# Bungee Jumping: A Mathematical Model

MA1033: Analysis of Differential Equations

Group 101

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# 1 Introduction

Bungee jumping is an attractive activity for risk takers, but it is rather extreme. Thinking that people would enjoy jumping from altitudinous places with just a rope seems unreal, and many people probably question how safe it is to jump.

Land diving originated as a rite of passage in Pentecost Island located in Vanuatu, this ancient ritual consisted of jumping from a platform with vines attached to their ankles which made the impact non-lethal. Further development of this activity started in 1979 but those involved were arrested, later on it became an extreme sport that is mostly safe, one enjoyed by hundreds in a multitude of ways, from wraping it around the ankles or the waits to even metal hoops inserted into the skin, certainly an extreme sport [1]. Even though to the untrained eye it seems like a simple motion, it's not that simple upon further inspection, since it implies hooke's law and drag force apart from Newton's second law which applies to all movement.

# 2 Theoretical Framework

## 2.1 Newton's Second Law of Motion

Newton's Second Law states that the sum of forces in a certain axis is equal to the product of mass and the second order differential of displacement in that same axis. Its behavior is shown by the following equation:

$$\sum_{k=1}^{N} \boldsymbol{F}_k = m\ddot{\boldsymbol{x}} \tag{2.1}$$

where the sum of forces represent every single one that interacts, meaning, friction, the force applied and other components to consider, the mass, which is the resistance an object has to a change in motion, also known as inertia, and finally, the second order differential of displacement, that represents the change in velocity with respect to time.

### 2.2 Hooke's Law

Hooke's law states that the force required to displace a point mass on a spring is directly proportional to the desired displacement from the equilibrium. This behavior is shown by the following equation:

$$\boldsymbol{F} = -k\boldsymbol{x}, \tag{2.2}$$

where $\boldsymbol{F}$ is the force, $k$ is the spring constant and $\boldsymbol{x}$ is the displacement from the equilibrium point [2, p. III-5].

### 2.3 Drag Force

Drag is a mechanical force that acts opposite to the motion of the object in motion, in this case a person willing to jump off a bridge. This drag, also known as air resistance opposes the movement of the jumper. This can be described by the following equation:

$$\boldsymbol{F} = -\beta\boldsymbol{v}, \tag{2.3}$$

which shows how the drag force and the velocity are directly proportional and opposite in direction [3].

## 3 Modelling the Jump

From the moment someone jumps, until the rope reaches its natural length, the individual is only subject to a gravitational field and drag. However, once the rope is streched, due to its elasticity, it will get enlongated and finally pull the jumper back. Recognizing this behaviour, it makes sense to model the position as a piece-wise function

$$s(t) = \begin{cases} s_1(t), & t \leq t_1 \\ s_2(t), & t > t_1 \end{cases} \tag{3.1}$$

such that $s(t)$ is the position at time $t$ and $t_1$ is the time it takes for the rope to reach its natural length $\ell$. Meaning $s_1(t)$ models the first stage of the jump and $s_2(t)$ the second.

## 3.1 Free Fall

The first stage of the jump is nothing more than free fall (considering air resistance). Let $\boldsymbol{F}$ be the total force experienced by the system, $\boldsymbol{F}_g$ the force due to gravity and $\boldsymbol{F}_d$ the drag force; from Newton's laws of motion we deduce

$$\boldsymbol{F} = \boldsymbol{F}_g + \boldsymbol{F}_d. \tag{3.2}$$

Figure 3.1 shows a free body diagram for this system.

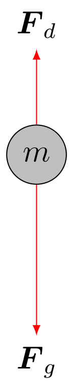


Figure 3.1: Free body diagram

If the system has a mass $m$ and the magnitude of the gravitational field strength is $g$ then the force due to the gravitational field is

$$\boldsymbol{F}_g = mg\hat{\boldsymbol{r}}, \tag{3.3}$$

such that $\hat{\boldsymbol{r}}$ is a unit vector in the radial direction of a massive body, in this case the Earth.

