

8. Calculus with curves

8.1. Parametrized curves

8.1.1 Simple regular parametrizations

Definition 8.1.1 (Simple regular parametrization) Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of a curve C . We say γ is a simple regular parametrization if γ is injective and $\gamma'(t) \neq 0$ for all $t \in [a, b]$.

Example 8.1.1 (Simple regular parametrization) The unit circle C in \mathbb{R}^2 is parametrized by $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. This is a simple regular parametrization.

Example 8.1.2 (Simple regular parametrization) The helix C in \mathbb{R}^3 is parametrized by $\gamma(t) = (\cos t, \sin t, t)$ for $t \in [0, 2\pi]$. This is a simple regular parametrization.

Example 8.1.3 (Simple regular parametrization) The curve C in \mathbb{R}^3 is parametrized by $\gamma(t) = (t, t^2, t^3)$ for $t \in [0, 1]$. This is a simple regular parametrization.

8.1.2 Curves and piecewise curves

Definition 8.1.2 (Curve and piecewise curve) A curve C in \mathbb{R}^n is a set of points in \mathbb{R}^n that can be parametrized by a continuous function $\gamma: [a, b] \rightarrow \mathbb{R}^n$. A piecewise curve is a curve that is the union of finitely many simple regular curves.

Example 8.1.2 (Curve and piecewise curve) The unit circle C in \mathbb{R}^2 is a curve. The helix C in \mathbb{R}^3 is a curve. The curve C in \mathbb{R}^3 is a piecewise curve.

8.1.3 Reparametrizations and orientation

Definition 8.1.3 (Reparametrization and orientation) Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of a curve C . A reparametrization of γ is a function $\tilde{\gamma}: [\tilde{a}, \tilde{b}] \rightarrow \mathbb{R}^n$ such that $\tilde{\gamma}(\tilde{t}) = \gamma(t)$ for some strictly increasing function $t = t(\tilde{t})$. The orientation of a curve is the direction in which the curve is traversed as the parameter increases.

Example 8.1.3 (Reparametrization and orientation) The unit circle C in \mathbb{R}^2 is parametrized by $\gamma(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. A reparametrization of γ is $\tilde{\gamma}(\tilde{t}) = (\cos 2\tilde{t}, \sin 2\tilde{t})$ for $\tilde{t} \in [0, \pi]$. The orientation of C is counter-clockwise.

8.2. Arc length

8.2.1 Definition and invariance

Definition 8.2.1 (Arc length) Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of a curve C . The arc length of C is the length of the curve, measured along the curve. It is invariant under reparametrization.

Example 8.2.1 (Arc length) The arc length of the unit circle C in \mathbb{R}^2 is 2π .

8.2.2 Arc length parametrization

Definition 8.2.2 (Arc length parametrization) Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of a curve C . An arc length parametrization of C is a parametrization $\tilde{\gamma}: [0, L] \rightarrow \mathbb{R}^n$ such that the arc length of $\tilde{\gamma}$ from \tilde{a} to \tilde{b} is $|\tilde{b} - \tilde{a}|$.

Example 8.2.2 (Arc length parametrization) The arc length parametrization of the unit circle C in \mathbb{R}^2 is $\tilde{\gamma}(\tilde{t}) = (\cos \tilde{t}, \sin \tilde{t})$ for $\tilde{t} \in [0, 2\pi]$.

8.2.3 Derivation of arc length

Theorem 8.2.3 (Derivation of arc length) Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of a curve C . The arc length of C is given by the formula $L = \int_a^b \|\gamma'(t)\| dt$.

Example 8.2.3 (Derivation of arc length) The arc length of the unit circle C in \mathbb{R}^2 is $L = \int_0^{2\pi} \|\gamma'(t)\| dt = \int_0^{2\pi} 1 dt = 2\pi$.

8.2.4 Line integrals of scalar functions

Definition 8.2.4 (Line integral of scalar functions) Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of a curve C . The line integral of a scalar function f over C is given by the formula $\int_C f d\mathbf{s} = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt$.

Example 8.2.4 (Line integrals of scalar functions) The line integral of the function $f(x, y) = x^2 + y^2$ over the unit circle C in \mathbb{R}^2 is $\int_C f d\mathbf{s} = \int_0^{2\pi} 1 dt = 2\pi$.

