The Derived Deligne Conjecture

Doctoral thesis by
Javier Aguilar Martín

Supervised by
Constanze Roitzheim

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School of Mathematics, Statistics and Actuarial Sciences
University of Kent

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But mathematics is the sister, as well as the servant, of the arts and is touched by the same madness and genius.

Marston Morse quoted in S. Gudder's *A Mathematical Journey*

The real voyage of discovery consists not in seeking out new landscapes but in having new eyes.

Marcel Proust in *La Prisionnière*

Folk in those stories had lots of chances of turning back only they didn't. Because they were holding on to something.

Samwise Gamgee in the film *The Lord of The Rings: The Two Towers*

Abstract

We study the operad of derived A_∞-algebras from a new point of view in order to find a derived version of the Deligne conjecture. We start by defining the brace structure on an operad of graded R-modules using operadic suspension, which we describe in depth for the first time as a functor, and use it to define A_∞-algebra structures on certain operads, with the endomorphism operad as our main example. This construction provides us with an operadic context from which A_∞-algebras arise in a natural way and allows us to generalize the Lie algebra structure on the Hochschild complex of an A_∞-algebra. Next, we generalize these constructions to operads of bigraded R-modules, introducing a totalization functor. This allows us to generalize a Lie algebra structure on the total complex of a derived A_∞-algebra. This construction and the use of some enriched categories allow us to obtain new versions of the Deligne conjecture.

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Contents

Chapter 1

Introduction

There are a number of mathematical fields in which A_∞-structures arise, ranging from topology to mathematical physics. To study these structures, different interpretations of A_∞-algebras have been given. From the original definition in 1963 [Sta63], to alternative definitions in terms of tensor coalgebras [Kel01], [Pen01], many approaches use the machinery of operads [LRW13], [LV12] or certain Lie brackets [RW11] to obtain these objects.

Another technique to describe A_∞-structures comes from brace algebras [GV95],[LM05], which often involves unwieldy sign calculations that are difficult to describe in a conceptual way.

Here we used an operadic approach to obtain these signs in a more conceptual and consistent way. As a consequence, we will generalize the Lie bracket used in [RW11] and will give a very simple interpretation of A_∞-algebras. The difference between our operadic approach and others mentioned before is that ours uses much more elementary tools and can be used to talk about A_∞-structures on any operad. We hope that this provides a useful way of thinking about A_∞-structures. A first application of this simple formulation is the generalization of the Deligne conjecture. The classical Deligne conjecture states that the Hochschild complex of an associative algebra has the structure of a homotopy G-algebra [GV95]. This result has its roots in the theory of topological operads [Kon99]. Since A_∞-algebras generalize associative algebras, it is natural to ask what sort of algebraic structure arises on their Hochschild complex. Thanks to the tools we develop, we are able to answer this question.

Later in 2009, derived A_∞-algebras were introduced by Sagave [Sag10] as a bigraded generalization of A_∞-algebras in order to bypass the projectivity requirements that are often imposed when working with classical A_∞-algebras. We generalize the operadic description of classical A_∞-algebras to the derived case by means of an operadic totalization inspired by the totalization functor described in [CESLW18]. This way we obtain an operation similar to the star operation in [LRW13] and generalize the construction that has been done for A_∞-algebras to more general derived A_∞-algebras. This allows us to generalize the Deligne conjecture even further to obtain a *derived* Deligne conjecture.

The text is organized as follows. In Chapter 2 we recall some basic definitions and results, and establish some conventions for both the classical and the derived cases. In Chapter 3 we define a device called *operadic suspension* that will help us obtain the signs that we want and link this device to the classical operadic approach to A_∞-algebras. We also take this construction to the level of the underlying collections of the operads to also obtain a nice description of ∞-morphisms of A_∞-algebras. We then explore the functorial properties of operadic suspension, being monoidality (Proposition 3.1.15) the most remarkable of them. In Section 3.2 we study the brace algebra induced by operadic suspension and obtain a relevant result, Proposition 3.2.3, which establishes a relation between the canonical brace structure on an operad and the one induced by its operadic suspension. We show that as a particular case of this result we obtain the Lie bracket from [RW11].

Following the terminology of [GV95], if $\mathcal{O}$ is an operad with an A_∞-multiplication $m \in \mathcal{O}$, it is natural to ask whether there are linear maps $M_j : \mathcal{O}^{\otimes j} \to \mathcal{O}$ satisfying the A_∞-algebra axioms. In Section 3.3 we use the aforementioned brace structure to define such linear maps on a shifted version of the operadic suspension. We then iterate this process in Section 3.3.1 to define an A_∞-structure on the Hochschild complex of an operad with A_∞-multiplication. This iteration process was inspired by the work of Getzler in [Get93].

Next, we prove our first main result, Theorem 3.3.9, which relates the A_∞-structure on an operad with the one induced on its Hochschild complex. More precisely, we have the following.

Theorem A. There is a morphism of A_∞-algebras $\Phi : S\mathfrak{s}\mathcal{O} \to S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$ lifting the canonical brace map $\mathfrak{s}\mathcal{O} \to \mathrm{End}_{\mathfrak{s}\mathcal{O}}$.

This result was hinted at by Gerstenhaber and Voronov in [GV95], but here we introduce a suitable context and prove it as Theorem 3.3.9.

We also draw a connection between our framework and the one from Gerstenhaber and Voronov. As a consequence of this theorem, if A is an A_∞-algebra and $\mathcal{O} = \mathrm{End}_A$ its endomorphism operad, we obtain the following A_∞-version of the Deligne conjecture in Corollary 3.3.14.

Theorem B. The Hochschild complex $S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$ of an operad with an A_∞-multiplication has a structure of J-algebra.

In the above theorem, J-algebras play the role of homotopy G-algebras in the classical case [GV95]. After this, we move to the bigraded case. The goal here is showing that an operad $\mathcal{O}$ with a derived A_∞-multiplication $m \in \mathcal{O}$ can be endowed with the structure of a derived A_∞-algebra, just like in the classical case. We start Chapter 4 recalling some definitions of derived A_∞-algebras and filtered A_∞-algebras. In Section 4.3, we define the totalization functor for operads and then the bigraded version of operadic suspension. We combine these two constructions to define an operation that allows us to understand a derived A_∞-multiplication as a Maurer-Cartan element. As a consequence we obtain the star operation that was introduced in [LRW13], which also defines a Lie Bracket. From this, we obtain in Section 4.4 a brace structure from which we can obtain a classical A_∞-algebra on the graded operad $S\,\mathrm{Tot}(\mathfrak{s}\mathcal{O})$. Finally, in Section 4.5, we prove our main results about derived A_∞-algebras. The first one is Theorem 4.5.3, which shows that, under mild boundedness assumptions, the A_∞-structure on totalization is equivalent to a derived A_∞-algebra on $S\mathfrak{s}\mathcal{O}$. The statement can be summarised as follows.

Theorem C. For any sufficiently bounded operad $\mathcal{O}$ with a derived A_∞-multiplication there are linear maps $M_{ij} : (S\mathfrak{s}\mathcal{O})^{\otimes j} \to S\mathfrak{s}\mathcal{O}$ satisfying the derived A_∞-algebra axioms.

The next result is Theorem 4.5.8, which generalizes Theorem 3.3.9 to the derived setting. More precisely,

Theorem D. There is a morphism $\Phi : S\mathfrak{s}\mathcal{O} \to S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$ of derived A_∞-algebras lifting the canonical brace map $\mathfrak{s}\mathcal{O} \to \mathrm{End}_{\mathfrak{s}\mathcal{O}}$.

As a consequence of this theorem we obtain a new version of the Deligne conjecture, Corollary 4.5.10, this time in the setting of derived A_∞-algebras. For this we also introduce a derived version of J-algebras.

Theorem E. The Hochschild complex $S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$ of an operad with a derived A_∞-multiplication has a structure of derived J-algebra.

We finish the thesis in Chapter 5 by outlining some open question that arise from our research. The first question is related to the boundedness assumptions that we need to make in order to obtain the derived Deligne conjecture. The other one would be the natural continuation of our research. In the classical case, the homotopy G-algebra structure on the Hochschild complex induced a Gerstenhaber algebra structure on cohomology [GV95]. We would like to know what structure there is on the Hochschild cohomology of a derived A_∞-algebra.

Chapter 2

Background and conventions

In this initial chapter we establish the necessary background and conventions for the rest of the thesis. We start Section 2.1 with some category theory background and results, including notions of enriched categories that will play an essential role connecting derived A_∞-algebras with classical A_∞-algebras. We recall the motivation for the study of A_∞-algebras as well as some definitions and well-known results in Section 2.2. In Section 2.3 we recall the main definitions regarding operads, since that is the framework in which we will work with derived A_∞-algebras. At last, in Section 2.4 we list several categories that we will use in our study and introduce the totalization functor, which is essential to encode derived A_∞-algebras.

2.1 Symmetric monoidal categories and enrichments

We assume that the reader is familiar with the basic terminology of category theory. For an introduction to this topic we refer the reader to [Mac71]. Here we briefly recall the notion of symmetric monoidal categories and several versions of monoidal functors. The detailed definitions with all the precise diagrams can also be found in [Mac71] and in [Bor94].

Definition 2.1.1. *A* symmetric monoidal category *is a category $\mathcal{C}$ equipped with a functor*

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

called tensor product*, an object $1 \in \mathcal{C}$ called* unit object*, natural isomorphisms called* associators

$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

for all objects $A, B, C \in \mathcal{C}$, a natural isomorphism called left unitor

$$\lambda_A : 1 \otimes A \to A$$

for every $A \in \mathcal{C}$, a natural isomorphism called right unitor

$$\rho_A : A \otimes 1 \to A$$

for every $A \in \mathcal{C}$, and a natural isomorphism called braiding *or* symmetry isomorphism

$$\tau_{A,B} : A \otimes B \to B \otimes A$$

for all $A, B \in \mathcal{C}$. These morphisms satisfy natural unitality and associativity axioms.

Remark 2.1.2. If we drop the symmetry isomorphism we get what is simply called a *monoidal category*.

Definition 2.1.3. *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ be symmetric monoidal categories. A* lax monoidal functor *is a functor $F : \mathcal{C} \to \mathcal{D}$ with a morphism $\varepsilon : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ and a natural transformation*

$$\mu_{A,B} : F(A) \otimes_{\mathcal{D}} F(B) \to F(A \otimes_{\mathcal{C}} B)$$

for all $A, B \in \mathcal{C}$ satisfying natural unitality, associativity and symmetry axioms. A lax monoidal functor is called strong monoidal *if ε and $\mu_{A,B}$ are isomorphisms for all $A, B \in \mathcal{C}$.*

Definition 2.1.4. *Suppose (F, μ, ε) and (F, ν, ϵ) are monoidal functors between the symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$. Then a* natural transformation *$\alpha : F \to G$ is* monoidal *if the following diagrams commute.*

$$\begin{tikzcd} F(A) \otimes_{\mathcal{D}} F(B) \arrow[r, "\alpha_A \otimes_{\mathcal{D}} \alpha_B"] \arrow[d, "\mu_{A,B}"] & G(A) \otimes_{\mathcal{D}} G(B) \arrow[d, "\nu_{A,B}"] \\ F(A \otimes_{\mathcal{C}} B) \arrow[r, "\alpha_{A \otimes_{\mathcal{C}} B}"] & G(A \otimes_{\mathcal{C}} B) \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathcal{D}} \arrow[d, "\varepsilon"] \arrow[dr, "\epsilon"] & \\ F(1) \arrow[r, "\alpha_1"] & G(1) \end{tikzcd}$$

Definition 2.1.5. *If $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories, a lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is a* monoidal equivalence *if there is a lax monoidal functor $G : \mathcal{D} \to \mathcal{C}$ such that there exist monoidal natural isomorphisms $\alpha : FG \Rightarrow id_{\mathcal{C}}$, $\beta : GF \Rightarrow id_{\mathcal{D}}$.*

Definition 2.1.6. *A symmetric monoidal category $\mathcal{C}$ is* closed *if for every object $A \in \mathcal{C}$ the tensor product functor $A \otimes - : \mathcal{C} \to \mathcal{C}$ has a right adjoint functor $[A, -] : \mathcal{C} \to \mathcal{C}$. In other words, for all $A, B, C \in \mathcal{C}$ we have a natural bijection between the morphism sets*

$$\mathrm{Hom}_{\mathcal{C}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathcal{C}}(A, [B, C])$$

natural in all arguments. The object $[A, B]$ is called the internal hom.

2.1.1 Monoidal categories over a base

We collect some results about enriched categories from [Rie14] and [CESLW18, §4.2] that we will need as a categorical setting for our results on derived A_∞-algebras. Here we combine the idea of enriched category with that of symmetric monoidal category.

Definition 2.1.7. *Let $\mathscr{V}$ a monoidal category. A $\mathscr{V}$*-category *$\mathcal{C}$, also called $\mathscr{V}$*- enriched category *or* category enriched over *$\mathscr{V}$, consists of*

- *a set* $\mathrm{Ob}(\mathcal{C})$ *of objects in $\mathcal{C}$,*
- *for each pair (A, B) of objects in $\mathcal{C}$ an object $\mathcal{C}(A, B) \in \mathscr{V}$ called the* hom-object *or* object of morphisms *from A to B,*

- *for every triple* (A, B, C) *of objects in* $\mathcal{C}$ *a morphism*

$$\circ_{A,B,C} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C)$$

  in $\mathscr{V}$ *called* composition morphism*,*

- *and for each object* A *in* $\mathcal{C}$ *a morphism* $u_A : 1 \to \mathcal{C}(A, A)$ *in* $\mathscr{V}$ *called the* identity element*.*

All this data is subject to associativity and unitality constrains that can be seen in detail in [Bor94].

Definition 2.1.8. *Let* $(\mathscr{V}, \otimes, 1)$ *be a symmetric monoidal category and let* $(\mathcal{C}, \otimes, 1)$ *be a monoidal category. We say that* $\mathcal{C}$ *is a* monoidal category over $\mathscr{V}$ *if we have an* external tensor product $* : \mathscr{V} \times \mathcal{C} \to \mathcal{C}$ *such that we have the following natural isomorphisms.*

- $1 * X \cong X$ *for all* $X \in \mathcal{C}$*,*
- $(C \otimes D) * X \cong C * (D * X)$ *for all* $C, D \in \mathscr{V}$ *and* $X \in \mathcal{C}$*,*
- $C * (X \otimes Y) \cong (C * X) \otimes Y \cong X \otimes (C * Y)$ *for all* $C \in \mathscr{V}$ *and* $X, Y \in \mathcal{C}$*.*

Remark 2.1.9. Throughout the thesis we will also assume that there is a bifunctor $\underline{\mathscr{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathscr{V}$ such that we have natural bijections

$$\mathrm{Hom}_{\mathcal{C}}(C * X, Y) \cong \mathrm{Hom}_{\mathscr{V}}(C, \underline{\mathscr{C}}(X, Y)).$$

Under this assumption we get a $\mathscr{V}$-enriched category $\underline{\mathscr{C}}$ with the same objects as $\mathcal{C}$ and with hom-objects given by $\underline{\mathscr{C}}(-,-)$. The unit morphism

$u_X : 1 \to \underline{\mathscr{C}}(X,X)$ corresponds to the identity map in $\mathcal{C}$ under the adjunction, and the composition morphism is given by the adjoint of the composite

$$\begin{tikzcd}
(\underline{\mathscr{C}}(Z,Y) \otimes \underline{\mathscr{C}}(X,Z)) * X \arrow[r, phantom, "\cong"] & \underline{\mathscr{C}}(Z,Y) * (\underline{\mathscr{C}}(X,Z) * X) \arrow[d, "id * ev_{XZ}"] & \\
& \underline{\mathscr{C}}(Z,Y) * Z \arrow[r, "ev_{ZY}"] & Y
\end{tikzcd}$$

where ev_{XZ} is the adjoint of the identity $\underline{\mathscr{C}}(X,Z) \to \underline{\mathscr{C}}(X,Z)$. Furthermore, $\underline{\mathscr{C}}$ is a monoidal $\mathscr{V}$-enriched category, namely we have an enriched functor

$$\underline{\otimes} : \underline{\mathscr{C}} \times \underline{\mathscr{C}} \to \underline{\mathscr{C}}$$

where $\underline{\mathscr{C}} \times \underline{\mathscr{C}}$ is the enriched category with objects $\mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C})$ and hom-objects

$$(\underline{\mathscr{C}} \times \underline{\mathscr{C}})((X,Y),(W,Z)) \coloneqq \underline{\mathscr{C}}(X,W) \otimes \underline{\mathscr{C}}(Y,Z).$$

In particular, we get maps in $\mathscr{V}$

$$\underline{\mathscr{C}}(X,W) \otimes \underline{\mathscr{C}}(Y,Z) \to \underline{\mathscr{C}}(X \otimes Y, W \otimes Z),$$

given by the adjoint of the composite

$$\begin{tikzcd}
(\underline{\mathscr{C}}(X,W) \otimes \underline{\mathscr{C}}(Y,Z)) * (X \otimes Y) \arrow[r, phantom, "\cong"] & (\underline{\mathscr{C}}(X,W) * X) \otimes (\underline{\mathscr{C}}(Y,Z) * Y) \arrow[d, "ev_{XW} \otimes ev_{YZ}"] \\
& W \otimes Z
\end{tikzcd}$$

Definition 2.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories over $\mathscr{V}$. A* lax functor over $\mathscr{V}$ *consists of a functor $F : \mathcal{C} \to \mathcal{D}$ together with a natural transformation*

$$\nu_F : - *_{\mathcal{D}} F(-) \Rightarrow F(- *_{\mathcal{C}} -)$$

which is associative and unital with respect to the monoidal structures over $\mathscr{V}$ of $\mathcal{C}$ and $\mathcal{D}$. See [Rie14, Proposition 10.1.5] for explicit diagrams stating the coherence axioms. If ν_F is a natural isomorphism we say F is a functor over $\mathscr{V}$. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be lax functors over $\mathscr{V}$. A* natural transformation over $\mathscr{V}$ *is a natural transformation $\mu : F \Rightarrow G$ such that for any $C \in \mathscr{V}$ and for any $X \in \mathcal{C}$ we have*

$$\nu_G \circ (1 *_{\mathcal{D}} \mu_X) = \mu_{C *_{\mathcal{C}} X} \circ \nu_F.$$

A lax monoidal functor over $\mathscr{V}$ *is a triple (F, ϵ, μ), where $F : \mathcal{C} \to \mathcal{D}$ is a lax functor over $\mathscr{V}$, $\epsilon : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ is a morphism in $\mathcal{D}$ and*

$$\mu : F(-) \otimes F(-) \Rightarrow F(- \otimes -)$$

is a natural transformation over $\mathscr{V}$ satisfying the standard unit and associativity conditions. If ν_F and μ are natural isomorphisms then we say that F is monoidal over $\mathscr{V}$.

Another notion of natural transformation in the enriched setting is given below, see [Rie14, Definition 3.5.8] for the detailed diagrams.

Definition 2.1.11. *A* $\mathscr{V}$-enriched natural transformation $\underline{\mu} : \underline{F} \Rightarrow \underline{G}$ *between a pair of $\mathscr{V}$-enriched functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of a morphism*

$\underline{\mu}_X : 1 \to \mathcal{D}(FX, GX)$ *in* $\mathscr{V}$ *for each* $X \in \mathcal{C}$ *satisfying certain naturality conditions with respect to the external product and enriched composition.*

The following is [CESLW18, Proposition 4.11].

Proposition 2.1.12. *Let* $F, G : \mathcal{C} \to \mathcal{D}$ *be lax functors over* $\mathscr{V}$. *Then* F *and* G *extend to* $\mathscr{V}$*-enriched functors*

$$\underline{F}, \underline{G} : \underline{\mathscr{C}} \to \underline{\mathscr{D}}$$

where $\underline{\mathscr{C}}$ *and* $\underline{\mathscr{D}}$ *denote the* $\mathscr{V}$*-enriched categories corresponding to* $\mathcal{C}$ *and* $\mathcal{D}$ *as described in Remark 2.1.9. Moreover, any natural transformation* $\mu : F \Rightarrow G$ *over* $\mathscr{V}$ *also extends to a* $\mathscr{V}$*-enriched natural transformation*

$$\underline{\mu} : \underline{F} \Rightarrow \underline{G}.$$

In particular, if F *is lax monoidal over* $\mathscr{V}$, *then* $\underline{F}$ *is lax monoidal in the enriched sense, where the monoidal structure on* $\underline{\mathscr{C}} \times \underline{\mathscr{C}}$ *is described in Remark 2.1.9.*

Lemma 2.1.13. *Let* $F, G : \mathcal{C} \to \mathcal{D}$ *lax functors over* $\mathscr{V}$ *and let* $\mu : F \Rightarrow G$ *a natural transformation over* $\mathscr{V}$. *For every* $X \in \mathcal{C}$ *and* $Y \in \mathcal{D}$ *there is a map*

$$\underline{\mathscr{D}}(GX, Y) \to \underline{\mathscr{D}}(FX, Y)$$

that is an isomorphism if μ *is an isomorphism.*

We would like to thank Sarah Whitehouse for her contribution to the proof of this result.

Proof. By Proposition 2.1.11 μ extends to a $\mathscr{V}$-enriched natural transformation

$$\underline{\mu} : \underline{F} \Rightarrow \underline{G}$$

that at each object X evaluates to

$$\underline{\mu}_X : 1 \to \underline{\mathscr{D}}(FX, GX)$$

defined to be the adjoint of $\mu_X : FX \to GX$. We define the map $\underline{\mathscr{D}}(GX, Y) \to \underline{\mathscr{D}}(FX, Y)$ as the composite

$$\begin{tikzcd} \underline{\mathscr{D}}(GX,Y) \cong \underline{\mathscr{D}}(GX,Y) \otimes 1 \arrow[r, "1\otimes\underline{\mu}_X"] & \underline{\mathscr{D}}(GX,Y) \otimes \underline{\mathscr{D}}(FX,GX) \arrow[d, "c"] \\ & \underline{\mathscr{D}}(FX,Y) \end{tikzcd} \tag{2.1}$$

where c is the composition map in the enriched setting.

When μ is an isomorphism we may analogously define the following map

$$\begin{tikzcd} \underline{\mathscr{D}}(FX,Y) \cong \underline{\mathscr{D}}(FX,Y) \otimes 1 \arrow[r, "1\otimes\underline{\mu}_X^{-1}"] & \underline{\mathscr{D}}(FX,Y) \otimes \underline{\mathscr{D}}(GX,FX) \arrow[d, "c"] \\ & \underline{\mathscr{D}}(GX,Y) \end{tikzcd} \tag{2.2}$$

We show that the above map is the inverse of the map (2.1). Consider the following diagram where the external arrows are the composition of (2.1) and (2.2).

$$
\begin{tikzcd}
\underline{\mathscr{D}}(GX,Y) \arrow[r, "\cong"] \arrow[dr, phantom, "(5)"] & \underline{\mathscr{D}}(GX,Y)\otimes 1 \arrow[r, "1\otimes\underline{\mu}_X"] \arrow[d, "1\otimes\alpha_X"] \arrow[dr, phantom, "(4)"] & \underline{\mathscr{D}}(GX,Y)\otimes\underline{\mathscr{D}}(FX,GX) \arrow[r, "c"] \arrow[d, "\cong"] \arrow[dr, phantom, "(1)"] & \underline{\mathscr{D}}(FX,Y) \arrow[ddl, bend left, "\cong"] \\
\arrow[d, phantom, "(3)"] & \underline{\mathscr{D}}(GX,Y)\otimes\underline{\mathscr{D}}(GX,GX) \arrow[ul, "c"'] & \underline{\mathscr{D}}(GX,Y)\otimes\underline{\mathscr{D}}(FX,GX)\otimes 1 \arrow[dl, "1\otimes 1\otimes\underline{\mu}_X^{-1}"] \arrow[dd, "c\otimes 1"] & \\
 & \underline{\mathscr{D}}(GX,Y)\otimes\underline{\mathscr{D}}(FX,GX)\otimes\underline{\mathscr{D}}(GX,FX) \arrow[u, "1\otimes c"] \arrow[d, "c\otimes 1"] \arrow[dr, phantom, "(2)"] & & \\
 & \underline{\mathscr{D}}(FX,Y)\otimes\underline{\mathscr{D}}(GX,FX) \arrow[uuul, bend left, "c"] & \underline{\mathscr{D}}(FX,Y)\otimes 1 \arrow[l, "1\otimes\underline{\mu}_X^{-1}"] &
\end{tikzcd}
\tag{2.3}
$$

In the above diagram (2.3), α_X is adjoint to $1_{GX} : GX \to GX$. Diagrams (1) and (2) clearly commute. Diagram (3) commutes by associativity of c. Diagram (4) commutes because $\underline{\mu}_X^{-1}$ and $\underline{\mu}_X$ are adjoint to mutual inverses, so their composition results in the adjoint of the identity. Finally, diagram (5) commutes because we are composing with an isomorphism. In particular, diagram (5) is a decomposition of the identity map on $\underline{\mathscr{D}}(GX,Y)$. By commutativity, this means that the overall diagram composes to the identity, showing that the maps (2.1) and (2.2) are mutually inverse. □

2.2 A_∞-algebras

In the early sixties, J. Stasheff introduced A_∞-spaces and A_∞-algebras [Sta61], [Sta63] as a tool in the study of "group-like" spaces. We are going to motivate them by explaining their topological origin and later we will give precise definitions. We will also recall minimal models to further motivate the study of A_∞-algebras and their limitations. A more detailed survey can be found in [Kel01] and in [LV12].

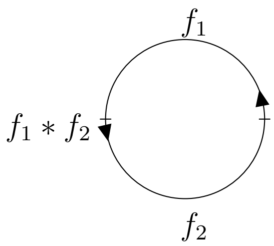
2.2.1 Topological origin

Let us consider the basic example. Let $(X, *)$ be a topological space with a base point $*$ and let ΩX denote the space of based loops in X: a point of ΩX is thus a continuous map $f : S^1 \to X$ taking the base point of the circle to the base point $*$.

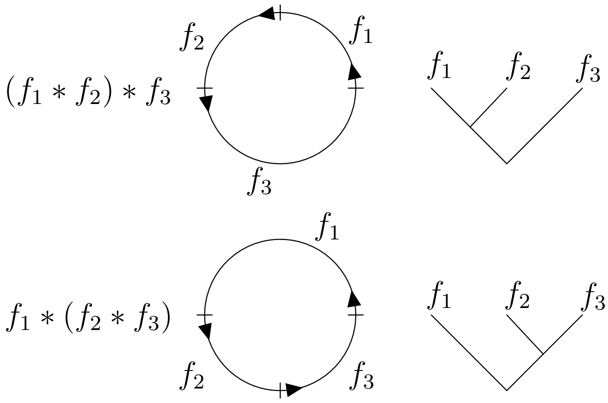
We have the composition map

$$m_2 : \Omega X \times \Omega X \to \Omega X$$

sending a pair of loops (f_1, f_2) to the loop $f_1 * f_2 = m_2(f_1, f_2)$ obtained by running through f_1 on the first half of the circle at twice the speed and through f_2 on the second half.



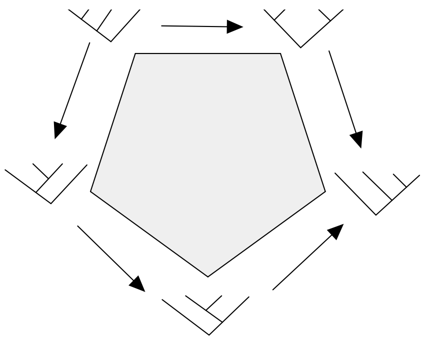
This composition is not associative: for three loops f_1, f_2, f_3, the composition $(f_1 * f_2) * f_3$ runs through f_1 on the first quarter of the circle whereas the composition $f_1 * (f_2 * f_3)$ runs through f_1 on the first half of the circle. We symbolize the two possibilities by two binary trees with three leaves.



There is a homotopy

$$m_3 : [0,1] \times \Omega X \times \Omega X \to \Omega X$$

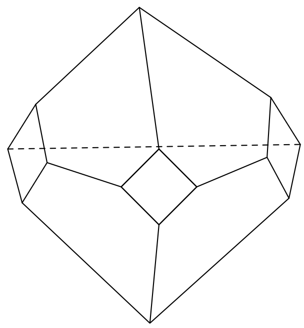
joining the two possibilities of composing three loops by a reparametrization. When we want to compose four loops, there are five possibilities corresponding to the five binary trees with four leaves. Using m_3, we obtain two concatenations of homotopies linking the compositions $(f_1, f_2, f_3, f_4) \mapsto ((f_1 * f_2) * f_3) * f_4$ and $(f_1, f_2, f_3, f_4) \mapsto f_1 * (f_2 * (f_3 * f_4))$. This results in the picture below.



These two concatenations are homotopic. Denote a homotopy by

$$m_4 : K_4 \times (\Omega X)^4 \to X$$

where K_4 denotes the pentagon bounded by the two paths. When we want to compose five loops, there are fourteen possibilities corresponding to the fourteen binary trees with five leaves. Using m_4 and m_3, we obtain homotopies linking the compositions and faces linking the homotopies. The resulting object is the boundary of the polytope K_5 depicted below.



The pentagonal faces are copies of K_4. More generally, Stasheff [Sta61] defined polytopes K_n of dimension $n-2$ for all $n \geq 2$, including $K_2 = *$ and $K_3 = [0,1]$. He defined an *A_∞-space* to be a topological space Y endowed with maps $m_n : K_n \times Y^n \to Y$, for $n \geq 2$, satisfying suitable compatibility conditions and admitting a "strict unit". The loop space ΩX is the prime example of such a space Y. Conversely [Ada78], a topological space that admits the structure of an A_∞-space and whose connected components form a group is homotopy equivalent to a loop space. If Y is an A_∞-space, the singular chain complex of Y is the paradigmatic example of an A_∞-algebra [Sta63], which will formally introduce in the next section using cohomological grading.

2.2.2 Definitions

In this section we first establish notation and assumptions about graded modules and sign conventions. We then briefly recall some of the basic definitions regarding A_∞-algebras.

Our base category is the category of $\mathbb{Z}$-graded R-modules and linear maps, where R is a commutative ring with unit of characteristic distinct from 2. All tensor products are taken over R. We denote the i-th degree component of A as A^i. If $x \in A^i$ we write $\deg(x) = i$ and we use cohomological grading. The symmetry isomorphism is given by the following Koszul sign convention.

$$\begin{aligned} \tau_{A,B} : A \otimes B &\to B \otimes A \\ x \otimes y &\mapsto (-1)^{\deg(x)\deg(y)} y \otimes x \end{aligned}$$

A map $f : A \to B$ of degree i satisfies $f(A^n) \subseteq B^{n+i}$ for all n. The R-modules $\mathrm{Hom}_R(A, B)$ are naturally graded by

$$\mathrm{Hom}_R(A, B)^i = \prod_k \mathrm{Hom}_R(A^k, B^{k+i}).$$

As a consequence of the above sign convention, we also adopt the following Koszul sign convention: for $x \in A$, $y \in B$, $f \in \mathrm{Hom}_R(A, C)$ and $g \in \mathrm{Hom}_R(B, D)$,

$$(f \otimes g)(x \otimes y) = (-1)^{\deg(x)\deg(g)} f(x) \otimes g(y).$$

Recall that if (A, ∂^A) and (B, ∂^B) are (co)chain complexes, the module $\mathrm{Hom}_R(A, B)$ also becomes a (co)chain complex with differential

$$\partial(f) = \partial^B \circ f + (-1)^{\deg(f)} f \circ \partial^A.$$

With all the notations and conventions established we can now introduce A_∞-algebras.

Definition 2.2.1. *An* A_∞-algebra *is a graded* R*-module* A *together with a family of maps* $m_n : A^{\otimes n} \to A$ *of degree* $2 - n$ *satisfying the equation*

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0 \tag{2.4}$$

for all $n \geq 1$.

The above equation will sometimes be referred to as the A_∞-*equation*. The signs are related to the orientation given to the Stasheff polytopes, see Section 2.2.1. This seemingly obscure definition captures the idea of an algebra that is associative up to homotopy. To see this, let us have a look at the first few cases.

- We have $m_1 m_1 = 0$, so (A, m_1) is a cochain complex.

- We have the Leibniz rule

$$m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1)$$

as maps $A^{\otimes 2} \to A$. Here 1 denotes the identity map on A. So m_1 is a graded derivation with respect to the multiplication m_2.

- We have

$$m_2(1 \otimes m_2 - m_2 \otimes 1) = m_1 m_3 + m_3(m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$$

as maps $A^{\otimes 3} \to A$. Note that the left hand side is the associator for m_2 and that the right hand side may be viewed as the boundary of m_3 in the morphism complex $\mathrm{Hom}_R(A^{\otimes 3}, A)$. This implies that m_2 is associative up to homotopy.

For more details about this the reader is referred to [Kel01] and to [LV12, §9.2], where they use a different sign convention but the concepts are the same.

Definition 2.2.2. *An* ∞-morphism *of A_∞-algebras $A \to B$ is a family of maps*

$$f_n : A^{\otimes n} \to B$$

of degree $1-n$ satisfying for all $n \geq 1$ the equation

$$\sum_{r+s+t=n} (-1)^{rs+t} f_{r+1+t}(1^{\otimes r} \otimes m_s^A \otimes 1^{\otimes t}) = \sum_{i_1+\cdots+i_k=n} (-1)^s m_k^B (f_{i_1} \otimes \cdots \otimes f_{i_k}),$$

where

$$s = \sum_{\alpha<\beta} i_\alpha (1 - i_\beta).$$

The composition of ∞-morphisms $f : A \to B$ and $g : B \to C$ is given by

$$(gf)_n = \sum_r \sum_{i_1+\cdots+i_r=n} (-1)^s g_r(f_{i_1} \otimes \cdots \otimes f_{i_r}).$$

Similarly to the A_∞-equation (2.4), the above definition captures the idea of a map that is a morphism of algebras up to homotopy. This can be observed in the first few cases.

- We have $f_1 m_1 = m_1 f_1$, i.e. f_1 is a morphism of complexes.
- We have

  $$f_1 m_2 = m_2(f_1 \otimes f_1) + m_1 f_2 + f_2(m_1 \otimes 1 + 1 \otimes m_1),$$

  which means that f_1 commutes with the multiplication m_2 up to a homotopy given by f_2.

Again, the reader can find more details in [Kel01] and [LV12, §9.2]. We also need another, perhaps more intuitive, notion of morphism of A_∞-algebras.

Definition 2.2.3. *A* morphism *of A_∞-algebras is a map of R-modules $f : A \to B$ such that*

$$f(m_j^A) = m_j^B \circ f^{\otimes j}.$$

Notice that working with $\mathbb{Z}$-graded modules forces every morphism of A_∞-algebras to be of degree 0.

We will discuss the topics of this section in the language of operads, which will be introduced in a later section. Before that, let us motivate the importance of A_∞-algebras through the theory of minimal models.

2.2.3 Minimal models

We now recall a definition and a theorem about minimal models of A_∞-algebras. The theorem relates differential graded algebras to A_∞-structures on their homology. This theorem is the main reason why A_∞-algebras became a relevant subject of study.

Definition 2.2.4. *An A_∞-algebra is called* minimal *if $m_1 = 0$.*

Over a field, one can replace any A_∞-algebra by a quasi-isomorphic minimal one, where by quasi-isomorphic we mean that there is a map that induces an isomorphism on cohomology with respect to m_1. This gives a very convenient way to describe a quasi-isomorphism class of an A_∞-algebra. More precisely we have the following.

Theorem 2.2.5 (Kadeishvili)**.** *Let A be a differential graded algebra over a field k, and let $H^*(A)$ be its cohomology module. Then $H^*(A)$ has an A_∞-structure such that*

- *$m_1 = 0$ and the multiplication m_2 is induced by the multiplication on A,*
- *there is an ∞-morphism of A_∞-algebras $f : H^*(A) \to A$ such that f_1 is a quasi-isomorphism.*

This A_∞-algebra $H^*(A)$ is called the minimal model *of A.*

Using this result it is also possible to show that under some conditions any other dga A' with $H^*(A') \cong H^*(A)$ is quasi-isomorphic to A. For more details see [Kad80].

2.3 Operads

In this section we recall the notion of operad, an object that is particularly useful to study algebraic structures given by multilinear maps. Operads will allow us to formulate definitions and results concerning derived A_∞-algebras in a very convenient way. The main references for this section are [LV12] and [KWZ15].

We will be working in the category of graded R-modules, but all the definitions and results in this section generalize with no substantial changes to any symmetric monoidal category like the ones we see in Section 2.4.

2.3.1 Definitions

We first give the main definitions that we will be using throughout the thesis. We start defining the underlying object of an operad.

Definition 2.3.1. *A* collection *is a family $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ of graded R-modules. We call the integer n the* arity*. When there is an action of the symmetric group Σ_n on each $\mathcal{O}(n)$ we say that the collection is an* $\mathbb{S}-$module*. A* map of collections *$f : \mathcal{O} \to \mathcal{P}$ is a family of maps $f_n : \mathcal{O}(n) \to \mathcal{P}(n)$. A map of collection is a* map of $\mathbb{S}-$modules *when it preserves the symmetric group action.*

We will mostly focus on the non-symmetric case, but our general results about operads generalize to the symmetric case as well. On a collection we can define an operad by adding some extra structure as in the following definition.

Definition 2.3.2. *A* (non-symmetric) operad *is a collection* $\mathcal{O} = \{\mathcal{O}(n)\}$ *where there is a distinguished* identity *element* $1 \in \mathcal{O}(1)$ *and with* insertion maps

$$\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(m+n-1)$$

for each $1 \leq i \leq n$ *satisfying natural unitality and associativity axioms, see [KWZ15, §1.1.2].*

Insertion maps can be iterated to define composition maps

$$\gamma(x; x_1, \ldots, x_n) = (\cdots(x \circ_1 x_1) \circ_2 x_2 \cdots) \circ_n x_n).$$

If $\mathcal{O}$ *is an* $\mathbb{S}-$*module and the insertion maps satisfy some additional invariance axioms regarding the symmetric group action, we say that* $\mathcal{O}$ *is a* symmetric operad, *see [LV12] for more details.*

A map of operads *(resp.* symmetric operads*) is a map of collections (resp.* $\mathbb{S}-$*modules) that is compatible with insertions.*

Collections also come with the following algebraic operation that will provide an alternative way of describing operads.

