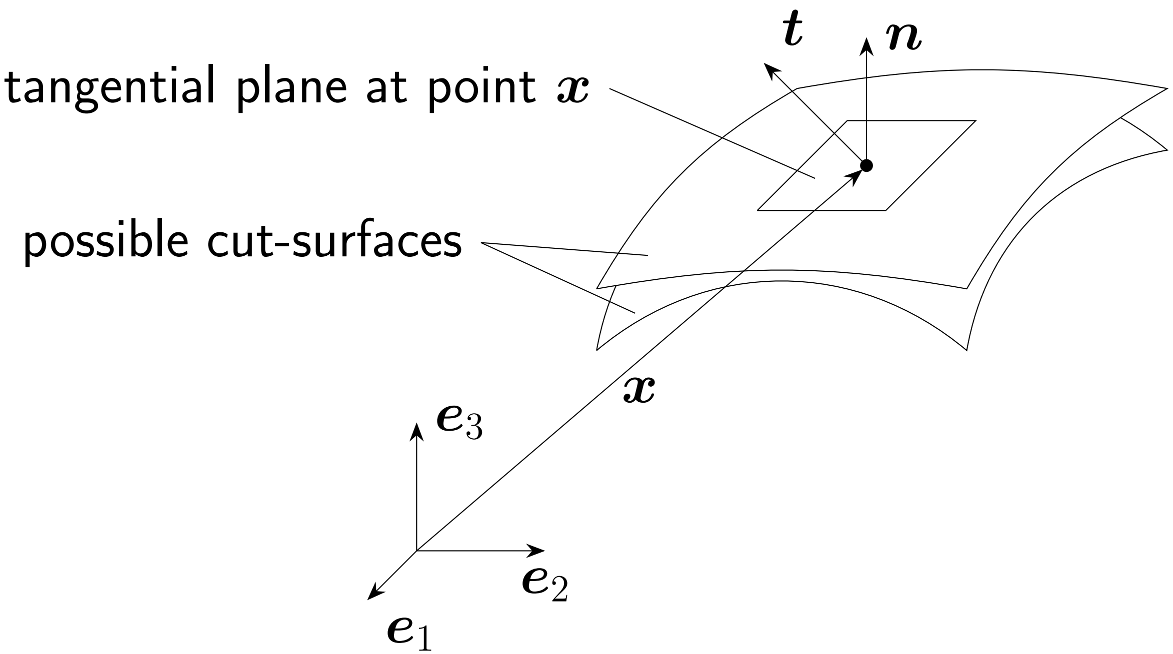
   Postulate (or axiom) is a proposition that is not proved or demonstrated but considered to be either self-evident, or subject to necessary decision.

2. **Lemma**

   A lemma is a proven proposition which is used as a stepping stone to a larger result rather than as a statement in-and-of itself.

3. **Theorem**

   A theorem is a statement which has been proved on the basis of previously established statements, such as other theorems, and previously accepted statements, such as axioms.



1. Cauchy postulate

   $\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{n}\,,\,\boldsymbol{x})$

2. Cauchy lemma

   $\boldsymbol{t}(-\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{n})$

3. Cauchy theorem

   $\boldsymbol{t} = \boldsymbol{\sigma}^t \cdot \boldsymbol{n} \qquad t_j = \sigma_{ij}\, n_i$

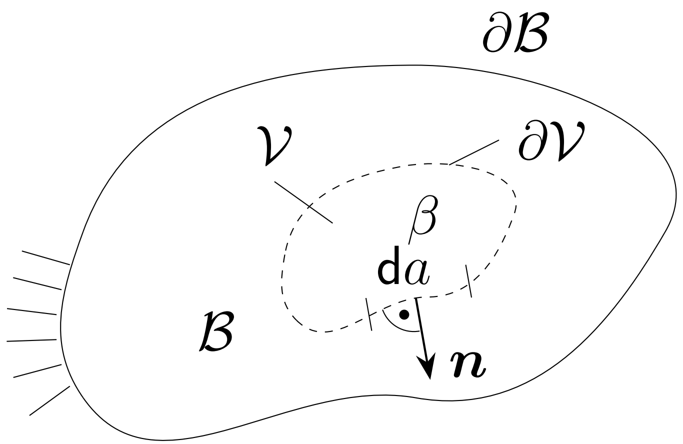
For the proof of the Cauchy lemma consider the Cauchy postulate for a cut-surface with

$$\boldsymbol{t}_1 = \boldsymbol{t}(\boldsymbol{n}_1) = \boldsymbol{t}(\boldsymbol{n}) \qquad \boldsymbol{t}_2 = \boldsymbol{t}(\boldsymbol{n}_2) = \boldsymbol{t}(-\boldsymbol{n})$$

and equilibrium condition

$$\boldsymbol{t}_1\, \mathrm{d}a + \boldsymbol{t}_2\, \mathrm{d}a = \boldsymbol{0}$$

$$\boldsymbol{t}_1 = -\boldsymbol{t}_2 \quad \Longrightarrow \quad \boldsymbol{t}(\boldsymbol{n}) = -\boldsymbol{t}(-\boldsymbol{n})$$



Surface theorem

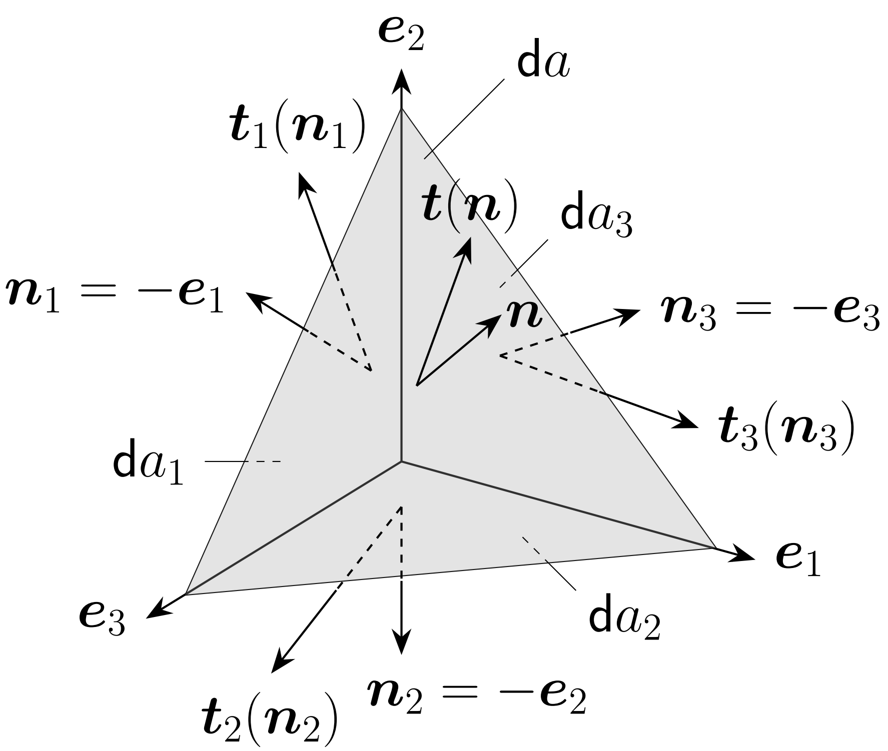
$$\int_{\partial\mathcal{V}} \mathrm{d}\boldsymbol{a} = \boldsymbol{0}$$

The integral of the vectorial area element over a closed surface vanishes due to the Gauss theorem.

Proof:

$$\begin{aligned}\int_{\partial\mathcal{V}} \mathrm{d}\boldsymbol{a} &= \int_{\partial\mathcal{V}} \boldsymbol{n}\,\mathrm{d}a \\ &= \int_{\partial\mathcal{V}} \boldsymbol{I}\cdot\boldsymbol{n}\,\mathrm{d}a \\ &= \int_{\mathcal{V}} \operatorname{div}\boldsymbol{I}\,\mathrm{d}v \\ &= \boldsymbol{0}\end{aligned}$$

Cauchy tetraeder argument



$$\boldsymbol{n}\,\mathrm{d}a + \sum_{i=1}^{3} \boldsymbol{n}_i\,\mathrm{d}a_i = \boldsymbol{0} \quad \text{surface theorem}$$

$$\boldsymbol{n}\,\mathrm{d}a = \sum_{i=1}^{3} \boldsymbol{e}_i\,\mathrm{d}a_i \quad \text{application to tetraeder}$$

$$\frac{\mathrm{d}a_i}{\mathrm{d}a} = \boldsymbol{e}_i \cdot \boldsymbol{n} = \cos\angle(\boldsymbol{e}_i\,,\,\boldsymbol{n})$$

$$\boldsymbol{t}\,\mathrm{d}a + \sum_{i=1}^{3} \boldsymbol{t}(\boldsymbol{n}_i)\,\mathrm{d}a_i = \boldsymbol{0} \quad \text{equilibrium of forces}$$

$$\boldsymbol{t}(\boldsymbol{n}_i) = -\boldsymbol{t}(\boldsymbol{e}_i) = -\boldsymbol{t}_i \quad \forall i$$

$$\boldsymbol{t} = \sum_{i=1}^{3} \boldsymbol{t}_i \frac{\mathrm{d}a_i}{\mathrm{d}a} = \sum_{i=1}^{3} \boldsymbol{t}_i [\boldsymbol{n} \cdot \boldsymbol{e}_i]$$

Resultant due to volume forces and inertia distributed within the volume of the tetraeder vanishes in the limit $\Delta v \to \mathrm{d}v$.

Cauchy stress

$$\boldsymbol{t} = [\boldsymbol{t}_i \otimes \boldsymbol{e}_i] \cdot \boldsymbol{n} = \boldsymbol{\sigma}^t \cdot \boldsymbol{n} \quad \text{(summation convention)}$$

Cauchy stress

$$\boldsymbol{\sigma}^t = \boldsymbol{t}_i \otimes \boldsymbol{e}_i = \sigma_{ij}\, \boldsymbol{e}_j \otimes \boldsymbol{e}_i \quad \text{with} \quad \boldsymbol{t}_i = \sigma_{ij} \boldsymbol{e}_j$$

$$\boldsymbol{\sigma} = \sigma_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

Matrix representation

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{matrix} = [t_{1j}] \\ = [t_{2j}] \\ = [t_{3j}] \end{matrix}$$

Symmetry due to of angular momentum

1. index: cut-surface normal

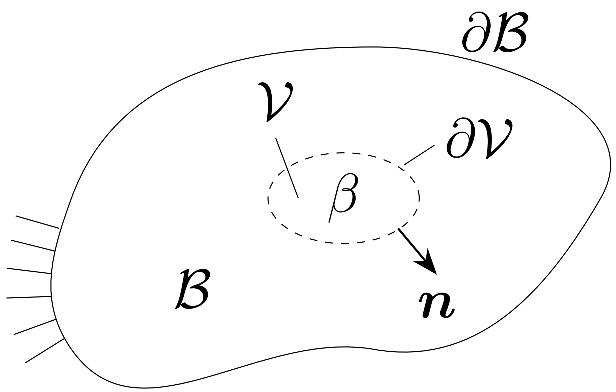
2. index: coordinate of traction vector

$$\boldsymbol{t} = \boldsymbol{\sigma}^t \cdot \boldsymbol{n} = [\sigma_{ij}\boldsymbol{e}_j \otimes \boldsymbol{e}_i] \cdot [n_k \boldsymbol{e}_k] = \sigma_{ij}\, n_i\, \boldsymbol{e}_j = t_j \boldsymbol{e}_j$$

with $\boldsymbol{e}_i \cdot \boldsymbol{e}_k = \delta_{ik}$ Kronecker delta

$$t_j = \sigma_{ij}\, n_i$$

Balance equations



β is a (generic) quantity that we want to balance in $\mathcal{V}$
time change of β in $\mathcal{V}$ due to:
sources in $\mathcal{V}$,
flux across $\partial\mathcal{V}$ and (possibly)
production in $\mathcal{V}$

$\mathcal{V}$ is a cut-out region of $\mathcal{B}$ with boundary $\partial\mathcal{V}$

β is a density per unit volume, its total amount in $\mathcal{V}$ is

$$\int_{\mathcal{V}} \beta \, \mathrm{d}v$$

Time change of β in $\mathcal{V}$

$$\dot{\overline{\int_{\mathcal{V}} \beta \, \mathrm{d}v}} = \int_{\mathcal{V}} \dot{\beta} \, \mathrm{d}v$$ due to the assumption of geometric linearity

Balance equation for β

$$\int_{\mathcal{V}} \dot{\beta} \; \mathsf{d}v = \int_{\mathcal{V}} \varsigma \; \mathsf{d}v + \int_{\partial\mathcal{V}} \Phi \; \mathsf{d}a + \int_{\mathcal{V}} \pi \; \mathsf{d}v$$

with $\int_{\mathcal{V}} \varsigma \; \mathsf{d}v$ source

$\int_{\partial\mathcal{V}} \Phi \; \mathsf{d}a$ flux

$\int_{\mathcal{V}} \pi \; \mathsf{d}v$ production with constraint $\pi \geq 0$

Balance equations

Problem: Balance equation has to hold for **arbitrary** cut-outs $\mathcal{V}$, but:

there is an integral on the boundary of the cut-out $\partial\mathcal{V}$

Suppose the Cauchy theorem holds (more details later)

$$\Phi = \boldsymbol{\Phi} \cdot \boldsymbol{n}$$

Remember Gauss theorem

$$\int_{\partial\mathcal{V}} \Phi \, \mathrm{d}a = \int_{\mathcal{V}} \operatorname{div} \boldsymbol{\Phi} \, \mathrm{d}v$$

Global statement of balance equation

$$\int_{\mathcal{V}} \left[\dot{\beta} - \varsigma - \operatorname{div} \boldsymbol{\Phi} - \pi \right] \mathrm{d}v \stackrel{!}{=} 0 \quad \forall \, \mathcal{V} \subset \mathcal{B}$$

Local statement of balance equation

$$\dot{\beta} - \varsigma - \operatorname{div} \boldsymbol{\Phi} - \pi = 0 \quad \forall \, \boldsymbol{x} \in \mathcal{B}$$

Note: we say β is conserved if $\dot{\beta} = 0$

Balance equations

Examples for β that we will discuss in the sequel:

Mechanical balance equations

1. Mass (density ρ)
2. Linear momentum (density $\boldsymbol{p} = \rho\boldsymbol{v}$)
3. Angular momentum (density $\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p}$)

Thermo-dynamical balance equations

4. Energy (density e)
5. Entropy (density s)

1. Mass

β	ς	$\boldsymbol{\Phi}$	Φ	π
ρ	0	$\mathbf{0}$	0	0

Mass density

$$\rho$$

Total mass in $\mathcal{V}$

$$m = \int_{\mathcal{V}} \rho \, \mathrm{d}v$$

Balance of mass (mass conservation)

local statement

$$\dot{\rho} = 0$$

global statement

$$\dot{m} = 0$$

2. Linear momentum

β	ς	$\boldsymbol{\Phi}$	Φ	π
$\boldsymbol{p}$	$\boldsymbol{b}$	$\boldsymbol{\sigma}^t$	$\boldsymbol{t} = \boldsymbol{\sigma}^t \cdot \boldsymbol{n}$	$\mathbf{0}$

