PHIL12A

Section questions, 25 April 2011

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1 Formal proofs

Decide whether the following arguments are valid or not. If they are, give formal proofs for them; otherwise give counterexamples.

1. **(Ex 13.2)**

1	`∀x(Cube(x) ↔ Small(x))`	
2	`∀xCube(x)`	
	———	
3	`∀xSmall(x)`	

Proof:

1	`∀x(Cube(x) ↔ Small(x))`	
2	`∀xCube(x)`	
	———	
3	| $\boxed{a}$	
	| ———	
4	| `Cube(a)`	∀Elim: 2
5	| `Cube(a) ↔ Small(a)`	∀Elim: 1
6	| `Small(a)`	↔Elim: 5, 4
7	`∀xSmall(x)`	∀Intro: 3-6

2. **(Ex 13.3)**

```
1 | ∀xCube(x)
2 | ∀xSmall(x)
  |___
3 | ∀x(Cube(x) ∧ Small(x))
```

Proof:

```
1 | ∀xCube(x)
2 | ∀xSmall(x)
  |___
3 | | [e]
  | |___
4 | | Cube(e)                  ∀Elim: 1
5 | | Small(e)                 ∀Elim: 2
6 | | Cube(e) ∧ Small(e)       ∧Intro: 4, 5
7 | ∀x(Cube(x) ∧ Small(x)      ∀Intro: 3-6
```

3. **(Ex 13.4)**

```
1 | ¬∀xCube(x)
  |___
2 | ¬∀x(Cube(x) ∧ Small(x))
```

Proof:

```
1 | ¬∀xCube(x)
  |___
2 | | ∀x(Cube(x) ∧ Small(x))
  | |___
3 | | | [a]
  | | |___
4 | | | Cube(a) ∧ Small(a)     ∀Intro: 2
5 | | | Cube(a)                ∧Elim: 4
6 | | ∀xCube(x)                ∀Intro: 3-5
7 | | ⊥                        ⊥Intro: 1, 6
8 | ¬∀x(Cube(x) ∧ Small(x))    ¬Intro: 2-7
```

4.

1	∀xCube(x)	
2	∃xCube(x)	

Proof:

1	∀xCube(x)	
2	Cube(e)	∀Elim: 1
3	∃xCube(x)	∃Intro: 2

5.

1	∃xCube(x)	
2	¬∀x¬Cube(x)	

Proof:

1	∃xCube(x)	
2	∀x¬Cube(x)	
3	$\boxed{e}$ Cube(e)	
4	¬Cube(e)	∀Elim: 2
5	⊥	⊥Intro: 3, 4
6	⊥	∃Elim: 1, 3-5
7	¬∀x¬Cube(x)	¬Intro: 2-6

6. **(Ex 13.12)**

1	$\forall x(\mathrm{Cube}(x) \lor \mathrm{Tet}(x))$	
2	$\exists x \neg \mathrm{Cube}(x)$	
3	$\exists x \mathrm{Tet}(x)$	

Proof:

1	$\forall x(\mathrm{Cube}(x) \lor \mathrm{Tet}(x))$	
2	$\exists x \neg \mathrm{Cube}(x)$	
3	$\boxed{e}$ $\neg \mathrm{Cube}(e)$	$\exists$Elim: 2
4	$\mathrm{Cube}(e) \lor \mathrm{Tet}(e)$	$\forall$Elim: 1
5	$\mathrm{Cube}(e)$	
6	$\perp$	$\perp$Intro: 3, 5
7	$\mathrm{Tet}(e)$	$\perp$Elim: 6
8	$\mathrm{Tet}(e)$	
9	$\mathrm{Tet}(e)$	Reit: 8
10	$\mathrm{Tet}(e)$	$\lor$Elim: 4, 5-7, 8-9
11	$\exists x \mathrm{Tet}(x)$	$\exists$Intro: 10
12	$\exists x \mathrm{Tet}(x)$	$\exists$Elim: 2, 3-11

