2. Topology

2.1. Sets

2.1.1 Balls and spheres

- **Definition 2.1.1** Let $r \geq 0$ and $a \in \mathbb{R}^n$.
  - The **open ball of radius r centred at a** is the set
    $$B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}.$$
    - **Remark 2.1.2** Other notations for the open ball include $B(a, r)$, $B(a; r)$, or $B_a(r)$. None of these will appear in this text, but they may appear in other sources.
    - **Remark 2.1.3** An open ball (or closed ball) is **punctured** if it excludes the centre. For example, $B_r(a) \setminus \{a\}$ is a punctured open ball.
  - The **closed ball of radius r centered at a** is the set
    $$\{x \in \mathbb{R}^n : \|x - a\| \leq r\}.$$
  - The **sphere of radius r centred at a** is the set
    $$\{x \in \mathbb{R}^n : \|x - a\| = r\}.$$
- **Definition 2.1.7** The **$(n-1)$-dimensional unit sphere** in $\mathbb{R}^n$ is the sphere of radius 1 centred at the origin and is denoted S^{n-1}. In other words,
  $$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

2.1.2 Rectangles

- **Definition 2.1.9** A **(closed) rectangle** in $\mathbb{R}^n$ is a set R of the form
  $$R = [a_1, b_1] \times \cdots \times [a_n, b_n] = \{(x_1, \ldots, x_n) : x_i \in [a_i, b_i] \text{ for } 1 \leq i \leq n\},$$
  where $a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$ and $a_i < b_i$ for all $1 \leq i \leq n$.
  - **Remark 2.1.10** The set $(a_1, b_1) \times \cdots \times (a_n, b_n)$ is an open rectangle. In general, a "rectangle" refers to a closed rectangle unless specified otherwise.
- **Definition 2.1.14** A **(n-dimensional) hypercube** is a set in $\mathbb{R}^n$ of the form
  $$[a, b]^n = \underbrace{[a, b] \times \cdots \times [a, b]}_{n \text{ times}}$$
  The **(n-dimensional) unit hypercube** is the set $[0, 1]^n$.

2.2. Interior, boundary, and closure

2.2.1 Interior

- **Definition 2.2.1** Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is an **interior point** of A if there exists $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq A$.
- **Definition 2.2.4** Let $A \subseteq \mathbb{R}^n$ be a set. The **interior** of A, denoted A° or $\mathrm{int}(A)$, is the set of interior points of A.
  - Other texts use $\mathring{A}$ or A^{int} to denote the interior.
- **Lemma 2.2.10** Let A and B be sets of $\mathbb{R}^n$. Then
  (a) $A^\circ \subseteq A$
  (b) $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$
  (c) $(A \cap B)^\circ = A^\circ \cap B^\circ$
  (d) $(A \times B)^\circ = A^\circ \times B^\circ$

2.2.2 Boundary

- **Definition 2.2.11** Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is a **boundary point** of A if for every $\varepsilon > 0$, the sets $B_\varepsilon(p) \cap A$ and $B_\varepsilon(p) \cap A^c$ are both non-empty.
- **Definition 2.2.14** Let $A \subseteq \mathbb{R}^n$ be a set. The **(topological) boundary** of A, denoted ∂A, is the set of boundary points of A.
- **Lemma 2.2.21** For any set $A \subseteq \mathbb{R}^n$, its interior A° and boundary ∂A are disjoint.

2.2.3 Closure

- **Definition 2.2.22** Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is a **limit point** of A if for every $\varepsilon > 0$, the set $B_\varepsilon(p) \setminus \{p\}$ contains points in A.
- **Definition 2.2.26** Let $A \subseteq \mathbb{R}^n$. The **closure** of A, denoted $\bar{A}$ or $\mathrm{cl}(A)$, is the the set A along with its limit points.
  - **Remark 2.2.27** The set of limit points of A is often denoted A' so $\bar{A} = A \cup A'$. Other texts may use A' for the set of limit points though you will not see that convention here.
- **Lemma 2.2.31** Let A and B be sets in $\mathbb{R}^n$. Then
  (a) $A \subseteq \bar{A}$
  (b) $\mathrm{cl}(A \cup B) = \bar{A} \cup \bar{B}$
  (c) $\mathrm{cl}(A \cap B) \subseteq \bar{A} \cap \bar{B}$
  (d) $\mathrm{cl}(A \times B) = \bar{A} \times \bar{B}$
- **Lemma 2.2.32** Let $A \subseteq \mathbb{R}^n$ be a set. Then $\bar{A} = A^\circ \cup \partial A$ and $\partial A = \bar{A} \setminus A^\circ$.

