Topics in Optimization

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Course Structure

- 8 weeks
- 1h lecture + 1h exercises

Github Page

https://github.com/Joao-Dionisio/Minicurso

- Slides (unfinished)
- Lecture Notes (unfinished)
- Some Code (unfinished)
- Competition Details

Course Contents

1. Convexity Theory
2. Linear Programming
3. Complexity Theory
4. Integer Programming
5. Decomposition Methods

Convex sets

Definition

A set $X \in \mathbb{R}^n$ is said to be convex if

$$\forall x, y \in X, (1-t)x + ty \in X, t \in [0,1]$$

Convex sets

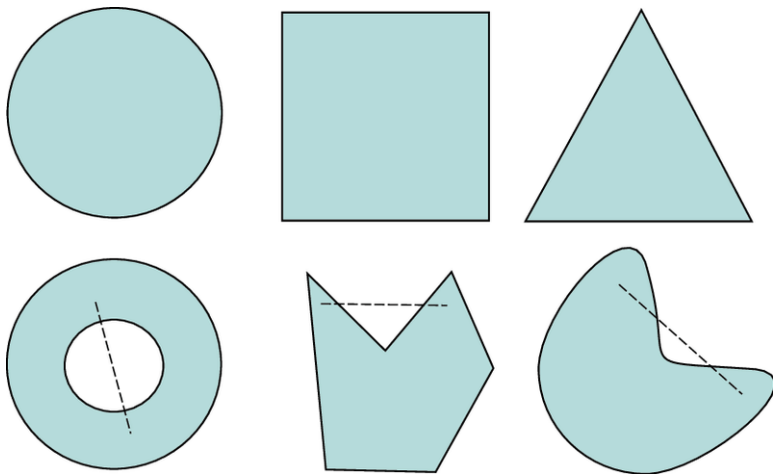Figure: Examples of convex and non-convex sets

Convex Functions

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if
$\forall x, y \in \mathbb{R}^n, \forall \lambda \in [0, 1]$ we have

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$$
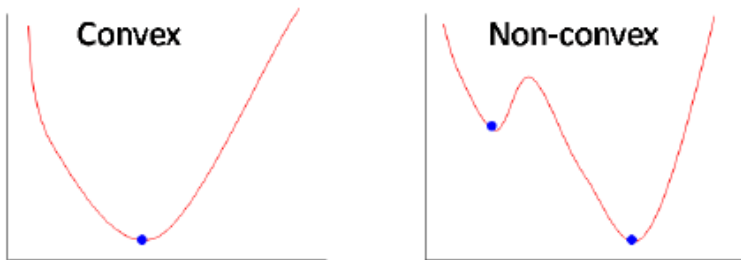
Convex Functions

Figure: Examples of convex and non-convex functions

Epigraph

Definition

We call the epigraph of a function $f : X \to \mathbb{R}$, denoted by epi(f), to the following set:

$$epi(f) = \{(x, y) \in X \times \mathbb{R} \mid f(x) \leq y\}$$

Epigraph

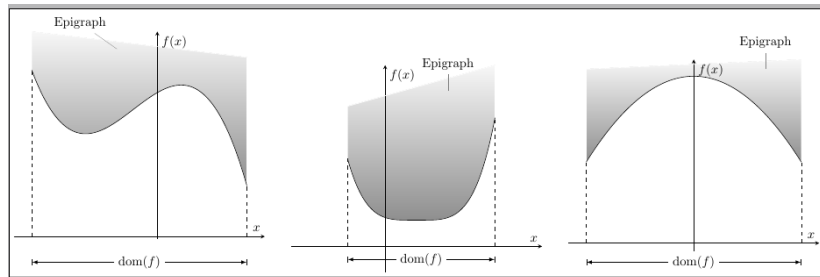Figure: Examples of epigraph

Convex sets and convex functions

Proposition

The following two statements are equivalent:

1. *f is convex;*
2. *epi(f) is a convex set.*

Convex Optimization

Definition

A convex optimization problem is

$$\min_x \quad f(x)$$
$$\text{s.t.} \quad g_i(x) \geq 0, i \in [m]$$

$f, g_i, i = 1 \ldots, m$ convex functions. It implies that $\bigcap_{i \in [m]} \text{epi}(g_i)$ forms a convex set.

Sufficient Optimality Condition

Theorem

Let $\mathcal{P}$ be a convex optimization problem, and without loss of generality, assume it is a minimization problem. Then, if x^* is a local minimizer, then x^* is a global minimizer.

Linear Programming
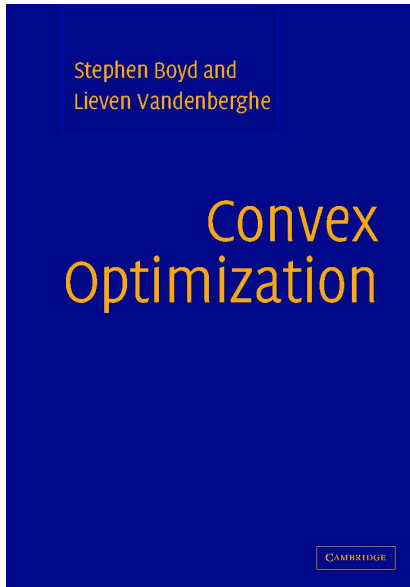
$$\min_{x} \quad c^{\mathsf{T}}x$$
$$\text{s.t.} \quad Ax \leq b$$
$$\qquad x \geq 0$$

The feasible region is a polyhedron.

Small Example

Dunder Mifflin can produce two types of industrial-sized sheets, type A and type B. Type A *can be produced* at a ratio of **200m** per hour, while type B can be produced at a ratio of **140m** per hour. *The profits* from each type of paper are **25** cents per meter and **30** cents per meter, respectively. Taking the market demand into account, next week's production schedule *cannot exceed* **6000m** for paper of type A and **4000m** for paper of type B. If on that week there is a *limit of* **40** production hours, how many meters of each product should be produced to maximize the profit?

Small Example

$$\max_{A,B} \quad 25A + 30B$$

$$\text{s.t.} \quad A/200 + B/140 \leq 40$$

$$A \leq 6000$$

$$B \leq 4000$$

$$A, B \geq 0$$

Example - Max flow

Suppose you have a network of pipes and receive money based on the amount of a valuable liquid that reaches a single destination, coming from a single source. Assume that the pipes have limited capacity and none of the liquid is gained or lost along the way. This is the max-flow problem, whose linear programming model follows.

Example - Max Flow

Definition

Let $G(V, E)$ be a graph with $s, t \in V$ being defined as the source and target. The **max-flow problem** is the following LP:

$$\max \quad \sum_{v:(s,v)\in E} f_{sv}$$

$$\text{s.t.} \quad f_{uv} \leq c_{uv}, \forall (u,v) \in E$$

$$\sum_{u} f_{uv} - \sum_{w} f_{vw} = 0, \forall v \in V \setminus \{s, t\}$$

Supporting hyperplane theorem

Theorem (Supporting Hyperplanes)

Let $C \subset \mathbb{R}^n$ be a convex set, and $x \in \partial C$. Then, there exists a hyperplane H, s.t. $x \in H \cap C$ and C is contained in one of the half-spaces bounded by H.

It justifies the importance of LPs in the more general Convex Optimization.
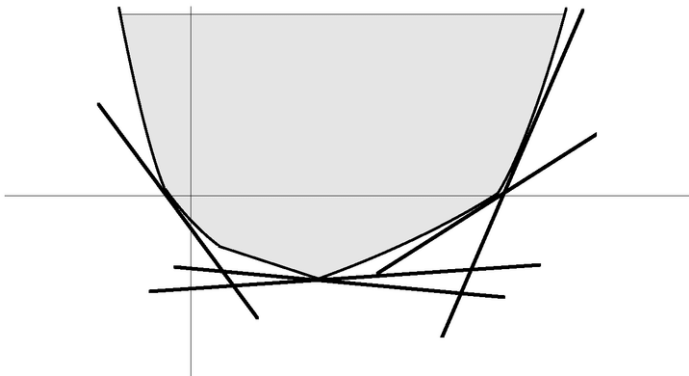
Supporting hyperplane theorem

Figure: Visual representation

How can we know if we are close to the optimal solution? Can we derive bounds?

Duality

Definition

The dual problem of an LP of the form 16 (which we now call the primal) is:

$$\max_{y} \quad b^{\mathsf{T}} y$$
$$\text{s.t.} \quad A^{\mathsf{T}} y \geq c$$
$$y \geq 0$$

Every primal constraint has an associated dual variable, and vice-versa.

Strong Duality

Theorem (Strong Duality for LPs)

Let P be an LP, D the corresponding dual, and x^*, y^* be the respective optimal solutions. We have that $b^\mathsf{T} y^* = c^\mathsf{T} x^*$.

Interpretations of the Dual - Example

The value of the dual variable can be seen as the maximum amount of money that the decision maker is willing to spend to buy an additional resource. See the following example.

A furniture company makes desks, tables, and chairs. The making of each type of furniture requires wood and two types of specialized work: finishes and carpentry. The quantity required of each of these resources for each of the types of furniture is in the table in the next slide.

Interpretation of the dual - Example

Resource	Desks	Tables	Chairs
Wood	8 boards	6 boards	1 board
Finishes	4 hours	2 hours	1.5 hours
Carpentry	2 hours	1.5 hours	0.5 hours

The company can expend 40 boards, 20h of finishes, and 8h of carpentry. The selling price is 60€ for desks, 30€ for tables, and 20€ for chairs.

Which has the following linear programming formulation:

$$\begin{aligned} \max \quad & 60x + 30y + 20z \\ \text{s.t.} \quad & 8x + 6y + 1z \leq 40 \\ & 4x + 2y + 1.5z \leq 20 \\ & 2x + 1.5y + 0.5z \leq 8 \\ & x, y, z \geq 0 \end{aligned}$$

And optimal solution $x = 2, y = 0, z = 8$, with an objective equal to 280€.

Interpretation of the dual - Example

The dual of the above problem is

$$\min_{\pi} \quad 40\pi_1 + 20\pi_2 + 8\pi_3$$

$$\text{s.t.} \quad 8\pi_1 + 4\pi_2 + 2\pi_3 \geq 60$$

$$6\pi_1 + 2\pi_2 + 1.5\pi_3 \geq 30$$

$$\pi_1 + 1.5\pi_2 + 0.5\pi_3 \geq 20$$

$$\pi_1, \pi_2, \pi_3 \geq 0$$

Variable π_1 is associated with the constraints limiting the boards. It tells us the impact of having one additional board on the optimum of the primal.

The optimal solution of the dual problem is
$\pi_1 = 0, \pi_2 = 10, \pi_3 = 10$. What is the objective value? Same as the primal's.

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$\pi_1 = 0, \pi_2 = 10, \pi_3 = 10$. What is the objective value? Same as the primal's.

This is telling us that the carpenter is willing to pay at most 10€ to have an additional hour of carpentry, since it would allow him to improve the objective by 10.
With 9 hours of carpentry instead of 8, the optimal objective is 290€.

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With 9 hours of carpentry instead of 8, the optimal objective is 290€.

What can we say about the boards, since the corresponding variable is 0?

Suppose now that the carpenter considers making stools. The design would require 0.5 boards, 1.8 hours of finishing touches, and 0.5 hours of carpentry. The carpenter would then sell it for 22€. Is it profitable to build stools?

Reduced Cost - Example

Suppose now that the carpenter considers making stools. The design would require 0.5 boards, 1.8 hours of finishing touches, and 0.5 hours of carpentry. The carpenter would then sell it for 22€. Is it profitable to build stools?

We know the benefit per stool, 22. If the resources are being spent on stools, they aren't being spent elsewhere. We can calculate the *opportunity cost* with the dual variables:
$0.5 * 0 + 1.8 * 10 + 0.5 * 10.$

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$0.5 * 0 + 1.8 * 10 + 0.5 * 10$.

$22 - 0.5 * 0 + 1.8 * 10 + 0.5 * 10 = -1 < 0$. Hence, the cost of building the stool outweighs the benefit.

Reduced cost - Example

Adding the possibility of building stools in the initial LP

$$
\begin{aligned}
\max \quad & 60x + 30y + 20z + 22 * w \\
\text{s.t.} \quad & 8x + 6y + 1z + 0.5 * w \leq 40 \\
& 4x + 2y + 1.5z + 1.8 * w \leq 20 \\
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& x, y, z \geq 0
\end{aligned}
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The optimal solution is $x = 2, y = 0, z = 8, w = 0$ - it isn't worth it to build stools.

