BUSN41916 Presentation: Ranking Poker Players with Bayesian Plackett-Luce Ranking Models

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Contents

Introduction

- Game ranking data: 24 players play 134 poker games. For each game, there is a ranking for the participants;
- Plackett-Luce model for multiple players, generalization of Bradley-Terry model for two players, likelihood, and choice of prior;
- Apply MCMC for Bayesian inference on this dataset;
- In particular, we want to understand whether the performance of the players depends on two covariates: seniority (age) and skill.

Introduction Ctn.

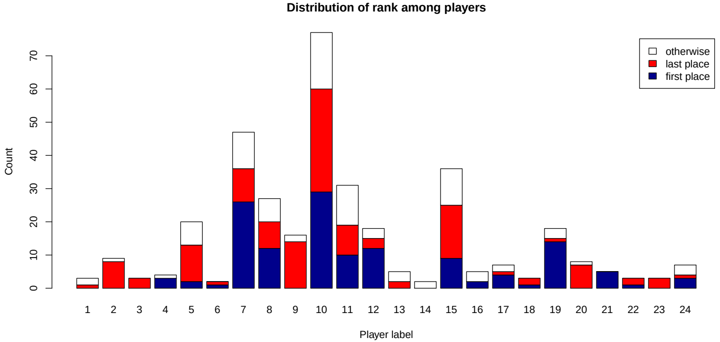
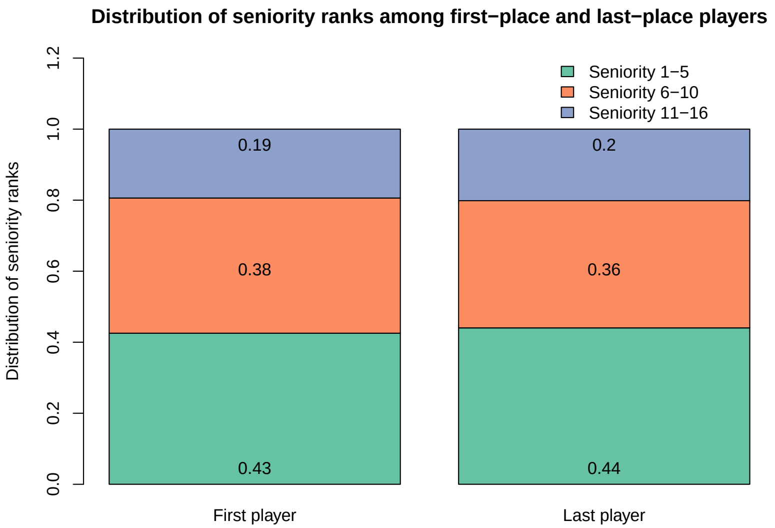


Figure: The distribution of ranking of the 24 players.

- Player 10 takes part in the largest number of games (77);
- Player 6 and player 14 only participate in 2 games;
- Player 19 has frequency 77% of first place (good player?);
- Player 9 has frequency 87% of last place (bad player?).

Introduction Ctn.



Model Description: Plackett-Luce observation model

Model 1 (without seniority covariates)

- skills: $\lambda_i \in \mathbb{R}$ for player $i \in 1, \ldots, n$
- m participated players: $\{i_1, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$ for some $1 \leq m \leq n$
- random rank outcome O: random vector $O = (O_1, \ldots, O_m)$ takes values in $\mathcal{P}_{i_1,\ldots,i_m}$, the permutation set of $\{i_1, \ldots, i_m\}$
- likelihood: for any possible outcome $o = (o_1, \ldots, o_m) \in \mathcal{P}_{i_1,\ldots,i_m}$,

$$\Pr\{O = o \mid \lambda\} = \prod_{i=1}^{m} \frac{\exp\left(\lambda_{o_i}\right)}{\sum_{j=i}^{m} \exp\left(\lambda_{o_j}\right)}$$

Model Description: Plackett-Luce observation model

Model 2 (with seniority covariates)

