Planarity

Definition 1. A graph is said to be planar (планарный) or embeddable in the plane (укладываем на плоскости), if it can be drawn in the plane so that its edges intersect only at their ends.

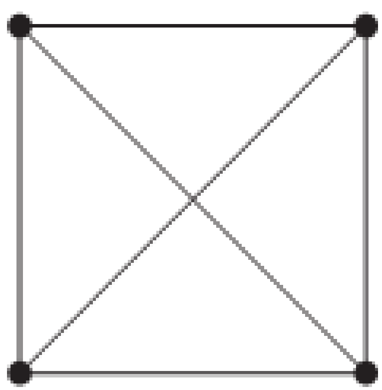
Such a drawing is called a planar embedding (планарная укладка) of the graph.

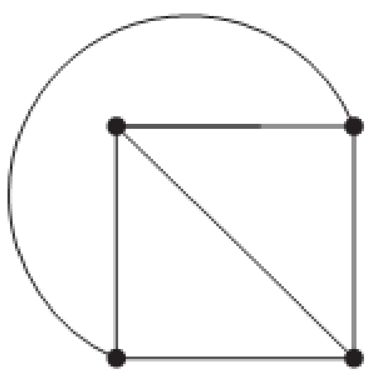
A graph may be planar even if it is usually drawn with crossings, because it may be possible to draw it in a different way without crossings.

EXAMPLE 1 Is K_4 (shown bellow with 2 edges crossing) planar?

Solution: K_4 is planar because there exists a planar drawing (embedding) as shown bellow.

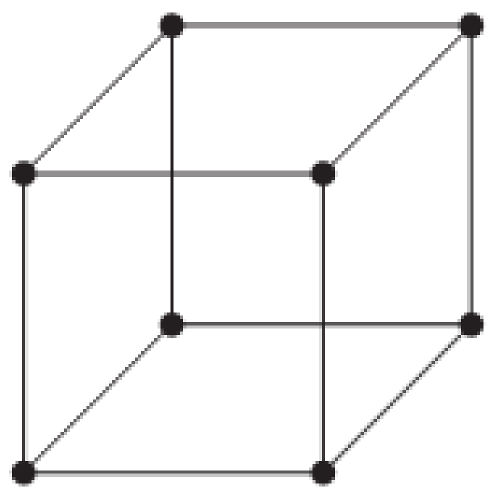
Plane graph (плоский граф) – a graph that is drawn without edge crossings.

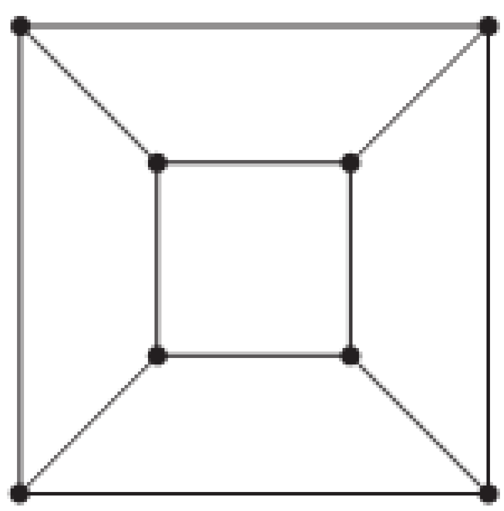




EXAMPLE 2 Is Q_3, shown bellow, planar?

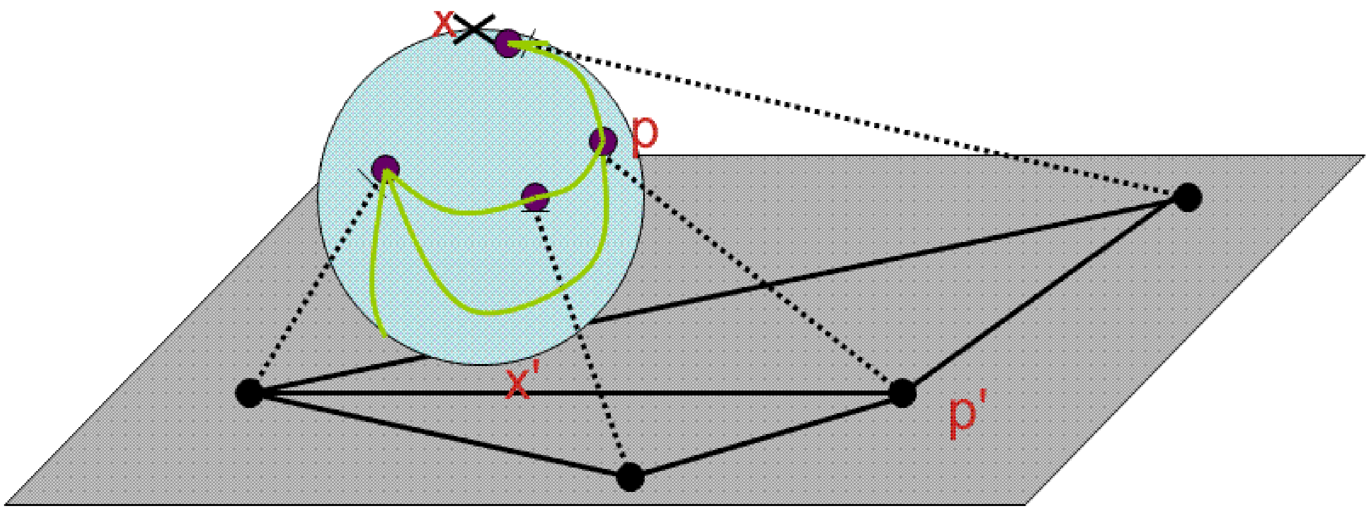
Solution: Q_3 is planar, because there exists a planar drawing (embedding), as shown.