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The optimal solution is $x = 2, y = 0, z = 8, w = 0$ - it isn't worth it to build stools.

If however, they were more expensive, say 25€, then the previous calculation would give us $25 - 0.5 * 0 - 1.8 * 10 - 0.5 * 10 = 2 > 0$, then the optimal solution would involve stools $(x = 2.75, y = 0, z = 0, w = 5)$.

Reduced cost

Definition

The **reduced cost** is the amount by which the coefficient in the objective function of a variable would have to improve in order for the variable to be non-zero in the optimal solution. Calculated by

$$c - A^{\mathsf{T}}\pi$$

The reduced cost can help us understand why the variables take the value that they do in the optimal solution.

Optimal solution at vertex

Theorem

Consider the following LP:

$$\min_{x} \quad c^\mathsf{T} x$$
$$s.t. \quad Ax \leq b \quad (P)$$
$$x \geq 0$$

Suppose it has at least one vertex. Then, if an optimal solution exists, there is also an optimal solution at a vertex.

Preliminaries

Definition

A point x of a convex set C is an **extreme point** (vertex) if $\nexists y, z \in C, \lambda \in]0, 1[\mid x = \lambda y + (1 - \lambda)z$.

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A polyhedron P contains a **line** if
$\exists d \in \mathbb{R}^n, x \in P \mid \forall \lambda \in \mathbb{R}, x + \lambda d \in P$.

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Definition

A polyhedron P contains a **line** if
$\exists d \in \mathbb{R}^n, x \in P \mid \forall \lambda \in \mathbb{R}, x + \lambda d \in P.$

Lemma

P has a line $\iff$ P does not have an extreme point.

P has an extreme point $\implies$ P has no line.

Proof Sketch

P has an extreme point $\implies$ P has no line. Q the set of optimal points has no line ($Q \subseteq P$), so it has an extreme point.

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$x^* = \lambda y + (1 - \lambda)z$.

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With this theorem, we have can create an algorithm to solve LPs.
How?

With this theorem, we have can create an algorithm to solve LPs. How? Need to check all vertices (an exponential number).

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Simplex Motivation

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João, say stuff about Dantzig, Von Neumann, and WWII.
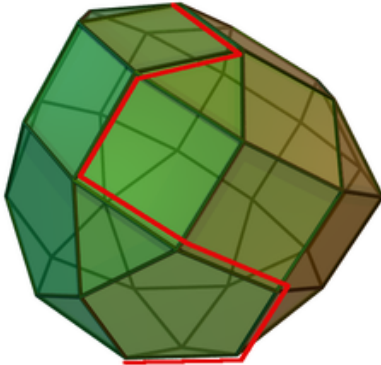
Figure: Example of simplex - only moves forward

Simplex Complexity

For the vast majority of problems, the Simplex Method runs in polynomial time, but so-called pathological examples have been found that ensure that the Simplex Method has to visit an exponential number of vertices.

Can be used on general NLPs.

$$\min_{x} \quad c^{\mathsf{T}}x$$
$$\text{s.t.} \quad c_i(x) \geq 0, i = 1, \ldots, m$$

IPM Motivation

We replace the constraints with what is called a *barrier function*, most commonly a logarithm, to discourage solutions close to the border of the feasible region.

$$\min_{x} \quad c^{\mathsf{T}}x - \mu \sum_{i=1}^{m} log(c_i(x))$$

Successive iterations decrease the value of μ.

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$$\min_x \quad c^\mathsf{T} x - \mu \sum_{i=1}^{m} log(c_i(x))$$

Successive iterations decrease the value of μ. IPM has a

polynomial running time ($O(n^{3.5} log(1/\varepsilon))$, in fact).

IPM visualization
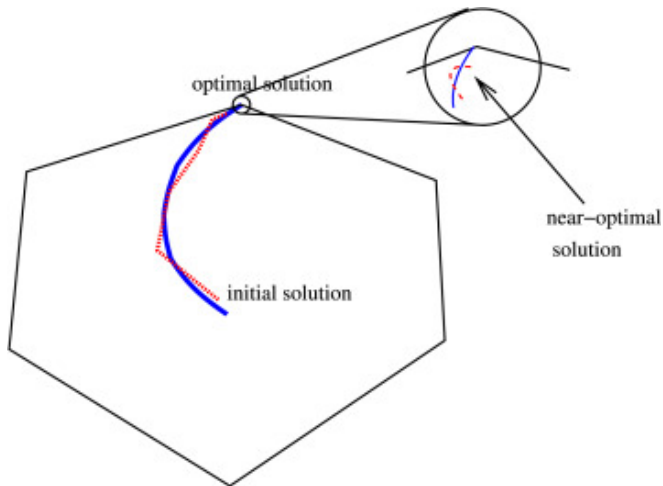
Figure: Example of IPM iterations

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Big-O
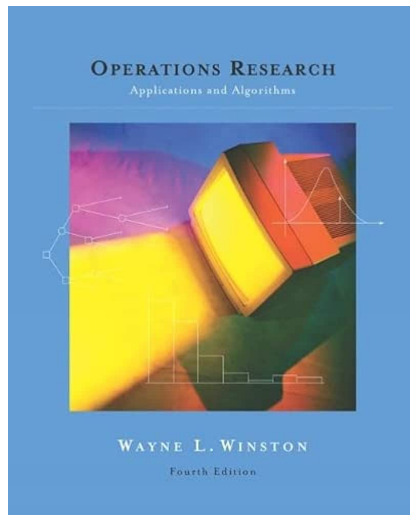
Definition

Given functions $f, g : \mathbb{R}^n \to \mathbb{R}$, we say that $f \in O(g(x))$ if

$$\forall x \geq x_0, |f(x)| \leq Mg(x)$$

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Algorithmic complexity

- Writing every number from 1 to n requires us to write $O(n)$ numbers. What about 1 to $2n$?
- Writing every element of the power set of size n require us to write $O(2^n)$ numbers. What about $2n$?

Complexity Classes

Skipping over a lot of details, (time) complexity classes are sets of problems characterized by the difficulty (time) of solving them. E.g.: **P**, **EXP**, . . .

P and NP

P is the set of problems for which an algorithm that runs in polynomial time solves it. **NP** is the set of problems for which an algorithm that runs in polynomial time can verify if a given candidate solution is indeed a solution.

Need to add more slides here...

Min Vertex-Cover

Given a graph $G(V, E)$ find the minimum number of vertices that cover all edges

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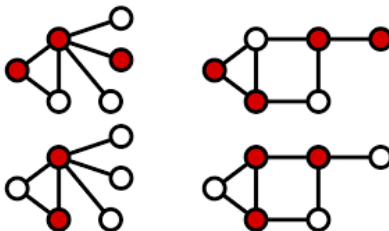
Figure: Example of Vertex Cover solutions

Min Vertex-Cover formulation

$$\min \quad \sum_{v \in V} x_v$$
$$\text{s.t.} \quad x_u + x_v \geq 1, \forall (uv) \in E$$
$$x_v \in \{0, 1\}$$

Given items with value and weight, fit them into a bag such that the total weight does not exceed W and the value is maximized.

Max Knapsack

Given items with value and weight, fit them into a bag such that the total weight does not exceed W and the value is maximized.

$$\max \quad \sum_{i=1}^{n} v_i x_i$$
$$\text{s.t.} \quad \sum_{i=1}^{n} w_i x_i \leq W$$
$$x_i \in \{0, 1\}$$

Given a graph $G(V, E)$ with weighted edges, find the least costly Hamiltonian cycle.

Traveling-Salesman
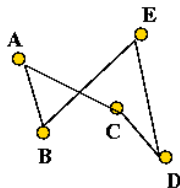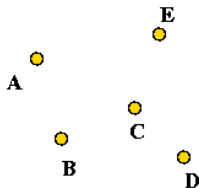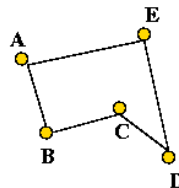
Given a graph $G(V, E)$ with weighted edges, find the least costly Hamiltonian cycle.

Figure: TSP example

TSP formulation

$$\min \quad \sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{i \neq j, i=1}^{n} x_{ij} = 1, j \in [n]$$

$$\sum_{i \neq j, j=1}^{n} x_{ij} = 1, i \in [n]$$

$$\sum_{i \in Q} \sum_{j \neq i, j \in Q} x_{ij} \leq |Q| - 1, \forall Q \subsetneq \{1, \ldots, n\}, |Q| \geq 2$$

$$x_{ij} \in \{0, 1\}$$

Cutting-Stock

Minimize the number of sheets of metal that, when cut into smaller sheets, satisfies a demand.

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Minimize the number of sheets of metal that, when cut into smaller sheets, satisfies a demand.

Figure: Cutting stock example

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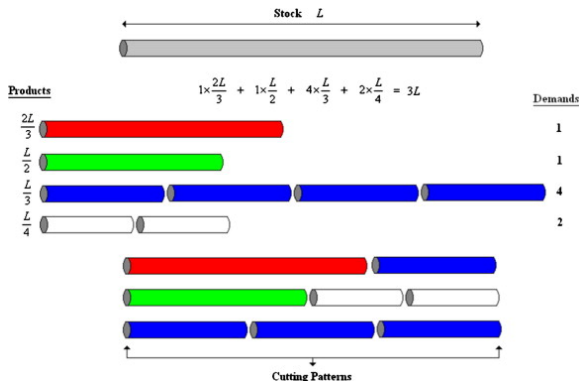
Cutting-Stock Formulation

$$\min \quad \sum_{j=1}^{M} y_j$$

$$\text{s.t.} \quad \sum_{j=1}^{M} x_{ij} = d_i, \qquad\qquad i \in [n]$$

$$\sum_{i=1}^{n} l_i x_{ij} \leq L, \qquad\qquad j \in [m]$$

$$x_{ij} \leq d_i y_j, \qquad i \in [n], j \in [m]$$

$$x_{ij} \in \mathbb{Z}^{+}, \qquad i \in [n], j \in [m]$$

$$y_j \in \{0, 1\}, \qquad\qquad j \in [m]$$

Knowing the solution to some optimization problems can give us the solution to other optimization problems.

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1. Vehicle routing and TSP

Reductions

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2. Min Vertex Cover and Max Independent set

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1. Vehicle routing and TSP
2. Min Vertex Cover and Max Independent set
3. Cutting Stock and Bin packing
4. Knapsack and Cutting Stock

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Exercises

Integer Programming

$$\min_{x,y} \quad c^{\mathsf{T}}(xy)^{\mathsf{T}}$$
$$\text{s.t.} \quad f(x, y) \leq 0$$
$$x \geq 0$$
$$y \in \mathbb{Z}$$

Innumerous applications

- Health Industry (Kidney Exchange, INEM stuff)
- Scheduling (Sports, Classes)
- Game Theory (Chess, Go, Economy)

Integer programming is not convex. Which strategies can we employ to solve it?

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Explicit enumeration?

Non-convex

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Explicit enumeration? Generally out of the question. An exponential number of solutions.

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Pick the integer point closest to the optimum for the LP?

Non-convex

Integer programming is not convex. Which strategies can we employ to solve it?

Explicit enumeration? Generally out of the question. An exponential number of solutions.

Pick the integer point closest to the optimum for the LP? Integer optimum can be arbitrarily far away.

LP vs. IP example

$$\max_{x,y} \quad 10x + y$$

$$\text{s.t.} \quad y \le 5x + 4$$

$$x, y \in \mathbb{N}_0$$
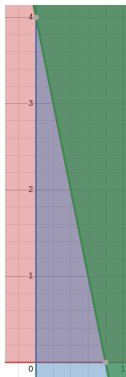
LP vs. IP example

Figure: Feasible region

LP vs. IP example
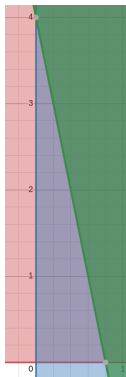
Figure: Feasible region

The optimal solution is $(0, 4)$. With $x, y \geq 0$ instead, optimal solution is $(\frac{4}{5}, 0)$.

Relaxations

$$\max_{x} \quad c^\mathsf{T} x$$
$$\text{s.t.} \quad Ax \leq b$$
$$Dx \leq e$$

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$$\max_{x} \quad c^{\mathsf{T}}x$$
$$\text{s.t.} \quad Ax \leq b$$

What can we deduce about these two problems?