Definition 2.3.3. *The* plethysm *or* composite $\mathcal{O} \circ \mathcal{P}$ *of two collections* $\mathcal{O}$ *and* $\mathcal{P}$ *given by*

$$(\mathcal{O} \circ \mathcal{P})(n) = \bigoplus_{N \geq 0} \mathcal{O}(N) \otimes \left(\bigoplus_{a_1 + \cdots + a_k = n} \mathcal{P}(a_1) \otimes \cdots \otimes \mathcal{P}(a_k) \right).$$

There is a definition of plethysm for $\mathbb{S}$-modules that requires some tools from the representation theory of symmetric groups that we are not going to introduce here. The reader is referred to [LV12] for the details.

Definition 2.3.4. *The* plethysm *or* composite $f \circ g$ *of maps* $f : \mathcal{O} \to \mathcal{O}'$ *and* $g : \mathcal{P} \to \mathcal{P}'$ *is given by*

$$(f \circ g)(x_0 \otimes x_1 \otimes \cdots \otimes x_k) = (-1)^{\varepsilon} f(x_0) \otimes g(x_1) \otimes \cdots \otimes g(x_k),$$

where $\varepsilon = \deg(g) \sum_{i=0}^{k} \deg(x_i)(k-i)$ *is the Koszul sign obtained from swapping each* g *by the corresponding elements.*

It is known that the category of collections with plethysm is a monoidal category, where the unit is the collection $I(1) = R$ and $I(n) = 0$ for $n \neq 1$. The following lemma is a well-known fact that describes operads in terms of this monoidal structure, see [LV12, §5] for more details.

Lemma 2.3.5. *An operad* $\mathcal{O}$ *is equivalent to a monoid in the monoidal category of collections with plethysm, where the multiplication map is given precisely by the composition* $\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$*.*

Definition 2.3.6. *An operad* $\mathcal{O}$ *is called* reduced *if* $\mathcal{O}(0) = 0$*.*

Definition 2.3.7. *The* Hadamard product $\mathcal{O} \otimes \mathcal{P}$ *of two operads* $\mathcal{O}$ *and* $\mathcal{P}$ *is given on each arity component by* $(\mathcal{O} \otimes \mathcal{P})(n) = \mathcal{O}(n) \otimes \mathcal{P}(n)$*. The structure maps are given by diagonal composition and diagonal symmetric group action in the case of symmetric operads.*

The next definition is the most important and common example of an operad. It is very useful to keep it in mind when intuitively thinking about operads.

Definition 2.3.8. *The* endomorphism operad End_A *of a graded R-module A is given by the modules*

$$\mathrm{End}_A(n) = \mathrm{Hom}_R(A^{\otimes n}, A).$$

Insertion maps are given by

$$f \circ_i g = f(1^{\otimes i-1} \otimes g \otimes 1^{\otimes n-i})$$

for $f \in \mathrm{End}_A(n)$ and $g \in \mathrm{End}_A(m)$. The identity element is given by the identity map and there is a symmetric group action given by permuting the inputs.

The endomorphism operad also allows us to define algebras over any operad.

Definition 2.3.9. *An* algebra over an operad $\mathcal{O}$*, or* $\mathcal{O}$-algebra*, is a map of operads $\mathcal{O} \to \mathrm{End}_A$ for some R-module A. By adjunction, this is equivalent to a collection of maps $\mathcal{O}(n) \otimes A^{\otimes n} \to A$ for each $n \geq 0$.*

Definition 2.3.10. *A* morphism of $\mathcal{O}$-algebras *A and B is a map of operads $\mathrm{End}_A \to \mathrm{End}_B$ so that the diagram*

$$\begin{tikzcd}
\mathcal{O} \arrow[r] \arrow[dr] & \mathrm{End}_A \arrow[d] \\
& \mathrm{End}_B
\end{tikzcd}$$

commutes. By adjunction, this is equivalent to a map $f : A \to B$ of

R-modules such that the diagram

$$\begin{tikzcd}
\mathcal{O}(n) \otimes A^{\otimes n} \arrow[r] \arrow[d, "id\otimes f^{\otimes n}"] & A \arrow[d, "f"] \\
\mathcal{O}(n) \otimes B^{\otimes n} \arrow[r] & B
\end{tikzcd}$$

commutes for all n.

Definition 2.3.11. *The $\mathcal{A}_\infty$-*operad *is the non-symmetric operad whose algebras are A_∞-algebras, see Definition 2.2.1. Therefore, it is generated by elements $\mu_i \in \mathcal{A}_\infty(i)$ satisfying the operadic version of the A_∞-equation (2.4).*

$$\sum_{r+s+t=n} (-1)^{rs+t} \mu_{r+t+1} \circ_{r+1} \mu_s = 0. \tag{2.5}$$

More details about this operad can be found in [LV12, Chapter 9].

Notice that for the $\mathcal{A}_\infty$-operad, a morphism of $\mathcal{A}_\infty$-algebras, Definition 2.3.10, is the same thing as a morphism of A_∞-algebras, Definition 2.2.3. We will also provide a new operadic interpretation of ∞-morphisms, Definition 2.2.2, and relate it to an existing interpretation in Section 3.2.2.

Remark 2.3.12. If one considers m_1 as an internal differential of the algebra A, Equation (2.4) reads for each j as

$$\begin{aligned}
&m_1(m_j) - (-1)^s \sum_{r+t+1=j} m_j(1^{\otimes r} \otimes m_1 \otimes 1^{\otimes t}) \\
&= - \sum_{\substack{r+s+t=j \\ s>1, r+t>0}} (-1)^{rs+t} m_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}).
\end{aligned}$$

This leads to a definition of the operad $\mathcal{A}_\infty$ in the category of cochain complexes as the operad generated by $\mu_i \in \mathcal{A}_\infty(i)$ for $i > 1$ and with differential given by

$$\partial_\infty(\mu_j) = -\sum_{r+s+t=j} (-1)^{rs+t} \mu_{r+t+1} \circ_{r+1} \mu_s.$$

Notice that this operad can now be described with no other relations than the differential. This is an example of what is called a *quasi-free* operad in the literature, see [LV12, §6.3.3]

2.3.2 Operads and monoidality

The monoidal definition of operad from Lemma 2.3.5 allows to define the dual notion of a cooperad.

Definition 2.3.13. *Let $\mathcal{O}$ be a collection. A* cooperad *is a structure of comonoid on $\mathcal{O}$ in the monoidal category* $(\mathrm{Col}, \bar{\circ}, I)$*, where*

$$(\mathcal{P} \bar{\circ} \mathcal{Q})(n) \coloneqq \bigoplus_r \bigoplus_{n=i_1+\cdots+i_r} (\mathcal{P}(r) \otimes \mathcal{Q}(i_1) \otimes \cdots \otimes \mathcal{Q}(i_r)),$$

and I is the collection such that $I(0) = R$ and is trivial elsewhere. See [LV12, §5.7.1] for more details and the symmetric version.

Note that this is not the exact dual notion of an operad. To define the exact dual of the notion of operad, one should instead consider the monoidal product

$$(\mathcal{P}\hat{\circ}\mathcal{Q})(n) = \prod_{r\geq 0}(\mathcal{P}(r) \otimes \prod_{n=i_1+\cdots+i_r} (\mathcal{Q}(i_1) \otimes \cdots \otimes \mathcal{Q}(i_r)))$$

in the category of collections, where the sums are replaced by products. In that case, a cooperad is defined as a comonoid $\Delta : \mathcal{O} \to \mathcal{O}\hat{\circ}\mathcal{O}$. When $\mathcal{O}(0) = 0$ (the operad is reduced), the right-hand side product is equal to a sum. In this case, we are back to the previous definition.

The following result is stated and proved in [Fre17, Proposition 3.1.1(a)], but the proof omits many details, so we are writing down the full proof here.

Proposition 2.3.14. *Any symmetric lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $\underline{F} : \mathrm{Op}_{\mathcal{C}} \to \mathrm{Op}_{\mathcal{D}}$ between the categories of operads in $\mathcal{C}$ and $\mathcal{D}$, respectively. The result is also true for cooperads.*

Proof. We prove the result for operads, since for cooperads is analogous. Let $\mathcal{O}$ be an operad in $\mathcal{C}$ and let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric lax monoidal functor. On objects, we define $\underline{F}(\mathcal{O})(n) = F(\mathcal{O}(n))$ and on morphisms we define $\underline{F}(f)_n = F(f_n)$ for $f : \mathcal{O} \to \mathcal{P}$.

Let $\varepsilon : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ and $\mu := \mu_{A,B} : F(A) \otimes F(B) \to F(A \otimes B)$ be the structure maps of the lax monoidal functor F.

Let us first define the structure maps for the operad $F(\mathcal{O})$ in terms of insertions. Let $e : 1_{\mathcal{C}} \to \mathcal{O}(1)$ be the unit of $\mathcal{O}$. We define the unit $e_F : 1_{\mathcal{D}} \to F(\mathcal{O}(1))$ as the composite

$$1_{\mathcal{D}} \xrightarrow{\varepsilon} F(1) \xrightarrow{F(e)} F(\mathcal{O}(1)).$$

Let $\circ_i : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n+m-1)$ be the insertion map on $\mathcal{O}$. We define the insertion map $\circ_i^F : F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m)) \to F(\mathcal{O}(n+m-1))$ as the composite

$$F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m)) \xrightarrow{\mu} F(\mathcal{O}(n) \otimes \mathcal{O}(m)) \xrightarrow{F(\circ_i)} F(\mathcal{O}(n+m-1)).$$

We show now that $F(\mathcal{O})$ satisfies the unit axioms with the above structure maps. We only show the unit axiom with respect to the right unitor, the axiom with respect to the left unitor being analogous.

Let $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{D}}$ be the right unitors of $\mathcal{C}$ and $\mathcal{D}$ respectively. Since $\mathcal{O}$ is an operad, by the unit axiom we have that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{O}(n) \otimes 1_{\mathcal{C}} \arrow[r, "\lambda_{\mathcal{C}}"] \arrow[d, "id\otimes e"] & \mathcal{O}(n) \\
\mathcal{O}(n) \otimes \mathcal{O}(1) \arrow[ur, bend right, "\circ_i"'] &
\end{tikzcd}$$

Applying F and introducing μ we get the following commutative diagram.

$$\begin{tikzcd}
F(\mathcal{O}(n)) \otimes F(1_{\mathcal{C}}) \arrow[r, "\mu"] \arrow[d, "id\otimes F(e)"] & F(\mathcal{O}(n) \otimes 1_{\mathcal{C}}) \arrow[r, "F(\lambda_{\mathcal{C}})"] \arrow[d, "F(id\otimes e)"] & F(\mathcal{O}(n)) \\
F(\mathcal{O}(n)) \otimes F(\mathcal{O}(1)) \arrow[r, "\mu"] & F(\mathcal{O}(n) \otimes \mathcal{O}(1)) \arrow[ur, bend right, "F(\circ_i)"'] &
\end{tikzcd} \tag{2.6}$$

We need to show that the following diagram commutes.

$$\begin{tikzcd}
F(\mathcal{O}(n)) \otimes 1_{\mathcal{D}} \arrow[r, "\lambda_{\mathcal{D}}"] \arrow[d, "id\otimes e_F"] & F(\mathcal{O}(n)) \\
F(\mathcal{O}(n)) \otimes F(\mathcal{O}(1)) \arrow[ur, bend right, "\circ_i^F"'] &
\end{tikzcd}$$

By monoidality of F we know that $\lambda_{\mathcal{D}}$ satisfies the following commutative diagram.

$$\begin{tikzcd}
F(\mathcal{O}(n)) \otimes 1_{\mathcal{D}} \arrow[r, "id\otimes\varepsilon"] \arrow[d, "\lambda_{\mathcal{D}}"] & F(\mathcal{O}(n)) \otimes F(1) \arrow[d, "\mu"] \\
F(\mathcal{O}(n)) & F(\mathcal{O}(n) \otimes 1_{\mathcal{C}}) \arrow[l, "F(\lambda)"']
\end{tikzcd}$$

Or, in other words, $\lambda_{\mathcal{D}} = F(\lambda)\circ\mu(id\otimes\varepsilon)$. On the other hand, by diagram (2.6) we have that $F(\lambda) \circ \mu = (F(\circ_i) \circ \mu)(id \otimes F(e))$, meaning that

$$\lambda_{\mathcal{D}} = F(\circ_i) \circ \mu \circ (id \otimes F(e)) \circ (id \otimes \varepsilon) = \circ_i^F(id \otimes e_F)$$

as we wanted to show.

Next we need to show that the associativity axioms of operads hold for $F(\mathcal{O})$, we refer the reader to [KWZ15, §1.1.2] to recall them. Let us first prove the one that does not involve the symmetry isomorphism.

Let $a_{\mathcal{C}}$ and $a_{\mathcal{D}}$ the associators for $\mathcal{C}$ and $\mathcal{D}$, respectively. We have the following commutative diagram from the associativity axioms of the operad $\mathcal{O}$ for $i \leq j \leq i+m-1$.

$$\begin{tikzcd}
(\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(l) \arrow[r, "a_{\mathcal{C}}"] \arrow[d, "\circ_i\otimes"] & \mathcal{O}(n) \otimes (\mathcal{O}(m) \otimes \mathcal{O}(l)) \arrow[d, "id\otimes\circ_{j-i+1}"] \\
\mathcal{O}(n+m-1) \otimes \mathcal{O}(l) \arrow[d, "\circ_j"] & \mathcal{O}(n) \otimes \mathcal{O}(m+l-1) \arrow[dl, bend left, "\circ_i"] \\
\mathcal{O}(n+m+l-2) &
\end{tikzcd}$$

Applying F we obtain the following commutative diagram.

$$
\begin{tikzcd}
F((\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(l)) \arrow[r, "F(a_{\mathcal{C}})"] \arrow[d, "F(\circ_i \otimes id)"] & F(\mathcal{O}(n) \otimes (\mathcal{O}(m) \otimes \mathcal{O}(l))) \arrow[d, "F(id \otimes \circ_{j-i+1})"] \\
F(\mathcal{O}(n+m-1) \otimes \mathcal{O}(l)) \arrow[d, "F(\circ_j)"] & F(\mathcal{O}(n) \otimes \mathcal{O}(m+l-1)) \arrow[dl, bend left, "F(\circ_i)"] \\
F(\mathcal{O}(n+m+l-2)) &
\end{tikzcd}
\tag{2.7}
$$

According to the definition of $\circ_i^F$, we need to show that the following diagram commutes.

$$
\begin{tikzcd}
(F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m))) \otimes F(\mathcal{O}(l)) \arrow[r, "a_{\mathcal{D}}"] \arrow[d, "\mu \otimes id"] & F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(m)) \otimes F(\mathcal{O}(l))) \arrow[d, "id \otimes \mu"] \\
F(\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes F(\mathcal{O}(l)) \arrow[d, "F(\circ_i) \otimes id"] & F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m) \otimes \mathcal{O}(l)) \arrow[d, "id \otimes F(\circ_{j+i-1})"] \\
F(\mathcal{O}(n+m-1)) \otimes F(\mathcal{O}(l)) \arrow[d, "\mu"] & F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m+l-1)) \arrow[d, "\mu"] \\
F(\mathcal{O}(n+m-1) \otimes \mathcal{O}(l)) \arrow[d, "F(\circ_j)"] & F(\mathcal{O}(n) \otimes \mathcal{O}(m+l-1)) \arrow[dl, bend left, "F(\circ_i)"] \\
F(\mathcal{O}(n+m+l-2)) &
\end{tikzcd}
\tag{2.8}
$$

By naturality of μ we have

$$
\mu \circ (F(\circ_i) \otimes id) = F(\circ_i \otimes id) \circ \mu \tag{2.9}
$$

and

$$
\mu \circ (id \otimes F(\circ_{j-i+1})) = F(id \otimes \circ_{j-i+1}) \circ \mu.
$$

Therefore we can replace the above compositions in diagram (2.8) accordingly. We can also subdivide the above diagram into two by using $F(a_{\mathcal{C}})$ as follows.

$$\begin{tikzcd}
(F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m))) \otimes F(\mathcal{O}(l)) \arrow[r, "a_{\mathcal{D}}"] \arrow[d, "\mu\otimes id"] & F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(m)) \otimes F(\mathcal{O}(l))) \arrow[d, "id\otimes\mu"] \\
F(\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes F(\mathcal{O}(l)) \arrow[d, "\mu"] & F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m) \otimes \mathcal{O}(l)) \arrow[d, "\mu"] \\
F((\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(l)) \arrow[r, dashed, "F(a_{\mathcal{C}})"] \arrow[d, "F(\circ_i\otimes id)"] & F(\mathcal{O}(n)) \otimes F(\mathcal{O}(n) \otimes (\mathcal{O}(m) \otimes \mathcal{O}(l))) \arrow[d, "F(id\otimes\circ_{j-1+1})"] \\
F(\mathcal{O}(n+m-1) \otimes \mathcal{O}(l)) \arrow[d, "F(\circ_j)"] & F(\mathcal{O}(n) \otimes \mathcal{O}(m+l-1)) \arrow[dl, bend left, "F(\circ_i)"] \\
F(\mathcal{O}(n+m+l-2)) &
\end{tikzcd}$$

Now, the top diagram commutes because it is the associativity axiom of lax monoidal functors. The bottom diagram is precisely diagram (2.7), so it commutes and we get the desired associativity axiom.

Finally, we need to show that the associativity axioms involving the symmetry isomorphism hold for $F(\mathcal{O})$. Since they are analogous to each other, we only prove the first one.

Let $B_{\mathcal{C}} := B_{\mathcal{C}}^{X,Y} : X \otimes Y \to Y \otimes X$ the symmetry isomorphism on $\mathcal{C}$ and similarly denote by $B_{\mathcal{D}}$ the symmetry isomorphism on $\mathcal{D}$.

We have the following associativity commutative diagram for $j < i$.

$$\begin{tikzcd}
(\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(l) \arrow[r, "a_{\mathcal{C}}"] \arrow[d, "\circ_i\otimes id"] & \mathcal{O}(n) \otimes (\mathcal{O}(m) \otimes \mathcal{O}(l)) \arrow[d, "id\otimes B_{\mathcal{C}}"] \\
\mathcal{O}(n+m-1) \otimes \mathcal{O}(l) \arrow[d, "\circ_j"] & \mathcal{O}(n) \otimes (\mathcal{O}(l) \otimes \mathcal{O}(m)) \arrow[d, "a_{\mathcal{C}}^{-1}"] \\
\mathcal{O}(n+m+l-2) & (\mathcal{O}(n) \otimes \mathcal{O}(l)) \otimes \mathcal{O}(m) \arrow[d, "\circ_j\otimes id"] \\
& \mathcal{O}(n+l-1) \otimes \mathcal{O}(m) \arrow[ul, bend left, "\circ_i"]
\end{tikzcd}$$

Applying F we get the following commutative diagram.

$$
\begin{tikzcd}
F((\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(l)) \arrow[r, "F(a_{\mathcal{C}})"] \arrow[d, "F(\circ_i \otimes id)"] & F(\mathcal{O}(n) \otimes (\mathcal{O}(m) \otimes \mathcal{O}(l))) \arrow[d, "F(id \otimes B^{\mathcal{C}})"] \\
F(\mathcal{O}(n+m-1) \otimes \mathcal{O}(l)) \arrow[d, "F(\circ_j)"] & \mathcal{O}(n) \otimes (\mathcal{O}(l) \otimes \mathcal{O}(m)) \arrow[d, "F(a_{\mathcal{C}})^{-1}"] \\
F(\mathcal{O}(n+m+l-2)) & F((\mathcal{O}(n) \otimes \mathcal{O}(l)) \otimes \mathcal{O}(m)) \arrow[d, "F(\circ_j \otimes id)"] \\
 & F(\mathcal{O}(n+l-1) \otimes \mathcal{O}(m)) \arrow[ul, bend left, "F(\circ_i)"]
\end{tikzcd}
\tag{2.10}
$$

We need to show that the following diagram commutes.

$$
\begin{tikzcd}
(F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m))) \otimes F(\mathcal{O}(l)) \arrow[r, "a_{\mathcal{D}}"] \arrow[d, "\mu \otimes id"] & F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(m)) \otimes F(\mathcal{O}(l))) \arrow[d, "id \otimes B_{\mathcal{D}}"] \\
F(\mathcal{O}(n) \otimes \mathcal{O}(n)) \otimes F(\mathcal{O}(l)) \arrow[d, "F(\circ_i) \otimes id"] & F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(l)) \otimes F(\mathcal{O}(m))) \arrow[d, "a_{\mathcal{D}}^{-1}"] \\
F(\mathcal{O}(n+m-1) \otimes F(\mathcal{O}(l)) \arrow[d, "\mu \otimes id"] & (F(\mathcal{O}(n)) \otimes F(\mathcal{O}(l))) \otimes F(\mathcal{O}(m)) \arrow[d, "\mu \otimes id"] \\
F(\mathcal{O}(m+n-1) \otimes \mathcal{O}(l)) \arrow[d, "F(\circ_j)"] & F(\mathcal{O}(n) \otimes \mathcal{O}(l)) \otimes F(\mathcal{O}(m)) \arrow[d, "F(\circ_j) \otimes id"] \\
F((\mathcal{O}(n+m+l-2)) & F(\mathcal{O}(n+l-1)) \otimes F(\mathcal{O}(m)) \arrow[d, "\mu"] \\
 & F(\mathcal{O}(n+l-1) \otimes \mathcal{O}(m)) \arrow[ul, bend left, "F(\circ_i)"]
\end{tikzcd}
$$

We use naturality of μ, i.e Equation (2.9), as we have done before to rewrite some of the arrows. We also subdivide the diagram into two by factoring by $F(a_{\mathcal{C}})^{-1} \circ F(id \otimes B_{\mathcal{C}}) \circ F(a_{\mathcal{C}})$.

$$
\begin{tikzcd}
(F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m))) \otimes F(\mathcal{O}(l)) \arrow[r, "a_{\mathcal{D}}"] \arrow[d, "\mu\otimes id"'] \arrow[ddd, phantom] & F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(m)) \otimes F(\mathcal{O}(l))) \arrow[d, "id\otimes B_{\mathcal{D}}"] \\
F(\mathcal{O}(n) \otimes \mathcal{O}(n)) \otimes F(\mathcal{O}(l)) \arrow[d, "\mu"'] & F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(l)) \otimes F(\mathcal{O}(m))) \arrow[d, "a_{\mathcal{D}}^{-1}"] \\
F((\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(l)) \arrow[d, "F(\circ_i\otimes id)"'] \arrow[rddd, dashed, "F(a_{\mathcal{C}})^{-1}\circ F(id\otimes B_{\mathcal{C}})\circ F(a_{\mathcal{C}})"] & (F(\mathcal{O}(n)) \otimes F(\mathcal{O}(l))) \otimes F(\mathcal{O}(m)) \arrow[d, "\mu\otimes id"] \\
F(\mathcal{O}(m+n-1) \otimes \mathcal{O}(l)) \arrow[d, "F(\circ_j)"'] & F(\mathcal{O}(n) \otimes \mathcal{O}(l)) \otimes F(\mathcal{O}(m)) \arrow[d, "\mu"] \\
F((\mathcal{O}(n+m+l-2)) & F((\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(m)) \arrow[d, "F(\circ_j\otimes id)"] \\
 & F(\mathcal{O}(n+l-1) \otimes \mathcal{O}(m)) \arrow[lu, bend left, "F(\circ_i)"]
\end{tikzcd}
$$

The bottom diagram commutes as it is precisely diagram (2.10). We decompose the top diagram as follows.

$$
\begin{tikzcd}
F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(m)) \otimes F(\mathcal{O}(l))) \arrow[r, "id\otimes B_{\mathcal{D}}"] \arrow[dddd, dashed, bend left, "(id\otimes\mu)\circ\mu"] & F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(l)) \otimes F(\mathcal{O}(m))) \arrow[d, "a_{\mathcal{D}}^{-1}"] \arrow[dddd, dashed, bend right, "(id\otimes\mu)\circ\mu"'] \\
(F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m))) \otimes F(\mathcal{O}(l)) \arrow[u, "a_{\mathcal{D}}"] \arrow[d, "\mu\otimes id"] & (F(\mathcal{O}(n)) \otimes F(\mathcal{O}(l))) \otimes F(\mathcal{O}(m)) \arrow[d, "\mu\otimes id"] \\
F(\mathcal{O}(n) \otimes \mathcal{O}(n)) \otimes F(\mathcal{O}(l)) \arrow[d, "\mu"] & F(\mathcal{O}(n) \otimes \mathcal{O}(l)) \otimes F(\mathcal{O}(m)) \arrow[d, "\mu"] \\
F((\mathcal{O}(n) \otimes \mathcal{O}(m)) \otimes \mathcal{O}(l)) \arrow[d, "F(a)"] & F((\mathcal{O}(n) \otimes \mathcal{O}(l)) \otimes \mathcal{O}(m)) \\
F(\mathcal{O}(n) \otimes (\mathcal{O}(m) \otimes \mathcal{O}(l))) \arrow[r, "F(id\otimes B_{\mathcal{C}})"] & F(\mathcal{O}(n) \otimes (\mathcal{O}(l) \otimes \mathcal{O}(m))) \arrow[u, "F(a_{\mathcal{C}})^{-1}"']
\end{tikzcd}
$$

The left and right subdiagrams commute because of the associativity axiom of lax monoidal functors. We decompose the central subdiagram further as

$$\begin{tikzcd}
F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(m)) \otimes F(\mathcal{O}(l)) \arrow[r, "id\otimes B_{\mathcal{D}}"] \arrow[d, "id\otimes\mu"] & F(\mathcal{O}(n)) \otimes (F(\mathcal{O}(l)) \otimes F(\mathcal{O}(m))) \arrow[d, "id\otimes\mu"] \\
F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m) \otimes \mathcal{O}(l)) \arrow[r, dashed, "id\otimes F(B_{\mathcal{C}})"] \arrow[d, "\mu"] & F(\mathcal{O}(n)) \otimes F(\mathcal{O}(l) \otimes \mathcal{O}(m)) \arrow[d, "\mu"] \\
F(\mathcal{O}(n) \otimes (\mathcal{O}(m) \otimes \mathcal{O}(l))) \arrow[r, "F(id\otimes B_{\mathcal{C}})"] & F(\mathcal{O}(n) \otimes (\mathcal{O}(l) \otimes \mathcal{O}(m)))
\end{tikzcd}$$

The top part commutes because F is symmetric lax monoidal and the bottom part commutes by naturality of μ. This proves that $F(\mathcal{O})$ is an operad in $\mathcal{D}$.

Lastly, we are only left with the proof that $F(f)$ is a map of operads. Since f is a map of operads, we have for all n the following commutative diagram.

$$\begin{tikzcd}
\mathcal{O}(n) \otimes \mathcal{O}(m) \arrow[r, "f_n\otimes f_m"] \arrow[d, "\circ_i^{\mathcal{O}}"] & \mathcal{P}(n) \otimes \mathcal{P}(m) \arrow[d, "\circ_i^{\mathcal{P}}"] \\
\mathcal{O}(n+m-1) \arrow[r, "f_{n+m-1}"] & \mathcal{P}(n+m-1)
\end{tikzcd}$$

After applying F we get the following commutative diagram.

$$\begin{tikzcd}
F(\mathcal{O}(n) \otimes \mathcal{O}(m)) \arrow[r, "F(f_n\otimes f_m)"] \arrow[d, "F(\circ_i^{\mathcal{O}})"] & F(\mathcal{P}(n) \otimes \mathcal{P}(m)) \arrow[d, "F(\circ_i^{\mathcal{P}})"] \\
F(\mathcal{O}(n+m-1)) \arrow[r, "F(f_{n+m-1})"] & F(\mathcal{P}(n+m-1))
\end{tikzcd} \tag{2.11}$$

We need to show that the following diagram commutes.

$$\begin{tikzcd}
F(\mathcal{O}(n)) \otimes F(\mathcal{O}(m)) \arrow[r, "F(f_n)\otimes F(f_m)"] \arrow[d, "\mu"] & F(\mathcal{P}(n)) \otimes F(\mathcal{P}(m)) \arrow[d, "\mu"] \\
F(\mathcal{O}(n) \otimes \mathcal{O}(m)) \arrow[r, dashed, "F(f_n\otimes f_m)"] \arrow[d, "F(\circ_i^{\mathcal{O}})"] & F(\mathcal{P}(n) \otimes \mathcal{P}(m)) \arrow[d, "F(\circ_i^{\mathcal{P}})"] \\
F(\mathcal{O}(n+m-1)) \arrow[r, "F(f_{n+m-1})"] & F(\mathcal{P}(n+m-1))
\end{tikzcd}$$

The top subdiagram commutes because μ is natural and the bottom part is precisely diagram (2.11), which commutes. This finishes the proof. $\square$

2.4 Base categories and totalization

Now introduce some categories and conventions that we need in order to study derived A_∞-algebras. Many results of A_∞-algebras need R to be a field because of projectivity, see Section 2.2.3. Thus, it is necessary to build in projective resolutions. In particular we need an extra degree compatible with derived A_∞-setting. In order to do that, we need a way to connect a single graded category with a bigraded category. This is usually done through totalization. But in order to properly translate A_∞-algebras into totalized derived A_∞-algebras we need to go through several suitably enriched categories that are defined in this section. Most of the definitions come from [CESLW18, §2] but we adapt them here to our conventions.

2.4.1 Filtered modules and complexes

First, we collect some definitions about filtered modules and filtered complexes. Filtrations will allow to add an extra degree to single-graded objects that will be necessary in order to connect them with bigraded objects.

Definition 2.4.1. *A* filtered R-module (A, F) *is given by a family of R-modules $\{F_pA\}_{p\in\mathbb{Z}}$ indexed by the integers such that $F_pA \subseteq F_{p-1}A$ for all $p \in \mathbb{Z}$ and $A = \bigcup_p F_pA$.*

A morphism of filtered modules *is a morphism $f : A \to B$ of R-modules which is compatible with filtrations, namely*

$$f(F_pA) \subset F_pB \text{ for all } p \in \mathbb{Z}.$$

Note that some other sources may reverse inclusions in the above definition and consider $F_pA \subseteq F_{p+1}A$ instead. We denote by C_R the category of cochain complexes of R-modules.

Definition 2.4.2. *A* filtered complex (K, d, F) *is a cochain complex $(K, d) \in \mathrm{C}_R$ together with a filtration F of each R-module K^n such that $d(F_pK^n) \subset F_pK^{n+1}$ for all $p, n \in \mathbb{Z}$. A* morphism *of filtered complexes is given by a morphism of complexes $f : K \to L$ compatible with filtrations, namely*

$$f(F_pK) \subset F_pL \text{ for all } p \in \mathbb{Z}.$$

We denote by fMod_R and fC_R the categories of filtered modules and filtered complexes of R-modules, respectively.

Definition 2.4.3. *The* tensor product of two filtered R-modules (A, F) *and (B, F) is the filtered R-module with*

$$F_p(A \otimes B) := \sum_{i+j=p} \mathrm{Im}(F_iA \otimes F_jB \to A \otimes B).$$

This makes the category of filtered R-modules into a symmetric monoidal category, where the unit is given by R with the trivial filtration

$$0 = F_1R \subset F_0R = R.$$

Definition 2.4.4. *Let K and L be filtered complexes. We define $\underline{\mathrm{Hom}}(K, L)$ to be the filtered complex whose underlying cochain complex is $\mathrm{Hom}_{\mathrm{C}_R}(K, L)$ and the filtration F given by*

$$F_p\underline{\mathrm{Hom}}(K, L) = \{f : K \to L \mid f(F_qK) \subset F_{q+p}L \text{ for all } q \in \mathbb{Z}\}.$$

In particular, $\mathrm{Hom}_{\mathrm{fMod}_R}(K, L) = F_0\underline{\mathrm{Hom}}(K, L)$.

2.4.2 Bigraded modules, vertical bicomplexes, twisted complexes and sign conventions

We collect some basic definitions of bigraded categories that we need to use, and we establish some conventions.

Definition 2.4.5. *We consider $(\mathbb{Z}, \mathbb{Z})$-bigraded R-modules $A = \{A_i^j\}$, where elements of A_i^j are said to have* bidegree *(i, j). We sometimes refer to i as the* horizontal degree *and j the* vertical degree. *The* total degree *of an element $x \in A_i^j$ is $i + j$ and is sometimes denoted by $|x|$.*

Definition 2.4.6. *A* morphism of bidegree *(p, q) maps A_i^j to A_{i+p}^{j+q}. The tensor product of two bigraded R-modules A and B is the bigraded R-module $A \otimes B$ given by*

$$(A \otimes B)_i^j \coloneqq \bigoplus_{p,q} A_p^q \otimes B_{i-p}^{j-q}.$$

We denote by bgMod_R the category whose objects are bigraded R-modules and whose morphisms are morphisms of bigraded R-modules of bidegree $(0,0)$. It is symmetric monoidal with the above tensor product.

We introduce the following scalar product notation for bidegrees: for x, y of bidegree (x_1, x_2), (y_1, y_2) respectively, we let $\langle x, y\rangle = x_1y_1 + x_2y_2$.

The symmetry isomorphism

$$\tau_{A\otimes B} : A \otimes B \to B \otimes A$$

is given by

$$x \otimes y \mapsto (-1)^{\langle x,y\rangle} y \otimes x.$$

We follow the Koszul sign rule: if $f : A \to B$ and $g : C \to D$ are bigraded morphisms, then the morphism $f \otimes g : A \otimes C \to B \otimes D$ is defined by

$$(f \otimes g)(x \otimes z) \coloneqq (-1)^{\langle g,x\rangle} f(x) \otimes g(z).$$

Definition 2.4.7. *A* vertical bicomplex *is a bigraded R-module A equipped with a vertical differential $d^A : A \to A$ of bidegree $(0,1)$. A* morphism of vertical bicomplexes *is a morphism of bigraded modules of bidegree $(0,0)$ commuting with the vertical differential.*

We denote by vbC_R the category of vertical bicomplexes. The tensor product of two vertical bicomplexes A and B is given by endowing the tensor product of underlying bigraded modules with vertical differential

$$d^{A\otimes B} := d^A \otimes 1 + 1 \otimes d^B : (A \otimes B)^v_u \to (A \otimes B)^{v+1}_u.$$

This makes vbC_R into a symmetric monoidal category. The symmetric monoidal categories $(\mathrm{C}_R, \otimes, R)$, $(\mathrm{bgMod}_R, \otimes, R)$ and $(\mathrm{vbC}_R, \otimes, R)$ are related by embeddings $\mathrm{C}_R \to \mathrm{vbC}_R$ and $\mathrm{bgMod}_R \to \mathrm{vbC}_R$ which are monoidal and full.

The following is [CESLW18, Lemma 4.15].

Lemma 2.4.8. *The category* fC_R *is monoidal over* vbC_R*. By restriction,* fMod_R *is monoidal over* bgMod_R.

Definition 2.4.9. *Let A, B be bigraded modules. We define $[A, B]_*^*$ to be the bigraded module of morphisms of bigraded modules $A \to B$. Furthermore, if A, B are vertical bicomplexes, and $f \in [A, B]_u^v$, we define*

$$\delta(f) := d^B f - (-1)^v f d^A.$$

Lemma 2.4.10. *If A, B are vertical bicomplexes, then $([A, B]_*^*, \delta)$ is a vertical bicomplex.*

Proof. Direct computation shows $\delta^2 = 0$. □

Definition 2.4.11. *A* twisted complex (A, d_m) *is a bigraded R-module $A = \{A_i^j\}$ together with a family of morphisms $\{d_m : A \to A\}_{m \geq 0}$ of bidegree $(m, 1-m)$ such that for all $m \geq 0$,*

$$\sum_{i+j=m} (-1)^i d_i d_j = 0.$$

Definition 2.4.12. *Let (A, d_m^A) and (B, d_m^B) be twisted complexes. A* morphism of twisted complexes $f : (A, d_m^A) \to (B, d_m^B)$ *is given by a family of morphisms of R-modules $\{f_m : A \to B\}_{m \geq 0}$ of bidegree $(m, -m)$ such* *that for all* $m \geq 0$*,*

$$\sum_{i+j=m} d_i^B f_j = \sum_{i+j=m} (-1)^i f_i d_j^A.$$

The composition of morphisms is given by $(g \circ f)_m := \sum_{i+j=m} g_i f_j$*.*

A morphism $f = \{f_m\}_{m\geq 0}$ *is said to be* strict *if* $f_i = 0$ *for all* $i > 0$*. The* identity *morphism* $1_A : A \to A$ *is the strict morphism given by* $(1_A)_0(x) = x$*. A morphism* $f = \{f_i\}$ *is an* isomorphism *if and only if* f_0 *is an isomorphism of bigraded* R*-modules.*

Note that if f is an isomorphism, then an inverse of f is obtained from an inverse of f_0 by solving a triangular system of linear equations.

Denote by tC_R the category of twisted complexes. The following construction endows tC_R with a symmetric monoidal structure, see [CESLW18, Lemma 3.3] for a proof.

Lemma 2.4.13. *The category* $(\mathrm{tC}_R, \otimes, R)$ *is symmetric monoidal, where the monoidal structure is given by the bifunctor*

$$\otimes : \mathrm{tC}_R \times \mathrm{tC}_R \to \mathrm{tC}_R.$$

On objects it is given by

$$((A, d_m^A), (B, d_m^B)) \to (A \otimes B, d_m^A \otimes 1 + 1 \otimes d_m^B)$$

and on morphisms it is given by $(f, g) \to f \otimes g$*, where*

$$(f \otimes g)_m := \sum_{i+j=m} f_i \otimes g_j.$$

In particular, by the Koszul sign rule we have that

$$(f_i \otimes g_j)(x \otimes z) = (-1)^{\langle g_j, x\rangle} f_i(x) \otimes g_j(z).$$

The symmetry isomorphism is given by the strict morphism of twisted complexes

$$\begin{aligned} \tau_{A\otimes B} \colon A \otimes B &\to B \otimes A \\ x \otimes y &\mapsto (-1)^{\langle x,y\rangle} y \otimes x. \end{aligned}$$

The internal hom on bigraded modules can be extended to twisted complexes via the following lemma whose proof is in [CESLW18, Lemma 3.4].

Lemma 2.4.14. *Let A, B be twisted complexes. For $f \in [A, B]_u^v$, setting*

$$(d_i f) := (-1)^{i(u+v)} d_i^B f - (-1)^v f d_i^A$$

for $i \geq 0$ endows $[A, B]_*^*$ with the structure of a twisted complex.

2.4.3 Totalization

Here we recall the definition of the totalization functor from [CESLW18] and some of the structure that it comes with. This functor and its enriched versions are key to establish a correspondence between A_∞-algebras and derived A_∞-algebras.