One may consider embeddings of graphs on surfaces other than the plane.

Theorem 1 A graph is embeddable in the sphere ⇔ it is embeddable in the plane.



Proof. We show this by using a mapping known as stereographic projection.

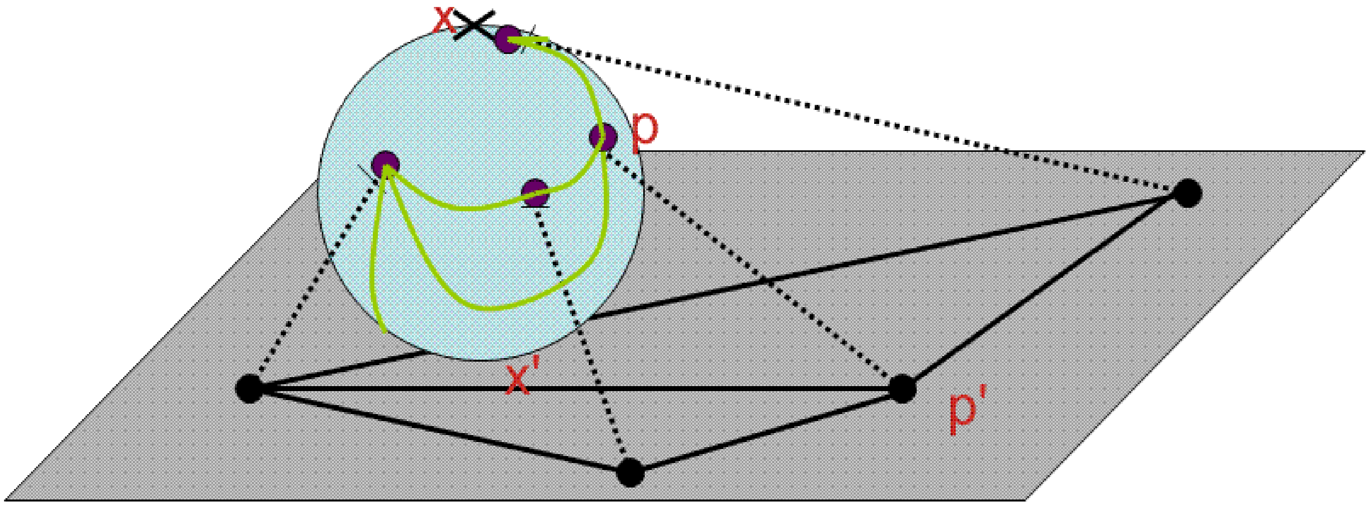
Consider a spherical surface *S*, touching a plane *P* at the point *x'* (called south pole).

The point *x* (called the point of projection or north pole) is on *S* and diametrically opposite *x'*.

Any point *p* on *P* can be projected uniquely onto *S* at *p*' by making *x*, *p* and *p'* collinear.

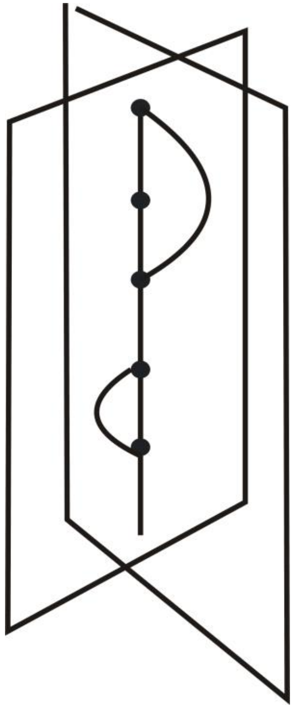
In this way any graph embedded in *P* can be projected onto *S*.

Conversely, we can project any graph embedded in *S* onto *P* , choosing *x* so as not to lie an any vertex or edge of the graph.



Theorem 2 Every graph is embeddable in R^3

Proof. See figure bellow



APPLICATIONS OF PLANAR GRAPHS

Planarity of graphs plays an important role in the design of electronic circuits.

We can model a circuit with a graph by representing components of the circuit by vertices and connections between them by edges.

We can print a circuit on a single board with no connections crossing if the graph representing the circuit is planar.

When this graph is not planar, we must turn to more expensive options. For example, we can partition the vertices in the graph representing the circuit into planar subgraphs.

We then construct the circuit using multiple layers.

We can construct the circuit using insulated wires whenever connections cross.

In this case, drawing the graph with the fewest possible crossings is important.

