Quantum Mechanics I

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1 Vectors and Dirac Notation

Remark 1.1. *This course will use two books, Shanker is more advanced and will be assigned with reading, Griffiths has more problem and easy-going in terms of difficulty.*

Definition 1.2. A vector space V on a field $\mathbb{F}$ is a set of vectors that are closed under addition and multiplication. That is, for any $v, w \in V$, $a, b \in \mathbb{F}$, $av + bw \in V$.

Remark 1.3. *From now on we will use V to denote a vector space, $a, b, c, \ldots$ to denote elements of $\mathbb{F}$, and $u, v, w, \ldots$ to denote elements of V.*

Definition 1.4. Vector addition and multiplication follow a set of axioms, which can be found in Shanker 1.1. Some of the most important ones are listed below:

1. $v + w = w + v$
2. $(u + v) + w = u + (v + w)$
3. $a \cdot (v + w) = a \cdot v + a \cdot w$

Definition 1.5. A real number space is a V over $\mathbb{R}$. A complex number space is a V over $\mathbb{C}$.

Remark 1.6. *From now on we will be limiting our discussion to complex number spaces.*

Example 1.7. Complex column vectors:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \tag{1}$$

Each $x_i \in \mathbb{C}$. Check that both operations are closed.

Example 1.8. Complex function vectors: $F = \{f : [0, L] \subset \mathbb{R} \to \mathbb{C}\}$. Define addition and scalar multiplication as function addition and multiplying function with scalar. Check function is closed under both operations.

Definition 1.9. $v_1, \ldots, v_n$ are linearly dependent if exists $a_1, \ldots, a_n$, at least one of them non-zero, such that

$$\sum_{i=1}^{n} a_i \cdot v_i = 0$$

$v_1, \ldots, v_n$ are linearly independent otherwise.

Definition 1.10. The dimension of a vector field V is the maximum number of linearly independent vectors subset to V.

Remark 1.11. *Example 1.7 has dimension k, example 1.8 has infinite dimension.*

Definition 1.12. $v_1, \ldots, v_n$ span V if and only if exists $a_1, \ldots, a_n$, at least one nonzero, such that

$$a_1 v_1 + \cdots + a_n v_n = 0$$

Definition 1.13. $v_1, \ldots, v_n$ is a basis of V if and only if they span V and are linearly independent.

Corollary 1.14. If $v_1, \ldots, v_n$ are a basis for V, then any $v \in V$ can be uniquely expressed by a linear combination of $v_1, \ldots, v_n$.

Definition 1.15. The complex conjugate of some complex number $a + bi$ is $a - bi$.

Definition 1.16. An inner product function is $(\cdot) : V \times V \to \mathbb{C}$, denoted as (w, v) such that

1. $(w, a_1 v_1 + a_2 v_2) = a_1 \cdot (w, v_1) + a_2 \cdot (w, v_2)$
2. $(v, w) \geq 0$, and only equal if v or w equals 0.
3. $(w, v) = (v, w)^*$

Corollary 1.17. $(a_1 w_1 + a_2 w_2, v) = a_1^* \cdot (w_1, v) + a_2^* \cdot (w_2, v)$

Proof. First note that $(c_1 \cdot c_2)* = (c_1)^* \cdot (c_2)^*$.

$$\begin{aligned} ((a+bi)(c+di))^* &= (ac - bd + (ad+bc)i)* \\ &= ac - bd - (ad+bc)i = (a-bi)(c-di) \\ &= (a+bi)^*(c+di)^* \end{aligned}$$

Hence the proof:

$$\begin{aligned} (a_1 w_1 + a_2 w_2, v) &= (v, a_1 w_1 + a_2 w_2)^* \\ &= (a_1(v, w_1) + a_2(v, w_2))^* = (a_1(v, w_1))^* + (a_2(v, w_2))^* \\ &= a_1^*(v, w_1)^* + a_2 * (v, w_2)^* = a_1^*(w_1, v) + a_2^*(w_2, v) \end{aligned}$$

$\square$

Example 1.18. The inner product of example 1.7 is $(w, v) = w_1^* v_1 + \cdots + w_k^* v_k$.

Example 1.19. The inner product of example 1.8 is

$$(f, g) = \int_0^L f^*(x) \cdot g(x) dx$$

Definition 1.20. The norm (sometimes called length) of v denoted $||v||$ is

$$||v|| = \sqrt{(v, v)}$$

Remark 1.21. *The inner product is a function that maps to $\mathbb{C}$, but $(v, v) \in \mathbb{R}$, so the square root for Definition 1.20 is the real number square root.*

Definition 1.22. v, w are orthogonal if and only if $(v, w) = 0$.

Definition 1.23. $v_1, \ldots, v_n$ are orthonormal if and only if for any v_i, v_j,

$$(v_i, v_j) = \delta_{ij} = bool(m == n)$$

Where δ_{ij} is called the 'kronecker delta.'

Corollary 1.24. $v_1, \ldots, v_n$ are orthonoral if and only if they are orthognal and have length 1.

Remark 1.25. *From now on we will limit the discussion to vector spaces with well-defined inner-products.*

Definition 1.26. A linear functional F is a linear function $F : V \to \mathbb{C}$.

Corollary 1.27. The set of linear functionals $\{F\}$ form a vector space.

Proof. Check closedness over addition and multiplication of linear functions. $\square$

Definition 1.28. Given any vector space V, exists a corresponding vector space V', named the dual space, which is the set of all linear functions on V.