Definition 2.4.15. *The* totalization *of a bigraded R-module $A = \{A_i^j\}$ is the graded R-module* $\mathrm{Tot}(A)$ *given by*

$$\mathrm{Tot}(A)^n \coloneqq \bigoplus_{i<0} A_i^{n-i} \oplus \prod_{i\geq 0} A_i^{n-i}.$$

The column filtration *of* $\mathrm{Tot}(A)$ *is the filtration given by*

$$F_p \,\mathrm{Tot}(A)^n \coloneqq \prod_{i\geq p} A_i^{n-i}.$$

Given a twisted complex (A, d_m), define a map $d : \mathrm{Tot}(A) \to \mathrm{Tot}(A)$ of degree 1 by letting

$$d(x)_j \coloneqq \sum_{m\geq 0} (-1)^{mn} d_m(x_{j-m})$$

for $x = (x_i)_{i\in\mathbb{Z}} \in \mathrm{Tot}(A)^n$. Here $x_i \in A_i^{n-i}$ denotes the i-th component of x, and $d(x)_j$ denotes the j-th component of $d(x)$. Note that, for a given $j \in \mathbb{Z}$ there is a sufficiently large $m \geq 0$ such that $x_{j-m'} = 0$ for all $m' \geq m$. Hence $d(x)_j$ is given by a finite sum. Also, for negative j sufficiently large, one has $x_{j-m} = 0$ for all $m \geq 0$, which implies $d(x)_j = 0$.

Given a morphism $f : (A, d_m) \to (B, d_m)$ of twisted complexes, let the *totalization of f* be the map $\mathrm{Tot}(f) : \mathrm{Tot}(A) \to \mathrm{Tot}(B)$ of degree 0 defined by

$$(\mathrm{Tot}(f)(x))_j \coloneqq \sum_{m\geq 0} (-1)^{mn} f_m(x_{j-m})$$

for $x = (x_i)_{i\in\mathbb{Z}} \in \mathrm{Tot}(A)^n$.

The following is [CESLW18, Theorem 3.8].

Theorem 2.4.16. *The assignments* $(A, d_m) \mapsto (\mathrm{Tot}(A), d, F)$*, where* F *is the column filtration of* $\mathrm{Tot}(A)$*, and* $f \mapsto \mathrm{Tot}(f)$ *define a functor* $\mathrm{Tot} : \mathrm{tC}_R \to \mathrm{fC}_R$ *which is an isomorphism when restricted to its image.*

For a filtered complex of the form $(\mathrm{Tot}(A), d, F)$ where $A = \{A_i^j\}$ is a bigraded R-module, we can recover the twisted complex structure on A as follows. For all $m \geq 0$, let $d_m : A \to A$ be the morphism of bidegree $(m, 1-m)$ defined by

$$d_m(x) = (-1)^{nm} d(x)_{i+m},$$

where $x \in A_i^{n-i}$ and $d(x)_k$ denotes the k-th component of $d(x)$. Note that $d(x)_k$ lies in A_k^{n+1-k}.

We will consider the following bounded categories since the totalization functor has better monoidal properties when restricted to them.

Definition 2.4.17. *We let* tC_R^b*,* vbC_R^b *and* bgMod_R^b *be the full subcategories of* horizontally bounded on the right *graded twisted complexes, vertical bicomplexes and bigraded modules respectively. This means that if* $A = \{A_i^j\}$ *is an object of any of these categories, then there exists* i *such that* $A_{i'}^j = 0$ *for* $i' > i$*. We let* fMod_R^b *and* fC_R^b *be the full subcategories of bounded filtered modules, respectively complexes, i.e. the full subcategories of objects* (K, F) *such that there exists some* p *with the property that* $F_{p'} K^n = 0$ *for all* $p' > p$*. We refer to all of these as the* bounded subcategories *of* tC_R*,* vbC_R*,* bgMod_R*,* fMod_R *and* fC_R *respectively.*

The following is [CESLW18, Proposition 3.11].

Proposition 2.4.18. *The totalization functors* $\mathrm{Tot} : \mathrm{bgMod}_R \to \mathrm{fMod}_R$ *and* $\mathrm{Tot} : \mathrm{tC}_R \to \mathrm{fC}_R$ *are lax symmetric monoidal with structure maps*

$$\epsilon : R \to \mathrm{Tot}(R) \text{ and } \mu = \mu_{A,B} : \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \to \mathrm{Tot}(A \otimes B)$$

given by $\epsilon = 1_R$. *For* $x = (x_i)_i \in \mathrm{Tot}(A)^{n_1}$ *and* $y = (y_j)_j \in \mathrm{Tot}(B)^{n_2}$,

$$\mu(x \otimes y)_k := \sum_{k_1+k_2=k} (-1)^{k_1 n_2} x_{k_1} \otimes y_{k_2}. \tag{2.12}$$

When restricted to the bounded case, $\mathrm{Tot} : \mathrm{bgMod}_R^b \to \mathrm{fMod}_R^b$ *and* $\mathrm{Tot} : \mathrm{tC}_R^b \to \mathrm{fC}_R^b$ *are strong symmetric monoidal functors.*

Remark 2.4.19. There is a certain heuristic to obtain the sign appearing in the definition of μ in Proposition 2.4.18. In the bounded case, we can write

$$\mathrm{Tot}(A) = \bigoplus_i A_i^{n-i}.$$

As direct sums commute with tensor products, we have

$$\mathrm{Tot}(A) \otimes \mathrm{Tot}(B) = (\bigoplus A_i^{n-i}) \otimes \mathrm{Tot}(B) \cong \bigoplus_i (A_i^{n-i} \otimes \mathrm{Tot}(B)).$$

In the isomorphism above we can interpret that each A_i^{n-i} passes by $\mathrm{Tot}(B)$. Since $\mathrm{Tot}(B)$ uses total grading, we can think of this degree as being the horizontal degree, while having 0 vertical degree. Thus, using the Koszul sign rule we would get precisely the sign from Proposition 2.4.18. This explanation is just an intuition, and opens the door for other possible sign choices: what if we decide to distribute $\mathrm{Tot}(A)$ over $\bigoplus_i B_i^{n-i}$ instead, or if we consider the total degree as the vertical degree? These alternatives lead to other valid definitions of μ, and we will explore the consequences of some of them in Remark 4.3.7.

Lemma 2.4.20. *In the conditions of Proposition 2.4.18 for the bounded case, the inverse*

$$\mu^{-1} : \mathrm{Tot}(A_{(1)} \otimes \cdots \otimes A_{(m)}) \to \mathrm{Tot}(A_{(1)}) \otimes \cdots \otimes \mathrm{Tot}(A_{(m)})$$

is given on pure tensors (for notational convenience) as

$$\mu^{-1}(x_{(1)} \otimes \cdots \otimes x_{(m)}) = (-1)^{\sum_{j=2}^{m} n_j \sum_{i=1}^{j-1} k_i} x_{(1)} \otimes \cdots \otimes x_{(m)}, \tag{2.13}$$

where $x_{(l)} \in (A_{(m)})_{k_l}^{n_l - k_l}$.

Proof. For the case $m = 2$,

$$\mu^{-1} : \mathrm{Tot}(A \otimes B) \to \mathrm{Tot}(A) \otimes \mathrm{Tot}(B)$$

is computed explicitly as follows. Let $c \in \mathrm{Tot}(A \otimes B)^n$. By definition, we have

$$\mathrm{Tot}(A \otimes B)^n = \bigoplus_k (A \otimes B)_k^{n-k} = \bigoplus_k \bigoplus_{\substack{k_1 + k_2 = k \\ n_1 + n_2 = n}} A_{k_1}^{n_1 - k_1} \otimes B_{k_2}^{n_2 - k_2}.$$

And thus, $c = (c_k)_k$ may be written as a finite sum $c = \sum_k c_k$, where

$$c_k = \sum_{\substack{k_1+k_2=k \\ n_1+n_2=n}} x_{k_1}^{n_1-k_1} \otimes y_{k_2}^{n_2-k_2}.$$

Here, we introduced superscripts to indicate the vertical degree, which, unlike in the definition of μ (Equation (2.12)), is not solely determined by the horizontal degree since the total degree also varies. However we are going to omit them in what follows for simplicity of notation. Distributivity allows us to rewrite c as

$$\begin{aligned} c &= \sum_k \sum_{\substack{k_1+k_2=k \\ n_1+n_2=n}} x_{k_1} \otimes y_{k_2} \\ &= \sum_{n_1+n_2=n} \sum_{k_1} \sum_{k_2} (x_{k_1} \otimes y_{k_2}) \\ &= \sum_{n_1+n_2=n} \left(\sum_{k_1} x_{k_1} \right) \otimes \left(\sum_{k_2} y_{k_2} \right). \end{aligned}$$

Therefore, μ^{-1} can be defined as

$$\mu^{-1}(c) = \sum_{n_1+n_2=n} \left(\sum_{k_1} (-1)^{k_1 n_2} x_{k_1} \right) \otimes \left(\sum_{k_2} y_{k_2} \right).$$

The general case follows inductively. $\square$

2.5 Enriched categories and totalization

We define here some useful enriched categories and results from [CESLW18, §4.3 and §4.4]. Each of these categories will be a piece in Theorem 4.5.1, which establishes a connection between A_∞-algebras and derived A_∞-algebras. Some of them have been modified according to our conventions.

Definition 2.5.1. *Let A, B, C be bigraded modules. We denote by $\underline{\mathit{bgMod}_R}(A,B)$ the bigraded module given by*

$$\underline{\mathit{bgMod}_R}(A,B)^v_u := \prod_{j\geq 0}[A,B]^{v-j}_{u+j}$$

where $[A,B]$ is the internal hom-object of bigraded modules. More precisely, $g \in \underline{\mathit{bgMod}_R}(A,B)^v_u$ is given by $g := (g_0, g_1, g_2, \dots)$, where $g_j : A \to B$ is a map of bigraded modules of bidegree $(u+j, v-j)$.

Moreover, we define a composition morphism

$$c : \underline{\mathit{bgMod}_R}(B,C) \otimes \underline{\mathit{bgMod}_R}(A,B) \to \underline{\mathit{bgMod}_R}(A,C)$$

by

$$c(f,g)_m := \sum_{i+j=m}(-1)^{i|g|} f_i g_j.$$

Definition 2.5.2. *Let (A, d_i^A) and (B, d_i^B) be twisted complexes and let $f \in \underline{\mathit{bgMod}_R}(A,B)^v_u$. Consider also $d^A := (d_i^A)_i \in \underline{\mathit{bgMod}_R}(A,A)^1_0$ and $d^B := (d_i^B)_i \in \underline{\mathit{bgMod}_R}(B,B)^1_0$. We define*

$$\delta(f) := c(d^B, f) - (-1)^{\langle f, d^A\rangle} c(f, d^A) \in \underline{\mathit{bgMod}_R}(A,B)^{v+1}_u.$$

More precisely,

$$(\delta(f))_m := \sum_{i+j=m} (-1)^{i|f|} d_i^B f_j - (-1)^{v+i} f_i d_j^A.$$

The following lemma justifies the above definition. For a proof see [CESLW18, Lemma 4.18].

Lemma 2.5.3. *The following equations hold.*

$$c(d^A, d^A) = 0$$

$$\delta^2 = 0$$

$$\delta(c(f,g)) = c(\delta(f), g) + (-1)^v c(f, \delta(g))$$

where v is the vertical degree of f. Furthermore, $f \in \underline{\mathcal{bgMod}}_R(A, B)$ is a map of twisted complexes if and only if $\delta(f) = 0$. In particular, f is a morphism in tC_R if and only if the bidegree of f is $(0, 0)$ and $\delta(f) = 0$. Moreover, for f, g morphisms in tC_R, we have that $c(f, g) = f \circ g$, where the latter denotes composition in tC_R.

Definition 2.5.4. *For A, B twisted complexes, we define $\underline{t\mathcal{C}_R}(A, B)$ to be the vertical bicomplex $\underline{t\mathcal{C}_R}(A, B) := (\underline{\mathcal{bgMod}}_R(A, B), \delta)$.*

Definition 2.5.5. *We denote by $\underline{\mathcal{bgMod}}_R$ the* bgMod_R-enriched category of bigraded modules *given by the following data.*

(1) The objects of $\underline{\mathcal{bgMod}}_R$ are bigraded modules.

(2) For A, B bigraded modules the hom-object is the bigraded module $\underline{\mathcal{bgMod}}_R(A, B)$.

(3) The composition morphism

$$c : \underline{\mathscr{bgMod}_R}(B, C) \otimes \underline{\mathscr{bgMod}_R}(A, B) \to \underline{\mathscr{bgMod}_R}(A, C)$$

is given by Definition 2.5.1.

(4) The unit morphism $R \to \underline{\mathscr{bgMod}_R}(A, A)$ is given by the morphism of bigraded modules that sends $1 \in R$ to $1_A : A \to A$, the strict morphism given by the identity of A.

Definition 2.5.6. *The* vbC$_R$-enriched category of twisted complexes $\underline{t\mathcal{C}_R}$ *is the enriched category given by the following data.*

(1) The objects of $\underline{t\mathcal{C}_R}$ are twisted complexes.

(2) For A, B twisted complexes the hom-object is the vertical bicomplex $\underline{t\mathcal{C}_R}(A, B)$.

(3) The composition morphism $c : \underline{t\mathcal{C}_R}(B, C) \otimes \underline{t\mathcal{C}_R}(A, B) \to \underline{t\mathcal{C}_R}(A, C)$ is given by Definition 2.5.1.

(4) The unit morphism $R \to \underline{t\mathcal{C}_R}(A, A)$ is given by the morphism of vertical bicomplexes sending $1 \in R$ to $1_A : A \to A$, the strict morphism of twisted complexes given by the identity of A.

The next tensor corresponds to $\underline{\otimes}$ in the categorical setting of Remark 2.1.9, see [CESLW18, Lemma 4.27].

Lemma 2.5.7. *The monoidal structure of $\underline{t\mathcal{C}_R}$ is given by the following map of vertical bicomplexes.*

$$\underline{\otimes} : \underline{t\mathcal{C}_R}(A, B) \otimes \underline{t\mathcal{C}_R}(A', B') \to \underline{t\mathcal{C}_R}(A \otimes A', B \otimes B')$$

$$(f, g) \to (f \underline{\otimes} g)_m := \sum_{i+j=m} (-1)^{ij} f_i \otimes g_j$$

The monoidal structure of $\underline{\mathcal{bgMod}_R}$ is given by the restriction of this map.

Definition 2.5.8. *The* bgMod_R-enriched category of filtered modules $\underline{\mathcal{fMod}_R}$ *is the enriched category given by the following data.*

(1) The objects of $\underline{\mathcal{fMod}_R}$ are filtered modules.

(2) For filtered modules (K, F) and (L, F), the bigraded module $\underline{\mathcal{fMod}_R}(K, L)$ is given by

$$\underline{\mathcal{fMod}_R}(K, L)^v_u := \{f : K \to L | f(F_q K^m) \subset F_{q+u} L^{m+u+v}, \forall m, q \in \mathbb{Z}\}.$$

(3) The composition morphism is given by $c(f, g) = (-1)^{u|g|} fg$, where u is the horizontal degree of f.

(4) The unit morphism is given by the map $R \to \underline{\mathcal{fMod}_R}(K, K)$ given by $1 \to 1_K$.

Definition 2.5.9. *Let (K, d^K, F) and (L, d^L, F) be filtered complexes. We define $\underline{f\mathcal{C}_R}(K, L)$ to be the vertical bicomplex whose underlying bigraded module is $\underline{\mathcal{fMod}_R}(K, L)$ with vertical differential*

$$\delta(f) := c(d^L, f) - (-1)^{\langle f, d^K \rangle} c(f, d^K) = d^L f - (-1)^{|f|} f d^K$$

for $f \in \underline{fMod}_R(K, L)^v_u$*, where* c *is the composition map from Definition* 2.5.8*.*

Definition 2.5.10. *The* vbC_R-enriched category of filtered complexes $\underline{fC}_R$ *is the enriched category given by the following data.*

(1) The objects of $\underline{fC}_R$ *are filtered complexes.*

(2) For K, L *filtered complexes the hom-object is the vertical bicomplex* $\underline{fC}_R(K, L)$*.*

(3) The composition morphism is given as in $\underline{fMod}_R$ *in Definition* 2.5.8*.*

(4) The unit morphism is given by the map $R \to \underline{fC}_R(K, K)$ *given by* $1 \to 1_K$*. We denote by* $\underline{sfC}_R$ *the full subcategory of* $\underline{fC}_R$ *whose objects are split filtered complexes.*

The enriched monoidal structure is given as follows and can be found in [CESLW18, Lemma 4.36].

Definition 2.5.11. *The monoidal structure of* $\underline{fC}_R$ *is given by the following map of vertical bicomplexes.*

$$\underline{\otimes} : \underline{fC}_R(K, L) \otimes \underline{fC}_R(K', L') \to \underline{fC}_R(K \otimes K', L \otimes L'),$$

$$(f, g) \mapsto f \underline{\otimes} g := (-1)^{u|g|} f \otimes g$$

Here u *is the horizontal degree of* f*.*

The proof of the following lemma is included in the proof of [CESLW18, Lemma 4.35].

Lemma 2.5.12. *Let A be a vertical bicomplex that is horizontally bounded on the right and let K and L be filtered complexes. There is a natural bijection*

$$\mathrm{Hom}_{\mathrm{fC}_R}(\mathrm{Tot}(A)\otimes K, L) \cong \mathrm{Hom}_{\mathrm{vbC}_R}(A, \underline{f\mathcal{C}_R}(K,L))$$

given by $f \mapsto \tilde{f} : a \mapsto (k \mapsto f(a\otimes k))$.

We now define an enriched version of the totalization functor.

Definition 2.5.13. *Let A, B be bigraded modules. We define*

$$\mathrm{Tot}(f) \in \underline{f\mathcal{M}od_R}(\mathrm{Tot}(A), \mathrm{Tot}(B))_u^v$$

for $f \in \underline{bg\mathcal{M}od_R}(A,B)_u^v$ to be given on any $x \in \mathrm{Tot}(A)^n$ by

$$(\mathrm{Tot}(f)(x)))_{j+u} := \sum_{m\geq 0} (-1)^{(m+u)n} f_m(x_{j-m}) \in B_{j+u}^{n-j+v} \subset \mathrm{Tot}(B)^{n+u+v}.$$

Let $K = \mathrm{Tot}(A)$, $L = \mathrm{Tot}(B)$ and $g \in \underline{f\mathcal{M}od_R}(K,L)_u^v$. We define

$$f := \mathrm{Tot}^{-1}(g) \in \underline{bg\mathcal{M}od_R}(A,B)_u^v$$

to be $f := (f_0, f_1, \dots)$ where f_i is given on each A_j^{m+j} by the composite

$$\begin{aligned} f_i : A_j^{m-j} \hookrightarrow \prod_{k\geq j} A_k^{m-k} &= F_j(\mathrm{Tot}(A)^m) \xrightarrow{g} F_{j+u}(\mathrm{Tot}(B)^{m+u+v}) \\ &= \prod_{l\geq j+u} B_l^{m+u+v-l} \xrightarrow{\times(-1)^{(i+u)m}} B_{j+u+i}^{m-j+v-i}, \end{aligned}$$

where the last map is a projection and multiplication with the indicated sign.

The following is [CESLW18, Theorem 4.39].

Theorem 2.5.14. *Let A, B be twisted complexes. The assignments* $\mathfrak{Tot}(A) := \mathrm{Tot}(A)$ *and*

$$\mathfrak{Tot}_{A,B} : \underline{t\mathcal{C}_R}(A, B) \to \underline{f\mathcal{C}_R}(\mathrm{Tot}(A), \mathrm{Tot}(B))$$
$$f \mapsto \mathrm{Tot}(f)$$

define a vbC_R*-enriched functor* $\mathfrak{Tot} : \underline{t\mathcal{C}_R} \to \underline{f\mathcal{C}_R}$ *which restricts to an isomorphism onto its image. Furthermore, this functor restricts to a* bgMod_R*-enriched functor*

$$\mathfrak{Tot} : \underline{\mathit{bgMod}_R} \to \underline{\mathit{fMod}_R}$$

which also restricts to an isomorphism onto its image.

We now present an enriched endomorphism operad. The precise operad structure is shown in [CESLW18, Lemma 4.41].

Definition 2.5.15. *Let $\underline{\mathscr{C}}$ be a monoidal $\mathscr{V}$-enriched category and A an object of $\underline{\mathscr{C}}$. We define $\underline{\mathcal{E}nd}_A$ to be the collection in $\mathscr{V}$ given by*

$$\underline{\mathcal{E}nd}_A(n) := \underline{\mathscr{C}}(A^{\otimes n}, A) \text{ for } n \geq 1.$$

The following contains Proposition 4.40, Lemma 4.43 and Proposition 4.46 from [CESLW18].

Proposition 2.5.16.

- *The enriched functors*

$$\mathfrak{Tot} : \underline{bgMod}_R \to \underline{fMod}_R, \qquad \mathfrak{Tot} : \underline{t\mathcal{C}_R} \to \underline{f\mathcal{C}}_R$$

  are lax symmetric monoidal in the enriched sense and when restricted to the bounded case they are strong symmetric monoidal in the enriched sense.

- *For $A \in \underline{\mathscr{C}}$, the collection $\underline{End}_A$ defines an operad in $\mathscr{V}$.*

- *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories over $\mathscr{V}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a lax monoidal functor over $\mathscr{V}$. Then for any $X \in \mathcal{C}$ there is an operad morphism*

$$\underline{End}_X \to \underline{End}_{F(X)}.$$

Lemma 2.5.17. *Let A be a twisted complex. Consider the operads $\underline{End}_A(n) = \underline{t\mathcal{C}_R}(A^{\otimes n}, A)$ and $\underline{End}_{\mathrm{Tot}(A)}(n) = \underline{f\mathcal{C}}_R(\mathrm{Tot}(A)^{\otimes n}, \mathrm{Tot}(A))$. There is a morphism of operads*

$$\underline{End}_A \to \underline{End}_{\mathrm{Tot}(A)},$$

which is an isomorphism of operads if A is bounded. If A is just a bigraded module, the same holds for the operads $\underline{End}_A(n) = \underline{bgMod}_R(A^{\otimes n}, A)$ and $\underline{End}_{\mathrm{Tot}(A)}(n) = \underline{fMod}_R(\mathrm{Tot}(A)^{\otimes n}, \mathrm{Tot}(A))$.

Proof. The proof of in the case of a A being a twisted complex can be found in [CESLW18, Lemma 4.54]. For the bigraded module case, we are going to do it analogously. First, by Theorem 2.5.14 we know that the functor $\mathfrak{Tot} : \underline{bgMod}_R \to \underline{fMod}_R$ is bgMod_R-enriched. In fact, by Proposition 2.5.16 it is lax monoidal in the enriched sense. In addition, both bgMod_R and fMod_R are monoidal over bgMod_R. In the case of bgMod_R it is in the obvious way and for fMod_R is given by Lemma 2.4.8. With all of this we may apply Proposition 2.5.16 to the totalization functor $\mathfrak{Tot} : \underline{bgMod}_R \to \underline{fMod}_R$ to obtain the desired map

$$\underline{\mathcal{E}nd}_A \to \underline{\mathcal{E}nd}_{\mathrm{Tot}(A)}.$$

The fact that it is an isomorphism in the bounded case is analogous to the twisted complex case. □

We are going to construct the inverse in the bounded case explicitly from Equation (2.1). The construction for the direct map is analogue but here we just need the inverse. We do it for a twisted complex A, but it is done similarly for a bigraded module.

Lemma 2.5.18. *In the conditions of Lemma 2.5.17 for the bounded case, the inverse is given by the map*

$$\begin{aligned}\underline{\mathcal{E}nd}_{\mathrm{Tot}(A)} &\to \underline{\mathcal{E}nd}_A\\ f &\mapsto \mathrm{Tot}^{-1}(f \circ \mu^{-1}).\end{aligned}$$

Proof. The inverse is given by the following composite.

$$
\begin{tikzcd}
\underline{\mathcal{E}nd}_{\mathrm{Tot}(A)}(n) = \underline{f\mathcal{C}_R}(\mathrm{Tot}(A)^{\otimes n}, \mathrm{Tot}(A)) \arrow[d] & \\
\underline{f\mathcal{C}_R}(\mathrm{Tot}(A^{\otimes n}), \mathrm{Tot}(A)) \arrow[r] & \underline{t\mathcal{C}_R}(A^{\otimes n}, A) = \underline{\mathcal{E}nd}_A(n)
\end{tikzcd}
$$

The second map is given by $\mathfrak{Tot}^{-1}$, see Definition 2.5.13. To describe the first map, let R be concentrated in bidegree $(0,0)$ with trivial vertical differential. Then the first map is given by the following composite

$$
\begin{aligned}
\underline{f\mathcal{C}_R}(\mathrm{Tot}(A)^{\otimes n}, \mathrm{Tot}(A)) &\cong R \otimes \underline{f\mathcal{C}_R}(\mathrm{Tot}(A)^{\otimes n}, \mathrm{Tot}(A)) \\
\xrightarrow{\underline{\mu}^{-1}\otimes 1} \underline{f\mathcal{C}_R}(\mathrm{Tot}(A^{\otimes n}), \mathrm{Tot}(A)^{\otimes n}) &\otimes \underline{f\mathcal{C}_R}(\mathrm{Tot}(A)^{\otimes n}, \mathrm{Tot}(A)) \\
&\xrightarrow{c} \underline{f\mathcal{C}_R}(\mathrm{Tot}(A^{\otimes n}), \mathrm{Tot}(A)),
\end{aligned}
$$

where c is the composition in $\underline{f\mathcal{C}_R}$, see Definition 2.5.8. The map $\underline{\mu}^{-1}$ is the adjoint of μ^{-1} under the bijection from Lemma 2.5.12. Explicitly,

$$
\begin{aligned}
\underline{\mu}^{-1} : R &\to \underline{f\mathcal{C}_R}(\mathrm{Tot}(A^{\otimes n}), \mathrm{Tot}(A)^{\otimes n}) \\
1 &\mapsto (a \mapsto \mu^{-1}(a)).
\end{aligned}
$$

Putting all this together, we get the map

$$
\begin{aligned}
\underline{\mathcal{E}nd}_{\mathrm{Tot}(A)} &\to \underline{\mathcal{E}nd}_A \\
f &\mapsto \mathrm{Tot}^{-1}(c(f, \mu^{-1})).
\end{aligned}
$$

Since the total degree of μ^{-1} is 0, the composition map reduces to $c(f, \mu^{-1}) = f \circ \mu^{-1}$ and we get the desired map. $\square$

Chapter 3

A_∞-algebras on operads

In this chapter we aim to encode A_∞-algebras in a simple operadic way. To do that we use operadic suspension, following an approach similar to the one introduced by Ward [KWZ15]. We explore some properties of this construction and the connection to other ways of encoding A_∞-algebras that are found in the literature. We then describe a brace algebra structure on operadic suspension and construct A_∞-algebras on a certain family of operads. We use these structures to prove Theorem 3.3.9, which was originally claimed by Gerstenhaber and Voronov [GV95]. This finally leads us to our first new version of the Deligne conjecture, that we prove in Corollary 3.3.14.

3.1 Operadic suspension

In this section we define an operadic suspension, which is a slight modification of the one found in [KWZ15]. This construction will help us define A_∞-multiplications in a simple way. The motivation to introduce operadic suspension is that signs in A_∞-algebras and related Lie structures are known to arise from a sequence of shifts. In order to discuss derived structures later, we need to pin this down more generally and rigorously. We are going to work only with non-symmetric operads, although most of what we do is also valid in the symmetric case.

3.1.1 Operadic suspension and A_∞-algebras

First recall the notion of shift or suspension of modules, which is the building block our next construction.

Definition 3.1.1. *For a graded R-module A, the* shift *or* suspension *SA is given by $SA^i = A^{i-1}$. The n-fold application of this operation to A is denoted S^nA.*

Let $\Lambda(n) = S^{n-1}R$, so that $\Lambda(n)$ is the ring R concentrated in degree $n-1$. We view this module as the free R-module of rank one spanned by the exterior power $e^n = e_1 \wedge \cdots \wedge e_n$ of degree $n-1$, where e_i is the i-th element of the canonical basis of R^n. By convention, $\Lambda(0)$ is one-dimensional concentrated in degree -1 and generated by e^0.

Let us define an operad structure on $\Lambda = \{\Lambda(n)\}_{n\geq 0}$ via the following insertion maps

$$\Lambda(n) \otimes \Lambda(m) \xrightarrow{\quad\circ_i\quad} \Lambda(n+m-1)$$

$$(e_1 \wedge \cdots \wedge e_n) \otimes (e_1 \wedge \cdots \wedge e_m) \longmapsto (-1)^{(n-i)(m-1)} e_1 \wedge \cdots \wedge e_{n+m-1}.$$

We are inserting the second factor onto the first one, so the sign can be explained by moving the power e^m of degree $m-1$ to the i-th position of e^n passing by e_n through e_{i+1}. More compactly,

$$e^n \circ_i e^m = (-1)^{(n-i)(m-1)} e^{n+m-1}.$$

The unit of this operad is $e^1 \in \Lambda(1)$. It can be checked by direct computation that Λ satisfies the axioms of an operad of graded modules.

In a similar way we can define $\Lambda^-(n) = S^{1-n}R$, with the same insertion maps.

Definition 3.1.2. *Let $\mathcal{O}$ be an operad of graded modules. The* operadic suspension $\mathfrak{s}\mathcal{O}$ *of $\mathcal{O}$ is given arity-wise by the Hadamard product of the operads $\mathcal{O}$ and Λ, in other words, $\mathfrak{s}\mathcal{O}(n) = (\mathcal{O}\otimes\Lambda)(n) = \mathcal{O}(n)\otimes\Lambda(n)$ with diagonal composition. Similarly, we define the* operadic desuspension *arity-wise as $\mathfrak{s}^{-1}\mathcal{O}(n) = \mathcal{O}(n) \otimes \Lambda^-(n)$.*

Even though the elements of $\mathfrak{s}\mathcal{O}$ are tensor products of the form $x\otimes e^n$, we may identify the elements of $\mathcal{O}$ with the elements the elements of $\mathfrak{s}\mathcal{O}$ and simply write x as an abuse of notation.

Definition 3.1.3. *For $x \in \mathcal{O}(n)$ of degree* $\deg(x)$*, its* natural degree $|x|$ *is the degree of $x \otimes e^n$ as an element of $\mathfrak{s}\mathcal{O}$, namely, $|x| = \deg(x) + n - 1$. To distinguish both degrees we call* $\deg(x)$ *the* internal degree *of x, since this is the degree that x inherits from the grading of $\mathcal{O}$.*

If we write $\circ_i$ for the operadic insertion on $\mathcal{O}$ and $\tilde{\circ}_i$ for the operadic insertion on $\mathfrak{s}\mathcal{O}$, we may find a relation between the two insertion maps in the following way.

Lemma 3.1.4. *For $x \in \mathcal{O}(n)$ and $y \in \mathcal{O}(m)$ we have*

$$(x\otimes e^n)\tilde{\circ}_i(y\otimes e^m) = (-1)^{(n-1)(m-1)+(n-1)\deg(y)+(i-1)(m-1)}(x\circ_i y)\otimes e^{n+m-1},$$

or written more compactly,

$$x\tilde{\circ}_i y = (-1)^{(n-1)(m-1)+(n-1)\deg(y)+(i-1)(m-1)}x\circ_i y. \tag{3.1}$$

Proof. Let $x \in \mathcal{O}(n)$ and $y \in \mathcal{O}(m)$, and let us compute $(x\otimes e^n)\tilde{\circ}_i(y\otimes e^m)$.

$$\begin{aligned}\mathfrak{s}\mathcal{O}(n)\otimes\mathfrak{s}\mathcal{O}(m) &= (\mathcal{O}(n)\otimes\Lambda(n))\otimes(\mathcal{O}(m)\otimes\Lambda(m))\\ &\cong (\mathcal{O}(n)\otimes\mathcal{O}(m))\otimes(\Lambda(n)\otimes\Lambda(m))\\ &\xrightarrow{\circ_i\otimes\circ_i}\mathcal{O}(m+n-1)\otimes\Lambda(n+m-1)=\mathfrak{s}\mathcal{O}(n+m-1).\end{aligned}$$

The symmetric monoidal structure produces the sign $(-1)^{(n-1)\deg(y)}$ in the isomorphism $\Lambda(n)\otimes\mathcal{O}(m)\cong\mathcal{O}(m)\otimes\Lambda(n)$, and the operadic structure of Λ produces the sign $(-1)^{(n-i)(m-1)}$, so

$$(x\otimes e^n)\tilde{\circ}_i(y\otimes e^m) = (-1)^{(n-1)\deg(y)+(n-i)(m-1)}(x\circ_i y)\otimes e^{n+m-1}.$$

Abusing notation, this can be written as

$$x\tilde{\circ}_i y = (-1)^{(n-1)\deg(y)+(n-i)(m-1)}x\circ_i y.$$

Now we can rewrite the exponent using that we have mod 2

$$(n-i)(m-1) = (n-1-i-1)(m-1) = (n-1)(m-1)+(i-1)(m-1)$$

so we conclude

$$x\tilde{\circ}_i y = (-1)^{(n-1)(m-1)+(n-1)\deg(y)+(i-1)(m-1)} x \circ_i y.$$

$\square$

Remark 3.1.5. The sign from Lemma 3.1.4 is exactly the sign in [RW11, §1.2] from which the sign in the equation defining A_∞-algebras, i.e. Equation (2.4) is derived. This means that if $m_s \in \mathcal{O}(s)$ has degree $2-s$ and $m_{r+1+t} \in \mathcal{O}(r+1+t)$ has degree $1-r-t$, abusing notation we get

$$m_{r+1+t}\tilde{\circ}_{r+1} m_s = (-1)^{rs+t} m_{r+1+t} \circ_{r+1} m_s.$$

Next, we are going to use the above fact to obtain a way to describe A_∞-algebras in simplified operadic terms. We are also going to compare this description with a classical approach that is more general but requires heavier operadic machinery.

Definition 3.1.6. *An operad $\mathcal{O}$ has an A_∞-*multiplication *if there is a map $\mathcal{A}_\infty \to \mathcal{O}$ from the operad of A_∞-algebras.*

Therefore, we have the following.

Lemma 3.1.7. *An A_∞-multiplication on an operad $\mathcal{O}$ is equivalent to an element $m \in \mathfrak{s}\mathcal{O}$ of degree 1 trivial on arity 0 such that $m\tilde{\circ}m = 0$, where $x\tilde{\circ}y = \sum_i x\tilde{\circ}_i y$.*

Proof. By definition, an A_∞-multiplication on $\mathcal{O}$ corresponds to a map of operads

$$f : \mathcal{A}_\infty \to \mathcal{O}.$$

Such a map is determined by the images of the generators $\mu_i \in \mathcal{A}_\infty(i)$ of degree $2-i$. Whence, f it is determined by $m_i = f(\mu_i) \in \mathcal{O}(i)$.

Let $m = m_1 + m_2 + \cdots$. Since $\deg(m_i) = \deg(\mu_i) = 2 - i$, we have that the image of m_i in $\mathfrak{s}\mathcal{O}$ is of degree $2 - i + i - 1 = 1$. Therefore, $m \in \mathfrak{s}\mathcal{O}$ is homogeneous of degree 1. Now, let us check that $m\tilde{\circ}m = 0$. Note that by Equation (3.1) we have the operation $\tilde{\circ}$ defined as

$$x\tilde{\circ}y = \sum_{i=1}^{n}(-1)^{(n-1)(m-1)+(n-1)\deg(y)+(i-1)(m-1)}x \circ_i y$$

for $x \in \mathcal{O}(n)$ and $y \in \mathcal{O}(m)$. Therefore, applying this definition to m_{r+1+t} and m_s we obtain that

$$m_{r+1+t}\tilde{\circ}_{r+1}m_s = (-1)^{rs+t}m_{r+1+t} \circ_{r+1} m_s, \tag{3.2}$$

which is the sign appearing in the definition of an A_∞-algebra (Equation (2.4)). Since the elements μ_i satisfy the A_∞-equation and f is a map of operads, so do the elements $m_i = f(\mu_i)$. Therefore, we have

$$0 = \sum_{\substack{r+s+t \\ r,t\geq 0,\ s\geq 1}} (-1)^{rs+t}m_{r+1+t} \circ_{r+1} m_s = \sum_{\substack{r+s+t \\ r,t\geq 0,\ s\geq 1}} m_{r+1+t}\tilde{\circ}_{r+1}m_s = m\tilde{\circ}m.$$

Conversely, if $m \in \mathfrak{s}\mathcal{O}$ of degree 1 such that $m\tilde{\circ}m = 0$, let m_i be the component of m lying in arity i. We have $m = m_1 + m_2 + \cdots$. By the usual identification, m_i has degree $1 - i + 1 = 2 - i$ in $\mathcal{O}$. Now we can use Equation (3.2) to conclude that $m\tilde{\circ}m = 0$ implies

$$\sum_{\substack{r+s+t \\ r,t\geq 0,\ s\geq 1}} (-1)^{rs+t} m_{r+1+t} \circ_{r+1} m_s = 0.$$

This shows that the elements m_i determine a map $f : \mathcal{A}_\infty \to \mathcal{O}$ defined on generators by $f(\mu_i) = m_i$, as desired. $\square$

Remark 3.1.8. If one works with dg operads, then the definition of $\mathcal{A}_\infty$ as a quasi-free operad should be used, see Remark 2.3.12. In that case, similarly to Lemma 3.1.7, the equation that an A_∞-multiplication on $\mathcal{O}$ satisfies is $\partial(m) + m\tilde{\circ}m = 0$, where ∂ is the differential on $\mathcal{O}$ and m is concentrated on arity at least 2. A similar analysis can be carried out from here, but we will stick to operads of graded modules for the most part.