APPLICATIONS OF PLANAR GRAPHS

The planarity of graphs is also useful in the design of road networks.

Suppose we want to connect a group of cities by roads.

We can model a road network connecting these cities using a simple graph with vertices representing the cities and edges representing the highways connecting them.

We can built this road network without using underpasses or overpasses if the resulting graph is planar.

Faces (geometrical definition)

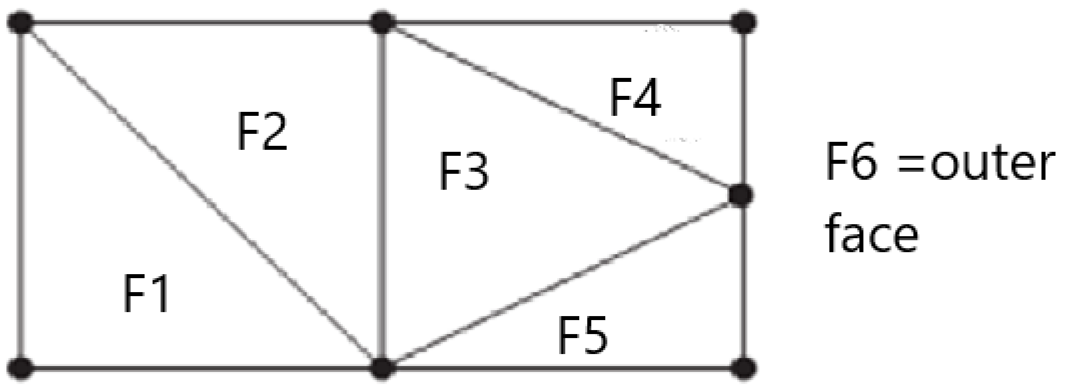
A planar drawing partitions the plane into connected regions called faces (Грани).

The unbounded face is usually called external face or outer face (внешняя грань).

Other faces are called internal faces (внутренняя грань)

We define the degree of a face $d(f)$, to be the number of edges bounding the face f.

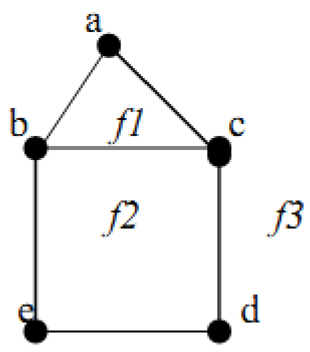
The graph bellow has 5 internal faces F_1, F_2, F_3, F_4, F_5 of degree 3, and an outer face F_6 of degree 7.

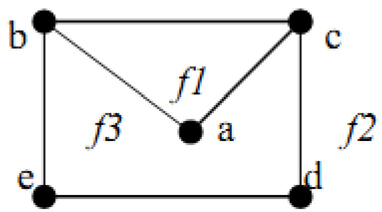


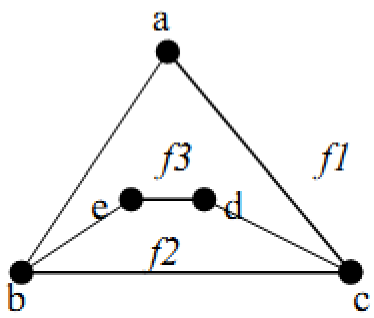
Theorem 3 A planar embedding G' of a graph G can be transformed into another embedding such that any specified face F becomes the outer face.

Proof N1 (geometrical). Map the drawing onto a sphere, and then map the sphere back into the plane in such a way that a point inside F becomes infinity.

The resulting drawing has the same planar embedding and F is the outer face.

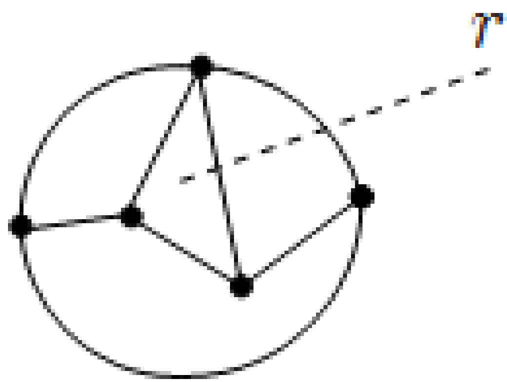






Proof N2 (combinatorial)

Let p be any point inside face F , and let r be a ray emanating from p that does not cross any vertex. (Since there are only finitely many vertices, there exists such a ray.)



(a)

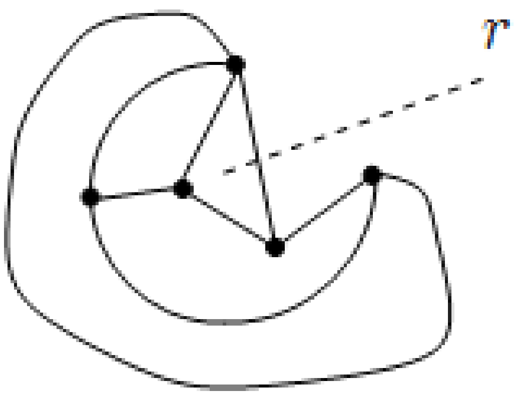
The proof is now by induction on the number of edges that r crosses.