Moreover, the drag force will be considered to vary linearly with velocity and since it is a resistive force its direction is opposite to velocity, thus

$$\boldsymbol{F}_d = -\beta \boldsymbol{v}, \quad 0 < \beta \in \mathbb{R} \tag{3.4}$$

such that $\boldsymbol{v}$ is velocity of the system and $\beta$ is a proportionality constant.

Finally, using Newton's second law of motion, we may write the net force $\boldsymbol{F}$ as the product of the mass $m$ and the acceleration $\boldsymbol{a}$ of the system. Noticing that the movement only occurs in the vertical direction $\hat{\boldsymbol{k}}$, we can analyse the system in 1 dimension and since vectors in this space are members of $\mathbb{R}^1$ these may be treated as scalars (the signs can be determined using Figure 3.1 and considering the direction of $v$ as the negative direction). Finally, since acceleration is the time derivative of velocity $\boldsymbol{a} = \dot{\boldsymbol{v}}$, we can deduce a first order differential equation that models the jump in one dimension

$$m\dot{v}_1 = -\beta v_1 - mg. \tag{3.5}$$

Adding $\beta v_1$ to both members of the equation and diving by $m$ we can clearly see that the equation is linear

$$\dot{v}_1 + \frac{\beta}{m} v_1 = -g. \tag{3.6}$$

We can solve this equation using an integrating factor

$$\mu(t) = e^{\int \frac{\beta}{m} dt} = e^{\frac{\beta}{m} t}, \tag{3.7}$$

which allows us to write Equation 3.6 in terms of the derivative of a product

$$\frac{d}{dt}\left[\mu(t) v_1(t)\right] = -\mu(t) g, \tag{3.8}$$

we can determine $v_1$ by integrating and dividing by $\mu(t)$ both sides of the equation

$$v_1(t) = \frac{\int \mu(t) g \, dt + C}{\mu(t)}. \tag{3.9}$$

If we approximate the gravitational field strength to be constant, we can deduce a general solution

$$\begin{aligned} v_1(t) &= \frac{-g \int e^{\frac{\beta}{m}t} \, dt + C}{e^{\frac{\beta}{m}t}} \\ &= \left( -\frac{gm}{\beta} e^{\frac{\beta}{m}t} + C \right) e^{-\frac{\beta}{m}t} \\ &= C e^{-\frac{\beta}{m}t} - \frac{gm}{\beta}. \end{aligned} \tag{3.10}$$

If we consider that at $t = 0$ we have an initial velocity $v_0$, we deduce

$$v_1(0) = C - \frac{gm}{\beta} = v_0 \tag{3.11}$$

$$\implies C = v_0 + \frac{gm}{\beta}. \tag{3.12}$$

Thus the 1-dimensional velocity of the system is

$$v_1(t) = \left( v_0 + \frac{gm}{\beta} \right) e^{-\frac{\beta}{m}t} - \frac{gm}{\beta}, \tag{3.13}$$

and in the particular case where $v_0 = 0$ the solution reduces to

$$v_1(t) = \frac{gm}{\beta} \left( e^{-\frac{\beta}{m}t} - 1 \right), \tag{3.14}$$

which occurs when the jumpers just let themselves fall. From this equation we can also determine the terminal velocity a jumper will have, which occurs in the limit to infinity

$$v_t = \lim_{t \to \infty} v_1(t) = \lim_{t \to \infty} \frac{gm}{\beta} \left( e^{-\frac{\beta}{m}t} - 1 \right), \tag{3.15}$$

since the limit of a product of functions equals the product of the limits of the functions (as long as they exist),

$$\begin{aligned} v_t &= \lim_{t \to \infty} \left( \frac{gm}{\beta} \right) \cdot \lim_{t \to \infty} \left( e^{-\frac{\beta}{m}t} - 1 \right) \\ &= \frac{gm}{\beta} \lim_{t \to \infty} \left( e^{-\frac{\beta}{m}t} - 1 \right), \end{aligned} \tag{3.16}$$

finally the limit of a sum of functions is the sum of the limits of the functions (iff the limits exist), therefore

$$\begin{aligned} v_t &= \frac{gm}{\beta}\left(\lim_{t\to\infty} e^{-\frac{\beta}{m}t} + \lim_{t\to\infty} -1\right) \\ &= \frac{gm}{\beta}(0-1) \\ &= -\frac{gm}{\beta}. \end{aligned} \tag{3.17}$$

Although the terminal velocity is a limiting behaviour, the convergence of the exponential towards 0 may be fast enough for velocity to converge quickly; therefore this kind of approximation could be used to give an approximate value for $\beta$ if the speed of a jumper is known after a certain time and if it's expected to be close to $v_t$.

