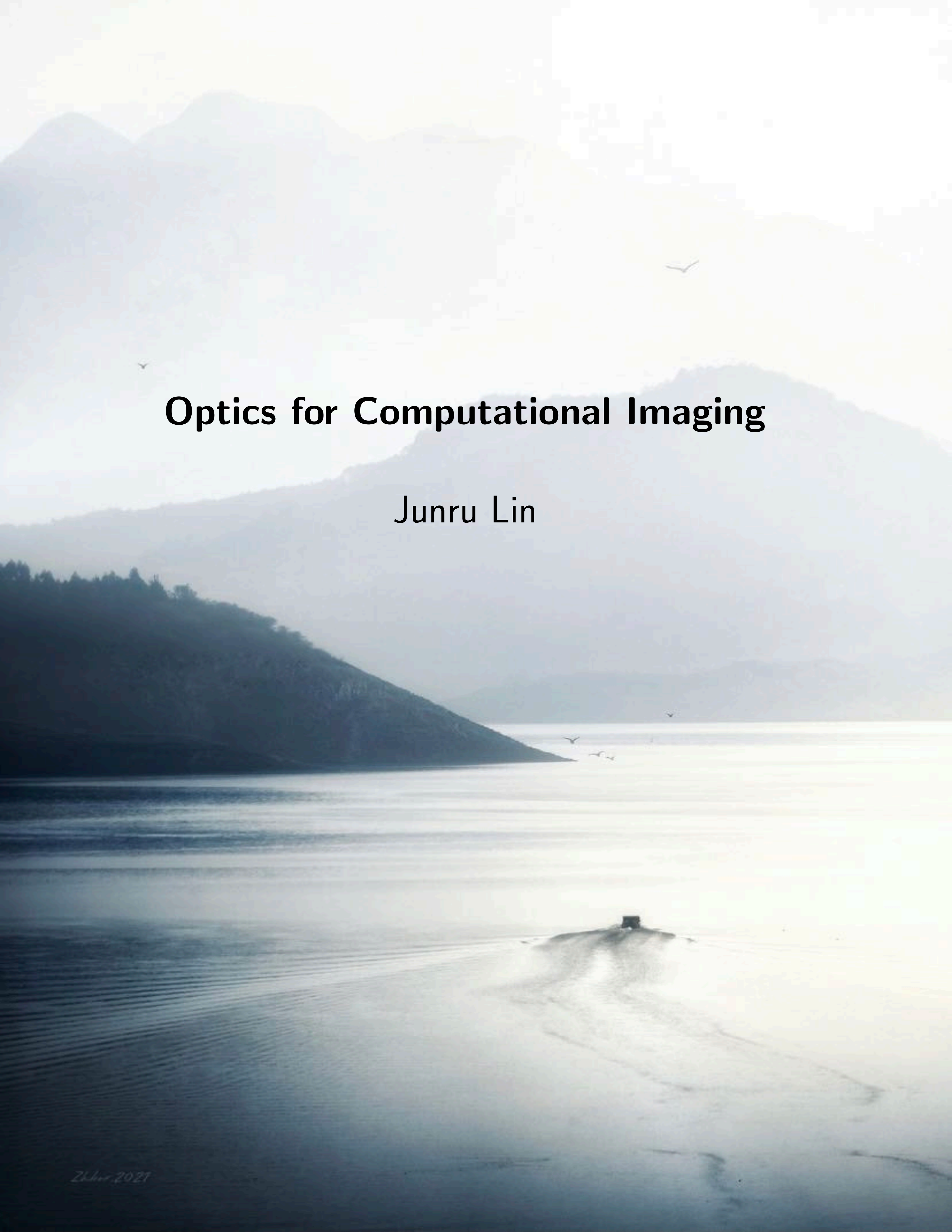


Optics for Computational Imaging

Junru Lin

Contents

Chapter 1

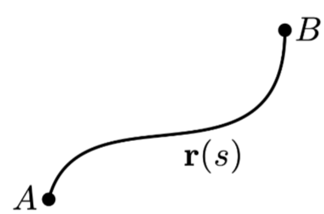
Ray Optics

1.1 Fermat's Principle

Index of refraction n

- Assumption: the media are lossless
- $n \geq 1$
  - vacuum: $n = 1$
  - Air: $n = 1.0003$
  - Water: $n = 1.33$
- Speed of light in the medium $c = c_0/n$
- Refractive indices that cary spatially: $n(r)$
- Vacuum: $n(r(s)) = r(s)$

Optical path length



- A corresponds tio $r(0)$ and B corresponding to $r(d)$

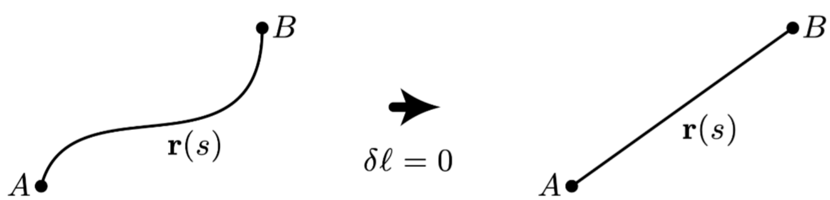
- Optical path length functional: $\ell[r] := \int_0^d n(r(s))\, ds$
- $\Delta t = \ell / c_0$

Fermat's Principle (Variational Principle)