If it crosses none, then F is already the unbounded face, i.e., the outer-face, and we are done.

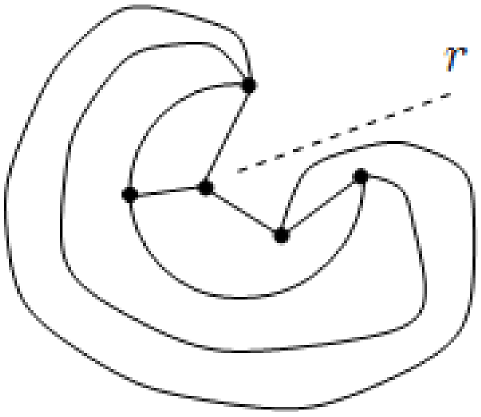
If ray r crosses some edges, then it must also cross an edge $e = (u, w)$ on the outer-face of the current drawing.

Making a face the outer-face.

Reroute the edge (*u, w*) such that it does not cross ray *r*
In this way we get planar embedding in which ray *r* crosses fewer edges, and are done by induction.



(b).



(c)

Euler's formula

Theorem 4 (Euler's formula) Let G be a connected, not necessarily simple, plane graph.

Then $n - m + f = 2$ where f is the number of faces.

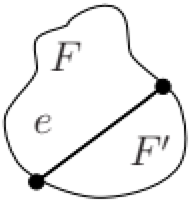
Proof: Let us start with a spanning tree of the graph G.

A tree with n vertices has exactly $n - 1$ edges, so

$n - m + f = n - (n - 1) + 1 = 2.$

Now we will add edges of the graph G one by one.

Each added edge e splits one of the faces of the graph into 2 faces.



Consequently, in this case,

$f = f + 1$,

$m = m + 1$, and

$n = n$.

Thus, each side of the formula relating the number of faces, edges, and vertices increases by exactly one, so this formula is still true.

In other words, $n - m + f = 2$

Lemma 5 Let G be a planar graph.

Then any planar drawing of G has the same number of faces.

Proof: $n - m + f = 2$ =>

The number of faces $f = m - n + 2$ by Euler's formula.

Since m and n are both independent of the planar drawing, the result follows

EXAMPLE 3 Suppose that a connected planar simple graph has 20 vertices, each of degree 3.

How many faces has a representation of this planar graph in the plane?

Solution: This graph has 20 vertices, each of degree 3, so $n = 20$.

Because the sum of the degrees of the vertices,

$3n = 3 \cdot 20 = 60$, is equal to twice the number of edges, we have $2m = 60$, or $m = 30$.

Consequently, from Euler's formula, the number of faces is

$f = m - n + 2 = 30 - 20 + 2 = 12$.

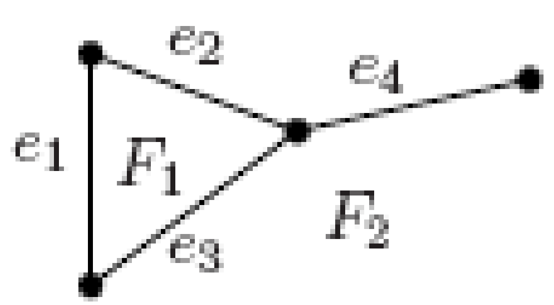
Lemma 6 Every simple connected planar graph with ≥ 3 vertices has $\leq 3n - 6$ edges.

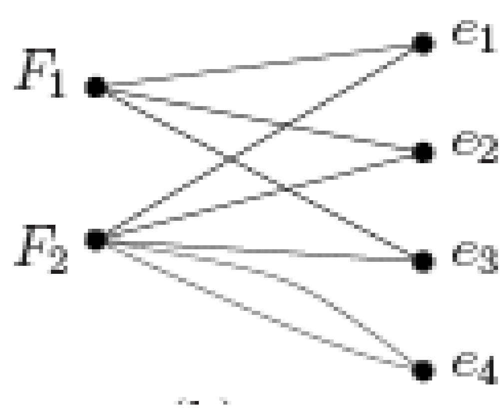
Proof: We will prove this statement by double-counting the edge-face incidences in some planar embedding of the graph.

We will make a list of faces of a graph in one column and a list of edges in the other column.

Then we draw a line from an edge to a face if they are incident.

We draw 2 lines between one pair if the edge is incident to this face twice.





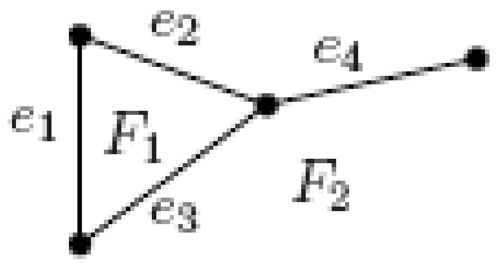
Now let us count the number L of lines twice.

