7. Integration methods

- **7.1. Fubini's theorem in 2D**
  - Lemma 7.1.1 Let $f : [a,b] \to \mathbb{R}$ be a bounded real-valued function on the interval $[a,b]$. If f is integrable on $[a,b]$ then $\int_{[a,b]} f\,dV = \int_a^b f(x)\,dx$.
  - **7.1.1 Integrals of slices**
    - Definition 7.1.2 Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a function.
      - A x-slice of f is a function $f^x : [c,d] \to \mathbb{R}$ of the form $f^x(y) = f(x,y)$ for some fixed $x \in [a,b]$.
      - A y-slice of f is a function $f^y : [a,b] \to \mathbb{R}$ of the form $f^y(x) = f(x,y)$ for some fixed $y \in [c,d]$.
    - Lemma 7.1.4 Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a function.
      (a) If f is bounded, then every slice of f is bounded.
      (b) If f is continuous, then every slice of f is continuous.
    - Theorem 7.1.6 If $f : [a,b]\times[c,d] \to \mathbb{R}$ is continuous, then both F and G respectively defined by (7.1.1) and (7.1.2) are continuous.
  - **7.1.2 Iterated double integrals**
    - Definition 7.1.8 Let $f : [a,b]\times[c,d] \to \mathbb{R}$ be a bounded function. The quantities
      $$\int_a^b \left(\int_c^d f(x,y)\,dy\right)dx, \qquad \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy$$
      are **iterated double integrals**.
  - **7.1.3 Fubini's theorem in two dimensions**
    - Theorem 7.1.14 (Fubini) Let $R = [a,b]\times[c,d]$ and let $f : R \to \mathbb{R}$ be bounded. For $x \in [a,b]$, define $f^x : [c,d] \to \mathbb{R}$ by $f^x(y) = f(x,y)$. Assume
      - f^x is integrable on $[c,d]$ for every $x \in [a,b]$,
      - f is integrable on $[a,b]\times[c,d]$.
      Then
      - $\int_c^d f(x,y)\,dy$ exists for every $x \in [a,b]$,
      - $\int_a^b \left(\int_c^d f(x,y)\,dy\right)dx$ exists and equals $\iint_R f\,dA$.
    - Corollary 7.1.16 (Fubini) If $f : [a,b]\times[c,d] \to \mathbb{R}$ is continuous, then
      $$\int_a^b \left(\int_c^d f(x,y)\,dy\right)dx \quad \text{and} \quad \int_c^d \left(\int_a^b f(x,y)\,dx\right)dy$$
      both exist and are equal to $\iint_{[a,b]\times[c,d]} f\,dA$.
  - **7.1.4 Proof of Fubini's theorem in 2D**
- **7.2. Fubini's theorem**
  - **7.2.1 Integrals of slices in 3D**
    - Definition 7.2.2 Let $R = [a,b]\times[c,d]\times[e,f]$ be a rectangle in $\mathbb{R}^3$. Let $\varphi : R \to \mathbb{R}$.
      A (x,y)-slice of φ is a function $\varphi^{x,y} : [e,f] \to \mathbb{R}$ of the form $\varphi^{x,y}(z) = \varphi(x,y,z)$ for some fixed $x \in [a,b], y \in [c,d]$.
      A (x,z)-slice of φ is a function $\varphi^{x,z} : [c,d] \to \mathbb{R}$ of the form $\varphi^{x,z}(y) = \varphi(x,y,z)$ for some fixed $x \in [a,b], z \in [e,f]$.
      A (y,z)-slice of φ is a function $\varphi^{y,z} : [a,b] \to \mathbb{R}$ of the form $\varphi^{y,z}(x) = \varphi(x,y,z)$ for some fixed $y \in [c,d], z \in [e,f]$.
    - Definition 7.2.4 Let $R = [a,b]\times[c,d]\times[e,f]$ be a rectangle in $\mathbb{R}^3$. Let $\varphi : R \to \mathbb{R}$.
      A x-slice of φ is a function $\varphi^x : [c,d]\times[e,f] \to \mathbb{R}$ of the form $\varphi^x(y,z) = \varphi(x,y,z)$ for some fixed $x \in [a,b]$.