Linear momentum density

$$\boldsymbol{p} = \rho \boldsymbol{v}$$

Total linear momentum in $\mathcal{V}$

$$\boldsymbol{P} = \int_{\mathcal{V}} \boldsymbol{p} \, \mathrm{d}v = \int_{\mathcal{V}} \rho \, \boldsymbol{v} \, \mathrm{d}v$$

Balance of linear momentum

local statement

$$\dot{\boldsymbol{p}} = \boldsymbol{b} + \mathrm{div}\, \boldsymbol{\sigma}^t$$

global statement

$$\dot{\boldsymbol{P}} = \boldsymbol{F} \quad \text{with} \quad \boldsymbol{F} = \int_{\mathcal{V}} \boldsymbol{b} \, \mathrm{d}v + \int_{\partial\mathcal{V}} \boldsymbol{t} \, \mathrm{d}a$$

Incorporation of balance of mass

$$\rho \dot{\boldsymbol{v}} = \boldsymbol{b} + \mathrm{div}\, \boldsymbol{\sigma}^t$$

3. Angular momentum

β	ς	$\boldsymbol{\Phi}$	Φ	π
$\boldsymbol{l}$	$\boldsymbol{r} \times \boldsymbol{b}$	$\boldsymbol{r} \times \boldsymbol{\sigma}^t$	$\boldsymbol{r} \times \boldsymbol{t}$	$\boldsymbol{0}$

Angular momentum density

$$\boldsymbol{l} = \boldsymbol{r} \times \boldsymbol{p} \quad \text{with} \quad \boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0$$

Total angular momentum in $\mathcal{V}$

$$\boldsymbol{L} = \int_{\mathcal{V}} \boldsymbol{l} \, \mathrm{d}v = \int_{\mathcal{V}} \boldsymbol{r} \times \boldsymbol{p} \, \mathrm{d}v = \int_{\mathcal{V}} \rho \, \boldsymbol{r} \times \boldsymbol{v} \, \mathrm{d}v$$

Balance of angular momentum

local statement

$$\dot{\boldsymbol{l}} = \boldsymbol{r} \times [\boldsymbol{b} + \mathrm{div}\, \boldsymbol{\sigma}^t] + \boldsymbol{e} : \boldsymbol{\sigma}$$

global statement

$$\dot{\boldsymbol{L}} = \boldsymbol{M} \quad \text{with} \quad \boldsymbol{M} = \int_{\mathcal{V}} \boldsymbol{r} \times \boldsymbol{b} \, \mathrm{d}v + \int_{\partial\mathcal{V}} \boldsymbol{r} \times \boldsymbol{t} \, \mathrm{d}a$$

Incorporation of balance of linear momentum

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^t$$

General case

Global statement for balance of angular momentum

$$\dot{\boldsymbol{L}} = \boldsymbol{M} \quad \text{with} \quad \boldsymbol{M} = \int_{\mathcal{V}} [\boldsymbol{r} \times \boldsymbol{b} + \boldsymbol{c}] \, \mathrm{d}v + \int_{\partial\mathcal{V}} [\boldsymbol{r} \times \boldsymbol{t} + \boldsymbol{m}] \, da$$

with $\boldsymbol{c}$ body couples: moments distributed in the volume

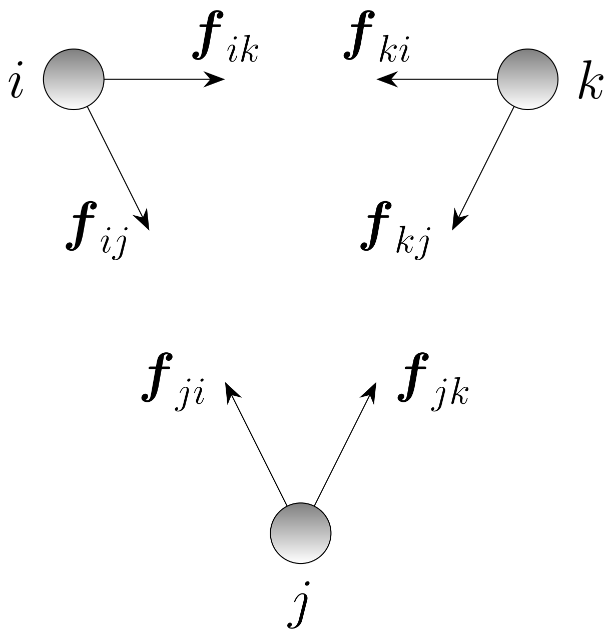
$\boldsymbol{m}$ couple tractions: moments distributed at the surface

Body couples and couple tractions are valid modelling choices,
for an example of non-central interaction forces think of
the Lorentz force on a charged particle in a magnetic field,
but in most of the cases we may neglect these quantities, thus we set

$$\boldsymbol{c} = \boldsymbol{0}, \quad \boldsymbol{m} = \boldsymbol{0}$$

(otherwise for $\boldsymbol{c} \neq \boldsymbol{0}$, $\boldsymbol{m} \neq \boldsymbol{0}$ we talk about a Cosserat or micropolar continuum)

Atomistic picture



Central interaction forces render symmetry of stress

Time change of angular momentum

$$
\begin{aligned}
\dot{\boldsymbol{L}} &= \int_{\mathcal{V}} \dot{\boldsymbol{l}} \, \mathrm{d}v \\
&= \int_{\mathcal{V}} [\dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}}] \, \mathrm{d}v \\
&= \int_{\mathcal{V}} \boldsymbol{r} \times \dot{\boldsymbol{p}} \, \mathrm{d}v
\end{aligned}
$$

with $\dot{\boldsymbol{r}} = \boldsymbol{v}$

$$\boldsymbol{p} = \rho \boldsymbol{v}$$

$$\dot{\boldsymbol{r}} \times \boldsymbol{p} = \boldsymbol{0}$$

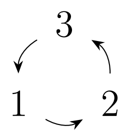
Balance equations

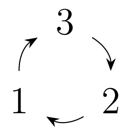
Cross product and permutation tensor

$$\boldsymbol{r} \times \boldsymbol{t} = \boldsymbol{e} : [\boldsymbol{r} \otimes \boldsymbol{t}] = e_{ijk} r_j t_k \, \boldsymbol{e}_i \qquad \text{with} \qquad [e_{ijk} r_j t_k] = \begin{bmatrix} r_2 t_3 - r_3 t_2 \\ r_3 t_1 - r_1 t_3 \\ r_1 t_2 - r_2 t_1 \end{bmatrix}$$

e_{ijk} are the coefficients of the permutation tensor (Levi-Civita symbol) $\boldsymbol{e}$:

$$e_{ijk} = \begin{cases} 1 & & & \text{even permutation of } (1,2,3) \\ -1 & \text{for} & (i,j,k) & \text{odd permutation of } (1,2,3) \\ 0 & & & \text{else} \end{cases}$$





Gauss theorem

$$
\begin{aligned}
\int_{\partial\mathcal{V}} \boldsymbol{r} \times \boldsymbol{t} \; \mathrm{d}a &= \int_{\partial\mathcal{V}} \boldsymbol{r} \times [\boldsymbol{\sigma}^t \cdot \boldsymbol{n}] \; \mathrm{d}a \\
&= \boldsymbol{e} : \int_{\partial\mathcal{V}} \left[\boldsymbol{r} \otimes \boldsymbol{\sigma}^t\right] \cdot \boldsymbol{n} \; \mathrm{d}a \\
&= \boldsymbol{e} : \int_{\mathcal{V}} \operatorname{div}\left(\boldsymbol{r} \otimes \boldsymbol{\sigma}^t\right) \; \mathrm{d}v \\
&= \boldsymbol{e} : \int_{\mathcal{V}} \left[\boldsymbol{\sigma} + \boldsymbol{r} \otimes \operatorname{div} \boldsymbol{\sigma}^t\right] \; \mathrm{d}v \\
&= \int_{\mathcal{V}} \left[\boldsymbol{e} : \boldsymbol{\sigma} + \boldsymbol{r} \times \operatorname{div} \boldsymbol{\sigma}^t\right] \; \mathrm{d}v
\end{aligned}
$$

with $\operatorname{div}(\boldsymbol{r} \otimes \boldsymbol{\sigma}^t) = [r_i \sigma_{kj}]_{,k} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = [r_{i,k}\sigma_{kj} + r_i \sigma_{kj,k}] \boldsymbol{e}_i \otimes \boldsymbol{e}_j$

$$= [\sigma_{ij} + r_i \sigma_{kj,k}] \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{\sigma} + \boldsymbol{r} \otimes \operatorname{div} \boldsymbol{\sigma}^t$$

Balance equations

Local statement balance of angular momentum

$$\boldsymbol{r} \times \dot{\boldsymbol{p}} = \boldsymbol{r} \times [\boldsymbol{b} + \operatorname{div} \boldsymbol{\sigma}^t] + \boldsymbol{e} : \boldsymbol{\sigma}$$

$\dot{\boldsymbol{p}} - [\boldsymbol{b} + \operatorname{div} \boldsymbol{\sigma}^t]$ vanishes if balance of linear momentum holds.

$$\boldsymbol{e} : \boldsymbol{\sigma} = \boldsymbol{0}$$

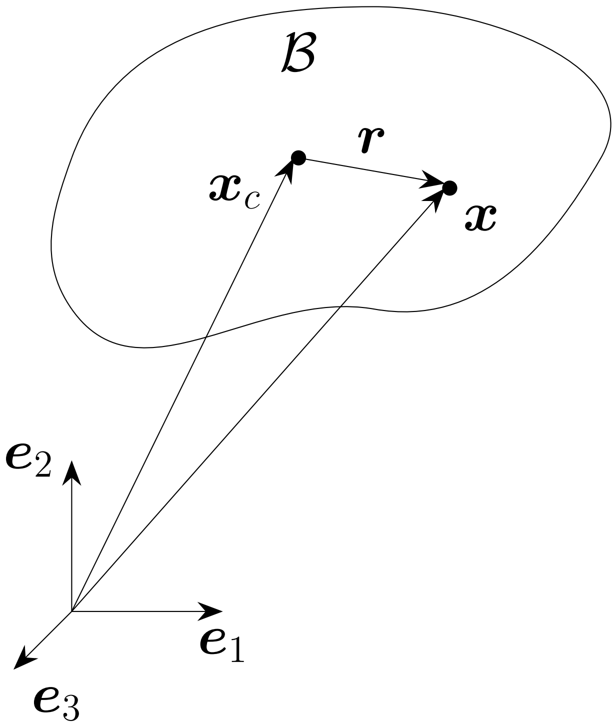
Index notation

$$[e_{ijk}\sigma_{jk}] = \begin{bmatrix} \sigma_{23} - \sigma_{32} \\ \sigma_{31} - \sigma_{13} \\ \sigma_{12} - \sigma_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boldsymbol{\sigma}$ is symmetric $\quad \Longrightarrow \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^t$

Consequence of balance of angular momentum and balance of linear momentum

Example: rigid body



Condition for $\boldsymbol{x}_c =$ center of gravity

$$\int_{\mathcal{B}} \rho \boldsymbol{r} \, \mathrm{d}v = \boldsymbol{0}$$

Rigid body kinematics (Euler)

$$\boldsymbol{v} = \boldsymbol{v}_c + \boldsymbol{\omega} \times \boldsymbol{r}$$

with $\boldsymbol{\omega}$ angular velocity

Total angular momentum of $\mathcal{B}$

$$\begin{aligned}
\boldsymbol{L} &= \int_{\mathcal{B}} \rho \boldsymbol{r} \times \boldsymbol{v} \,\mathrm{d}v \\
&= \int_{\mathcal{B}} \rho \boldsymbol{r} \times [\boldsymbol{v}_c + \boldsymbol{\omega} \times \boldsymbol{r}] \,\mathrm{d}v \\
&= \int_{\mathcal{B}} \rho \boldsymbol{r} \,\mathrm{d}v \times \boldsymbol{v}_c + \int_{\mathcal{B}} \rho \boldsymbol{r} \times [\boldsymbol{\omega} \times \boldsymbol{r}] \,\mathrm{d}v \\
&= \int_{\mathcal{B}} \rho \,[[\boldsymbol{r} \cdot \boldsymbol{r}]\boldsymbol{\omega} - [\boldsymbol{r} \cdot \boldsymbol{\omega}]\boldsymbol{r}] \,\mathrm{d}v \\
&= \int_{\mathcal{B}} \rho \,[[\boldsymbol{r} \cdot \boldsymbol{r}]\boldsymbol{I} - \boldsymbol{r} \otimes \boldsymbol{r}] \,\mathrm{d}v \cdot \boldsymbol{\omega} \\
&= \boldsymbol{\Theta} \cdot \boldsymbol{\omega}
\end{aligned}$$

Tensor of inertia (index notation)

$$\Theta_{ij} = \int_{\mathcal{B}} \rho [r_k r_k \delta_{ij} - r_i r_j] \,\mathrm{d}v$$

Balance equations

4. Energy

β	ς	$\boldsymbol{\Phi}$	Φ	π
e	$\boldsymbol{v} \cdot \boldsymbol{b} + r$	$\boldsymbol{v} \cdot \boldsymbol{\sigma}^t - \boldsymbol{q}$	$\boldsymbol{v} \cdot \boldsymbol{t} + q$	0

Energy density

$$e$$

Total energy in $\mathcal{V}$

$$E = \int_{\mathcal{V}} e \, \mathrm{d}v$$

Balance of energy

local statement

$$\dot{e} = \boldsymbol{\sigma}^t : \nabla \boldsymbol{v} + \boldsymbol{v} \cdot [\mathrm{div}\, \boldsymbol{\sigma}^t + \boldsymbol{b}] + r - \mathrm{div}\, \boldsymbol{q}$$

global statement

$$\dot{E} = \mathcal{E}^{\mathrm{inp}} \quad \text{with} \quad \mathcal{E}^{\mathrm{inp}} = \int_{\mathcal{V}} \varsigma \, \mathrm{d}v + \int_{\partial\mathcal{V}} \Phi \, \mathrm{d}a$$

Incorporation of balances of momentum

$$\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + r - \mathrm{div}\, \boldsymbol{q}$$

Balance of energy (1. Law of thermodynamics)

