MATH 578 LAB ASSIGNMENT 1

PART 1

In this exercise, we will do an experimental study of the growth factor for Gaussian elimination with partial pivoting. Given the factorization $PA = LU$, define the growth factor by

$$g(A) = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|},$$

where u_{ij} and a_{ij} are the elements of U and A, respectively.

1. A relevant paper is L.N. Trefethen and R.S. Schreiber. Average-case stability of Gaussian elimination. Please read this paper in sufficient detail to understand the main points.
2. Write a function that takes n as its parameter and generates an $n \times n$ matrix, whose entries are random numbers with a normal distribution. Pick the parameters (μ and σ) at your convenience.
3. Implement Gaussian elimination with partial pivoting, and test it on a number of cases to be sure of its correctness.
4. Plot the growth factor against the matrix size, in logarithmic scale, where the matrices are generated as in Step 2. The sample of matrices should be large enough to be a good representative of random matrices with n ranging between, say 10 and 1000 (larger sizes are welcome, if it does not take too long). In particular, for each n, one should experiment on a generous number of matrices. From the experimental plot, estimate the power α in the assumed dependence $g \approx cn^{\alpha}$, where g is the growth factor, n is the matrix size, and c is a constant. Compare this with the worst case scenario $g \approx 2^n$.
5. Now we study the probability distribution of the growth factor for a fixed n. Fix n, say $n = 10$ or $n = 16$, and generate an abundant number of random matrices, as in Step 2, to measure their growth factors. Then by subdividing the value-space of the growth factors into small subintervals of equal length, and by counting the number of matrices with growth factor lying in each of those subintervals, produce an approximation of the probability density function of the growth factor (the usual "histogram" technique). Plot it against the growth factor value, with the vertical axis in logarithmic scale. Make a conjecture on how the probability density decays as the growth factor becomes large. Note that the number of matrices and the length of the subintervals should be so that most of the subintervals individually contain a large number of matrices, and that there are enough subintervals to give a meaningful approximation of the probability density

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function (i.e., the width of bars in the histogram must be small, but most of the bars must still include a large number of cases). Repeat the experiment for several values of n, say $n = 20, 40, 80$, or $n = 32, 64, 128$.

PART 2

Let A be the 1000×1000 tridiagonal symmetric matrix with $1, 2, \ldots, 1000$ on the diagonal and 1 on the sub- and super-diagonals, and let $b = (1, 1, \ldots, 1)^T$.

1. Write a code that performs 1000 (or more) steps of the steepest descent iteration to approximately solve $Ax = b$.
2. Write a code that performs 1000 steps of the CG to approximately solve $Ax = b$.
3. Produce a plot with four curves on it:
   - the computed residual norms $\|r_i\|_2$ for CG,
   - the actual residual norms $\|b - Ax_i\|_2$ for CG,
   - the residual norms $\|r_i\|_2$ for steepest descent, and
   - the estimate $2(\sqrt{\kappa} - 1)^i / (\sqrt{\kappa} + 1)^i$, where $\kappa = \|A\|\|A^{-1}\|$ is the 2-norm condition number of A (which can be estimated by NumPy function `linalg.cond`), and i is the iteration number that is to be on the horizontal axis of the plot.

   Choose logarithmic scale for the vertical axis if appropriate.
4. Comment on your results.

HOMEWORK POLICY

Please submit a link to your Jupyter notebook on GitHub through the `Assignments` tab on MyCourses.
   - Include theory and commentaries in your Jupyter notebook, with Markdown cells.
   - Notebooks without sufficient commentaries will not be accepted.
   - You can update your notebook up to the time limit, but please do not commit any overdue updates.

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.