- $\delta\ell = 0$
- $\delta\ell$: the (first) variation of ℓ
- Meaning: nearby paths have the same path length

1.2 Fermat's Principle: Examples

Homogeneous Medium



- $\ell = n \int_A^B ds = nd$
- $\delta\ell = 0$ implies straight path from A to B

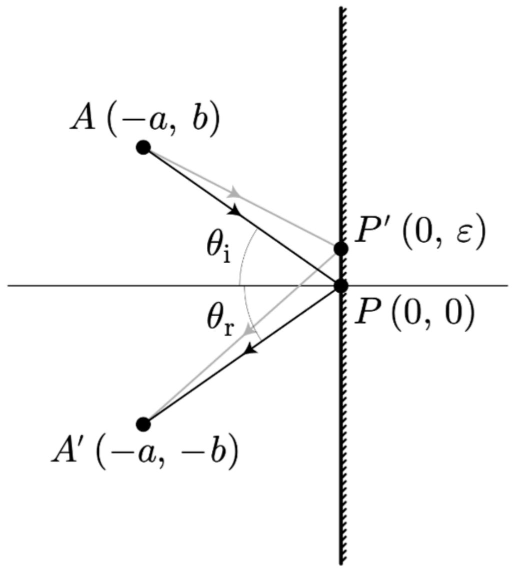
Plane Mirror (Law of Reflection)

- By symmetry, APA' is the minimum-length path. To prove it, consider a nearby point P'.
- The length of $AP'A'$ is

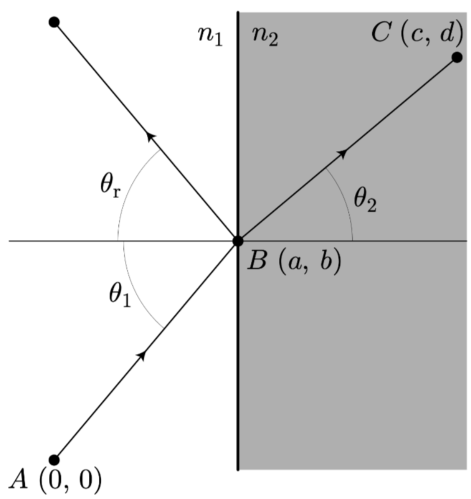
$$\frac{\ell}{n} = \sqrt{a^2 + (b - \varepsilon)^2} + \sqrt{a^2 + (b + \varepsilon)^2}$$

- Differentiation with respect to the perturbation ε,

$$\frac{\partial}{\partial \varepsilon}\left(\frac{\ell}{n}\right) = \frac{\varepsilon - b}{\sqrt{a^2 + (b - \varepsilon)^2}} + \frac{\varepsilon + b}{\sqrt{a^2 + (b + \varepsilon)^2}} = 0$$



$A\ (-a,\ b)$
$P'\ (0,\ \varepsilon)$
θ_{i}
θ_{r}
$P\ (0,\ 0)$
$A'\ (-a,\ -b)$



n_1
n_2
$C\ (c,\ d)$
θ_{r}
θ_2
$B\ (a,\ b)$
θ_1
$A\ (0,\ 0)$

- We get $\varepsilon = 0$ and thus
$$\theta_i = \theta_r$$

Refractive Interface (Snell's Law)

- Assume the refracted ray begins at A and ends at C. Find B by $\delta(ABC) = 0$.
- Path length is
$$\ell = n_1\sqrt{a^2 + b^2} + n_2\sqrt{(c-a)^2 + (d-b)^2}$$
- Differentiating with respect to the moveable coordinate b of B,
$$\frac{\partial \ell}{\partial b} = \frac{n_1 b}{\sqrt{a^2 + b^2}} - \frac{n_2(d-b)}{\sqrt{(c-a)^2 + (d-b)^2}} = 0$$
- Equivalently,
$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \qquad \text{(Snell's Law)}$$

Critical Angle and Total Internal Reflection

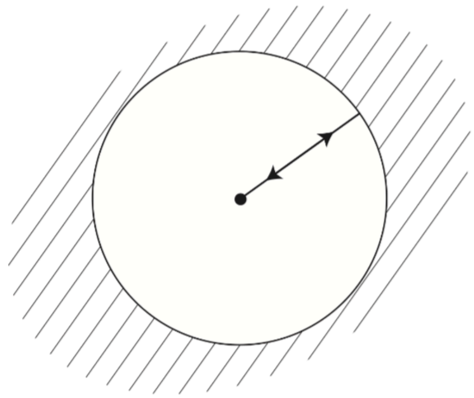
- Notice that
$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1 \leq 1$$
- If $n_1 > n_2$, critical angle θ_c is given by
$$\frac{n_1}{n_2}\sin\theta_c = 1$$
- Total internal reflection: If $\theta_1 > \theta_c$, then there is no possible transmitted ray and all the light is reflected.