Total energy

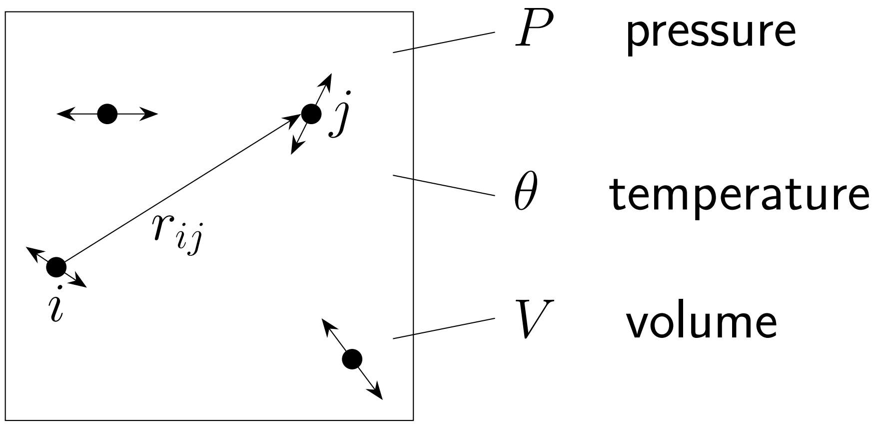
$$E = \int_{\mathcal{V}} e \; \mathrm{d}v = \int_{\mathcal{V}} [k + u] \; \mathrm{d}v = K + U$$

with e total energy density

k kinetic energy density $\quad k = \frac{1}{2}\rho|\boldsymbol{v}|^2 \quad$ with $\boldsymbol{v}$ macroscopic velocity

u internal energy density $\quad$ due to microscopic random velocity of molecules + other energy forms like for example interaction energies $\Phi_{ij} = \Phi(r_{ij})$

Classical PVT thermodynamics



Box corresponds to one continuum point $\boldsymbol{x}$

$$V_{\text{box}} = \sum_i \frac{1}{2} m_i |\tilde{\boldsymbol{v}}_i|^2 + \sum_{i,j} \frac{1}{2} \Phi_{ij}$$

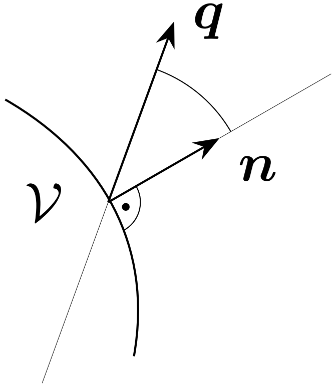
with $\tilde{\boldsymbol{v}}_i$ velocity fluctuations relative to $\boldsymbol{v} = \dfrac{1}{V_{\text{box}}} \sum_i \boldsymbol{v}_i$

$\boldsymbol{r}_{ij} = \boldsymbol{r}_j - \boldsymbol{r}_i$ with $r_{ij} = |\boldsymbol{r}_{ij}|$ distance between molecules

Balance equations

Mechanical power (energy input)

$$\mathcal{P}^{\mathsf{ext}} = \int_{\mathcal{V}} \boldsymbol{v} \cdot \boldsymbol{b} \, \mathsf{d}v + \int_{\partial\mathcal{V}} \boldsymbol{v} \cdot \boldsymbol{t} \, \mathsf{d}a = \int_{\mathcal{V}} [\boldsymbol{v} \cdot \boldsymbol{b} + \mathsf{div}(\boldsymbol{v} \cdot \boldsymbol{\sigma}^t)] \, \mathsf{d}v$$



Thermal power (energy input)

$$q = -\boldsymbol{q} \cdot \boldsymbol{n}$$

$$\mathcal{Q} = \int_{\mathcal{V}} r \, \mathsf{d}v + \int_{\partial\mathcal{V}} q \, \mathsf{d}a = \int_{\mathcal{V}} [r - \mathsf{div}\, \boldsymbol{q}] \, \mathsf{d}v$$

system egoistic view point

diminuitive "less than total differential of something"

Balance of total energy (global statement)

$$\dot{E} = \dot{K} + \dot{U} = \mathcal{P}^{\mathsf{ext}} + \mathcal{Q} = \mathcal{E}^{\mathsf{inp}}$$

Before we elaborate the localized format for the balance of energy, let us consider the local form of the balance of linear momentum

$$\rho\,\dot{\boldsymbol{v}} = [\operatorname{div}\boldsymbol{\sigma}^t + \boldsymbol{b}]$$
$$\int_{\mathcal{V}} \boldsymbol{v}\cdot[\rho\,\dot{\boldsymbol{v}}]\,\mathrm{d}v = \int_{\mathcal{V}} \boldsymbol{v}\cdot[\operatorname{div}\boldsymbol{\sigma}^t + \boldsymbol{b}]\,\mathrm{d}v$$
$$\dot{k} = \boldsymbol{v}\cdot[\rho\,\dot{\boldsymbol{v}}]$$

$$\begin{aligned}\boldsymbol{v}\cdot\operatorname{div}\boldsymbol{\sigma}^t &= \operatorname{div}(\boldsymbol{v}\cdot\boldsymbol{\sigma}^t) - \nabla\boldsymbol{v} : \boldsymbol{\sigma}^t\\ &= \operatorname{div}(\boldsymbol{v}\cdot\boldsymbol{\sigma}^t) - \nabla\boldsymbol{v} : \boldsymbol{\sigma}^{\mathsf{sym}}\\ &= \operatorname{div}(\boldsymbol{v}\cdot\boldsymbol{\sigma}^t) - \nabla^{\mathsf{sym}}\boldsymbol{v} : \boldsymbol{\sigma}\\ &= \operatorname{div}(\boldsymbol{v}\cdot\boldsymbol{\sigma}^t) - \dot{\boldsymbol{\varepsilon}} : \boldsymbol{\sigma}\end{aligned}$$

$$\dot{K} = \underbrace{\int_{\partial\mathcal{V}} \boldsymbol{v}\cdot\boldsymbol{\sigma}^t\cdot\boldsymbol{n}\,\mathrm{d}a + \int_{\mathcal{V}} \boldsymbol{v}\cdot\boldsymbol{b}\,\mathrm{d}v}_{\mathcal{P}^{\mathsf{ext}}} - \underbrace{\int_{\mathcal{V}} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\,\mathrm{d}v}_{\mathcal{P}^{\mathsf{int}}}$$

$$\dot{K} = \mathcal{P}^{\mathsf{ext}} - \mathcal{P}^{\mathsf{int}}$$

consequence of balance of linear momentum

Balance equations

Balance of total energy

$$\dot{K} + \dot{U} = \mathcal{P}^{\mathsf{ext}} + \mathcal{Q}$$

'Balance of kinetic energy'

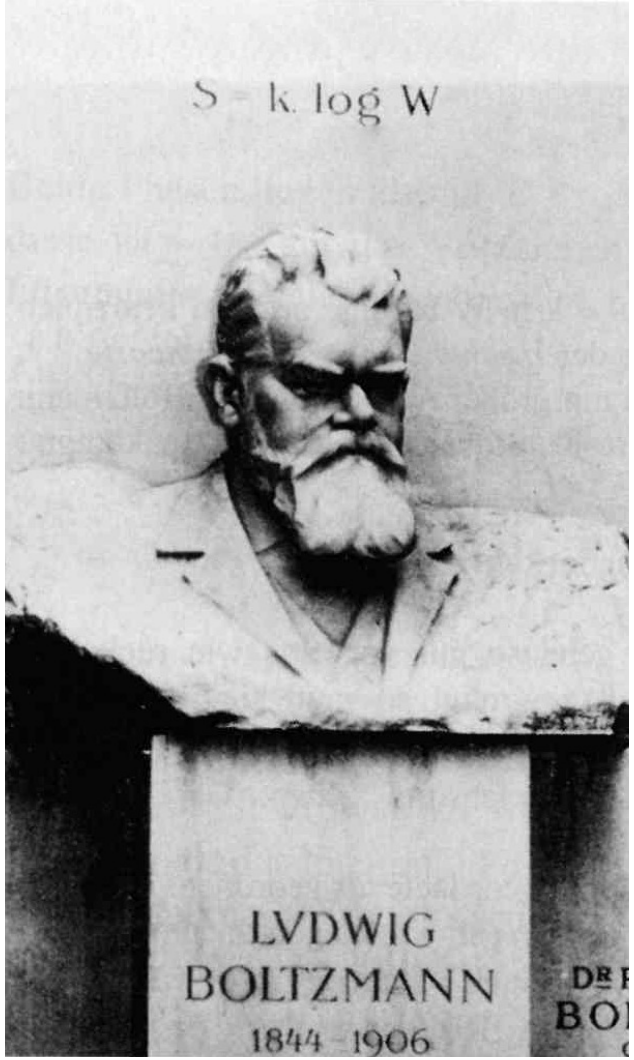
$$\dot{K} = \mathcal{P}^{\mathsf{ext}} - \mathcal{P}^{\mathsf{int}}$$

Balance of internal energy (global statement)

$$\dot{U} = \mathcal{P}^{\mathsf{int}} + \mathcal{Q}$$

Localisation

$$\dot{u} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + r - \operatorname{div} \boldsymbol{q}$$



$$S = k \ \ln w$$

Ludwig Eduard Boltzmann (1844–1906)

Boltzmann postulate of statistical mechanics

Entropy of a macrostate is proportional to the logarithm of its thermodynamical probability

$$S = k \ln w$$

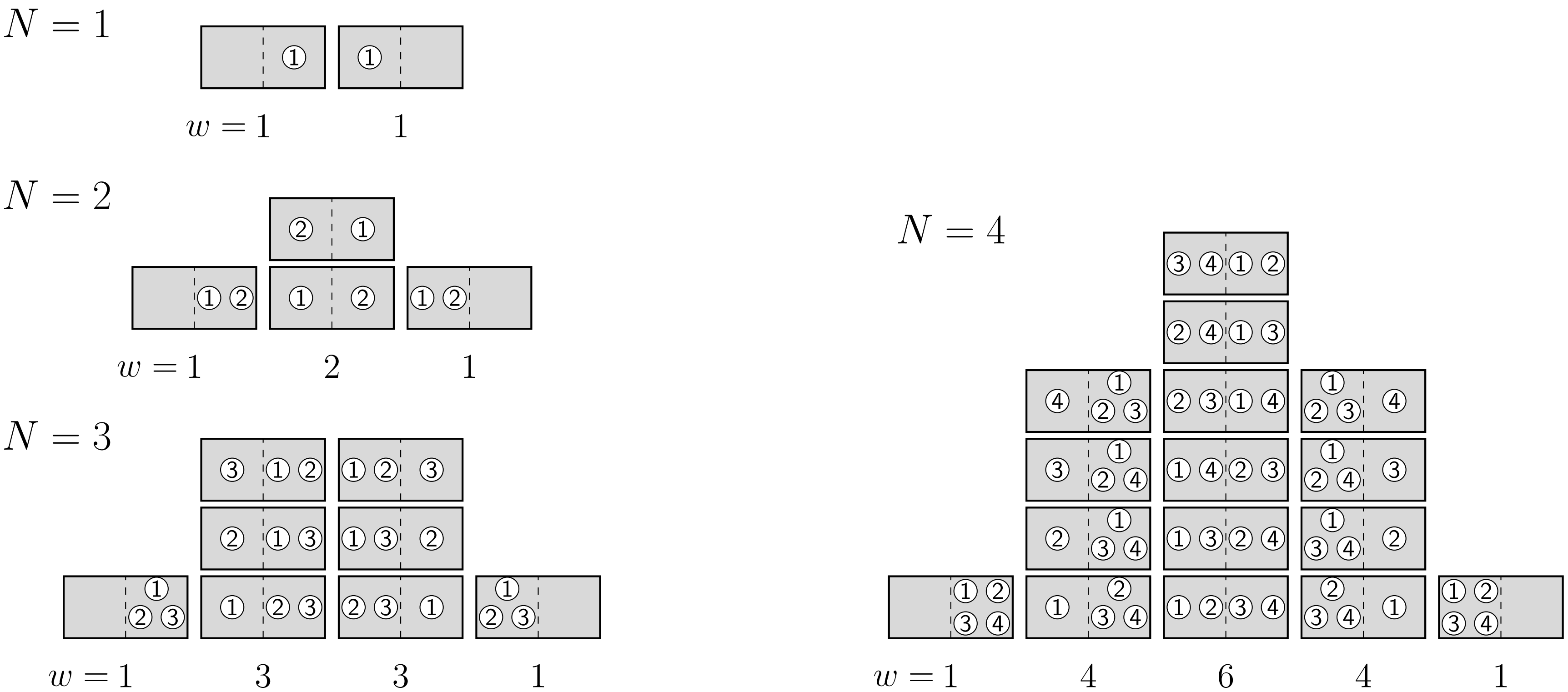
with Boltzmann constant $k = 1.38064852 \cdot 10^{-23}\,\mathrm{J/K}$

Evaluation for an ideal gas (Sommerfeld 1964, Müller 1998)

$$S = kN\left[\ln V + \frac{3}{2}\ln\theta\right] + \mathrm{const.}$$

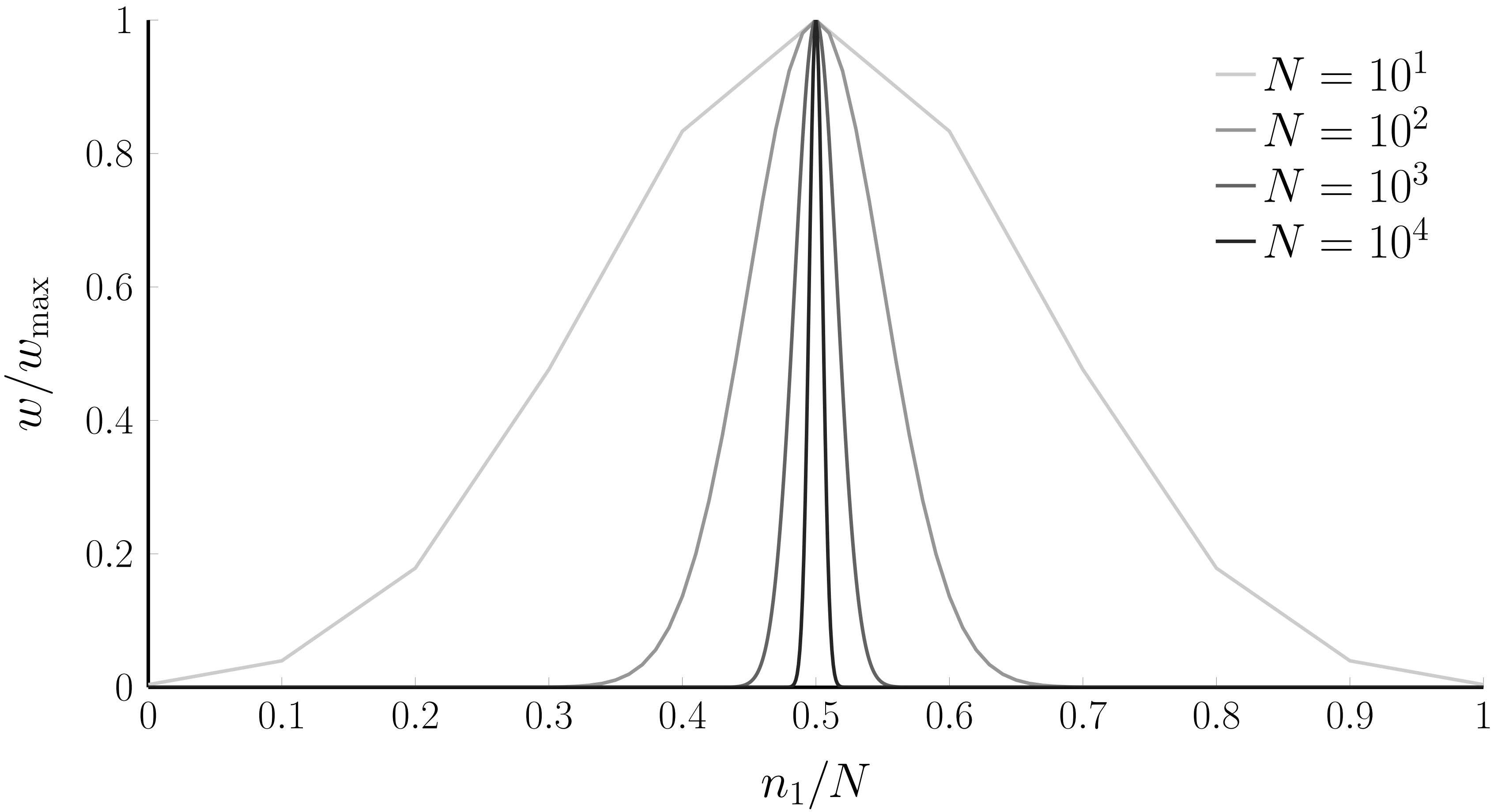
- Macrostate: indistiguishable microstates, uncertainty
- Microstate: possible arrangement of atoms/molecules of a thermodynamical system w.r.t. their positions and momenta
- Number of microstates M^N with number of cells (microscopic conditions) M and number of particles N
- Thermodynamical “probability” w: number of microstates corresponding to a given macrostate