2.3. Sequences

- **Definition 2.3.1** A **sequence** in $\mathbb{R}^n$ is a function with domain $\{k \in \mathbb{Z} : k \geq k_0\}$ for some fixed $k_0 \in \mathbb{Z}$ and codomain $\mathbb{R}^n$.
  - $(x(k)) \quad (x(k))_k \quad (x(k))_{k=k_0}^\infty$
  - $\{x(k)\} \quad \{x(k)\}_k \quad \{x(k)\}_{k=k_0}^\infty$
  - It is also common to write x_k instead of $x(k)$. Unless stated otherwise, you may assume the domain of a sequence is $\mathbb{N}^* = \{1, 2, 3, \ldots\}$.

2.3.1 Convergence of sequences

- **Definition 2.3.4** Let $\{x(k)\}_k$ be a sequence in $\mathbb{R}^n$. Then $\{x(k)\}_k$ **converges to** $p \in \mathbb{R}^n$ if
  $$\forall \varepsilon > 0, \exists K \in \mathbb{N} \text{ s.t. } \forall k \in \mathbb{N}, k \geq K \implies \|x(k) - p\| < \varepsilon.$$
  If the above holds, write $\lim_{k \to \infty} x(k) = p$ or equivalently $x(k) \to p$.
  The sequence $\{x(k)\}_k$ **converges** if there exists $p \in \mathbb{R}^n$ such that $\lim_{k \to \infty} x(k) = p$. Otherwise, the sequence $\{x(k)\}_k$ **diverges**.
- **Lemma 2.3.9** Let $\{x(k)\}_k$ be a sequence in $\mathbb{R}^n$ with $x(k) = (x_1(k), \ldots, x_n(k))$. The sequence $\{x(k)\}_k$ converges if and only if $\{x_i(k)\}_k$ converges for all $i = 1, 2, \ldots, n$.

2.3.2 Limit points, boundary points, and interior points

- **Lemma 2.3.12** Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is a limit point of A if and only if there exists a sequence of points in $A \setminus \{p\}$ which converges to p.
- **Lemma 2.3.14** Let $A \subseteq \mathbb{R}^n$ be a set. Let $p \in \mathbb{R}^n$ be a point.
  - The point p is an interior point of A if and only if for every sequence $\{x(k)\}_k$ of points converging to p, there exists $K \in \mathbb{N}^*$ such that $\{x(k)\}_{k=K}^\infty \subseteq A$.
  - The point p is a boundary point of A if and only if there exists a sequence of points in A converging to p and there exists a sequence of points in A^c converging to p.

2.4. Open sets and closed sets

2.4.1 Open sets

- **Definition 2.4.1** A set $A \subseteq \mathbb{R}^n$ is **open** if every point of A is an interior point of A.
- **Lemma 2.4.4** The interior of a set $A \subseteq \mathbb{R}^n$ is open.
- **Lemma 2.4.5** Let $A \subseteq \mathbb{R}^n$. Then all of the following are equivalent:
  (a) A is open.
  (b) $A = A^\circ$.
  (c) $A \cap \partial A = \varnothing$.

2.4.2 Closed sets

- **Definition 2.4.7** A set $A \subseteq \mathbb{R}^n$ is **closed**[5] if every limit point of A belongs to A.
- **Lemma 2.4.10** The closure of a set A is closed.
- **Lemma 2.4.11** Let $A \subseteq \mathbb{R}^n$. All of the following are equivalent:
  (a) A is closed.
  (b) $A = \bar{A}$.
  (c) $\partial A \subseteq A$.