8.2.5 Elements and infinitesimals

8.3. Line integrals

8.3.1 Oriented curves

Definition 8.3.1 (Oriented curve) Let C be a curve in \mathbb{R}^n . An orientation of C is a direction in which the curve is traversed. It is given by a parametrization $\gamma: [a, b] \rightarrow \mathbb{R}^n$ of C .

Example 8.3.1 (Oriented curve) The unit circle C in \mathbb{R}^2 is oriented counter-clockwise.

8.3.2 Line integrals of vector fields

Definition 8.3.2 (Line integrals of vector fields) Let $\gamma: [a, b] \rightarrow \mathbb{R}^n$ be a parametrization of a curve C . The line integral of a vector field F over C is given by the formula $\int_C F \cdot d\mathbf{s} = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$.

Example 8.3.2 (Line integrals of vector fields) The line integral of the vector field $F(x, y) = (x, y)$ over the unit circle C in \mathbb{R}^2 is $\int_C F \cdot d\mathbf{s} = 0$.

$$\int_C F \cdot d\mathbf{s} = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$$

8.3.3 Definition of work done along a curve

8.4. Fundamental theorem of line integrals

8.4.1 Statement and proof

Theorem 8.4.1 (Fundamental theorem of line integrals) Let F be a vector field in \mathbb{R}^n and C be a curve in \mathbb{R}^n parametrized by $\gamma: [a, b] \rightarrow \mathbb{R}^n$. If F is conservative, then the line integral of F over C is given by the difference of the potential function at the endpoints of C .

Example 8.4.1 (Fundamental theorem of line integrals) The line integral of the vector field $F(x, y) = (x, y)$ over the unit circle C in \mathbb{R}^2 is $\int_C F \cdot d\mathbf{s} = 0$.

8.4.2 Conservative vector fields and potentials

Definition 8.4.2 (Conservative vector fields and potentials) A vector field F in \mathbb{R}^n is conservative if it is the gradient of a scalar function ϕ , called a potential function. That is, $F = \nabla \phi$.

Example 8.4.2 (Conservative vector fields and potentials) The vector field $F(x, y) = (x, y)$ is conservative, with potential function $\phi(x, y) = \frac{1}{2}(x^2 + y^2)$.

8.4.3 Irrotational vector fields

Definition 8.4.3 (Irrotational vector fields) A vector field F in \mathbb{R}^n is irrotational if its curl is zero. That is, $\nabla \times F = 0$.

Example 8.4.3 (Irrotational vector fields) The vector field $F(x, y) = (x, y)$ is irrotational.

8.5. Conservative vector fields

8.5.1 Physical viewpoints

Definition 8.5.1 (Physical viewpoints) A conservative vector field is one that represents a force field. The line integral of a conservative vector field over a curve is the work done by the force along the curve.

Example 8.5.1 (Physical viewpoints) The vector field $F(x, y) = (x, y)$ represents a force field. The line integral of F over the unit circle C in \mathbb{R}^2 is the work done by the force along the circle.

8.5.2 Path independence

Definition 8.5.2 (Path independence) A vector field F in \mathbb{R}^n is path independent if the line integral of F over any curve depends only on the endpoints of the curve.

Example 8.5.2 (Path independence) The vector field $F(x, y) = (x, y)$ is path independent.

8.5.3 Irrotational vector fields and exact differential forms

Definition 8.5.3 (Irrotational vector fields and exact differential forms) A vector field F in \mathbb{R}^n is irrotational if and only if it is the gradient of a scalar function. This is equivalent to saying that the differential form $F \cdot d\mathbf{s}$ is exact.

Example 8.5.3 (Irrotational vector fields and exact differential forms) The vector field $F(x, y) = (x, y)$ is irrotational, and the differential form $F \cdot d\mathbf{s}$ is exact.

$$\int_C F \cdot d\mathbf{s} = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt$$

8.6. Circulation and flux in 2D

8.6.1 Circulation and curl in 2D

Definition 8.6.1 (Circulation and curl in 2D) Let C be a closed curve in \mathbb{R}^2 . The circulation of a vector field F over C is the line integral of F over C . The curl of F is a scalar function $\text{curl } F$ defined by $\text{curl } F = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$.

Example 8.6.1 (Circulation and curl in 2D) The circulation of the vector field $F(x, y) = (-y, x)$ over the unit circle C in \mathbb{R}^2 is 2π .