We can connect Lemma 3.1.7 with the existing literature. Recall that the Koszul dual cooperad $\mathcal{A}s^{\text{i}}$ of the associative operad $\mathcal{A}s$ is $k\mu_n$ in arity n, where μ_n has degree $n-1$ for $n \geq 1$, see [LV12, §7.1]. Thus, for a graded module A, we have the following operad isomorphisms, where the notation (≥ 1) means that we are taking the reduced sub-operad with trivial arity 0 component.

$$\operatorname{Hom}(\mathcal{A}s^{\text{i}}, \operatorname{End}_A) \cong \operatorname{End}_{S^{-1}A}(\geq 1) \cong \mathfrak{s}\operatorname{End}_A(\geq 1).$$

The first operad is the convolution operad, see [LV12, §6.4.1], for which

$$\operatorname{Hom}(\mathcal{A}s^{\text{i}}, \operatorname{End}_A)(n) = \operatorname{Hom}_R(\mathcal{A}s^{\text{i}}(n), \operatorname{End}_A(n)).$$

Explicitly, for $f \in \mathrm{End}_A(n)$ and $g \in \mathrm{End}_A(m)$, the convolution product is given by

$$f \star g = \sum_{i=1}^{n} (-1)^{(n-1)(m-1)+(n-1)\deg(b)+(i-1)(m-1)} f \circ_i g = \sum_{i=1}^{n} f \tilde{\circ}_i g = f \tilde{\circ} g.$$

It is known that A_∞-structures on A are determined by elements $\varphi \in \mathrm{Hom}(\mathcal{A}s^{\mathrm{i}}, \mathrm{End}_A)$ of degree 1 such that $\varphi \star \varphi = 0$ [LV12, Proposition 10.1.3]. Since the convolution product coincides with the operation $\tilde{\circ}$, such an element φ is sent via the above isomorphism to an element $m \in \mathfrak{s}\,\mathrm{End}_A(\geq 1)$ of degree 1 satisfying $m \tilde{\circ} m = 0$. Therefore, we see that this classical interpretation of A_∞-algebras is equivalent to the one that Lemma 3.1.7 provides in the case of the operad End_A. See [LV12, Proposition 10.1.11] for more details about convolution operads and the more classical operadic interpretation of A_∞-agebras, taking into account that in the dg-setting the definition has to be modified slightly, see Remark 3.1.8. There is also a difference in sign conventions that arises from the choice of the isomorphism $\mathrm{End}_{SA} \cong \mathfrak{s}^{-1}\,\mathrm{End}_A$, see Theorem 3.1.10.

What is more, replacing End_A by any operad $\mathcal{O}$ and doing similar calculations to [LV12, Proposition 10.1.11], we retrieve the notion of A_∞-multiplication on $\mathcal{O}$ given by Definition 3.1.6.

Remark 3.1.9. Above we needed to specify that only positive arity was considered. This is the case in many situations in literature, but for our purposes, we cannot assume that operads have trivial component in arity 0 in general, and this is what forces us to specify that A_∞-multiplications are trivial on arity 0.

When we obtain the signs for the full operadic composition on operadic suspension we will be able to also give a new interpretation of ∞-morphisms But before that, let us expose the relation between operadic suspension and the usual suspension or shift of graded modules.

Theorem 3.1.10. *([MSS07, Chapter 3, Lemma 3.16]) Given a graded R-module A, there is an isomorphism of operads $\sigma^{-1} : \operatorname{End}_{SA} \cong \mathfrak{s}^{-1}\operatorname{End}_A$, where End_A is the endomorphism operad of A.*

The original statement is about vector spaces, but it is still true when R is not a field. The proof in the original reference is not very explicit, see Theorem A.2 for a detailed proof. But in the case of the operadic suspension defined above, the isomorphism is given by

$$\sigma^{-1} : \operatorname{End}_{SA} \to \mathfrak{s}^{-1}\operatorname{End}_A,$$

where $\sigma^{-1}(F) = (-1)^{\binom{n}{2}} S^{-1} \circ F \circ S^{\otimes n}$ for $F \in \operatorname{End}_{SA}(n)$. The symbol $\circ$ here is just composition of maps. Note that we are using the identification of elements of End_A with those in $\mathfrak{s}^{-1}\operatorname{End}_A$. The notation σ^{-1} comes from [RW11], where a twisted version of this map is the inverse of a map σ. Here, we define $\sigma : \operatorname{End}_A(n) \to \operatorname{End}_{SA}(n)$ as the map of graded modules given by

$$\sigma(f) = S \circ f \circ (S^{-1})^{\otimes n}. \tag{3.3}$$

In [RW11] the sign for the insertion maps was obtained by computing $\sigma^{-1}(\sigma(x) \circ_i \sigma(y))$. This can be interpreted as sending x and y from End_A to End_{SA} via σ (which is a map of graded modules, not of operads), and then applying the isomorphism induced by σ^{-1}. In the end this is the same as simply sending x and y to their images in $\mathfrak{s}^{-1}\operatorname{End}_A$, which is what Theorem 3.1.10 does.

Even though σ is only a map of graded modules, it can be shown in a completely analogous way to Theorem 3.1.10 that $\overline{\sigma} = (-1)^{\binom{n}{2}}\sigma$ induces an isomorphism of operads

$$\overline{\sigma} : \operatorname{End}_A \cong \mathfrak{s}\operatorname{End}_{SA}\,. \tag{3.4}$$

This isomorphism can also be proved in a more direct way using the isomorphism

$$\mathfrak{s}\mathfrak{s}^{-1}\mathcal{O} \cong \mathcal{O}$$

from Lemma 3.1.11, namely, since $\operatorname{End}_{SA} \cong \mathfrak{s}^{-1}\operatorname{End}_A$, we have

$$\mathfrak{s}\operatorname{End}_{SA} \cong \mathfrak{s}\mathfrak{s}^{-1}\operatorname{End}_A \cong \operatorname{End}_A\,.$$

In this case, the isomorphism map that we obtain goes in the opposite direction to $\overline{\sigma}$, and it is precisely its inverse.

Lemma 3.1.11. *There are isomorphisms of operads $\mathfrak{s}^{-1}\mathfrak{s}\mathcal{O} \cong \mathcal{O}$ and $\mathcal{O} \cong \mathfrak{s}\mathfrak{s}^{-1}\mathcal{O}$.*

Proof. We are only showing the first isomorphism since the other one is analogous. Note that as graded R-modules,

$$\mathfrak{s}^{-1}\mathfrak{s}\mathcal{O}(n) = \mathcal{O}(n) \otimes S^{1-n}R \otimes S^{n-1}R \cong \mathcal{O}(n),$$

and any automorphism of $\mathcal{O}(n)$ determines such an isomorphism. Therefore, we are going to find an automorphism f of $\mathcal{O}(n)$ such that the above isomorphism induces a map of operads, i.e f induces a map that preserves insertions. Observe that the insertion in $\mathfrak{s}^{-1}\mathfrak{s}\mathcal{O}$ differs from that of $\mathcal{O}$ in just a sign. The insertion on $\mathfrak{s}^{-1}\mathfrak{s}\mathcal{O}$ is defined as the composition of the isomorphism

$$(\mathcal{O}(n) \otimes \Lambda(n) \otimes \Lambda^{-}(n)) \otimes (\mathcal{O}(m) \otimes \Lambda(m) \otimes \Lambda^{-}(m)) \cong$$
$$(\mathcal{O}(m) \otimes \mathcal{O}(m)) \otimes (\Lambda(n) \otimes \Lambda(m)) \otimes (\Lambda^{-}(n) \otimes \Lambda^{-}(m))$$

with the tensor product of the insertions corresponding to each operad. After cancellations, the only sign left is $(-1)^{(n-1)(m-1)}$. So we need to find an automorphism f of $\mathcal{O}$ such that, for $x \in \mathcal{O}(n)$ and $y \in \mathcal{O}(m)$,

$$f(x \circ_i y) = (-1)^{(n-1)(m-1)} f(x) \circ_i f(y).$$

By Lemma A.1, $f(x) = (-1)^{\binom{n}{2}} x$ is such an automorphism. $\square$

3.1.2 Functorial properties of operadic suspension

Here we study operadic suspension at the level of the underlying collections as an endofunctor. Recall from Definition 2.3.1 that a collection is a family $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 0}$ of graded R-modules.

We define the suspension of a collection $\mathcal{O}$ as $\mathfrak{s}\mathcal{O}(n) = \mathcal{O}(n) \otimes S^{n-1}R$, where $S^{n-1}R$ is the ground ring concentrated in degree $n-1$. We first show that $\mathfrak{s}$ is a functor both on collections and on operads. Given a morphism of collections $f : \mathcal{O} \to \mathcal{P}$, there is an obvious induced morphism

$$\mathfrak{s}f : \mathfrak{s}\mathcal{O} \to \mathfrak{s}\mathcal{P},\ \mathfrak{s}f(x \otimes e^n) = f(x) \otimes e^n. \tag{3.5}$$

Since morphisms of collections preserve arity, this map is well defined because e^n is the same for x and $f(x)$. Note that if f is homogeneous, the degree of $\mathfrak{s}f$ is the same as that of f.

Lemma 3.1.12. *The assignment $\mathcal{O} \mapsto \mathfrak{s}\mathcal{O}$ and $f \mapsto \mathfrak{s}f$ is a functor on both the category* Col *of collections and the category* Op *of operads.*

Proof. The assignment preserves composition of maps. Indeed, given any $g : \mathcal{P} \to \mathcal{C}$, by definition $\mathfrak{s}(g \circ f)(x \otimes e^n) = g(f(x)) \otimes e^n$, and also

$$(\mathfrak{s}g \circ \mathfrak{s}f)(x \otimes e^n) = \mathfrak{s}g(f(x) \otimes e^n) = g(f(x)) \otimes e^n.$$

This means that $\mathfrak{s}$ defines an endofunctor on the category Col of collections.

We know that when $\mathcal{O}$ is an operad, $\mathfrak{s}\mathcal{O}$ is again an operad. What is more, if f is a map of operads, then the map $\mathfrak{s}f$ is again a map of operads, since for $a \in \mathcal{O}(n)$ and $b \in \mathcal{O}(m)$ we have

$$\begin{aligned}
\mathfrak{s}f(x \tilde{\circ}_i y) &= \mathfrak{s}f((x \otimes e^n)\tilde{\circ}_i(y \otimes e^m)) \\
&= (-1)^{(n-1)\deg(y)+(n-i)(m-1)}\mathfrak{s}f((x \circ_i y) \otimes e^{n+m-1}) \\
&= (-1)^{(n-1)\deg(y)+(n-i)(m-1)}f(x \circ_i y) \otimes e^{n+m-1} \\
&= (-1)^{(n-1)\deg(y)+(n-i)(m-1)+\deg(f)\deg(x)}(f(x) \circ_i f(y)) \otimes e^{n+m-1}
\end{aligned}$$

By the definition of $\tilde{\circ}$ this equals

$$
\begin{aligned}
&(-1)^{(n-1)\deg(y)+(n-1)(\deg(y)+\deg(f))+\deg(f)\deg(x)}(f(x)\otimes e^n)\tilde{\circ}_i(f(y)\otimes e^m)\\
&=(-1)^{\deg(f)(\deg(x)+n-1)}\mathfrak{s}f(x)\tilde{\circ}_i\mathfrak{s}f(y).
\end{aligned}
$$

Note that $\deg(x)+n-1$ is the degree of $x\otimes e^n$, and as we said before $\deg(\mathfrak{s}f)=\deg(f)$, so the above relation is consistent with the Koszul sign rule. In any case, recall that a morphism of operads is necessarily of degree 0, but the above calculation hints at some monoidality properties of $\mathfrak{s}$ that we will study afterwards. Clearly $\mathfrak{s}f$ preserves the unit, so $\mathfrak{s}f$ is a morphism of operads. $\square$

The fact that $\mathfrak{s}$ is a functor allows to describe algebras over operads using operadic suspension. For instance, an A_∞-algebra is equivalent to a map of operads $\mathcal{O}\to\operatorname{End}_A$ where $\mathcal{O}$ is an operad with A_∞-multiplication. Since $\mathfrak{s}$ is a functor, this map corresponds to a map $\mathfrak{s}\mathcal{O}\to\mathfrak{s}\operatorname{End}_A$. Since in addition the map $\mathfrak{s}\mathcal{O}\to\mathfrak{s}\operatorname{End}_A$ is fully determined by the original map $\mathcal{O}\to\operatorname{End}_A$, this correspondence is bijective, and algebras over $\mathcal{O}$ are equivalent to algebras over $\mathfrak{s}\mathcal{O}$. In fact, using Lemma 3.1.11, it is not hard to show the following.

Proposition 3.1.13. *The functor $\mathfrak{s}$ is an equivalence of categories both at the level of collections and at the level of operads.* $\square$

In particular, for A_∞-algebras it is more convenient to work with $\mathfrak{s}\mathcal{O}$ since the formulation of an A_∞-multiplication on this operad is much simpler, yet we do not lose any information.

3.1.3 Monoidal properties of operadic suspension

Now we are going to explore the monoidal properties of operadic suspension. Since operads are precisely monoids on the category Col of collections, we have the following.

Proposition 3.1.14. *The endofunctor* $\mathfrak{s} : \mathrm{Col} \to \mathrm{Col}$ *sends monoids to monoids and morphisms of monoids to morphisms of monoids, in other words, it induces a well-defined endofunctor on the category of monoids* $\mathrm{Mon}(\mathrm{Col})$. □

In fact, we can show a stronger result.

Proposition 3.1.15. *The functor* $\mathfrak{s} : \mathrm{Col} \to \mathrm{Col}$ *defines a lax monoidal functor. When restricted to the subcategory of reduced operads, it is strong monoidal.*

Proof. Firstly, we need to define the structure maps of a lax monoidal functor. Namely, we define the unit morphism $\varepsilon : I \to \mathfrak{s}I$ to be the map $\varepsilon(n) : I(n) \to I(n) \otimes S^{n-1}R$ to be the identity for $n \neq 1$ and the isomorphism $R \cong R \otimes R$ for $n = 1$. We also need to define a natural transformation $\mu : \mathfrak{s}\mathcal{O} \circ \mathfrak{s}\mathcal{P} \to \mathfrak{s}(\mathcal{O} \circ \mathcal{P})$. To define it, observe that for $\mathcal{P} = \mathcal{O}$ we would want the map

$$\mathfrak{s}\mathcal{O} \circ \mathfrak{s}\mathcal{O} \xrightarrow{\mu} \mathfrak{s}(\mathcal{O} \circ \mathcal{O}) \xrightarrow{\mathfrak{s}\gamma} \mathfrak{s}\mathcal{O}$$

to coincide with the operadic composition $\tilde{\gamma}$ on $\mathfrak{s}\mathcal{O}$, where γ is the composition on $\mathcal{O}$.

We know that $\mathfrak{s}\gamma$ does not add any signs. Therefore, if $\tilde{\gamma} = (-1)^\eta \gamma$, with η explicitly computed in Proposition 3.2.3, the sign must come entirely from the map $\mathfrak{s}\mathcal{O} \circ \mathfrak{s}\mathcal{O} \to \mathfrak{s}(\mathcal{O} \circ \mathcal{O})$. Thus, we define the map

$$\mu : \mathfrak{s}\mathcal{O} \circ \mathfrak{s}\mathcal{P} \to \mathfrak{s}(\mathcal{O} \circ \mathcal{P})$$

as the map given by

$$x \otimes e^N \otimes x_1 \otimes e^{a_1} \otimes \cdots \otimes x_N \otimes e^{a_N} \mapsto (-1)^\eta x \otimes x_1 \otimes \cdots \otimes x_N \otimes e^n,$$

where $a_1 + \cdots + a_N = n$ and

$$\eta = \sum_{j<l} a_j \deg(b_l) + \sum_{j=1}^{N} (a_j + \deg(b_j) - 1)(N - j),$$

which is the case $k_0 = \cdots = k_n = 0$ in Proposition 3.2.3. Note that $(-1)^\eta$ only depends on degrees and arities, so the map is well defined. Another way to obtain this map is using the associativity isomorphisms and operadic composition on Λ to obtain a map $\mathfrak{s}\mathcal{O} \circ \mathfrak{s}\mathcal{P} \to \mathfrak{s}(\mathcal{O} \circ \mathcal{P})$.

We now show that μ is natural, or in other words, for $f : \mathcal{O} \to \mathcal{O}'$ and $g : \mathcal{P} \to \mathcal{P}'$, we show that the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{s}\mathcal{O} \circ \mathfrak{s}\mathcal{P} \arrow[r, "\mathfrak{s}f \circ \mathfrak{s}g"] \arrow[d, "\mu"'] & \mathfrak{s}\mathcal{O}' \circ \mathfrak{s}\mathcal{P}' \arrow[d, "\mu"] \\
\mathfrak{s}(\mathcal{O} \circ \mathcal{P}) \arrow[r, "\mathfrak{s}(f \circ g)"] & \mathfrak{s}(\mathcal{O}' \circ \mathcal{P}')
\end{tikzcd}$$

Let $c = x \otimes e^N \otimes x_1 \otimes e^{a_1} \otimes \cdots \otimes x_N \otimes e^{a_N} \in \mathfrak{s}\mathcal{O} \circ \mathfrak{s}\mathcal{P}$ and let us compute $\mathfrak{s}(f \circ g)(\mu(c))$. One has

$$
\begin{aligned}
\mathfrak{s}(f\circ g)(\mu(c)) &= \mathfrak{s}(f\circ g)((-1)^{\varepsilon} x\otimes x_1\otimes\cdots\otimes x_N\otimes e^n)\\
&= (-1)^{\varepsilon+\delta} f(x)\otimes g(x_1)\otimes\cdots\otimes g(x_N)\otimes e^n
\end{aligned}
$$

where

$$
\varepsilon = \sum_{j<l} a_j \deg(x_l) + \sum_{j=1}^{N} (\deg(x_j) + a_j - 1)(N-j).
$$

and

$$
\delta = N\deg(g)\deg(x) + \deg(g)\sum_{j=1^N} \deg(x_j)(N-j)
$$

Now, let us compute $\mu((\mathfrak{s}f\circ\mathfrak{s}g)(c))$. We have

$$
\mu((\mathfrak{s}f\circ\mathfrak{s}g)(c)) = (-1)^{\sigma} f(x)\otimes g(x_1)\otimes\cdots\otimes g(x_N)\otimes e^n,
$$

where

$$
\begin{aligned}
\sigma = &\deg(g)\sum_{j=1}^{N}(\deg(x_j)+a_j-1)(N-j) + N\deg(g)(\deg(x)+N-1)\\
&+\sum_{j<l} a_j(\deg(x_j)+\deg(g))\\
&+\sum_{j=1}^{N}(a_j+\deg(x_j)+\deg(g)-1)(N-j).
\end{aligned}
$$

Now we compare the two signs by computing $\varepsilon+\delta+\sigma \mod 2$. After some cancellations of common terms and using that $N(N-1)=0 \mod 2$ we get

$$\begin{aligned}
&\deg(g)\sum_{j=1}^{N}(a_j-1)(N-j)+\sum_{j<l}a_j\deg(g)+\sum_{j=1}^{N}\deg(g)(N-j)\\
&=\deg(g)\sum_{j=1}^{N}a_j(N-j)+\deg(g)\sum_{j<l}a_j\\
&=\deg(g)\left(\sum_{j=1}^{N}a_j(N-j)+\sum_{j=1}^{N}a_j(N-j)\right)\\
&=0 \mod 2.
\end{aligned}$$

This shows naturality of μ. Unitality follows directly from the definitions by direct computation. In the case of associativity, observe that by the definition of μ, the associativity axiom for μ is equivalent to the associativity of the operadic composition $\tilde{\gamma}$, which we know to be true. This shows that $\mathfrak{s}$ is a lax monoidal functor.

In the case where the operads have trivial arity 0 component, we may define an inverse to the operadic composition on Λ from Section 3.1.1 Namely, for $n>0$, we may define

$$\Lambda(n)\to\bigoplus_{N\geq 0}\Lambda(N)\otimes\left(\bigoplus_{a_1+\cdots+a_N=n}\Lambda(a_1)\otimes\cdots\otimes\Lambda(a_N)\right)$$

as the map

$$e^n\mapsto\sum_{a_1+\cdots+a_N=n}(-1)^{\delta}e^N\otimes e^{a_1}\otimes\cdots\otimes e^{a_N},$$

where δ is the same sign that appears in the operadic composition on Λ (Proposition 3.2.3) and where $a_1,\ldots,a_k>0$. Since there are only finitely many ways of decomposing n into N positive integers, the sum is finite and thus the map is well defined. In fact, this map defines a cooperad structure on the reduced sub-operad of Λ with trivial arity 0 component. This map induces the morphism $\mu^{-1} : \mathfrak{s}(\mathcal{O} \circ \mathcal{P}) \to \mathfrak{s}\mathcal{O} \circ \mathfrak{s}\mathcal{P}$ that we are looking for.

The unit morphism ε is always an isomorphism, so this shows $\mathfrak{s}$ is strong monoidal in the reduced case. □

Remark 3.1.16. If we decide to work with symmetric operads, we just need to introduce the sign action of the symmetric group on $\Lambda(n)$, turning it into the sign representation of the symmetric group. The action on tensor products is diagonal, and the results we have obtained follow similarly replacing Col by the category of $\mathbb{S}$-modules.

3.2 Brace algebras

Brace algebras appear naturally in the context of operads when we fix the first argument of operadic composition [GV95]. This simple idea gives rise to a very rich structure that is the building block of the derived A_∞-structures that we are going to construct.

In this section we define a brace algebra structure for an arbitrary operad using operadic suspension. The use of operadic suspension will have as a result a generalization of the Lie bracket defined in [RW11]. First recall the definition of a brace algebra.

Definition 3.2.1. *A* brace algebra *on a graded module A consists of a* *family of maps*

$$b_n : A^{\otimes 1+n} \to A$$

called braces, *that we evaluate on* $(x, x_1, \ldots, x_n)$ *as* $b_n(x; x_1, \ldots, x_n)$. *They must satisfy the* brace relation

$$b_m(b_n(x; x_1, \ldots, x_n); y_1, \ldots, y_m) =$$
$$\sum_{\substack{i_1,\ldots,i_n \\ j_1\ldots,j_n}} (-1)^{\varepsilon} b_l(x; y_1, \ldots, b_{j_1}(x_1; y_{i_1+1}, \ldots), \ldots, b_{j_n}(x_n; y_{i_n+1}, \ldots), \ldots, y_m)$$

where $l = m + n - \sum_{p=1}^{n} j_p$ *and* $\varepsilon = \sum_{p=1}^{n} \deg(x_p) \sum_{q=1}^{i_p} \deg(y_q)$, *i.e. the sign is picked up by the* x_i*'s passing by the* y_i*'s in the shuffle.*

Remark 3.2.2. Some authors might use the notation b_{1+n} instead of b_n, but the first element is usually going to have a different role from the others, so we found b_n more intuitive. A shorter notation for $b_n(x; x_1, \ldots, x_n)$ found in the literature ([GV95], [Get93]) is $x\{x_1, \ldots, x_n\}$.

We will also see a bigraded version of this kind of map in Section 4.4.

3.2.1 Brace algebra structure on an operad

Given an operad $\mathcal{O}$ with composition map $\gamma : \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ we can define a brace algebra on the underlying module of $\mathcal{O}$ by setting

$$b_n : \mathcal{O}(N) \otimes \mathcal{O}(a_1) \otimes \cdots \otimes \mathcal{O}(a_n) \to \mathcal{O}\left(N - n + \sum a_i\right)$$

$$b_n(x; x_1, \ldots, x_n) = \sum \gamma(x; 1, \ldots, 1, x_1, 1, \ldots, 1, x_n, 1, \ldots, 1),$$

where the sum runs over all possible order-preserving insertions. The brace $b_n(x; x_1, \ldots, x_n)$ vanishes whenever $n > N$ and $b_0(x) = x$. The brace relation follows from the associativity axiom of operads.

This construction can be used to define braces on $\mathfrak{s}\mathcal{O}$. More precisely, we define maps

$$b_n : \mathfrak{s}\mathcal{O}(N) \otimes \mathfrak{s}\mathcal{O}(a_1) \otimes \cdots \otimes \mathfrak{s}\mathcal{O}(a_n) \to \mathfrak{s}\mathcal{O}\left(N - n + \sum a_i\right)$$

using the operadic composition $\tilde{\gamma}$ on $\mathfrak{s}\mathcal{O}$ as

$$b_n(x; x_1, \ldots, x_n) = \sum \tilde{\gamma}(x; 1, \ldots, 1, x_1, 1, \ldots, 1, x_n, 1, \ldots, 1).$$

We have the following relation between the brace maps b_n defined on $\mathfrak{s}\mathcal{O}$ and the operadic composition γ on $\mathcal{O}$.

Proposition 3.2.3. *For $x \in \mathfrak{s}\mathcal{O}(N)$ and $x_i \in \mathfrak{s}\mathcal{O}(a_i)$ of internal degree q_i ($1 \le i \le n$), we have*

$$b_n(x; x_1, \ldots, x_n) = \sum_{N-n=k_0+\cdots+k_n} (-1)^\eta \gamma(x \otimes 1^{\otimes k_0} \otimes x_1 \otimes \cdots \otimes x_n \otimes 1^{\otimes k_n}),$$

where

$$\eta = \sum_{0 \le j < l \le n} k_j q_l + \sum_{1 \le j < l \le n} a_j q_l + \sum_{j=1}^{n} (a_j + q_j - 1)(n - j) + \sum_{1 \le j \le l \le n} (a_j + q_j - 1) k_l.$$

Proof. To obtain the signs that make $\tilde{\gamma}$ differ from γ, we must first look at the operadic composition on Λ. We are interested in compositions of the form

$$\tilde{\gamma}(x \otimes 1^{\otimes k_0} \otimes x_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes x_n \otimes 1^{\otimes k_n})$$

where $N - n = k_0 + \cdots + k_n$, x has arity N and each x_i has arity a_i and internal degree q_i. Therefore, let us consider the corresponding operadic composition

$$\Lambda(N) \otimes \Lambda(1)^{k_0} \otimes \Lambda(a_1) \otimes \cdots \otimes \Lambda(a_n) \otimes \Lambda(1)^{k_n} \longrightarrow \Lambda\left(N - n + \sum_{i=1}^{n} a_i\right).$$

This operadic composition can be described in terms of insertions in the obvious way, namely, if $f \in \mathfrak{s}\mathcal{O}(N)$ and $h_1, \ldots, h_N \in \mathfrak{s}\mathcal{O}$, then we have

$$\tilde{\gamma}(x; y_1, \ldots, y_N) = (\cdots(x \tilde{\circ}_1 y_1)\tilde{\circ}_{1+a(y_1)} y_2 \cdots)\tilde{\circ}_{1+\sum a(y_p)} y_N,$$

where $a(y_p)$ is the arity of y_p (in this case y_p is either 1 or some x_i). So we just have to find out the sign iterating the same argument as in the i-th insertion. In this case, each $\Lambda(a_i)$ produces a sign given by the exponent

$$(a_i - 1)(N - k_0 + \cdots - k_{i-1} - i).$$

For this, recall that the degree of $\Lambda(a_i)$ is $a_i - 1$ and that the generator of this space is inserted in the position $1 + \sum_{j=0}^{i-1} k_j + \sum_{j=1}^{i-1} a_j$ of a wedge of $N + \sum_{j=1}^{i-1} a_j - i + 1$ generators. Therefore, performing this insertion as described in the previous section yields the aforementioned sign. Now, since $N - n = k_0 + \cdots + k_n$, we have that

$$(a_i - 1)(N - k_0 + \cdots + k_{i-1} - i) = (a_i - 1)\left(n - i + \sum_{l=i}^{n} k_l\right).$$

Let us introduce for the rest of the proof the notation $a_0 = N$ for the sake of compactness of the formulas. Now we can compute the sign factor of a brace. For this, notice that the isomorphism

$$(\mathcal{O}(1) \otimes \Lambda(1))^{\otimes k} \cong \mathcal{O}(1)^{\otimes k} \otimes \Lambda(1)^{\otimes k}$$

does not produce any signs because of degree reasons. Therefore, the sign coming from the isomorphism

$$\begin{aligned}&\bigotimes_{i=0}^{n} (\mathcal{O}(a_i) \otimes \Lambda(a_i) \otimes (\mathcal{O}(1) \otimes \Lambda(1))^{\otimes k_i} \\ &\cong \left(\bigotimes_{i=0}^{n} \mathcal{O}(a_i) \otimes \mathcal{O}(1)^{\otimes k_i}\right) \otimes \left(\bigotimes_{i=0}^{n} \Lambda(a_i) \otimes \Lambda(1)^{\otimes k_i}\right)\end{aligned}$$

is determined by the exponent

$$(a_0 - 1)\sum_{i=1}^{n} q_i + \sum_{i=1}^{n}(a_i - 1)\sum_{l>i} q_l.$$

This equals

$$\left(\sum_{j=0}^{n} k_j + n - 1\right)\sum_{i=1}^{n} q_i + \sum_{i=1}^{n}(a_i - 1)\sum_{l>i} q_l.$$

After doing the operadic composition

$$\bigotimes_{i=0}^{n} \mathcal{O}(a_i) \otimes \bigotimes_{i=0}^{n} \Lambda(a_i) \to \mathcal{O}\left(\sum_{i=0}^{n} a_i - n\right) \otimes \Lambda\left(\sum_{i=0}^{n} a_i - n\right)$$

we can add the sign coming from the suspension, so all in all the sign $(-1)^\eta$ we were looking for is given by

$$\eta = \sum_{i=1}^{n}(a_i-1)(n-i+\sum_{l=i}^{n}k_l)+(\sum_{j=0}^{n}k_j+n-1)\sum_{i=1}^{n}q_i+\sum_{i=1}^{n}(a_i-1)\sum_{l>i}q_l.$$

It can be checked that this can be rewritten modulo 2 as

$$\eta = \sum_{0\leq j<l\leq n}k_jq_l+\sum_{1\leq j<l\leq n}a_jq_l+\sum_{j=1}^{n}(a_j+q_j-1)(n-j)+\sum_{1\leq j\leq l\leq n}(a_j+q_j-1)k_l$$

as we stated. □

Notice that for $\mathcal{O} = \mathrm{End}_A$, the brace on operadic suspension is precisely

$$b_n(f;g_1,\dots,g_n) = \sum(-1)^\eta f(1,\dots,1,g_1,1,\dots,1,g_n,1,\dots,1).$$

Using the brace structure on $\mathfrak{s}\,\mathrm{End}_A$, the sign η gives us in particular the the same sign of the Lie bracket defined in [RW11]. More precisely, we have the following.

Corollary 3.2.4. *The brace $b_1(f;g)$ is the operation $f\circ g$ defined in [RW11] that induces a Lie bracket on the Hochschild complex of an A_∞-algebra via*

$$[f,g] = b_1(f;g)-(-1)^{|f||g|}b_1(g;f).$$

However, we prefer to use $f\tilde{\circ}g$ to make clear that we are using the operadic composition in $\mathfrak{s}\mathcal{O}$. Note that

$$b_1(f;g) = \sum_i f\tilde{\circ}_i g = f\tilde{\circ}g,$$

so the notation $f \tilde{\circ} g$ is suggestive for operadic suspension. The notation $f \circ g$ will still be used whenever the insertion maps are denoted by $\circ_i$.

In [RW11], the sign is computed using a strategy that we generalize in Appendix C, see Lemma C.2. The approach we have followed here has the advantage that the brace relation follows immediately from the associativity axiom of operadic composition. This approach also works for any operad since the difference between γ and $\tilde{\gamma}$ is always the same sign.

3.2.2 Reinterpretation of ∞-morphisms

As we mentioned before, we can show an alternative description of ∞-morphisms of A_∞-algebras and their composition in terms of suspension of collections. Recall Definition 2.2.2 for the definition of these morphisms.

Defining the suspension $\mathfrak{s}$ at the level of collections as we did in Section 3.1.2 allows us to talk about ∞-morphisms of A_∞-algebras in this setting, since they live in collections of the form

$$\mathrm{End}_B^A = \{\mathrm{Hom}_R(A^{\otimes n}, B)\}_{n \geq 1}.$$

More precisely, there is a left module structure on End_B^A over the operad End_B

$$\mathrm{End}_B \circ \mathrm{End}_B^A \to \mathrm{End}_B^A$$

given by compostion of maps

$$f \otimes g_1 \otimes \cdots \otimes g_n \mapsto f(g_1 \otimes \cdots \otimes g_n)$$

for $f \in \operatorname{End}_B(n)$ and $g_i \in \operatorname{End}_B^A$, and also an infinitesimal right module structure over the operad End_A

$$\operatorname{End}_B^A \circ_{(1)} \operatorname{End}_A \to \operatorname{End}_B^A$$

given by insertion of maps

$$f \otimes 1^{\otimes r} \otimes g \otimes 1^{\otimes n-r-1} \mapsto f(1^{\otimes r} \otimes g \otimes 1^{\otimes n-r-1})$$

for $f \in \operatorname{End}_B^A(n)$ and $g \in \operatorname{End}_A$. In addition, we have a composition $\operatorname{End}_C^B \circ \operatorname{End}_B^A \to \operatorname{End}_C^A$ analogous to the left module described above. They induce maps on the respective operadic suspensions which differ from the original ones by some signs that can be calculated in an analogous way to what we did on Proposition 3.2.3. These induced maps will give us the characterization of ∞-morphisms in Lemma 3.2.5.

For these collections we also have $\mathfrak{s}^{-1} \operatorname{End}_B^A \cong \operatorname{End}_{SB}^{SA}$ in analogy with Theorem 3.1.10, and the proof is similar but shorter since we do not need to worry about insertions.

Lemma 3.2.5. *An ∞-morphism of A_∞-algebras $A \to B$ with respective structure maps m^A and m^B is equivalent to an element $f \in \mathfrak{s} \operatorname{End}_B^A$ of degree 0 trivial on arity 0 such that*

$$\rho(f \circ_{(1)} m^A) = \lambda(m^B \circ f), \tag{3.6}$$

where

$$\lambda : \mathfrak{s}\operatorname{End}_B \circ \mathfrak{s}\operatorname{End}_B^A \to \mathfrak{s}\operatorname{End}_B^A$$

is induced by the left module structure on End_B^A *and*

$$\rho : \mathfrak{s}\operatorname{End}_B \circ_{(1)} \mathfrak{s}\operatorname{End}_B^A \to \mathfrak{s}\operatorname{End}_B^A$$

is induced by the right infinitesimal module structure on End_B^A*.*

In addition, the composition of ∞*-morphisms is given by the natural composition*

$$\mathfrak{s}\operatorname{End}_C^B \circ \mathfrak{s}\operatorname{End}_B^A \to \mathfrak{s}\operatorname{End}_C^A .$$

Proof. From the definitions of the operations in Equation (3.6), we know that this equation coincides with the one defining ∞-morphisms of A_∞-algebras (Definition 2.2.2) up to sign. The signs that appear in the above equation are obtained in a similar way to that on $\tilde{\gamma}$, see the proof of Proposition 3.2.3. Thus, it is enough to plug into the sign η from Proposition 3.2.3 the corresponding degrees and arities to obtain the desired result. The composition of ∞-morphisms follows similarly. □

Notice the similarity between this definition and the definitions given in [LV12, §10.2.4], taking into account the minor modifications to accommodate the dg case.

In the case that $f : A \to A$ is an ∞-endomorphism, Equation (3.6) can be written in terms of operadic composition as $f \tilde{\circ} m = \tilde{\gamma}(m \circ f)$.

3.3 A_∞-algebra structures on operads

Let $\mathcal{O}$ be an operad of graded R-modules and $\mathfrak{s}\mathcal{O}$ its operadic suspension. Let us consider the underlying graded module of the operad $\mathfrak{s}\mathcal{O}$, which we call $\mathfrak{s}\mathcal{O}$ again by abuse of notation, i.e.

$$\mathfrak{s}\mathcal{O} = \prod_n \mathfrak{s}\mathcal{O}(n)$$

with grading given by its natural degree, i.e. $|x| = \deg(x) + n - 1$ for $x \in \mathfrak{s}\mathcal{O}(n)$, where $\deg(x)$ is its internal degree, the degree in $\mathcal{O}(n)$.

For any operad $\mathcal{O}$, recall the operation $\circ$ defined as

$$x \circ y = \sum_{i=1}^{n} x \circ_i y \in \mathcal{O}(n+m-1)$$

for $x \in \mathcal{O}(n)$ and $y \in \mathcal{O}(m)$. We write $x \tilde{\circ} y$ for the corresponding operation on $\mathfrak{s}\mathcal{O}$, namely

$$x \tilde{\circ} y = \sum_{i=1}^{n} x \tilde{\circ}_i y = b_1(x; y) \in \mathfrak{s}\mathcal{O}(n+m-1)$$

where

$$x \tilde{\circ}_i y = (-1)^{(n-1)\deg(y)+(n-i)(m-1)} x \circ_i y.$$

Definition 3.3.1. *Let $m \in \mathfrak{s}\mathcal{O}$ be of natural degree 1 and trivial on arity 0 such that $m \tilde{\circ} m = 0$, or equivalently $m = m_1 + m_2 + \cdots$ is a formal sum of maps $m_j \in \mathcal{O}(j)^{2-j}$ satisfying the usual A_∞-equation for all $n \geq 1$*

$$\sum_{r+s+t=n} (-1)^{rs+t} m_{r+1+t} \circ_{r+1} m_s = 0. \tag{3.7}$$

*Such m is an A_∞-*multiplication *on $\mathcal{O}$. As we saw in Lemma 3.1.7, its existence is equivalent to a map of operads $\mathcal{A}_\infty \to \mathcal{O}$ from the operad $\mathcal{A}_\infty$ of A_∞-algebras to $\mathcal{O}$. We may call each m_j the j-th* component *of m.*

Remark 3.3.2. An A_∞-multiplication on the operad End_A is equivalent to an A_∞-algebra structure on A.

Following [GV95] and [Get93], if $\mathcal{O}$ has an A_∞-multiplication m, one would define an A_∞-algebra structure on $\mathfrak{s}\mathcal{O}$ using the maps

$$\begin{aligned} M'_1(x) &\coloneqq [m, x] = m\tilde{\circ}x - (-1)^{|x|}x\tilde{\circ}m, \\ M'_j(x_1, \dots, x_j) &\coloneqq b_j(m; x_1, \dots, x_j), \qquad\qquad j > 1. \end{aligned}$$

The prime notation here is used to indicate that these are not the definitive maps that we are going to take. Getzler shows in [Get93] that $M' = M'_1 + M'_2 + \cdots$ satisfies the relation $M' \circ M' = 0$ using that $m \circ m = 0$, and the proof is independent of the operad in which m is defined, so it is still valid if $m\tilde{\circ}m = 0$. But we have two problems here. First, the equation $M' \circ M' = 0$ does depend on how the circle operation is defined. More precisely, this circle operation in [Get93] is the natural circle operation on the endomorphism operad, which does not have any additional signs, so M' is not an A_∞-structure under our convention. The other problem has to do with the degrees. We need M'_j to be homogeneous of degree $2 - j$ as a map $\mathfrak{s}\mathcal{O}^{\otimes j} \to \mathfrak{s}\mathcal{O}$, but we find that M'_j is homogeneous of degree 1 instead, as the following lemma shows.