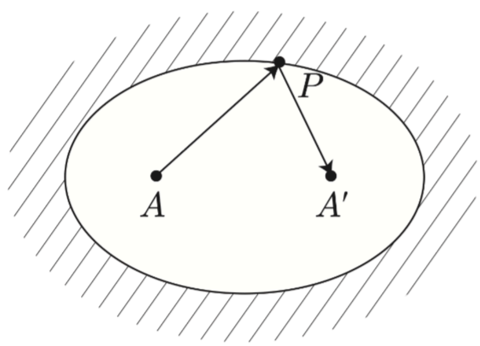
Spherical Mirror

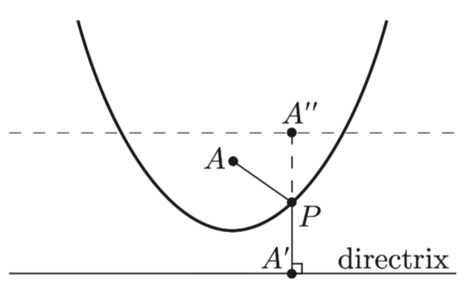
- All rays from the center to the outer edge and back have the same (minimum) optical path length.
- A spherical mirror focuses rays from an object at the center point back onto itself

Elliptical Mirror

- Ellipse: $\{P : APA' = d\}$ for some constant distance d
- A and A': the **foci** of the ellipse




P
A
A′


A″
A
P
A′
directrix

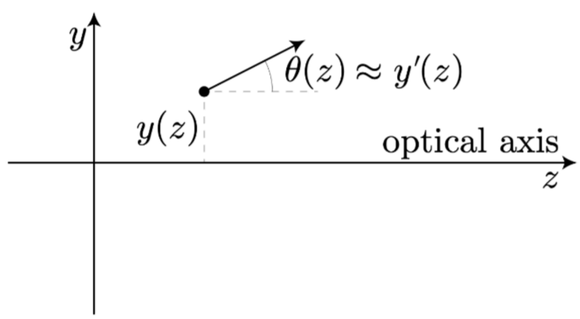
- Since APA' is constant for any P on the ellipse, rays starting at A will end at A'
- Therefore, an elliptical mirror images an object at A to A'

Parabolic Mirror

- Parabola: $\{P : AP = PA' \text{ where } PA' \perp \text{directrix}\}$
- A: the **focus** of the parabola
- For any P on the parabola, $AP + PA'' = PA' + PA''$ and thus $AP + PA'' = A'A''$, which is constant.
- Thus, a parabolic mirror collimates all rats starting at A.

1.3 Paraxial Rays

Vector Representation of a Ray



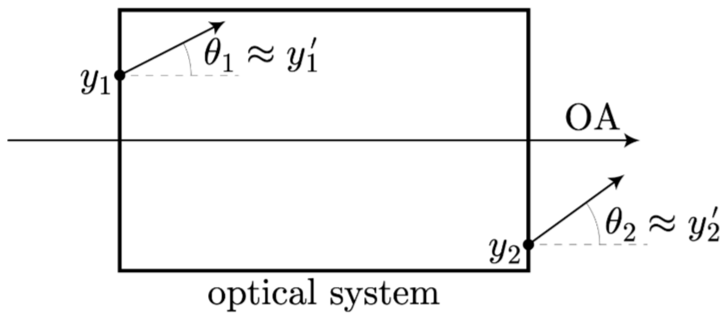
- Optical axis: the reference axis for the optical propagation
- y: displacement from the optical axis
- θ: direction
- In the paraxial approximation,

$$\begin{bmatrix} y \\ \theta \end{bmatrix} \approx \begin{bmatrix} y \\ y' \end{bmatrix}$$

Change in the Ray Vector

- Model the optical system as a transformation of ray vectors:

$$\begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = f \begin{bmatrix} y_1 \\ y_1' \end{bmatrix}$$



- Applying Taylor polynomial,

$$\begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = Df(0,0) \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} + \text{higher-order terms in } y_1, y_1'$$

  - Assumption 1: for $i = 1, 2$, y_i and y_i' are small so that $\theta_i \approx \sin\theta_i \approx \tan\theta_i = y_i'$.
  - Assumption 2: $f_1(0,0) = f_2(0,0) = 0$.

1.4 Matrix Optics

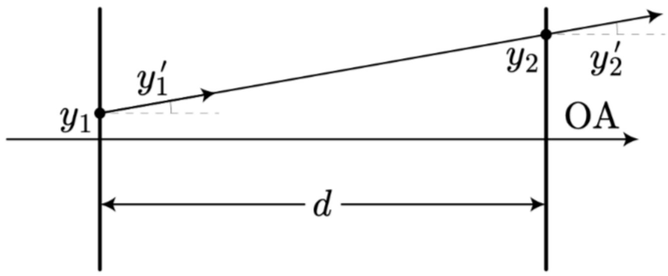
Ray-transfer Matrix

- In the last section, $Df(0,0)$ is a 2×2 matrix.
- For the general paraxial case we use the notation

$$\begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} + \text{higher-order terms in } y_1, y_1'$$

  - ray-transfer matrix
  - $ABCD$ matrix
  - ray matrix

Free-Space Propagation

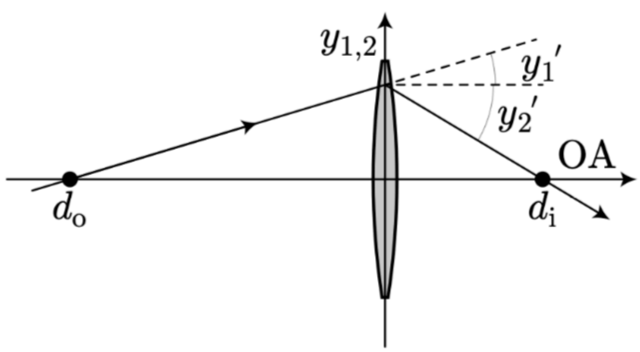


- The ray travels in a straight line,

$$\begin{aligned} y_2 = y_1 + y_1'd,\ y_2 = Ay_1 + By_1' \quad &\Rightarrow \quad A = 1, B = d \\ y_2' = y_1',\ y_2' = Cy_1 + Dy_1' \quad &\Rightarrow \quad C = 0, D = 1 \end{aligned}$$

