Report-1: Introduction to Quantum Error Correction

Mainak Roy and Jessica John Britto
Indian Institute of Technology Kharagpur

November 4th, 2022

Concepts covered - *Basics of Quantum Computing and Stabilizer Codes*

1 Introduction

In the *little endian* convention, the qubits are labelled from right-to-left where the rightmost qubit is the zeroth qubit. The opposite of this convention is *big endian* and this convention is followed generally. A single qubit can be written in this parametrized form - $\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ where $\theta\epsilon[0,\pi]$ and $\phi\epsilon[0,2\pi]$. Clearly, four complex terms are needed to represent qubit however, taking out a global phase from the qubit's representation equation, we have two imaginary coefficients and one real coefficient. Hence, a single qubit can be presented on a bloch sphere. For multi-qubit systems, bloch sphere cannot be used for their representation since there will be many parameters in the general equation of the multi-qubit systems.

Tensor/Kronecker product is valid for both the *kets* and *bras*. Tensor product is obtained by multiplying each term of the first matrix/vector by the entire second matrix/vector. For instance, in case of two qubits, its done the following way for *kets*-

$$|00\rangle = |0\rangle|0\rangle = |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

And with the bras, it is done the following way-

$$\langle 00| = \langle 0| \otimes \langle 0| = \begin{pmatrix}1 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1 & 0\end{pmatrix} & 0\begin{pmatrix}1 & 0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}$$

The inner product of $|01\rangle$ and $\langle 00|$ is done by matching up the qubits, for example,

$$\langle 00|01\rangle = \langle 0|0\rangle \cdot \langle 0|1\rangle = 1 \cdot 0 = 0$$

Single qubit gates are applied to a single qubit only and they are the hadamard, identity, X-, Y- and Z-, S- and T- gates and their properties in detail can be found here. Two qubit quantum states include SWAP gate, CNOT gate - is a quantum XOR, i.e, $CNOT\,|a\rangle\,|b\rangle = |a\rangle\,|a \oplus b\rangle$. Three-qubit gates such as Toffoli gate - $|a\rangle\,|b\rangle\,|c\rangle = |a\rangle\,|b\rangle\,|ab \oplus c\rangle$. Any one-qubit quantum state can be expressed in the following way -

$$U = e^{i\gamma}(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}(n_x X + n_y Y + n_z Z))$$

Quantum gates are linear maps that keep the total probability equal to 1 and are unitary matrices. Classical reversible logic gates can be quantum gates. Single-qubit quantum gates are associated with rotations by some angle about some axis on the bloch sphere. Bloch sphere corresponds to spherical coordinate system.

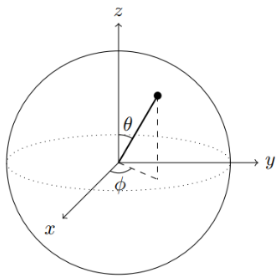


Figure 1: Bloch Sphere

Global phases are physically irrelevant since a qubit is a point on the bloch sphere and changing the global phases will point to the same qubit and does not affect the probability amplitude of each outcome. Relative phases are significant since these correspond to different points on the Bloch sphere.

Any classical gate can be made reversible by XORing the output with a third input, which is generally initialised to zero. Following table shows the list of classical gates with its corresponding quantum gates.

Classical		Reversible/Quantum	
NOT	A → $\overline{A}$	X-Gate	A – $\boxed{X}$ – $\overline{A}$
AND	A, B → AB	Toffoli	A, B, 0 → A, B, AB
OR	A, B → $A+B$	anti-Toffoli	A, B, 1 → A, B, $A+B$
XOR	A, B → $A \oplus B$	CNOTs	A, B, 0 → A, B, $A \oplus B$
NAND	A, B → $\overline{AB}$	Toffoli	A, B, 1 → A, B, $\overline{AB}$
NOR	A, B → $\overline{A+B}$	anti-Toffoli	A, B, 0 → A, B, $\overline{A+B}$