Lemma 3.3.3. *For $x \in \mathfrak{s}\mathcal{O}$ we have that the degree of the map of graded modules $b_j(x; -) : \mathfrak{s}\mathcal{O}^{\otimes j} \to \mathfrak{s}\mathcal{O}$ is precisely $|x|$.*

Proof. Let $a(x)$ denote the arity of x, i.e. $a(x) = n$ whenever $x \in \mathfrak{s}\mathcal{O}(n)$. Also, let $\deg(x)$ be its internal degree in $\mathcal{O}$. The natural degree of $b_j(x; x_1, \ldots, x_j)$ for $a(x) \geq j$ is computed as follows. By definition, we have that the natural degree of $b_j(x; x_1, \ldots, x_j)$ as an element of $\mathfrak{s}\mathcal{O}$ is

$$|b_j(x; x_1, \ldots, x_j)| = a(b_j(x; x_1, \ldots, x_j)) + \deg(b_j(x; x_1, \ldots, x_j)) - 1.$$

We have

$$a(b_j(x; x_1, \ldots, x_j)) = a(x) - j + \sum_i a(x_i)$$

and

$$\deg(b_j(x; x_1, \ldots, x_j) = \deg(x) + \sum_i \deg(x_i).$$

Combining these two we obtain

$$\begin{aligned}
a(b_j(x; x_1, \ldots, x_j)) + \deg(b_j(x; x_1, \ldots, x_j)) - 1 &= \\
a(x) - j + \sum_i a(x_i) + \deg(x) + \sum_i \deg(x_i) - 1 &= \\
a(x) + \deg(x) - 1 + \sum_i a(x_i) + \sum_i \deg(x_i) - j &= \\
a(x) + \deg(x) - 1 + \sum_i (a(x_i) + \deg(x_i) - 1) &= \\
|x| + \sum_i |x_i|.&
\end{aligned}$$

This means that the degree of the map $b_j(x; -) : \mathfrak{s}\mathcal{O}^{\otimes j} \to \mathfrak{s}\mathcal{O}$ equals $|x|$. $\square$

Corollary 3.3.4. *The maps*

$$M'_j : \mathfrak{s}\mathcal{O}^{\otimes j} \to \mathfrak{s}\mathcal{O},\ (x_1, \dots, x_j) \mapsto b_j(m; x_1, \dots, x_j)$$

for $j > 1$ and the map

$$M'_1 : \mathfrak{s}\mathcal{O} \to \mathfrak{s}\mathcal{O},\ x \mapsto b_1(m; x) - (-1)^{|x|} b_1(m; x)$$

are homogeneous of degree 1.

Proof. For $j > 1$ it is a direct consequence of Lemma 3.3.3. For $j = 1$ we have the summand $b_1(m; x)$ whose degree follows as well from Lemma 3.3.3. The degree of the other summand, $b_1(x; m)$, can be computed in a similar way as in the proof Lemma 3.3.3, giving that $|b_1(x; m)| = 1 + |x|$. This concludes the proof. □

The problem we have encountered with the degrees can be resolved using shift maps as the following proposition shows. Recall that the *shift* of a graded module A is given by $SA^i = A^{i-1}$ and that we have maps $A \to SA$ of degree 1 given by the identity.

Proposition 3.3.5. *If $\mathcal{O}$ is an operad with an A_∞-multiplication $m \in \mathcal{O}$, then there is an A_∞-algebra structure on the shifted module $S\mathfrak{s}\mathcal{O}$.*

Proof. Note in the proof of Lemma 3.3.3 that a way to turn M'_j into a map of degree $2 - j$ is introducing a grading on $\mathfrak{s}\mathcal{O}$ given by arity plus internal degree (without subtracting 1). This is equivalent to defining an A_∞-algebra structure M on $S\mathfrak{s}\mathcal{O}$ shifting the map $M' = M'_1 + M'_2 + \cdots$, where S is the shift of graded modules. Therefore, we define M_j to be the map making the following diagram commute.

$$\begin{tikzcd} (S\mathfrak{s}\mathcal{O})^{\otimes j} \arrow[r, "M_j"] \arrow[d, "(S^{\otimes j})^{-1}"'] & S\mathfrak{s}\mathcal{O} \\ \mathfrak{s}\mathcal{O}^{\otimes j} \arrow[r, "M'_j"] & \mathfrak{s}\mathcal{O} \arrow[u, "S"'] \end{tikzcd}$$

In other words, $M_j = \overline{\sigma}(M'_j)$, where $\overline{\sigma}(F) = S \circ F \circ (S^{\otimes n})^{-1}$ for $F \in \operatorname{End}_{\mathfrak{s}\mathcal{O}}(n)$ is the map inducing an isomorphism $\operatorname{End}_{\mathfrak{s}\mathcal{O}} \cong \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$, see Equation (3.4). Since $\overline{\sigma}$ is an operad morphism, for $M = M_1 + M_2 + \cdots$, we have

$$M \tilde{\circ} M = \overline{\sigma}(M') \tilde{\circ} \overline{\sigma}(M') = \overline{\sigma}(M' \circ M') = 0.$$

So now we have that $M \in \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$ is an element of natural degree 1 such that $M \tilde{\circ} M = 0$. Therefore, in light of Remark 3.3.2, M is the desired A_∞-algebra structure on $S\mathfrak{s}\mathcal{O}$. $\square$

Notice that M is defined as an structure map on $S\mathfrak{s}\mathcal{O}$. This kind of shifted operad is called *odd operad* in [KWZ15]. This means that $S\mathfrak{s}\mathcal{O}$ is not an operad anymore, since the associativity relation for graded operads involves signs that depend on the degrees, which are now shifted.

3.3.1 Iterating the process

We have defined A_∞-structure maps $M_j \in \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$. Now we can use the brace structure of the operad $\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$ to define a A_∞-algebra structure given by maps

$$\overline{M}_j : (S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}})^{\otimes j} \to S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}} \tag{3.8}$$

by applying $\overline{\sigma}$ to maps

$$\overline{M}'_j : (\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}})^{\otimes j} \to \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$$

defined as

$$\begin{aligned}
&\overline{M}'_j(f_1,\ldots,f_j) = \overline{B}_j(M;f_1,\ldots,f_j) && j>1,\\
&\overline{M}'_1(f) = \overline{B}_1(M;f) - (-1)^{|f|}\overline{B}_1(f;M),
\end{aligned}$$

where $\overline{B}_j$ denotes the brace map on $\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$.

We define the Hochschild complex as done by Ward in [KWZ15].

Definition 3.3.6. *The* Hochschild cochains *of a graded module A are defined to be the graded module $S\mathfrak{s}\operatorname{End}_A$. If (A,d) is a (co)chain complex, then $S\mathfrak{s}\operatorname{End}_A$ is endowed with a differential*

$$\partial(f) = [d,f] = d\circ f - (-1)^{|f|} f\circ d$$

where $|f|$ is the natural degree of f and $\circ$ is the plethysm operation given by insertions.

In particular, $S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$ is the module of Hochschild cochains of $S\mathfrak{s}\mathcal{O}$. If $\mathcal{O}$ has an A_∞-multiplication, then the differential of the Hochschild complex is $\overline{M}_1$ from Equation (3.8).

Remark 3.3.7. The functor $S\mathfrak{s}$ is called the “oddification” of an operad in the literature [War13]. The reader might find odd to define the Hochschild complex in this way instead of just End_A. The reason is that operadic suspension provides the necessary signs and the extra shift gives us the appropriate degrees. In addition, this definition allows the extra structure to arise naturally instead of having to define the signs by hand. For instance, if we have an associative multiplication $m_2 \in \mathrm{End}_A(2) = \mathrm{Hom}(A^{\otimes 2}, A)$, the element m_2 would not satisfy the equation $m_2 \circ m_2 = 0$ and thus cannot be used to induce a multiplication on End_A as we did above.

A natural question to ask is what relation there is between the A_∞-algebra structure on $S\mathfrak{s}\mathcal{O}$ and the one on $S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$. In [GV95] it is claimed that given an operad $\mathcal{O}$ with an A_∞-multiplication, the map

$$\begin{aligned}\mathcal{O} &\to \mathrm{End}_{\mathcal{O}}\\ x &\mapsto \sum_{n\geq 0} b_n(x;-)\end{aligned}$$

is a morphism of A_∞-algebras. In the associative case, this result leads to the definition of homotopy G-algebras, which connects with the classical Deligne conjecture. We are going to adapt the statement of this claim to our context and prove it. This way we will obtain an A_∞-version of homotopy G-algebras and consequently an A_∞-version of the Deligne conjecture. Let Φ' the map defined as above but on $\mathfrak{s}\mathcal{O}$, i.e.

$$\begin{aligned}\Phi' \colon \mathfrak{s}\mathcal{O} &\to \mathrm{End}_{\mathfrak{s}\mathcal{O}}\\ x &\mapsto \sum_{n\geq 0} b_n(x;-).\end{aligned}$$

Let $\Phi : S\mathfrak{s}\mathcal{O} \to S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$ the map making the following diagram commute

$$\begin{tikzcd}
S\mathfrak{s}\mathcal{O} \arrow[rr, "\Phi"] \arrow[d] & & S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}} \\
\mathfrak{s}\mathcal{O} \arrow[r, "\Phi'"] & \operatorname{End}_{\mathfrak{s}\mathcal{O}} \arrow[r, "\cong"] & \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}} \arrow[u]
\end{tikzcd} \tag{3.9}$$

where the isomorphism $\operatorname{End}_{\mathfrak{s}\mathcal{O}} \cong \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$ is given in Equation (3.4). Note that the degree of the map Φ is zero.

Remark 3.3.8. Notice that we have only used the operadic structure on $\mathfrak{s}\mathcal{O}$ to define an A_∞-algebra structure on $S\mathfrak{s}\mathcal{O}$, so the constructions and results in these sections are valid if we replace $\mathfrak{s}\mathcal{O}$ by any graded module A such that SA is an A_∞-algebra.

Theorem 3.3.9. *The map Φ defined in diagram (3.9) above is a morphism of A_∞-algebras, i.e. for all $j \geq 1$ the equation*

$$\Phi(M_j) = \overline{M}_j(\Phi^{\otimes j}) \tag{3.10}$$

holds, where the M_j is the j-th component of the A_∞-algebra structure on $S\mathfrak{s}\mathcal{O}$ and $\overline{M}_j$ is the j-th componnent of the A_∞-algebra structure on $S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$.

Proof. Let us have a look at the following diagram

$$
\begin{tikzcd}
(S\mathfrak{s}\mathcal{O})^{\otimes j} \arrow[rrrr, bend left, "\Phi^{\otimes j}"] \arrow[dd, "M_j"'] \arrow[dr] & & & & (S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}})^{\otimes j} \arrow[dd, "\overline{M}_j"] \\
& \mathfrak{s}\mathcal{O}^{\otimes j} \arrow[r, "(\Phi')^{\otimes j}"] \arrow[d, "M'_j"] & (\operatorname{End}_{\mathfrak{s}\mathcal{O}})^{\otimes j} \arrow[r, "\overline{\sigma}^{\otimes j}"] \arrow[d, dashed, "\mathcal{M}_j"] & (\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}})^{\otimes j} \arrow[ur] \arrow[d, "\overline{M}'_j"] & \\
& \mathfrak{s}\mathcal{O} \arrow[r, "\Phi'"] & \operatorname{End}_{\mathfrak{s}\mathcal{O}} \arrow[r, "\overline{\sigma}"] & \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}} \arrow[dr] & \\
S\mathfrak{s}\mathcal{O} \arrow[ur] \arrow[rrrr, bend right, "\Phi"'] & & & & S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}
\end{tikzcd}
\tag{3.11}
$$

where the diagonal red arrows are shifts of graded R-modules. We need to show that the diagram defined by the external black arrows commutes. But these arrows are defined so that they commute with the red and blue arrows, so it is enough to show that the inner blue diagram commutes, since the outer squares commute by definition. The blue diagram can be split into two different squares using the dashed arrow $\mathcal{M}_j$ that we are going to define next, so it will be enough to show that the two squares commute. The commutativity of the left square will be more involved as we will have to distinguish between different kinds of insertions.

The map

$$\mathcal{M}_j : (\operatorname{End}_{\mathfrak{s}\mathcal{O}})^{\otimes j} \to \operatorname{End}_{\mathfrak{s}\mathcal{O}}$$

is defined by

$$\mathcal{M}_j(f_1, \dots, f_j) = B_j(M'; f_1, \dots, f_j) \qquad\qquad \text{for } j > 1,$$
$$\mathcal{M}_1(f) = B_1(M'; f) - (-1)^{|f|} B_1(f; M'),$$

where B_j is the natural brace structure map on the operad $\mathrm{End}_{\mathfrak{s}\mathcal{O}}$, i.e. for $f \in \mathrm{End}_{\mathfrak{s}\mathcal{O}}(n)$,

$$B_j(f; f_1, \dots, f_j) = \sum_{k_0 + \cdots + k_j = n-j} f(1^{\otimes k_0} \otimes f_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes f_j \otimes 1^{\otimes k_j}).$$

The 1's in the braces are identity maps. In the above definition, $|f|$ denotes the degree of f as an element of $\mathrm{End}_{\mathfrak{s}\mathcal{O}}$, which is the same as the degree $\overline{\sigma}(f) \in \mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$ because $\overline{\sigma}$ is an isomorphism, as mentioned in Equation (3.4).

The inner square of diagram (3.11) is divided into two halves, so we divide the proof into two as well, showing the commutativity of each half independently.

Commutativity of the right blue square

Let us show now that the right square commutes. Recall that $\overline{\sigma}$ is an isomorphism of operads and $M = \overline{\sigma}(M')$. Then we have for $j > 1$

$$\begin{aligned}\overline{M}'_j(\overline{\sigma}(f_1), \dots, \overline{\sigma}(f_j)) &= \overline{B}_j(M; \overline{\sigma}(f_1), \dots, \overline{\sigma}(f_j)) \\ &= \overline{B}_j(\overline{\sigma}(M'); \overline{\sigma}(f_1), \dots, \overline{\sigma}(f_j)).\end{aligned}$$

Now, since the brace structure is defined as an operadic composition, it commutes with $\overline{\sigma}$, so

$$\begin{aligned}\overline{B}_j(\overline{\sigma}(M');\overline{\sigma}(f_1),\dots,\overline{\sigma}(f_j)) &= \overline{\sigma}(B_j(M';f_1,\dots,f_j))\\ &= \overline{\sigma}(\mathcal{M}_j(f_1,\dots,f_j))\end{aligned}$$

and therefore the right blue square commutes for $j>1$. For $j=1$ the result follows analogously taking into account that the degree of f in $\mathrm{End}_{\mathfrak{s}\mathcal{O}}$ is the same as the degree of $\overline{\sigma}(f)$ in $\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$.

The proof that the left blue square commutes consists of several lengthy calculations so we are going to devote the next section to that. However, it is worth noting that the commutativity of the left square does not depend on the particular operad $\mathfrak{s}\mathcal{O}$, so it is still valid if m satisfies $m\circ m=0$ for any circle operation defined in terms of insertions. This is essentially the original statement in [GV95].

Commutativity of the left blue square

We are going to show here that the left blue square in diagram (3.11) commutes, i.e. that

$$\Phi'(M'_j)=\mathcal{M}_j((\Phi')^{\otimes j}) \tag{3.12}$$

for all $j\geq 1$. First we prove the case $j>1$. Let $x_1,\dots,x_j\in\mathfrak{s}\mathcal{O}^{\otimes j}$. We have on the one hand

$$\Phi'(M'_j(x_1,\dots,x_j)) = \Phi'(b_j(m;x_1,\dots,x_j)) = \sum_{n\geq 0} b_n(b_j(m;x_1,\dots,x_j);-)$$
$$= \sum_n \sum_l \sum b_l(m;-,b_{i_1}(x_1;-),\cdots,b_{i_j}(x_j;-),-)$$

where $l = n-(i_1+\cdots+i_j)+j$. The sum with no subindex runs over all the possible order-preserving insertions. Note that $l \geq j$. Evaluating the above map on elements would yield Koszul signs coming from the brace relation. Also recall from Lemma 3.3.3 that $|b_j(x;-)| = |x|$. Now, fix some value of $l \geq j$ and let us compute the M'_l component of

$$\mathcal{M}_j(\Phi'(x_1),\dots,\Phi'(x_j)) = B_j(M';\Phi'(x_1),\dots,\Phi'(x_j))$$

that is, $B_j(M'_l;\Phi'(x_1),\dots,\Phi'(x_j))$. By definition, this equals

$$\sum M'_l(-,\Phi'(x_1),\cdots,\Phi'(x_j),-) =$$
$$\sum_{i_1,\dots,i_j}\sum M'_l(-,b_{i_1}(x_1;-),\cdots,b_{i_j}(x_j;-),-) =$$
$$\sum_{i_1,\dots,i_j}\sum b_l(m;-,b_{i_1}(x_1;-),\cdots,b_{i_j}(x_j;-),-).$$

We are using hyphens instead of 1's to make the equality of both sides of the Equation (3.12) more apparent, and to make clear that when evaluating on elements those are the places where the elements go.

For each tuple $(i_1,\dots,i_j)$ we can choose n such that

$$n-(i_1+\cdots+i_j)+j = l,$$

so the above sum equals

$$\sum_{\substack{n,i_1,\dots,i_j \\ n-(i_1+\cdots+i_j)+j=l}} \sum b_l(m;-,b_{i_1}(x_1;-),\cdots,b_{i_j}(x_j;-),-).$$

So each M'_l component for $l \geq j$ produces precisely the terms $b_l(m;\dots)$ appearing in $\Phi'(M'_j)$. Conversely, for every $n \geq 0$ there exists some tuple $(i_1,\dots,i_j)$ and some $l \geq j$ such that n is the that $n-(i_1+\cdots+i_j)+j=l$, so we do get all the summands from the left hand side of the Equation (3.12), and thus we have the equality $\Phi'(M'_j) = \mathcal{M}_j((\Phi')^{\otimes j})$ for all $j > 1$.

It is worth treating the case $n = 0$ separately since in that case we have the summand

$$b_0(b_j(m;x_1,\dots,x_j))$$

in $\Phi'(b_j(m;x_1,\dots,x_j))$, where we cannot apply the brace relation. This summand is equal to

$$\begin{aligned} B_j(M'_j;b_0(x_1),\dots,b_0(x_j)) &= M'_j(b_0(x_1),\dots,b_0(x_j)) \\ &= b_j(m;b_0(x_1),\dots,b_0(x_j)), \end{aligned}$$

since by definition $b_0(x) = x$.

Now we are going to show the case $j = 1$, that is

$$\Phi'(M'_1(x)) = \mathcal{M}_1(\Phi'(x)). \tag{3.13}$$

This is going to be divided into two parts, since M'_1 has two clearly distinct summands, one of them consisting of braces of the form $b_l(m;\cdots)$ (insertions in m) and another one consisting of braces of the form

$b_l(x;\cdots)$ (insertions in x). We will therefore show that both types of braces cancel on each side of Equation (3.13).

Insertions in m

Let us first focus on the insertions in m that appear in equation (3.13). Recall that

$$\Phi'(M_1'(x)) = \Phi'([m,x]) = \Phi'(b_1(m;x)) - (-1)^{|x|}\Phi'(b_1(x;m)) \tag{3.14}$$

so we focus on the first summand

$$\begin{aligned}\Phi'(b_1(m;x)) &= \sum_n b_n(b_1(m;x);-) \\ &= \sum_n \sum_{\substack{i\\ n\geq i}} \sum b_{n-i+1}(m;-,b_i(x;-),-) \\ &= \sum_{\substack{n,i\\ n-i+1>0}} \sum b_{n-i+1}(m;-,b_i(x;-),-)\end{aligned}$$

where the sum with no indices runs over all the positions in which $b_i(x;-)$ can be inserted (from 1 to $n-i+1$ in this case).

On the other hand, since $|\Phi'(x)| = |x|$, the right hand side of Equation (3.13) becomes

$$\mathcal{M}_1(\Phi'(x)) = B_1(M';\Phi'(x)) - (-1)^{|x|}B_1(\Phi'(x);M'). \tag{3.15}$$

Again, we are focusing now on the first summand, but with the exception of the part of M_1' that corresponds to $b_1(\Phi(x);m)$. From here the argument is a particular case of the proof for $j>1$, so the terms of the form $b_l(m;\cdots)$ are the same on both sides of Equation (3.13).

Insertions in x

And now, let us study the insertions in x that appear in Equation (3.13). For that we will check that insertions in x from the left hand side and right hand side cancel. Let us look first at the left hand side. From $\Phi'(M_1'(x))$ in Equation (3.14) we had

$$-(-1)^{|x|}\Phi'(b_1(x;m)) = -(-1)^{|x|}\sum_n b_n(b_1(x;m);-).$$

The factor $-(-1)^{|x|}$ is going to appear everywhere, so we may cancel it. Thus we just have

$$\Phi'(b_1(x;m)) = \sum_n b_n(b_1(x;m);-).$$

We are going to evaluate each term of the sum, so let $z_1,\dots,z_n \in \mathfrak{s}\mathcal{O}$. We have by the brace relation that

$$\begin{aligned} &b_n(b_1(x;m);z_1,\dots,z_n) = \\ &\quad\sum_{l+j=n+1}\sum_{i=1}^{n-j+1}(-1)^{\varepsilon}b_l(x;z_1,\dots,b_j(m;z_i,\dots,z_{i+j}),\dots,z_n) \\ &\quad\quad+\sum_{i=1}^{n+1}(-1)^{\varepsilon}b_{n+1}(x;z_1,\dots,z_{i-1},m,z_i,\dots,z_n), \end{aligned} \tag{3.16}$$

where ε is the usual Koszul sign with respect to the grading in $\mathfrak{s}\mathcal{O}$. We have to check that the insertions in x that appear in $\mathcal{M}_1(\Phi'(x))$ (right hand side of eq. (3.13)) are exactly those in Equation (3.16) above (left hand side of eq. (3.13)).

Therefore let us look at the right hand side of Equation (3.13). Here we will study the cancellations from each of the two summands that naturally appear. From Equation (3.15), i.e.

$$\mathcal{M}_1(\Phi'(x)) = B_1(M'; \Phi'(x)) - (-1)^{|x|} B_1(\Phi'(x); M')$$

we have

$$-(-1)^{|x|} b_1(\Phi'(x); m) = -(-1)^{|x|} \sum_n b_1(b_n(x; -); m)$$

coming from the first summand since $B_1(M'_1; \Phi'(x)) = M'_1(\Phi'(x))$. We are now only interested in insertions in x. Again, cancelling $-(-1)^{|x|}$ we get

$$b_1(\Phi'(x); m) = \sum_n b_1(b_n(x; -); m).$$

Each term of the sum can be evaluated on $(z_1, \ldots, z_n)$ to produce

$$\begin{aligned} b_1(b_n(x; z_1, \ldots, z_n); m) = & \\ & \sum_{i=1}^{n} (-1)^{\varepsilon + |z_i|} b_n(x; z_1, \ldots, b_1(z_i; m), \ldots, z_n) \\ & + \sum_{i=1}^{n+1} (-1)^{\varepsilon} b_{n+1}(x; z_1, \ldots, z_{i-1}, m, z_i, \ldots, z_n) \end{aligned} \tag{3.17}$$

Note that we have to apply the Koszul sign rule twice: once at evaluation, and once more to apply the brace relation. Now, from the second summand of $\mathcal{M}_1(\Phi'(x))$ in the right hand side of eq. (3.15), after cancelling $-(-1)^{|x|}$ we obtain

$$
\begin{aligned}
B_1(\Phi'(x); M') &= \sum_l B_1(b_l(x; -); M') \\
&= \sum_l \sum b_l(x; -, M', -) \\
&= \sum_l \sum_{j>1} \sum b_l(x; -, b_j(m; -), -) \\
&\quad + \sum_l \sum b_l(x; -, b_1(-; m), -).
\end{aligned}
$$

We are going to evaluate on $(z_1, \ldots, z_n)$ to make this map more explicit. This evaluation gives us the following

$$
\begin{aligned}
&\sum_{l+j=n+1} \sum_{i=1}^{n-j+1} (-1)^{\varepsilon} b_l(x; z_1, \ldots, b_j(m; z_i, \ldots, z_{i+j}), \ldots, z_n) \\
&\quad - \sum_{i=1}^{n} (-1)^{\varepsilon+|z_i|} b_n(x; z_1, \ldots, b_1(z_i; m), \ldots, z_n).
\end{aligned} \tag{3.18}
$$

The minus sign comes from the fact that $b_1(z_i; m)$ comes from $M_1'(z_i)$, so we apply the signs in the definition of $M_1'(z_i)$. We therefore have that the right hand side of eq. (3.15) is the result of adding equations (3.17) and (3.18). After this addition we can see that the first sum of eq. (3.17) cancels the second sum of eq. (3.18).

We also have that the second sum in eq. (3.17) is the same as the second sum in eq. (3.16), so we are left with only the first sum of eq. (3.18). This is the same as the first sum in eq. (3.16), so we have already checked that the equation $\Phi'(M_1') = \mathcal{M}_1(\Phi')$ holds.

In the case $n = 0$, we have to note that $B_1(b_0(x); m)$ vanishes because of arity reasons: $b_0(x)$ is a map of arity 0, so we cannot insert any inputs. And this finishes the proof. $\square$

3.3.2 Explicit structure and Deligne conjecture

We have given an implicit definition of the components of the A_∞-algebra structure on $S\mathfrak{s}\mathcal{O}$, namely,

$$M_j = \overline{\sigma}(M'_j) = (-1)^{\binom{j}{2}} S \circ M'_j \circ (S^{-1})^{\otimes j},$$

but it is useful to have an explicit expression that determines how it is evaluated on elements of $S\mathfrak{s}\mathcal{O}$. We will also need these expressions to state the A_∞-version of the Deligne conjecture in a precise way. Recall that the classical Deligne conjecture [GV95] states that the Hochschild complex of an associative algebra has a structure of homotopy G-algebra. Here, we will define J-algebras as the A_∞-generalization of homotopy G-algebras. We will do this in terms of the explicit expressions we give for the maps M_j. These explicit formulas will also clear up the connection with the work of Gerstenhaber and Voronov. We hope that these explicit expressions can be useful to perform calculations in other mathematical contexts where A_∞-algebras are used.

Lemma 3.3.10. *For $x, x_1, \dots, x_n \in \mathfrak{s}\mathcal{O}$, we have the following expressions.*

$$M_n(Sx_1, \dots, Sx_n) = (-1)^{\sum_{i=1}^{n}(n-i)|x_i|} Sb_n(m; x_1, \dots, x_n) \qquad n > 1,$$
$$M_1(Sx) = Sb_1(m; x) - (-1)^{|x|} Sb_1(x; m).$$

Here $|x|$ is the degree of x as an element of $\mathfrak{s}\mathcal{O}$, i.e. its natural degree.

Proof. The deduction of these explicit formulas is done as follows. Let

$n > 1$ and $x_1, \ldots, x_n \in \mathfrak{s}\mathcal{O}$. Then

$$\begin{aligned} M_n(Sx_1, \ldots, Sx_n) &= SM'_n((S^{\otimes n})^{-1})(Sx_1, \ldots, Sx_n) \\ &= (-1)^{\binom{n}{2}} SM'_n((S^{-1})^{\otimes n})(Sx_1, \ldots, Sx_n) \\ &= (-1)^{\binom{n}{2} + \sum_{i=1}^{n}(n-i)(|x_i|+1)} SM'_n(x_1, \ldots, x_n) \end{aligned} \tag{3.19}$$

Now, note that $\binom{n}{2}$ is even exactly when $n \equiv 0, 1 \mod 4$. In these cases, an even amount of $|x_i|$'s have an odd coefficient in the sum (when $n \equiv 0 \mod 4$ these are the $|x_i|$ with even index, and when $n \equiv 1 \mod 4$, the $|x_i|$ with odd index). This means that 1 is added on the exponent an even number of times, so the sign is not changed by the binomial coefficient nor by adding 1 on each term. Similarly, when $\binom{n}{2}$ is odd, i.e. when $n \equiv 2, 3 \mod 4$, there is an odd number of $|x_i|$ with odd coefficient, so the addition of 1 an odd number of times cancels the binomial coefficient. This means that Equation (3.19) can be simplified to

$$M_n(Sx_1, \ldots, Sx_n) = (-1)^{\sum_{i=1}^{n}(n-i)|x_i|} SM'_n(x_1, \ldots, x_n),$$

which by definition equals

$$(-1)^{\sum_{i=1}^{n}(n-i)|x_i|} Sb_n(m; x_1, \ldots, x_n).$$

The case $n = 1$ is analogous, one just has to note that

$$M'_1(x) = b_1(m; x) - (-1)^{|x|} b_1(x; m)$$

and that $\overline{\sigma}$ is linear. □

It is possible to show that the maps defined explicitly as we have just done satisfy the A_∞-equation without relying on the fact that $\overline{\sigma}$ is a map of operads, but it is a lengthy and tedious calculation.

Remark 3.3.11. In the case $n = 2$, omitting the shift symbols by abuse of notation, we obtain

$$M_2(x, y) = (-1)^{|x|} b_2(m; x, y).$$

Let M_2^{GV} be the product defined in [GV95] as

$$M_2^{GV}(x, y) = (-1)^{|x|+1} b_2(m; x, y).$$

We see that $M_2 = -M_2^{GV}$. Since the authors of [GV95] work in the associative case $m = m_2$, this minus sign does not affect the A_∞-relation, which in this case reduces to the associativity and differential relations. This difference in sign can be explained by the difference between $(S^{\otimes n})^{-1}$ and $(S^{-1})^{\otimes n}$, since any of these shift maps can be used to define a map $(S\mathfrak{s}\mathcal{O})^{\otimes n} \to \mathfrak{s}\mathcal{O}^{\otimes n}$.

Now that we have the explicit formulas for the A_∞-structure on $S\mathfrak{s}\mathcal{O}$ we can state and prove an A_∞-version of the Deligne conjecture. Let us first re-adapt the definition of homotopy G-algebra from [GV95, Definition 2] to our conventions.

Definition 3.3.12. *A* homotopy G-algebra *is differential graded algebra V with a differential M_1 and a product M_2 such that the shift $S^{-1}V$ is a brace algebra with brace maps b_n. The differential M_1 and* *the product M_2 must satisfy the following compatibility identities. Let $x, x_1, x_2, y_1, \dots, y_n \in S^{-1}V$. We demand*

$$
\begin{aligned}
&Sb_n(S^{-1}M_2(Sx_1, Sx_2); y_1, \dots, y_n) = \\
&\qquad \sum_{k=0}^{n} (-1)^{(|x_2|+1)\sum_{i=1}^{k}|y_i|} M_2(b_k(x_1; y_1, \dots, y_k), b_{n-k}(x_2; y_{k+1}, \dots, y_n))
\end{aligned}
$$

and

$$
\begin{aligned}
&Sb_n(S^{-1}M_1(Sx); y_1, \dots, y_n) - M_1(Sb_n(x; y_1, \dots, y_n)) \\
&-(-1)^{|x|+1}\sum_{p=1}^{n}(-1)^{\sum_{i=1}^{p}|y_i|} Sb_n(x; y_1, \dots, M_1(Sy_p), \dots, y_n) \\
&= -(-1)^{(|x|+1)|y_1|} M_2(Sy_1, Sb_{n-1}(x; y_2, \dots, y_n)) \\
&\quad + (-1)^{|x|+1}\sum_{p=1}^{n-1}(-1)^{n-1+\sum_{i=1}^{p}|y_i|} Sb_{n-1}(x; y_1, \dots, M_2(Sy_p, Sy_{p+1}), \dots y_n) \\
&\quad - (-1)^{|x|+\sum_{i=1}^{n-1}|y_i|} M_2(Sb_{n-1}(x; y_1, \dots, y_{n-1}), Sy_n)
\end{aligned}
$$

Notice that our signs are slightly different to those in [GV95] as a consequence of our conventions. Our signs will be a particular case of those in Definition 3.3.13, which are set so that Corollary 3.3.14 holds in consistent way with operadic suspension and all the shifts that the authors of [GV95] do not consider.

We now introduce J-algebras as an A_∞-generalization of homotopy G-algebras. This will allow us to generalize the Deligne conjecture to the A_∞-setting.

Definition 3.3.13. *A J-algebra V is an A_∞-algebra with structure maps $\{M_j\}_{j\geq 1}$ such that the shift $S^{-1}V$ is a brace algebra. Furthermore, the braces and the A_∞-structure satisfy the following compatibility relations. Let $x, x_1, \ldots, x_j, y_1, \ldots, y_n \in S^{-1}V$. For $n \geq 0$ we demand*

$$
\begin{aligned}
(-1)^{\sum_{i=1}^{n}(n-i)|y_i|} & Sb_n(S^{-1}M_1(Sx); y_1, \ldots, y_n) = \\
& \sum_{\substack{l+k-1=n \\ 1\leq i_1 \leq n-k+1}} (-1)^{\varepsilon} M_l(Sy_1, \ldots, Sb_k(x; y_{i_1}, \ldots), \ldots, Sy_n) \\
& -(-1)^{|x|} \sum_{\substack{l+k-1=n \\ 1\leq i_1 \leq n-k+1}} (-1)^{\eta} Sb_k(x; y_1, \ldots, S^{-1}M_l(Sy_{i_1}, \ldots,), \ldots, y_n)
\end{aligned}
$$

where

$$\varepsilon = \sum_{v=1}^{i_1-1} |y_v|(|x| - k + 1) + \sum_{v=1}^{k} |y_{i_1+v-1}|(k - v) + (l - i_1)|x|.$$

and

$$\eta = \sum_{v=1}^{i_1-1}(k-v)|y_v| + l \sum_{v=1}^{i_1-1} |y_v| + \sum_{v=i_1}^{i_1+l-1} (k - i_1)|y_v| + \sum_{v=i_1}^{n-l}(k-v)|y_{v+l}|$$

For $j > 1$ we demand

$$
\begin{aligned}
& (-1)^{\sum_{i=1}^{n}(n-i)|y_i|} Sb_n(S^{-1}M_j(Sx_1, \ldots, Sx_j); y_1, \ldots, y_n) = \\
& \sum (-1)^{\varepsilon} M_l(Sy_1, \ldots, Sb_{k_1}(x_1; y_{i_1}, \ldots), \ldots, Sb_{k_j}(x_j; y_{i_j}, \ldots), \ldots, Sy_n).
\end{aligned}
$$

The unindexed sum runs over all possible choices of non-negative integers that satisfy $l+k_1+\cdots+k_j-j = n$ and over all possible ordering-preserving insertions. The right hand side sign is given by

$$\varepsilon = \sum_{\substack{1\le t\le j\\ 1\le v\le k_t}} |y_{i_t+v-1}|(k_t - v) + \sum_{1\le v<l\le j} k_v|x_l| + \sum_{1\le v\le l\le j} |x_v|(i_{l+1} - i_l - k_l)$$
$$+ \sum_{\substack{0\le t<l\le j\\ i_t\le v<i_{t+1}}} (|y_v|+1)(|x_l| - k_l + 1) + \sum_{0\le v<l\le j} (i_{v+1} - i_v - k_v)(|x_l| - k_l + 1)$$

In the sums we are setting $i_0 = 0$ and $i_{j+1} = n+1$.

Corollary 3.3.14 (The A_∞-Deligne conjecture)**.** *If A is an A_∞-algebra, then its Hochschild complex $S\mathfrak{s}\operatorname{End}_A$ is a J-algebra.*

Proof. Clearly, $\mathfrak{s}\operatorname{End}_A$ is a brace algebra as it is an operad. Since A is an A_∞-algebra, the structure map $m = m_1 + m_2 + \cdots$ determines an A_∞-multiplication $m \in \mathfrak{s}\operatorname{End}_A$. It follows by Proposition 3.3.5 that $S\mathfrak{s}\operatorname{End}_A$ is an A_∞-algebra. Therefore, we need to show the compatibility relations. The result follows by direct computation from Theorem 3.3.9, expanding the definitions and taking into account the sign rules described in Appendix B. Let us do this in detail.

Recall that Theorem 3.3.9 states that $\Phi \circ M_j = \overline{M}_j \circ \Phi^{\otimes j}$. We start with the case $j > 1$. Let $Sx_1, \ldots, Sx_j \in S\mathfrak{s}\operatorname{End}_A$. Recall Diagram (3.9) for the definition of Φ. The left hand side of Equation (3.10) is by definition

$$\Phi(M_j(Sx_1, \ldots, Sx_j)) = S\overline{\sigma}\Phi'(S^{-1}M_j(Sx_1, \ldots, Sx_j)).$$

Notice that this map belongs to $S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$, where $\mathcal{O} = \operatorname{End}_A$, so let us consider just its arity n component. We are going to omit the external shift and consider the equation on $\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$ since this extra shift will cancel.

$$\overline{\sigma} b_n(S^{-1}M_j(Sx_1,\ldots,Sx_j);-) = (-1)^{\binom{n}{2}} Sb_n(S^{-1}M_j(Sx_1,\ldots,Sx_j);S^{-n}).$$

Now evaluate this on $Sy_1,\ldots,Sy_n \in S\mathfrak{s}\,\mathrm{End}_A$. Using the same calculation as in the proof of Lemma 3.3.10 we get

$$(-1)^{\sum_{i=1}^{n}(n-i)|y_i|} Sb_n(S^{-1}M_j(Sx_1,\ldots,Sx_j);y_1,\ldots,y_n). \tag{3.20}$$

This is already the left hand side in Definition 3.3.13. Let us now have a look at the right hand side of Equation (3.10). We evaluate again on $Sx_1,\ldots,Sx_j$ to obtain

$$\begin{aligned}\overline{M}_j(\Phi^{\otimes j})(Sx_1,\ldots,Sx_j) &= \overline{M}_j(\Phi(Sx_1),\ldots,\Phi(Sx_j))\\ &= \sum_{k_1,\ldots,k_j} \overline{M}_j(S\overline{\sigma}\Phi'(x_1),\ldots,S\overline{\sigma}\Phi'(x_j)).\end{aligned}$$

Expanding, this equals

$$\sum_{k_1,\ldots,k_j} \overline{M}_j(S(-1)^{\binom{k_1}{2}}Sb_{k_1}(x_1;S^{-k_1}),\ldots,S(-1)^{\binom{k_j}{2}}Sb_{k_j}(x_j;S^{-k_j})). \tag{3.21}$$

We now apply the definition of $\overline{M}_j$. Notice that by the isomorphism $\overline{\sigma}$ and Lemma 3.3.3 we have

$$|Sb_k(x;S^{-k})| = |\overline{\sigma}(b_k(x;-))| = |b_k(x;-)| = |x|.$$

Therefore, from Equation (3.21) we proceed again similarly as in

Lemma 3.3.10 to get

$$\sum_{k_1,\ldots,k_j} (-1)^{\sum_{v=1}^{j}\left[\binom{k_v}{2}+(j-v)|x_v|\right]}\overline{B}_j(M; Sb_{k_1}(x_1;S^{-k_1}),\ldots,Sb_{k_j}(x_j;S^{-k_j})).$$

Here we have omitted the extra shift just like on the left hand side. Now we need to use Proposition 3.2.3 to turn the above brace into composition of maps. Taking only the arity n component yields

$$\sum(-1)^{\sum_{v=1}^{j}\binom{k_v}{2}+\xi} M_l(-, Sb_{k_1}(x_1;S^{-k_1}),\ldots,Sb_{k_j}(x_j;S^{-k_j}),-).$$

where

$$\xi = \sum_{1\le v<l\le j} k_v|x_l| + \sum_{0\le v<l\le j} space_v(|x_l| - k_l + 1) + \sum_{1\le v\le l\le j} |x_v| space_l.$$

The variable $space_v$ represents the space between the v-th and the $(v+1)$-th brace. More precisely, if the v-th brace is inserted in the position i_v, $space_v = i_{v+1} - i_v - k_v$. The unindexed sum runs all possible ordering-preserving insertions and over all possible choices of integers that satisfy $l + k_1 + \cdots + k_j - j = n$. We have also used the fact that $k_v^2 \equiv k_v$ mod 2 to simplify the sign. Finally, we evaluate on $Sy_1,\ldots,Sy_n$. Here we need to take into account the sign rules explained in Appendix B. In particular, this means that we use the internal degree of $Sb_k(x;S^{-k})$ which is $|x| - k + 1$. This evaluation and some simplification produces the desired sign $(-1)^{\varepsilon}$.