Furthermore, to determine the position function $s_1(t)$ we can notice that velocity is the derivative of position, thus we may write Equation 3.14 as an ODE

$$\dot{s}_1 = \frac{gm}{\beta}\left(e^{-\frac{\beta}{m}t} - 1\right), \tag{3.18}$$

integrating both sides we deduce

$$\begin{aligned} s_1(t) &= \int \frac{gm}{\beta}\left(e^{-\frac{\beta}{m}t} - 1\right) dt + C \\ &= \frac{gm}{\beta}\int e^{-\frac{\beta}{m}t}\, dt - \frac{gm}{\beta}t + C \\ &= -\frac{gm^2}{\beta^2}e^{-\frac{\beta}{m}t} - \frac{gm}{\beta}t + C. \end{aligned} \tag{3.19}$$

Moreover, by referring to the height from which the bungee jump is done as $h_0$ we can deduce a particular solution

$$s_1(0) = -\frac{gm^2}{\beta^2} + C = h_0 \tag{3.20}$$

$$\implies C = \frac{gm^2}{\beta^2} + h_0. \tag{3.21}$$

Therefore, the first stage of the jump can be modelled by the function

$$s_1(t) = -\frac{gm^2}{\beta^2}e^{-\frac{\beta}{m}t} - \frac{gm}{\beta}t + \frac{gm^2}{\beta^2} + h_0. \tag{3.22}$$

The constants $m$ and $\beta$ depend on certain conditions of the jumper, $\beta$ also depends on the design of the bungee, so does $h_0$.

## 3.2 Damped Oscillator

Once the bungee rope reaches its natural length $\ell$, it will start streching, and according to Hooke's law, the system will experience an additional force $\boldsymbol{F}_e$ that is proportional to the displacement of the rope and opposite in direction

$$\boldsymbol{F}_e = -k\boldsymbol{\Delta x}, \quad 0 < k \in \mathbb{R} \tag{3.23}$$

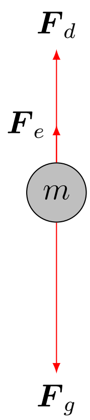


Figure 3.2: Free body diagram

Since the rope reaches its natural length at $s = h_0 - \ell$ this is its equilibrium position and the force exerted by the rope must be defined, in one dimension, as

$$F_e = -k\left(s - h_0 + \ell\right) \tag{3.24}$$

Thus, from Newton's laws of motion and using the free body diagram in Figure 3.2, the position of the jumper as a function of time can be described in one dimension by the second order differential equation

$$m\ddot{s} = -\beta\dot{s} - k\left(s - h_0 + \ell\right) - mg, \tag{3.25}$$

which is a non-homogeneous second order differential equation, since it can be written in the form $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = F(t)$ as shown in Equation 3.26

$$m\ddot{s} + \beta\dot{s} + ks = kh_0 - k\ell - mg. \tag{3.26}$$

To solve for $s$ we need to solve the complementary equation

$$m\ddot{s} + \beta\dot{s} + ks = 0, \tag{3.27}$$

which is homogeneous and therefore may be solved using the characteristic equation

$$\begin{aligned} mr^2 + \beta r + k &= 0 \\ \implies r &= \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}. \end{aligned} \tag{3.28}$$

Letting $\lambda = \frac{-\beta}{2m}$ and $\mu = \frac{\sqrt{|\beta^2 - 4mk|}}{2m}$ the solution to the complementary equation that models an underdamped oscillator is

$$s_c = (C_1 \cos \mu t + C_2 \sin \mu t)\, e^{\lambda t}, \tag{3.29}$$

if and only if $\beta^2 - 4mk < 0$. Which is the expected behaviour of the system and thus $k$ should be chosen for this condition to hold.