Example: binary cell system



Binomial "distribution" $(M = 2)$

$$w = \frac{N!}{n_1!\, n_2!}, \qquad N = n_1 + n_2$$



Binomial “distribution” $(M = 2)$

$$w = \frac{N!}{n_1!\, n_2!}, \qquad N = n_1 + n_2, \qquad w_{\max} = \frac{N!}{[N/2]!\,[N/2]!}$$

Interlude: Entropy

Balance of internal energy in local form with thermal power $\mathcal{Q}$ and stress power $\wp$

$$\mathcal{Q} = -\wp + \dot{u}$$

Stress power in hydrostatic stress state ($\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathsf{vol}} = p\boldsymbol{I}$)

$$\wp = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = \boldsymbol{\sigma}^{\mathsf{vol}} : \dot{\boldsymbol{\varepsilon}}^{\mathsf{vol}} = p\,\dot{d} = p\,\dot{V}/V_0$$

Ideal gas (sign convention from mechanics)

$$-p = kN\frac{\theta}{V} \quad \text{and} \quad u = u(\theta) \quad \text{with} \quad \frac{\partial u}{\partial \theta} = c_v,\; c_v = \frac{3}{2}\frac{kN}{V_0}$$

Entropy input (thermal power $\mathcal{Q}$ with integrating denominator θ)

$$\frac{\mathcal{Q}}{\theta} = -\frac{p\,\dot{V}}{\theta\,V_0} + \frac{\partial u}{\partial \theta}\frac{\dot{\theta}}{\theta} = \frac{kN}{V_0}\left[\frac{\dot{V}}{V} + \frac{3}{2}\frac{\dot{\theta}}{\theta}\right]$$

Balance of entropy for reversible processes

$$\dot{s} = \frac{\mathcal{Q}}{\theta} \quad \Longrightarrow \quad s = \frac{kN}{V_0}[\ln V + \frac{3}{2}\ln\theta] + \text{const.}$$

Remarks

- Thermal power input $\mathcal{Q}$ does not define total differential
- Entropy input $\mathcal{Q}/\theta$ defines total differential for reversible processes
- Entropy of statistical mechanics $\hat{=}$ Entropy of thermodynamics
- Motivation only valid for ideal gas, but extensions to solids possible
- Temperature of ideal gas (average kinetic energy):

$$3kN\theta = \sum_i m_i |\tilde{\boldsymbol{v}}_i|^2$$

Pressure in ideal gas:

$$-3pV = \sum_i m_i |\tilde{\boldsymbol{v}}_i|^2$$

5. Entropy

β	ς	$\boldsymbol{\Phi}$	Φ	π
s	$\dfrac{r}{\theta}$	$-\dfrac{\boldsymbol{q}}{\theta}$	$\dfrac{q}{\theta}$	$\dfrac{\mathcal{D}}{\theta} \geq 0$

Entropy density

$$s$$

Total entropy in $\mathcal{V}$

$$S = \int_{\mathcal{V}} s \ \mathrm{d}v$$

Balance of entropy

local statement

$$\theta \dot{s} = r - \operatorname{div} \boldsymbol{q} + \boldsymbol{q} \cdot \nabla \ln \theta + \mathcal{D}$$

global statement

$$\dot{S} = \mathcal{S}^{\text{inp}} + \mathcal{S}^{\text{pro}} \quad \text{with} \quad \mathcal{S}^{\text{pro}} \geq 0$$

Incorporation of balance of internal energy

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{u} + \theta \dot{s} - \boldsymbol{q} \cdot \nabla \ln \theta$$

Balance of entropy

$$S = \int_{\mathcal{V}} s \; \mathrm{d}v \qquad \dot{S} = \mathcal{S}^{\mathsf{inp}} + \mathcal{S}^{\mathsf{pro}}$$

Motivated by ideal gases, we assume

$$\mathcal{S}^{\mathsf{inp}} = \int_{\mathcal{V}} \frac{r}{\theta} \mathrm{d}v + \int_{\partial\mathcal{V}} \frac{q}{\theta} \mathrm{d}a = \int_{\mathcal{V}} \left[\frac{r}{\theta} - \mathrm{div}\left(\frac{\boldsymbol{q}}{\theta}\right)\right] \mathrm{d}v$$

$$= \int_{\mathcal{V}} \frac{1}{\theta} \left[r - \mathrm{div}\, \boldsymbol{q} + \boldsymbol{q} \cdot \nabla \ln \theta\right] \mathrm{d}v$$

with $\quad \theta > 0 \quad$ absolute temperature

$$\mathcal{S}^{\mathsf{pro}} = \int_{\mathcal{V}} \frac{\mathcal{D}}{\theta} \mathrm{d}v \quad \text{with} \quad \mathcal{D} \geq 0$$

local

$$\theta \dot{s} = r - \mathrm{div}\, \boldsymbol{q} + \boldsymbol{q} \cdot \nabla \ln \theta + \mathcal{D} \quad \text{with} \quad \mathcal{D} \geq 0$$

Dissipation power density

$$\mathcal{D} = \theta \dot{s} - [r - \operatorname{div} \boldsymbol{q}] - \boldsymbol{q} \cdot \nabla \ln \theta \geq 0$$

Clausius-Duhem inequality (CDI)

We can split dissipation power density into

$$\mathcal{D} = \mathcal{D}^{\text{loc}} + \mathcal{D}^{\text{con}} \geq 0$$

with $\mathcal{D}^{\text{loc}}$ local dissipation power density

$\mathcal{D}^{\text{con}}$ heat conduction dissipation power density

Clausius-Planck inequality

$$\mathcal{D}^{\text{loc}} = \theta \dot{s} - [r - \operatorname{div} \boldsymbol{q}] \geq 0$$

Fourier inequality

$$\mathcal{D}^{\text{con}} = -\boldsymbol{q} \cdot \nabla \ln \theta \geq 0$$

Example for thermodynamically consistent material modelling:

rigid heat conductor

$$\mathcal{D}^{\text{con}} = -\boldsymbol{q} \cdot \nabla \ln \theta \geq 0$$

classical Fourier law

$$\boldsymbol{q} = -\kappa \nabla \theta$$

with κ heat conduction coefficient

$$\mathcal{D}^{\text{con}} = \frac{\kappa}{\theta} |\nabla \theta|^2 \geq 0 \quad \Longrightarrow \quad \kappa > 0$$

Remember balance of internal energy

$$r - \operatorname{div} \boldsymbol{q} = \dot{u} - \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}$$

Clausius-Planck inequality reformulated by balance of internal energy

$$\mathcal{D}^{\mathsf{loc}} = \theta \dot{s} - \dot{u} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \geq 0$$

Good parametrization for $u = u(\boldsymbol{\varepsilon},\, s)$

$$\dot{u} = \frac{\partial u}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial u}{\partial s}\, \dot{s}$$

$$\mathcal{D}^{\mathsf{loc}} = \left[\theta - \frac{\partial u}{\partial s} \right] \dot{s} + \left[\boldsymbol{\sigma} - \frac{\partial u}{\partial \boldsymbol{\varepsilon}} \right] : \dot{\boldsymbol{\varepsilon}} \geq 0$$

Locally reversible case

$$\mathcal{D}^{\mathsf{loc}} = 0 \quad \Longrightarrow \quad \theta = \frac{\partial u}{\partial s} \quad \text{and} \quad \boldsymbol{\sigma} = \frac{\partial u}{\partial \boldsymbol{\varepsilon}} \quad \text{constitutive law}$$

Legendre transformations

Free (Helmholtz) energy

$$\Psi = \Psi(\boldsymbol{\varepsilon}, \theta) = \min_{s}\{u(\boldsymbol{\varepsilon}, s) - \theta s\} = u\big(\boldsymbol{\varepsilon}, s(\boldsymbol{\varepsilon}, \theta)\big) - \theta\, s(\boldsymbol{\varepsilon}, \theta)$$

Free (Gibbs) enthalpy

$$\varphi = \varphi(\boldsymbol{\sigma}, \theta) = \max_{\boldsymbol{\varepsilon}}\{\boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \Psi(\boldsymbol{\varepsilon}, \theta)\} = \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{\sigma}, \theta) - \Psi\big(\boldsymbol{\varepsilon}(\boldsymbol{\sigma}, \theta), \theta\big)$$

Enthalpy

$$h = h(\boldsymbol{\sigma}, s) = \min_{\theta}\{\varphi(\boldsymbol{\sigma}, \theta) - s\theta\} = \varphi\big(\boldsymbol{\sigma}, \theta(\boldsymbol{\sigma}, s)\big) - s\,\theta(\boldsymbol{\sigma}, s)$$

Internal energy

$$u = u(\boldsymbol{\varepsilon}, s) = \max_{\boldsymbol{\sigma}}\{\boldsymbol{\sigma} : \boldsymbol{\varepsilon} - h(\boldsymbol{\sigma}, s)\} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, s) : \boldsymbol{\varepsilon} - h\big(\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, s), s\big)$$

Rate of free energy versus rate of internal energy

$$\dot{\Psi} = \dot{u} - \dot{\theta} s - \theta \dot{s}$$

Clausius-Planck inequality

$$\mathcal{D}^{\mathsf{loc}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} - s\dot{\theta} \geq 0$$

Alternative formats

$$\mathcal{D}^{\mathsf{loc}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{u} + \theta \dot{s} \geq 0$$

$$\mathcal{D}^{\mathsf{loc}} = -\boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}} + \dot{\varphi} - s\dot{\theta} \geq 0$$

$$\mathcal{D}^{\mathsf{loc}} = -\boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}} + \dot{h} + \theta \dot{s} \geq 0$$

Colemann-Noll exploitation procedure for the dissipation inequality

$$\mathcal{D} = \mathcal{D}^{\mathsf{loc}} + \mathcal{D}^{\mathsf{con}} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} - s\dot{\theta} - \boldsymbol{q} \cdot \nabla \ln \theta \geq 0$$

1. Principle of equipresence:

$$\begin{aligned}
\boldsymbol{\sigma} &= \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \theta, \boldsymbol{\gamma}, \alpha) \\
\Psi &= \Psi(\boldsymbol{\varepsilon}, \theta, \boldsymbol{\gamma}, \alpha) \\
s &= s(\boldsymbol{\varepsilon}, \theta, \boldsymbol{\gamma}, \alpha) \\
\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{\varepsilon}, \theta, \boldsymbol{\gamma}, \alpha)
\end{aligned}$$

with $\quad \boldsymbol{\gamma} = \nabla\theta \quad$ and $\quad \alpha \quad$ internal variable(s)

Consequence

$$\dot{\Psi} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \Psi}{\partial \theta}\dot{\theta} + \frac{\partial \Psi}{\partial \boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} + \frac{\partial \Psi}{\partial \alpha}\dot{\alpha}$$

2. Positive dissipation power $\mathcal{D} \geq 0$ should hold for all processes $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ and $\theta = \theta(\boldsymbol{x}, t)$, whereby $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$, θ and $\dot{\theta}$, $\boldsymbol{\gamma}$ and $\dot{\boldsymbol{\gamma}}$ can be controlled independently:

$$\mathcal{D} = \left[\boldsymbol{\sigma} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}\right] : \dot{\boldsymbol{\varepsilon}} - \left[s + \frac{\partial \Psi}{\partial \theta}\right] \dot{\theta} - \boldsymbol{q} \cdot \nabla \ln \theta - \frac{\partial \Psi}{\partial \boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} - \frac{\partial \Psi}{\partial \alpha} \dot{\alpha} \geq 0$$

3. All terms linear in $\dot{\boldsymbol{\varepsilon}}, \dot{\theta}, \dot{\boldsymbol{\gamma}}$ have to vanish.

Why are ε and $\dot{\varepsilon}$ independent?