- seniority: $\beta_i \in \mathbb{R}$ for player $i \in 1, \ldots, n$
- skills: $\lambda_i \in \mathbb{R}$ for player $i \in 1, \ldots, n$
- m participated players: $\{i_1, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$ for some $1 \leq m \leq n$
- random rank outcome O: random vector $O = (O_1, \ldots, O_m)$ takes values in $\mathcal{P}_{i_1,\ldots,i_m}$, the permutation set of $\{i_1, \ldots, i_m\}$
- likelihood: for any possible outcome $o = (o_1, \ldots, o_m) \in \mathcal{P}_{i_1,\ldots,i_m}$,

$$\Pr\{O = o \mid \lambda\} = \prod_{i=1}^{m} \frac{\exp\left(\lambda_{o_i} + \beta_{e_i}\right)}{\sum_{j=i}^{m} \exp\left(\lambda_{o_j} + \beta_{e_j}\right)}$$

e_i is the seniority rank of player o_i

Specifying Priors on λ and β

We want the prior to have good properties

- non-informative with respect to the relative skills of the players and the advantage for seniority
- all permutations $o \in \mathcal{P}_{i_1,\dots,i_m}$ are equally probable
- the priors of λ_i, β_i for each player i defined on $\mathbb{R}$

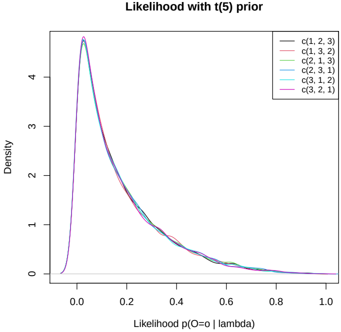
choose $\lambda_i, \beta_i \overset{\text{i.i.d}}{\sim}$ certain t distribution

Checking the Priors

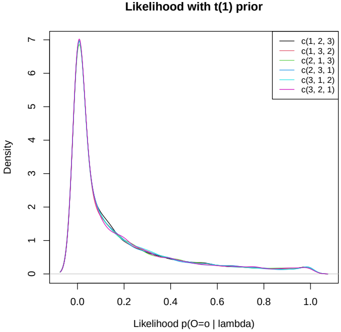
Simulation results suggest choosing the prior t_5.
Setup:

- focus on players 1, 2, 3
- simulate 10,000 samples from the priors $\lambda_{o_i} \overset{\text{i.i.d}}{\sim} t_5$
- for each random permutation $o \in \mathcal{P}_{1,2,3}$, plot the density of $\Pr\{O = o \mid \lambda\}$

Checking the Priors



(a) t_5



(b) t_1

Figure: Comparison of t_5 prior and t_1 prior in model 1.

- all density curves coincide $\implies$ the prior is non-informative & all permutations are equally probable
- t_5 less concentrated around zero than t_1 $\implies$ t_5 induces more randomness in the game, and hence more preferred

Posterior Inference with MCMC

Algorithm: sequential scan Metropolis-Hastings – we visit each component of the parameters in turn, and for each component j in iteration t, we propose

$$X_j \sim q_j(\cdot \mid X_1^{(t)}, \ldots, X_{j-1}^{(t)}, X_j^{(t-1)}, \ldots, X_d^{(t-1)})$$

- **Symmetric proposal q_j:** The proposal density cancels when evaluating the acceptance probability, and we are only left with the likelihood ratio (random Walk Metropolis-Hastings).
- For each run, we simulate a Markov chain of length $100,000$ and select a sub-sample for every 100 steps.

MCMC Output Diagnostics – Trace Plots

- **Trace Plots:** stationary from the beginning of the simulation, suggesting that no burn-in period is required and we can keep the entire chain.