Relaxations

$$\max_x \quad c^\mathsf{T} x$$
$$\text{s.t.} \quad Ax \le b$$
$$Dx \le e$$

$$\max_x \quad c^\mathsf{T} x$$
$$\text{s.t.} \quad Ax \le b$$

What can we deduce about these two problems? The optimal value of the second cannot be worse than that of the first. It provides an upper bound.

Relaxing constraints provides a bound on the optimal solution.
Can this help us solve IPs?

Relaxations

Relaxing constraints provides a bound on the optimal solution.
Can this help us solve IPs? Yes! We can easily derive bounds.

$$\max_{x} \quad c^{\mathsf{T}}x$$
$$\text{s.t.} \quad Ax \leq b$$
$$x \in \mathbb{Z}$$

$$\max_{x} \quad c^{\mathsf{T}}x$$
$$\text{s.t.} \quad Ax \leq b$$
$$x \geq 0$$

Branching

We can fix a binary variable x to 0 and 1, and solve the resulting two subproblems. The best optimal value will be the optimal value of the problem. This is called **branching**.

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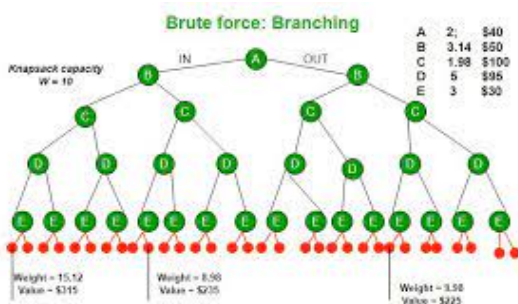
Figure: Branching on all variables (explicit enumeration)

Branch-and-Bound Idea

- Pick a variable and branch on it. Creates problems P_1, P_2
- If the linear relaxation of P_1 is worse than our best solution, ignore it
- If it is better, pick a variable in P_1 and branch on it
- If all variables are integer, save the solution if it's better
- Repeat until all subproblems have been explored

Branch-and-Bound

1   $L \leftarrow \{X\}$ // List of unexplored subproblems
2   $\overline{z} = \infty$ // Upper bound
3   **while** $L \neq \{\}$ **do**
4   select a subproblem S from L // search strategy
5   solve LP relaxation of S, with solution $x\prime$ and objective z'
6   remove S from L
7   **if** S *infeasible or* $z' \geq \overline{z}$ **then**
8   **continue**
9   **if** x' *is integer and* $z' < \overline{z}$ **then**
10         $\overline{z} \leftarrow z'$
11         $x^* = x'$ // x' becomes new best solution
12   **continue**
13   // otherwise S is not pruned, x' is feasible and fractional
14   split S into subproblems S_1, S_2
15   insert S_1, S_2 into L
16 **return** x^*

Branch-and-Bound
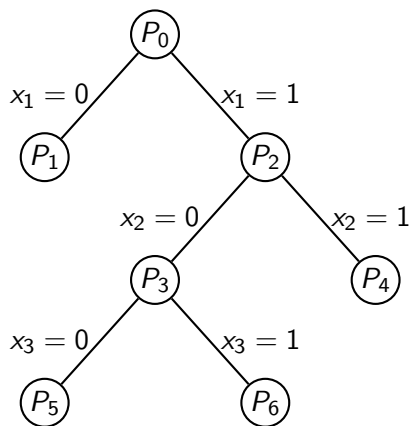
Revisiting Knapsack

$$\max \quad \sum_{i=1}^{n} v_i x_i$$

$$\text{s.t.} \quad \sum_{i=1}^{n} w_i x_i \leq W$$

$$x_i \in \{0, 1\}$$

It has an easy LP-relaxation

1. Sort items by price density (price/weight)
2. Pick items until capacity is exceeded
3. Remove the excess of the last item

Knapsack instance

Item	Weight	Value	Value/Weight
A	3	5	1.67
B	2	3	1.5
C	1	1	1

Knapsack capacity: 4

Branch and Bound
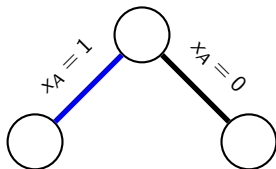
Branch and Bound
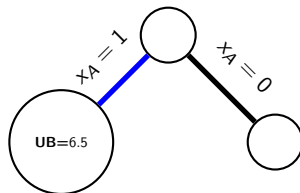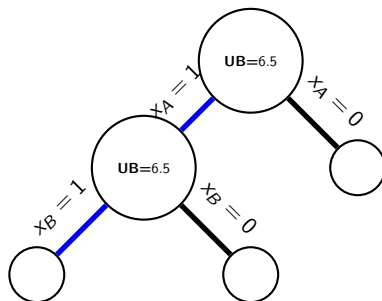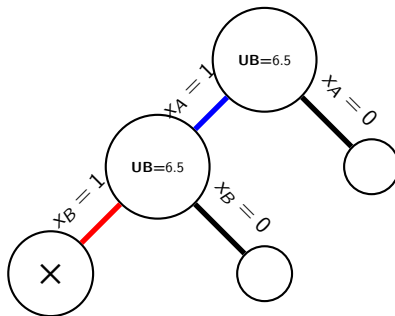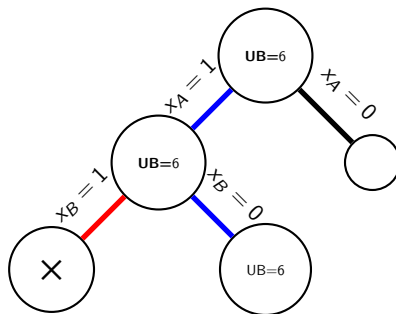
Branch and Bound

Branch and Bound

Branch and Bound
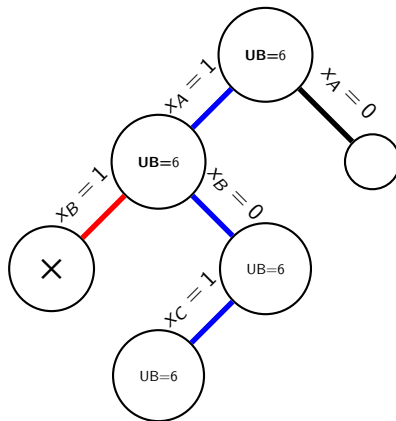
Branch and Bound
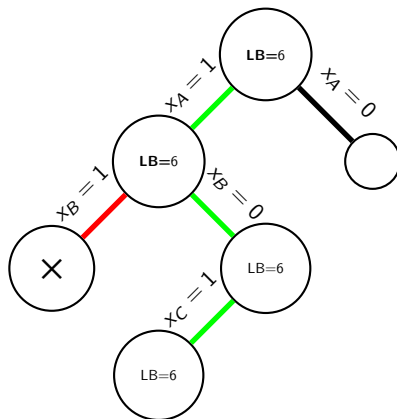
Branch and Bound
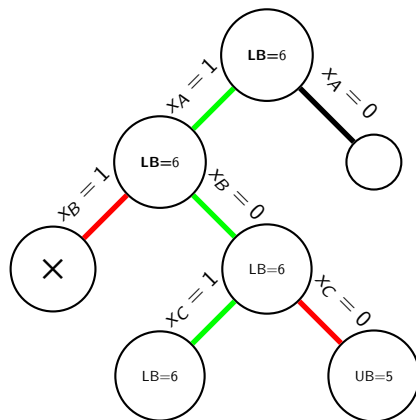
Branch and Bound
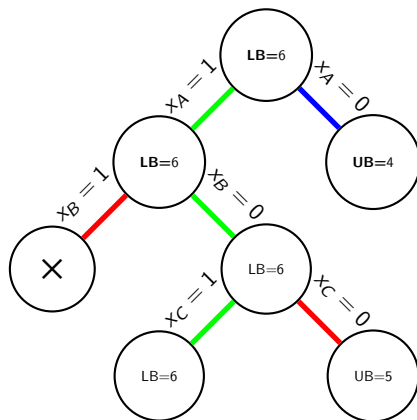
Branch and Bound
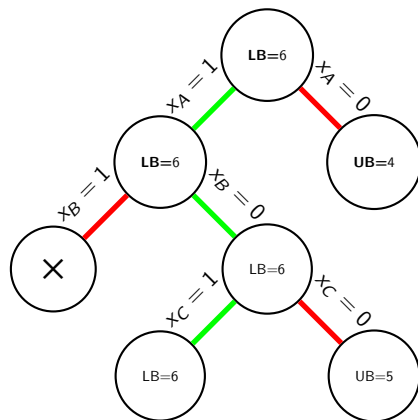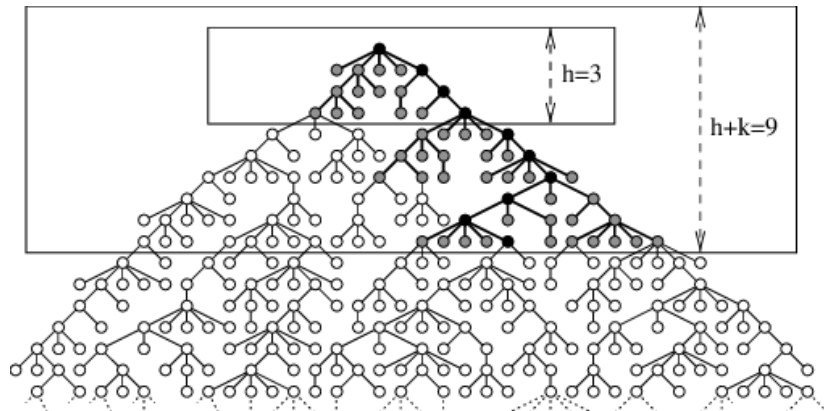
Heuristics

Figure: Branch and Bound trees can get very big. How big?

Heuristics

We will focus on heuristics for the TSP.

$$\min \quad \sum_{i=1}^{n} \sum_{j \neq i, j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{i \neq j, i=1}^{n} x_{ij} = 1, j \in [n]$$

$$\sum_{i \neq j, j=1}^{n} x_{ij} = 1, i \in [n]$$

$$\sum_{i \in Q} \sum_{j \neq i, j \in Q} x_{ij} \leq |Q| - 1, \forall Q \subsetneq \{1, \ldots, n\}, |Q| \geq 2$$