The solution to the non-homogeneous equation is the linear combination of the complementary solution $s_c$ and a particular solution $s_p$

$$s = s_c + s_p, \tag{3.30}$$

since the forcing function is just a constant, the particular solution has the form $s_p = C_3$. Substituting $s_p$ and its derivatives in Equation 3.26 we get the equation

$$\begin{aligned} m \cdot 0 + \beta \cdot 0 + kC_3 &= kh_0 - k\ell - mg \\ C_3 &= h_0 - \ell - \frac{mg}{k}. \end{aligned} \tag{3.31}$$

Finally we deduce the general solution to the non-homogeneous equation using the principle of superposition

$$s_2(t) = (C_1 \cos \mu t + C_2 \sin \mu t)\, e^{\lambda t} + h_0 - \ell - \frac{mg}{k}. \tag{3.32}$$

To determine the unique solution we need 2 initial conditions, one for the position and the other for the velocity. These can be determined from the previous model for the jump, since we expect the functions to be continuous

$$\begin{aligned} s_2(t_1) &= s_1(t_1) = h_0 - \ell \\ v_2(t_1) &= v_1(t_1) \approx v_t. \end{aligned} \tag{3.33}$$

By differentiating Equation 3.32 we deduce the velocity function $v_2$

$$v_2(t) = \lambda\, (C_1 \cos \mu t + C_2 \sin \mu t)\, e^{\lambda t} + \mu\, (C_2 \cos \mu t - C_1 \sin \mu t)\, e^{\lambda t}. \tag{3.34}$$

Thus leading to the system of equations

$$\begin{cases} h_0 - \ell = (C_1 \cos \mu t_1 + C_2 \sin \mu t_1)\, e^{\lambda t_1} + h_0 - \ell - \frac{mg}{k} \\ v_1(t_1) = \lambda\, (C_1 \cos \mu t_1 + C_2 \sin \mu t_1)\, e^{\lambda t_1} + \mu\, (C_2 \cos \mu t_1 - C_1 \sin \mu t_1)\, e^{\lambda t_1} \end{cases} \tag{3.35}$$

which is linear once particular values for the parameters $h_0$, $\ell$, $t_1$, $v_1\,(t_1)$, $\mu$ and $\lambda$ are determined. The system may therefore be represented with matrices and solved with a computer

$$\begin{bmatrix} \cos \mu t_1 & \sin \mu t_1 \\ \lambda \cos \mu t_1 - \mu \sin \mu t_1 & \lambda \sin \mu t_1 + \mu \cos \mu t_1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \frac{mg}{ke^{\lambda t_1}} \\ \frac{v_1(t_1)}{e^{\lambda t_1}} \end{bmatrix} \tag{3.36}$$

$$\implies \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \cos \mu t_1 & \sin \mu t_1 \\ \lambda \cos \mu t_1 - \mu \sin \mu t_1 & \lambda \sin \mu t_1 + \mu \cos \mu t_1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{mg}{ke^{\lambda t_1}} \\ \frac{v_1(t_1)}{e^{\lambda t_1}} \end{bmatrix} \tag{3.37}$$

Finally, to analyze the limiting behaviour of the system, Equation 3.32 may also be written in the form

$$s_2(t) = R \cos(\mu t - \phi) e^{\lambda t} + h_0 - \ell - \frac{mg}{k}, \tag{3.38}$$

such that $R = \sqrt{C_1^2 + C_2^2}$ and $\phi$ is the angle formed by the vector $(C_1, C_2)$ with the horizontal, that is

$$\phi = \begin{cases} \arctan\left(\frac{C_2}{C_1}\right), & C_1 > 0 \\ \arctan\left(\frac{C_2}{C_1}\right) + \pi, & (C_1 < 0) \text{ and } (C_2 \geq 0) \\ \arctan\left(\frac{C_2}{C_1}\right) - \pi, & (C_1 < 0) \text{ and } (C_2 < 0) \\ \frac{\pi}{2}, & (C_1 = 0) \text{ and } (C_2 > 0) \\ -\frac{\pi}{2}, & (C_1 = 0) \text{ and } (C_2 < 0) \end{cases}$$