      A y-slice of φ is a function $\varphi^y : [a,b]\times[e,f] \to \mathbb{R}$ of the form $\varphi^y(x,z) = \varphi(x,y,z)$ for some fixed $y \in [c,d]$.
      A z-slice of φ is a function $\varphi^z : [a,b]\times[c,d] \to \mathbb{R}$ of the form $\varphi^z(x,y) = \varphi(x,y,z)$ for some fixed $z \in [e,f]$.
  - **7.2.2 Iterated triple integrals and Fubini's theorem in 3D**
    - Definition 7.2.7 Let $\varphi : [a,b]\times[c,d]\times[e,f] \to \mathbb{R}$ be a bounded function. The quantity
      $$\int_a^b \int_c^d \int_e^f \varphi(x,y,z)\,dz\,dy\,dx$$
      is an **iterated triple integral**.
    - Theorem 7.2.10 (Fubini) Let $R = [a,b]\times[c,d]\times[e,f]$ and let $\varphi : R \to \mathbb{R}$ be bounded. If
      - For every $x \in [a,b]$, $y \in [c,d]$, the (x,y)-slice $\varphi^{x,y}$ is integrable on $[e,f]$.
      - For every $x \in [a,b]$, the x-slice φ^x is integrable on $[c,d]\times[e,f]$.
      - φ is integrable on $R = [a,b]\times[c,d]\times[e,f]$.
      Then the iterated triple integral
      $$\int_a^b \int_c^d \int_e^f \varphi(x,y,z)\,dz\,dy\,dx$$
      exists and is equal to the integral $\iiint_R \varphi\,dV$.
    - Corollary 7.2.12 Let $R = [a,b]\times[c,d]\times[e,f]$ be a rectangle in $\mathbb{R}^3$. If $\varphi : R \to \mathbb{R}$ is continuous, then every iterated triple integral of φ on R exists and they are all equal to $\iiint_R \varphi\,dV$.
  - **7.2.3 Iterated integrals and Fubini's theorem in any dimension**
    - Definition 7.2.14 Let R be a rectangle in $\mathbb{R}^n$. Let $f : R \to \mathbb{R}$ be a real-valued function. A function g is a **slice** of f on R if g is defined by fixing one or more coordinates of f in R.
      - Remark 7.2.15 Notice f is not a slice of itself. Also, all coordinates cannot be fixed, otherwise, g would have empty domain.
    - Lemma 7.2.18 Let R be a rectangle in $\mathbb{R}^n$. Let $f : R \to \mathbb{R}$ be a real-valued function.
      (a) If f is bounded, then every slice of f is bounded.
      (b) If f is continuous, then every slice of f is continuous.
    - Definition 7.2.20 Let $R = [a_1,b_1]\times\cdots\times[a_n,b_n]$ be a rectangle in $\mathbb{R}^n$. Let $\varphi : R \to \mathbb{R}$ be a bounded function. The quantity
      $$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n} \varphi(x_1,\ldots,x_n)\,dx_n\cdots dx_2\,dx_1$$
      is an **iterated n-fold integral**.
    - Theorem 7.2.22 (Fubini) Let $R = [a_1,b_1]\times\cdots\times[a_n,b_n]$ be a rectangle in $\mathbb{R}^n$. Let $f : R \to \mathbb{R}$ be a bounded real-valued function. If f is integrable on R and every slice of f on R is integrable on its domain, then every iterated integral of f on R exists and they are all equal to the integral of f on R. That is, the iterated n-fold integral
      $$\int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} f(x_1,\ldots,x_n)\,dx_n\cdots dx_1$$
      exists and all $n!$ orderings of this iterated integral exist and are all equal to $\int_R f\,dV$.
    - Corollary 7.2.23 Let R be a rectangle in $\mathbb{R}^n$. If $f : R \to \mathbb{R}$ is continuous, then every iterated integral of f on R exists and they are all equal to the integral of f on R.