- $M = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

Thin Lens



- Lens is thin and ray is continuous:

$$y_2 = y_1 \quad \Rightarrow \quad A = 1, B = 0$$

- By thin lens law:

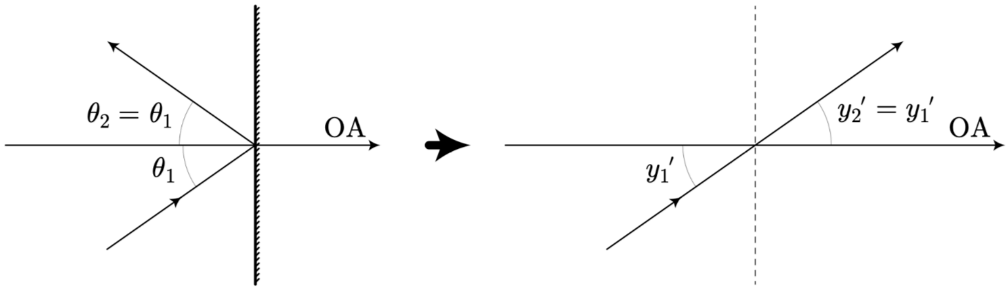
$$\frac{1}{d_0} + \frac{1}{d_i} = \frac{1}{f}$$

- Notice

$$y_1' = \frac{y_1}{d_0}, y_2' = -\frac{y_2}{d_i} \quad \Rightarrow \quad C = -\frac{1}{f}, D = 1$$

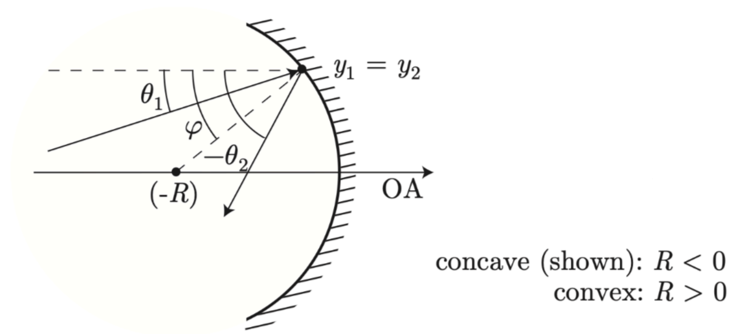
- $M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$

Plane Mirror



- $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Spherical Mirror



- The mirror is thin:

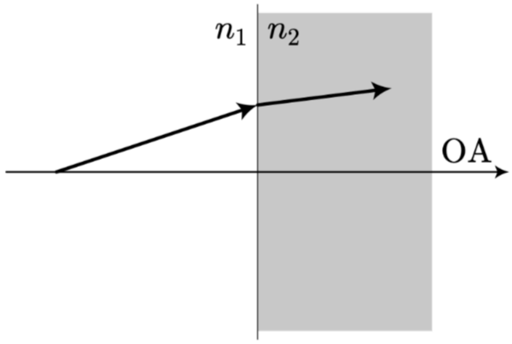
$$y_2 = y_1 \quad \Rightarrow \quad A = 1, B = 0$$

- By Law of Reflection

$$\varphi - \theta_1 = -\theta_2 - \varphi, \ \varphi = \frac{y_1}{-R} \quad \Rightarrow \quad \theta_2 = \theta_1 + \frac{2y_1}{R} \quad \Rightarrow \quad C = \frac{2}{R}, D = 1$$

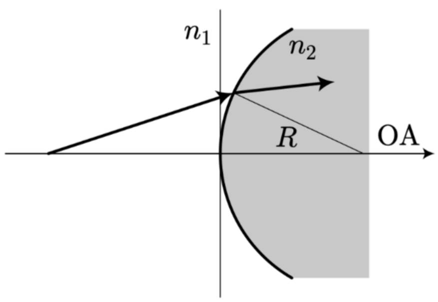
- $M = \begin{bmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{bmatrix}$

Planar Refractive Interface



- $M = \begin{bmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{bmatrix}$

Spherical Refractive Interface

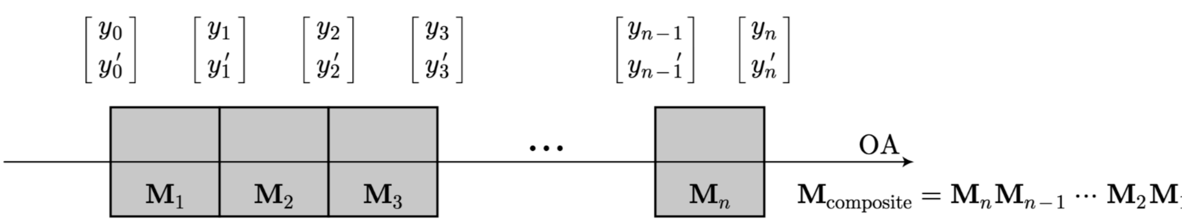


convex (shown): $R > 0$
concave: $R < 0$

- $M = \begin{bmatrix} 1 & 0 \\ -\frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$