2.4.3 Set operations

- **Lemma 2.4.13** A set $A \subseteq \mathbb{R}^n$ is open if and only if its complement $A^c = \mathbb{R}^n \setminus A$ is closed.
- **Lemma 2.4.17** All of the following are true for sets in $\mathbb{R}^n$:
  (a) A finite intersection of open sets is open.
  (b) A finite or infinite union of open sets is open.
  (c) A finite union of closed sets is closed.
  (d) A finite or infinite intersection of closed sets is closed.
  (e) A finite Cartesian product of open sets is open.
  (f) A finite Cartesian product of closed sets is closed.

2.5. Compact sets

2.5.1 Definitions of compactness

- **Definition 2.5.2** Let $x : \mathbb{N}^* \to \mathbb{R}^n$ be a sequence and let $m : \mathbb{N}^* \to \mathbb{N}^*$ be a strictly increasing function. The sequence $\{x(m(k))\}_{k=1}^\infty$ is a **subsequence** of the sequence $\{x(k)\}_{k=1}^\infty$.
- **Definition 2.5.5** A set $A \subseteq \mathbb{R}^n$ is **compact**[6] if every sequence of A has a subsequence which converges to a point lying inside A.
- **Definition 2.5.7** A set $A \subseteq \mathbb{R}^n$ is **bounded** if $\exists R > 0$ such that $A \subseteq \{x \in \mathbb{R}^n : \|x\| < R\}$. If a set is not bounded, then it is **unbounded**.
- **Theorem 2.5.10** (Bolzano–Weierstrass) A set in $\mathbb{R}^n$ is compact if and only if it is both closed and bounded.

2.5.2 Set operations and subsets

- **Lemma 2.5.14** All of the following are true for sets in $\mathbb{R}^n$.
  (a) A finite union of compact sets is compact.
  (b) A finite or infinite intersection of compact sets is compact.
  (c) A finite Cartesian product of compact sets is compact.
- **Lemma 2.5.17** Let A be a compact set in $\mathbb{R}^n$. If $B \subseteq A$ and B is closed then B is compact.

2.5.3 Proof of Bolzano–Weierstrass

2.6. Limits

2.6.1 Formal definitions

- **Definition 2.6.1** Let $f : A \to \mathbb{R}^m$ be a function with $A \subseteq \mathbb{R}^n$. Let $a \in \mathbb{R}^n$ be a limit point of A and let $b \in \mathbb{R}^m$. Define b to be the **limit of f at a** provided
  $$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, 0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon.$$
  If the above holds, then write $\lim_{x \to a} f(x) = b$ or write $f(x) \to b$ as $x \to a$.
  - **Remark 2.6.2** If there exists $b \in \mathbb{R}^m$ such that $\lim_{x \to a} f(x) = b$ then the **limit of f at a exists**. Otherwise, the limit does not exist.
- **Lemma 2.6.8** Let $A \subseteq \mathbb{R}^n$ be a set and let $f : A \to \mathbb{R}^m$ be a function. Let $a \in \mathbb{R}^n$ be a limit point of A and let $b \in \mathbb{R}^m$. Then $\lim_{x \to a} f(x) = b$ if and only if for every sequence of points $\{x(k)\}_k$ in $A \setminus \{a\}$ with $x(k) \to a$, the sequence of points $\{f(x(k))\}_k$ in $\mathbb{R}^m$ converges to b; that is, $f(x(k)) \to b$.