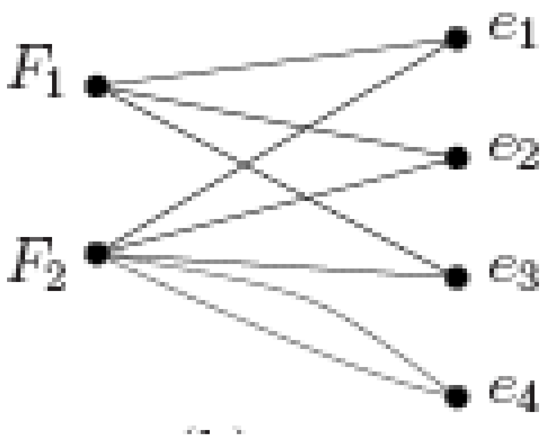
We know that each edge is incident to 2 faces (not necessarily different) and thus there are 2 lines leading from each edge.

Therefore $L = 2m$.

We can prove that each face is incident to at least three edges.

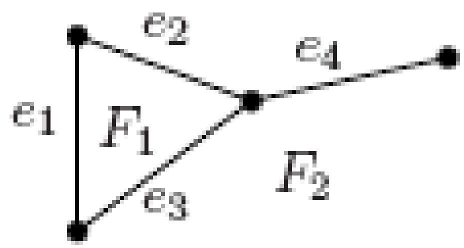
If there were a face incident to only one edge, then it would be enclosed in this edge, i.e., the edge would be a loop, which contradicts that the graph is simple.

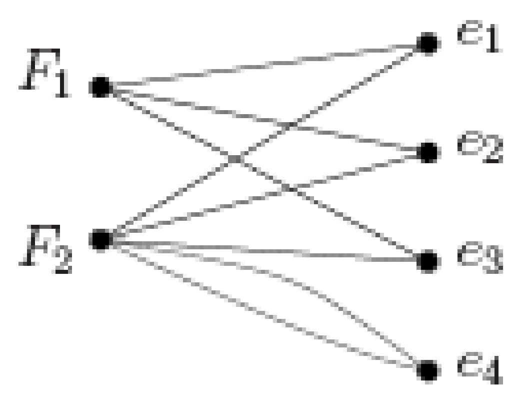




If there were a face incident to only 1 edge twice, this edge would be a connected component by itself, which contradicts that the graph has at least 3 vertices and is connected.

If a face were incident to exactly 2 different edges, then these edges would form a cycle, i.e., they are multiple edges, which contradicts that the graph is simple.





So each face is incident to ≥ 3 edges, therefore ≥ 3 lines leave from each face and $L \geq 3f$.

We have counted the number of lines L twice and if we put the results together we get $3f \leq 2m$.

To finish the proof we multiply Euler's formula by 3 and plug the inequality $3f \leq 2m$ into it to obtain

$$6 = 3n - 3m + 3f \leq 3n - 3m + 2m = 3n - m.$$

Observe that the bound of $m \leq 3n - 6$ edges does not hold for graphs with 1 or 2 vertices.

On the other hand, the bound of $m \leq 3n - 3$ edges holds for any simple planar graph, even if it is not connected, which one can show easily by induction on the number of connected components.

Thus, every planar simple graph has $O(n)$ edges.

On the other hand, no such bound holds for planar graphs that are not simple: we could have 2 vertices, and arbitrarily many edges between them.

Lemma 7. If G is a connected planar simple graph, then G has a vertex of degree ≤ 5.

Proof: If G has 1 or 2 vertices, the result is true.

If G has ≥ 3 vertices, by Lemma 6 we know that

$m \leq 3n - 6$, so $2m \leq 6n - 12$.

If the degree of every vertex were ≥ 6, then because

$$2m = \sum_{v \in V} \deg(v)$$

(by the handshaking theorem), we would have $2m \geq 6n$.

But this contradicts the inequality $2m \leq 6n - 12$.

It follows that there must be a vertex with degree ≤ 5.

Lemma 8 Any simple planar bipartite graph with at least 3 vertices has $m \leq 2n - 4$ edges.

Proof: The proof is almost identical to the proof of Lemma 6, except for the following observation:

If a graph G is bipartite, then all cycles in G have even length, which means in particular that G has no triangle.

Therefore, every face must be incident to ≥ 4 edges.

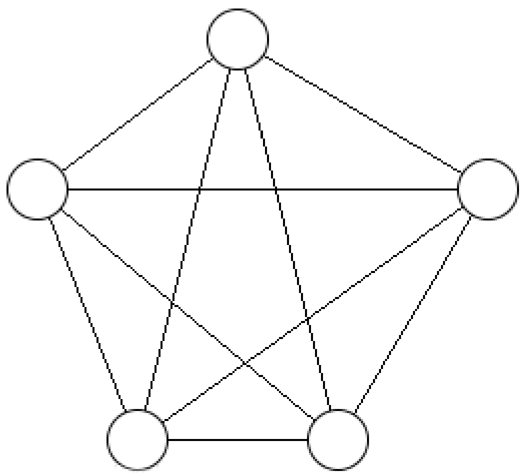
Using again double-counting, we obtain $4f \leq 2m$, which in conjunction with Euler's formula yields the result.