$$x_{ij} \in \{0, 1\}$$
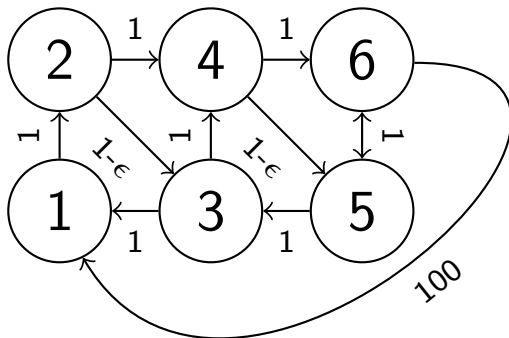
Nearest Neighbor
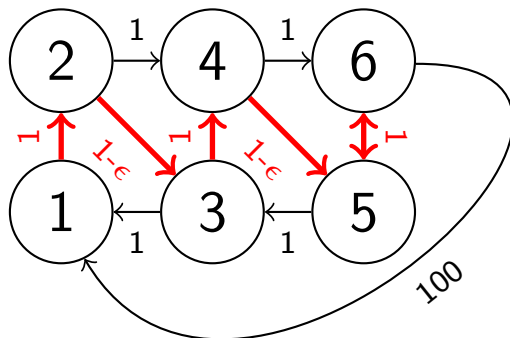
Nearest Neighbor
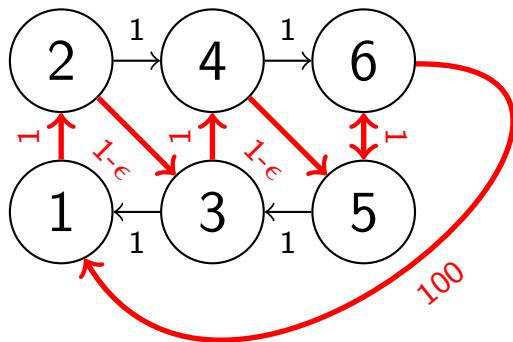
Greedy Algorithm
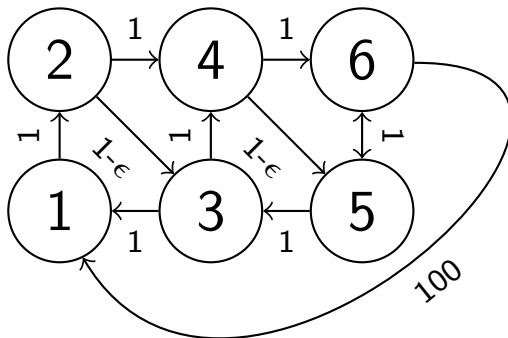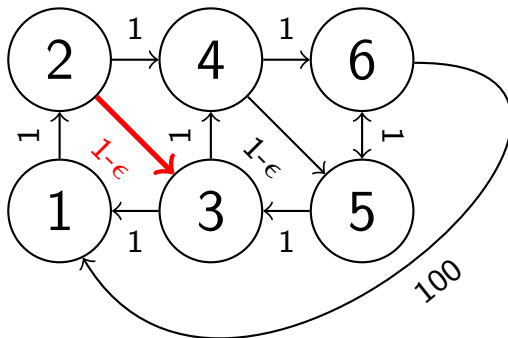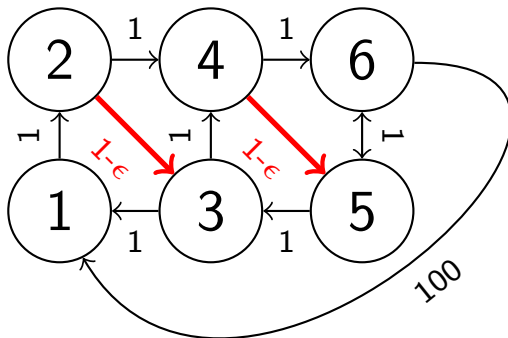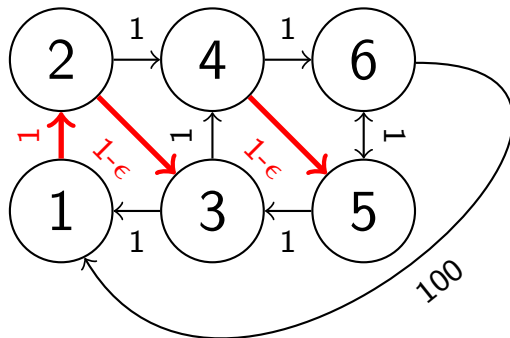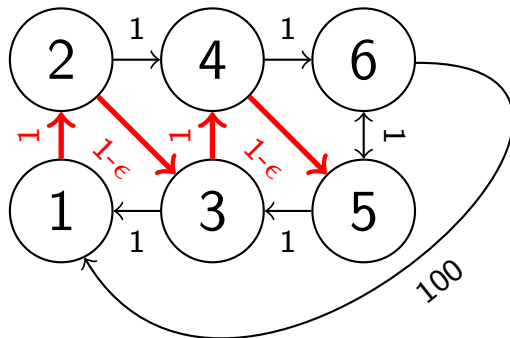
Greedy Algorithm
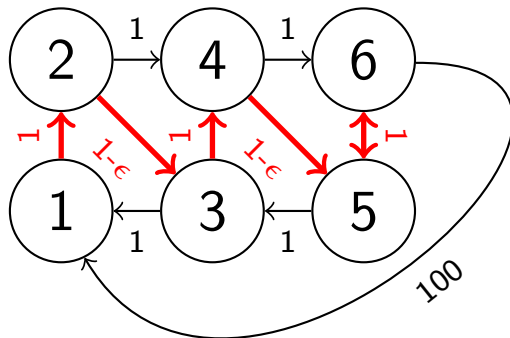
Greedy Algorithm

Greedy Algorithm
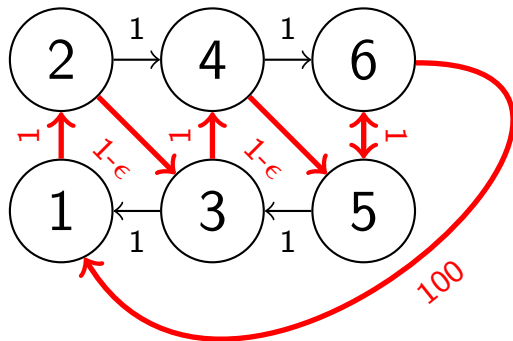
2-OPT algorithm
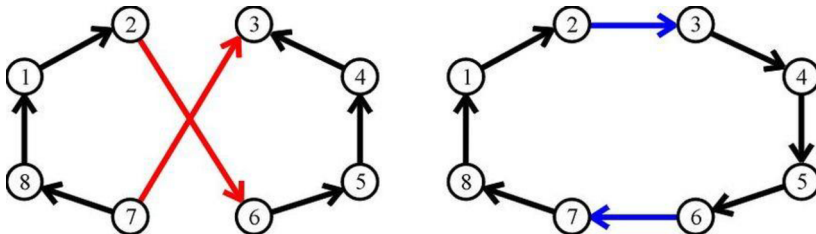
Figure: If two edges cross, we can find a strictly better solution

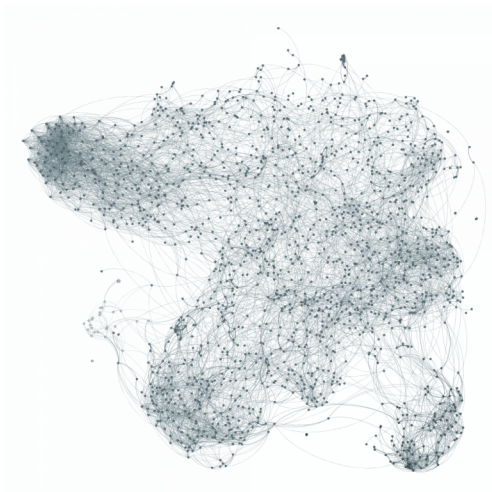TSP: 2-opt visualization

Large TSP

Figure: Heuristics are for large instances

Approximation algorithms

Some heuristics can have theoretical guarantees of their solution quality.

We will study an approximation algorithm for TSP where the cost function satisfies the triangle inequality.

Approximation algorithms

Some heuristics can have theoretical guarantees of their solution quality.

We will study an approximation algorithm for TSP where the cost function satisfies the triangle inequality.

But first, some preliminaries.

Trees

Definition

A **tree** is an undirected graph where any two vertices are connected by exactly one path. Equivalently, it is an undirected graph without cycles.
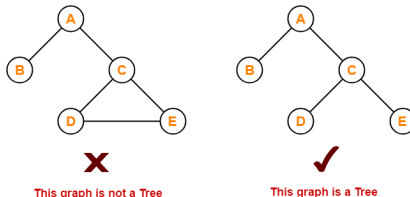
Trees

Definition

A **tree** is an undirected graph where any two vertices are connected by exactly one path. Equivalently, it is an undirected graph without cycles.

X

This graph is not a Tree

✓

This graph is a Tree

Figure: Example of tree and non-tree

Minimum spanning tree

Definition

A **minimum spanning tree** is a subset of the edges of a graph that connects all vertices, without any cycles, such that total edge weight is minimum.
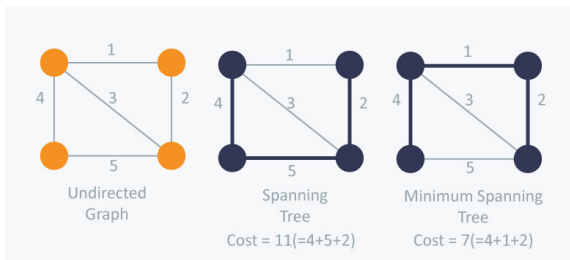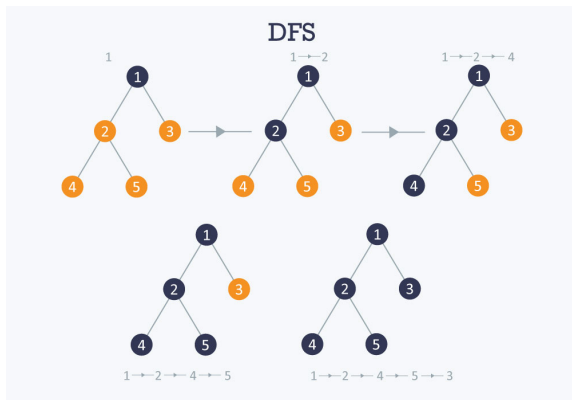
Minimum spanning tree

Definition

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Figure: Spanning Tree vs Minimum spanning tree

Depth-First Search

Depth-First Search is an algorithm for traversing trees.

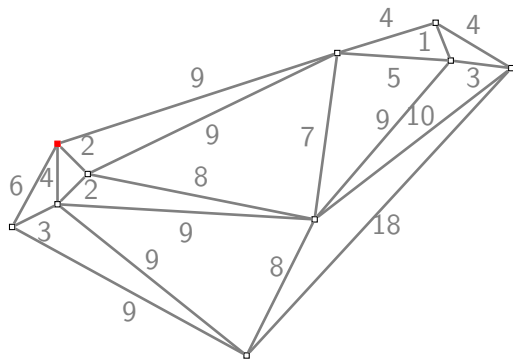Figure: Example of a Depth-First Search

Minimum spanning tree
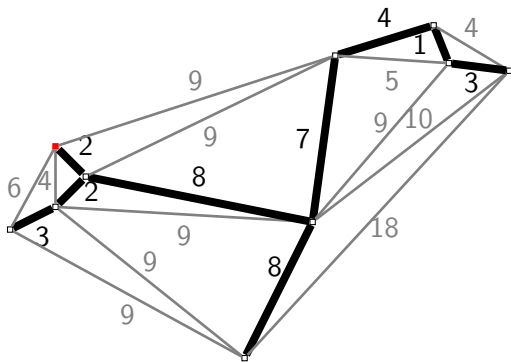

Figure: TSP instance

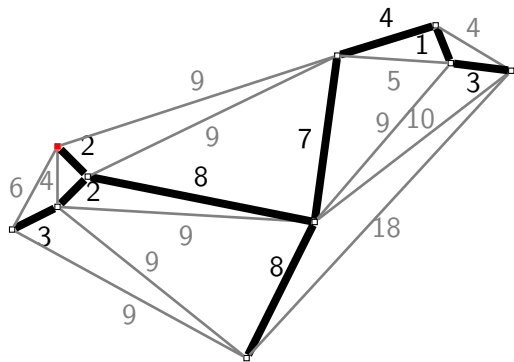# Minimum spanning tree

Figure: Minimal spanning tree

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Figure: DFS

Depth-first search

Figure: DFS

Figure: DFS

Figure: DFS

Figure: DFS

Figure: DFS

Figure: DFS

Figure: DFS
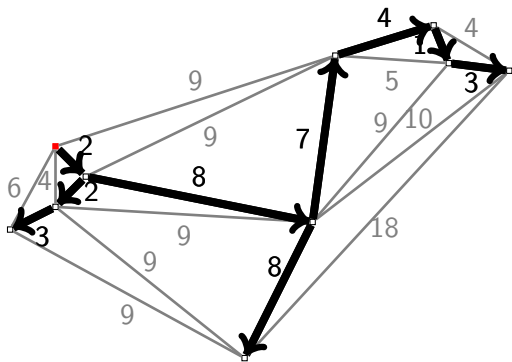
Figure: DFS

Figure: DFS

Approx algorithm
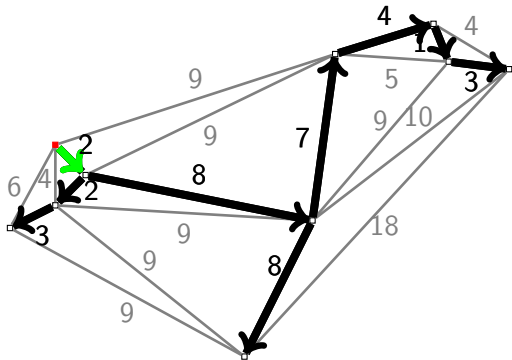
Figure: Approximation algorithm

Approx algorithm

Figure: Approximation algorithm

Figure: Approximation algorithm

Figure: Approximation algorithm

Figure: Approximation algorithm

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The solution given by this heuristic is at most twice as bad as the optimal solution. Why?

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MST is a lower bound. The solution is less than twice the MST, because of the assumption of the triangle inequality.

Approximation ratio

The solution given by this heuristic is at most twice as bad as the optimal solution. Why?

MST is a lower bound. The solution is less than twice the MST, because of the assumption of the triangle inequality. We say it is a 2-approx algorithm.

SCIP heuristics

Modern solvers employ a lot of heuristics.

Heuristics can often get stuck in local optima. How can we deal with this?

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Potentially accept worsening solutions

Metaheuristics

Heuristics can often get stuck in local optima. How can we deal with this?