1.5 Composite Systems

Ray-transfer Matrix for Composite Optical System



- For the first component:

$$\begin{bmatrix} y_1 \\ y_1' \end{bmatrix} = M_1 \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

- For the second component:

$$\begin{bmatrix} y_2 \\ y_2' \end{bmatrix} = M_2 \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} = M_2 M_1 \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

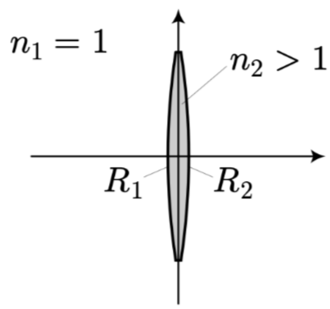
- For the entire system:

$$\begin{bmatrix} y_n \\ y_n' \end{bmatrix} = M_n M_{n-1} \cdots M_2 M_1 \begin{bmatrix} y_0 \\ y_0' \end{bmatrix} =: M_{composite} \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

- Ray-transfer Matrix for Composite Optical System: right-to-left ordering

$$M_{composite} = M_n M_{n-1} \cdots M_2 M_1$$

Thin Lens (Lensmaker's Formula)



- The composite matrix is

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{n_1 - n_2}{n_2 R_2} & \frac{n_2}{n_1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{n_2 - n_1}{n_2 R_1} & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_1}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) & 1 \end{bmatrix}$$

- Taking $n_1 = 1$ for simplification,

$$M = \begin{bmatrix} 1 & 0 \\ (n_2 - 1)\left(\frac{1}{R_2} - \frac{1}{R_1}\right) & 1 \end{bmatrix}$$

- Standard thin-lens matrix:

$$M = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix}$$

- Therefore,

$$\frac{1}{f} = -(n_2 - 1)\left(\frac{1}{R_2} - \frac{1}{R_1}\right) \qquad \text{(Lensmaker's formula)}$$

Chapter 2

Fourier Analysis

2.1 Periodic Functions: Fourier Series

Period Function

- $f(t) = f(t+T)$
- **Period:** T
- **Frequency (Hz):** ν
- **Angular Frequency (rad/s):** ω
- **Relation:**

$$\omega = 2\pi\nu = \frac{2\pi}{T}$$

Fourier Series

$$f(t) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos(n\omega t) + 2\sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

Complex Fourier Series

- Rewrite Fourier series:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n(e^{in\omega t} + e^{-in\omega t}) - i\sum_{n=1}^{\infty} b_n(e^{in\omega t} - e^{-in\omega t})$$

- For non-negative n, define

$$a_{-n} := a_n, \quad b_{-n} := -b_n, \quad c_n = a_n + ib_n$$

- Notice that $b_0 = 0$. Therefore,

$$f(t) = \sum_{n=-\infty}^{\infty} a_n e^{-in\omega t} + \sum_{n=-\infty}^{\infty} b_n e^{-in\omega t} = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega t}$$

- **Complex Fourier Series:**

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega t}$$

- Relation between real and complex coefficients:

$$\begin{aligned} a_n &= \frac{1}{2}(c_n + c_n^*) = \frac{1}{2}(c_n + c_{-n}) \\ b_n &= \frac{1}{2}(c_n - c_n^*) = \frac{1}{2i}(c_n - c_{-n}) \end{aligned}$$

Orthonormality of Harmonic Functions

- **Kronecker Delta**:

$$\delta_{nn'} = \frac{1}{T}\int_0^T \left(e^{-in'\omega t}\right)^* e^{-in\omega t}\, dt$$

- **Property of Kronecker Delta:** $\delta_{nn'} = 1$ if $n = n'$ and 0 otherwise.

- **Proof:**

$$\begin{aligned} \delta_{nn'} &= \frac{1}{T}\int_0^T \left(e^{-in'\omega t}\right)^* e^{-in\omega t}\, dt \\ &= \frac{1}{T}\int_0^T e^{-in\omega T} e^{in'\omega T}\, dt \\ &= \frac{1}{T}\int_0^T e^{-i(n-n')\omega t}\, dt \end{aligned}$$

  - If $n = n'$:

$$\delta_{nn'} = \frac{1}{T}\int_0^T e^0\, dt = 1$$

  - If $n \neq n'$:

$$\delta_{nn'} = \frac{1}{T}\int_0^T \Big(\cos(-i(n-n')\omega t) + i\sin(-i(n-n')\omega t)\Big)\, dt = 0$$

