1 Introduction

From [1]:

> The central idea of the reduced basis approach is the identification of a suitable problem-dependent basis to effectively represent parametrized solutions to partial differential equations.

- Difference between local and global support.
- Decomposition techniques.

2 Needs

We aim at obtaining efficiently the solution of parametrized parabolic PDEs with moving boundaries in time. Others have solved a problem with a moving mesh in time, but only using DEIM [2].

- Build a ROM system equivalent to the FEM discretization of the weak form of the PDE.
  - Solve the ROM for each time-step and project back to physical domain.
  - Functional evaluation of the solution.
- Do not assemble and project any high fidelity operators to build the ROM operators.
  - Sampling strategies across the parameter space are crucial.
    * Ensure convergence of the reduced basis.
    * Computational efficiency.
  - Create a POD-basis for:
    * The solution space.
    * Each operator, matrix or vector, of the system.
    * The "snapshots method" is used [3], [4].
  - Parameter Separability of the ROM operators:

$$A(\mu, t) = \sum_q \theta_q(\mu, t) A_q$$

- Certify the ROM solution with a posteriori error bounds.

2.1 Scope

- Prescribed deformation of the domain at the boundary:
  - No FSI problem to be solved.
  - Separable geometrical and time parametrization of the domain deformation.
  - Interpolate deformation through the mesh with a Laplacian operator for each time-step (to ensure smoothness).
- Linear operators:
  - Heat equation. See [5], parametrized domain but constant in time.
  - Include a non-linear term (bonus).
  - Convection-diffusion with known velocity field (bonus).

How to deal with inhomogeneous boundary conditions, [6].

  - Lifting function:
    * Split the solution into arbitrary function honoring Dirichlet b.c. + solution to the homogeneous problem.

$$u = u_D + \hat{u}$$

    * The homogeneous problem is reduced.
    * The lifting operators (they arise from the application of the operators upon u_D) are reduced.

3 What Others Have Done

- POD-Galerkin projection.
- Operators reduction:
  - (DEIM) Vector reduction and interpolation [7].
  - (MDEIM) Matrix reduction and interpolation [8].
  - Reduction of non-linear operators [2].
- Possible problem parametrizations:
  - Geometrical deformation of the domain.
  - PDE coefficients.
  - Boundary conditions.

4 What We Intend To Do

- Validate code/bounds with a non-parametrized time deforming mesh.
- Reduce a parametrized time-deforming mesh. This includes creating a reduced model for the domain deformation problem too.

4.1 Error Bounds

Means to certify the construction of the RB model: a posteriori error bounds.

They should be:

1) Computable (they often use continuity and coercivity constants).
2) Rigorous (i.e. provable).
3) Effective/Sharp (they should not arbitrarily overestimate the error).

Adapt [8], [9] for time-changing domains.

4.2 Research Questions

- How does the parameter sampling strategy affect the goodness of the POD-basis?
- How does the parameter sampling strategy affect the goodness of the Discrete Empirical Interpolation?
- Can we predict the minimum viable number of basis for a given error?
- What is a representative snapshot, quantitatively?
- What does it mean for a basis to be *rich*? When can we consider the basis sufficiently rich?
- How does the selection of the inner product change the resultant basis of the POD?

- Can we use information from the PDE to improve this inner product?
- How do we deal with boundary conditions?
- How do the empirical interpolation error and the reduced basis error affect the final approximation?
- How does the domain deformation problem reduction affect the quality of the reduced solution?

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