Potentially accept worsening solutions

Metaheuristics are abstractions that work with sets of solutions. Eg: TSP paths instead of cities.

Main idea
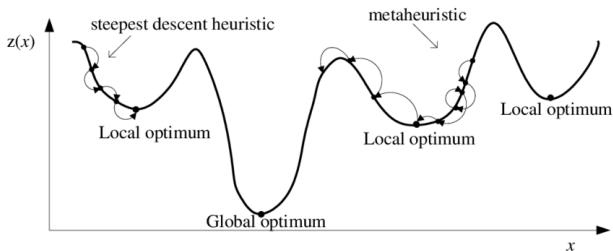
Figure: Escape local optima by accepting worsening moves

Simulated Annealing

Randomly decide to move to a neighboring solution based on its fitness. The probabilities decrease with time.

Traveling Salesman Problem Visualization

Simulated Annealing
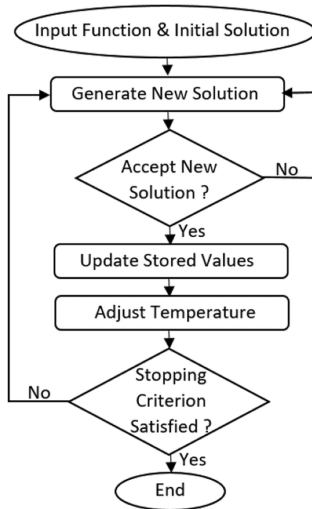
Figure: Simulated Annealing Flowchart

Tabu Search

Initially accept worsening solutions.
Keep a tabu list that forbids recently visited solutions.

Tabu Search
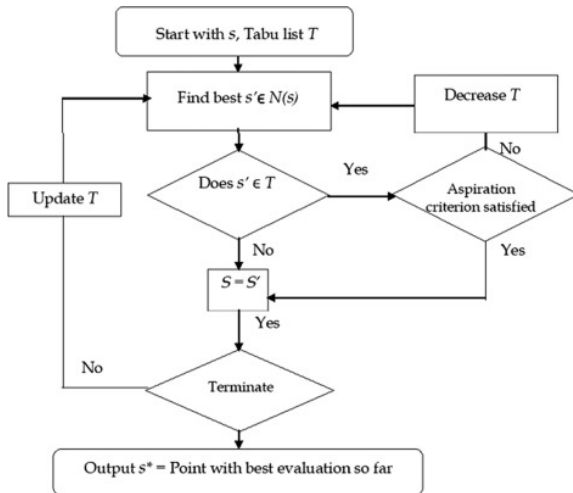
Figure: Tabu Search Flowchart

Genetic Algorithm

Inspired by natural processes, keeps a population of candidate solutions. Iteratively combines them to create new solutions.

Genetic Algorithm
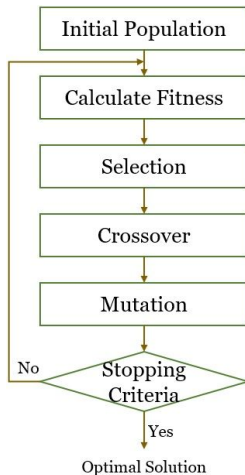
Figure: Genetic Algorithm Flowchart

Visualization of metaheuristics for the traveling salesman problem

Reformulations
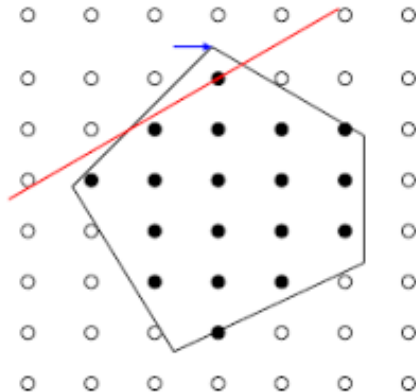
Figure: Integer programs admit infinite formulations

If formulation A is contained in formulation B, then the former is preferred. Why is that?

Reformulations

If formulation A is contained in formulation B, then the former is preferred. Why is that?

The provided bound is better, can prune branch and bound tree earlier.

Convex Hull

Definition

Let $X \subseteq \mathbb{R}^n$ be a set. The **convex hull** of X, denoted by $conv(X)$, is the intersection of all convex sets containing X. Equivalently, it is the set of all convex combinations of X.

Perfect Formulation

(a) Input.

(b) Output.
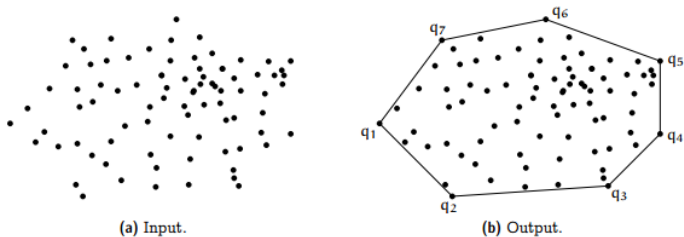
Figure: Linear relaxation equal to convex hull

Perfect Formulation
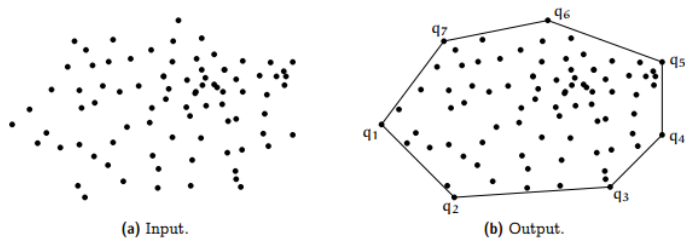
Figure: Linear relaxation equal to convex hull

Which points do we need to check?

Perfect Formulation
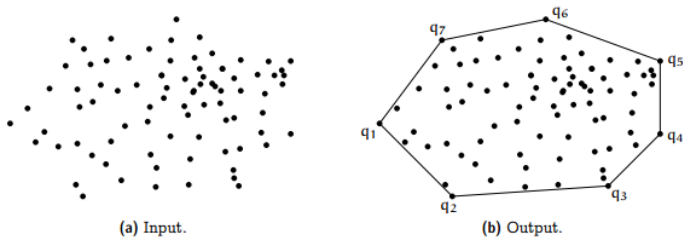
Figure: Linear relaxation equal to convex hull

Which points do we need to check? Why?

In theory, every integer program has an equivalent linear programming formulation. Why?

Cutting planes

In theory, every integer program has an equivalent linear programming formulation. Why?

Convex hull, linear relaxation is upper bound, hence optimal solution at vertex.

Throughout the solving process, we keep adding cuts to contract the feasible region.

However, a weaker formulation may sometimes be beneficial. Why?

However, a weaker formulation may sometimes be beneficial. Why? Stronger formulations tend to require more constraints, hence the linear relaxation is harder. A trade-off is needed.

Symmetry

The solution space can exhibit a lot of symmetry (equivalent solutions modulo a permutation of the variables, for example). Especially damaging in integer optimization.

Symmetry
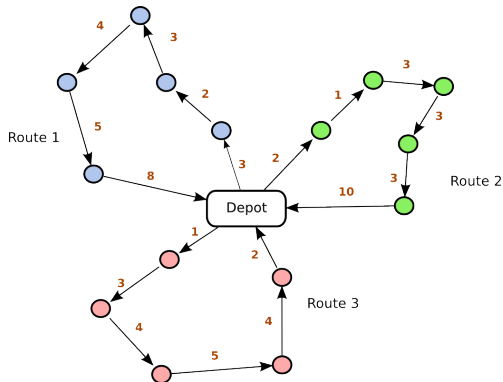
Figure: The vehicle routing problem has a lot of symmetry

Symmetry in chess
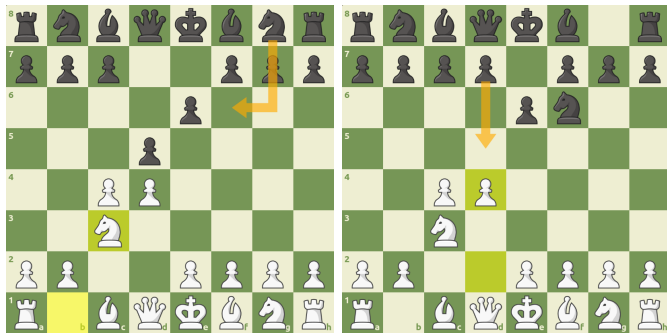
Figure: The same position can be reached from the Queen's Gambit and the English

Symmetry is common in chess. Engines need to record previously studied board positions to avoid wasting time with transpositions.

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Presolving

Presolving: Analyze the problems and reduce the solution space by identifying symmetry, redundant or implicit variables, etc.

For example:

- $x \le 1.5, y \ge 0.5, x + y \le 1 \implies x \le 0.5$
- $x \le 1.5, x \in \mathbb{Z} \implies x \le 1$

Presolving

```
presolving (26 rounds: 26 fast, 3 medium, 3 exhaustive):
 46 deleted vars, 569 deleted constraints, 0 added constraints, 12680 tightened bounds, 0 add
 937 implications, 100 cliques
presolved problem has 2982 variables (120 bin, 0 int, 0 impl, 2862 cont) and 5338 constraints
```

Figure: Example of SCIP presolving

Logical Constraints

With binary variables, we can model some logical constraints.

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$$\neg x$$

Logical Constraints

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$$\begin{array}{c|c} \neg x & 1 - x \\ x \implies y & \end{array}$$

Logical Constraints

With binary variables, we can model some logical constraints.

$$
\begin{array}{c|c}
\neg x & 1 - x \\
x \implies y & x \leq y \\
x \wedge y &
\end{array}
$$

Logical Constraints

With binary variables, we can model some logical constraints.

$$
\begin{array}{c|c}
\neg x & 1 - x \\
x \implies y & x \leq y \\
x \wedge y & x + y = 2 \\
x \vee y & x + y \geq 1 \\
x \,\dot{\vee}\, y & x + y = 1 \\
\exists x &
\end{array}
$$

Logical Constraints

With binary variables, we can model some logical constraints.

$$
\begin{array}{c|c}
\neg x & 1 - x \\
x \implies y & x \leq y \\
x \wedge y & x + y = 2 \\
x \vee y & x + y \geq 1 \\
x \mathbin{\dot\vee} y & x + y = 1 \\
\exists x & \sum_{i=1}^{n} x_i \geq 1 \\
\exists! x & \sum_{i=1}^{n} x_i = 1
\end{array}
$$

Table: Formulating logical expressions with integer programming

Suggestions
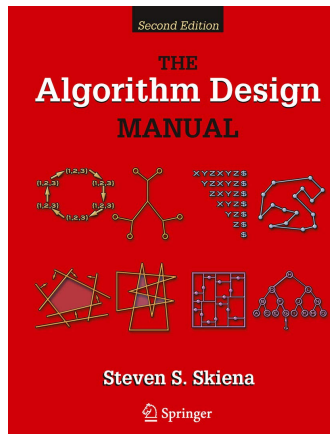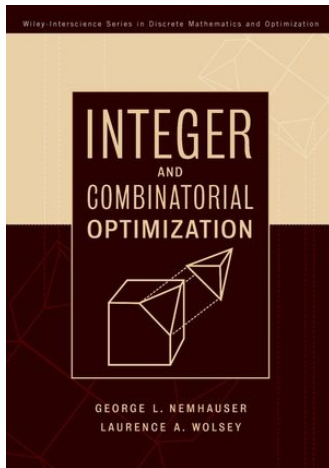
Light read on integer programming: Link

Discrete Optimization with Professor Pascal Van Hentenryck, on Coursera: Link

Exercises

Revisiting Cutting-Stock

$$\min \quad \sum_{j=1}^{M} y_j$$

$$\text{s.t.} \quad \sum_{j=1}^{M} x_{ij} = d_i, \qquad\qquad i \in [n]$$

$$\sum_{i=1}^{n} l_i x_{ij} \leq L, \qquad\qquad j \in [m]$$

$$x_{ij} \leq d_i y_j, \qquad i \in [n], j \in [m]$$

$$x_{ij} \in \mathbb{Z}^+, \qquad i \in [n], j \in [m]$$

$$y_j \in \{0, 1\}, \qquad\qquad j \in [m]$$

Revisiting Cutting-Stock
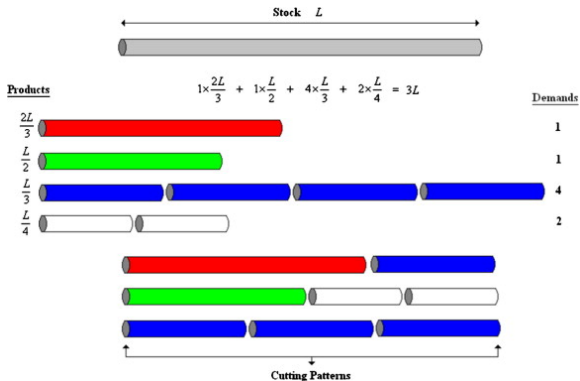
Figure: Cutting stock example