2 Quantum Error Correction

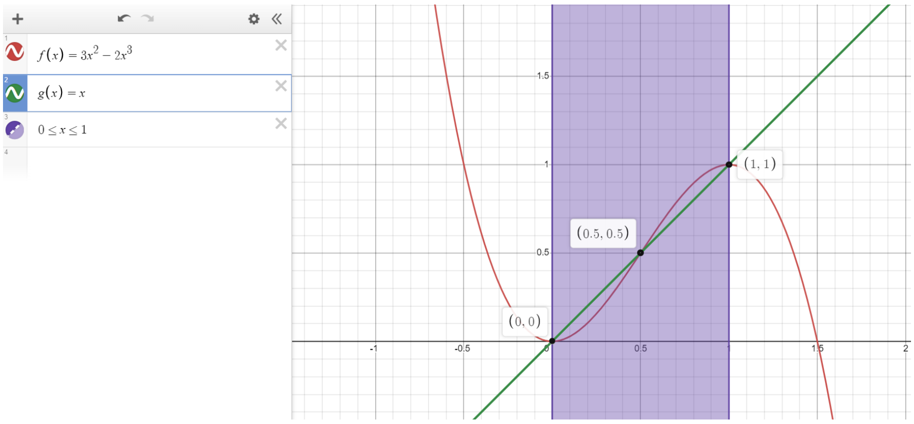
2.1 Classical Error Correcting Codes and Digitisation of Quantum Errors

Digitisation of Quantum Errors - Coherent errors can be expressed in terms of Pauli matrices. Let U be the unitary operator which describes the coherent error - $U(\delta\theta, \delta\phi)\,|\chi\rangle = (\alpha_I I + \alpha_X X + \alpha_Z Z + \alpha_{XZ} XZ)\,|\chi\rangle$, where $\delta\theta, \delta\phi$ are the amount of angles by which a qubit has been rotated. Coherent errors are caused by slow noise processes unlike incoherent errors that are caused by fast noise processes.

Linear block codes - Hamming codes belong to the family of linear block coeds which are capable of correcting a single-error and detecting two-errors in linear data blocks, and are represented by [n, k, d] where n is the total number of bits needed to encode a message bit, k is the number of information bits and d is the minimum distance needed to go from codeword to another. The relation between d and t (which is the number of correctable errors) is $d = 2t + 1$.

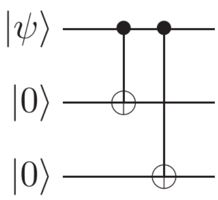
In classical communication, we make use of repetition codes to encode the bits before transmitting so as to protect them from the effects of noise in a binary symmetric channel and we can decode the bits by considering the majority voting. Majority voting fails when two or more bits are flipped in case of 3-repetition bit encodings. Let p be the probability of a bit getting flipped in the classical communication channel.Therefore, in this case, the transmission of information becomes reliable and useful to do error correction on the corrupted codeword provided $p_e < p$ where $p_e = 3p^2 - 2p^3$, and this happens when $p < 0.5$ as we can see in the graph below, and p_e is the probability of an uncorrectable error occurring.

We can apply the same logic to QEC since the no-cloning theorem is not violated as we will be comparing the parities of qubits instead of measuring them, thereby not using the majority voting procedure. So, we will not be disturbing the state as we will not know the values of α and β. We add quantum redundancy to qubits by expanding the Hilbert Space they belong to. For example, two qubits belong to H_2, on encoding the qubits, they belong to the codespace C which has elements - 00, 11 and C is a subset of H_4 which is the Hilbert Space consisting of elements - 00, 01, 10, 11. When an error operator occurs on any of the qubits, it will rotate the qubits into the error space F, which is mutually perpendicular to C and both are subsets of H_4, and



we can distinguish elements in F and C through the use of projective measurements, and this process is called Stabilizer measurement. For example - Z_1Z_2, when acted on an uncorrupted logical state gives a (+1) eigenvalue and (-1) on an corrupted logical state. Z_1Z_2 stabilizes a logical state when it leaves the state unchanged. Syndrome extraction is done using controlled stabilizer measurements on the ancilla qubits with the encoded qubits as the target. When we perform syndrome extraction on an encoded qubit, the state collapses to a syndrome measurement and the error probability associated with the collapsed state is less than that of an unencoded state in which the probability of an error occurring is more. The distance of a quantum code is defined as the minimum size error that will go undetected. Since we have phase flips as well, we get minimum quantum distance as 1 since $Z\left|+\right\rangle = \left|-\right\rangle$.