Or:

```
1   │ ∀x(Cube(x) ∨ Tet(x))
2   │ ∃x¬Cube(x)
    ├───
3   │ │ [e] ¬Cube(e)            ∃Elim: 2
    │ ├───
4   │ │ Cube(e) ∨ Tet(e)        ∀Elim: 1
5   │ │ │ Cube(e)
    │ │ ├───
6   │ │ │ ⊥                     ⊥Intro: 3, 5
7   │ │ │ ∃xTet(x)              ⊥Elim: 6
8   │ │ │ Tet(e)
    │ │ ├───
9   │ │ │ ∃xTet(x)              ∃Intro: 8
10  │ │ ∃xTet(x)                ∨Elim: 4, 5-7, 8-9
11  │ ∃xTet(x)                  ∃Elim: 2, 3-10
```

7. **(Ex 13.14)**

1	$\forall x(\text{Cube}(x) \leftrightarrow \text{Small}(x))$	
2	$\exists x \neg \text{Cube}(x)$	
3	$\exists x \neg \text{Small}(x)$	

Proof:

1	$\forall x(\text{Cube}(x) \leftrightarrow \text{Small}(x))$	
2	$\exists x \neg \text{Cube}(x)$	
3	$\boxed{e}$ $\neg \text{Cube}(e)$	
4	$\text{Small}(e)$	
5	$\text{Cube}(e) \leftrightarrow \text{Small}(e)$	$\forall$Intro: 1
6	$\text{Cube}(e)$	$\leftrightarrow$Elim 4, 5
7	$\perp$	$\perp$Intro: 3, 6
8	$\neg \text{Small}(e)$	$\neg$Intro: 4-7
9	$\exists x \neg \text{Small}(x)$	$\exists$Intro: 8
10	$\exists x \neg \text{Small}(x)$	$\exists$Elim: 2, 3-9

2 Informal proofs

Are the following informal proofs correct or not?

1. **(Ex 12.11)**

1	
2	There is a number greater than every other number.

> **Purported proof:** Let n be an arbitrary number. Then n is less than some other number, $n + 1$ for example. Let m be any such number. Thus $n \leq m$. But n is an arbitrary number, so every number is less or equal m. Hence there is a number that is greater than every other number.

No. We pick n arbitrary, find a number bigger than n and existentially generalize, concluding that there is some m such that m is greater than n. Now we cannot use universal generalization to say that everything is smaller than m, since the existence of m depended on n.

2. **(Ex 12.15)**

1	$\forall \mathtt{x} \forall \mathtt{y} \forall \mathtt{z}[(\mathtt{Outgrabe(x,y)} \land \mathtt{Outgrabe(y,z)}) \to \mathtt{Outgrabe(x,z)}]$
2	$\forall \mathtt{x} \forall \mathtt{y}[\mathtt{Outgrabe(x,y)} \to \mathtt{Outgrabe(y,x)}]$
3	$\exists \mathtt{x} \exists \mathtt{y}\mathtt{Outgrabe(x,y)}$
4	$\forall \mathtt{x}\mathtt{Outgrabe(x,x)}$

> **Purported proof:** Applying existential instantiation to the third premise, let $\mathtt{b}$ and $\mathtt{c}$ be arbitrary objects in the domain of discourse such that $\mathtt{b}$ outgrabes $\mathtt{c}$. By the second premise, we also know that $\mathtt{c}$ outgrabes $\mathtt{b}$. Applying the first premise (with $\mathtt{x} = \mathtt{z} = \mathtt{b}$ and $\mathtt{y} = \mathtt{c}$) we see that $\mathtt{b}$ outgrabes itself. But $\mathtt{b}$ was arbitrary. Thus by universal generalization, $\forall \mathtt{x}\mathtt{Outgrabe(x,x)}$.

No. We introduced $\mathtt{b}$ by existential instantiation on premise 3. So $\mathtt{b}$ is not completely arbitrary, and we cannot universally generalize on a sentence containing it.