Choose $\boldsymbol{\varepsilon}_0 = \mathrm{const}$, $\boldsymbol{\kappa}_0 = \mathrm{const}$ arbitrarily prescribed, and

$$\delta(t) \text{ with } \begin{cases} \delta(0) = 0, \\ \dot{\delta}(0) = 1. \end{cases}$$

Set $\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}_0 + \delta(t)\boldsymbol{\kappa}_0$.

$$\Rightarrow \begin{cases} \boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}_0 = \mathrm{const} & \text{prescribed} \\ \dot{\boldsymbol{\varepsilon}}(0) = \boldsymbol{\kappa}_0 = \mathrm{const} & \text{prescribed} \end{cases}$$

$\Rightarrow$ Independent control of $\boldsymbol{\varepsilon}$ and $\dot{\boldsymbol{\varepsilon}}$

Constitutive equations

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}$$

$$s = -\frac{\partial \Psi}{\partial \theta}$$

$$\mathbf{0} = \frac{\partial \Psi}{\partial \boldsymbol{\gamma}} \quad \Longrightarrow \quad \Psi \neq \Psi(\boldsymbol{\gamma})$$

Reduced disspation inequality

$$\mathcal{D}^{\mathsf{red}} = -\frac{\partial \Psi}{\partial \alpha}\dot{\alpha} - \boldsymbol{q} \cdot \nabla \ln \theta \geq 0 \quad \text{with} \quad \boldsymbol{q} = \boldsymbol{q}(\boldsymbol{\varepsilon}, \theta, \boldsymbol{\gamma}, \alpha)$$

e.g. generalized Fourier Law

$$\boldsymbol{q} = \boldsymbol{q}(\theta, \boldsymbol{\gamma}) = -\kappa(\theta)\boldsymbol{\gamma}, \; \kappa(\theta) > 0$$

Constitutive modelling

In order for constitutive equations to be **thermodynamically consistent** (i.e. we do not violate the 2nd Law of thermodynamics) we have to derive $\boldsymbol{\sigma}$ and s from a free energy density $\Psi = \Psi(\boldsymbol{\varepsilon}, \theta, \alpha)$ so that $\boldsymbol{\sigma} = \dfrac{\partial \Psi}{\partial \boldsymbol{\varepsilon}}$ and $s = -\dfrac{\partial \Psi}{\partial \theta}$ and to respect the reduced dissipation inequality

$$\mathcal{D}^{\mathsf{red}} = -\frac{\partial \Psi}{\partial \alpha}\dot{\alpha} - \boldsymbol{q} \cdot \nabla \ln \theta \geq 0$$

Example: $\quad A = -\dfrac{\partial \Psi}{\partial \alpha}\quad$ thermodynamic driving force

$$\mathcal{D}^{\mathsf{red}}_{\mathsf{loc}} = A\dot{\alpha} \geq 0$$

Thermodynamically consistent evolution equation for internal variable(s)

$$\dot{\alpha} = kA, \quad k > 0$$

Special cases:

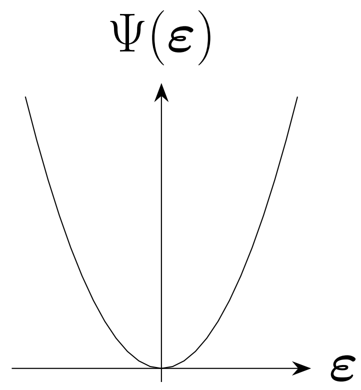
1. Linear isothermal elasticity

Linear elasticity derives from a quadratic stored energy density

$$\Psi = \Psi(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{E} : \boldsymbol{\varepsilon}$$

Stress computation

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{E} : \boldsymbol{\varepsilon}$$



Elasticity tensor (4^{th}-order tensor)

$$\boldsymbol{E} = \frac{\partial^2 \Psi}{\partial \boldsymbol{\varepsilon} \otimes \partial \boldsymbol{\varepsilon}}$$

Complementary stored energy density (Legendre-transformation)

$$\varphi = \varphi(\boldsymbol{\sigma}) = \max_{\boldsymbol{\varepsilon}}\{\boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \Psi(\boldsymbol{\varepsilon})\} = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{C} : \boldsymbol{\sigma} \quad \text{with} \quad \boldsymbol{C} = \boldsymbol{E}^{-1}$$

1. a) Isotropy

$$\Psi = \Psi(\boldsymbol{\varepsilon}) = \Psi(\boldsymbol{Q} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{Q}^t) \quad \forall \quad \boldsymbol{Q} \in \mathsf{SO}(3) = \{\boldsymbol{Q} \mid \boldsymbol{Q}^{-1} = \boldsymbol{Q}^t,\ \det \boldsymbol{Q} = +1\}$$

Representation theorem for isotropic tensor functions

$$\Psi = \Psi(\mathsf{inv}_{\boldsymbol{\varepsilon}})$$

$\mathsf{inv}_{\boldsymbol{\varepsilon}}$ invariants of $\boldsymbol{\varepsilon}$

$$\mathsf{inv}_{\boldsymbol{\varepsilon}} = \begin{cases} \text{principal} & \{I_1,\ I_2,\ I_3\} \\ \text{basic} & \{\bar{I}_1,\ \bar{I}_2,\ \bar{I}_3\} = \{\boldsymbol{\varepsilon} : \boldsymbol{I},\ \boldsymbol{\varepsilon}^2 : \boldsymbol{I},\ \boldsymbol{\varepsilon}^3 : \boldsymbol{I}\} \\ \text{spectral} & \{\lambda_1,\ \lambda_2,\ \lambda_3\} \end{cases}$$

$$\Psi(\boldsymbol{\varepsilon}) \overset{\text{isotropy}}{=} \Psi(\mathsf{inv}_{\boldsymbol{\varepsilon}}) \overset{\text{e.g.}}{=} \Psi(\boldsymbol{\varepsilon} : \boldsymbol{I},\ \boldsymbol{\varepsilon}^2 : \boldsymbol{I},\ \boldsymbol{\varepsilon}^3 : \boldsymbol{I})$$

How to take derivatives of $\bar{I}_m = \boldsymbol{\varepsilon}^m : \boldsymbol{I}$, $m = 1, 2, 3$

$$\frac{\partial \bar{I}_m}{\partial \boldsymbol{\varepsilon}} = m\, \boldsymbol{\varepsilon}^{m-1}$$

$$\frac{\partial \bar{I}_1}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{I}$$

$$\frac{\partial \bar{I}_2}{\partial \boldsymbol{\varepsilon}} = 2\boldsymbol{\varepsilon}$$

$$\frac{\partial \bar{I}_3}{\partial \boldsymbol{\varepsilon}} = 3\boldsymbol{\varepsilon}^2$$

Stress computation

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \Psi}{\partial \bar{I}_1} \boldsymbol{I} + 2\frac{\partial \Psi}{\partial \bar{I}_2} \boldsymbol{\varepsilon} + 3\frac{\partial \Psi}{\partial \bar{I}_3} \boldsymbol{\varepsilon}^2$$
$$= \Phi_1 \boldsymbol{I} + \Phi_2 \boldsymbol{\varepsilon} + \Phi_3 \boldsymbol{\varepsilon}^2$$

Let's assume linearity

$$\Phi_1 = \Phi_{11} \bar{I}_1$$
$$\Phi_2 = \text{const.}$$
$$\Phi_3 = 0$$

Linear stress-strain relation: Hooke's Law ($\Phi_{11} = \lambda,\ \Phi_2 = 2\mu;\quad \mu,\ \lambda$ - Lamé constants)

$$\boldsymbol{\sigma} = \Phi_{11} \bar{I}_1 \boldsymbol{I} + \Phi_2 \boldsymbol{\varepsilon} = \lambda [\boldsymbol{\varepsilon} : \boldsymbol{I}] \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon} = \boldsymbol{E} : \boldsymbol{\varepsilon}$$

Only two constants for isotropic linear elasticity!

Isotropic elasticity tensor (4th order tensor)

$$\boldsymbol{E} \;=\; \lambda \boldsymbol{I} \otimes \boldsymbol{I} + 2\mu \mathbf{1}^{\mathsf{sym}}$$

$$E_{ijkl} \;=\; \lambda \delta_{ij}\delta_{kl} + \mu[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$$

Alternative representation

$$\boldsymbol{E} \;=\; K\,\boldsymbol{I} \otimes \boldsymbol{I} + 2\mu \mathbf{1}^{\mathsf{dev}}$$

$$E_{ijkl} \;=\; K\delta_{ij}\delta_{kl} + \mu\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right]$$

with $K = \lambda + \dfrac{2}{3}\mu$ bulk (compression) modulus and $\mathbf{1}^{\mathsf{dev}} = \mathbf{1}^{\mathsf{sym}} - \dfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I}$

Voigt vector matrix notation

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1123} & E_{1113} & E_{1112} \\ E_{2211} & E_{2222} & E_{2233} & E_{2223} & E_{2213} & E_{2212} \\ E_{3311} & E_{3322} & E_{3333} & E_{3323} & E_{3313} & E_{3312} \\ E_{2311} & E_{2322} & E_{2333} & E_{2323} & E_{2313} & E_{2312} \\ E_{1311} & E_{1322} & E_{1333} & E_{1323} & E_{1313} & E_{1312} \\ E_{1211} & E_{1222} & E_{1233} & E_{1223} & E_{1213} & E_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} = u_{1,1} \\ \varepsilon_{22} = u_{2,2} \\ \varepsilon_{33} = u_{3,3} \\ 2\varepsilon_{23} = \gamma_{23} = u_{2,3} + u_{3,2} \\ 2\varepsilon_{13} = \gamma_{13} = u_{1,3} + u_{3,1} \\ 2\varepsilon_{12} = \gamma_{12} = u_{1,2} + u_{2,1} \end{bmatrix}$$

$\varepsilon_{ij;\, i\neq j}$ tensor shear strains

$\gamma_{ij;\, i\neq j}$ engineering shear strains

Reduction from $9^2 = 81$ by symmetry of strain / stress to $6^2 = 36$ and

by symmetry of second partial derivative of Ψ to 21 and by isotropy to only 2!

Elasticity tensor for Hooke's law in Voigt notation

$$
\boldsymbol{E} \;\hat{=}\; \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}
$$

$$
\hat{=}\; \begin{bmatrix} K+\frac{4}{3}\mu & K-\frac{2}{3}\mu & K-\frac{2}{3}\mu & 0 & 0 & 0 \\ K-\frac{2}{3}\mu & K+\frac{4}{3}\mu & K-\frac{2}{3}\mu & 0 & 0 & 0 \\ K-\frac{2}{3}\mu & K-\frac{2}{3}\mu & K+\frac{4}{3}\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}
$$

$$\Psi(\boldsymbol{\varepsilon}) = \frac{1}{2}\lambda\,[\boldsymbol{\varepsilon} : \boldsymbol{I}]^2 + \mu[\boldsymbol{\varepsilon}^2 : \boldsymbol{I}] = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{E} : \boldsymbol{\varepsilon}$$

$$0 < \Psi(\boldsymbol{\varepsilon}) = \frac{1}{2}K[\boldsymbol{\varepsilon}^{\mathsf{vol}} : \boldsymbol{I}]^2 + \mu\big[[\boldsymbol{\varepsilon}^{\mathsf{dev}}]^2 : \boldsymbol{I}\big] < \infty \quad \forall \boldsymbol{\varepsilon} \neq \boldsymbol{0}$$

$$0 < K < \infty, \quad 0 < \mu < \infty \quad \Longrightarrow \quad -\frac{2}{3}\mu < \lambda < \infty$$

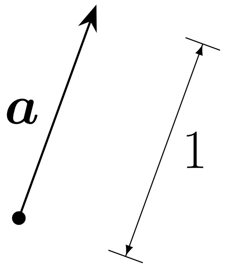
Alternative constants: elasticity modulus E and Poisson ratio ν

$$K = \frac{E}{3[1-2\nu]} \qquad\qquad \mu = \frac{E}{2[1+\nu]}$$

Limits for Poisson ratio

(auxetic limit) $-1 < \nu < \dfrac{1}{2}$ (incompressible limit)

1. b) Transversal isotropy



with $|\boldsymbol{a}| = 1$ fibre direction

$\boldsymbol{\alpha} = \boldsymbol{a} \otimes \boldsymbol{a}$ structure tensor

$$
\begin{aligned}
\Psi &= \Psi(\boldsymbol{\varepsilon}, \pm\boldsymbol{a}) = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}) \\
&= \Psi(\boldsymbol{Q} \cdot \boldsymbol{\varepsilon} \cdot \boldsymbol{Q}^t,\ \boldsymbol{Q} \cdot \boldsymbol{\alpha} \cdot \boldsymbol{Q}^t) \quad \forall \quad \boldsymbol{Q} \in \mathsf{SO}(3)
\end{aligned}
$$

Representation theorem

$$\Psi = \Psi\left(\mathsf{inv}_{\boldsymbol{\varepsilon}},\ \mathsf{inv}_{\boldsymbol{\alpha}}, \mathsf{inv}_{\boldsymbol{\varepsilon}\,\boldsymbol{\alpha}}\right)$$

Structure tensor is idempotent

$$
\begin{aligned}
\mathsf{inv}_{\boldsymbol{\alpha}} &= \{\boldsymbol{\alpha} : \boldsymbol{I}(=1),\ \boldsymbol{\alpha}^2 : \boldsymbol{I}(=1),\ \boldsymbol{\alpha}^3 : \boldsymbol{I}(=1)\} \\
\mathsf{inv}_{\boldsymbol{\varepsilon}\,\boldsymbol{\alpha}} &= \{\boldsymbol{\varepsilon} : \boldsymbol{\alpha},\ \boldsymbol{\varepsilon}^2 : \boldsymbol{\alpha},\ \boldsymbol{\varepsilon} : \boldsymbol{\alpha}^2(= \boldsymbol{\varepsilon} : \boldsymbol{\alpha}),\ \boldsymbol{\varepsilon}^2 : \boldsymbol{\alpha}^2(= \boldsymbol{\varepsilon}^2 : \boldsymbol{\alpha})\} = \{\bar{I}_4,\ \bar{I}_5\}
\end{aligned}
$$

Constitutive modelling

Stress computation

$$\boldsymbol{\sigma} = \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} = \frac{\partial \Psi}{\partial \bar{I}_1} \boldsymbol{I} + 2\frac{\partial \Psi}{\partial \bar{I}_2} \boldsymbol{\varepsilon} + 3\frac{\partial \Psi}{\partial \bar{I}_3} \boldsymbol{\varepsilon}^2 + \frac{\partial \Psi}{\partial \bar{I}_4} \boldsymbol{\alpha} + 2\frac{\partial \Psi}{\partial \bar{I}_5} [\boldsymbol{\varepsilon} \cdot \boldsymbol{\alpha}]^{\mathsf{sym}}$$

$$= \Phi_1 \boldsymbol{I} + \Phi_2 \boldsymbol{\varepsilon} + \Phi_3 \boldsymbol{\varepsilon}^2 + \Phi_4 \boldsymbol{\alpha} + 2\Phi_5 [\boldsymbol{\varepsilon} \cdot \boldsymbol{\alpha}]^{\mathsf{sym}}$$

Let's assume linearity

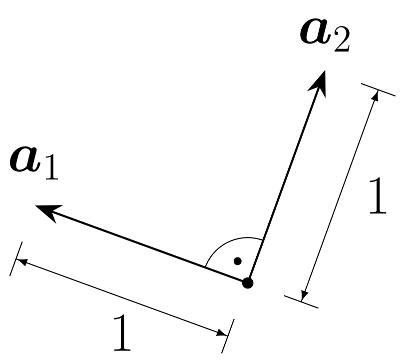
$$\begin{aligned} \Phi_1 &= \Phi_{11} \bar{I}_1 + \Phi_{14} \bar{I}_4 & \Phi_4 &= \Phi_{41} \bar{I}_1 + \Phi_{44} \bar{I}_4 \\ \Phi_2 &= \text{const.} & \Phi_5 &= \text{const.} \\ \Phi_3 &= 0 \end{aligned}$$

Linear stress-strain relation

$$\boldsymbol{\sigma} = [\Phi_{11} \bar{I}_1 + \Phi_{14} \bar{I}_4] \boldsymbol{I} + \Phi_2 \boldsymbol{\varepsilon} + [\Phi_{41} \bar{I}_1 + \Phi_{44} \bar{I}_4] \boldsymbol{\alpha} + 2\Phi_5 [\boldsymbol{\varepsilon} \cdot \boldsymbol{\alpha}]^{\mathsf{sym}} = \boldsymbol{E} : \boldsymbol{\varepsilon}$$

Major symmetry of $\boldsymbol{E}$: only five constants $(\Phi_{14} = \Phi_{41})$ for transversal isotropic linear elasticity!

