Derivation of Posterior for Normal Likelihood and Normal-Gamma Prior

Any prior for θ_1 and θ_2 is such that

$$p(\theta_1, \theta_2) = p(\theta_1 | \theta_2)\, p(\theta_2).$$

For the normal-gamma prior

$$p(\theta_2) \propto \theta_2^{a-1} e^{-b\theta_2} I_{(0,\infty)}(\theta_2)$$

and

$$p(\theta_1 | \theta_2) \propto \sqrt{\theta_2}\, \exp\left(-\frac{\tau\theta_2}{2}(\theta_1 - \mu)^2\right).$$

From the table of distributions we know that

$$p(\theta_2) = \frac{b^a}{\Gamma(a)} \theta_2^{a-1} e^{-b\theta_2} I_{(0,\infty)}(\theta_2)$$

and

$$p(\theta_1 | \theta_2) = \frac{\sqrt{\tau\theta_2}}{\sqrt{2\pi}} \exp\left(-\frac{\tau\theta_2}{2}(\theta_1 - \mu)^2\right).$$

Note that the constant multiplier,

$$\frac{b^a}{\Gamma(a)} \cdot \frac{\sqrt{\tau}}{\sqrt{2\pi}},$$

in the normal-gamma distribution is completely determined by a, b and τ. The form of the constant is the same for <u>every</u> normal-gamma distribution. Therefore, if we know that the posterior is proportional to a normal-gamma density with parameters a', b', μ' and τ', then we automatically know that the correct constant multiplier is

$$\frac{(b')^{a'}}{\Gamma(a')} \cdot \frac{\sqrt{\tau'}}{\sqrt{2\pi}}.$$

The likelihood is proportional to

$$\theta_2^{n/2} \exp\left(-\frac{\theta_2}{2}\sum_{i=1}^{n}(y_i-\theta_1)^2\right) =$$

$$\theta_2^{n/2} \exp\left(-\frac{\theta_2}{2}\sum_{i=1}^{n}(y_i-\bar{y})^2\right)\exp\left(-\frac{n\theta_2}{2}(\theta_1-\bar{y})^2\right).$$

$$p(\theta_1, \theta_2 | y) \propto \theta_2^{n/2+a-1} \exp\left(-\theta_2\left[b + \frac{1}{2}\sum_{i=1}^{n}(y_i - \bar{y})^2\right]\right)$$
$$\times \sqrt{\theta_2} \exp\left(-\frac{\theta_2}{2}\left[n(\theta_1 - \bar{y})^2 + \tau(\theta_1 - \mu)^2\right]\right)$$

The next step is to rewrite the second term in brackets so that it has the form

$$A(\theta_1 - \mu')^2 + B, \qquad (*)$$

where A and B don't depend on either θ_1 or θ_2. We do this to identify the normal part of the posterior.

$n(\theta_1 - \bar{y})^2 + \tau(\theta_1 - \mu)^2 =$

$n(\theta_1^2 - 2\theta_1\bar{y} + \bar{y}^2) + \tau(\theta_1^2 - 2\mu\theta_1 + \mu^2) =$

$(n+\tau)\theta_1^2 - 2\theta_1(n\bar{y} + \tau\mu) + n\bar{y}^2 + \tau\mu^2 =$

$(n+\tau)\left[\theta_1^2 - 2\theta_1\frac{(n\bar{y}+\tau\mu)}{(n+\tau)}\right] + n\bar{y}^2 + \tau\mu^2. \quad (**)$

To simplify notation, let

$$m = \frac{n\bar{y}+\tau\mu}{(n+\tau)}.$$

So, (**) is equal to

$$(n+\tau)\left[\theta_1^2 - 2\theta_1 m + m^2 - m^2\right] + n\bar{y}^2 + \tau u^2 =$$

$$(n+\tau)(\theta_1 - m)^2 - m^2(n+\tau) + n\bar{y}^2 + \tau u^2.$$

So we have succeeded in doing (*).

Substituting into the expression at the top of p. 3,

$$p(\theta_1, \theta_2 | y) \propto$$

$$\theta_2^{n/2+a-1} \exp\left(-\theta_2\left[b + \frac{1}{2}\sum_{i=1}^{n}(y_i - \bar{y})^2 + \frac{1}{2}\left\{n\bar{y}^2 + \tau u^2 - m^2(n+\tau)\right\}\right]\right)$$

$$\times \sqrt{\theta_2} \exp\left(-\frac{(n+\tau)\theta_2}{2}(\theta_1 - m)^2\right). \quad (***)$$

The general normal-gamma density with parameters a, b, τ and u is proportional to

$$\theta_2^{a-1} e^{-b\theta_2} \sqrt{\theta_2} \exp\left(-\frac{\tau\theta_2}{2}(\theta_1 - u)^2\right).$$

Notice that (***) has <u>exactly</u> this

form but with different parameters.
We have

$$a' = \frac{n}{2} + a,$$

$$b' = b + \frac{1}{2}\sum_{i=1}^{n}(y_i - \bar{y})^2 + \frac{1}{2}\left\{n\bar{y}^2 + \tau\mu^2 - m^2(n+\tau)\right\},$$

$$\tau' = \tau + n \quad \text{and}$$

$$\mu' = m = \frac{n\bar{y} + \tau\mu}{n+\tau}.$$

The term in curly brackets is

$$n\bar{y}^2 + \tau\mu^2 - \frac{(n\bar{y} + \tau\mu)^2}{(n+\tau)} =$$

$$n\bar{y}^2 + \tau\mu^2 - \frac{n^2\bar{y}^2}{(n+\tau)} - \frac{2n\tau\bar{y}\mu}{(n+\tau)} - \frac{\tau^2\mu^2}{(n+\tau)} =$$

$$(n+\tau)^{-1}\Big[(n^2 + n\tau)\bar{y}^2 + (n\tau + \tau^2)\mu^2 - n^2\bar{y}^2 - 2n\tau\bar{y}\mu$$
$$- \tau^2\mu^2\Big] =$$

$$(n+\tau)^{-1}\left[n\tau\bar{y}^2 + n\tau\mu^2 - 2n\tau\bar{y}\mu\right] =$$

$$\frac{n\tau(\bar{y} - \mu)^2}{(n+\tau)}.$$ So we see that

the parameters match those on p. 97 of the notes.