2.6.2 Basic properties

- **Theorem 2.6.10** Let $f : A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$. Let a be a limit point of A and let $b = (b_1, \ldots, b_m) \in \mathbb{R}^m$. Let $f_1, \ldots, f_m$ be the coordinate functions of f so $f = (f_1, \ldots, f_m)$. Then
  $$\lim_{x \to a} f(x) = b$$
  if and only if for all $i \in \{1, \ldots, m\}$,
  $$\lim_{x \to a} f_i(x) = b_i$$
- **Theorem 2.6.12** Let $A \subseteq \mathbb{R}^n$ be a set and let a be a limit point of A. Let f and g be $\mathbb{R}^m$-valued functions defined on A. Let ϕ be a real-valued function defined on A. Let $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}^m$ be constants. All of the following hold:
  - (a) (*Constants*) $\lim_{x \to a} b = b$ and $\lim_{x \to a} x = a$.
  - (b) (*Linearity*) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist then $\lim_{x \to a} (f(x) + \lambda g(x))$ exists and
    $$\lim_{x \to a} (f(x) + \lambda g(x)) = \lim_{x \to a} f(x) + \lambda \lim_{x \to a} g(x).$$
  - (c) (*Dot product*) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist then $\lim_{x \to a} (f(x) \cdot g(x))$ exists and
    $$\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right).$$
  - (d) (*Scalar product*) If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} \phi(x)$ exist then $\lim_{x \to a} (\phi(x) f(x))$ exists and
    $$\lim_{x \to a} (\phi(x) f(x)) = \left(\lim_{x \to a} \phi(x)\right)\left(\lim_{x \to a} f(x)\right).$$
- **Theorem 2.6.13** (Squeeze theorem) Let $A \subseteq \mathbb{R}^n$ be a set and let a be a limit point of A. Let f, g, h be real-valued functions with domain A. Assume there exists $\delta > 0$ such that
  $$\forall x \in A, 0 < \|x - a\| < \delta \implies f(x) \leq g(x) \leq h(x)$$
  If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = b$ for some $b \in \mathbb{R}$ then $\lim_{x \to a} g(x) = b$.

2.6.3 Limits with infinity

- **Definition 2.6.14** Let $A \subseteq \mathbb{R}^n$ be unbounded. Let $f : A \to \mathbb{R}^m$ and let $b \in \mathbb{R}^m$. Define b to be the **limit of $f(x)$ as $\|x\| \to \infty$** provided
  $$\forall \varepsilon > 0, \exists M > 0 \text{ s.t. } \forall x \in A, \|x\| > M \implies \|f(x) - b\| < \varepsilon.$$
  If the above holds, then write $\lim_{\|x\| \to \infty} f(x) = b$ or write $f(x) \to b$ as $\|x\| \to \infty$.
  If the above does not hold, the limit $\lim_{\|x\| \to \infty} f(x)$ does not exist.
- **Definition 2.6.16** Let $A \subseteq \mathbb{R}^n$ be a set. Let a be a limit point of A. Let $f : A \to \mathbb{R}$ be a real-valued function. The **limit of f at a diverges to $+\infty$** provided
  $$\forall M > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, 0 < \|x - a\| < \delta \implies f(x) > M.$$
  If the above holds, then write $\lim_{x \to a} f(x) = +\infty$ or write $f(x) \to +\infty$ as $x \to a$.
  The definition for $f(x) \to -\infty$ as $x \to a$ is similar.

2.7. Continuity

2.7.1 Formal definitions

- **Definition 2.7.1** Let $f : A \to \mathbb{R}^m$ be a function with domain $A \subseteq \mathbb{R}^n$. Let $a \in A$ be a point. The function f is **continuous at a** provided
  $$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, \|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.$$
- **Lemma 2.7.2** Let $f : A \to \mathbb{R}^m$ be a function with domain $A \subseteq \mathbb{R}^n$. Let $a \in A$ be a point. Both of the following hold:
  (a) If a is an isolated point of A, then f is continuous at a.
  (b) If a is a limit point of A, then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.
- **Lemma 2.7.4** Let $f : A \to \mathbb{R}^m$ be a function with domain $A \subseteq \mathbb{R}^n$. Let $a \in A$ be a point. Then f is continuous at a if and only if for every sequence $\{x(k)\}_k$ in A converging to a, the sequence $\{f(x(k))\}_k$ in $\mathbb{R}^m$ converges to $f(a)$.
- **Definition 2.7.5** Let $f : A \to \mathbb{R}^m$ be a function with domain $A \subseteq \mathbb{R}^n$. For a subset $S \subseteq A$, the function f is **continuous on S** if f is continuous at a for every $a \in S$. The function f is **continuous** if f is continuous on its domain A.

