Presentation-1

Theory of Open Quantum Systems and Summary of the Paper

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Theory of Open Quantum Systems

Density Matrix

- It is hermitian, positive semi-definite; its trace is one.
- A density matrix is used for describing physical systems whose information is not completely known (such as the statistical mixtures) or for those systems which are entangled with another system.
- The time-evolution of the density matrix is given by the von Neumann equation and is given by $i\hbar\frac{\partial\rho}{\partial t} = [H, \rho]$.
- For a general Hamiltonian, it is given by - $\rho(t) = U(t)\rho(0)U(t)^{\dagger}$ where $U(t)$ is a wavefunction propagator over some interval, and it is a unitary operator.
- In the geometric sense, a set of density operators is a convex set, and the pure states are the extremal points of the set. The mixed states are in the interior of the Bloch sphere.

Positive and Completely Positive Maps

- A quantum channel is a type of map which takes states to states. A positive map takes positive elements to positive elements, and it is a monotone map.
- A linear map is Hermicity-preserving iff $\phi(A^{\dagger}) = [\phi(A)]^{\dagger}$, positive iff $\phi(M_n^+(C)) \subset M_n^+(C)$, trace-preserving iff $Tr\phi(A) = TrA$, unital iff $\phi(I_n) = I_n$.
- If two maps are positive, their combination need not be positive, unlike completely positive maps.
- CP maps are the most general representation of quantum evolutions, however they do not exhaust all possibilities.
- For instance, P-maps fail in the case of entangled systems. All CP-maps are a subset of P-maps.

Positive and Completely Positive Maps

- Let $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ be a two-qubit entangled system, then its density matrix is given by ρ-

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

When we take partial transpose over qubit-1, we get the following density matrix ρ^{T_2} -

$$\rho^{T_2} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly, not all of its eigenvalues are positive, hence, P-maps fail at this point.

Positive and Completely Positive Maps

- In general, when we extend P-map to a higher dimension, i.e from $\phi : H(A) \rightarrow H(B)$ to $Id \otimes \phi : C^{k\times k} \otimes H(A) \rightarrow C^{k\times k} \otimes H(B)$, therefore, ϕ is k-positive if $Id \otimes \phi$ else ϕ is CP if ϕ is k-positive for all k. If $dim(H) = n$, then ϕ is CP if and only if ϕ is n-positive.
- k-positivity means l-positivity if $l < k$.
- A CP-map can be representad by Kraus Operators.

Born-Markov Master Equation

In Born-Markov approximation, we consider there's a weak coupling between the system and the environment and the correlation between them is for a very short interval and this is called short memory effect.
Born-Markov Master equation-

$$i\hbar\frac{d\rho_S(t)}{dt} = -\alpha^2\frac{i}{\hbar}tr_B[H(t), \int_0^\infty [H(t^{'}), \rho(t^{'})]dt^{'}]$$

Markovian reservoir engineering

In this example, they simulate a semigroup Markovian master equation which has the Bell state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ as its stationary state.
The 4 Bell states can be identified as the eigenstates of the operators $\sigma_x^{(1)} \otimes \sigma_x^{(2)}$ and $\sigma_z^{(1)} \otimes \sigma_z^{(2)}$ as shown below:

Bell state	Eigenvalue wrt $\sigma_x^{(1)} \otimes \sigma_x^{(2)}$	Eigenvalue wrt $\sigma_z^{(1)} \otimes \sigma_z^{(2)}$
$\|\phi^+\rangle$	$+1$	$+1$
$\|\phi^-\rangle$	-1	$+1$
$\|\psi^+\rangle$	$+1$	-1
$\|\psi^-\rangle$	-1	-1

Markovian reservoir engineering

Now, they design two channels which they name the XX pump and the ZZ pump. The action of each is to pump states from the $+1$ eigenspace to the -1 eigenspace of the corresponding operator. From the table above, we can see that the only state belonging to the -1 eigenspace of both $\sigma_x^{(1)} \otimes \sigma_x^{(2)}$ and $\sigma_z^{(1)} \otimes \sigma_z^{(2)}$ is $|\psi^-\rangle$. So if we compose the two channels and apply them together, we should be able to pump everything to a single state, which is $|\psi^-\rangle$. Also, applying any one channel by itself will obviously pump states to a mixture of the states in the corresponding -1 eigenspace (which is $\{|\phi^-\rangle, |\psi^-\rangle\}$ for the XX pump and $\{|\psi^+\rangle, |\psi^-\rangle\}$ for the ZZ pump).

Markovian reservoir engineering

The form of the XX pump and ZZ pump are given as

$$\Phi_{xx}\rho_s = E_{1x}\rho_s E_{1x}^\dagger + E_{2x}\rho_s E_{2x}^\dagger$$
$$\Phi_{zz}\rho_s = E_{1z}\rho_s E_{1z}^\dagger + E_{2z}\rho_s E_{2z}^\dagger$$

with

$$E_{1z} = \sqrt{p}\mathbb{I}^{(1)} \otimes \sigma_x^{(2)} \frac{1}{2}(\mathbb{I} + \sigma_z^{(1)} \otimes \sigma_z^{(2)})$$
$$E_{1z} = \frac{1}{2}(\mathbb{I} - \sigma_z^{(1)} \otimes \sigma_z^{(2)}) + \frac{\sqrt{1-p}}{2}(\mathbb{I} + \sigma_z^{(1)} \otimes \sigma_z^{(2)})$$

and E_{1x} and E_{2x} are the same with the replacements $\sigma_z^{(2)} \to \sigma_x^{(2)}$ and $\sigma_z^{(1)} \otimes \sigma_z^{(2)} \to \sigma_x^{(1)} \otimes \sigma_x^{(2)}$.