    - Theorem 7.2.24 (Fubini) Let R be a rectangle in $\mathbb{R}^n$. Let $\varphi : R\times[a,b] \to \mathbb{R}$ be bounded. For every $t \in [a,b]$, define the slice $\varphi^t : R \to \mathbb{R}$ by $\varphi^t(x) = \varphi(x,t)$. If the function φ is integrable on $R\times[a,b]$, and for every $t \in [a,b]$ the slice φ^t is integrable on R, then the function $t \mapsto \int_R \varphi^t\,dV$ is integrable on $[a,b]$ and
      $$\int_{R\times[a,b]} \varphi\,dV = \int_a^b \left(\int_R \varphi^t\,dV\right)dt.$$
- **7.3. Double integrals**
  - Definition 7.3.6 A set $S \subseteq \mathbb{R}^2$ is x-simple if there exist continuous $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ such that
    $$S = \{(x,y) \in \mathbb{R}^2 : a \le x \le b, f(x) \le y \le g(x)\}.$$
    A set $T \subseteq \mathbb{R}^2$ is y-simple if there exist continuous $p : [c,d] \to \mathbb{R}$ and $q : [c,d] \to \mathbb{R}$ such that
    $$T = \{(x,y) \in \mathbb{R}^2 : c \le y \le d, p(y) \le x \le q(y)\}.$$
    - you can express integrals over x-simple sets S as an iterated double integral $dy\,dx$.
      $$\iint_S \varphi\,dA = \int_a^b \int_{f(x)}^{g(x)} \varphi(x,y)\,dy\,dx$$
    - Similarly, you can express integrals over y-simple sets T as an iterated double integral $dx\,dy$.
      $$\iint_T \varphi\,dA = \int_c^d \int_{p(y)}^{q(y)} \varphi(x,y)\,dx\,dy.$$
  - Sketch the region and describe it in several ways
  - directly calculate with FTC
  - swap the order of integration with Fubini's theorem
    - Example 7.3.2
  - apply symmetries of the integrand or region
    - Example 7.3.10
  - break up the region into simpler pieces
    - Example 7.3.9
  - interpret geometrically as a volume or area of a classic object
    - Example 7.3.12
- **7.4. Double integrals in polar coordinates**
  - **7.4.1 Regions in polar coordinates**
  - **7.4.2 Derivation of integrals in polar coordinates**
    - Theorem 7.4.5 Let $\Omega \subseteq \mathbb{R}^2$ be a Jordan measurable set such that the restricted polar coordinate transformation $g_p : \Omega \to g(\Omega)$ given by (7.4.1) is a bijection. If $f : g(\Omega) \to \mathbb{R}$ is integrable on $g(\Omega)$, then $F : \Omega \to \mathbb{R}$ given by $F(r,\theta) = (f \circ g)(r,\theta) \cdot |r|$ is integrable on Ω and
      $$\iint_{g(\Omega)} f\,dA = \iint_\Omega F\,dA.$$
  - **7.4.3 Examples of integrals in polar coordinates**
- **7.5. Triple integrals**
  - *projection method*
    - Example 7.5.1
  - *slicing method*
    - Example 7.5.3
- **7.6. Triple integrals in cylindrical coordinates**
  - **7.6.1 Regions in cylindrical coordinates**
  - **7.6.2 Derivation of integrals in cylindrical coordinates**
    - Theorem 7.6.4 Let $\Omega \subseteq \mathbb{R}^3$ be a Jordan measurable set such that the restricted cylindrical coordinate transformation $g_c : \Omega \to g(\Omega)$ given by (7.6.1) is a bijection. If the real-valued function $f : g(\Omega) \to \mathbb{R}$ is integrable on $g(\Omega)$, then the real-valued function $F : \Omega \to \mathbb{R}$ given by $F(r,\theta,z) = (f \circ g)(r,\theta,z) \cdot |r|$ is integrable on Ω and
      $$\iiint_{g(\Omega)} f\,dV = \iiint_\Omega F\,dV.$$
  - **7.6.3 Examples of integrals in cylindrical coordinates**
- **7.7. Triple integrals in spherical coordinates**
  - **7.7.1 Regions in spherical coordinates**
  - **7.7.2 Derivation of integrals in spherical coordinates**
    - Theorem 7.7.3 Let $\Omega \subseteq \mathbb{R}^3$ be a Jordan measurable set such that the restricted spherical coordinate transformation $g_s : \Omega \to g(\Omega)$ given by (7.7.1) is a bijection. If the real-valued function $f : g(\Omega) \to \mathbb{R}$ is integrable on $g(\Omega)$, then the real-valued function $F : \Omega \to \mathbb{R}$ given by $F(\rho,\theta,\phi) = (f \circ g)(\rho,\theta,\phi) \cdot |\rho^2 \sin\phi|$ is integrable on Ω and