EXAMPLE 4 Show that K_5 is nonplanar using Lemma 6.

Solution: The graph K_5 has 5 vertices and 10 edges.

However, the inequality $m \leq 3n - 6$ is not satisfied for this graph because $m = 10$ and $3n - 6 = 9$.

Therefore, K_5 is not planar.



Example 5 : show that $K_{3,3}$ is not planar.

Note, that this graph has 6 vertices and 9 edges.

This means that the inequality $m = 9 \leq 12 = 3 \cdot 6 - 6$ is satisfied.

Consequently, the fact that the inequality $m \leq 3n - 6$ is satisfied does not imply that a graph is planar.

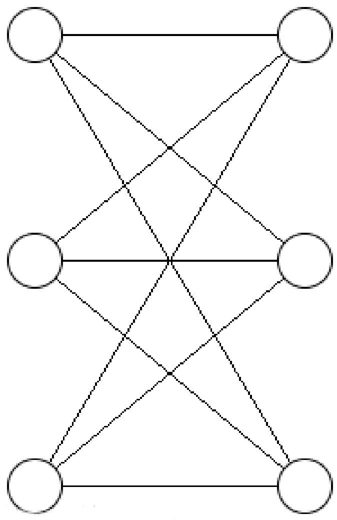
However, Lemma 8 can be used to show that $K_{3,3}$ is nonplanar.

Because $K_{3,3}$ has no circuits of length 3 (this is easy to see because it is bipartite), Corollary 3 can be used.

$K_{3,3}$ has 6 vertices and 9 edges.

Because $m = 9$ and $2n - 4 = 8$,

Corollary 3 shows that $K_{3,3}$ is nonplanar.



Kuratowski's Theorem

We have seen that $K_{3,3}$ and K_5 are not planar.

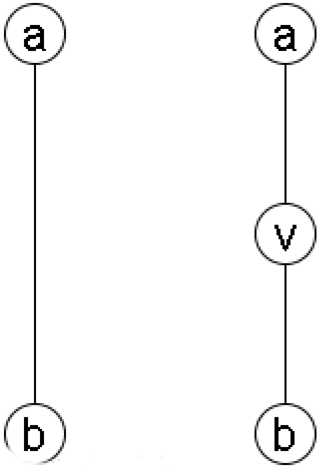
Clearly, a graph is not planar if it contains either of these 2 graphs as a subgraph.

Surprisingly, all nonplanar graphs must contain a subgraph that can be obtained from $K_{3,3}$ or K_5 using certain permitted operations.

If a graph is planar, so will be any graph obtained by removing an edge $\{a, b\}$ and adding a new vertex v together with edges $\{a, v\}$ and $\{v, b\}$.

Such an operation is called an elementary subdivision.

The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisions.



Kuratowski's Theorem

The Polish mathematician Kazimierz Kuratowski established Theorem 9 in 1930, which characterizes planar graphs using the concept of graph homeomorphism.

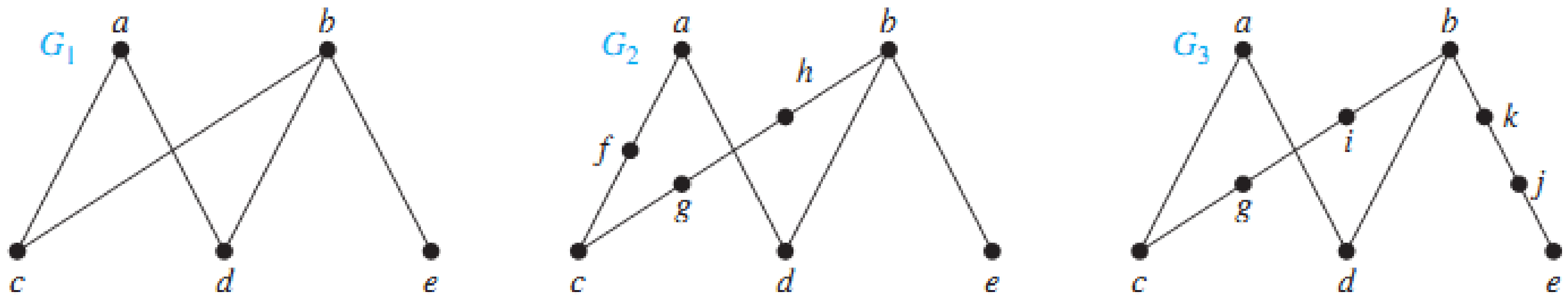
THEOREM 9 A graph is nonplanar ⇔ it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

It is clear that a graph containing a subgraph homeomorphic to $K_{3,3}$ or K_5 is nonplanar.

However, the proof of the converse, namely that every nonplanar graph contains a subgraph homeomorphic to $K_{3,3}$ or K_5, is complicated and will not be given here.

Examples 7 and 8 illustrate how Kuratowski's theorem is used.

