1. Maps

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- **1.1. Curves** 10
  - **parametric curves** 10
    - **vector-valued functions of a real variable**, especially when $n \geq 2$. 10
  - **1.1.1 Motion** 11
    - **component functions** 12

      $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t))$
    - **unit tangent vector**, 12

      $$T(t) = \frac{\gamma'(t)}{||\gamma'(t)||}.$$
  - **1.1.2 Frenet frame in three dimensions** 13
    - **(principal) unit normal** 13

      $$N(t) = \frac{T'(t)}{||T'(t)||}.$$
  - **1.1.3 Geometry of curves** 15
    - The **trace** of a parametric curve $\gamma : I \to \mathbb{R}^n$ is the image of γ, that is, $\gamma(I)$. 15
    - **Definition 1.1.14** Let $C \subseteq \mathbb{R}^n$ be a set. We say C is a **curve** in $\mathbb{R}^n$ if C is the trace of a continuous[2] parametric curve $\gamma : I \to \mathbb{R}^n$. 16
- **1.2. Real-valued functions** 17
  - **1.2.1 Scalar fields and densities** 17
  - **1.2.2 Graphs, level sets, and slices** 20
    - **Definition 1.2.11** Let $A \subseteq \mathbb{R}^n$. The **graph of a function** $f : A \to \mathbb{R}$ is the set in $\mathbb{R}^{n+1}$ given by $$\{(x, f(x)) : x \in A\}.$$ 20
    - **Definition 1.2.14** Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ be a real-valued function. Fix $k \in \mathbb{R}$. The **level set** of f at k is the set $\{x \in \mathbb{R}^n : f(x) = k\}$. This is also called the k-**level set**. 21
      - A level set in $\mathbb{R}^2$ is also called a **contour**. 21
    - **Definition 1.2.19** Let $A \subseteq \mathbb{R}^2$ and $f : A \to \mathbb{R}$ be a real-valued function. 23
      - For fixed $a \in \mathbb{R}$, the x-**slice at** a **of the graph of** f is the set $\{(y, z) : z = f(a, y)\}$.
      - For fixed $b \in \mathbb{R}$, the y-**slice at** b **of the graph of** f is the set $\{(x, z) : z = f(x, b)\}$.
    - **Definition 1.2.21** Let $A \subseteq \mathbb{R}^3$ and $f : A \to \mathbb{R}$ be a real-valued function. 24
      - For fixed $a \in \mathbb{R}$, the x-**slice at** a **of the graph of** f is the set $\{(y, z, w) : w = f(a, y, z)\}$.
      - For fixed $b \in \mathbb{R}$, the y-**slice at** b **of the graph of** f is the set $\{(x, z, w) : w = f(x, b, z)\}$.
      - For fixed $c \in \mathbb{R}$, the z-**slice at** c **of the graph of** f is the set $\{(x, y, w) : w = f(x, y, c)\}$.
- **1.3. Vector fields** 25
  - **Definition 1.3.1** A **(n-dimensional) vector field** is a function F with domain and codomain lying in $\mathbb{R}^n$. 25
    - F **is a vector field in** $\mathbb{R}^n$. 25