1. c) Orthotropy



with $|\boldsymbol{a}_{1,2}| = 1$ fibre direction

$\boldsymbol{\alpha}_{1,2} = \boldsymbol{a}_{1,2} \otimes \boldsymbol{a}_{1,2}$ structure tensor

$$\Psi = \Psi(\boldsymbol{\varepsilon}, \pm\boldsymbol{a}_1, \pm\boldsymbol{a}_2) = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2)$$
$$= \Psi(\boldsymbol{Q}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{Q}^t,\ \boldsymbol{Q}\cdot\boldsymbol{\alpha}_1\cdot\boldsymbol{Q}^t,\ \boldsymbol{Q}\cdot\boldsymbol{\alpha}_2\cdot\boldsymbol{Q}^t) \quad \forall \quad \boldsymbol{Q} \in \mathsf{SO}(3)$$

Representation theorem

$$\Psi = \Psi\left(\mathsf{inv}_{\boldsymbol{\varepsilon}},\ \mathsf{inv}_{\boldsymbol{\alpha}_1},\ \mathsf{inv}_{\boldsymbol{\alpha}_2},\ \mathsf{inv}_{\boldsymbol{\varepsilon}\,\boldsymbol{\alpha}_1},\ \mathsf{inv}_{\boldsymbol{\varepsilon}\,\boldsymbol{\alpha}_2}, \mathsf{inv}_{\boldsymbol{\alpha}_1\,\boldsymbol{\alpha}_2}\right)$$

Structure tensors are idempotent

$$\mathsf{inv}_{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2} = \{\boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2 = 0,\ \boldsymbol{\alpha}_1^2 : \boldsymbol{\alpha}_2 = 0,\ \boldsymbol{\alpha}_1 : \boldsymbol{\alpha}_2^2 = 0,\ \boldsymbol{\alpha}_1^2 : \boldsymbol{\alpha}_2^2 = 0\}$$

Stress computation

$$\boldsymbol{\sigma} = \frac{\partial\Psi}{\partial\bar{I}_1}\boldsymbol{I} + 2\frac{\partial\Psi}{\partial\bar{I}_2}\boldsymbol{\varepsilon} + 3\frac{\partial\Psi}{\partial\bar{I}_3}\boldsymbol{\varepsilon}^2 + \frac{\partial\Psi}{\partial\bar{I}_4}\boldsymbol{\alpha}_1 + 2\frac{\partial\Psi}{\partial\bar{I}_5}[\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha}_1]^{\mathsf{sym}} + \frac{\partial\Psi}{\partial\bar{I}_6}\boldsymbol{\alpha}_2 + 2\frac{\partial\Psi}{\partial\bar{I}_7}[\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha}_2]^{\mathsf{sym}}$$

$$= \Phi_1\boldsymbol{I} + \Phi_2\boldsymbol{\varepsilon} + \Phi_3\boldsymbol{\varepsilon}^2 + \Phi_4\boldsymbol{\alpha}_1 + 2\Phi_5[\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha}_1]^{\mathsf{sym}} + \Phi_6\boldsymbol{\alpha}_2 + 2\Phi_7[\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha}_2]^{\mathsf{sym}}$$

Let's assume linearity

$$\Phi_1 = \Phi_{11}\bar{I}_1 + \Phi_{14}\bar{I}_4 + \Phi_{16}\bar{I}_6 \qquad \Phi_4 = \Phi_{41}\bar{I}_1 + \Phi_{44}\bar{I}_4 + \Phi_{46}\bar{I}_6 \qquad \Phi_6 = \Phi_{61}\bar{I}_1 + \Phi_{64}\bar{I}_4 + \Phi_{66}\bar{I}_6$$

$$\Phi_2 = \text{const.} \qquad \Phi_5 = \text{const.} \qquad \Phi_7 = \text{const.}$$

$$\Phi_3 = 0$$

Linear stress-strain relation

$$\boldsymbol{\sigma} = [\Phi_{11}\bar{I}_1 + \Phi_{14}\bar{I}_4 + \Phi_{16}\bar{I}_6]\boldsymbol{I} + \Phi_2\boldsymbol{\varepsilon} + [\Phi_{41}\bar{I}_1 + \Phi_{44}\bar{I}_4 + \Phi_{46}\bar{I}_6]\,\boldsymbol{\alpha}_1 +$$

$$2\Phi_5[\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha}_1]^{\mathsf{sym}} + [\Phi_{61}\bar{I}_1 + \Phi_{64}\bar{I}_4 + \Phi_{66}\bar{I}_6]\,\boldsymbol{\alpha}_2 + 2\Phi_7[\boldsymbol{\varepsilon}\cdot\boldsymbol{\alpha}_2]^{\mathsf{sym}} = \boldsymbol{E}:\boldsymbol{\varepsilon}$$

Major symmetry of $\boldsymbol{E}$: only nine constants $(\Phi_{14} = \Phi_{41},\ \Phi_{16} = \Phi_{61},\ \Phi_{46} = \Phi_{64})$ for orthotropic linear elasticity!

2. Linear thermoelasticity

Remember: 1. and 2. Law render entropy evolution equation

$$\theta \dot{s} = r - \mathsf{div}\, \boldsymbol{q} + \mathcal{D}^{\mathsf{loc}}$$

Exploitation of dissipation inequality

$$s = -\frac{\partial \Psi}{\partial \theta}$$

Thermoelasticity

$$\Psi = \Psi(\boldsymbol{\varepsilon},\, \theta) \quad \text{and} \quad \mathcal{D}^{\mathsf{loc}} = 0$$

Entropy rate

$$\dot{s} = -\frac{\partial^2 \Psi}{\partial \theta \partial \theta}\, \dot{\theta} - \frac{\partial^2 \Psi}{\partial \theta \partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} = \frac{c_v}{\theta}\, \dot{\theta} + \boldsymbol{\beta} : \dot{\boldsymbol{\varepsilon}}$$

Constitutive modelling

Isometric heat capacity and thermal stress

$$c_v = -\theta \frac{\partial^2 \Psi}{\partial \theta \, \partial \theta} \qquad \boldsymbol{\beta} = -\frac{\partial \boldsymbol{\sigma}}{\partial \theta} = -\frac{\partial^2 \Psi}{\partial \theta \partial \boldsymbol{\varepsilon}}$$

Temperature evolution equation

$$c_v \dot{\theta} = r - \operatorname{div} \boldsymbol{q} - \theta \boldsymbol{\beta} : \dot{\boldsymbol{\varepsilon}}$$

$\theta \boldsymbol{\beta} : \dot{\boldsymbol{\varepsilon}}$ thermoelastic heating / cooling (Gough-Joule effect)

Discussion:

adiabatic process: $c_v \dot{\theta} = -\theta \boldsymbol{\beta} : \dot{\boldsymbol{\varepsilon}}$

isometric process: $c_v \dot{\theta} = r - \operatorname{div} \boldsymbol{q}$

(rigid heat conductor)

Free energy density: separation ansatz

$$\Psi = \Psi(\boldsymbol{\varepsilon},\, \theta) = W(\boldsymbol{\varepsilon}) + T(\theta) + M(\boldsymbol{\varepsilon},\, \theta)$$

Assumption: $c_v =$ const.

$$\frac{\partial^2 \Psi}{\partial \theta \, \partial \theta} = -\frac{c_v}{\theta}$$

Integration

$$\frac{\partial \Psi}{\partial \theta} = -c_v \ln \frac{\theta}{\theta_0} - G(\boldsymbol{\varepsilon})$$

Free energy density

$$\Psi = W(\boldsymbol{\varepsilon}) + c_v \left[[\theta - \theta_0] - \theta \ln \frac{\theta}{\theta_0} \right] - [\theta - \theta_0]\, G(\boldsymbol{\varepsilon})$$

Stored energy contribution

$$W(\boldsymbol{\varepsilon}) = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{E} : \boldsymbol{\varepsilon}$$

Capacitive contribution

$$T(\theta) = c_v \left[[\theta - \theta_0] - \theta \ln \frac{\theta}{\theta_0} \right]$$

$$T(\theta) \overset{\text{quadratic expansion}}{=} -\frac{c_v}{\theta_0}\frac{[\theta - \theta_0]^2}{2} = -\frac{c_v}{\theta_0}\frac{\vartheta^2}{2}$$

$$\left(\text{for } x > \tfrac{1}{2} : \quad \ln x = \frac{x-1}{x} + \frac{[x-1]^2}{2x^2} + \ldots\right)$$

Coupling contribution

$$M(\boldsymbol{\varepsilon}, \theta) = -[\theta - \theta_0]\, G(\boldsymbol{\varepsilon})$$

Constitutive modelling

Quadratic free energy

$$\Psi = \frac{1}{2}\boldsymbol{\varepsilon} : \boldsymbol{E} : \boldsymbol{\varepsilon} - \frac{c_v}{\theta_0}\frac{\vartheta^2}{2} - \vartheta\boldsymbol{\beta} : \boldsymbol{\varepsilon} \qquad \text{with} \qquad \vartheta = \theta - \theta_0$$

Stress

$$\boldsymbol{\sigma} = \frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}} = \boldsymbol{E} : \boldsymbol{\varepsilon} - \vartheta\boldsymbol{\beta} = \boldsymbol{E} : [\boldsymbol{\varepsilon} - \vartheta\boldsymbol{\alpha}] = \boldsymbol{E} : \boldsymbol{\varepsilon}_e$$

with heat expansion $\boldsymbol{\alpha} = \boldsymbol{E}^{-1} : \boldsymbol{\beta}$, thermal strain $\boldsymbol{\varepsilon}_\vartheta = \vartheta\boldsymbol{\alpha}$, and elastic strain $\boldsymbol{\varepsilon}_e = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_\vartheta$

Entropy

$$s = -\frac{\partial\Psi}{\partial\theta} = \frac{c_v}{\theta_0}\vartheta + \boldsymbol{\beta} : \boldsymbol{\varepsilon}$$

Linearised temperature evolution equation

$$c_v\dot{\vartheta} = r - \operatorname{div}\boldsymbol{q} - \theta_0\,\boldsymbol{\beta} : \dot{\boldsymbol{\varepsilon}}$$

Special cases

Isotropy

$$\boldsymbol{\alpha} \overset{\text{iso}}{=} \alpha\,\boldsymbol{I} \qquad \text{with } \beta \overset{\text{iso}}{=} 3K\alpha \text{ and } \alpha \text{ isotropic heat expansion coefficient}$$

$$\boldsymbol{\beta} \overset{\text{iso}}{=} \beta\,\boldsymbol{I} \qquad \Longrightarrow \qquad \boldsymbol{\beta} : \dot{\boldsymbol{\varepsilon}} = \beta\,\dot{\boldsymbol{\varepsilon}} : \boldsymbol{I} = \beta\dot{d} \quad (d = \boldsymbol{\varepsilon} : \boldsymbol{I}), \quad \dot{\boldsymbol{\varepsilon}}_\vartheta = \dot{\vartheta}\alpha\boldsymbol{I}$$

- isotropy + adiabatic process $(r - \operatorname{div}\boldsymbol{q} = 0)$

  $c_v\dot{\vartheta} = -\theta_0\beta\dot{d}$ Gough-Joule effect

- isotropy + isometric process $(\dot{d} = 0)$

  $c_v\dot{\vartheta} = r - \operatorname{div}\boldsymbol{q}$

- isotropy + free thermal expansion $(\dot{d} = 3\alpha\dot{\vartheta})$

  $c_p\dot{\vartheta} = r - \operatorname{div}\boldsymbol{q}$ with $c_p = c_v + 3\theta_0\beta\alpha$

Abstract toy problem

Find solution $u : (0, \ell) \to \mathbb{R}$ for differential equation

$$-[h(x; u, u')]' + f(x; u, u') = 0 \quad \text{in } (0, \ell)$$

with boundary conditions

$$u(0) = u_0, \quad u(\ell) = u_\ell$$

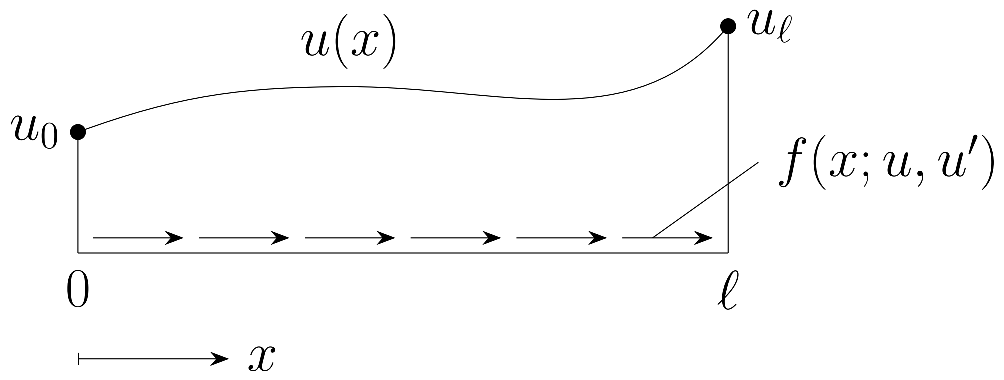
Examples:

- linear elastic bar

$$[EA(x)u'(x)]' + b(x) = 0$$

- stationary heat equation

$$[\kappa(x)\theta'(x)]' + r(x) = 0$$



(D) differential equation: strong form

$$r(u, u', u'') = f(u, u') - [h(u, u')]' = 0 \quad \text{in } (0, \ell)$$

with boundary conditions $\quad u(0) = u_0, \quad u(\ell) = u_\ell$

(M) minimization problem

$$u = \underset{w \in \mathcal{W}}{\text{argmin}}\, \Pi(w) \quad \text{with functional} \quad \Pi(w) = \int_0^\ell \pi(w, w')\, \mathrm{d}x$$

$$\mathcal{W} = \left\{ w \in H^1(0, \ell) \mid w(0) = u_0,\ w(\ell) = u_\ell \right\} \quad \text{(space of trial functions)}$$

(V) variational formulation: weak form

$$u \in \mathcal{W} \quad \text{with} \quad \int_0^\ell [f(u, u')\, v + g(u, u')\, v']\, \mathrm{d}x = 0 \quad \forall v \in \mathcal{V}$$

$$\mathcal{V} = \left\{ v \in H^1(0, \ell) \mid v(0) = v(\ell) = 0 \right\} \quad \text{(space of test functions)}$$

Note: Here we assume $g(\bullet, u') \sim u'$ and $\pi(\bullet, u') \sim |u'|^2$.