Let us now treat the $j = 1$ case, where we have $\Phi \circ M_1 = \overline{M}_1 \circ \Phi$.

The left hand side is analogous to the general case, so we have

$$(-1)^{\sum_{i=1}^{n}(n-i)|y_i|} Sb_n(S^{-1}M_1(Sx); y_1, \dots, y_n). \tag{3.22}$$

On the right hand side we have

$$\begin{aligned}\overline{M}_1(\Phi(Sx)) &= \sum_k \overline{M}_1(S\overline{\sigma}\Phi'(x)) \\ &= \sum_k \overline{M}_1(S(-1)^{\binom{k}{2}} Sb_k(x; S^{-k})).\end{aligned}$$

Recalling that $|Sb_k(x; S^{-k})| = |x|$ and cancelling again the extra shift we may expand the above expression to obtain

$$\sum_k (-1)^{\binom{k}{2}} \left(\overline{B}_1(M; Sb_k(x; S^{-k})) - (-1)^{|x|}\overline{B}_1(Sb_k(x; S^{-k}); M)\right) \tag{3.23}$$

The first term is analogous to the general case, yielding

$$\sum_{\substack{l+k-1=n \\ 1\leq i_1 \leq n-k+1}} (-1)^{\varepsilon} M_l(Sy_1, \dots, Sb_k(x; y_{i_1}, \dots), \dots, Sy_n) \tag{3.24}$$

upon evaluation and cancelling $(-1)^{\binom{k}{2}}$, where

$$\varepsilon = \sum_{v=1}^{i_1-1}(|y_v|+1)(|x|-k+1) + \sum_{v=1}^{k}|y_{i_1+v-1}|(k-v) + (l-1)|x| + (i_1-1)(k-1).$$

Let us now focus on the second term of Equation (3.23) and let us omit the sign $(-1)^{\binom{k}{2}+|x|}$ for now. On arity n we have

$$\overline{B}_1(Sb_k(x;S^{-k});M) =$$
$$\sum_{\substack{l+k-1=n\\1\le i_1\le n-k+1}} (-1)^{l(k-1)+k-i_1} Sb_k(x;S^{-(i_1-1)},S^{-1}M_l,S^{-(k-i_1)}).$$

The sign is computed using Proposition 3.2.3 and the Koszul sign rule for the shifts. Notice that here we need to use the internal degree of $M_l \in \mathfrak{s}\,\mathrm{End}_A$, that is, $2-l$. Finally, evaluating at $Sy_1,\dots,Sy_n$ and combining the resulting signs with the factor $(-1)^{\binom{k}{2}+|x|}$ produces the result. $\square$

Remark 3.3.15. In Corollary 3.3.14, when A is just an associative algebra, we recover the homotopy G-algebra structure on its Hochschild complex according to Definition 3.3.12, as described in [GV95].

Chapter 4

Derived A_∞-algebras on operads

A lot of the research on A_∞-algebras relies on the existence and uniqueness of minimal models for dgas. This is guaranteed by the results of Kadeishvili [Kad80] when the dgas and their homologies are assumed to be degree-wise projective. In practice, this is implied by assuming a ground field. However, there are important examples arising from homotopy theory where projectivity cannot be guaranteed. In 2008, Sagave introduced the notion of derived A_∞-algebras, providing a framework for not necessarily projective modules over an arbitrary commutative ground ring [Sag10].

In this chapter we recall some definitions about derived A_∞-algebras and their motivation through minimal models. We also present some new ways of interpreting them in terms of operads and collections. We then recall the notion of filtered A_∞-algebra, since it will play a role in obtaining derived A_∞-algebras from A_∞-algebras on totalization. We combine a bigraded operadic suspension with totalization to encode derived A_∞-algebras. Using suitable brace structures we are able to define derived A_∞-algebra structures on certain operads and in turn show Theorem 4.5.8, which generalizes Theorem 3.3.9 to the derived setting. From this follows our major result, Corollary 4.5.10, a derived version of the Deligne conjecture.

4.1 Derived A_∞-algebras

In this section we introduce derived A_∞-algebras. We first give some definitions and then motivate them by explaining how they generalize the theory of minimal models that we saw in Section 2.2.3.

4.1.1 Definitions

Here we recall some definitions about derived A_∞-algebras from [Sag10] and provide some operadic interpretations. We also refer to Section 2.4.2 to recall some definitions and sign conventions.

Definition 4.1.1. *Using the notation in [RW11], a* derived A_∞-algebra *on a $(\mathbb{Z}, \mathbb{Z})$-bigraded R-module A consist of a family of R-linear maps*

$$m_{ij} : A^{\otimes j} \to A$$

of bidegree $(i, 2-(i+j))$ for each $j \geq 1$, $i \geq 0$, satisfying the equation

$$\sum_{\substack{u=i+p\\ v=j+q-1\\ j=r+1+t}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0 \tag{4.1}$$

for all $u \geq 0$ *and* $v \geq 1$.

According to the above definition, there are two equivalent ways of defining the operad of derived A_∞-algebras $d\mathcal{A}_\infty$ depending on the underlying category. One of them works on the category of bigraded modules bgMod_R and the other one is suitable for the category of vertical bicomplexes vbC_R. This is similar to the alternative definition of the $\mathcal{A}_\infty$-operad in Remark 2.3.12. We give the two definitions here as we are going to use both.

Definition 4.1.2. *The operad* $d\mathcal{A}_\infty$ *in* bgMod_R *is the operad generated by* $\{m_{ij}\}_{i\geq 0, j\geq 1}$ *subject to the derived* A_∞*-relation*

$$\sum_{\substack{u=i+p\\ v=j+q-1\\ j=r+1+t}} (-1)^{rq+t+pj} m_{ij} \circ_{r+1} m_{pq} = 0$$

for all $u \geq 0$ *and* $v \geq 1$.

The operad $d\mathcal{A}_\infty$ *in* vbC_R *is the dg operad generated by* $\{m_{ij}\}_{(i,j)\neq(0,1)}$ *with vertical differential given by*

$$\partial_\infty(m_{uv}) = -\sum_{\substack{u=i+p, v=j+q-1\\ j=r+1+t\\ (i,j)\neq(0,1)\neq(p,q)}} (-1)^{rq+t+pj} m_{ij} \circ_{r+1} m_{pq}.$$

Definition 4.1.3. *Let A and B be derived A_∞-algebras with respective structure maps m^A and m^B. An* ∞-morphism of derived A_∞-algebras *$f : A \to B$ is a family of maps $f_{st} : A^{\otimes t} \to B$ of bidegree $(s, 1-s-t)$ satisfying*

$$\sum_{\substack{u=i+p\\ v=j+q-1\\ j=r+1+t}} (-1)^{rq+t+pj} f_{ij}(1^{\otimes r} \otimes m^A_{pq} \otimes 1^{\otimes s}) = \sum_{\substack{u=i+p_1+\cdots+p_j\\ v=q_1+\cdots+q_j}} (-1)^{\epsilon} m^B_{ij}(f_{p_1q_1} \otimes \cdots \otimes f_{p_jq_j}) \tag{4.2}$$

for all $u \geq 0$ and $v \geq 1$, where

$$\epsilon = u + \sum_{1 \leq w < l \leq j} q_w(1 - p_l - q_l) + \sum_{w=1}^{j} p_w(j-w).$$

Example 4.1.4.

1. *An A_∞-algebra is the same as a derived A_∞-algebra such that m_{ij} vanishes for all $i > 0$.*

2. *One can check that, on any derived A_∞-algebra A, the maps $d_i = (-1)^i m_{i1}$ define a twisted complex structure. This leads to the possibility of defining a derived A_∞-algebra as a twisted complex with some extra structure, see Remark 4.5.4.*

Analogously to Definition 3.3.1, we have the following.

Definition 4.1.5. *A* derived A_∞-multiplication *on a bigraded operad $\mathcal{O}$ is a map of operads $d\mathcal{A}_\infty \to \mathcal{O}$.*

4.1.2 Minimal models

We would like to motivate the introduction of derived A_∞-algebras by stating the derived version of minimal models that we saw for A_∞-algebras in Section 2.2.3. In order to do that, we need some previous definitions that we take from [Sag10]. We also refer to this paper for all the technical details.

Definition 4.1.6. *A* bidga *is a monoid in the category of bicomplexes. Equivalently, a bidga is a derived A_∞-algebra with $m_{ij} = 0$ for $i + j \geq 3$.*

Recall that a quasi-isomorphism of A_∞-algebras is a morphism of A_∞-algebras that induces a quasi-isomorphism of complexes with respect to m_1. In the case of derived A_∞-algebras, the role of the quasi-isomorphisms is played by the so called E_2-equivalences, see [McC01] for more details about these equivalences.

Remark 4.1.7. The equations (4.1) defining a derived A_∞-structure include $m_{01}m_{01} = 0$. For a derived A_∞-algebra A, let $H^*_{ver}(A)$ denote its homology with respect to the *vertical differential* m_{01}. The map m_{01} is called the vertical differential because it raises the vertical degree.

Since the equations defining a derived A_∞-algebra also include

$$m_{21}m_{01} - m_{11}m_{11} + m_{01}m_{21} = 0,$$

it follows that the map m_{11} becomes a differential in horizontal direction on the bigraded module $H^*_{ver}(A)$. Therefore, we can form $H^*_{hor}(H^*_{ver}(A)) = H^*(H^*_{ver}(A); m_{11})$.

Definition 4.1.8. *An ∞-morphism $f : A \to B$ of derived A_∞-algebras is called an E_2*-equivalence *if $H^*_{hor}(H^*_{ver}(f_{01}))$ is an isomorphism of R-modules.*

We would like to extend some applications of A_∞-algebras to differential graded algebras that are not necessarily projective over the ground ring R or whose homology is not projective. The problem we encounter is that not all differential graded algebras possess a minimal model as an A_∞-algebra. However, Sagave showed that dgas have reasonable minimal models in the world of derived A_∞-algebras. For this, one has to apply a special projective resolution.

Definition 4.1.9. *Let A be a graded algebra. A* termwise R-projective resolution *of A is a termwise R-projective bidga P with $m_{01} = 0$ together with an E_2-equivalence $P \to A$.*

The following definition is analogue to Definition 2.2.4

Definition 4.1.10. *A derived A_∞-algebra is called* minimal *if $m_{01} = 0$.*

Finally, we can state the derived version of Theorem 2.2.5.

Theorem 4.1.11. *[Sag10, Theorem 1.1] Let A be a dga over R. Then there is a degree-wise R-projective derived A_∞-algebra E together with an E_2-equivalence $E \to A$ such that*

- *E is minimal,*
- *E is well-defined up to E_2-equivalence,*
- *together with the differential m_{11} and the multiplication m_{02}, E is a termwise R-projective resolution of the graded algebra $H^*(A)$.*

4.2 Filtered A_∞-algebras

We will make use of the filtration induced by the totalization functor in order to relate classical A_∞-algebras to derived A_∞-algebras. For this reason, we recall the notion of filtered A_∞-algebras.

Definition 4.2.1. *A* filtered A_∞*-algebra is an A_∞-algebra (A, m_i) together with a filtration $\{F_pA^i\}_{p\in\mathbb{Z}}$ on each R-module A^i such that for all $i \geq 1$ and all $p_1, \dots, p_i \in \mathbb{Z}$ and $n_1, \dots, n_i \geq 0$,*

$$m_i(F_{p_1}A^{n_1} \otimes \cdots \otimes F_{p_i}A^{n_i}) \subseteq F_{p_1+\cdots+p_i}A^{n_1+\cdots+n_i+2-i}.$$

Remark 4.2.2. Consider $\mathcal{A}_\infty$ as an operad in filtered complexes with the trivial filtration and let K be a filtered complex. There is a one-to-one correspondence between filtered A_∞-algebra structures on K and morphisms of operads in filtered complexes $\mathcal{A}_\infty \to \underline{\mathrm{End}}_K$ (recall $\underline{\mathrm{Hom}}$ from Definition 2.4.4). To see this, notice that if one forgets the filtrations such a map of operads gives an A_∞-algebra structure on K. The fact that this is a map of operads in filtered complexes implies that all the m_i's respect the filtrations.

Since the image of $\mathcal{A}_\infty$ lies in $\mathrm{End}_K = F_0\underline{\mathrm{End}}_K$, if we regard $\mathcal{A}_\infty$ as an operad in cochain complexes, then we get a one-to-one correspondence between filtered A_∞-algebra structures on K and morphisms of operads in cochain complexes $\mathcal{A}_\infty \to \mathrm{End}_K$.

We will not need to distinguish between morphisms and ∞-morphisms in the filtered case, so we give the following definition.

Definition 4.2.3. *A* morphism of filtered A_∞-algebras *from* (A, m_i, F) *to* (B, m_i, F) *is an* ∞*-morphism* $f : (A, m_i) \to (B, m_i)$ *of* A_∞*-algebras such that each map* $f_j : A^{\otimes j} \to A$ *is compatible with filtrations, i.e.*

$$f_j(F_{p_1}A^{n_1} \otimes \cdots \otimes F_{p_j}A^{n_j}) \subseteq F_{p_1+\cdots+p_j}B^{n_1+\cdots+n_j+1-j},$$

for all $j \geq 1$*,* $p_1, \dots p_j \in \mathbb{Z}$ *and* $n_1, \dots, n_j \geq 0$*.*

We will study the notions from this section from an operadic point of view. For this purpose we introduce some useful constructions in the next section.

4.3 Operadic totalization and vertical operadic suspension

We extend the totalization functor from Section 2.4.3 to the category of bigraded operads. We then extend operadic suspension from Chapter 3 to the bigraded setting. Combining these two devices we can use results of classical A_∞-algebras to study derived A_∞-algebras.

4.3.1 Operadic totalization

By Proposition 2.4.18 and the fact that the image of an operad under a lax monoidal functor is also an operad [Fre17, Proposition 3.1.1(a)], the totalization functor defined in Section 2.4.3 will define a functor from operads in brigraded modules (resp. twisted complexes) to operads in graded modules (resp. cochain complexes).

Therefore, let $\mathcal{O}$ be either a bigraded operad, i.e. an operad in the category of bigraded R-modules or an operad in twisted complexes. We define $\mathrm{Tot}(\mathcal{O})$ as the operad of graded R-modules (or cochain complexes) for which

$$\mathrm{Tot}(\mathcal{O}(n))^d = \bigoplus_{i<0} \mathcal{O}(n)_i^{d-i} \oplus \prod_{i\geq 0} \mathcal{O}(n)_i^{d-i}$$

is the image of $\mathcal{O}(n)$ under the totalization functor. The differential would be as described after Definition 2.4.15. The insertion maps $\bar{\circ}_r$ are given by the composition

$$\mathrm{Tot}(\mathcal{O}(n))\otimes\mathrm{Tot}(\mathcal{O}(m)) \xrightarrow{\mu} \mathrm{Tot}(\mathcal{O}(n)\otimes\mathcal{O}(m)) \xrightarrow{\mathrm{Tot}(\circ_r)} \mathrm{Tot}(\mathcal{O}(n+m-1)), \tag{4.3}$$

that is explicitly

$$(x\bar{\circ}_r y)_k \coloneqq (\mathrm{Tot}(\circ_r)\circ\mu(x,y))_k = \sum_{k_1+k_2=k} (-1)^{k_1 d_2} x_{k_1} \circ_r y_{k_2}$$

for $x = (x_i)_i \in \mathrm{Tot}(\mathcal{O}(n))^{d_1}$ and $y = (y_j)_j \in \mathrm{Tot}(\mathcal{O}(m))^{d_2}$.

More generally, operadic composition $\bar{\gamma}$ is defined by the composite

$$\begin{array}{l} \mathrm{Tot}(\mathcal{O}(N)) \otimes \mathrm{Tot}(\mathcal{O}(a_1)) \otimes \cdots \otimes \mathrm{Tot}(\mathcal{O}(a_N)) \\ \qquad\qquad\qquad \big\downarrow \mu \\ \qquad \mathrm{Tot}(\mathcal{O}(N) \otimes \mathcal{O}(a_1) \otimes \cdots \otimes \mathcal{O}(a_N)) \xrightarrow{\mathrm{Tot}(\gamma)} \mathrm{Tot}\left(\mathcal{O}\left(\sum a_i\right)\right), \end{array}$$

This map can be computed explicitly by iteration of the insertions $\bar{\circ}_r$, giving the following.

Lemma 4.3.1. *The operadic composition* $\bar{\gamma}$ *on* $\mathrm{Tot}(\mathcal{O})$ *is given by*

$$\bar{\gamma}(x; x^1, \dots, x^N)_k = \sum_{k_0+k_1+\cdots+k_N=k} (-1)^{\varepsilon} \gamma(x_{k_0}; x^1_{k_1}, \dots, x^N_{k_N})$$

for $x = (x_k)_k \in \mathrm{Tot}(\mathcal{O}(N))^{d_0}$ *and* $x^i = (x^i_k)_k \in \mathrm{Tot}(\mathcal{O}(a_i))^{d_i}$*, where*

$$\varepsilon = \sum_{j=1}^{m} d_j \sum_{i=0}^{j-1} k_i \tag{4.4}$$

and γ *is the operadic composition on* $\mathcal{O}$*.* □

Notice that the sign is precisely the same appearing in Equation (2.13).

4.3.2 Vertical operadic suspension

On a bigraded operad we can use operadic suspension on the vertical degree with analogue results to those of the graded case that we explored in Chapter 3.

We define $\Lambda(n) = S^{n-1}R$, where S is a vertical shift of degree so that $\Lambda(n)$ is the underlying ring R concentrated in bidegree $(0, n-1)$. As in the single-graded case, we express the basis element of $\Lambda(n)$ as $e^n = e_1 \wedge \cdots \wedge e_n$. Similarly, $\Lambda^-(n) = S^{1-n}R$ is defined.

The operad structure on the bigraded $\Lambda = \{\Lambda(n)\}_{n \geq 0}$ is the same as in the graded case, namely

$$\Lambda(n) \otimes \Lambda(m) \xrightarrow{\quad\circ_{r+1}\quad} \Lambda(n+m-1)$$

$$(e_1 \wedge \cdots \wedge e_n) \otimes (e_1 \wedge \cdots \wedge e_m) \longmapsto (-1)^{(n-r-1)(m-1)} e_1 \wedge \cdots \wedge e_{n+m-1}.$$

Definition 4.3.2. *Let $\mathcal{O}$ be a bigraded linear operad, i.e. an operad on the category of bigraded R-modules. The* vertical operadic suspension $\mathfrak{s}\mathcal{O}$ *of $\mathcal{O}$ is given arity-wise by the Hadamard product of the operads $\mathcal{O}$ and Λ, in other words, $\mathfrak{s}\mathcal{O}(n) = (\mathcal{O} \otimes \Lambda)(n) = \mathcal{O}(n) \otimes \Lambda(n)$ with diagonal composition. Similarly, we define the* vertical operadic desuspension $\mathfrak{s}^{-1}\mathcal{O}(n) = \mathcal{O}(n) \otimes \Lambda^{-}(n)$.

We may identify the elements of $\mathcal{O}$ with the elements of $\mathfrak{s}\mathcal{O}$.

Definition 4.3.3. *For $x \in \mathcal{O}(n)$ of bidegree $(k, d-k)$, its* natural bidegree *in $\mathfrak{s}\mathcal{O}$ is the pair $(k, d+n-k-1)$. To distinguish both degrees we call $(k, d-k)$ the* internal bidegree *of x, since this is the degree that x inherits from the grading of $\mathcal{O}$.*

If we write $\circ_{r+1}$ for the operadic insertion on $\mathcal{O}$ and $\tilde{\circ}_{r+1}$ for the operadic insertion on $\mathfrak{s}\mathcal{O}$, we may find a relation between the two insertion maps in a completely analogous way to Lemma 3.1.4.

Lemma 4.3.4. *For $x \in \mathcal{O}(n)$ and $y \in \mathcal{O}(m)^q_l$ we have*

$$x \tilde{\circ}_{r+1} y = (-1)^{(n-1)q+(n-1)(m-1)+r(m-1)} x \circ_{r+1} y. \tag{4.5}$$

□

As can be seen, this is the same sign as the single-graded operadic suspension but with vertical degree. We will see that this is also the case more generally for bigraded braces in Section 4.4. As a consequence we have the following theorem with similar proof to the single-graded case, where all the suspensions are vertical.

Theorem 4.3.5. *Given a bigraded R-module A, there is an isomorphism of operads* $\mathrm{End}_A \cong \mathfrak{s}\,\mathrm{End}_{SA}$. $\square$

Another consequence of Lemma 4.3.4 is that $\tilde{\circ}$ leads to the Lie bracket from [RW11], which implies that $m = \sum_{i,j} m_{ij}$ is a derived A_∞-multiplication if and only if for all $u \geq 0$

$$\sum_{i+j=u} \sum_{l,k} (-1)^i m_{jl} \tilde{\circ} m_{ik} = 0. \tag{4.6}$$

In [RW11, Proposition 2.15] this equation is described in terms of a sharp operator $\sharp$.

We also get the functorial properties that we studied for the single-graded case in Section 3.1.2 and Proposition 3.1.15.

4.3.3 Vertical suspension and totalization

Now we are going to combine vertical operadic suspension and totalization. More precisely, the *totalized vertical suspension* of a bigraded operad $\mathcal{O}$ is the graded operad $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$. This operad has an insertion map explicitly given by

$$(x \star_{r+1} y)_k = \sum_{k_1+k_2=k} (-1)^{(n-1)(d_2-k_2-m+1)+(n-1)(m-1)+r(m-1)+k_1 d_2} x_{k_1} \circ_{r+1} y_{k_2} \tag{4.7}$$

for $x = (x_i)_i \in \mathrm{Tot}(\mathfrak{s}\mathcal{O}(n))^{d_1}$ and $x = (x_j)_j \in \mathrm{Tot}(\mathfrak{s}\mathcal{O}(m))^{d_2}$. As usual, denote

$$x \star y = \sum_{r=0}^{m-1} x \star_{r+1} y. \tag{4.8}$$

This star operation is precisely the star operation from [LRW13, §5.1], i.e. the convolution operation on $\mathrm{Hom}((dAs)^{\mathrm{i}}, \mathrm{End}_A)$. In particular, we can recover the Lie bracket from in [LRW13]. We will do this in Corollary 4.4.3.

Before continuing, let us show a lemma that allows us to work only with the single-graded operadic suspension if needed.

Proposition 4.3.6. *For a bigraded operad $\mathcal{O}$ we have an isomorphism $\mathrm{Tot}(\mathfrak{s}\mathcal{O}) \cong \mathfrak{s}\,\mathrm{Tot}(\mathcal{O})$, where the suspension on the left hand side is the bigraded version and on the right hand side is the single-graded version.*

Proof. We may identify each element $x = (x_k \otimes e^n)_k \in \mathrm{Tot}(\mathfrak{s}\mathcal{O}(n))$ with the element $x = (x_k)_k \otimes e^n \in \mathfrak{s}\,\mathrm{Tot}(\mathcal{O}(n))$. Thus, abusing of notation, for an element $(x_k)_k \in \mathrm{Tot}(\mathfrak{s}\mathcal{O}(n))$ the isomorphism is given by

$$\begin{aligned} f : \mathrm{Tot}(\mathfrak{s}\mathcal{O}(n)) &\cong \mathfrak{s}\,\mathrm{Tot}(\mathcal{O}(n)) \\ (x_k)_k &\mapsto ((-1)^{kn} x_k)_k \end{aligned}$$

Clearly, this map is bijective so we just need to check that it commutes with insertions. Recall from Equation (4.7) that the insertion on $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$ is given by

$$(x \star_{r+1} y)_k = \sum_{k_1+k_2=k} (-1)^{(n-1)(d_2-k_2-n+1)+(n-1)(m-1)+r(m-1)+k_1 d_2} x_{k_1} \circ_{r+1} y_{k_2}$$

for $x = (x_i)_i \in \mathrm{Tot}(\mathfrak{s}\mathcal{O}(n))^{d_1}$ and $y = (y_j)_j \in \mathrm{Tot}(\mathfrak{s}\mathcal{O}(m))^{d_2}$. Similarly, we may compute the insertion on $\mathfrak{s}\,\mathrm{Tot}(\mathcal{O})$ by combining the sign produced first by Tot and then by $\mathfrak{s}$. This results in the following insertion map

$$(x \star'_{r+1} y)_k = \sum_{k_1+k_2=k} (-1)^{(n-1)(d_2-n+1)+(n-1)(m-1)+r(m-1)+k_1(d_2-m+1)} x_{k_1} \circ_{r+1} y_{k_2}$$

for $x = (x_i)_i \in \mathfrak{s}\,\mathrm{Tot}(\mathcal{O}(n))^{d_1}$ and $y = (y_j)_j \in \mathfrak{s}\,\mathrm{Tot}(\mathcal{O}(m))^{d_2}$. Now let us show that $f(x \star y) = f(x) \star f(y)$. We do this by showing that all the insertions are equal on both sides. By definition, we have that $f((x \star_{r+1} y))_k$ equals the following.

$$\begin{aligned}
&\sum_{k_1+k_2=k} (-1)^{k(n+m-1)+(n-1)(d_2-k_2-n+1)+(n-1)(m-1)+r(m-1)+k_1 d_2} x_{k_1} \circ_{r+1} y_{k_2} \\
&= \sum_{k_1+k_2=k} (-1)^{(n-1)(d_2-n+1)+(n-1)(m-1)+r(m-1)+k_1(d_2-m+1)} f(x_{k_1}) \circ_{r+1} f(y_{k_2}) \\
&= (f(x) \star_{r+1} f(y))_k
\end{aligned}$$

as desired. □

Remark 4.3.7. As we mentioned in Remark 2.4.19, there exist other possible ways of totalizing by varying the natural transformation μ. For instance, we can choose the totalization functor Tot' which is the same as Tot but with a natural transformation μ' defined in such a way that the insertion on $\mathrm{Tot}'(\mathcal{O})$ is defined by

$$(x \hat{\circ} y)_k = \sum_{k_1+k_2=k} (-1)^{k_2 n_1} x_{k_1} \circ y_{k_2}.$$

This is also a valid approach for our purposes and there is simply a sign difference. But we have chosen our convention to be consistent with other conventions, such as the derived A_∞-equation. However, it can be verified that $\mathrm{Tot}'(\mathfrak{s}\mathcal{O}) = \mathfrak{s}\,\mathrm{Tot}'(\mathcal{O})$. With the original totalization we have a non identity isomorphism given by Proposition 4.3.6. Similar relations can be found among the other alternatives mentioned in Remark 2.4.19.

Using the operadic structure on $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$, we can describe derived A_∞-multiplications in a new conceptual way.

Lemma 4.3.8. *A derived A_∞-multiplication on a bigraded operad $\mathcal{O}$ is equivalent to an element $m \in \mathrm{Tot}(\mathfrak{s}\mathcal{O})$ of degree 1 concentrated in positive arity such that $m \star m = 0$.*

Proof. A derived A_∞-multiplication on $\mathcal{O}$ is by Definition 4.1.5 a map

$$f : d\mathcal{A}_\infty \to \mathcal{O}.$$

Since $\mathcal{A}_\infty$ is generated by elements μ_{ij} of bidegree $(i, 2-i-j)$, such a map is determined by the elements $m_{ij} = f(\mu_{ij}) \in \mathcal{O}_i^{2-i-j}(j)$. Consider the A_∞-multiplication $m_j = (m_{ij})_i \in \mathrm{Tot}(\mathfrak{s}\mathcal{O}(j))$. We have that $\deg(m_j) = 1$ for all j. Therefore, let $m = m_1 + m_2 + \cdots \in \mathrm{Tot}(\mathfrak{s}\mathcal{O})$. We may check that $m \star m = 0$. For that we just need to check Equation (4.7). On arity n, this amounts to computing

$$(m \star m)_k = \sum_{r=0}^{n-1} \sum_{\substack{i+p=k \\ j+q=n-1}} (-1)^{rp+j-r-1+pj} m_{ij} \circ_{r+1} m_{pq} = 0.$$

The above expression vanishes precisely because the elements m_{ij} satisfy the derived A_∞-equation.

Conversely, let $m \in \mathrm{Tot}(\mathfrak{s}\mathcal{O})$ of degree 1, is concentrated in positive arity and satisfying $m \star m = 0$. We can split m into its arity and horizontal degree components as $m = \sum_{i,j} m_{ij}$. As we have seen, the fact that $m \star m = 0$ is equivalent to the elements m_{ij} satisfying the derived A_∞-equation, and therefore, a map $f : d\mathcal{A}_\infty \to \mathcal{O}$ is determined by the images $f(\mu_{ij}) = m_{ij}$, which are of bidegree $(i, 2-i-j)$. $\square$

Remark 4.3.9. Similarly to Remark 3.1.8, one can use the definition of $d\mathcal{A}_\infty$ as an operad of vertical bicomplexes from Definition 4.1.2. In that case, one obtains in an analogous way to Lemma 4.3.8, that a derived A_∞-multiplication on of vertical bicomplexes $\mathcal{O}$ with vertical differential ∂ is equivalent to an element of degree 1 concentrated on arity at least 2 satisfying the equation $\partial(m) + m \star m = 0$. However, we stick to operads of bigraded modules for the sake of consistency.

From Lemma 4.3.8, since now m is an A_∞-multiplication on a single-graded operad, we can proceed as in the proof of Proposition 3.3.5 to show that m determines an A_∞-algebra structure on $S\operatorname{Tot}(\mathfrak{s}\mathcal{O}) \cong S\mathfrak{s}\operatorname{Tot}(\mathcal{O})$. The goal now is showing that this A_∞-structure on $S\operatorname{Tot}(\mathfrak{s}\mathcal{O})$ is equivalent to a derived A_∞-structure on $S\mathfrak{s}\mathcal{O}$ and compute the structure maps explicitly. We will do this in Section 4.5.

Before that, let us explore the brace structures that appear from this new operadic constructions and use them to reinterpret derived ∞-morphisms and their composition.

4.4 Bigraded braces and totalized braces

In this section we generalize the brace algebras we saw in Section 3.2 to bigraded and totalized operads. This will allow to continue our generalization towards a derived version of the Deligne conjecture by following similar steps to the A_∞-case. We also use this generalization to reinterpret derived ∞-morphisms in a similar way to Section 3.2.2.

4.4.1 Braces

We are going to define a brace structure on $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$ using totalization. First note that one can define bigraded braces just like in the single-graded case, only changing the sign ε in Definition 3.2.1 to be $\varepsilon = \sum_{p=1}^{n}\sum_{q=i}^{i_p}\langle x_p, y_q\rangle$ according to the bigraded sign convention.

As one might expect, we can define bigraded brace maps b_n on a bigraded operad $\mathcal{O}$ and also on its operadic suspension $\mathfrak{s}\mathcal{O}$, obtaining similar signs as in the single-graded case, but with vertical internal degrees, see Proposition 3.2.3.

We can also define braces on $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$ via operadic composition. In this case, these are usual single-graded braces. More precisely, we define the maps

$$b_n^\star : \mathrm{Tot}(\mathfrak{s}\mathcal{O}(N)) \otimes \bigotimes_{j=1}^{n} \mathrm{Tot}(\mathfrak{s}\mathcal{O}(a_j)) \longrightarrow \mathrm{Tot}\left(\mathfrak{s}\mathcal{O}\left(N-n+\sum a_i\right)\right)$$

using the operadic composition $\gamma^\star$ on $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$ as

$$b_n^\star(x; x_1, \dots, x_n) = \sum \gamma^\star(x; 1, \dots, 1, x_1, 1, \dots, 1, x_n, 1, \dots, 1),$$

where the sum runs over all possible ordering preserving insertions. The brace map $b^{\star}_n(x;x_1,\dots,x_n)$ vanishes whenever $n>N$ and $b^{\star}_0(x)=x$.

Operadic composition can be described in terms of insertions in the obvious way, namely

$$\gamma^{\star}(x;y_1,\dots,y_N)=(\cdots(x\star_1 y_1)\star_{1+a(y_1)} y_2\cdots)\star_{1+\sum a(y_p)} y_N, \tag{4.9}$$

where $a(y_p)$ is the arity of y_p (in the case of a brace y_p is either 1 or some x_i). If we want to express this composition in terms of the composition γ in $\mathcal{O}$ we just have to find out the sign factor applying the same strategy as in the single-graded case. In fact, as we said, there is a sign factor that comes from vertical operadic suspension that is identical to the graded case, but replacing internal degree by internal vertical degree. This is the sign that determines the brace b_n on $\mathfrak{s}\mathcal{O}$. Explicitly, it is given by the following lemma, whose proof is identical to the single-graded case, see Proposition 3.2.3.

Lemma 4.4.1. *For $x\in\mathfrak{s}\mathcal{O}(N)$ and $x_i\in\mathfrak{s}\mathcal{O}(a_i)$ of internal vertical degree q_i $(1\le i\le n)$, we have*

$$b_n(x;x_1,\dots,x_n)=\sum_{N-n=h_0+\cdots+h_n}(-1)^{\eta}\gamma(x\otimes 1^{\otimes h_0}\otimes x_1\otimes\cdots\otimes x_n\otimes 1^{\otimes h_n}),$$

where

$$\eta=\sum_{0\le j<l\le n}h_jq_l+\sum_{1\le j<l\le n}a_jq_l+\sum_{j=1}^{n}(a_j+q_j-1)(n-j)+\sum_{1\le j\le l\le n}(a_j+q_j-1)h_l.$$

The other sign factor is produced by totalization. This was computed in Lemma 4.3.1. Combining both factors we obtain the following.

Lemma 4.4.2. *We have*

$$b_j^\star(x;x^1,\dots,x^N)_k = \sum_{\substack{k_0+k_1+\cdots+k_N=k\\ h_0+h_1+\cdots+h_N=j-N}} (-1)^{\eta+\sum_{j=1}^m d_j\sum_{i=0}^{j-1}k_i}\gamma(x_{k_0};1^{h_0},x^1_{k_1},1^{h_1},\dots,x^N_{k_N},1^{h_N}) \tag{4.10}$$

for $x=(x_k)_k\in \mathrm{Tot}(\mathfrak{s}\mathcal{O}(N))^{d_0}$ and $x^i=(x^i_k)_k\in \mathrm{Tot}(\mathfrak{s}\mathcal{O}(a_i))^{d_i}$, where η is defined in Lemma 4.4.1. □

Corollary 4.4.3. *For $\mathcal{O}=\mathrm{End}_A$, where A is a bigraded module, the brace $b_1^\star(f;g)$ is the operation $f\star g$ defined in [LRW13]. As a consequence,*

$$[f,g]=b_1^\star(f;g)-(-1)^{NM}b_1^\star(g;f)$$

for $f\in\mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_A)^N$ and $g\in\mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_A)^M$, is the same Lie bracket that was defined in [LRW13]. □

Notice that in [LRW13] the sign in the bracket is $(-1)^{(N+1)(M+1)}$, but this is because their total degree differs by one with respect to ours.

4.4.2 Reinterpretation of derived ∞-morphisms

Just like we did for graded modules in Section 3.2.2, for bigraded modules A and B we may define the collection $\mathrm{End}_B^A=\{\mathrm{Hom}_R(A^{\otimes n},B)\}_{n\geq 1}$ of bigraded modules. Recall that this collection has a left module structure over End_B

$$\mathrm{End}_B\circ\mathrm{End}_B^A\to\mathrm{End}_B^A$$

given by composition of maps. Similarly, given a bigraded module C, we can define composition maps

$$\mathrm{End}_C^B \circ \mathrm{End}_B^A \to \mathrm{End}_C^A .$$

The collection End_B^A also has an infinitesimal right module structure over End_A

$$\mathrm{End}_B^A \circ_{(1)} \mathrm{End}_A \to \mathrm{End}_B^A$$

given by insertion of maps.

Similarly to the single-graded case, we may describe derived ∞-morphisms in terms of the above operations.

Lemma 4.4.4. *A derived ∞-morphism of A_∞-algebras $A \to B$ with respective structure maps m^A and m^B is equivalent to a degree 0 element $f \in \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_B^A)$ concentrated in positive arity such that*

$$\rho(f \circ_{(1)} m^A) = \lambda(m^B \circ f),$$

where

$$\lambda : \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_B) \circ \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_B^A) \to \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_B^A)$$

is induced by the left module structure on End_B^A, and

$$\rho : \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_B) \circ_{(1)} \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_B^A) \to \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_B^A)$$

is induced by the right infinitesimal module structure on End_B^A.