- turns counter-clockwise, $(\text{curl } F)(p) > 0$.
- turns clockwise, $(\text{curl } F)(p) < 0$.
- does not turn at all, $(\text{curl } F)(p) = 0$.

8.6.2 Unit normal and positive orientation

Definition 8.6.2 (Unit normal and positive orientation) Let C be a closed curve in \mathbb{R}^2 . A unit normal vector \mathbf{n} to C is a vector of length 1 perpendicular to the tangent vector $\gamma'(t)$. Positive orientation is the orientation in which the curve is traversed such that the region inside the curve is to the left.

Example 8.6.2 (Unit normal and positive orientation) The unit normal vector to the unit circle C in \mathbb{R}^2 is $\mathbf{n} = (x, y)$.

- $\mathbf{n} = (x, y)$ for the unit circle C in \mathbb{R}^2 .
- $\mathbf{n} = (-y, x)$ for the unit circle C in \mathbb{R}^2 .
- $\mathbf{n} = (x, y)$ for the unit circle C in \mathbb{R}^2 .

8.6.3 Flux and divergence in 2D

Definition 8.6.3 (Flux and divergence in 2D) Let C be a closed curve in \mathbb{R}^2 . The flux of a vector field F over C is the line integral of $F \cdot \mathbf{n}$ over C . The divergence of F is a scalar function $\text{div } F$ defined by $\text{div } F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}$.

Example 8.6.3 (Flux and divergence in 2D) The flux of the vector field $F(x, y) = (x, y)$ over the unit circle C in \mathbb{R}^2 is 2π .

$$\text{div}(G) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}$$

8.7. Green's theorem and curl

8.7.1 Regular regions and orienting the boundary

Definition 8.7.1 (Regular regions and orienting the boundary) A region R in \mathbb{R}^2 is regular if its boundary C is a piecewise smooth curve. The boundary C is oriented positively if it is traversed in the counter-clockwise direction.

Example 8.7.1 (Regular regions and orienting the boundary) The unit circle C in \mathbb{R}^2 is a regular region, oriented positively.

8.7.2 Statement and proof

Theorem 8.7.2 (Green's theorem) Let R be a regular region in \mathbb{R}^2 with boundary C . If F is a vector field in \mathbb{R}^2 , then the line integral of F over C is equal to the double integral of the curl of F over R .

Example 8.7.2 (Statement and proof) The line integral of the vector field $F(x, y) = (-y, x)$ over the unit circle C in \mathbb{R}^2 is 2π .

8.7.3 Examples with Green's theorem

Example 8.7.3 (Examples with Green's theorem) Use Green's theorem to compute the line integral of the vector field $F(x, y) = (-y, x)$ over the unit circle C in \mathbb{R}^2 .

Example 8.7.4 (Examples with Green's theorem) Use Green's theorem to compute the line integral of the vector field $F(x, y) = (x, y)$ over the unit circle C in \mathbb{R}^2 .

- calculate a line integral directly from the definition.
- find closed curves of line integrals (Theorem 8.4.1).
- apply your newly learned Green's Theorem.

8.8. Green's theorem and divergence

8.8.1 Regular regions and orienting the boundary

Definition 8.8.1 (Regular regions and orienting the boundary) A region R in \mathbb{R}^2 is regular if its boundary C is a piecewise smooth curve. The boundary C is oriented positively if it is traversed in the counter-clockwise direction.

Example 8.8.1 (Regular regions and orienting the boundary) The unit circle C in \mathbb{R}^2 is a regular region, oriented positively.

8.8.2 Statement and proof

Theorem 8.8.2 (Green's theorem) Let R be a regular region in \mathbb{R}^2 with boundary C . If F is a vector field in \mathbb{R}^2 , then the line integral of F over C is equal to the double integral of the divergence of F over R .

Example 8.8.2 (Statement and proof) The line integral of the vector field $F(x, y) = (x, y)$ over the unit circle C in \mathbb{R}^2 is 2π .

8.8.3 Examples with Green's theorem

Example 8.8.3 (Examples with Green's theorem) Use Green's theorem to compute the line integral of the vector field $F(x, y) = (x, y)$ over the unit circle C in \mathbb{R}^2 .

Example 8.8.4 (Examples with Green's theorem) Use Green's theorem to compute the line integral of the vector field $F(x, y) = (-y, x)$ over the unit circle C in \mathbb{R}^2 .

$$\text{div}(G) = \frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}$$