Sobolev spaces

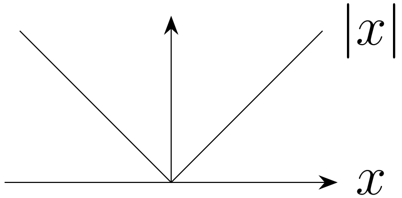
$$H^m(a,b) = \{v : (a,b) \to \mathbb{R} \mid \exists \text{ 'weak' derivatives } v', \dots, v^{(m)} \text{ and } \sum_{k=0}^{m} \|v^{(k)}\|_{L^2(a,b)} < \infty\}$$

where:

- L^2-norm: $\quad \|v\|_{L^2(a,b)} = \left[\int_a^b |v|^2 \, \mathrm{d}x\right]^{\frac{1}{2}}$
- v' is weak derivative of v if

$$\int_a^b v'\phi \, \mathrm{d}x = -\int_a^b v\,\phi' \, \mathrm{d}x \qquad \forall \phi \in C^1(a,b) \text{ with } \phi(a) = \phi(b) = 0$$

Example: $v(x) = |x|$



$$v \in H^1(-1,1) \text{ with } v'(x) = \operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

but $v \notin C^1(-1,1)$!

$$u \text{ solution of (M)} \implies u \text{ solution of (V)?}$$

$$\Pi(u) \leq \Pi(u+\alpha\, v) \qquad \forall \alpha \in \mathbb{R},\ v \in \mathcal{V} \quad (\text{i.e. } u+\alpha\, v \in \mathcal{W})$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\Pi(u+\alpha\, v)\Big|_{\alpha=0} = \int_0^\ell \left[\frac{\partial \pi(u,u')}{\partial u}\, v + \frac{\partial \pi(u,u')}{\partial u'}\, v'\right] \mathrm{d}x = 0 \quad \forall v \in \mathcal{V}$$

Any solution of (M) is also a solution of (V)!

$$u \text{ solution of (V)} \implies u \text{ solution of (M)?}$$

Condition: $\exists\ \pi(u,u')$ such that $f(u,u') = \dfrac{\partial \pi}{\partial u}$ and $g(u,u') = \dfrac{\partial \pi}{\partial u'}$

u solution of (D) $\implies$ u solution of (V)?

$$\int_0^\ell r(u, u', u'')\, v \,\mathsf{d}x = 0 \quad \forall v \in \mathcal{V}$$

$$-\int_0^\ell [h(u,u')]'\, v \;\mathsf{d}x = \int_0^\ell h(u,u')\, v' \,\mathsf{d}x - \Big[h(u,u')\, v\Big]_0^\ell \quad \text{with} \quad v(0) = v(\ell) = 0$$

$$\int_0^\ell \big[f(u,u')\, v + h(u,u')\, v'\big] \,\mathsf{d}x = 0 \quad \forall v \in \mathcal{V}$$

Any solution of (D) is also a solution of (V)!

u solution of (V) $\implies$ u solution of (D)?

Condition: $\exists\, [g(u,u')]'$ such that $[g(u,u')]' = [h(u,u')]'$

In particular, this requires $\exists\, u''$!

Conclusion

$$(\mathsf{M}) \;\Rightarrow\; (\mathsf{V})$$

$$\text{if } \exists\,\pi \quad (\mathsf{M}) \;\Leftrightarrow\; (\mathsf{V})$$

$$(\mathsf{V}) \;\Leftarrow\; (\mathsf{D})$$

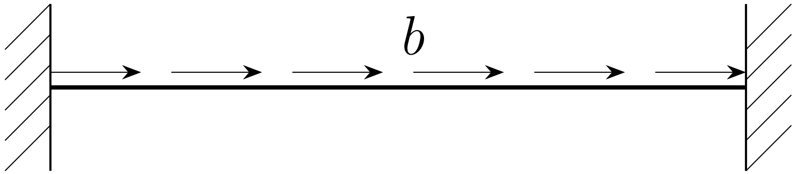
$$(\mathsf{V}) \;\Leftrightarrow\; (\mathsf{D}) \quad \text{if } \exists\, g'$$

If $\exists\,\pi$ and if $\exists\,g'$, all three formulations are equivalent!

$$(\mathsf{M}) \Leftrightarrow (\mathsf{V}) \Leftrightarrow (\mathsf{D})$$

In general (V) requires weaker regularity than (D): weak form (V), strong form (D)

Example: linear elastic bar



(D) $\qquad N' + b = 0 \qquad$ in $(0, \ell)$ $\qquad$ ODE for stress resultant

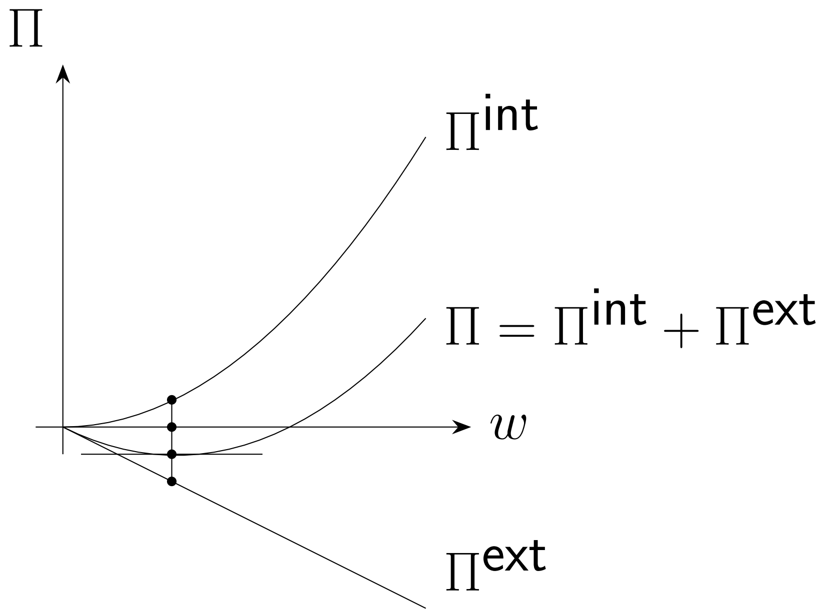
(M) $\qquad \Pi(w) = \int_0^\ell \left[\frac{1}{2} [E A\, w']\, w' - b\, w \right] \mathrm{d}x \quad \rightarrow \quad \min_{w \in \mathcal{W}}$

(V) $\qquad \int_0^\ell N\, v'\, \mathrm{d}x = \int_0^\ell b\, v\, \mathrm{d}x \qquad \forall v \in \mathcal{V}$ $\qquad$ Principle of virtual work

With $N = E A\, u'$ all three cases are equivalent!

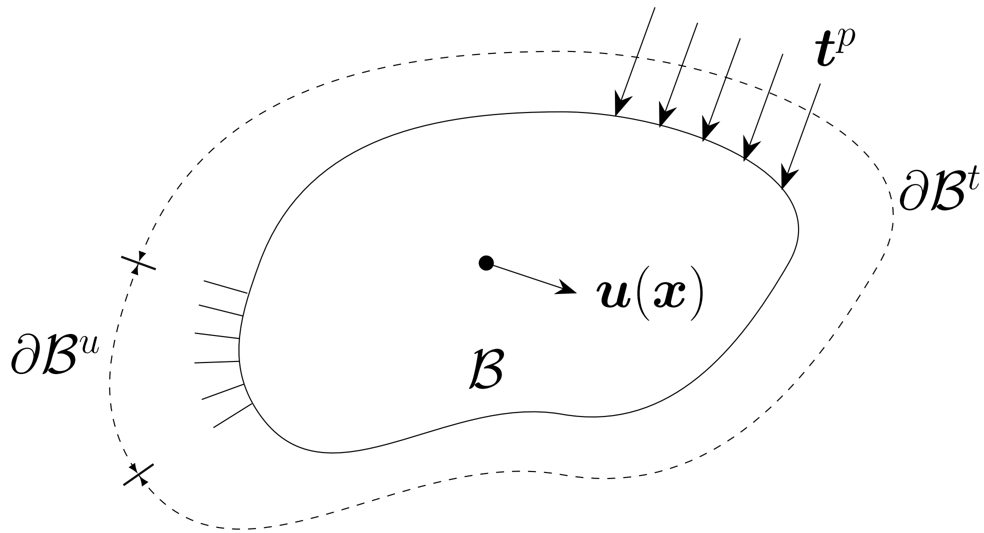
Potential energy at equilibrium

$$\Pi(u) = \int_0^\ell \left[\frac{1}{2}[E A\, u']\, u' - b\, u\right] \mathsf{d}x \overset{\text{(D)}}{=} -\int_0^\ell \frac{1}{2}\, b\, u\, \mathsf{d}x$$



At the equilibrium solution the potential energy of a linear elastic system takes a minimum with value $\frac{1}{2}$ of the external potential.

Linear elasticity



Find solution $\boldsymbol{u} : \mathcal{B} \to \mathbb{R}^3$ for system of partial differential equations

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \quad \text{in} \quad \mathcal{B}$$

with boundary conditions

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{u}^p \quad && \text{on} \quad \partial\mathcal{B}^u \\ \boldsymbol{\sigma}^t \cdot \boldsymbol{n} &= \boldsymbol{t}^p \quad && \text{on} \quad \partial\mathcal{B}^t \end{aligned}$$

(D) differential equation: strong form

$$\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \quad \text{in} \quad \mathcal{B}$$

and $\quad \boldsymbol{u} = \boldsymbol{u}^p \quad \text{on} \quad \partial\mathcal{B}^u, \quad \boldsymbol{\sigma}^t \cdot \boldsymbol{n} = \boldsymbol{t}^p \quad \text{on} \quad \partial\mathcal{B}^t$

(M) minimization problem

$$\boldsymbol{u} = \underset{\boldsymbol{w}\in\mathcal{W}}{\operatorname{argmin}}\, \Pi(\boldsymbol{w})$$

$$\Pi(\boldsymbol{w}) = \Pi^{\text{int}}(\boldsymbol{w}) + \Pi^{\text{ext}}(\boldsymbol{w}) = \int_{\mathcal{B}} W\left(\nabla^{\text{sym}}\boldsymbol{w}\right)\,\mathrm{d}v - \int_{\mathcal{B}} \boldsymbol{w}\cdot\boldsymbol{b}\,\mathrm{d}v - \int_{\partial\mathcal{B}^t} \boldsymbol{w}\cdot\boldsymbol{t}^p\,\mathrm{d}a$$

$$\mathcal{W} = \left\{\boldsymbol{w} \in H^1(\mathcal{B})^3 \mid \boldsymbol{w} = \boldsymbol{u}^p \text{ on } \partial\mathcal{B}^u\right\}$$

(V) variational formulation: weak form

$$\boldsymbol{u} \in \mathcal{W} \quad \text{with} \quad \int_{\mathcal{B}} \boldsymbol{\sigma} : \nabla^{\text{sym}}\boldsymbol{v}\,\mathrm{d}v = \int_{\mathcal{B}} \boldsymbol{v}\cdot\boldsymbol{b}\,\mathrm{d}v + \int_{\partial\mathcal{B}^t} \boldsymbol{v}\cdot\boldsymbol{t}^p\,\mathrm{d}a \quad \forall\boldsymbol{v} \in \mathcal{V}$$

$$\mathcal{V} = \left\{\boldsymbol{v} \in H^1(\mathcal{B})^3 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\mathcal{B}^u\right\}$$

$$\boldsymbol{u} \text{ solution of (M)} \implies \boldsymbol{u} \text{ solution of (V)?}$$

$$\Pi(\boldsymbol{u}) \leq \Pi(\boldsymbol{u} + \alpha\, \boldsymbol{v}) \qquad \forall \alpha \in \mathbb{R},\ \boldsymbol{v} \in \mathcal{V} \quad (\text{i.e. } \boldsymbol{u} + \alpha\, \boldsymbol{v} \in \mathcal{W})$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \Pi(\boldsymbol{u} + \alpha\, \boldsymbol{v})\Big|_{\alpha=0} = \int_{\mathcal{B}} \left[\frac{\partial W}{\partial \nabla^{\mathsf{sym}} \boldsymbol{u}} : \nabla^{\mathsf{sym}} \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{b} \right] \mathrm{d}v - \int_{\partial \mathcal{B}^t} \boldsymbol{v} \cdot \boldsymbol{t}^p \, \mathrm{d}a = 0 \quad \forall \boldsymbol{v} \in \mathcal{V}$$

Any solution of (M) is also a solution of (V)!

$$\boldsymbol{u} \text{ solution of (V)} \implies \boldsymbol{u} \text{ solution of (M)?}$$

Condition: $\exists\ W(\nabla^{\mathsf{sym}}\, \boldsymbol{u})$ such that $\boldsymbol{\sigma} = \dfrac{\partial W}{\partial \nabla^{\mathsf{sym}} \boldsymbol{u}}$

$$\boldsymbol{u} \text{ solution of (D)} \implies \boldsymbol{u} \text{ solution of (V)?}$$

$$\int_{\mathcal{B}} \boldsymbol{v} \cdot [\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{b}] \, \mathrm{d}v = 0 \quad \text{and} \quad \int_{\partial \mathcal{B}^t} \boldsymbol{v} \cdot [\boldsymbol{\sigma} \cdot \boldsymbol{n} - \boldsymbol{t}^p] \, \mathrm{d}a = 0 \quad \forall \boldsymbol{v} \in \mathcal{V}$$

$$\int_{\mathcal{B}} \boldsymbol{v} \cdot \operatorname{div} \boldsymbol{\sigma} \, \mathrm{d}v = \int_{\partial \mathcal{B}^t} \boldsymbol{v} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} \, \mathrm{d}a - \int_{\mathcal{B}} \boldsymbol{\sigma} : \nabla^{\text{sym}} \boldsymbol{v} \, \mathrm{d}v \quad \text{with} \quad \boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \partial \mathcal{B}^u$$

$$\int_{\mathcal{B}} \boldsymbol{\sigma} : \nabla^{\text{sym}} \boldsymbol{v} \, \mathrm{d}v = \int_{\mathcal{B}} \boldsymbol{v} \cdot \boldsymbol{b} \, \mathrm{d}v + \int_{\partial \mathcal{B}^t} \boldsymbol{v} \cdot \boldsymbol{t}^p \, \mathrm{d}a = 0 \quad \forall \boldsymbol{v} \in \mathcal{V}$$

Any solution of (D) is also a solution of (V)!

$$\boldsymbol{u} \text{ solution of (V)} \implies \boldsymbol{u} \text{ solution of (D)?}$$

Condition: $\exists \ \operatorname{div} \boldsymbol{\sigma}$

FEM generates only an approximation to the true solution!