EXAMPLE 6 Show that the graphs G_1, G_2, and G_3 displayed bellow are all homeomorphic.
Solution: These 3 graphs are homeomorphic because all 3 can be obtained from G_1 by elementary subdivisions.
G_1 can be obtained from itself by an empty sequence of elementary subdivisions.
To obtain G_2 from G_1 we can use the following sequence of elementary subdivisions:
(i) remove the edge {a, c}, add the vertex f , and add the edges {a,f } and {f, c};
(ii) remove the edge {b, c}, add the vertex g, and add the edges {b, g} and {g, c};
(iii) remove the edge {b, g}, add the vertex h, and add the edges {g,h} and {b, h}.
Exercise: determine the sequence of elementary subdivisions needed to obtain G_3 from G_1.



EXAMPLE 7 Determine whether the graph G shown bellow is planar.

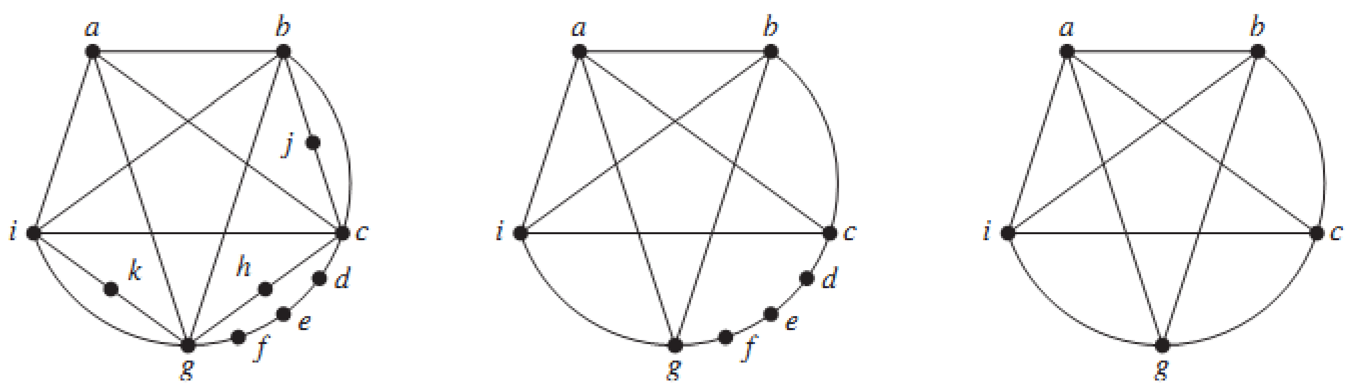
Solution: G has a subgraph H homeomorphic to K_5.

H is obtained by deleting h, j , and k and all edges incident with these vertices.

H is homeomorphic to K_5 because it can be obtained from K_5 (with vertices a, b, c, g, and i) by a sequence of elementary subdivisions, adding the vertices d, e, and f .

(Exercise: Construct a sequence of elementary subdivisions .)

Hence, G is nonplanar.



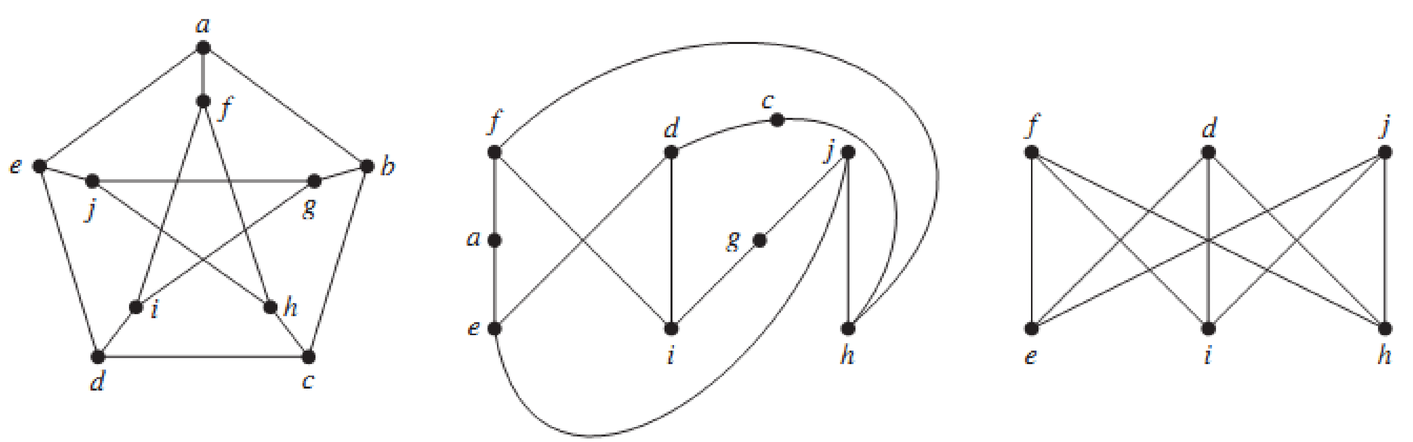
G H K_5

EXAMPLE 8 Is the Petersen graph, shown bellow, planar? (The Danish mathematician Julius Petersen studied this graph in 1891; it is often used to illustrate various theoretical properties of graphs.)