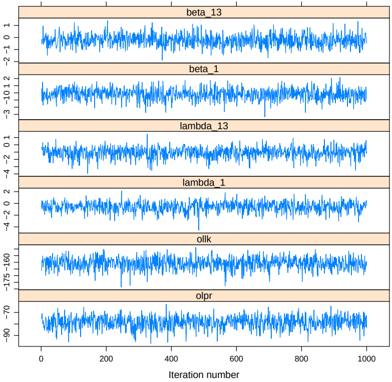


Figure: Trace plots of $\lambda^{(t)}$, $\beta^{(t)}$, the log-likelihood, and the log-prior

MCMC Output Diagnostics – Autocorrelations of Traces

- **Autocorrelations of Traces:** fall off to zero quickly, so the effective sample size is close to the number of sub-samples 1000.

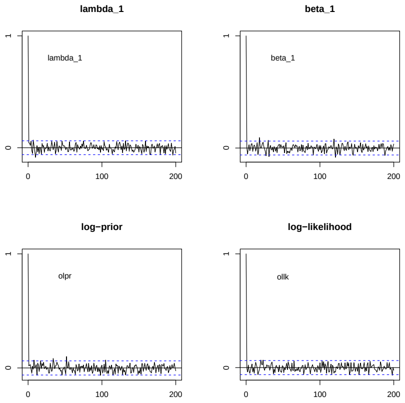


Figure: ACFs of the trace plots fall off to zero quickly

MCMC Output Diagnostics – Marginal Distributions

- **Marginal Distributions:** perform multiple runs from different start states and check those marginal agree

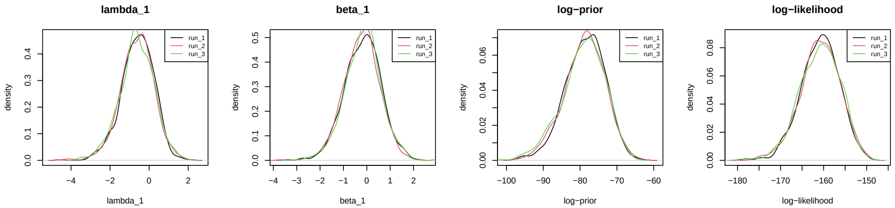


Figure: Density plots of the variables and key functions in three runs with different start states

From the diagnostic plots:

- Good convergence of the chain to the target distribution
- Good explorations of the parameter space of the target posterior

Player Skills

From the posterior mean, we can conclude that players $21, 19, 4$ have the strongest skills, whereas players $20, 2, 9$ have the weakest.

Param.	Post. Mean	95% HPD interval	ESS
λ_1	-0.67	(-2.18, 0.85)	1132
λ_2	-2.31	(-4.09, -0.44)	884
λ_3	-1.35	(-3.68, 0.52)	1000
λ_4	1.35	(-0.11, 2.97)	1000
λ_5	-0.45	(-1.41, 0.42)	1000
λ_6	-0.11	(-1.75, 1.50)	1000
λ_7	0.76	(0.07, 1.51)	1000
λ_8	0.30	(-0.54, 1.13)	1230
λ_9	-2.25	(-3.58, -1.01)	1000
λ_{10}	0.39	(-0.19, 1.05)	1000
λ_{11}	0.20	(-0.51, 0.94)	1000
λ_{12}	0.90	(-0.26, 1.81)	1000
λ_{13}	-0.82	(-2.08, 0.45)	960

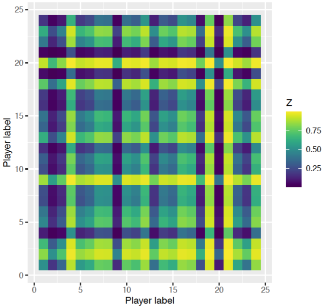
Player Skills

Param.	Post. Mean	95% HPD interval	ESS
λ_{14}	0.09	(-1.15, 1.44)	1000
λ_{15}	-0.26	(-1.07, 0.40)	1000
λ_{16}	1.17	(-0.23, 2.56)	1000
λ_{17}	0.91	(-0.28, 2.04)	1077
λ_{18}	-1.44	(-3.41, 0.05)	798
λ_{19}	2.03	(0.79, 3.36)	1000
λ_{20}	-2.36	(-4.57, -0.57)	825
λ_{21}	2.91	(0.23, 6.02)	879
λ_{22}	-0.19	(-1.84, 1.44)	1000
λ_{23}	-0.97	(-3.05, 1.26)	1000
λ_{24}	0.81	(-0.42, 2.12)	1000
olpr	-44.39	(-52.62, -36.49)	1000
ollk	-161.1	(-168.27, -153.29)	1000