How can we cut the sheet?
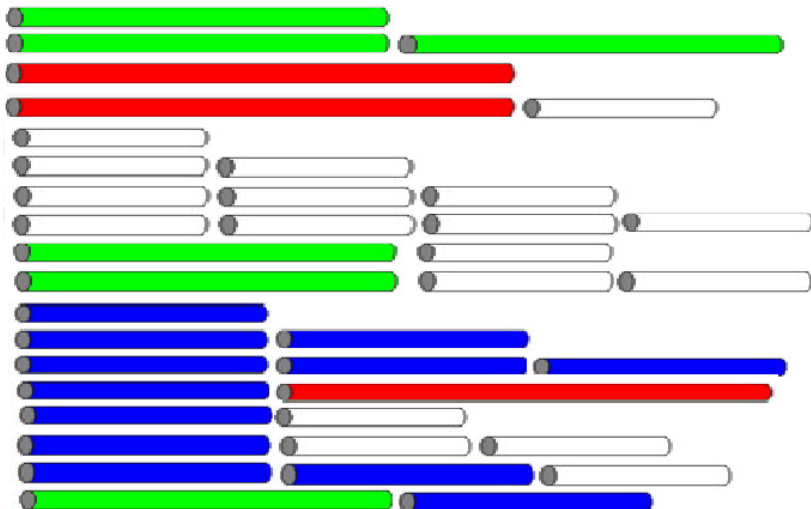
Figure: All possibilities of cutting the sheet into the orders (with Paint!)

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Equivalent formulation

Idea: Of all possible combinations, pick the best ones.

Let X be the set of all feasible patterns, and $p_{i,x}$ the number of orders of type i in pattern x.

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$$\min_x \quad \sum_{x \in X} x$$
$$\text{s.t.} \quad \sum_{x \in X} x p_{i,x} = d_i, \forall i$$
$$x \in \mathbb{Z}_0^+$$

Equivalent formulation

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Let X be the set of all feasible patterns, and $p_{i,x}$ the number of orders of type i in pattern x.

$$
\begin{aligned}
\min_{x} \quad & \sum_{x \in X} x \\
\text{s.t.} \quad & \sum_{x \in X} x p_{i,x} = d_i, \forall i \\
& x \in \mathbb{Z}_0^+
\end{aligned}
$$

Problem: This has an exponential number of possibilities. The problem is even harder!

Notice that the majority of patterns are not used (not a
coincidence...)

Restricted Master Problem

Notice that the majority of patterns are not used (not a coincidence...)

Main idea: Start with a small subset of patterns and solve the easy problem. Add new patterns that will improve the solution.

Which patterns should we add?

Suppose we are restricted to these patterns:

Figure: Example of a bad set of patterns

The blue orders would contribute to the solution more than they should! We need better patterns with blue orders.

Pricing Problem

Which new pattern should we add? We want patterns that:

- Prioritize orders that are harming the solution
- Do not exceed the length of the metal sheet

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- Priority orders $\rightarrow$ Valuable items

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- Priority orders $\rightarrow$ Valuable items
- Order length $\rightarrow$ Item weight

Pricing Problem

Which new pattern should we add? We want patterns that:

- Prioritize orders that are harming the solution
- Do not exceed the length of the metal sheet

This is precisely a knapsack problem!

- Priority orders $\rightarrow$ Valuable items
- Order length $\rightarrow$ Item weight
- Sheet length $\rightarrow$ Knapsack capacity

Cutting-Stock and Knapsack
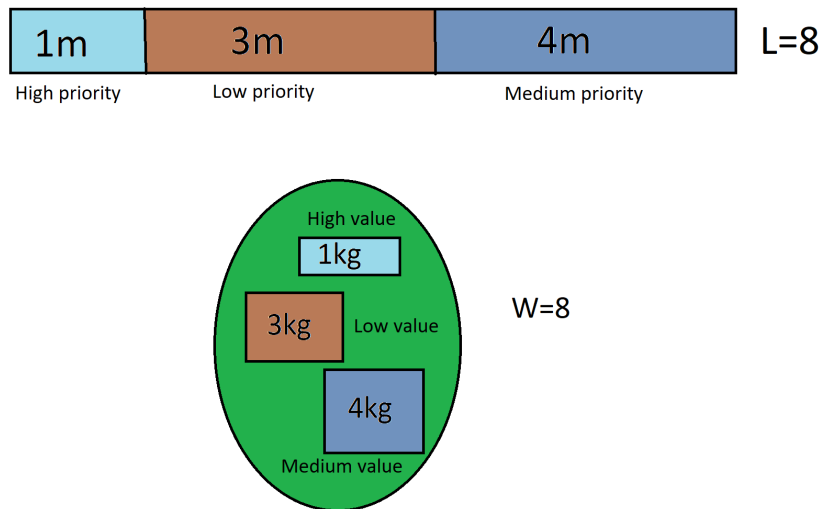
Figure: Cutting stock pattern and knapsack instance

How should we decide the prices for the items?

How should we decide the prices for the items?

The dual values of the constraints can give us the answer! In the context of this problem, the dual associated with the demand of item i tells us the cost that a slightly increased demand would have on the number of used sheets.

$$\max_{y} \quad b^{\mathsf{T}}\pi$$
$$\text{s.t.} \quad A^{\mathsf{T}}\pi \geq c$$
$$\pi \geq 0$$

Recall that the dual variable associated with a constraint gives us the infinitesimal increase of the objective per infinitesimal increase of the right-hand side of the constraint.

E.g.: constraint $f(z) \leq b$ having a dual of -0.03 means that (more or less), $f(z) \leq b + 1$ would decrease the objective by 0.03.

When do we stop?

We know (very informally) what the benefit of using the generated pattern would be ($\pi^\mathsf{T} x$). We know what the cost of using it would be as well. By using a new pattern, we are using a new roll, thus worsening the solution by 1.

We stop the process when the benefit of the best new pattern doesn't outweigh the cost of using it. So, when $1 - \pi^\mathsf{T} x \geq 0$.

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1. Initialize problem with a small subset of patterns

The idea

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2. Optimize the RMP to get dual values

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The idea

1. Initialize problem with a small subset of patterns
2. Optimize the RMP to get dual values
3. Solve the knapsack subproblem
4. If the best pattern cannot improve RMP, optimal solution found
5. Else, add the new pattern to RMP and go to step 2

I lied a little bit here, but I will explain things later.

PySCIPOpt

```
9 orders  | Time original: 0.35 | Time col. generation: 0.04 | Time difference: 0.31
18 orders | Time original: 1.90 | Time col. generation: 0.02 | Time difference: 1.88
19 orders | Time original: 0.77 | Time col. generation: 0.02 | Time difference: 0.75
22 orders | Time original: 4.59 | Time col. generation: 0.01 | Time difference: 4.58
11 orders | Time original: 0.58 | Time col. generation: 0.01 | Time difference: 0.57
11 orders | Time original: 0.52 | Time col. generation: 0.04 | Time difference: 0.48
20 orders | Time original: 7.31 | Time col. generation: 0.03 | Time difference: 7.28
22 orders | Time original: 7.56 | Time col. generation: 0.02 | Time difference: 7.55
14 orders | Time original: 0.40 | Time col. generation: 0.05 | Time difference: 0.36
20 orders | Time original: 1.07 | Time col. generation: 0.05 | Time difference: 1.02
```

Figure: Example of speed of reformulation in some instances

Big linear programs tend to only use a small subset of columns in the optimal solution.

Column Generation

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Column generation idea:

- Start with a small subset of columns. Optimize the resulting problem.

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Column generation idea:

- Start with a small subset of columns. Optimize the resulting problem.
- Use dual values of the optimal solution to generate new columns
- If no columns can improve the solution, it is optimal

Column generation
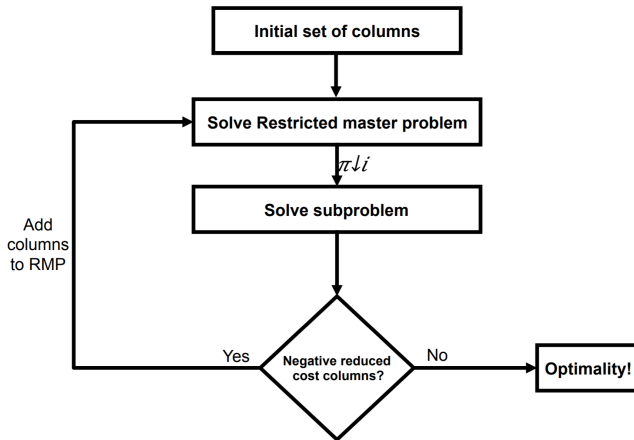
Figure: Column generation Flowchart

Master Problem

With the available columns, pick the ones that optimize the objective.

$$\begin{aligned} \min \quad & c^{\mathsf{T}} x \\ \text{s.t.} \quad & \text{Column satisfies original constraints} \end{aligned}$$

Out of all available columns, pick one that improves the solution of the restricted master problem. How? Pick the one with the most negative reduced cost.

$$\min \quad c - A^\mathsf{T}\pi$$
$$\text{s.t.} \quad \text{Column satisfies original constraints}$$

Column generation Tips

- The pricing problem should be easy.
- Solve the pricing problem heuristically. How many times does it need to be solved optimally?
- Add multiple columns per iteration.
- Remove columns from the RMP that have been inactive for many iterations.

Column generation is only applicable to linear problems[1] (duality).
We then need to branch on fractional variables/columns/patterns.
We do this by embedding column generation in a
Branch-and-Bound tree.

One of the reasons it works so well in the cutting-stock problem is
that the linear relaxation of the knapsack is so easy.

---

[1]This was the lie I talked about earlier.

Branch-and-Price diagram
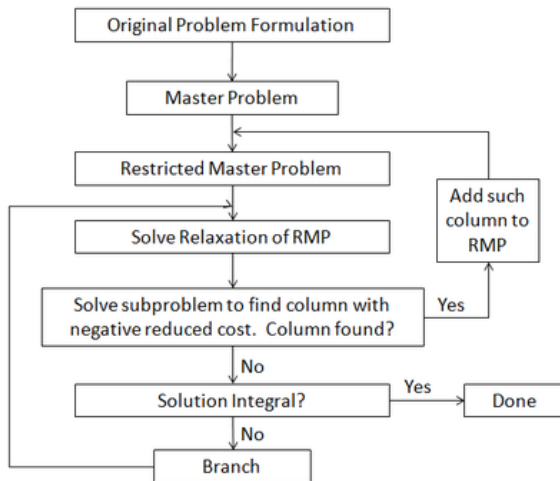
Figure: Branch-and-price diagram

Problem structure

Sometimes difficult problems have a structure that can be explored.

For example, there may be a set of constraints that are difficulting the problem:

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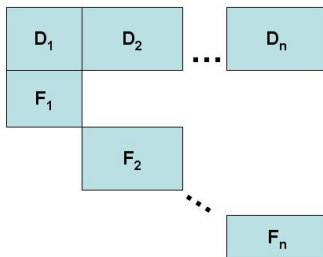
Problem structure

Other times it's a set of variables that are leading to a difficult problem.

$$
\begin{array}{llll}
\min & a^{\mathsf T}x & +b^{\mathsf T}y + c^{\mathsf T}z & \\
\text{s.t.} & Ax & & \leq B \\
& Cx & +Dy & \leq E \\
& Fx & +Gy + Hz & \leq I \\
& & x, y, z \geq 0 &
\end{array}
$$

Dantzig-Wolfe Decomposition

When can we use column generation? Is there a systematic way to do it?

Dantzig-Wolfe Decomposition

When can we use column generation? Is there a systematic way to do it?

Yes, with a reformulation of the problem called Dantzig-Wolfe decomposition. First, some preliminaries.

Minkowsi-Weyl Theorem

Theorem

Let $P \subseteq \mathbb{R}^n$ be a bounded convex polyhedron. Then there is a finite set Q such that $P = conv(Q)$.

Minkowsi-Weyl Theorem

Theorem

Let $P \subseteq \mathbb{R}^n$ be a bounded convex polyhedron. Then there is a finite set Q such that $P = conv(Q)$.