Then, taking the limit as $t$ approaches infinity we observe

$$\begin{aligned} \lim_{t\to\infty} s_2(t) &= \lim_{t\to\infty} \left(R\cos(\mu t - \phi)e^{\lambda t} + h_0 - \ell - \frac{mg}{k}\right) \\ &= \lim_{t\to\infty} \left(R\cos(\mu t - \phi)e^{\lambda t}\right) + \lim_{t\to\infty} \left(h_0 - \ell - \frac{mg}{k}\right) \\ &= \lim_{t\to\infty} \left(R\cos(\mu t - \phi)e^{\lambda t}\right) + h_0 - \ell - \frac{mg}{k}. \end{aligned} \tag{3.39}$$

Since $\lambda = -\frac{\beta}{2m}$ is strictly negative ($\beta$ and $m$ are both strictly positive), we can use the squeeze theorem to determine the limit

$$\begin{aligned} -R \leq & R\cos(\mu t - \phi) \leq R \\ -Re^{\lambda t} \leq & R\cos(\mu t - \phi)e^{\lambda t} \leq Re^{\lambda t} \\ \lim_{t\to\infty} -Re^{\lambda t} \leq & \lim_{t\to\infty} R\cos(\mu t - \phi)e^{\lambda t} \leq \lim_{t\to\infty} Re^{\lambda t} \\ 0 \leq & \lim_{t\to\infty} R\cos(\mu t - \phi)e^{\lambda t} \leq 0 \\ \implies & \lim_{t\to\infty} R\cos(\mu t - \phi)e^{\lambda t} = 0. \end{aligned} \tag{3.40}$$

Hence, substituting the limit in Equation 3.39, we can see that the rope will try to keep the jumper's position fixed at $h_0 - \ell - \frac{mg}{k}$. Choosing a particular spring constant $k$ could allow the system to stop near a desired point, for instance a low value of $k$ could allow us to retrieve the jumper near ground level (notice the minimum point should be considered separetly to verify the jumper is not in danger); however, as mentioned previously, $k$ should at least be greater than $\frac{\beta^2}{4m}$ for the underdamped oscillation to occur.

## 4 Numerical Simulation

Throughout this research we have been treating the model abstractly. The purpose of such an approach was to be able to create a computer program that adjusts to different jumpers and conditions and thus a MATLAB program was designed for this particular reason (Program 1)

```
clc; clear all; close all;

% PARAMETERS (SI Units)
m = 80;
g = 9.81;
b = 14;
h0 = 200;
l = 80;
k = 15;

% SIMULATION
syms v1(t) s1(t) s2(t) s(t)
v1(t) = g*m/b * (exp(-b/m*t) - 1);
s1(t) = -g*(m^2)/(b^2) * exp(-b/m * t) - g*m/b * t + g*m^2/b^2 + h0;
t1 = solve(s1(t) == h0-l, t, "Real",true,"PrincipalValue",true);

lambda = - b/(2*m);
mu = sqrt(abs(b^2-4*m*k))/(2*m);

a11 = cos(mu*t1);
a12 = sin(mu*t1);
a21 = lambda*cos(mu*t1) - mu*sin(mu*t1);
a22 = lambda*sin(mu*t1) + mu*cos(mu*t1);
```

```
A = [a11, a12; a21, a22];
B = [m*g/(k*exp(lambda*t1)); v1(t1)/exp(lambda*t1)];
C = A \ B;
s2(t) = (C(1)*cos(mu*t) + C(2)*sin(mu*t))*exp(lambda*t) + h0 - l - m*g/k;

% PLOT
s(t) = piecewise(t <= t1,s1(t),t > t1,s2(t));
figure
hold on
grid on
title("Vertical position vs time")
xlabel("Time (s)")
ylabel("Position (m)")
fplot(s(t), [0, 80], "LineWidth",1.2)
```