2.7.2 Basic properties

- **Theorem 2.7.12** The map $f = (f_1, \ldots, f_m) : A \to \mathbb{R}^m$ is continuous at $a \in A$ if and only if for each $i \in \{1, \ldots, m\}$, the component function f_i is continuous at a.
- **Lemma 2.7.14** Every linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ is continuous.
- **Theorem 2.7.16** Let $A \subseteq \mathbb{R}^n$ and let $a \in A$. Let f and g be $\mathbb{R}^m$-valued functions defined on A. Let ϕ be a real-valued function defined on A. Let $\lambda \in \mathbb{R}$. All of the following hold:
  (a) If f and g are continuous at a then the function $f + \lambda g$ is continuous at a.
  (b) If f and g are continuous at a then their dot product $f \cdot g$ is continuous at a.
  (c) If f and ϕ are continuous at a then their scalar product ϕf is continuous at a.
- **Corollary 2.7.18** Let $f : A \to B$ where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Let $g : B \to \mathbb{R}^k$. Let $a \in A$. If f is continuous at a and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.
- **Theorem 2.7.19** Let $f : A \to B$ where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Let $g : B \to \mathbb{R}^k$. Let a be a limit point of A and let $b \in B$. If $\lim_{x \to a} f(x) = b$ and g is continuous at b then $\lim_{x \to a} g \circ f(x) = g(b)$.
- **Definition 2.7.22** A **monomial** in the n variables $x_1, \ldots, x_n$ is a function of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$. A **polynomial** in the n variables $x_1, \ldots, x_n$ is a linear combination of monomials in n variables with real coefficients.
- **Lemma 2.7.24** All polynomials in n variables are continuous on $\mathbb{R}^n$.

2.7.3 Topological properties

- **Theorem 2.7.25** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. The following are equivalent:
  (a) f is continuous on $\mathbb{R}^n$
  (b) The preimage $f^{-1}(U)$ is open for every open set $U \subseteq \mathbb{R}^m$
  (c) The preimage $f^{-1}(V)$ is closed for every closed set $V \subseteq \mathbb{R}^m$.
- **Theorem 2.7.29** If A is a compact subset of $\mathbb{R}^n$ and f is an $\mathbb{R}^m$-valued function that is continuous on A then $f(A)$ is a compact subset of $\mathbb{R}^m$.

2.8. Path-connected sets

- **Definition 2.8.1** A set $S \subseteq \mathbb{R}^n$ is **path-connected**[8] if for every pair of points $p, q \in S$ there exists a continuous function $\gamma : [a, b] \to \mathbb{R}^n$ such that $\gamma(a) = p$ and $\gamma(b) = q$ and $\mathrm{img}(\gamma) \subseteq S$.
- **Definition 2.8.4** A set $S \subseteq \mathbb{R}^n$ is **convex** if the line segment between any two points $p, q \in S$ lies inside S.
- **Theorem 2.8.7** Let $S \subseteq \mathbb{R}^n$ be a path-connected set. Let f be a $\mathbb{R}^m$-valued function defined on S. If f is continuous on S then $f(S)$ is path-connected.
- **Corollary 2.8.9** (Intermediate value theorem) Let f be a real-valued function defined on $[a, b]$. If f is continuous on $[a, b]$ then $f([a, b])$ is path-connected.

2.9. Global extrema

2.9.1 Definitions of global extrema

- **Definition 2.9.1** Let $A \subseteq \mathbb{R}^n$ and let f be a real-valued function defined on A.
  - A point $p \in A$ is a **(global) maximum point of f on A** if $f(p) \geq f(x)$ for all $x \in A$. If so, the value $f(p)$ is the **(global) maximum value of f on A**.
  - If a maximum point of f on A exists, then f **attains a (global) maximum on A**.
  The definitions of minimum point, minimum value, and attaining a minimum are similar.
  - **Remark 2.9.2** The word "global" is often omitted. The phrase "on A" is often omitted when A is the domain of the function. A maximum value is also called a **maximum**. The word **extremum** is used to refer to either a minimum or a maximum.

2.9.2 Extreme value theorem

- **Theorem 2.9.7** (Extreme value theorem) If $A \subseteq \mathbb{R}^n$ is a non-empty compact set and the function $f : A \to \mathbb{R}$ is continuous, then f attains maximum and minimum values at points of A.
- **Lemma 2.9.12** Let $A \subseteq \mathbb{R}^n$ be closed and unbounded. Let f be a continuous real-valued function on A. If $f(x) \to -\infty$ as $\|x\| \to \infty$ in A, then f attains a maximum on A.