Complex Fourier Coefficient

- **Inner Product of Periodic Functions**

$$< f_1, f_2 >:= \frac{1}{T} \int_0^T f_1^*(t) f_2(t)\, dt$$

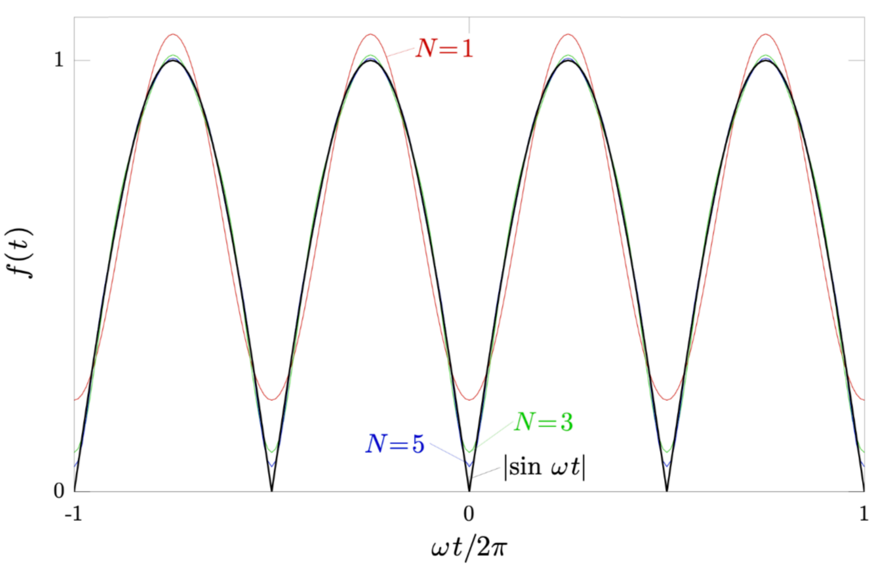
- **Complex Fourier Coefficient:**

$$\begin{aligned} c_n &= \frac{1}{T} \int_0^T \left(e^{-in\omega t}\right)^* f(t)\, dt \\ &= \frac{1}{T} \int_0^T e^{in\omega t} f(t)\, dt \end{aligned}$$

- **Proof from Inner Product:**

$$\begin{aligned} < e^{-in\omega t}, f > &= \frac{1}{T} \int_0^T \left(e^{-in\omega t}\right)^* f(t)\, dt \\ &= \sum_{n'=-\infty}^{\infty} \frac{c_{n'}}{T} \int_0^T \left(e^{-in\omega t}\right)^* e^{-in'\omega t}\, dt \\ &= \sum_{n'=-\infty}^{\infty} c_{n'} \delta_{n'n} \\ &= c_n \qquad \text{(by property of Kronecker Delta)} \end{aligned}$$

Example: Rectified Sine Wave $|sin\omega t|$



- Because of "rectification", the effective frequency is 2ω, and period is $T = \frac{\pi}{\omega}$.

- By formula, the complex Fourier coefficient satisfies

$$\begin{aligned}
c_n &= \frac{1}{T}\int_0^T |\sin\omega t| e^{in(2\omega)t}\,dt \\
&= \frac{1}{T}\int_0^T \frac{1}{2i}(e^{i\omega t} - e^{-i\omega t})\,e^{in(2\omega)t}\,dt \\
&= \frac{1}{2iT}\int_0^T \left(e^{i(2n+1)\omega t} - e^{i(2n-1)\omega t}\right)dt \\
&= \frac{1}{2iT}\int_0^\pi \left(e^{i(2n+1)x} - e^{i(2n-1)x}\right)dx \qquad (x := \omega t) \\
&= \frac{2}{\pi(1-4n^2)}
\end{aligned}$$

- Therefore,

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{2}{\pi(1-4n^2)} e^{-i2n\omega t}$$

2.2 Aperiodic Functions: Fourier Transform

Conditions

- f is a real function
- $\int_{-\infty}^{\infty} f(t)\,dt$ exists
- f has a finite number of discontinuities
- f has a finite number of maxima and minima in any finite interval
- f has no infinite discontinuities

Generalization to Inverse Fourier Transform

- Harmonic (discrete): $\Delta\omega = \frac{2\pi}{T}$.
- Aperiodic (continuous): $T \to \infty$, thus $\Delta\omega \to 0$ is needed.
- Inverse Fourier Transform:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{-in\omega t} \longrightarrow f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t}\,d\omega$$

- $\tilde{f}(\omega)/2\pi$: Amplitude of component $e^{-i\omega t}$

- $\tilde{f}(\omega)$: **Fourier transform** of $f(t)$.
- **Property:** If f is a real function, then

$$\tilde{f}(\omega) = \tilde{f}^*(-\omega)$$

- **Inverse:** finding $f(t)$ from its Fourier transform

Generalization to Fourier Transform

- By projection,

$$c_n = \frac{1}{T}\int_0^T e^{in\omega t} f(t)\, dt \longrightarrow \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t}\, dt$$

- **Fourier Transform Pair**: $f(t)$ and $\tilde{f}(\omega)$

Two Forms of Fourier Transforms

- In ω form:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t}\, d\omega$$
$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

- Define

$$\bar{f}(\nu) := \tilde{f}(\omega/2\pi)$$

- In $\nu - t$ form:

$$f(t) = \int_{-\infty}^{\infty} \bar{f}(\nu)\, e^{-i2\pi\nu t}\, d\omega$$
$$\bar{f}(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{i2\pi\nu t}\, dt$$

Connection to Linear Algebra

- Denote the Fourier transform by the symbol $\mathscr{F}$. Then

$$\tilde{f}(\omega) = \mathscr{F}[f(t)]$$
$$f(t) = \mathscr{F}^{-1}[\tilde{f}(\omega)]$$

- The Fourier transform is a linear transformation between vector spaces of functions

- By linearity of the Fourier transform,

$$\mathscr{F}[\alpha f(t) + \beta g(t)] = \alpha \tilde{f}(\omega) + \beta \tilde{g}(\omega)$$
$$\text{iff} \quad \mathscr{F}[f(t)] = \tilde{f}(\omega), \quad \mathscr{F}[g(t)] = \tilde{g}(\omega)$$

