1 FOM Calibration

1.1 Artificial Viscosity

As stated before, we included an artificial viscosity term in the final formulation, to go around the need for more involved stabilization schemes such as SUPG[1] methods or others alike.

A value needs to be choosen for the viscosity constant, ε. Initially, it was set to be the square of the mesh size,

$$\varepsilon \sim (\Delta x)^2 \sim 10^{-6}.$$

However, a parametric sweep for different viscosity values showed that this value, no matter how small, was introducing measurable damping into the system.

The actual value that seems correct, keeping the system stable whilst purely convective (at least for the time-scale for which we are solving) seems to be a much smaller one,

$$\varepsilon \sim 10^{-10}. \tag{1}$$

In Figure **??** we present a comparison for these two viscosity values. We show the motion of the fluid at the outflow and a phase plot, to ease the analysis of the correlation between the weak shock wave at the outflow and the piston motion. The smallest viscosity value, (10^{-10}) leads to a convection-like phase plot; the shape of the input is distorted but the maximum and mimimum values honored, and the same path is repeated over and over. Instead, the solution with a higher viscosity value (or at least the one that we thought would do the job) shows a diffusive pattern, reducing the extrema and changing its path on every pass. Luckily, the complete reduction scheme[2] seems to withstand such a small numerical viscosity term without incurring into round-off errors[3].

research_project/piston/figures/artificial_viscosit

Fig. 1. Artificial viscosity comparison. (Top) Outflow velocities for two different values of the viscosity term. In dashed, the piston motion, to help in the visualization of the nonlinear distortion. (Bottom) Phase plot between the piston motion (x-axis) and the response at the outflow (y-axis) for the same artificial viscosity values. Due to the creation of a weak shock wave, the piston motion is distorted. However, only the smallest viscosity value (10^{-10}) presents a stable phase plot, going over and over the same path.

1.2 BDF Scheme Convergence Rates

This requires a more detailed investigation.

The next concept we are going to look at to check the quality of our simulation is the convergence rate for the two time-integration schemes, BDF-1 and BDF-2, see Figure **??**.

For some reason, the BDF-2 scheme does not decrease at the expected rate of 2. Nevertheless, since for the calculation of the mass defect figure a space integral and a time derivative are used, this could reveal a weakness in this specific calculation, rather than a bug in the overall simulation.

It is worthy pointing out how the BDF-2 scheme is actually superior to the BDF-1 scheme for most of the period, as we can see from Figure **??** and **??**. However, for some reason yet unknown, in the neighborhood of the shock wave creation, mass preservation decreases, rising up the average error.

1.3 Parameter Range

For the construction of the reduced basis we are randomly sampling the parameter space. Hence, we need to determine an acceptable range for each parameter.

To do so, we sample a large space and visually check each of the solutions. Visual inspection reveals that certain solutions contain wiggles when a shock wave occurs, see Figure **??**. Naturally, wiggles were not expected to be captured by this simple model, so they must be due to the fact that the shock wave is becoming thinner than the mesh size.

Check for different mesh sizes for a configuration with wiggles.

To prevent these unrealistic solutions from polluting our reference solutions, we need to determine a safety margin in the sampling range prevent using a parametrization for which wiggles will take place.

1.3.1 *System Forcing And Nonlinear Response*

On the one hand, when we think about the equations and their response, we could hint at the fact that shock wave strength will be driven by the magnitude of the piston's velocity. This variable, although time dependent, is scaled by the combination of the speed of sound, piston frequency and displacement, namely the piston peak velocity

$$u_p \sim \frac{\delta \omega L_0}{a_0}. \tag{2}$$

1. Note that we could have used the same hyperreduction scheme, since the approach is purely algebraic.
2. Namely snapshot SVD compression and interpolation coefficients calculation.
3. If there were, we could always reduce the plain vanilla Laplace operator $\int \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, \mathrm{d}\Omega$ and scale it by ε right before solving the system.

research_project/piston/figures/bdf_convergence/convergence_rates_mass.png

Fig. 2. Convergence rates for mass defect. The BDF-1 scheme decreases at the expected rate, whereas the BDF-2 does not. This could be due to the calculation procedure for the mass defect and not to a bug in the calculation itself.

research_project/piston/figures/bdf_convergence/mass_FOM_8_bdf_1.png

Fig. 3. (BDF-1, $dt \sim 10^{-4}$). Mass conservation, visual comparison (top) and numerical error (bottom).

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Fig. 4. (BDF-2, $dt \sim 10^{-4}$). Mass conservation, visual comparison (top) and numerical error (bottom).

research_project/piston/figures/wiggles/smooth_vs_w

Fig. 5. Smooth and wiggled solutions comparison. Each solution has a different scale and frequency because they were obtained for different parametrizations.

Such peak velocity will be maximum (minimum) for maximum (minimum) values of displacement and frequency, and minimum (maximum) values for the speed of sound.

On the other hand, we need a procedure to quantify the nonlinearity of the system response. In the absence of the nonlinear term, the equations would reduce to a linear convection model. In such model we would expect the piston motion to be perfectly convected towards the outflow. But due to the presence of the nonlinear term, we know this will not be the case. Hence, we suggest and use as a nonlinearity metric η the ratio between the time between two peaks at the ouflow and the piston,

$$\eta := \frac{T}{T_0}, \tag{3}$$

where T and T_0 are respectively defined as

$$T_0 := t_+^L - t_-^L, \tag{4a}$$
$$T := t_+^{\text{out}} - t_-^{\text{out}}. \tag{4b}$$

The symbols $t_+^{(\cdot)}$ and $t_-^{(\cdot)}$ denote the first positive and the second[4] negative peaks in the velocity function at their respective locations.

The closer η is to one, the more linear the response is. As it tends to zero, the response is becoming nonlinear, since a shock wave is building up at the outflow;

$$\eta \sim 1 : \text{Linear response,}$$
$$\eta \to 0 : \text{Nonlinear response.}$$

In Figure **??** we show the response of the system for different parametrization values, alongside with the forcing u_p and nonlinearity measure η. We have manually tagged which combinations present wiggles. There are several observations to be made about this pair plot.

First of all, looking the raw parameters a_0, ω and δ, we observe how none of them individually explain how nonlinear the response of the system will be. There is certainly a trend, low values of a_0 and large values of δ and ω contain wiggles (as shown by the density plots). However, points with low values of a_0 belong to the set without wiggles too. Therefore, it must be the combination of them which explains the existence of wiggles.

When we look at the correlation between u_p and η, we observe a linear trend between them: the higher the forcing, the stronger the nonlinearity in the response. And clearly, beyond a threshold in the forcing, the system seems to break and most solutions contain wiggles. This critical value u_M for the forcing seems to be in the neighborhood of $u_M \sim 0.4$.

With this empirical study of the system's response (and after running some tests) we choose the following ranges for our parameters:

TABLE 1
Acceptable parameter range for random sampling. This configuration should prevent the existence of wiggles at the outflow.

Variable	Minimum	Maximum	Units
a_0	18	25	m/s
ω	15	30	1/s
δ	0.15	0.3	[-]

4. We take the second negative peak and not the first one to remove any possible distortion due to the initial transient from rest to harmonic movement.

research_project/piston/figures/wiggles/wiggle_correlation.png

Fig. 6. Parameter space split by wiggle presence. We observe how wiggles are prone to appear for low values in the speed of sound, large values in displacement and frequency; that is, when the forcing into the system is larger than a given threshold u_M.