In other words

$$P = \{x \in R^n \mid x = \sum_{q \in Q} \lambda_q x_q, \sum_q \lambda_q = 1, \lambda \geq 0\}$$

So, a bounded convex polyhedron can either be represented by the intersection of half-spaces (constraints), or by a convex combination of its extreme points. These representations are equivalent.

Note

Both the theorem and the following methods work for unbounded
polyhedra, when including conic combinations of extreme rays.
However, this would introduce unnecessary complexity, so I will
present things like this. Nevertheless, I leave here the definition of
extreme rays.

Definition

A ray (semirreta) r of a polyhedron P is said to be an **extreme
ray** if

$$\nexists r_1, r_2 \in P, \lambda \in]0, 1[|\ r = \lambda r_1 + (1 - \lambda) r_2$$

Where r_1, r_2 are rays.
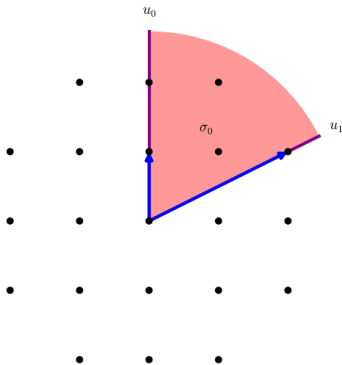
Figure: Example of extreme rays

Imagine an LP with a set of easy and a set of complicating constraints:

$$\begin{aligned} \min \quad & c^\mathsf{T}x \\ \text{s.t.} \quad & Ax \geq b \\ & Dx \geq d \\ & x \geq 0 \end{aligned}$$

For example, a problem with linking constraints fits this description.

Figure: Example of a problem with hard and easy constraints

Both the hard and the easy constraints define a polyhedron. Their intersection is the feasible region.

Dantzig-Wolfe problem visualization
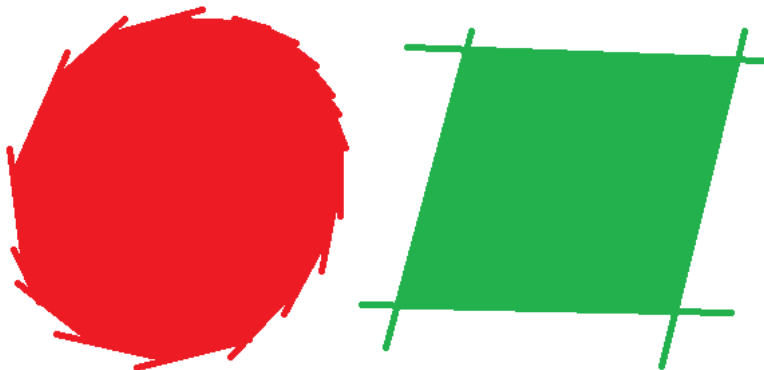
Figure: The corresponding polyhedrons

Dantzig-Wolfe reformulation

Dantzig-Wolfe reformulation is the reformulation of the easy constraints of the LP above to the extreme points representation. We do this for the set $X = \{x \geq 0 \mid Dx \geq d\}$.

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With extreme points $Q = \{x_1, \ldots, x_{|Q|}\}$

$$x = \sum_{q \in Q} \lambda_q x_q$$

$$\sum_{q \in Q} \lambda_q = 1$$

$$\lambda_q \geq 0$$

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With extreme points $Q = \{x_1, \ldots, x_{|Q|}\}$

$$x = \sum_{q \in Q} \lambda_q x_q$$

$$\sum_{q \in Q} \lambda_q = 1$$

$$\lambda_q \geq 0$$

Remember, this is characterizing every point in X.

Replacing x in the original LP with this representation we get

$$
\begin{aligned}
\min \quad & c^\mathsf{T} \left(\sum_{q \in Q} \lambda_q x_q \right) \\
\text{s.t.} \quad & A \left(\sum_{q \in Q} \lambda_q x_q \right) \geq b \\
& \sum_{q \in Q} \lambda_q = 1 \\
& \lambda_q \geq 0
\end{aligned}
$$

Dantzig-Wolfe master problem

Replacing x in the original LP with this representation we get

$$
\begin{aligned}
\min \quad & c^{\mathsf{T}} \left(\sum_{q \in Q} \lambda_q x_q \right) \\
\text{s.t.} \quad & A \left(\sum_{q \in Q} \lambda_q x_q \right) \geq b \\
& \sum_{q \in Q} \lambda_q = 1 \\
& \lambda_q \geq 0
\end{aligned}
$$

This is equivalent to the original problem. Why?

Dantzig-Wolfe master problem

Rearranging the terms and defining $c_q := c^\mathsf{T} x_q$, $a_q := A x_q$ we get the Dantzig-Wolfe master problem:

$$
\begin{aligned}
\min \quad & \sum_{q \in Q} c_q \lambda_q \\
\text{s.t.} \quad & \sum_{q \in Q} a_q \lambda_q \geq b \\
& \sum_{q \in Q} \lambda_q = 1 \\
& \lambda_q \geq 0
\end{aligned}
$$

Restricted master problem

This problem is gigantic (there is a very large number of extreme points). So we need to apply column generation. So, we start with a small subset of extreme points.

$$
\begin{aligned}
\min \quad & \sum_{q \in Q'} c_q \lambda_q \\
\text{s.t.} \quad & \sum_{q \in Q'} a_q \lambda_q \geq b \quad (\pi) \\
& \sum_{q \in Q'} \lambda_q = 1 \quad (\pi_0) \\
& \lambda_q \geq 0, \\
& \text{with } |Q'| << |Q|.
\end{aligned}
$$

DW-decomposition pricing problem

Which column should we add? The one with most negative
reduced cost:

$$\min_{x_q \in Q} \left(c^\mathsf{T} - \pi^\mathsf{T} A \right) x_q$$

DW-decomposition pricing problem

Which column should we add? The one with most negative reduced cost:

$$\min_{x_q \in Q} (c^\mathsf{T} - \pi^\mathsf{T} A) \, x_q$$

Which is equivalent to:

$$
\begin{aligned}
\min \quad & (c^\mathsf{T} - \pi^\mathsf{T} A) \, x \\
\text{s.t.} \quad & Dx \geq d \\
& x \geq 0
\end{aligned}
$$

This is an easy problem!

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Cases for pricing problem

- Optimal solution < 0. We identified an extreme point that improves the solution. Add column $(A x_{q^*} \; 1)^\mathsf{T}$ to master problem.
- Optimal solution ≥ 0. We found the optimal solution.

The restricted master problem is very easy (very few columns).
Generating new columns is also easy. Why?

Why does this work?

The restricted master problem is very easy (very few columns). Generating new columns is also easy. Why?

Because we generate columns by solving the pricing problem, which corresponds to the easy constraints!

Solving the DW-reformulation, we get the solution in terms of the extreme points. Projecting back to the original variables is easy:

$$x = \sum_{q \in Q} \lambda_q x_q$$

Suggestions

Light read on column-generation: Link

A more in-depth explanation of column-generation/Dantzig-Wolfe:
Link

Exercises

What if instead of easy subproblems linked by a few constraints, we have a difficult problem that becomes easy if we fix a few variables?

Benders' Decomposition

What if instead of easy subproblems linked by a few constraints, we have a difficult problem that becomes easy if we fix a few variables?

For example, a MIP with integer variables fixed, or a problem with linking variables.

Divide the problem into two subproblems. The first problem (the outer problem), keeps picking the values to fix the difficult variable.

Benders' Decomposition Idea

Divide the problem into two subproblems. The first problem (the outer problem), keeps picking the values to fix the difficult variable. The second problem (the inner problem), uses duality to tell the outer problem which values for the variable fixing will work best.

Benders' Decomposition Idea

Divide the problem into two subproblems. The first problem (the outer problem), keeps picking the values to fix the difficult variable. The second problem (the inner problem), uses duality to tell the outer problem which values for the variable fixing will work best.

They keep interacting until they find an ϵ-optimal solution.

Algorithm Idea
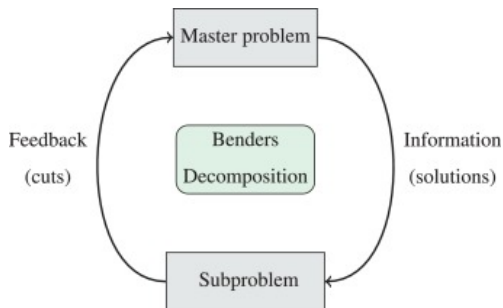
Figure: Benders' Decomposition Idea

Benders' Decomposition

Consider the following problem:

$$\min_{x,y} \quad c^\mathsf{T}x + d^\mathsf{T}y$$
$$\text{s.t.} \quad Ax + By \geq b$$
$$y \in Y$$
$$x \geq 0$$

Suppose y are variables that difficult the problem.

Min-Max formulation

This problem is equivalent to optimizing over the y variables first, and then over the x variables (a theorem).

$$\min_{y \in Y}[d^\mathsf{T} y + \min_{x \geq 0}\{c^\mathsf{T} x \mid Ax + By \geq b\}]$$

And thus we have two problems, the outer and the inner optimization problem.

Fixing y to $\overline{y}$ in the inner problem, we get the following easy problem (LP).

$$
\begin{aligned}
\min_{x} \quad & c^{\mathsf{T}}x + d^{\mathsf{T}}\overline{y} \\
\text{s.t.} \quad & Ax + B\overline{y} \geq b \\
& x \geq 0
\end{aligned}
$$

Benders' Decomposition

The inner problem with a fixed y has the following dual:

$$
\min_{u} \quad (b - B\overline{y})^{\mathsf{T}} u + d^{\mathsf{T}} \overline{y}
$$
$$
\text{s.t.} \quad A^{\mathsf{T}} u \leq c
$$
$$
u \geq 0
$$

Benders' Decomposition

The inner problem with a fixed y has the following dual:

$$\min_{u} \quad (b - B\overline{y})^{\mathsf{T}} u + d^{\mathsf{T}} \overline{y}$$
$$\text{s.t.} \quad A^{\mathsf{T}} u \leq c$$
$$u \geq 0$$

Recall that in LP the dual and the primal have the same optimal solution.

Benders' Decomposition

Thus, the min-max formulation is equivalent to:

$$\min_{y \in Y}[d^\mathsf{T}y + \max_{u \geq 0}\{(b - By)^\mathsf{T}u \mid A^\mathsf{T}u \leq c\}]$$

Outer and inner problem

Outer problem: Pick y's from Y, initially with no regard for feasibility or optimality.

Inner Problem: With a fixed y use duality theory to add constraints to the outer problem, saying which y's have been infeasible, which have led to bad solutions, and which have led to good solutions.

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Outer problem: Pick y's from Y, initially with no regard for feasibility or optimality.

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How can the inner problem provide that information?

Duality revisited

Proposition

Let P be an LP and D the corresponding dual. We have the following:

1. *D is infeasible $\implies$ P is unbounded*
2. *D is unbounded $\implies$ P is infeasible*
3. *D has an optimal solution y^* $\implies$ P has an optimal solution x^* and $b^\mathsf{T} y \le c^\mathsf{T} x^*$*

Proof

3 is simply the weak duality theorem, already proven. As the dual of the dual is the primal problem, then we have that $1 \iff 2$. Thus, all we need to prove is that $3 \implies 2$

Proof

3 is simply the weak duality theorem, already proven. As the dual of the dual is the primal problem, then we have that $1 \iff 2$. Thus, all we need to prove is that $3 \implies 2$

P a primal problem, and D its unbounded dual. Suppose P has a feasible solution x. By weak duality, $c^\mathsf{T} x \geq b^\mathsf{T} y$ for all feasible y. But then D is not unbounded. Absurd.

Optimality cuts

If the inner problem is feasible, we know that the optimal solution is at least as good as this one.

$$z \geq (b - By)^{\mathsf{T}}\overline{u} + d^{\mathsf{T}}y$$

Add an *optimality cut* to ensure the outer problem picks betters y's.

Feasibility cuts

If the inner problem is unbounded, there is no x such that the original problem is feasible. We know of an extreme ray u that achieves this unboundedness.

$$(b - By)^{\mathsf{T}}\overline{u} \leq 0$$

Add a *feasibility cut* to ensure that the outer problem doesn't pick this or related y's.

When does it stop?

The inner problem gives us lower bounds LB - actual solutions. The outer problem gives us upper bounds UB. Stop when their difference is smaller than a predetermined tolerance, $UB - LB \leq \varepsilon$.