Example: Fourier Transform of a Gaussian Pulse

- **Gaussian pulse:**

$$f(t) = Ae^{-\alpha t}$$

- By definition of Fourier transform,

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} Ae^{-\alpha t^2 + i\omega t}\, dt = \int_{-\infty}^{\infty} Ae^{-a(t-b)^2 + c}\, dt$$

where

$$t^2 : -a = -\alpha \Rightarrow a = \alpha$$
$$t^1 : 2ab = i\omega \Rightarrow b = \frac{i\omega}{2\alpha}$$
$$t^0 : c - ab^2 = 0 \Rightarrow c = -\frac{\omega^2}{4\alpha}$$

- Substitute $t - b$ with t:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} Ae^{-at^2 + c}\, dt$$

- Compare with the standard normalized form of the Gaussian:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2} \right) dx = 1$$

- Taking $\sigma = 1/\sqrt{2\alpha}$,we get

$$\tilde{f}(\omega) = Ae^c \sqrt{\frac{\pi}{a}} = A\sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\omega^2}{4\alpha} \right)$$

- Therefore, **the Fourier transform of a Gaussian is a Gaussian**.

Uncertainty Principle

$$\delta t \sim \frac{1}{\delta \omega}$$

- δt: “width” of $f(t)$
- $\delta \omega$: “width” of $\tilde{f}(\omega)$

- From the example of Gaussian Pulse, for the original Gaussian

$$\sigma_t = \frac{1}{\sqrt{2\alpha}}$$

- For the Fourier transform Gaussian

$$\sigma_\omega = \sqrt{2\alpha}$$

2.3 The Fourier Transform in Optics

Gaussian

- The Fourier transform of a Gaussian is a Gaussian

$$\mathscr{F}\left[e^{-t^2/2}\right] = e^{-\omega^2/2}$$

Exponential

- The Fourier transform of an exponential is a Lorentzian

$$\mathscr{F}\left[e^{-|t|}\right] = \frac{2}{1+\omega^2}$$

- Atoms decay exponentially due to spontaneous emission

$$N_e(t) = N_e(0)\exp(-\Gamma t)$$

Square Pulse

- The **rectangular function** is defined by

$$rect(t) := \begin{cases} 1 & \text{if } |t| < 1/2 \\ 1/2 & \text{if } |t| = 1/2 \\ 0 & \text{if } |t| > 1/2 \end{cases}$$

- The Fourier transform of a square pulse is a **sinc function**

$$\mathscr{F}\left[rect(t)\right] = \text{sinc}(\omega/2) := \frac{\sin(\omega/2)}{\omega/2}$$

Constant

- The Fourier transform of a constant is a delta function

$$\mathscr{F}\left[\frac{1}{2\pi}\right] = \delta(\omega)$$

- This relation is the extreme limit of the uncertainty principle.

Delta Function

- The Fourier transform of a delta function is a constant

$$\mathscr{F}\Big[\delta(t)\Big] = 1$$

- This is the opposite extreme of the uncertainty principle.

2.4 Delta Function

Introduction

- A delta function is an idealized limit of a very short pulse.
- Construct a sequence of functions $h_n(t)$ such that
  - The h_n are "reasonable" (e.g., simply peaked around $t = 0$).
  - The width of h_n converges to zero as $n \to \infty$.
  - The h_n are normalized : $\int_{-\infty}^{\infty} h_n(t)\, dt = 1$ for all n.
- Example: Gaussian Functions.
  - Normalized form of Gaussian Function:

$$h_n(t) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left(-\frac{t^2}{2\sigma_n^2}\right)$$

  - Taking $\sigma_n = 1/\sqrt{2\pi}n$,

$$h_n(t) = n \exp\left(-\pi n^2 t^2\right)$$

Definition of Delta Function

- Integral form:

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt := \lim_{n\to\infty} h_n(t) f(t)\, dt$$

- Inner-product form:
$$< \delta, f >:= \lim_{n\to\infty} < \delta, h_n >$$

- Note: The delta function only makes sense as part of the argument of an integral since $\lim_{n\to\infty} h_n(t)$ DNE.

- But as a shorthand of the definition, we often write
$$\delta(t) = \lim_{n\to\infty} h_n(t)$$

- For all $t \neq 0$,
$$\lim_{n\to\infty} h_n(t) = 0$$

2.5 Properties of Delta Function

Normalization of Delta Function

$$\int_{-\infty}^{\infty} \delta(t)\; dt = 1$$

Projection Property of Delta Function

$$\int_{-\infty}^{\infty} \delta(t) f(t)\; dt = f(0)$$

Shifted Projection Property of Delta Function

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\; dt = f(a)$$

Integral Representation of Delta Function

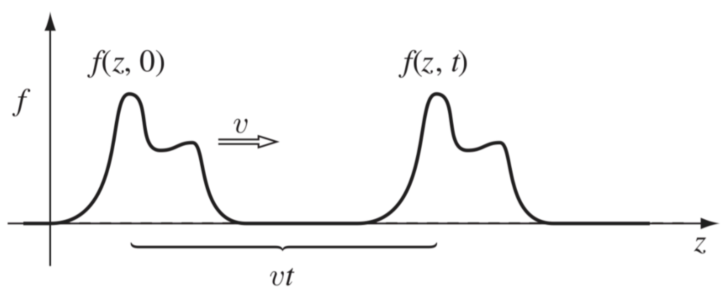
$$\delta(t-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega(t-a)}\; d\omega$$