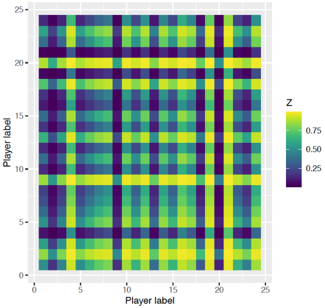
Table: Summary of the posterior mean, HPD intervals and the ESS

Perform a comparison of the relative skills of each pair $\{i, j\}$ of players by considering games involving only the two players

$$\Pr\{i \text{ win}\} = \frac{\exp\left(\hat{\lambda}_{o_i}\right)}{\exp\left(\hat{\lambda}_{o_i}\right) + \exp\left(\hat{\lambda}_{o_j}\right)}$$



(a) Model 1 pairwise skill comparisons



(b) Model 2 pairwise skill comparisons

Model selection

We use the Bayes factor $B_{1,2}$ to perform a model comparison between Model 1 (without seniority) and Model 2 (with seniority)

$$B_{1,2} = \frac{p\,(y \mid m=1)}{p\,(y \mid m=2)} = \frac{\int p\,(y \mid \theta, m=1)\,\pi_{m=1}(\theta) d\theta}{\int p(y \mid \theta, m=2)\pi_{m=2}(\theta) d\theta}$$

- Compute the marginal $p(y \mid m)$ using the Bridge estimator

$$\widehat{p} = \frac{\sum_t \pi\left(\theta^{(1,t)}\right) p\left(y \mid \theta^{(1,t)}\right) h\left(\theta^{(1,t)}\right)}{\sum_t \pi\left(\theta^{(2,t)}\right) h\left(\theta^{(2,t)}\right)}$$

- Choose $h(\theta) = \pi(\theta)^{-1} p(y \mid \theta)^{-1/2}$ that minimises the MSE

$$\widehat{p} = \frac{\sum_t p\left(y \mid \theta^{(1,t)}\right)^{1/2}}{\sum_t p\left(y \mid \theta^{(2,t)}\right)^{-1/2}}$$

- $\theta^{(1,t)} \sim \pi(\theta)$ are random samples simulated from the prior
- $\theta^{(2,t)} \sim \pi(\theta \mid y)$ are simulated from the posterior

Model selection

Conclusion: The estimated Bayes factor $\widehat{B}_{1,2}$ has the order of magnitude of 10^5 and does not vary upon multiple computations, providing substantial evidence in favour of the model without seniority covariates than the model with.

Interesting Counter Example

Consider a simple example where game 68 involving nine players $s = \{4, 7, 11, 5, 10, 14, 15, 13, 9\}$ was replayed (i.e. with the same players and same seniority values), we are interested in the probability that player 7 wins. We estimate this by summing the probabilities of all 8! permutations where player 7 comes at the top

$$\sum_{o_1=7} E_{\lambda,\beta|model}\left[\Pr\{O = o \mid \lambda, \beta\}\right] \quad \text{where} \quad O \in \mathcal{P}_s$$

- The marginals are estimated using the Bridge estimator
- In model one without seniority covariates, the probability is 17.4%
- With model two, when the seniority covariates are added, the probability that player 7 wins drops to 4.1%
- Seniority has **non-trivial** effects on the outcomes

Conclusions

- Our analysis suggests the given data provide more evidence in favour of the simpler observation model (model 1) without seniority rank covariate.
- Potential Limitations:
  - Not all players directly play with each other.
  - Relatively lower number of observations for several players.
- Potential Extensions:
  - A wider class of prior models.
  - Better computational methodology.

Thank You!