At each iteration, the outer problem becomes more and more constrained. Thus, this procedure is called *row generation*.

Benders' Decomposition Diagram
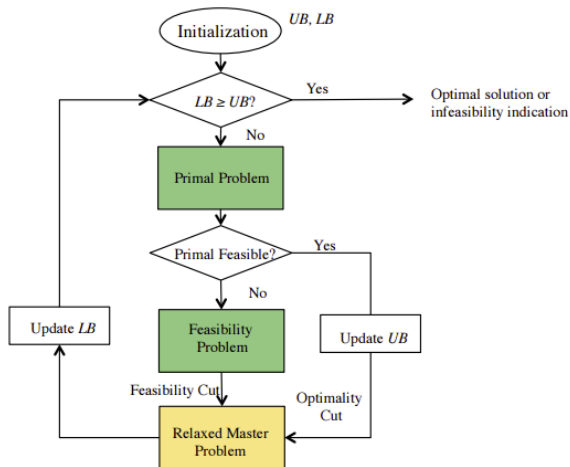
Let us exemplify the method by solving a simple problem. The example is taken from this webpage: Link.
Consider the MIP in the following slide.

Example

$$
\begin{aligned}
\max_{x,y} \quad & 8x_1 + 9x_2 + 5x_3 + 6x_4 - 15y_1 - 10y_2 \\
\text{s.t.} \quad & x1 \qquad\quad + x_3 \qquad\qquad\qquad\qquad\qquad \leq 1 \\
& x1 \qquad\qquad\qquad\quad +x_4 \qquad\qquad\qquad \leq 1 \\
& \qquad\quad x_2 + x_3 \qquad\qquad\qquad\qquad\quad \leq 1 \\
& \qquad\quad x_2 \qquad\qquad +x_4 \qquad\qquad\qquad \leq 1 \\
& \qquad\qquad\qquad\qquad\qquad\quad - y_1 \quad -y_2 \leq -1 \\
& x_1 \qquad\qquad\qquad\qquad\qquad - y_1 \qquad\quad \leq 0 \\
& \qquad\quad x_2 \qquad\qquad\qquad\qquad - y_1 \qquad\quad \leq 0 \\
& \qquad\qquad x_3 \qquad\qquad\qquad\qquad\qquad -y_2 \leq 0 \\
& \qquad\qquad\qquad\qquad x_4 \qquad\qquad\qquad -y_2 \leq 0 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x \geq 0 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\quad y \in \{0,1\}
\end{aligned}
$$

Example

Hard variables: y. Initial restricted master problem (RMP):

$$\max_{y,z} \quad z - 15y_1 - 10y_2$$
$$\text{s.t.} \quad z \leq 28$$
$$y \in \{0, 1\}$$

Optimal solution is $z = 28, y_1 = y_2 = 0$, with objective 28. Upper bound ≤ 28.

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Example

Dual of original problem

$$
\begin{aligned}
\min_{u} \quad & u_1 + u_2 + u_3 + u_4 - u_5 \\
\text{s.t.} \quad & u_1 + u_2 && + u_6 && && \geq 8 \\
& u_3 + u_4 && + u_7 && && \geq 9 \\
& u_1 + u_3 && && + u_8 && \geq 5 \\
& u_2 + u_4 && && u_9 && \geq 6 \\
& && && && u \geq 0
\end{aligned}
$$

Unbounded solution, with ray $r^1 = (0,0,0,0,1,0,0,0,0)^\mathsf{T}$. Lower Bound $\geq -\infty$.

The chosen y leads to an unbounded dual inner problem. Thus, the inner problem is infeasible. So we must exclude the chosen y. How?

The chosen y leads to an unbounded dual inner problem. Thus, the inner problem is infeasible. So we must exclude the chosen y. How?

We know a direction that makes the problem unbounded. Forbid it by adding constraint $r^1 By \leq b^\intercal r^1$ to RMP.

Example

$$
\begin{aligned}
\max_{z} \quad & z \; -15y_1 - 10y_2 \\
\text{s.t.} \quad & z && \leq 28 \\
& -y_1 - y_2 && \leq -1 \\
& && y \in \{0,1\}
\end{aligned}
$$

Optimal solution is $z = 28, y_1 = 0, y_2 = 1$, with objective 18.
Upper Bound ≤ 18. Update the objective of the inner problem as
$(b - H^\intercal y)^\intercal u$.

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Example

$$
\begin{array}{rllll}
\min_{u} & u_1 & +u_2 + u_3 & +u_4 & & +u_8 + u_9 \\
\text{s.t.} & u_1 & +u_2 & +u_6 & & \geq 8 \\
& & u_3 & +u_4 & +u_7 & \geq 9 \\
& u_1 & +u_3 & & +u_8 & \geq 5 \\
& & u_2 & +u_4 & u_9 & \geq 6 \\
& & & & u \geq 0
\end{array}
$$

Optimal solution $u^1 = (0, 0, 0, 0, 0, 8, 9, 5, 6)^\intercal$, with objective 11.
Add constraint $z + (u^1)^\intercal By \leq b^\intercal u^1$ to RMP.

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Example

The optimal of the dual inner problem being 11 means that for $y_1 = 0, y_2 = 1$, the optimum is 11 in the primal as well. We can recover the solution in terms of the x variables, meaning that we have a feasible solution for the original problem. Its objective is $1(11 - 15\overline{y_1} - 10\overline{y_2})$. Thus, objective ≥ 1.

Exercise

$$\max_{z} \quad z \qquad -15y_1 - 10y_2$$

$$\text{s.t.} \quad z \qquad\qquad\qquad\qquad \le 28$$

$$\qquad\qquad\qquad -y_1 - y_2 \qquad \le -1$$

$$\qquad z \qquad -17y_1 - 11y_2 \qquad \le 0$$

$$\qquad y \in \{0, 1\}$$

Optimal solution is $z = 28, y_1 = 1, y_2 = 1$, with objective 3.

Update the objective of the inner problem as $(b - H^\mathsf{T} y)^\mathsf{T} u$.

Objective ≤ 3.

Exercise

$$\min_{u} \quad u_1 + u_2 + u_3 + u_4 + u_5 + u_6 + u_7 + u_8 + u_9$$

$$
\begin{array}{llllll}
\text{s.t.} & u_1 + u_2 & & + u_6 & & \geq 8 \\
& u_3 + u_4 & & + u_7 & & \geq 9 \\
& u_1 + u_3 & & & + u_8 & \geq 5 \\
& u_2 + u_4 & & & u_9 & \geq 6 \\
& & & & u \geq 0 &
\end{array}
$$

Optimal solution $u^2 = (8, 0, 9, 0, 0, 0, 0, 0, 0)^\intercal$, with objective 17. Add constraint $z + (u^2)^\intercal By \leq b^\intercal u^2$ to RMP. Found solution for the initial problem with value $-8 (= 17 - 15\overline{y_1} - 10\overline{y_2})$.

Exercise

$$
\begin{aligned}
\max_{z} \quad & z \quad -15y_1 - 10y_2 \\
\text{s.t.} \quad & z && \leq 28 \\
& -y_1 - y_2 && \leq -1 \\
& z \quad -17y_1 - 11y_2 && \leq 0 \\
& z && \leq 17 \\
& && y \in \{0, 1\}
\end{aligned}
$$

Optimal solution is $z = 17, y_1 = 1, y_2 = 0$, with objective 2.
Update the objective of the inner problem as $(b - H^\mathsf{T} y)^\mathsf{T} u$.
Objective ≤ 2.

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Exercise

$$\min_{u} \quad u_1 + u_2 + u_3 + u_4 + u_6 + u_7$$

$$
\begin{array}{lllllll}
\text{s.t.} & u_1 & +u_2 & & & +u_6 & & & & \geq 8 \\
& & & u_3 & +u_4 & & +u_7 & & & \geq 9 \\
& u_1 & & +u_3 & & & & +u_8 & & \geq 5 \\
& & u_2 & & +u_4 & & & & u_9 & \geq 6 \\
& & & & & & & & u \geq 0 &
\end{array}
$$

Solution u^2 still optimal, with objective 17. Thus the optimal objective is indeed $2 (= 17 - 15\overline{y_1} - 10\overline{y_2})$.

Light read on Benders Decomposition: Link.

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Annex A - Dynamic Programming

Let us take a quick detour to talk about the standard method for solving the knapsack problem.

$$\max \quad \sum_{i=1}^{n} v_i x_i$$
$$\text{s.t.} \quad \sum_{i=1}^{n} w_i x_i \leq W$$
$$x_i \in \{0, 1\}$$

Pick which items to take in order to maximize total value while not exceeding the knapsack's capacity.

Dynamic Programming

- Look at the problem recursively.
- Use a table to store previously found solutions.
- See whether using a new item improves the result.

The table B will store the best solution using items from 1 to n which uses at most 1 to W total weight. The entry $B_{i,j}$ refers to the best solution using items 1 to i, and at most weight j.

Dynamic Programming in Knapsack

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For example, entry $B_{3,10}$ tells us the best solution which uses items $1, 2, 3$ whose weight does not exceed 10.

Where can we find the optimal solution?

Dynamic Programming in Knapsack

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Where can we find the optimal solution? The entry at $B_{n,W}$ is the optimal solution since we can use every item and the knapsack capacity.

How can we calculate an entry in the table?
Notation: w_i, p_i the weight and price of item i, and W the knapsack capacity.

Calculating entries

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Entries $B_{0,j} = 0$, since we can pick no items. The same is true for entries $B_{i,0}$, as we can use no weight.
If we can use an additional item, either it belongs to the solution or it doesn't. The optimal solution will be the maximum of these two possibilities.

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$$B_{i,j} = \max(B_{i-1,j}, V_i + B_{i-1,j-w_i})$$

(Terrible) Visualization

Either i+1 is not used

If we can also use item i+1

Or i+1 is used

Best solution using items 1, 2, ..., i and weight w

Item i+1

Best solution using items 1, 2, ..., i and weight w - w_{i+1}
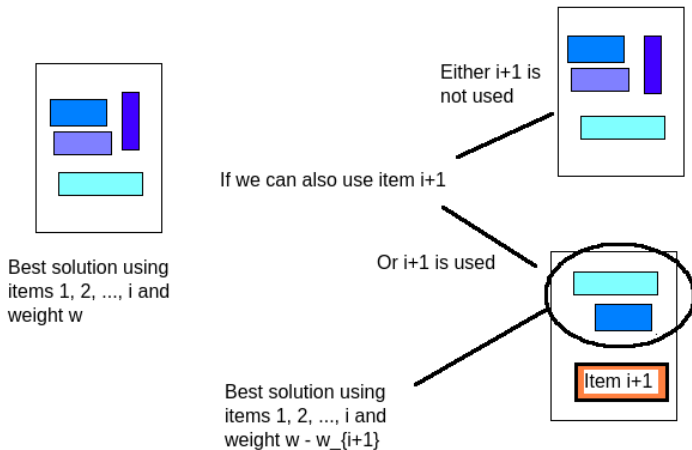
Figure: Example of dynamic programming decision

Example

Item	Weight	Value
1	1	10
2	2	15
3	3	40

Knapsack capacity: 6

Example

If no items can be used, the best objective is 0. If no weight can be used, the best objective is also 0.

Item\Weight	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0						
2	0						
3	0						

Example

If we can use item 1, then the best objective is 0 until we can use w_1 weight, and p_1 from then on.

Item\Weight	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	10	10	10	10	10	10
2	0						
3	0						

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Example

Now we can use items 1 and 2.

Item\Weight	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	10	10	10	10	10	10
2	0	10	15	25	25	25	25
3	0						

To get entry $B_{2,3}$, we have to calculate if using item 2 improves the previous best solution of 10, at $B_{1,3}$. If we pick 2, then we must use $w_2 = 2$ weight. The best solution with weight 3 (the current available weight) minus 2 (item 2 weight) is at $B_{1,3-2} = 10$. $10 + v_2 = 10 + 15 > 10$. So using item 2 is better.

Example

Item\Weight	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	10	10	10	10	10	10
2	0	10	15	25	25	25	25
3	0	10	15	40	50	55	65

How can we deduce the optimal solution?

Start at the bottom right cell ($B_{n,W}$)

- If current cell $B_{i,j} = B_{i-1,j}$, we didn't take item i. Go to $B_{i-1,j}$.
- If $B_{i,j} \neq B_{i-1,j}$ we took item i. Go to $B_{i-1,j-w_i}$.
- Repeat until we reach $B_{0,0}$.