Chapter 3

Electromagnetic Waves

3.1 The Wave Equation

Wave



- **Definition:** A wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity.
- $f(z,t)$: represents the displacement of the string at the point z, at time t.
- Initial shape of the string: $g(z) \equiv f(z,0)$
- fixed shape traveling in the z direction at speed v

$$f(z,t) = f(z-vt,0) = g(z-vt)$$

Examples of Waves

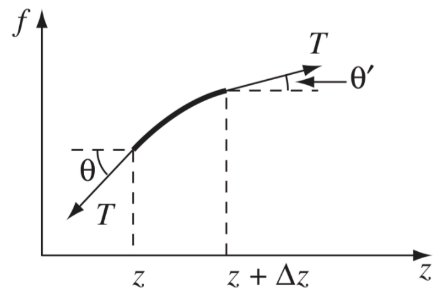
- $f_1(z,t) = Ae^{-b(z-vt)^2}$
- $f_2(z,t) = A\sin[b(z-vt)]$

- $f_3(z,t) = \frac{A}{b(z-vt)^2+1}$

Examples of Non-Waves

- $f_4(z,t) = Ae^{-b(bz^2+vt)}$
- $f_5(z,t) = A\sin(bz)\cos(bvt)^3$

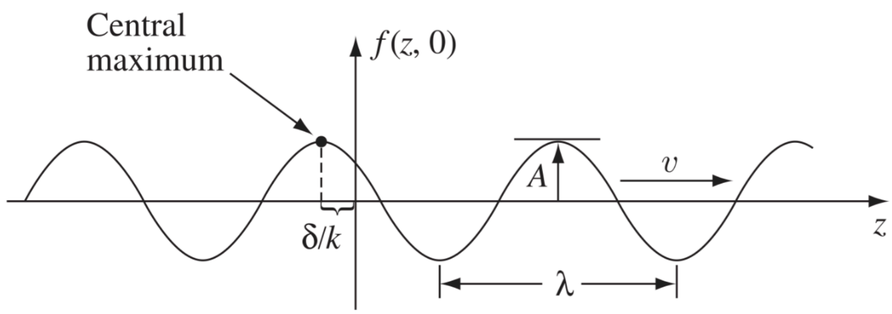
Wave Equation



- Net transverse force
$$\Delta F = T\sin\theta' - T\sin\theta$$
- Provided the distortion of the string of the string is not too great, θ' and θ are small,
$$\Delta F \approx T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \approx T\frac{\partial^2 f}{\partial z^2}\Delta z$$
- μ: the mass per unit length. By Newton's second law,
$$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2}$$
- Therefore,
$$\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}$$
- Let $v = \sqrt{\frac{T}{\mu}}$,
$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad \text{(Wave Equation)}$$

3.2 Sinusoidal Waves

Terminology



- **Sinusoidal Wave**
$$f(z,t) = A\cos[k(z - vt) + \delta]$$
- **Amplitude (maximum displacement from equilibrium, positive):**
$$A$$
- **Phase:**
$$k(z - vt) + \delta$$
- **Phase Constant:**
$$\delta$$
ordinarily,
$$0 \leq \delta < 2\pi$$
- **Wave Number:**
$$k$$
- **Wavelength:**
$$\lambda = \frac{2\pi}{k}$$
- **Period:**
$$T = \frac{2\pi}{kv}$$
- **Frequency (number of oscillations per unit time):**
$$\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}$$
- **Angular Frequency (number of radians swept out per unit time):**
$$\omega = 2\pi\nu = kv$$

- **Sinusoidal Waves in terms of ω:**
$$f(z,t) = A\cos(kz - \omega t + \delta)$$
- **Traveling to the left:**
$$f(z,t) = A\cos(kz + \omega t - \delta) = A\cos(-kz - \omega t + \delta)$$

Complex Notation

- **Advantage:** Exponentials are much easier to manipulate than sins and cosines
- **Euler's Formula**
$$e^{i\theta} = \cos\theta + i\sin\theta$$
- Therefore,
$$f(z,t) = A\cos(kz - \omega t + \delta) = \text{Re}[Ae^{i(kz-\omega t+\delta)}]$$
- **Real part of the complex number ξ:**
$$\text{Re}(\xi)$$
- **Complex Wave Function:**
$$\tilde{f}(z,t) \equiv \tilde{A}e^{i(zk-\omega t)}$$
- **Complex Amplitude:**
$$\tilde{A} \equiv Ae^{i\delta}$$
- **Actual Wave function:**
$$f(z,t) = \text{Re}[\tilde{f}(z,t)]$$

Example: Combination of Two Sinusoidal Waves

- Given
$$\tilde{f}_1 = \tilde{A}_1 e^{i(kz-\omega t)}$$
$$\tilde{f}_2 = \tilde{A}_2 e^{i(kz-\omega t)}$$
- The combination of the two waves is
$$\tilde{f}_1 + \tilde{f}_2 = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)} = (\tilde{A}_1 + \tilde{A}_2)e^{i(kz-\omega t)}$$
- In other words, we only need to add the complex amplitudes.

Linear Combinations of Sinusoidal Waves

- Any wave can be expressed as a linear combination of sinusoidal ones:
$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}\,dk$$
- If we know how sinusoidal waves behave, we know in principle how any wave behaves.