Program 1: Bungee Jump Simulation

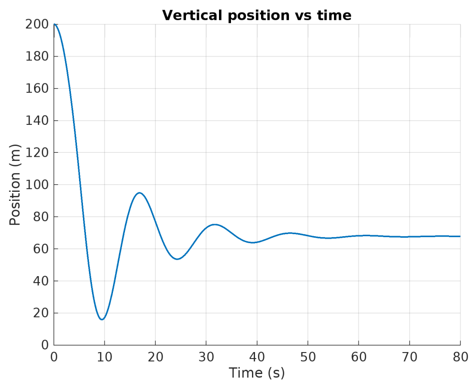


Figure 4.1: $s(t)$ for $0 \leq t \leq 80\,\mathrm{s}$

The parameters in the source code can be changed as desired, however to run one simulation we used a mass $m$ of $80\,\mathrm{kg}$, a drag coefficient $\beta$ of $14\,\mathrm{kgs}^{-1}$, an initial height $h_0$ of $200\,\mathrm{m}$ and a bungee rope whose length is $80\,\mathrm{m}$ with a spring constant $k$ of $15\,\mathrm{Nm}^{-1}$. The value that could be questionable is the drag coefficient; nevertheless, as mentioned previously, according to our model, the terminal velocity of a jumper in free fall is $-\frac{gm}{\beta}$, and after evaluating with the parameters that turns out to be $56.06\,\mathrm{ms}^{-1}$, which is congruent with real data [4].

After executing the program Figure 4.1 was generated, which shows the position of the jumper as a function of time. This plot shows that it would take about 10 seconds for the jumper to reach the lowest point, after descending around 180 m; moreover, it would take around a minute for the bungee rope to stay fixed in place, thus concluding the jump. Another aspect one might notice from the plot is that the length of the rope was chosen to be 80 m and thus it might be risky to expect a rope to extend almost 100 m from its natural length without breaking. To compensate for such possibility we could use a system of ropes of the same length to increase the spring constant and due to the increase in the force reduce the expected lowest point.

# 5 Conclusion

From this research we found out that the position of an individual in a bungee jump can be modelled as a piecewise continuous function

$$s(t) = \begin{cases} -\frac{gm^2}{\beta^2} e^{-\frac{\beta}{m}t} - \frac{gm}{\beta}t + \frac{gm^2}{\beta^2} + h_0, & t \leq t_1 \\ \left(C_1 \cos \mu t + C_2 \sin \mu t\right) e^{\lambda t} + h_0 - \ell - \frac{mg}{k}, & t > t_1, \end{cases}$$

such that $t_1$ is the time taken for the rope to extend to its natural length $\ell$ and the constants $h_0, C_1, C_2, \mu$ and $\lambda$ depend on the design of the bungee.

After testing the model with vaues based on recorded data for humans, we can see the model predicts the expected position function: an underdamped oscillator; however, for this to occur the spring constant of the rope must be varied to adjust to certain conditions of the jumper and the place where they are jumping from, to avoid the system to be risky and to at least satisfy the condition $\beta^2 < 4mk$.

Moreover it would be interesting to test the model with experimental data in a controlled and safe way, since this can allow us to know the limitiations of such model and avoid people from putting themselves in a dangerous position. Good maintenance for the bungee is of vital importance and the ropes should be checked constantly to avoid the chance of them breaking during a jump.

# References

[1] Attenborough, D. (Director). (n.d.). The Land Divers of Pentecost (Season 1, Episode 1) [TV series episode]. In *The People of Paradise*. BBC. https://www.bbc.co.uk/iplayer/episode/p00y207q/the-people-of-paradise-1-the-land-divers-of-pentecost

[2] Morin, D. (2004). *Introduction to Classical Mechanics*. Cambridge: Cambridge University Press.

[3] National Aeronautics and Space Administration. (n.d.). *What is Drag?* NASA Glenn Research Center. https://www.grc.nasa.gov/www/k-12/airplane/drag1.html

[4] The Physics Factbook (n.d.). *Speed of a Skydiver (Terminal Velocity)*. https://hypertextbook.com/facts/1998/JianHuang.shtml