      $$\iiint_{g(\Omega)} f\,dV = \iiint_\Omega F\,dV.$$
  - **7.7.3 Examples of integrals in spherical coordinates**
- **7.8. Change of variables**
  - **7.8.1 Derivation of change of variables**
    - Theorem 7.8.1 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. For every rectangle R in $\mathbb{R}^n$,
      $$\mathrm{vol}(T(R)) = |\det(T)|\,\mathrm{vol}(R).$$
  - **7.8.2 Statement and consequences**
    - Theorem 7.8.3 (Change of variables) Let $U, V \subseteq \mathbb{R}^n$ be open sets. Let $g : U \to V$ be a diffeomorphism. Let $\Omega \subseteq U$ be a compact Jordan measurable set. The function f is integrable on $g(\Omega)$ if and only if $(f \circ g)\,|\det Dg|$ is integrable on Ω. If so,
      $$\int_{g(\Omega)} f\,dV = \int_\Omega (f \circ g)\,|\det Dg|\,dV.$$
      If additionally Fubini's theorem is satisfied for both integrals, then
      $$\int \cdots \int_{g(\Omega)} f(x)\,dx_1\cdots dx_n = \int \cdots \int_\Omega (f \circ g)(u)\,|\det Dg(u)|\,du_1\cdots du_n.$$
    - Corollary 7.8.5 Let $U, V \subseteq \mathbb{R}$ be open sets with $[a,b] \subseteq U$. Let $g : U \to V$ be a C^1 increasing function. If f is integrable on $[g(a), g(b)]$, then $f \circ g$ is integrable on $[a,b]$ and
      $$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du.$$
    - Corollary 7.8.6 Let $U, V \subseteq \mathbb{R}^n$ be open sets. Let $g : U \to V$ be a diffeomorphism. For any compact Jordan measurable set $\Omega \subseteq U$,
      $$\mathrm{vol}(g(\Omega)) = \int_\Omega |\det Dg|\,dV.$$
    - Corollary 7.8.7 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map. For any compact Jordan measurable set $\Omega \subseteq \mathbb{R}^n$,
      $$\mathrm{vol}(T(\Omega)) = |\det(T)|\,\mathrm{vol}(\Omega).$$
    - Lemma 7.8.8 Let $U, V \subseteq \mathbb{R}^n$ be open sets. If $g : U \to V$ is C^1, bijective, and $\det Dg(x) \ne 0$ for all $x \in U$, then g is a diffeomorphism.
  - **7.8.3 Examples with change of variables**
- **7.9. Improper integrals**
  - **7.9.1 Local integrability**
    - Definition 7.9.2 Let $\Omega \subseteq \mathbb{R}^n$ be a set. A real-valued function f is **locally integrable** on Ω if f is integrable on every compact Jordan measurable subset of Ω.
    - Lemma 7.9.4 Let $\Omega \subseteq \mathbb{R}^n$ be a set. If a real-valued function f is continuous on Ω, then f is locally integrable on Ω.
    - Lemma 7.9.5 Let $\Omega \subseteq \mathbb{R}^n$ be a Jordan measurable set. Let f be a bounded real-valued function on Ω. If f is integrable on Ω, then f is locally integrable on Ω.
  - **7.9.2 Exhaustions**
    - Definition 7.9.6 Let $\Omega \subseteq \mathbb{R}^n$ be a set. A sequence of compact Jordan measurable sets $\{\Omega_k\}_{k=1}^\infty$ is an **exhaustion** of Ω if $\Omega = \bigcup_{k=1}^\infty \Omega_k$ and $\forall k \ge 1$, $\Omega_k \subseteq \Omega_{k+1}^\circ$.
    - Lemma 7.9.11 Let $\Omega \subseteq \mathbb{R}^n$ be a set. If there exists an exhaustion of Ω by compact Jordan measurable sets $\{\Omega_k\}_{k=1}^\infty$, then Ω is open.
  - **7.9.3 Improper integrals and monotone convergence**
    - Definition 7.9.13 Let $\Omega \subseteq \mathbb{R}^n$ be a set. Let $\{\Omega_k\}_{k=1}^\infty$ be an exhaustion of Ω by compact Jordan measurable sets. Let $f : \Omega \to \mathbb{R}$ be locally integrable. The **improper integral** of f over Ω is given by
      $$\int_\Omega f\,dV = \lim_{k\to\infty} \int_{\Omega_k} f\,dV$$
      provided the limit does not depend on the choice of exhaustion and, if so,
      - the improper integral **converges** when the limit exists.