We can encode a single qubit state in three qubit states as shown in the diagram below and by doing we will get - $\alpha\left|0\right\rangle + \beta\left|1\right\rangle \longrightarrow \alpha\left|000\right\rangle + \beta\left|111\right\rangle$



Now detecting the error and recovering the state is done using the error-detection/ syndrome diagnosis. In this, we have four error syndromes, corresponding to the four projection operators,i.e,

$$\begin{aligned}
P_0 &= |000\rangle \langle 000| + |111\rangle \langle 111| - (1) \\
P_1 &= |100\rangle \langle 100| + |011\rangle \langle 011| - (2) \\
P_2 &= |010\rangle \langle 010| + |101\rangle \langle 101| - (3) \\
P_2 &= |001\rangle \langle 001| + |110\rangle \langle 110| - (4)
\end{aligned} \tag{1}$$

(1) means that no error has occurred on any qubit, (2) indicates error on qubit-1, (3) indicates error on qubit-2 and (4) indicates error on qubit-3. If the error has occurred on qubit-1, then $\langle\chi| P_1 |\chi\rangle = 1$. This measurement is called error syndrome and since the measurement gives 1, it means error has occurred on qubit-1. The state χ is recovered applying the relevant pauli operators on the qubit which has been flipped. Quantum errors can be modelled as undesired, random applications of the X, Z operators to the qubit state. This method is not completely efficient since the quantum states are continuous in nature, so the error may occur partially on qubit. We can compare if two consecutive qubits are same, by applying this measurement on the qubits - $Z_j Z_k = (-1)^{b_j+b_k}$. If, for example, $Z_1 Z_2 = 1$ and $Z_2 Z_3 = 1$, then no error has occurred, else if $Z_1 Z_2 = -1$ and $Z_2 Z_3 = -1$, then on qubit-2, error has occurred, else if $Z_1 Z_2 = -1$ and $Z_2 Z_3 = 1$, then on qubit-1, error has occurred or else if $Z_1 Z_2 = 1$ and $Z_2 Z_3 = -1$, then on qubit-3, error has occurred. This related to classical bit parity.

Similar approach can be applied to the phase-flips of quantum states, however, this is done with respect to the X-basis instead of the Z-basis. This is obtained by following the same circuit diagram for the the bit-flips. Following this, we add hadamard gates to each of the three qubit encodings and we get - $|0\rangle \longrightarrow |+++\rangle$ and $|1\rangle \longrightarrow |---\rangle$. In this, error is detected by applying this measurement on the qubits - $X_j X_k = (-1)^{b_j+b_k}$ which is equivalent to $H^{\otimes 3} Z_j Z_k H^{\otimes 3} = X_j X_k$ and the state is recovered by applying Z gate on the qubit whose phase has been flipped.

2.2 Stabilizer Codes

If stabilizer P_i commutes with the error, we get the measurement as 0 and when it anti-commutes, we get it as 1. Two Pauli operators will commute with one another if they intersect non-trivially on an even number of qubits, and anti-commute if otherwise. Properties of Stabilizer codes - they must be Pauli group elements, stabilizes all logical states, all stabilizers must commute with each other.

2.3 The Shor code

This is obtained by encoding the single-qubit quantum state using the phase flip code and then encoding each of these qubits using the bit flip code. The Shor code is the concatenation of the bit flip and phase flip error correcting codes. This code can detect and recover the quantum state from any arbitrary errors on any one of the 9 qubits. We obtain the following upon doing so as shown in the figure 5. For phase flips on any one qubit, we make use of the syndrome measurement on the first two blocks of three qubits, and on the last two blocks. And, we then compare the signs to find which block of three qubits has a phase flip, and then revert to its original state.

$$|0\rangle \rightarrow |0_L\rangle \equiv \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow |1_L\rangle \equiv \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