1. Strong form (D)

Mechanical problem

$$\rho \dot{\boldsymbol{v}} = \operatorname{div} \boldsymbol{\sigma} + \boldsymbol{b} \quad \text{in} \quad \mathcal{B} \qquad \text{and} \qquad \begin{aligned} \boldsymbol{u} &= \boldsymbol{u}^p && \text{on} \quad \partial\mathcal{B}^u \\ \boldsymbol{\sigma}^t \cdot \boldsymbol{n} &= \boldsymbol{t}^p && \text{on} \quad \partial\mathcal{B}^t \end{aligned}$$

Thermal problem

$$c_v \dot{\vartheta} = R - \operatorname{div} \boldsymbol{q} \quad \text{in} \quad \mathcal{B} \qquad \text{and} \qquad \begin{aligned} \vartheta &= \vartheta^p && \text{on} \quad \partial\mathcal{B}^\vartheta \\ -\boldsymbol{q} \cdot \boldsymbol{n} &= q^p && \text{on} \quad \partial\mathcal{B}^q \end{aligned}$$

with $\quad R = r - \theta_0 \boldsymbol{\beta} : \dot{\boldsymbol{\varepsilon}}$ generalised heat source

2. Variational formulation (V)

Mechanical problem

$$\int_{\mathcal{B}} \rho\,\delta\boldsymbol{u}\cdot\dot{\boldsymbol{v}}\,\mathrm{d}v \;=\; \int_{\mathcal{B}} \delta\boldsymbol{u}\cdot\boldsymbol{b}\,\mathrm{d}v + \int_{\partial\mathcal{B}^t} \delta\boldsymbol{u}\cdot\boldsymbol{t}^p\,\mathrm{d}a - \int_{\mathcal{B}} \boldsymbol{\sigma} : \nabla^{\mathsf{sym}}\delta\boldsymbol{u}\,\mathrm{d}v \qquad \forall\delta\boldsymbol{u}\in\mathcal{V}^u$$

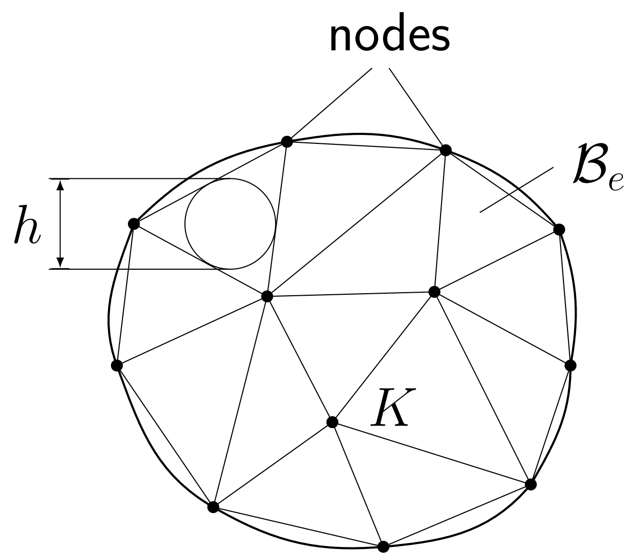
$$\mathcal{V}^u \;=\; \left\{\delta\boldsymbol{u}\in H^1(\mathcal{B})^3 \mid \delta\boldsymbol{u}=\boldsymbol{0} \text{ on } \partial\mathcal{B}^u\right\}$$

Thermal problem

$$\int_{\mathcal{B}} c_v\,\delta\vartheta\,\dot{\vartheta}\,\mathrm{d}v \;=\; \int_{\mathcal{B}} \delta\vartheta\,R\,\mathrm{d}v + \int_{\partial\mathcal{B}^q} \delta\vartheta\,q^p\,\mathrm{d}a + \int_{\mathcal{B}} \boldsymbol{q}\cdot\nabla\delta\vartheta\,\mathrm{d}v \qquad \forall\delta\vartheta\in\mathcal{V}^\vartheta$$

$$\mathcal{V}^\vartheta \;=\; \left\{\delta\vartheta\in H^1(\mathcal{B}) \mid \delta\vartheta=0 \text{ on } \partial\mathcal{B}^\vartheta\right\}$$

3. Spatial discretisation



$$\mathcal{B} \approx \mathcal{B}^h = \bigcup_{e=1}^{n_{el}} \mathcal{B}_e$$

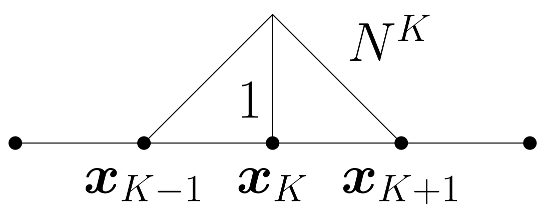
Test functions

$$\delta\boldsymbol{u}^h = \sum_{K=1}^{n_{np}} N^K(\boldsymbol{x})\,\delta\boldsymbol{u}_K \qquad \delta\vartheta^h = \sum_{K=1}^{n_{np}} N^K(\boldsymbol{x})\,\delta\vartheta_K$$

Trial functions

$$\boldsymbol{u}^h = \sum_{K=1}^{n_{np}} N^K(\boldsymbol{x})\,\boldsymbol{u}_K \qquad \vartheta^h = \sum_{K=1}^{n_{np}} N^K(\boldsymbol{x})\,\vartheta_K$$

h indicates the resolution of the discretisation



$N^K(\boldsymbol{x}_L) = \delta^K_L$ denote global ansatz functions with local support (hat functions)

4. Temporal discretisation

Approximation of velocities

$$\dot{\boldsymbol{u}} \approx \frac{\boldsymbol{u}_{n+1} - \boldsymbol{u}_n}{\Delta t} \qquad \left(\dot{\boldsymbol{\varepsilon}} \approx \frac{\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n}{\Delta t} \right)$$

$$\dot{\boldsymbol{v}} \approx \frac{\boldsymbol{v}_{n+1} - \boldsymbol{v}_n}{\Delta t}$$

$$\dot{\vartheta} \approx \frac{\vartheta_{n+1} - \vartheta_n}{\Delta t}$$

Generalised midpoint rule

$$\dot{x} = f(x) \quad \Longrightarrow \quad x_{n+1} - x_n = \Delta t \, f\left(x_{n+\alpha}\right) \quad \text{with} \quad x_{n+\alpha} = \alpha x_{n+1} + [1-\alpha] x_n$$

- $\alpha = 0$ Euler forward (1. order, conditionally stable)
- $\alpha = \frac{1}{2}$ midpoint rule (2. order, conditionally stable)
- $\alpha = 1$ Euler backward (1. order, unconditionally stable)

5. Algebraic set of equations

Mechanical problem

$$\frac{1}{\Delta t}\sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} N^K \rho\,[\boldsymbol{v}_{n+1}-\boldsymbol{v}_n]\,\mathrm{d}v = \sum_{e=1}^{n_{el}}\left[\int_{\mathcal{B}_e} N^K \boldsymbol{b}_{n+\alpha}\,\mathrm{d}v + \int_{\partial\mathcal{B}_e^t} N^K \boldsymbol{t}^p_{n+\alpha}\,\mathrm{d}a - \int_{\mathcal{B}_e} \boldsymbol{\sigma}^h_{n+\alpha}\cdot\nabla N^K\,\mathrm{d}v\right] \quad \forall K$$

with $\boldsymbol{u}_{n+1} - \boldsymbol{u}_n = \Delta t\,\boldsymbol{v}_{n+\beta}$

Thermal problem

$$\frac{1}{\Delta t}\sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} N^K c_v\,[\vartheta_{n+1}-\vartheta_n]\,\mathrm{d}v = \sum_{e=1}^{n_{el}}\left[\int_{\mathcal{B}_e} N^K R_{n+\gamma}\,\mathrm{d}v + \int_{\partial\mathcal{B}_e^q} N^K q^p_{n+\gamma}\,\mathrm{d}a - \int_{\mathcal{B}_e} \boldsymbol{q}^h_{n+\gamma}\cdot\nabla N^K\,\mathrm{d}v\right] \quad \forall K$$

with $R_{n+\gamma} = r_{n+\gamma} - \theta_0 \boldsymbol{\beta} : \dfrac{\boldsymbol{\varepsilon}_{n+1} - \boldsymbol{\varepsilon}_n}{\Delta t}$

We may select different values for $\alpha, \beta, \gamma \in (0, 1]$ in the case of implicit time integration

Nodal vectors

$$
\begin{aligned}
\boldsymbol{F}^{K,u}_{\text{dyn}} &= -\frac{1}{\Delta t}\sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} N^K \rho \frac{1}{\beta}\left[\frac{\boldsymbol{u}_{n+1}-\boldsymbol{u}_n}{\Delta t}-\boldsymbol{v}_n\right]\,\mathrm{d}v \\
\boldsymbol{F}^{K,u}_{\text{ext}} &= \sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} N^K \boldsymbol{b}_{n+\alpha}\,\mathrm{d}v + \sum_{e=1}^{n_{el}}\int_{\partial\mathcal{B}_e^t} N^K \boldsymbol{t}^p_{n+\alpha}\,\mathrm{d}a \\
\boldsymbol{F}^{K,u}_{\text{int}} &= \sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} \boldsymbol{\sigma}^h_{n+\alpha}\cdot\nabla N^K\,\mathrm{d}v \\
F^{K,\vartheta}_{\text{dyn}} &= -\frac{1}{\Delta t}\sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} N^K c_v\left[\vartheta_{n+1}-\vartheta_n\right]\,\mathrm{d}v \\
F^{K,\vartheta}_{\text{ext}} &= \sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} N^K R_{n+\gamma}\,\mathrm{d}v + \sum_{e=1}^{n_{el}}\int_{\partial\mathcal{B}_e^q} N^K q^p_{n+\gamma}\,\mathrm{d}a \\
F^{K,\vartheta}_{\text{int}} &= \sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} \boldsymbol{q}^h_{n+\gamma}\cdot\nabla N^K\,\mathrm{d}v
\end{aligned}
$$

Global vectors

$$\boldsymbol{F}^{u}_{\mathsf{dyn}} = \left[\boldsymbol{F}^{1,u}_{\mathsf{dyn},u} \cdots \boldsymbol{F}^{K,u}_{\mathsf{dyn},u} \cdots \boldsymbol{F}^{n_{np},u}_{\mathsf{dyn}}\right]^{t}$$

$$\boldsymbol{F}^{u}_{\mathsf{ext}} = \left[\boldsymbol{F}^{1,u}_{\mathsf{ext},u} \cdots \boldsymbol{F}^{K,u}_{\mathsf{ext},u} \cdots \boldsymbol{F}^{n_{np},u}_{\mathsf{ext}}\right]^{t}$$

$$\boldsymbol{F}^{u}_{\mathsf{int}} = \left[\boldsymbol{F}^{1,u}_{\mathsf{int}} \cdots \boldsymbol{F}^{K,u}_{\mathsf{int}} \cdots \boldsymbol{F}^{n_{np},u}_{\mathsf{int}}\right]^{t}$$

$$\boldsymbol{F}^{\vartheta}_{\mathsf{dyn}} = \left[F^{1,\vartheta}_{\mathsf{dyn}} \cdots F^{K,\vartheta}_{\mathsf{dyn}} \cdots F^{n_{np},\vartheta}_{\mathsf{dyn}}\right]^{t}$$

$$\boldsymbol{F}^{\vartheta}_{\mathsf{ext}} = \left[F^{1,\vartheta}_{\mathsf{ext}} \cdots F^{K,\vartheta}_{\mathsf{ext}} \cdots F^{n_{np},\vartheta}_{\mathsf{ext}}\right]^{t}$$

$$\boldsymbol{F}^{\vartheta}_{\mathsf{int}} = \left[F^{1,\vartheta}_{\mathsf{int}} \cdots F^{K,\vartheta}_{\mathsf{int}} \cdots F^{n_{np},\vartheta}_{\mathsf{int}}\right]^{t}$$

Residual statement

$$\underbrace{\begin{aligned} \boldsymbol{R}^u &= \boldsymbol{F}^u_{\text{dyn}} + \boldsymbol{F}^u_{\text{ext}} - \boldsymbol{F}^u_{\text{int}} = \boldsymbol{0} \\ \boldsymbol{R}^\vartheta &= \boldsymbol{F}^\vartheta_{\text{dyn}} + \boldsymbol{F}^\vartheta_{\text{ext}} - \boldsymbol{F}^\vartheta_{\text{int}} = \boldsymbol{0} \end{aligned}}$$

$$\boldsymbol{R} = \boldsymbol{F}_{\text{dyn}} + \boldsymbol{F}_{\text{ext}} - \boldsymbol{F}_{\text{int}} = \boldsymbol{0}$$

with

$$\left.\begin{aligned} \boldsymbol{F}_{\text{dyn}} &= \left[\boldsymbol{F}^u_{\text{dyn}} \,,\, \boldsymbol{F}^\vartheta_{\text{dyn}}\right]^t \\ \boldsymbol{F}_{\text{ext}} &= \left[\boldsymbol{F}^u_{\text{ext}} \,,\, \boldsymbol{F}^\vartheta_{\text{ext}}\right]^t \\ \boldsymbol{F}_{\text{int}} &= \left[\boldsymbol{F}^u_{\text{int}} \,,\, \boldsymbol{F}^\vartheta_{\text{int}}\right]^t \end{aligned}\right\} \quad \boldsymbol{R} = \left[\boldsymbol{R}^u \,,\, \boldsymbol{R}^\vartheta\right]^t$$

6. Linearisation

Iteration step (Newton method)

$$\boldsymbol{R}_k + \mathsf{d}\boldsymbol{R} = \boldsymbol{0} \qquad\qquad k = 1 \cdots m_{it}$$

Iteration matrix (stiffness)

$$-\,\mathsf{d}\boldsymbol{R} = -\begin{bmatrix} \mathsf{d}\boldsymbol{R}^u \\ \mathsf{d}\boldsymbol{R}^\vartheta \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}^{uu} & \boldsymbol{K}^{u\vartheta} \\ \boldsymbol{K}^{\vartheta u} & \boldsymbol{K}^{\vartheta\vartheta} \end{bmatrix} \begin{bmatrix} \mathsf{d}\boldsymbol{u}_u \\ \mathsf{d}\boldsymbol{u}_\vartheta \end{bmatrix}$$

Iterative update

$$\begin{aligned} \boldsymbol{u}_u^{k+1} &= \boldsymbol{u}_u^k + \mathsf{d}\boldsymbol{u}_u \\ \boldsymbol{u}_\vartheta^{k+1} &= \boldsymbol{u}_\vartheta^k + \mathsf{d}\boldsymbol{u}_\vartheta \end{aligned}$$

Global vectors

$$\begin{aligned} \boldsymbol{u}_u &= \left[\boldsymbol{u}_1 \cdots \boldsymbol{u}_K \cdots \boldsymbol{u}_{n_{np}}\right]^t \\ \boldsymbol{u}_\vartheta &= \left[\vartheta_1 \cdots \vartheta_K \cdots \vartheta_{n_{np}}\right]^t \end{aligned}$$

Nodal iteration matrices

$$
\begin{aligned}
\boldsymbol{K}^{uu}_{KL} &= \frac{1}{\beta\Delta t^2}\sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} N^K \rho N^L \,\mathrm{d}v \\
&\quad + \sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} \frac{\partial[\boldsymbol{\sigma}^h_{n+\alpha}\cdot\nabla N^K]}{\partial\boldsymbol{\varepsilon}}\cdot\Delta N^L \,\mathrm{d}v \\
\boldsymbol{K}^{\vartheta u}_{KL} &= \frac{1}{\Delta t}\sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} N^K c_v N^L \,\mathrm{d}v \\
&\quad + \sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} \frac{\partial[\boldsymbol{q}^h_{n+\gamma}\cdot\nabla N^K]}{\partial\boldsymbol{\gamma}}\cdot\Delta N^L \,\mathrm{d}v \\
\boldsymbol{K}^{u\vartheta}_{KL} &= \sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} \frac{\partial[\boldsymbol{\sigma}^h_{n+\alpha}\cdot\nabla N^K]}{\partial\vartheta} N^L \,\mathrm{d}v \\
\boldsymbol{K}^{\vartheta\vartheta}_{KL} &= \frac{1}{\Delta t}\sum_{e=1}^{n_{el}}\int_{\mathcal{B}_e} N^K \theta_0\,\boldsymbol{\beta}\cdot\nabla N^L \,\mathrm{d}v
\end{aligned}
$$