      - the improper integral **diverges** when the limit does not exist.
      - the improper integral **diverges to** ∞ when the limit is ∞.
      - the improper integral **diverges to** $-\infty$ when the limit is $-\infty$.
      If the limit depends on the choice of exhaustion, then the improper integral **diverges**.
    - Theorem 7.9.16 (Monotone convergence theorem) Let $\Omega \subseteq \mathbb{R}^n$ be a set with an exhaustion by compact Jordan measurable sets. Let f be a real-valued locally integrable function on Ω. If $f \ge 0$ on Ω then the improper integral $\int_\Omega f\,dV$ either converges or diverges to ∞.
    - Lemma 7.9.19 Let $\Omega \subseteq \mathbb{R}^n$ be a set with an exhaustion by compact Jordan measurable sets. If f is integrable on Ω, then the improper integral of f on Ω converges and its value is equal to the integral of f on Ω.
    - Lemma 7.9.20 Let $\Omega \subseteq \mathbb{R}^n$ be a set with an exhaustion by compact Jordan measurable sets. Let $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ be locally integrable. Fix $\lambda \in \mathbb{R}$. If the improper integrals $\int_\Omega f\,dV$ and $\int_\Omega g\,dV$ both converge, then the improper integral $\int_\Omega (f + \lambda g)\,dV$ converges and
      $$\int_\Omega (f + \lambda g)\,dV = \int_\Omega f\,dV + \lambda \int_\Omega g\,dV.$$
  - **7.9.4 A family of improper integrals**
    - Theorem 7.9.25 Fix $p \in \mathbb{R}$. For the given improper integrals in $\mathbb{R}^n$, one has
      $$\int_{\|x\| \ge 1} \frac{1}{\|x\|^p}\,dV \quad \begin{cases} \text{converges if } p > n \\ \text{diverges to } \infty \text{ if } p \le n \end{cases}$$
      $$\int_{0 < \|x\| \le 1} \frac{1}{\|x\|^p}\,dV \quad \begin{cases} \text{diverges to } \infty \text{ if } p \ge n \\ \text{converges if } p < n \end{cases}$$
  - **7.9.5 Basic comparison test**
    - Theorem 7.9.28 (Basic comparison test) Let $\Omega \subseteq \mathbb{R}^n$ be open. Let f and g be real-valued locally integrable functions on Ω.
      - If $0 \le f \le g$ on Ω and $\int_\Omega g\,dV$ converges then $\int_\Omega f\,dV$ converges.
      - If $0 \le f \le g$ on Ω and $\int_\Omega f\,dV$ diverges then $\int_\Omega g\,dV$ diverges.
  - **7.9.6 Absolute convergence**
    - Lemma 7.9.31 Let $\Omega \subseteq \mathbb{R}^n$. If f is locally integrable on Ω, then $|f|$ is locally integrable on Ω.
    - Definition 7.9.32 Let $\Omega \subseteq \mathbb{R}^n$ be a set with an exhaustion by compact Jordan measurable sets. Let f be a real-valued locally integrable function on Ω. The improper integral $\int_\Omega f\,dV$ **absolutely converges** if the improper integral $\int_\Omega |f|\,dV$ converges.
    - Theorem 7.9.34 Let $\Omega \subseteq \mathbb{R}^n$ be a set with an exhaustion by compact Jordan measurable sets. Let f be a real-valued locally integrable function on Ω. If the improper integral $\int_\Omega |f|\,dV$ converges, then the improper integral $\int_\Omega f\,dV$ converges.
  - **7.9.7 Proof of the monotone convergence theorem**
    - Theorem 7.9.39 (Heine-Borel) Let $K \subseteq \mathbb{R}^n$ be compact. Let $\{V_i\}_{i=1}^\infty$ be a sequence of open sets such that $V_i \subseteq V_{i+1}$ for $i \ge 1$. If $K \subseteq \bigcup_{i=1}^\infty V_i$, then there exists $k \in \mathbb{N}$ such that $K \subseteq V_k$.