In addition, the composition of ∞-morphisms is given by the natural composition

$$\mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_C^B) \circ \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_B^A) \to \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_C^A).$$

Proof. Since $f_j = (f_{ij})_i \in \mathrm{Tot}(\mathfrak{s}\,\mathrm{End}_B^A(j))$ is of degree 0, we have that that f_{ij} is of bidegree $(i, 1-i-j)$. Thus, the equation

$$\rho(f \circ_{(1)} m^A) = \lambda(m^B \circ f)$$

coincides up to signs with with the Equation (4.2), the equation defining derived ∞-morphisms of derived A_∞-algebras. The signs that appear in the above equation are obtained in a similar way to that on the brace $b_j^\star$, see Equation (4.10). Thus, it is enough to plug in the sign provided by Equation (4.10) from the corresponding degrees and arities to obtain the desired result. The composition of derived ∞-morphisms follows similarly. □

In the case that $f : A \to A$ is an ∞-endomorphism, the above definition can be written in terms of operadic composition as $f \star m = \gamma^\star(m \circ f)$, where $\gamma^\star$ is the composition map derived from the operation $\star$, see Equation (4.9). Here, $\circ$ is the plethysm of maps of collections, not to be confused with composition of maps.

4.5 The derived A_∞-structure on an operad

In this section we show that, under some reasonable assumptions, an operad with a derived A_∞-multiplication is a derived A_∞-algebra and compute the structure maps explicitly. From this structure we obtain a derived version of the Deligne conjecture for the Hochschild complex of a derived A_∞-algebra.

4.5.1 Derived A_∞-structures

As in the single-graded case, we identify $\mathfrak{s}\mathcal{O} = \prod_n \mathfrak{s}\mathcal{O}(n)$. We follow a strategy inspired by the proof of the following theorem to show that there is a derived A_∞-structure on $S\mathfrak{s}\mathcal{O}$. We refer the reader to Section 2.4 to recall the definitions of the categories used.

Theorem 4.5.1. *([CESLW18, Proposition 4.55]) Let $(A, d^A) \in \mathrm{tC}_R^b$ be a twisted complex horizontally bounded on the right and A its underlying cochain complex. We have natural bijections*

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{vbOp},d^A}(d\mathcal{A}_\infty, \mathrm{End}_A) &\cong \mathrm{Hom}_{\mathrm{vbOp}}(\mathcal{A}_\infty, \underline{\mathcal{E}nd}_A) \\
&\cong \mathrm{Hom}_{\mathrm{vbOp}}(\mathcal{A}_\infty, \underline{\mathcal{E}nd}_{\mathrm{Tot}(A)}) \\
&\cong \mathrm{Hom}_{\mathrm{fCOp}}(\mathcal{A}_\infty, \underline{\mathrm{End}}_{\mathrm{Tot}(A)}),
\end{aligned}$$

where vbOp *and* fCOp *denote the categories of operads in* vbC_R *and* fC_R *respectively, and* $\mathrm{Hom}_{\mathrm{vbOp},d^A}$ *denotes the subset of morphisms which send μ_{i1} to d_i^A. We view $\mathcal{A}_\infty$ as an operad in* vbC_R *sitting in horizontal degree zero or as an operad in filtered complexes with trivial filtration.*

Remark 4.5.2. According to Remark 4.2.2, the last isomorphism can be replaced by

$$\mathrm{Hom}_{\mathrm{vbOp}}(\mathcal{A}_\infty, \underline{\mathcal{E}nd}_{\mathrm{Tot}(A)}) \cong \mathrm{Hom}_{\mathrm{COp}}(\mathcal{A}_\infty, \mathrm{End}_{\mathrm{Tot}(A)}),$$

where COp is the category of operads in cochain complexes.

There are several important assumptions to make in order to use the theorem. First of all, we need A to be horizontally bounded on the right, meaning that there exists some integer i such that $A_k^{d-k} = 0$ for all $k > i$. In our case, $A = S\mathfrak{s}\mathcal{O}$ for $\mathcal{O}$ an operad with a derived A_∞-multiplication m, so being horizontally bounded on the right implies that, for each $j > 0$, we can only have at most a finite number of non-zero components m_{ij}. This situation happens in practice in all known examples of derived A_∞-algebras so far, some of them are in [MM21, Remark 6.5], [RW11], and [ARLR+15, §5]. Under this assumption we may replace all direct products by direct sums, implying thus extra monoidality properties.

We also need to provide A with a twisted complex structure. The reason for this is that Theorem 4.5.1 uses the definition of derived A_∞-algebras on an underlying twisted complex, see Remark 4.5.4. We show explicitly the existence of a twisted complex structure on an operad with derived A_∞-multiplication in Appendix D, but it also follows from Corollary 4.5.6. We also provide another version of this theorem that works for bigraded modules, Corollary 4.5.5.

With these assumption, by Theorem 4.5.1 we can guarantee the existence of a derived A_∞-algebra structure on A provided that $\mathrm{Tot}(A)$ has an A_∞-algebra structure.

Theorem 4.5.3. *Let $A = S\mathfrak{s}\mathcal{O}$ where $\mathcal{O}$ is an operad horizontally bounded on the right with a derived A_∞-multiplication $m = \sum_{ij} m_{ij} \in \mathcal{O}$. Let $x_1 \otimes \cdots \otimes x_j \in (A^{\otimes j})_k^{d-k}$ and let $x_v = Sy_v$ for $v = 1, \ldots, j$ and y_v be of bidegree $(k_v, d_v - k_v)$. The following maps M_{ij} for $j \geq 2$ determine a derived A_∞-algebra structure on A.*

$$M_{ij}(x_1, \ldots, x_j) = (-1)^{\sum_{v=1}^{j}(j-v)(d_v-k_v)} \sum_l Sb_j(m_{il}; y_1, \ldots, y_j).$$

Note that we abuse of notation and identify $x_1 \otimes \cdots \otimes x_j$ with an element of $\mathrm{Tot}(A^{\otimes j})$ with only one non-zero component. For a general element, extend linearly.

Proof. Since m is a derived A_∞-multiplication $\mathcal{O}$, we have that $m \star m = 0$ when we view m as an element of $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$. By Proposition 3.3.5, this defines an A_∞-algebra structure on $S\,\mathrm{Tot}(\mathfrak{s}\mathcal{O})$ given by maps

$$M_j : (S\,\mathrm{Tot}(\mathfrak{s}\mathcal{O}))^{\otimes j} \to S\,\mathrm{Tot}(\mathfrak{s}\mathcal{O})$$

induced by shifting brace maps

$$b_j^\star(m; -) : (\mathrm{Tot}(\mathfrak{s}\mathcal{O}))^{\otimes j} \to \mathrm{Tot}(\mathfrak{s}\mathcal{O}).$$

The graded module $S\,\mathrm{Tot}(\mathfrak{s}\mathcal{O})$ is endowed with the structure of a filtered complex with differential M_1 and filtration induced by the column filtration on $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$. Note that $b_j^\star(m; -)$ preserves the column filtration since every component $b_j^\star(m_{ij}; -)$ has positive horizontal degree.

Since $S\,\mathrm{Tot}(\mathfrak{s}\mathcal{O}) \cong \mathrm{Tot}(S\mathfrak{s}\mathcal{O})$, we can view M_j as the image of a morphism of operads of filtered complexes $f : \mathcal{A}_\infty \to \mathrm{End}_{\mathrm{Tot}(S\mathfrak{s}\mathcal{O})}$ in such a way that $M_j = f(\mu_j)$ for $\mu_j \in \mathcal{A}_\infty(j)$.

We now work our way backwards using the strategy also employed by the proof of Theorem 4.5.1. The isomorphism

$$\mathrm{Hom}_{\mathrm{vbOp}}(\mathcal{A}_\infty, \underline{\mathcal{E}nd}_{\mathrm{Tot}(A)}) \cong \mathrm{Hom}_{\mathrm{COp}}(\mathcal{A}_\infty, \mathrm{End}_{\mathrm{Tot}(A)})$$

does not modify the map M_j at all but allows us to see it as a element of $\underline{\mathcal{E}nd}_{\mathrm{Tot}(A)}$ of bidegree $(0, 2-j)$.

The isomorphism

$$\mathrm{Hom}_{\mathrm{vbOp}}(\mathcal{A}_\infty, \underline{\mathcal{E}nd}_A) \cong \mathrm{Hom}_{\mathrm{vbOp}}(\mathcal{A}_\infty, \underline{\mathcal{E}nd}_{\mathrm{Tot}(A)})$$

in the direction we are following is the result of applying $\mathrm{Hom}_{\mathrm{vbOp}}(\mathcal{A}_\infty, -)$ to the map described in Lemma 2.5.18. Under this isomorphism, f is sent to the map

$$\mu_j \mapsto \mathfrak{Tot}^{-1} \circ c(M_j, \mu^{-1}) = \mathfrak{Tot}^{-1} \circ M_j \circ \mu^{-1},$$

where c is the composition in $\underline{f\mathcal{C}_R}$. The functor $\mathfrak{Tot}^{-1}$ decomposes M_j into a sum of maps $M_j = \sum_i \widetilde{M}_{ij}$, where each $\widetilde{M}_{ij}$ is of bidegree $(i, 2-j-i)$. More explicitly, let $A = S\mathfrak{s}\mathcal{O}$ and let $x_1 \otimes \cdots \otimes x_j \in (A^{\otimes j})^{d-k}_k$. We abuse of notation and identify $x_1 \otimes \cdots \otimes x_j$ with an element of $\mathrm{Tot}(A^{\otimes j})$ with only one non-zero component. For a general element, extend linearly. Then we have

$$\begin{aligned}
\mathfrak{Tot}^{-1}(M_j(\mu^{-1}(x_1 \otimes \cdots \otimes x_j))) &= \\
\mathfrak{Tot}^{-1}(Sb_j^{\star}(m; (S^{-1})^{\otimes j}(\mu^{-1}(x_1 \otimes \cdots \otimes x_j)))) &= \\
\sum_i (-1)^{id} \sum_l Sb_j^{\star}(m_{il}; (S^{-1})^{\otimes j}(\mu^{-1}(x_1 \otimes \cdots \otimes x_j))) &= \\
\sum_i (-1)^{id} \sum_l (-1)^{\varepsilon} Sb_j(m_{il}; (S^{-1})^{\otimes j}(\mu^{-1}(x_1 \otimes \cdots \otimes x_j))) &= \\
\sum_i \sum_l (-1)^{id+\varepsilon} Sb_j(m_{il}; (S^{-1})^{\otimes j}(\mu^{-1}(x_1 \otimes \cdots \otimes x_j)))
\end{aligned} \tag{4.11}$$

so that

$$\widetilde{M}_{ij}(x_1, \ldots, x_j) = \sum_l (-1)^{id+\varepsilon} Sb_j(m_{il}; (S^{-1})^{\otimes j}(\mu^{-1}(x_1 \otimes \cdots \otimes x_j))),$$

where b_j is the brace on $\mathfrak{s}\mathcal{O}$ and ε is given in Lemma 4.3.1.

According to the isomorphism

$$\mathrm{Hom}_{\mathrm{vbOp},d^A}(d\mathcal{A}_\infty, \mathrm{End}_A) \cong \mathrm{Hom}_{\mathrm{vbOp}}(\mathcal{A}_\infty, \underline{\mathcal{E}nd}_A), \tag{4.12}$$

the maps $M_{ij} = (-1)^{ij}\widetilde{M}_{ij}$ define an A_∞-structure on $S\mathfrak{s}\mathcal{O}$. Therefore we now just have to work out the signs. Notice that d_v is the total degree of y_v as an element of $\mathfrak{s}\mathcal{O}$ and recall that d is the total degree of $x_1 \otimes \cdots \otimes x_j \in A^{\otimes j}$. Therefore, ε can be written as

$$\varepsilon = i(d-j) + \sum_{1 \le v < w \le j} k_v d_w.$$

The sign produced by μ^{-1}, as we saw in Lemma 2.4.20, is precisely determined by the exponent

$$\sum_{w=2}^{j} d_w \sum_{v=1}^{w-1} k_v = \sum_{1\leq v<w\leq j} k_v d_w,$$

so this sign cancels the right hand summand of ε. This cancellation was expected since this sign comes from μ^{-1}, and operadic composition is defined using μ, see Equation (4.3). Finally, the sign $(-1)^{i(d-j)}$ left from ε cancels with $(-1)^{id}$ in Equation (4.11) and $(-1)^{ij}$ from the isomorphism (4.12). This means that we only need to consider signs produced by vertical shifts. This calculation has been done previously in Lemma 3.3.10 and as we claimed the result is

$$M_{ij}(x_1, \dots, x_j) = (-1)^{\sum_{v=1}^{j}(j-v)(d_v-k_v)} \sum_{l} Sb_j(m_{il}; y_1, \dots, y_j).$$

□

Remark 4.5.4. As in the case of A_∞-algebras in C_R, see Remark 2.3.12, it can be seen that we have two equivalent descriptions of A_∞-algebras in tC_R, see [CESLW18].

(1) A twisted complex (A, d^A) together with a morphism $\mathcal{A}_\infty \to \underline{\mathcal{E}nd}_A$ of operads in vbC_R, which is determined by a family of elements $M_i \in \underline{t\mathcal{C}_R}(A^{\otimes i}, A)_0^{2-i}$ for $i \geq 2$ for which the A_∞-relations holds for $i \geq 2$, see Equation (2.4). The composition is the one prescribed by the composition morphisms of $\underline{t\mathcal{C}_R}$.

(2) A bigraded module A with elements $M_i \in \underline{\mathcal{bgMod}}_R(A^{\otimes i}, A)_0^{2-i}$ for $i \geq 1$ for which all the A_∞-relations hold, see Equation (2.4). The composition is prescribed by the composition morphisms of $\underline{\mathcal{bgMod}}_R$.

Since the composition morphism in $\underline{\mathcal{bgMod}}_R$ is induced from the one in $\underline{t\mathcal{C}_R}$ by forgetting the differential, these two presentations are equivalent.

This equivalence allows us to formulate the following alternative version of Theorem 4.5.1.

Corollary 4.5.5. *Given a bigraded module A horizontally bounded on the right we have isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathrm{bgOp}}(d\mathcal{A}_\infty, \mathrm{End}_A) &\cong \mathrm{Hom}_{\mathrm{bgOp}}(\mathcal{A}_\infty, \underline{\mathcal{End}}_A)\\ &\cong \mathrm{Hom}_{\mathrm{bgOp}}(\mathcal{A}_\infty, \underline{\mathcal{End}}_{\mathrm{Tot}(A)})\\ &\cong \mathrm{Hom}_{\mathrm{fOp}}(\mathcal{A}_\infty, \underline{\mathrm{End}}_{\mathrm{Tot}(A)}),\end{aligned}$$

where bgOp *is the category of operads of bigraded modules and* fOp *is the category of operads of filtered modules.*

Proof. Let us look at the first isomorphism

$$\mathrm{Hom}_{\mathrm{bgOp}}(\mathcal{A}_\infty, \underline{\mathcal{End}}_A) \cong \mathrm{Hom}_{\mathrm{bgOp}}(d\mathcal{A}_\infty, \mathrm{End}_A).$$

Let $f : \mathcal{A}_\infty \to \underline{\mathcal{End}}_A$ be a map of operads in bgOp. This is equivalent to maps in bgOp

$$\mathcal{A}_\infty(j) \to \underline{\mathcal{End}}_A(j)$$

for each $j \geq 1$, which are determined by elements $M_j := f(\mu_j) \in \underline{\mathcal{End}}_A(j)$ for $v \geq 1$ of bidegree $(0, 2-j)$ satisfying the A_∞-equation with respect to the composition in $\underline{\mathcal{bgMod}}_R$. Moreover, $M_j := (\tilde{m}_{0j}, \tilde{m}_{1j}, \dots)$ where $\tilde{m}_{ij} := (M_j)_i : A^{\otimes n} \to A$ is a map of bidegree $(i, 2-i-j)$. Since the composition in $\underline{\mathcal{bgMod}}_R$ is the same as in $\underline{t\mathcal{C}_R}$, the computation of the A_∞-equation becomes analogous to the computation done in [CESLW18, Prop 4.47], showing that the maps $m_{ij} = (-1)^i \tilde{m}_{ij}$ for $i \geq 0$ and $j \geq 0$ define a derived A_∞-algebra structure on A.

The second isomorphism

$$\mathrm{Hom}_{\mathrm{bgOp}}(\mathcal{A}_\infty, \underline{\mathcal{End}}_A) \cong \mathrm{Hom}_{\mathrm{bgOp}}(\mathcal{A}_\infty, \underline{\mathcal{End}}_{\mathrm{Tot}(A)})$$

follows from the bigraded module case of Lemma 2.5.17. Finally, the isomorphism

$$\mathrm{Hom}_{\mathrm{bgOp}}(\mathcal{A}_\infty, \underline{\mathcal{End}}_{\mathrm{Tot}(A)}) \cong \mathrm{Hom}_{\mathrm{fOp}}(\mathcal{A}_\infty, \underline{\mathrm{End}}_{\mathrm{Tot}(A)})$$

is analogous to the last isomorphism of Theorem 4.5.1, replacing the quasi-free relation by the full A_∞-algebra relations. □

According to Corollary 4.5.5, if we have an A_∞-algebra structure on $A = S\mathfrak{s}\mathcal{O}$, we can consider its arity 1 component $M_1 \in \underline{\mathrm{End}}_{\mathrm{Tot}(A)}$ and split it into maps $M_{i1} \in \mathrm{End}_A$. Since these maps must satisfy the derived A_∞-relations, they define a twisted complex structure on A. The next corollary describes the maps M_{i1} explicitly.

Corollary 4.5.6. *Let $\mathcal{O}$ be a bigraded operad with a derived A_∞-multiplication and let*

$$M_{i1} : S\mathfrak{s}\mathcal{O} \to S\mathfrak{s}\mathcal{O}$$

be the arity 1 derived A_∞-algebra maps induced by Corollary 4.5.5 from

$$M_1 : \mathrm{Tot}(S\mathfrak{s}\mathcal{O}) \to \mathrm{Tot}(S\mathfrak{s}\mathcal{O}).$$

Then

$$M_{i1}(x) = \sum_l (Sb_1(m_{il}; S^{-1}x) - (-1)^{\langle x, m_{il}\rangle} Sb_1(S^{-1}x; m_{il})),$$

where $x \in (S\mathfrak{s}\mathcal{O})_k^{d-k}$ and $\langle x, m_{il}\rangle = ik + (1-i)(d-1-k)$.

Proof. Notice that the proof of Corollary 4.5.5 is essentially the same as the proof Theorem 4.5.1. This means that the proof of this result is an arity 1 restriction of the proof of Theorem 4.5.3. Thus, we apply Equation (4.11) to the case $j = 1$. Recall that for $x \in (S\mathfrak{s}\mathcal{O})_k^{d-k}$,

$$M_1(x) = b_1^\star(m; S^{-1}x) - (-1)^{n-1} b_1^\star(S^{-1}x; m).$$

In this case, there is no μ involved. Therefore, introducing the final extra sign $(-1)^i$ from the proof of Theorem 4.5.3, we get from Equation (4.11) that

$$\begin{aligned}\widetilde{M}_{i1}(x) = (-1)^i \sum_l ((-1)^{id+i(d-1)} Sb_1(m_{il}; S^{-1}x) \\ -(-1)^i \sum_l (-1)^{d-1+id+k} Sb_1(S^{-1}x; m_{il})),\end{aligned}$$

where b_1 is the brace on $\mathfrak{s}\mathcal{O}$. Simplifying signs we obtain

$$\widetilde{M}_{i1}(x) = \sum_l Sb_1(m_{il}; S^{-1}x) - (-1)^{\langle m_{il}, x\rangle} Sb_1(m_{il}; S^{-1}x)) = M_{i1}(x),$$

where $\langle m_{il}, x\rangle = ik + (1-i)(d-1-k)$, as claimed. □

4.5.2 The derived Deligne conjecture

Note that the maps given by Theorem 4.5.3 and Corollary 4.5.6 formally look the same as their single-graded analogues in Lemma 3.3.10 but with an extra index that is fixed for each M_{ij}. This means that we can follow the same procedure as in Section 3.3.1 to define higher derived A_∞-maps on the Hochschild complex of a derived A_∞-algebra. More precisely, given an operad $\mathcal{O}$ with a derived multiplication and $A = S\mathfrak{s}\mathcal{O}$, we will define a derived A_∞-algebra structure on $S\mathfrak{s}\operatorname{End}_A$. We will then connect the algebraic structure on A with the structure on $S\mathfrak{s}\operatorname{End}_A$ through braces. This connection will allow us to formulate and show a new, more general version of the Deligne conjecture that generalizes the one that we obtained in Corollary 3.3.14.

Let $\overline{B}_j$ be the bigraded brace map on $\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$ and consider the maps

$$\overline{M}'_{ij} : (\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}})^{\otimes j} \to \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}} \tag{4.13}$$

defined as

$$\overline{M}'_{ij}(f_1, \dots, f_j) = \overline{B}_j(M_{i\bullet}; f_1, \dots, f_j) \qquad j > 1,$$
$$\overline{M}'_{i1}(f) = \overline{B}_1(M_{i\bullet}; f) - (-1)^{ip+(1-i)q}\overline{B}_1(f; M_{i\bullet}),$$

for f of natural bidegree (p,q), where $M_{i\bullet} = \sum_j M_{ij}$. We define

$$\overline{M}_{ij} : (S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}})^{\otimes j} \to S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}},$$
$$\overline{M}_{ij} := \overline{\sigma}(M'_{ij}) = S \circ M'_{ij} \circ (S^{\otimes n})^{-1}.$$

As in the single-graded case we can define a map

$$\Phi : S\mathfrak{s}\mathcal{O} \to S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$$

as the map making the following diagram commute

$$\begin{tikzcd} S\mathfrak{s}\mathcal{O} \arrow[rr, "\Phi"] \arrow[d] & & S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}} \\ \mathfrak{s}\mathcal{O} \arrow[r, "\Phi'"] & \operatorname{End}_{\mathfrak{s}\mathcal{O}} \arrow[r, "\cong"] & \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}} \arrow[u] \end{tikzcd} \tag{4.14}$$

where

$$\Phi' \colon \mathfrak{s}\mathcal{O} \to \operatorname{End}_{\mathfrak{s}\mathcal{O}}$$
$$x \mapsto \sum_{n\geq 0} b_n(x; -).$$

The isomorphism $\operatorname{End}_{\mathfrak{s}\mathcal{O}} \cong \mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$ is given by $\overline{\sigma}$.

In this setting we have the bigraded version of Theorem 3.3.9. But before stating the theorem, for the sake of completeness let us state the definition of the Hochschild complex of a bigraded module.

Definition 4.5.7. *We define the* Hochschild cochain complex *of a bigraded module A to be the bigraded module $S\mathfrak{s}\operatorname{End}_A$. If (A,d) is a vertical bicomplex, then the Hochschild complex has a vertical differential given* *by $\partial(f) = [d, f] = d \circ f - (-1)^q f \circ d$, where f has natural vertical degree q and $\circ$ is the plethysm corresponding to operadic insertions.*

In particular, $S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$ is the Hochschild cochain complex of $S\mathfrak{s}\mathcal{O}$. If $\mathcal{O}$ has a derived A_∞-multiplication, then the differential of the Hochschild complex $S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$ is given by $\overline{M}_{01}$ from Equation (4.13).

The following is the same as Theorem 3.3.9 but carrying the extra index i and using the bigraded sign conventions.

Theorem 4.5.8. *The map Φ defined in diagram (4.14) above is a morphism of $d\mathcal{A}_\infty$-algebras, i.e. for all $i \geq 0$ and $j \geq 1$ the equation*

$$\Phi(M_{ij}) = \overline{M}_{ij}(\Phi^{\otimes j})$$

holds. $\square$

As a consequence of this theorem, we can obtain a derived version of the Deligne conjecture. In order to formulate this new Deligne conjecture, we need to introduce the notion of *derived J-algebra*, as a derived version of J-algebras introduced in Definition 3.3.13. To have a more succinct formulation we use the notation $\mathrm{vdeg}(x)$ for the vertical degree of x.

Definition 4.5.9. *A* derived J-algebra V *is a derived A_∞-algebra with structure maps $\{M_{ij}\}_{i\geq 0, j\geq 1}$ such that the shift $S^{-1}V$ is a brace algebra. Furthermore, the braces and the derived A_∞-structure satisfy the following compatibility relations. Let $x, x_1, \ldots, x_j, y_1, \ldots, y_n \in S^{-1}V$. For all $n, i \geq 0$ we demand*

$$(-1)^{\sum_{i=1}^{n}(n-v)\mathrm{vdeg}(y_v)} Sb_n(S^{-1}M_{i1}(Sx); y_1, \dots, y_n) =$$

$$\sum_{\substack{l+k-1=n \\ 1\le i_1 \le n-k+1}} (-1)^{\varepsilon} M_{il}(Sy_1, \dots, Sb_k(x; y_{i_1}, \dots), \dots, Sy_n)$$

$$-(-1)^{\langle x, M_{il}\rangle} \sum_{\substack{l+k-1=n \\ 1\le i_1 \le n-k+1}} (-1)^{\eta} Sb_k(x; y_1, \dots, S^{-1}M_{il}(Sy_{i_1}, \dots,), \dots, y_n)$$

where

$$\varepsilon = \sum_{v=1}^{i_1-1} \langle Sy_v, S^{1-k}x\rangle + \sum_{v=1}^{k} \mathrm{vdeg}(y_{i_1+v-1})(k-v) + (l-i_1)\mathrm{vdeg}(x).$$

and

$$\eta = \sum_{v=1}^{i_1-1}(k-v)\mathrm{vdeg}(y_v) + l\sum_{v=1}^{i_1-1}\mathrm{vdeg}(y_v) + \sum_{v=i_1}^{i_1+l-1}(k-i_1)\mathrm{vdeg}(y_v) + \sum_{v=i_1}^{n-l}(k-v)\mathrm{vdeg}(y_{v+l})$$

For $j > 1$ we demand

$$(-1)^{\sum_{i=1}^{n}(n-v)\mathrm{vdeg}(y_v)} Sb_n(S^{-1}M_{ij}(Sx_1, \dots, Sx_j); y_1, \dots, y_n) =$$

$$\sum (-1)^{\varepsilon} M_{il}(Sy_1, \dots, Sb_{k_1}(x_1; y_{i_1}, \dots), \dots, Sb_{k_j}(x_j; y_{i_j}, \dots), \dots, Sy_n).$$

The unindexed sum runs over all possible choices of non-negative integers that satisfy $l+k_1+\dots+k_j-j = n$ and over all possible ordering preserving insertions. The right hand side sign is given by

$$\varepsilon = \sum_{\substack{1\le t\le j \\ 1\le v\le k_t}} \mathrm{vdeg}(y_{i_t+v-1})(k_v - v) + \sum_{1\le i<l\le j} k_v \mathrm{vdeg}(x_l) + \sum_{\substack{0\le t<l\le j \\ i_t\le v<i_{t+1}}} \langle S y_v, S^{1-k_l} x_l\rangle$$
$$+ \sum_{0\le v<l\le j} (i_{v+1} - i_v - k_v)\mathrm{vdeg}(S^{1-k_l} x_l) + \sum_{1\le v\le l\le j} \mathrm{vdeg}(x_v)(i_{l+1} - i_l - k_l)$$

All the above shifts are vertical and we are setting $i_0 = 0$, $i_{j+1} = n+1$.

Corollary 4.5.10 (The derived Deligne conjecture)**.** *If A is a derived A_∞-algebra horizontally bounded on the right, then its Hochschild complex $S\mathfrak{s}\,\mathrm{End}_A$ is a derived J-algebra.*

Proof. The result follows from Theorem 4.5.8 analogously to Corollary 3.3.14 using the explicit expressions and signs given by Theorem 4.5.3, Corollary 4.5.6 and Lemma 4.4.1. □

Chapter 5

Future research

We finish by outlining some questions that remain open after our research and that would be interesting to investigate in the future. These questions arise naturally from the work done with derived A_∞-algebras and from the classical results by Gerstenhaber and Voronov [GV95]. First, we recall the boundedness assumptions we needed to make on derived A_∞-algebras, see Remark 4.5.2, and wonder how we can either guarantee or bypass them. Then we recall the implications of the classical Deligne conjecture on the Hochschild complex of an associative algebra to try to formulate a generalization for derived A_∞-algebras.

5.1 Boundedness conditions

In Theorem 4.5.3 we obtained a derived A_∞-algebra structure on the bigraded module $A = S\mathfrak{s}\mathcal{O}$ for an operad $\mathcal{O}$ with a derived A_∞-multiplication. Since this structure was obtained from Theorem 4.5.1, a crucial assumption for it to exist is that A is horizontally bounded on the right. This was necessary to apply strong monoidality on $\mathrm{Tot}(A^{\otimes n})$. As a consequence, the components m_{ij} of the derived A_∞-multiplication (Definition 4.1.5) vanish for sufficiently large i.

As we mentioned in Remark 4.5.2, this condition is satisfied in all known examples of derived A_∞-algebras [MM21, Remark 6.5], [RW11], and [ARLR+15, §5]. These examples usually come as minimal models of dgas. So a first question that arises is the following.

Question 1. *Are there any conditions on a dga that guarantee that its minimal model is horizontally bounded on the right?*

An answer to this question would give us a better understanding of how general our results are. In fact, it is open whether a derived A_∞-structure can be obtained for a more general operad. Even though we needed to use some monoidality results that require boundedness, the explicit maps that we obtained in Theorem 4.5.3 can be defined for any operad with a derived A_∞-multiplication. A first idea would be attempting a direct computation to see if they satisfy the derived A_∞-equation, see Equation (4.1). Of course, we would like to use a more conceptual approach. So more generally the question would be the following.

Question 2. *Can we define a derived A_∞-structure on any operad with a derived A_∞-multiplication?*

5.2 Hochschild cohomology

The classical Deligne conjecture states that the Hochschild complex of an associative algebra has a structure of homotopy G-algebra [GV95]. This has implications on the Hochschild cohomology of the associative algebra, namely the homotopy G-algebra structure on the Hochschild complex induces a Gerstenhaber algebra structure on cohomology. We would like to extend this result to derived A_∞-algebras.

Let us review the structure on the Hochschild complex of an associative operad in order to understand the question that we will be asking about the derived A_∞-case.

Let $\mathcal{O}$ be an operad with an associative multiplication m, i.e. an A_∞-multiplication m such that $m = m_2$, see Definition 3.1.6. In this case, as a consequence of Proposition 3.3.5 or by [GV95, Proposition 2], we have a dg-algebra structure on $S\mathfrak{s}\mathcal{O}$ given by the differential

$$d(Sx) = Sb_1(m; x) - (-1)^{|x|} Sb_1(x; m) \tag{5.1}$$

and the multiplication

$$m(Sx, Sy) = Sb_2(m; x, y). \tag{5.2}$$

In particular, if $\mathcal{O} = \mathrm{End}_A$ is the endomorphism operad of an associative algebra A, these maps provide a dg-algebra structure on the Hochschild complex of A. But this is not all the structure that we get. Since any operad is a brace algebra, we have an interaction between the dg-algebra and the brace structure. More precisely, $\mathcal{O}$ has a structure of *homotopy G-algebra*, see Definition 2 and Theorem 3 of [GV95] for the original statements and Definition 3.3.12 for our adapted definition.

Given the algebraic structure described above on the Hochschild complex of an associative algebra, we can then take cohomology with respect to d, eq. (5.1), to compute the Hochschild cohomology of A, denoted by $HH^*(A)$. It is known that m, eq. (5.2), and the bracket

$$[x, y] = Sb_1(x; y) - (-1)^{|x||y|} Sb_1(y; x)$$

induce a structure of a Gerstenhaber algebra on $HH^*(A)$ [GV95, Corollary 5]. The proof relies on some identities that can be deduced from the definition of homotopy G-algebra, such as graded homotopy commutativity.

If we try to replicate this argument for A_∞-algebras, the structure we get on the Hochschild complex is that of a J-algebra, see Definition 3.3.13. In this case, we have to compute cohomology with respect to M_1, see Lemma 3.3.10. In the definition of J-algebras, we encounter an explosion in the number and complexity of relations and maps involved with respect to homotopy G-algebras. Therefore, the resulting structure has not been feasible to manipulate and it is not very clear what kind of algebraic structure is induced on cohomology. The derived case is of course even more difficult to handle as we would need to work with the even more complex derived J-algebras, see Definition 4.5.9. In addition, as we explained in Section 4.1.2, it is possible to consider vertical and horizontal cohomologies. These should be taken with respect to M_{01} and M_{11} respectively, see Corollary 4.5.6. So the natural question to ask is the following.

Question 3. *What algebraic structure do derived J-algebras induce on the vertical and horizontal cohomologies of a derived A_∞-algebra?*

Appendix

A Some proofs and details

In this appendix we prove some results that rely on sign calculations and combinatorics.

Lemma A.1. *For any integers n and m, the following equality holds mod 2.*

$$\binom{n+m-1}{2} + \binom{n}{2} + \binom{m}{2} = (n-1)(m-1).$$

Proof. Let us compute

$$\binom{n+m-1}{2} + \binom{n}{2} + \binom{m}{2} + (n-1)(m-1) \mod 2.$$

By definition, this equals

$$\frac{(n+m-1)(n+m-2)}{2} + \frac{n(n-1)}{2} + \frac{m(m-1)}{2} + (n-1)(m-1)$$

Let us expand the above expression into the following.

$$\begin{aligned}\frac{n^2+2nm-2n+m^2-2m-n-m+2}{2}+\\ \frac{n^2-n+m^2-m+2(nm-n-m+1)}{2}=\\ n^2+2nm-3n+m^2-3m+2=\\ n^2+m+m^2+m=\\ 0 \mod 2\end{aligned}$$

as desired, because $n^2 = n \mod 2$.

$\square$

Recall that we define the *suspension* or *shift* of a graded module A as the graded module SA having degree components $(SA)^i = A^{i-1}$.

Theorem A.2. *There is an isomorphism of (symmetric) operads* $\mathrm{End}_{SA} \cong \mathfrak{s}^{-1}\,\mathrm{End}_A$.

Proof. For each n, we clearly have an isomorphism of graded modules

$$\begin{aligned}\mathrm{End}_{SA}(n) &= \mathrm{Hom}_R((SA)^{\otimes n}, SA)\\ &\cong \mathrm{Hom}_R(A^{\otimes n}, A) \otimes S^{1-n} sig_n\\ &= \mathfrak{s}^{-1}\,\mathrm{End}_A(n)\end{aligned}$$

given by the map σ^{-1} defined before as

$$\sigma^{-1}(F) = (-1)^{\binom{n}{2}} S^{-1} \circ F \circ S^{\otimes n},$$

where $\circ$ denotes the composition of maps. We must show that this map is an isomorphism of operads, in other words, it commutes with insertions and with the symmetric group action.

Let us first check that σ^{-1} commutes with insertions. For that, let $F \in \mathrm{End}_{SA}(n)$ and $G \in \mathrm{End}_{SA}(m)$. On the one had we have

$$\sigma^{-1}(F \circ_i G) = (-1)^{\binom{n+m-1}{2}+\deg(G)(i-1)} S^{-1} \circ F(S^{\otimes i-1} \otimes G(S^{\otimes m}) \otimes S^{\otimes n-i}),$$

and on the other hand

$$\begin{aligned} \sigma^{-1}(F)\tilde{\circ}_i\sigma^{-1}(G) = \\ (-1)^{(n-1)(m-1)+(n-1)(\deg(G)+m-1)+(i-1)(m-1)}\sigma^{-1}(F) \circ_i \sigma^{-1}(G) = \\ (-1)^{\varepsilon} S^{-1} \circ F(S^{\otimes i-1} \otimes G(S^{\otimes m}) \otimes S^{\otimes n-i}), \end{aligned}$$

where

$$\begin{aligned} \varepsilon = &\binom{n}{2} + \binom{m}{2} + (n-1)(m-1) + (n-1)(\deg(G) + m - 1) \\ &+ (i-1)(m-1) + (\deg(G) + m - 1)(n-i). \end{aligned}$$

By Lemma A.1,

$$\binom{n+m-1}{2} = \binom{n}{2} + \binom{m}{2} + (n-1)(m-1) \mod 2,$$

so we only need to check that $\deg(G)(i-1) \mod 2$ equals

$$(n-1)(\deg(G) + m - 1) + (i-1)(m-1) + (\deg(G) + m - 1)(n-i).$$

This can be done by direct computation.

Now we are going to show that σ^{-1} commutes with the action of the symmetric group. Recall that on End_{SA} we have the usual permutation action, whilst on $\mathfrak{s}^{-1}\operatorname{End}_A$ the action is twisted by the sign of the permutation. It is enough to show this for transpositions of the form $\tau = (i\ i+1)$ since they generate the symmetric group.

Let us write $(-1)^v$ for $(-1)^{\deg(v)}$. On the one hand,

$$\sigma^{-1}(F\tau)(v_1 \otimes \cdots \otimes v_n) = (-1)^{\sum_{j=1}^n (n-j)v_j} S^{-1} \circ (F\tau)(Sv_1 \otimes \cdots \otimes Sv_n)$$

Applying τ we obtain

$$(-1)^{\sum_{j=1}^n (n-j)v_j + (v_i - 1)(v_{i+1}-1)} S^{-1} \circ F(Sv_1 \otimes \cdots \otimes Sv_{i+1} \otimes Sv_i \otimes \cdots \otimes Sv_n). \tag{A1}$$

The sign $(-1)^{\sum_{j=1}^n (n-j)v_j}$ comes from swapping the shift maps S past the v_j's, and the sign $(-1)^{(v_i-1)(v_{i+1}-1)}$ comes from permuting v_i and v_{i+1}. On the other hand, performing similar sign computations we have

$$\begin{aligned}
&(\sigma^{-1}(F)\tau)(v_1 \otimes \cdots \otimes v_n) && \text{(A2)}\\
&= (-1)^{v_i v_{i+1} - 1} S^{-1} \circ F \circ S^{\otimes n}(v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n)\\
&= (-1)^{\delta} S^{-1} \circ f(Sv_1 \otimes \cdots \otimes Sv_{i+1} \otimes Sv_i \otimes \cdots \otimes Sv_n)
\end{aligned}$$

where $\delta = v_i v_{i+1} - 1 + \sum_{j \neq i, i+1} (n-j)v_j + (n-i-1)v_i + (n-i)v_{i+1}$.