      $F(x) \in \mathbb{R}^n$ is a vector at the point $x \in \mathbb{R}^n$.
    - **Notation** 27
      - $F(x, y, z) = (x^2, yx, -z),$ 27
      - $F(x, y, z) = \langle x^2, yx, -z \rangle,$ 27
      - $F = [x^2, yx, -z],$ 27
      - $F = x^2\hat{i} + yx\hat{j} - z\hat{k}.$ 27
- **1.4. Coordinate transformations** 28
  - **coordinate transformation** 28
  - **1.4.1 Polar coordinates** 29
    - Formally, define the **polar coordinate transformation** $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $$T(r, \theta) = (r\cos\theta, r\sin\theta)$$ 29
    - **Lemma 1.4.10** The polar coordinate transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $$T(r, \theta) = (r\cos\theta, r\sin\theta)$$ maps the subset $(0, \infty) \times (-\pi, \pi)$ bijectively to the subset $\mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\}$. 33
  - **1.4.2 Cylindrical coordinates** 34
    - Define the **cylindrical coordinate transformation** $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $$T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$ 34
    - **Lemma 1.4.15** The cylindrical coordinate transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $$T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$ maps the subset $(0, \infty) \times (-\pi, \pi) \times \mathbb{R}$ bijectively to the subset $\mathbb{R}^3 \setminus \{(x, 0, z) : x \leq 0, z \in \mathbb{R}\}$. 35
  - **1.4.3 Spherical coordinates** 36
    - Define the **spherical coordinate transformation** $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $$T(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$ 36
    - **Lemma 1.4.20** The spherical coordinate transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $$T(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$ maps the subset $(0, \infty) \times (-\pi, \pi) \times (0, \pi)$ bijectively to the subset $\mathbb{R}^3 \setminus \{(x, 0, z) : x \leq 0, z \in \mathbb{R}\}$. 38
- **1.5. Surfaces** 39
  - **1.5.1 Parametric surfaces** 39
    - **parametric surfaces.** 39

      $$\mathbb{R}^n \longrightarrow \mathbb{R}^m$$

      when $n < m$.
    - The **unit sphere** is the sphere of radius 1 centred at the origin. 39
    - **Definition 1.5.3** Let $m, n \in \mathbb{N}^+$ with $n < m$. Let $S \subseteq \mathbb{R}^m$ be a set. Let $A \subseteq \mathbb{R}^n$ be a set and let $g : A \to \mathbb{R}^m$ be a continuous map. If $S = \{g(x) : x \in A\} = \mathrm{img}(g)$, then the pair (S, g) is a **parametric surface** or, equivalently, the set S **is parametrized by** g. 40
  - **1.5.2 Explicit surfaces** 40
    - **Definition 1.5.7** Let $m, n \in \mathbb{N}^+$. Let $A \subseteq \mathbb{R}^n$. The **graph** of a function $f : A \to \mathbb{R}^m$ is the set $$S = \{(x, f(x)) : x \in A\} \subseteq \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{m+n}.$$ A set $S' \subseteq \mathbb{R}^{m+n}$ is a graph of f if S' is the same as S after reordering the variables. 41
      - **Remark 1.5.8** "Reordering variables" can be formally expressed as $S' = \pi(S)$ for a linear transformation $\pi(x) = Px$ where P is a $(m+n) \times (m+n)$ permutation matrix. 41
    - **Definition 1.5.9** A set $S \subseteq \mathbb{R}^n$ is an **explicit surface** if S is a graph of a continuous function. 42
  - **1.5.3 Implicit surfaces** 42
    - **Definition 1.5.14** A set $S \subseteq \mathbb{R}^n$ is an **implicit surface** if there exists a constant $c \in \mathbb{R}^m$, a set $A \subseteq \mathbb{R}^n$, and a continuous function $f : A \to \mathbb{R}^m$ such that $$S = f^{-1}(\{c\}) = \{x \in \mathbb{R}^n : f(x) = c\}.$$ 43
      - **Remark 1.5.15** The notation f^{-1} is *not* the inverse function of f unless specified otherwise. In fact, it is not a function at all. It is an operation on sets. The **preimage** of a set C under a function $f : A \to B$ is the set $$f^{-1}(C) = \{x \in A : f(x) \in C\}.$$ If the set $C = \{c\}$ is a singleton, then the condition $f(x) \in C$ becomes $f(x) = c$. It is incorrect to write $f^{-1}(c)$ instead of $f^{-1}(\{c\})$ because the expression $f^{-1}(c)$ only makes sense if f^{-1} is the inverse function of f. 43
- **1.6. Projections** 45
  - **Example 1.6.3** For $i \in \{1, \ldots, n\}$, the map $\pi_i : \mathbb{R}^n \to \mathbb{R}$ given by $$\pi_i(x_1, \ldots, x_n) = x_i$$ is the i^{th} **coordinate map**. These are convenient for proofs whe 45
    - **Remark 1.6.4** The Greek letter π is often used for projections. Since π is used for other reasons, this is an *abuse of notation*. You are permitted to abuse notation provided the context makes your notation unambiguous. 46