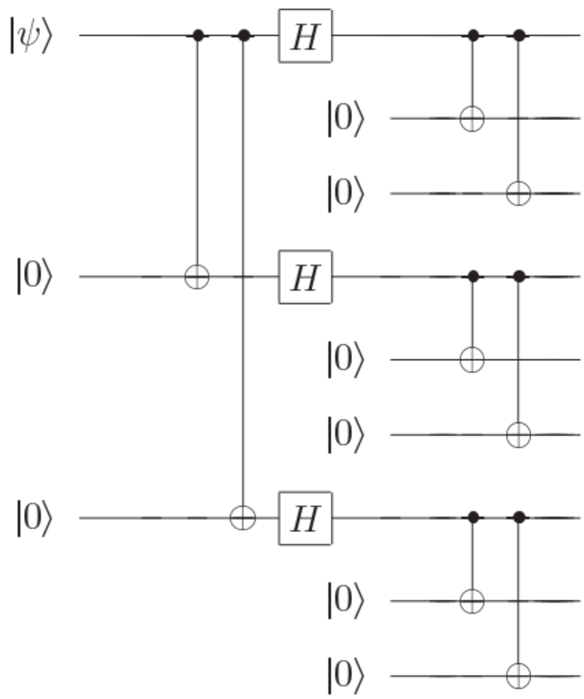


Figure 2: Shor Code Circuit Diagram

For example, the fourth qubit is flipped and a phase in the second block of the encoded state of $\chi = \alpha |0\rangle + \beta |1\rangle$. Then we apply the bit-flip stabilizers to each of the three blocks (or to all nine qubits, in general). On doing so, we obtain -1 for g_3 and 1 for the remaining first six stabilizers. For this bit-flip, we apply X_4 to the fourth qubit. For the phase-flip, we apply the last two stablilizers and we obtain -1 in both the cases, indicating that the phase has been changed in the second block. Therefore, for this, we apply Z gate to any one of the qubits in the second block or to all of them. Below is the table consisting of the eight stabilizers needed to detect and recovering the encoded state.

Name	Stabilizer
g_1	$Z_1Z_2I_3I_4I_5I_6I_7I_8I_9$
g_2	$I_1Z_2Z_3I_4I_5I_6I_7I_8I_9$
g_3	$I_1I_2I_3Z_4Z_5I_6I_7I_8I_9$
g_4	$I_1I_2I_3I_4Z_5Z_6I_7I_8I_9$
g_5	$I_1I_2I_3I_4I_5I_6Z_7Z_8I_9$
g_6	$I_1I_2I_3I_4I_5I_6I_7Z_8Z_9$
g_7	$X_1X_2X_3X_4X_5X_6I_7I_8I_9$
g_8	$I_1I_2I_3X_4X_5X_6X_7X_8X_9$

Using shor code, we can correct any arbitrary errors and this is done by expanding a trace-preserving quantum operation ϵ in an operator-sum representation with operation elements as E_i. To understand how this works, we can consider only a single term in this sum - $\epsilon(|\chi\rangle \langle\chi|) = \sum_i E_i |\chi\rangle \langle\chi| E_i^\dagger$. As an operator on a single qubit, E_i can be written as a linear combination I, X, Z and XZ, i.e, $E_i = e_{i0}I + e_{i1}X_1 + e_{i2}Z_1 + e_{i3}X_1Z_1$. $E_i |\chi\rangle$ is an un-normalized quantum state. Measuring the error syndrome, the state will collapse into any one of the four states. Depending on the result of the measurement, we can apply relevant operation on the result we obtained to get the original state χ.

Supposing the noise is affecting more than one qubit and we are assuming that the noise's effect on qubits is acting independently, then in this case, we can consider its effect as a sum of error on no qubits, one qubit and so on, with the terms of errors on the first two cases dominating the higher terms. Therefore, we can correct the zeroth and first order terms properly and leaving out the higher order errors, hence achieving a net suppression of error.

2.4 Surface code

The primary challenge in constructing quantum error-correcting codes lies in creating commuting sets of stabilizers to detect errors without disturbing the encoded information. Many special code constructions have been developed for this task. The surface code is one of them.

3 References

1. Quantum Error Correction: An Introductory Guide
2. Lecture Notes by Professor D K Ghosh
3. Nielson and Chuang
4. Spring School on Quantum Error Correction
5. Chapter 4 from "Introduction to Classical and Quantum Computing"