Now we just have to check that the signs are the same. Modulo 2, the sign on Equation (A1) is

$$v_i v_{i+1} + v_i + v_{i+1} - 1 + \sum_{j=1}^n (n-j)v_j =$$

$$v_i v_{i+1} - 1 + \sum_{j \neq i, i+1}^n (n-j)v_j + (n-i-1)v_i + (n-i)v_{i+1},$$

which indeed coincides with the sign on Equation (A2). $\square$

Remark A.3. If in the proof above we replace S with S^{-1}, we have that the map

$$\sigma^{-1}(F) = (-1)^{\binom{n}{2}} S^{-1} \circ F \circ S^{\otimes n}$$

transforms into $(-1)^{\binom{n}{2}} S \circ F \circ (S^{-1})^{\otimes n} = S \circ F \circ (S^{\otimes n})^{-1}$. This is the map $\overline{\sigma}(F)$ from page 9 of [RW11], and following the same proof we have done above but with this change of S into S^{-1} we get the isomorphism of operads

$$\overline{\sigma} : \mathrm{End}_A \cong \mathfrak{s}\,\mathrm{End}_{SA}\,.$$

B Koszul sign on operadic suspension

The purpose of this section is to clear up the procedure to apply the Koszul sign rule in situations in which operadic suspension is involved.

Let End_A be the endomorphism operad of some R-module A and consider the operadic suspension $\mathfrak{s}\,\mathrm{End}_A$. We are going to make a few comments on the application of the Koszul rule when applying maps from $\mathfrak{s}\,\mathrm{End}_A(n)$ to elements of $A^{\otimes n}$. Let $f \otimes e^n \in \mathfrak{s}\,\mathrm{End}_A(n)$ be of degree $\deg(f) + n - 1$. For $a \in A^{\otimes n}$ we have

$$(f \otimes e^n)(a) = (-1)^{\deg(a)(n-1)} f(a) \otimes e^n$$

because $\deg(e^n) = n-1$. Note that $f \otimes e^n = g \otimes e^n$ if and only if $f = g$. In addition, it is not possible that $f \otimes e^n = g \otimes e^m$ for $n \neq m$. The reader may notice that $f(a) \otimes e^n \notin A$, but it can be identified with an element of $S^{n-1}A$. This is a reminiscence of the isomorphism $\mathfrak{s}^{-1}\operatorname{End}_A \cong \operatorname{End}_{SA}$.

If we take the tensor product of two maps $f \otimes e^n \in \mathfrak{s}\operatorname{End}_A(n)$ and $g \otimes e^m \in \mathfrak{s}\operatorname{End}_A(m)$ and apply it to $a \otimes b \in A^{\otimes n} \otimes A^{\otimes m}$, we have

$$\begin{aligned}
&((f \otimes e^n)\otimes(g \otimes e^m))(a \otimes b)\\
&\qquad\qquad = (-1)^{\deg(a)(\deg(g)+m-1)}(f \otimes e^n)(a) \otimes (g \otimes e^m)(b)\\
&\qquad\qquad = (-1)^{\varepsilon}(f(a) \otimes e^n) \otimes (f(b) \otimes e^m),
\end{aligned}$$

where $\varepsilon = \deg(a)(\deg(g) + m - 1) + \deg(a)(n - 1) + \deg(b)(m - 1)$. The last remark that we want to make is the case of a map of the form

$$f(1^{\otimes k-1} \otimes g \otimes 1^{\otimes n-k}) \otimes e^{m+n-1} \in \mathfrak{s}\operatorname{End}_A(n + m - 1),$$

such as those produced by the operadic insertion $\mathfrak{s}f\tilde{\circ}_k\mathfrak{s}g$. In this case, this map applied to $a_{k-1} \otimes b \otimes a_{n-k} \in A^{\otimes k-1} \otimes A^{\otimes m} \otimes A^{\otimes n-k}$ results in

$$\begin{aligned}
(f(1^{\otimes k-1} \otimes g \otimes 1^{\otimes n-k}) \otimes e^{m+n-1})(a_{k-1} \otimes b \otimes a_{n-k}) =\\
(-1)^{\nu} f(1^{\otimes k-1} \otimes g \otimes 1^{\otimes n-k}(a_{k-1} \otimes b \otimes a_{n-k})) \otimes e^{m+n-1} =\\
(-1)^{\nu+\deg(a_{k-1})\deg(g)} f(a_{k-1} \otimes g(b) \otimes a_{n-k}) \otimes e^{m+n-1}.
\end{aligned}$$

where $\nu = (m + n)(\deg(a_{k-1}) + \deg(b) + \deg(a_{n-k}))$. To go from the first line to the second, we switch e^{m+n-1} of degree $m + n - 2$ with $a_{k-1} \otimes b \otimes a_{n-k}$. To go from the second line to the third we apply the usual sign rule for graded maps.

The purpose of this last remark is not only review the Koszul sign rule but also remind that the insertion $\mathfrak{s}f\tilde{\circ}_k\mathfrak{s}g$ is of the above form, so that the e^{m+n-1} component is always at the end and does not play a role in the application of the sign rule with the composed maps. In other words, it does not affect the individual degrees of the maps, just the degree of the overall composition.

C Sign of the braces

In order to find the sign of the braces on $\mathfrak{s}\,\mathrm{End}_A$, let us use an analogous strategy to the one used in [RW11, Appendix] to find the signs of the Lie bracket $[f, g]$ on End_A.

Let A be a graded module. Let SA be the graded module with $SA^v = A^{v+1}$, and so the *suspension* or *shift* map $S : A \to SA$ given by the identity map has degree -1.

Let $f \in \mathrm{End}_A(N)^i = \mathrm{Hom}_R(A^{\otimes N}, A)^i$. Recall that σ is the inverse of the map from Theorem 3.1.10, so that $\sigma(f)$ is defined as the map making the following diagram commute.

$$\begin{tikzcd} SA^{\otimes N} \arrow[r, "\sigma(f)"] \arrow[d, "(S^{-1})^{\otimes N}"'] & SA \\ A^{\otimes N} \arrow[r, "f"] & A \arrow[u, "S"'] \end{tikzcd}$$

Explicitly, $\sigma(f) = S \circ f \circ (S^{-1})^{\otimes N} \in \mathrm{End}_A(N)^{i+N-1}$.

Remark C.1. In [RW11] there is a sign $(-1)^{N+i-1}$ in front of f, but it seems to be irrelevant for our purposes. Another fact to remark on is that the suspension of graded modules used here and in [RW11] is the opposite that we have used to define the operadic suspension in the sense that in Section 3.1.1 we used $SA^v = A^{v-1}$. This does not change the signs or the procedure, but in the statement of Theorem 3.1.10, operadic desuspension should be changed to operadic suspension.

Notice that by the Koszul sign rule

$$\begin{aligned}(S^{-1})^{\otimes N} \circ S^{\otimes N} &= (-1)^{\sum_{j=1}^{N-1} j} 1\\ &= (-1)^{\frac{N(N-1)}{2}} 1\\ &= (-1)^{\binom{N}{2}} 1,\end{aligned}$$

so $(S^{-1})^{\otimes N} = (-1)^{\binom{N}{2}} (S^{\otimes N})^{-1}$. For this reason, given $F \in \mathrm{End}_{S(A)}(m)^j$, we have

$$\sigma^{-1}(F) = (-1)^{\binom{m}{2}} S^{-1} \circ F \circ S^{\otimes m} \in \mathrm{End}_A(m)^{j-m+1}.$$

For $g_j \in \mathrm{End}_A(a_j)^{q_j}$, let us write $f[g_1, \dots, g_n]$ for the map

$$\sum_{k_0+\cdots+k_n=N-n} f(1^{\otimes k_0} \otimes g_1 \otimes 1^{\otimes k_1} \otimes \cdots \otimes g_n \otimes 1^{\otimes k_n}) \in \mathrm{End}_A(N - n + \sum a_j)^{i+\sum q_j}$$

We define $b_n(f; g_1, \dots, g_n) \in \mathrm{End}_A(N - n + \sum a_j)^{i+\sum q_j}$ as

$$b_n(f; g_1, \dots, g_n) = \sigma^{-1}(\sigma(f)[\sigma(g_1), \dots, \sigma(g_n)]).$$

With this the definition we can prove the following.

Lemma C.2. *We have*

$$b_n(f;g_1,\dots,g_n) = \sum_{N-n=k_0+\cdots+k_n} (-1)^{\eta} f(1^{\otimes k_0}\otimes g_1\otimes\cdots\otimes g_n\otimes 1^{\otimes k_n}),$$

where

$$\eta = \sum_{0\le j<l\le n} k_j q_l + \sum_{1\le j<l\le n} a_j q_l + \sum_{j=1}^{n}(a_j+q_j-1)(n-j) + \sum_{1\le j\le l\le n}(a_j+q_j-1)k_l.$$

Proof. Let us compute η using the definition of b_n.

$$\begin{aligned}
&\sigma^{-1}(\sigma(f)[\sigma(g_1),\dots,\sigma(g_n)])\\
&= (-1)^{\binom{N-n+\sum a_j}{2}} S^{-1}\circ\\
&(\sigma(f)(1^{\otimes k_0}\otimes\sigma(g_1)\otimes 1^{\otimes k_1}\otimes\cdots\otimes\sigma(g_n)\otimes 1^{\otimes k_n}))\circ S^{\otimes N-n+\sum a_j}\\
&= (-1)^{\binom{N-n+\sum a_j}{2}} S^{-1}\circ S\circ f\circ (S^{-1})^{\otimes N}\circ\\
&\left(1^{\otimes k_0}\bigotimes((S\circ g_i\circ (S^{-1})^{\otimes a_i})\otimes 1^{\otimes k_i})\right)\circ S^{\otimes N-n+\sum a_j}\\
&= (-1)^{\binom{N-n+\sum a_j}{2}} f\circ((S^{-1})^{k_0}\otimes S^{-1}\otimes\cdots\otimes S^{-1}\otimes (S^{-1})^{k_n})\circ\\
&\left(1^{\otimes k_0}\bigotimes((S\circ g_i\circ (S^{-1})^{\otimes a_i})\otimes 1^{\otimes k_i})\right)\circ S^{\otimes N-n+\sum a_j}
\end{aligned}$$

Now we move each $1^{\otimes k_{j-1}}\otimes S\circ g_j\circ (S^{-1})^{a_j}$ to apply $(S^{-1})^{k_{j-1}}\otimes S^{-1}$ to it. Doing this for all $j=1,\dots,n$ produces a sign

$$\begin{aligned}
&(-1)^{(a_1+q_1-1)(n-1+\sum k_l)+(a_2+q_2-1)(n-2+\sum_2^n k_l)+\cdots+(a_n+q_n-1)k_n}\\
&\qquad = (-1)^{\sum_{j=1}^n (a_j+q_j-1)(n-j+\sum_j^n k_l)},
\end{aligned}$$

and we denote the exponent by

$$\varepsilon = \sum_{j=1}^{n}(a_j + q_j - 1)\left(n - j + \sum_{j}^{n} k_l\right).$$

So now we have that, decomposing $S^{\otimes N-n+\sum a_j}$, the last map up to multiplication by $(-1)^{\binom{N-n+\sum a_j}{2}+\varepsilon}$ is

$$(-1)^{\binom{N-n+\sum a_j}{2}+\varepsilon} f \circ ((S^{-1})^{k_0} \otimes g_1 \circ (S^{-1})^{\otimes a_1} \otimes \cdots \otimes g_n \circ$$
$$(S^{-1})^{\otimes a_n} \otimes (S^{-1})^{k_n}) \circ (S^{\otimes k_0} \otimes S^{\otimes a_1} \otimes \cdots \otimes S^{\otimes a_n} \otimes S^{\otimes k_n}).$$

Now we turn the tensor of inverses into inverses of tensors by introducing the appropriate signs. More precisely, we introduce the sign

$$(-1)^{\delta} = (-1)^{\binom{k_0}{2}+\sum_j\left(\binom{a_j}{2}+\binom{k_j}{2}\right)}. \tag{C3}$$

Therefore we have up to multiplication by $(-1)^{\binom{N-n+\sum a_j}{2}+\varepsilon+\delta}$ the map

$$f \circ ((S^{k_0})^{-1} \otimes g_1 \circ (S^{\otimes a_1})^{-1} \otimes \cdots \otimes g_n \circ (S^{\otimes a_n})^{-1} \otimes (S^{k_n})^{-1}) \circ$$
$$(S^{\otimes k_0} \otimes S^{\otimes a_1} \otimes \cdots \otimes S^{\otimes a_n} \otimes S^{\otimes k_n}).$$

The next step is moving each component of the last tensor product in front of its inverse. This will produce the sign $(-1)^{\gamma}$, where

$$\gamma = -k_0 \sum_{1}^{n}(k_j + a_j + q_j) - a_1\left(\sum_{1}^{n} k_j + \sum_{2}^{n}(a_j + q_j)\right) - \cdots - a_n k_n \tag{C4}$$
$$= \sum_{j=0}^{n} k_j \sum_{l=j+1}^{n}(k_l + a_l + q_l) + \sum_{j=1}^{n} a_j \left(\sum_{l=j}^{n} k_l + \sum_{l=j+1}^{n}(a_l + q_l)\right) \mod 2.$$

So in the end we have

$$b_n(f; g_1, \ldots, g_n) = \sum (-1)^{\binom{N-n+\sum a_j}{2}+\varepsilon+\delta+\gamma} f(1^{\otimes k_0} \otimes g_1 \otimes \cdots \otimes g_n \otimes 1^{\otimes k_n}).$$

This means that

$$\eta = \binom{N - n + \sum a_j}{2} + \varepsilon + \delta + \gamma.$$

Next, we are going to simplify this sign to get rid of the binomial coefficients.

Remark C.3. If the top number of a binomial coefficient is less than 2, then the coefficient is 0. In the case of arities or k_j this is because $(S^{-1})^{\otimes 1} = (S^{\otimes 1})^{-1}$, and if the tensor is taken 0 times then it is the identity and the equality also holds, so there are no signs.

We are now going to simplify the sign to obtain the desired result. Notice that $N - n + \sum_j a_j = \sum_i k_i + \sum_j a_j$. In general, consider a finite sum $\sum_i b_i$. We can simplify the binomial coefficients mod 2

$$\binom{\sum_i b_i}{2} + \sum_i \binom{b_i}{2}$$

in the following way. Note that all the b_i's will appear squared once in the big binomial coefficient and once in the sum, as so will do the terms themselves, so they will cancel. This will leave the double products which cancel out the 2 in the denominator. More precisely, we have the following equality mod 2.

$$\binom{\sum b_i}{2} + \sum \binom{b_i}{2} = \sum_{i<j} b_i b_j \mod 2.$$

The result of applying this to $\binom{N-n+\sum a_j}{2}$ and adding δ from Equation (C3) in our sign η is

$$\sum_{0\leq i<l\leq n} k_i k_l + \sum_{1\leq j<l\leq n} a_j a_l + \sum_{i,j} k_i a_j. \tag{C5}$$

Recall γ in the sign from Equation (C4).

$$\gamma = \sum_{j=0}^{n} k_j \sum_{l=j+1}^{n} (k_l + a_l + q_l) + \sum_{j=1}^{n} a_j \left(\sum_{l=j}^{n} k_l + \sum_{l=j+1}^{n} (a_l + q_l) \right).$$

As we see, all the sums in the previous simplification appear in γ so we can cancel them. Let us rewrite γ in a way that this becomes more clear.

$$\begin{aligned}\gamma = &\sum_{0\leq j<l\leq n} k_j k_l + \sum_{0\leq j<l\leq n} k_j a_l + \sum_{0\leq j<l\leq n} k_j q_l + \sum_{1\leq j\leq l\leq n} a_j k_l \\ &+ \sum_{1\leq j<l\leq n} a_j a_l + \sum_{1\leq j<l\leq n} a_j q_l.\end{aligned}$$

So after adding the expression (C5) modulo 2 we have only the terms that include the internal degrees, i.e.

$$\sum_{0\leq j<l\leq n} k_j q_l + \sum_{1\leq j<l\leq n} a_j q_l. \tag{C6}$$

Let us move now to the ε term in the sign to rewrite it.

$$\begin{aligned}\varepsilon &= \sum_{j=1}^{n}(a_j+q_j-1)(n-j+\sum_{j}^{n}k_l)\\ &= \sum_{j=1}^{n}(a_j+q_j-1)(n-j)+\sum_{1\leq j\leq l\leq n}(a_j+q_j-1)k_l\end{aligned}$$

We may add this to what we had in (C6) in such a way that the brace sign becomes

$$\eta = \sum_{0\leq j<l\leq n}k_jq_l+\sum_{1\leq j<l\leq n}a_jq_l+\sum_{j=1}^{n}(a_j+q_j-1)(n-j)+\sum_{1\leq j\leq l\leq n}(a_j+q_j-1)k_l. \tag{C7}$$

as announced at the end of Section 3.2. $\square$

D Twisted complex on an operad

In this section we provide a description of the twisted complex structure on an operad $\mathcal{O}$ with a derived A_∞-multiplication. More precisely, we show by hand that the maps found in Corollary 4.5.6 define a twisted complex structure on $S\mathfrak{s}\mathcal{O}$.

Lemma D.1. *Let $\mathcal{O}$ be an operad and $m\in\mathfrak{s}\mathcal{O}$ a derived A_∞-multiplication. Then $S\mathfrak{s}\mathcal{O}$ becomes a twisted complex with structure maps*

$$M_{i1}(x)=\sum_{l}(Sb_1(m_{il};S^{-1}x)-(-1)^{\langle x,m_{il}\rangle}Sb_1(S^{-1}x;m_{il})),$$

where $x\in(S\mathfrak{s}\mathcal{O})_k^{n-k}$ and $\langle x,m_{il}\rangle = ik+(1-i)(n-1-k)$.

Proof. Throughout the proof we omit the shift maps. Let us first check the twisted complex equation up to signs, to give a conceptual proof before introducing the signs. Up to sign, the maps $\{M_{i1}\}_{i\geq 0}$ must satisfy the equation

$$\sum_{i+j=u} M_{i1} \circ M_{j1} = 0,$$

for all u, where $\circ$ is composition of maps.

Therefore, up to signs we have to compute

$$\begin{aligned}\sum_{i+j=u} M_{i1}(M_{j1}(x)) &= \sum_{i+j=u} M_{i1}\left(\sum_{l} b_1(m_{jl};x) + b_1(x;m_{jl})\right) \\ &= \sum_{i+j=u}\sum_{l,k} \left(b_1(m_{ik};b_1(m_{jl};x)) + b_1(m_{ik};b_1(x;m_{jl}))\right. \\ &\quad \left. + b_1(b_1(m_{jl};x);m_{ik}) + b_1(b_1(x;m_{jl});m_{ik})\right).\end{aligned}$$

Applying the brace relation we obtain

$$\begin{gathered}\sum_{i+j=u}\sum_{l,k}(b_1(m_{ik};b_1(m_{jl};x)) + b_1(m_{ik};b_1(x;m_{jl}))+ \\ b_2(m_{jl};x,m_{ik}) + b_1(m_{jl};b_1(x;m_{ik})) + b_2(m_{jl};m_{ik},x)+ \\ b_2(x;m_{jl},m_{ik}) + b_1(x;b_1(m_{jl};m_{ik})) + b_2(x;m_{ik},m_{jl})).\end{gathered}$$

In the sum, all terms of the form $b_1(x;b_1(m_{jl};m_{ik}))$ that can be seen in the last line should add up to vanish provided that m is a dA_∞-multiplication, meaning that up to sign $b_1(m;m)=0$. Since i and j are interchangeable, i.e. for each pair (i,j) there is the pair (j,i), the terms $b_2(x;m_{jl},m_{ik}) + b_2(x;m_{ik},m_{jl})$ in the last line should cancel as well. For this, we should have the pair (j,i) with the opposite sign. Here it is also relevant that the sum runs through all possible values of k and l, so that the pair (j,i) appears with l and k interchanged as well. So far the entire last line vanishes up to sign.

Then $b_1(m_{ik}; b_1(x; m_{jl}))$ on the first line should cancel with $b_1(m_{jl}; b_1(x; m_{ik}))$ on the second line, but from a different summand: the one where i and j are interchanged. Finally, the remaining terms $b_1(m_{ik}; b_1(m_{jl}; x)) + b_2(m_{jl}; x, m_{ik}) + b_2(m_{jl}; m_{ik}, x)$ add up to $b_1(b_1(m; m); x)$ up to sign. That would cancel everything.

Let us now introduce the signs. We now compute for all u the sum

$$\sum_{i+j=u} (-1)^i M_{i1} \circ M_{j1}.$$

For $x \in \mathfrak{s}\mathcal{O}$, by definition, we have

$$\begin{aligned}
&\sum_{i+j=u} (-1)^i M_{i1}(M_{j1}(x)) = \\
&\sum_{i+j=u} (-1)^i M_{i1} \left(\sum_l b_1(m_{jl}; x) - (-1)^{\langle x, m_{jl} \rangle} b_1(x; m_{jl}) \right) = \\
&\sum_{i+j=u} (-1)^i \sum_{l,k} \big(b_1(m_{ik}; b_1(m_{jl}; x)) - (-1)^{\langle x, m_{jl} \rangle} b_1(m_{ik}; b_1(x; m_{jl})) + \\
&\qquad -(-1)^{\langle b_1(m_{jl};x), m_{ik} \rangle} b_1(b_1(m_{jl}; x); m_{ik}) \\
&\qquad +(-1)^{\langle b_1(m_{jl};x), m_{ik} \rangle + \langle x, m_{jl} \rangle} b_1(b_1(x; m_{jl}); m_{ik}) \big).
\end{aligned}$$

Observe that $\langle b_1(m_{jl}; x), m_{ik} \rangle = \langle m_{ij}, m_{ik} \rangle + \langle x, m_{ik} \rangle$.

Applying the brace relation we obtain

$$
\begin{aligned}
\sum_{i+j=u}\sum_{l,k}((-1)^i b_1(m_{ik};b_1(m_{jl};x)) - (-1)^{i+\langle x,m_{jl}\rangle} b_1(m_{ik};b_1(x;m_{jl}))+ \\
-(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle}(b_2(m_{jl};x,m_{ik}) + (-1)^{\langle x,m_{ik}\rangle} b_2(m_{jl};m_{ik},x)) \\
-(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle} b_1(m_{jl};b_1(x;m_{ik})) \\
+(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle+\langle x,m_{jl}\rangle}(b_2(x;m_{jl},m_{ik}) + (-1)^{\langle m_{ik},m_{jl}\rangle} b_2(x;m_{ik},m_{jl})) \\
+(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle+\langle x,m_{jl}\rangle} b_1(x;b_1(m_{jl};m_{ik}))).
\end{aligned}
\tag{D8}
$$

Recall from Equation (4.6) that m being a dA_∞-multiplication means that

$$\sum_{i+j=u}\sum_{k,l}(-1)^i b_1(m_{jl};m_{ik}) = 0.$$

Let us check now the cancellations with the signs. First, let us check that the terms

$$(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle+\langle x,m_{jl}\rangle} b_1(x;b_1(m_{jl};m_{ik})))$$

can be added up to vanish. For that, we compute the sign

$$\langle b_1(m_{jl};x),m_{ik}\rangle + \langle x,m_{jl}\rangle = \langle m_{jl},m_{ik}\rangle + \langle x,m_{ik}\rangle + \langle x,m_{jl}\rangle.$$

Recall that the braces are defined on the operadic suspension, so that the bidegree of m_{ik} is $(i,1-i)$. Therefore, writing the bidegree of x as $(k,n-k)$, so that the total degree is $|x|=n$, the above equals

$$
\begin{aligned}
&ji + (1-i)(1-j) + ki + (n-k)(1-i) + kj + (n-k)(1-j) \\
&= 1 + i + j + (i+j)k + (i+j)(n-k) \mod 2 \\
&= 1 + (i+j)(1+n) = 1 + u(1+|x|).
\end{aligned}
$$

Since this sign is constant for all terms $b_1(m_{ik}; m_{ij})$ that share the same horizontal degree $i + j = u$, we can rewrite

$$
(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle+\langle x,m_{jl}\rangle} b_1(x; b_1(m_{jl}; m_{ik})))
$$

as

$$
-(-1)^{u(1+|x|)} b_1(x; (-1)^i b_1(m_{ik}; m_{jl})).
$$

Hence,

$$
\sum_{i+j=u} \sum_{k,l} -(-1)^{u(1+|x|)} b_1(x; (-1)^i b_1(m_{ik}; m_{jl})) = 0.
$$

Therefore, after applying the brace relation, expression (D8) reduces to

$$
\begin{aligned}
\sum_{i+j=u} \sum_{l,k} &((-1)^i b_1(m_{ik}; b_1(m_{jl}; x)) - (-1)^{i+\langle x,m_{jl}\rangle} b_1(m_{ik}; b_1(x; m_{jl}))+ \\
&-(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle} (b_2(m_{jl}; x, m_{ik}) + (-1)^{\langle x,m_{ik}\rangle} b_2(m_{jl}; m_{ik}, x)) \\
&-(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle} b_1(m_{jl}; b_1(x; m_{ik})) \\
&+(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle+\langle x,m_{jl}\rangle} (b_2(x; m_{jl}, m_{ik}) + (-1)^{\langle m_{ik},m_{jl}\rangle} b_2(x; m_{ik}, m_{jl})).
\end{aligned} \tag{D9}
$$

Let us focus on the last line. For each pair (i, j) we should have cancellation with the pair (j, i), which adds the same elements, but with different signs. We also need to consider the pairs (k,l) and (l,k) to get a cancellation. Let us compare the signs. For the pair $((i,j),(k,l))$ we have precisely the last line of the above equation

$$(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle+\langle x,m_{jl}\rangle}(b_2(x;m_{jl},m_{ik})+(-1)^{\langle m_{ik},m_{jl}\rangle}b_2(x;m_{ik},m_{jl}))$$

For the pair $((j,i),(l,k))$ we have

$$(-1)^{j+\langle b_1(m_{ik};x),m_{jl}\rangle+\langle x,m_{ik}\rangle}(b_2(x;m_{ik},m_{jl})+(-1)^{\langle m_{jl},m_{ik}\rangle}b_2(x;m_{jl},m_{ik})).$$

Comparing the sign of $b_2(x;m_{jl},m_{ik})$ we find that for $((i,j),(k,l))$ we have

$$-(-1)^{i+(i+j)(1+|x|)}b_2(x;m_{jl},m_{ik})=-(-1)^{j+u|x|}b_2(x;m_{jl},m_{ik})$$

and for the pair $((j,i),(l,k))$ we have

$$(-1)^{j+u|x|}b_2(x;m_{jl},m_{ik}).$$

As we see, we get opposite signs and thus cancellation. For $b_2(x;m_{ik},m_{jl})$ it is completely analogous. Thus, we have reduced expression (D9) to

$$\begin{aligned}\sum_{i+j=u}\sum_{l,k}((-1)^{i}b_1(m_{ik};b_1(m_{jl};x))-(-1)^{i+\langle x,m_{jl}\rangle}b_1(m_{ik};b_1(x;m_{jl}))+\\-(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle}(b_2(m_{jl};x,m_{ik})+(-1)^{\langle x,m_{ik}\rangle}b_2(m_{jl};m_{ik},x))\\-(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle}b_1(m_{jl};b_1(x;m_{ik})).\end{aligned}\tag{D10}$$

In a similar fashion to the previous calculation, we are going to cancel $b_1(m_{ik}; b_1(x; m_{jl}))$ in the first line with $b_1(m_{jl}; b_1(x; m_{ik}))$ in the last line by considering switched pairs. For the pair $((i,j),(k,l))$, the term in the first line is

$$-(-1)^{i+\langle x,m_{jl}\rangle} b_1(m_{ik}; b_1(x; m_{jl}))$$

and for the pair $((j,i),(l,k))$ the term in the last line is

$$\begin{aligned} -(-1)^{j+\langle b_1(m_{ik};x),m_{jl}\rangle} b_1(m_{ik}; b_1(x; m_{jl})) &= \\ (-1)^{1+j+\langle m_{ik},m_{jl}\rangle+\langle x,m_{jl}\rangle} b_1(m_{ik}; b_1(x; m_{jl})) &= \\ (-1)^{i+\langle x,m_{jl}\rangle} b_1(m_{ik}; b_1(x; m_{jl})), \end{aligned}$$

which has precisely the opposite sign to the other pair, and thus cancels. This reduces expression (D10) to

$$\sum_{i+j=u} \sum_{l,k} ((-1)^i b_1(m_{ik}; b_1(m_{jl}; x)) \\ -(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle} (b_2(m_{jl}; x, m_{ik}) + (-1)^{i+\langle m_{jl},m_{ik}\rangle} b_2(m_{jl}; m_{ik}, x)). \tag{D11}$$

We want these terms to add up to something of the form $b_1(b_1(m;m);x)$. Notice that for this we need to switch some pairs. For simplicity, we switch the pair of the first term and rewrite the sum as

$$\sum_{i+j=u} \sum_{l,k} ((-1)^j b_1(m_{jl}; b_1(m_{ik}; x)) \\ -(-1)^{i+\langle b_1(m_{jl};x),m_{ik}\rangle} b_2(m_{jl}; x, m_{ik}) + (-1)^{i+\langle m_{jl},m_{ik}\rangle} b_2(m_{jl}; m_{ik}, x)).$$

Simplifying the signs we get

$$\sum_{i+j=u}\sum_{l,k}((-1)^j b_1(m_{jl}; b_1(m_{ik}; x)) + (-1)^{j+\langle x,m_{ik}\rangle} b_2(m_{jl}; x, m_{ik})$$
$$+(-1)^j b_2(m_{jl}; m_{ik}, x)).$$

By the brace relation and Equation (4.6) this equals

$$\sum_{i+j=u}\sum_{l,k}(-1)^j b_1(b_1(m_{jl}; m_{ik}); x) = 0.$$

□

The reader can see that the twisted complex structure given by the above Lemma is the same as the one given by Corollary 4.5.6.

Bibliography

[Ada78] John Frank Adams. *Infinite loop spaces.* Annals of Mathematics Studies, No. 90. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[ARLR+15] Camil I. Aponte Román, Muriel Livernet, Marcy Robertson, Sarah Whitehouse, and Stephanie Ziegenhagen. Representations of derived A-infinity algebras. In *Women in topology: collaborations in homotopy theory*, volume 641 of *Contemp. Math.*, pages 1–27. Amer. Math. Soc., Providence, RI, 2015.

[Bor94] Francis Borceux. *Handbook of Categorical Algebra*, volume 2 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, 1994.

[CESLW18] Joana Cirici, Daniela Egas Santander, Muriel Livernet, and Sarah Whitehouse. Derived A-infinity algebras and their homotopies. *Topology Appl.*, 235:214–268, 2018.

[Fre17] Benoit Fresse. *Homotopy of operads and Grothendieck-Teichmüller groups. Part 1*, volume 217 of *Mathematical*

Surveys and Monographs. American Mathematical Society, Providence, RI, 2017. The algebraic theory and its topological background.

[Get93] Ezra Getzler. Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology. In *In Quantum Deformations of Algebras and Their Representations, Israel Math. Conf. Proc. 7*, pages 65–78, 1993.

[GV95] Murray Gerstenhaber and Alexander A. Voronov. Homotopy G -algebras and moduli space operad. *International Mathematics Research Notices*, 1995(3):141–153, January 1995.

[Kad80] T. V. Kadeišvili. On the theory of homology of fiber spaces. *Uspekhi Mat. Nauk*, 35(3(213)):183–188, 1980. International Topology Conference (Moscow State Univ., Moscow, 1979).

[Kel01] Bernhard Keller. Introdution to A_∞ algebras and modules. *Homology Homotopy Appl.*, 3(1):35, 2001.

[Kon99] Maxim Kontsevich. Operads and motives in deformation quantization. volume 48, pages 35–72. 1999. Moshé Flato (1937–1998).

[KWZ15] Ralph M. Kaufmann, Benjamin C. Ward, and J. Javier Zúñiga. The odd origin of Gerstenhaber brackets, Batalin-Vilkovisky operators, and master equations. *Journal of*

Mathematical Physics, 56(10):103504, October 2015. Publisher: American Institute of Physics.

[LM05] Tom Lada and Martin Markl. Symmetric Brace Algebras. *Appl Categor Struct*, 13(4):351–370, August 2005.

[LRW13] Muriel Livernet, Constanze Roitzheim, and Sarah Whitehouse. Derived A_∞–algebras in an operadic context. *Algebraic & Geometric Topology*, 13(1):409 – 440, 2013.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic Operads*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin Heidelberg, 2012.

[Mac71] Saunders MacLane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.

[McC01] John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[MM21] Jeroen Maes and Fernando Muro. Derived homotopy algebras. *arXiv:math/2106.14987*, June 2021.

[MSS07] Martin Markl, Steve Shnider, and Jim Stasheff. *Operads in Algebra, Topology and Physics*, volume 96 of *Mathematical*

Surveys and Monographs. American Mathematical Society, Providence, Rhode Island, July 2007.

[Pen01] Michael Penkava. Infinity Algebras, Cohomology and Cyclic Cohomology, and Infinitesimal Deformations. *arXiv:math/0111088*, November 2001.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs.* Cambridge University Press, Cambridge, 2014.

[RW11] Constanze Roitzheim and Sarah Whitehouse. Uniqueness of A_∞-structures and Hochschild cohomology. *Algebr. Geom. Topol.*, 11(1):107–143, January 2011.

[Sag10] Steffen Sagave. DG-algebras and derived A_∞-algebras. *J. Reine Angew. Math.*, 639:73–105, 2010.

[Sta61] James Dillon Stasheff. *HOMOTOPY ASSOCIATIVITY OF H-SPACES.* ProQuest LLC, Ann Arbor, MI, 1961. Thesis (Ph.D.)–Princeton University.

[Sta63] James Dillon Stasheff. Homotopy associativity of H-spaces. II. *Trans. Amer. Math. Soc. 108*, pages 275–292, 1963.

[War13] Benjamin C. Ward. *Cohomology of Operad Algebras and Deligne's Conjecture.* PhD thesis, 2013.

Glossary

R	A ring of non-zero characteristic
$\mathrm{Hom}_R(A,B)$	The (bi)graded module of linear maps
$\deg(x)$	Degree of an element x in a graded module
$\mathbb{Z}$	The group of integers
S	Shift map for single-graded modules,
	vertical shift for bigraded modules
$\mathcal{O}$	A linear (bi)graded operad
$\otimes$	Tensor product of (bi)graded R-modules,
	also Hadamard product of operads
End_A	Endomorphism operad
End_B^A	Collection $\{\mathrm{Hom}_R(A^{\otimes n}, B)\}_{n\geq 1}$
$\mathcal{A}_\infty$	Operad of A_∞-algebras
$d\mathcal{A}_\infty$	Operad of derived A_∞-algebras
$m = m_1 + m_2 + \cdots$	A_∞-structure maps on an R-module,
	or A_∞-multiplication in an operad
$M = M_1 + M_2 + \cdots$	A_∞-structure maps on $S\mathfrak{s}\mathcal{O}$
$\overline{M} = \overline{M}_1 + \overline{M}_2 + \cdots$	A_∞-structure maps on $S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$

$m = \sum_{ij} m_{ij}$	Derived A_∞-structure maps,
	or derived A_∞-multiplication in $\mathcal{O}$
$M = \sum_{ij} M_{ij}$	derived A_∞-structure maps on $S\mathfrak{s}\mathcal{O}$
$\overline{M} = \sum_{ij} \overline{M}_{ij}$	derived A_∞-structure maps on $S\mathfrak{s}\operatorname{End}_{S\mathfrak{s}\mathcal{O}}$
γ	Operadic composition
	sometimes used as an exponent in signs
$\circ_i$	Operadic insertion
$\circ$	Plethysm of operads,
	also circle operation
	and composition of maps
Λ	Operad structure on the shifts of R
$\mathfrak{s}\mathcal{O} = \mathcal{O} \otimes \Lambda$	Operadic suspension of $\mathcal{O}$
	also its underlying (bi)graded module
$\|x\|$	Natural degree of x in $\mathfrak{s}\mathcal{O}$ if single-graded
	also total degree of x in a bigraded module
$\operatorname{vdeg}(x)$	Vertical degree of a bigraded element x
$\langle , \rangle$	Dot product of bidegrees
$\tilde{\gamma}$	Operadic composition on $\mathfrak{s}\mathcal{O}$
$\tilde{\circ}_i$	Operadic insertion on $\mathfrak{s}\mathcal{O}$
$\tilde{\circ}$	Circle operation on $\mathfrak{s}\mathcal{O}$
$[,]$	Lie bracket
	also internal hom
b_n	Brace map on $\mathcal{O}$ or $\mathfrak{s}\mathcal{O}$
B_n	Brace map on $\operatorname{End}_{\mathfrak{s}\mathcal{O}}$

$\overline{B}_n$	Brace map on $\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$
$\overline{\sigma}$	Isomorphism $\mathrm{End}_A \cong \mathfrak{s}\,\mathrm{End}_{SA}$
$\Phi : S\mathfrak{s}\mathcal{O} \to S\mathfrak{s}\,\mathrm{End}_{S\mathfrak{s}\mathcal{O}}$	Morphism of (derived) A_∞-algebras
$\mathcal{C} = (\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$	A (closed) (symmetric) monoidal category
$\mathrm{Hom}_{\mathcal{C}}(A, B)$	Set of morphism from A to B
$\mathcal{C}^b$	Category horizontally bounded on the right
C_R	Category of cochain complexes
fC_R	Category of filtered complexes
$\underline{\mathrm{Hom}}(A, B)$	Filtered hom complex
bgMod_R	Category of bigraded modules
vbC_R	Category of vertical bicomplexes
tC_R	Twisted complexes
$\underline{\mathscr{C}}$	Enriched category
$\underline{\otimes}$	Enriched tensor product
$\underline{\mathcal{E}nd}_A$	Enriched endomorphism operad
$\underline{bgMod}_R$	bgMod_R-enriched category of bigraded modules
$\underline{t\mathcal{C}_R}$	vbC_R-enriched category of twisted complexes
$\underline{fMod}_R$	bgMod_R-enriched category of filtered modules
$\underline{f\mathcal{C}_R}$	vbC_R-enriched category of filtered complexes
Tot	Totalization functor

$\mathfrak{Tot}$	Enriched totalization functor
$\bar{\circ}_i$	Operadic insertion on $\mathrm{Tot}(\mathcal{O})$
$\star_i$	Operadic insertion on $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$
$\bar{\gamma}$	Operadic composition on $\mathrm{Tot}(\mathcal{O})$
$\gamma^\star$	Operadic composition on $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$
$b_n^\star$	Brace map on $\mathrm{Tot}(\mathfrak{s}\mathcal{O})$
$\mu = \mu_{A,B}$	The map $\mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \to \mathrm{Tot}(A \otimes B)$