Solution: The subgraph H of the Petersen graph obtained by deleting vertex b and the 3 edges that have b as an endpoint, shown bellow (b), is homeomorphic to $K_{3,3}$, with vertex sets $\{f, d, j\}$ and $\{e, i, h\}$, because it can be obtained by a sequence of elementary subdivisions,

- deleting $\{d, h\}$ and adding $\{c, h\}$ and $\{c, d\}$, deleting $\{e, f\}$ and adding $\{a, e\}$ and $\{a, f\}$, and
- deleting $\{i, j\}$ and adding $\{g, i\}$ and $\{g, j\}$.

Hence, the Petersen graph is not planar.



(a) (b) H (c) $K_{3,3}$

Planarity testing

Directly applying Kuratowski’s characterization of planar graphs based on subdivisions would yield an exponential-time algorithm.

The first polynomial-time algorithms (APG) for planarity are due to Auslander and Parter [1961], Goldstein [1963], and, independently, Bader [1964].
In 1974 Hopcroft and Tarjan proposed the first linear-time planarity testing algorithm.

However, the algorithm is very complex and difficult to implement.

Planarity testing

A different approach has its starting point in the algorithm presented by Lempel, Even, and Cederbaum [LEC 67].

This algorithm, also called "vertex-addition algorithm," is based on considering the vertices one-by-one, following an st numbering; it has been shown to be implementable in linear time by Booth and Lueker [BL76],

Also in this case, a further contribution by Chiba, Nishizeki, Abe, and Ozawa [CNAO85] has been needed for showing how to construct an embedding of a graph that is found planar.

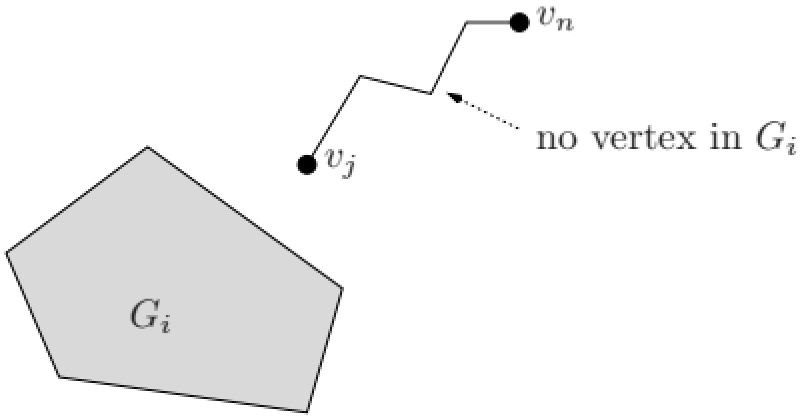
The algorithm by Lempel, Even and Cederbaum

Now assume that G is a plane graph with an *st*-order $v_1, \dots, v_n$, and edge (v_1, v_n) is on the outer-face.

This tells us a lot about the location of other vertices.

Let G_i be the graph induced by $v_1, \dots, v_i$, and use for G_i the planar embedding induced by G.

Each v_j, $j > i$ must be in the outer-face of G_i.

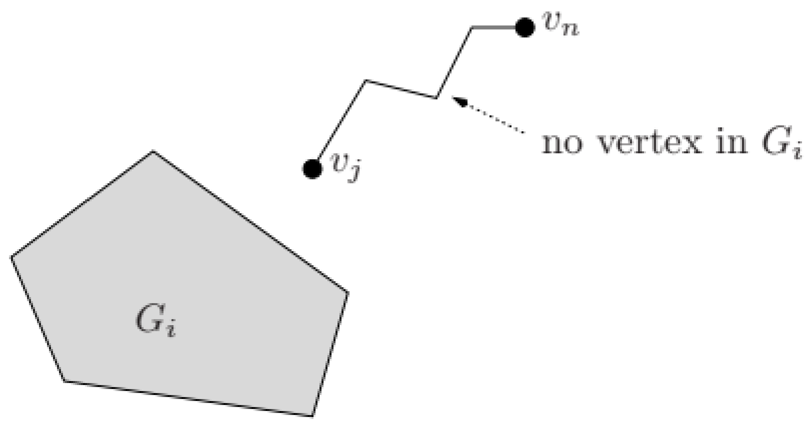


Since v_n is on the outer-face, for each G_i, $i < n$, vertex v_n is located inside the outer-face of G_i.

But even more can be said.

For any v_j, $j > i$, there exists a path from $v_j \sim> v_n$ that only uses vertices not in G_i (this holds because we can get from v_j to a successor, to a successor, to a successor, and so on, until we must finally stop at v_n.)

Therefore, this whole path must be in the outer-face of G_i, and in particular, all v_j , $j > i$ must be in the outer-face of G_i.



Bush form (кустовая форма) B_i.

Now assume that G is a graph with an *st*-order $v_1, \ldots, v_n$ and we want to test whether G is planar.

The idea is to build planar drawings of G_i in such a way that $v_{i+1}, \ldots, v_n$ can be in the outer-face of G_i.

To do so, we define a slightly modified graph called the *bush form* B_i.

Bush form (кустовая форма) B_k .

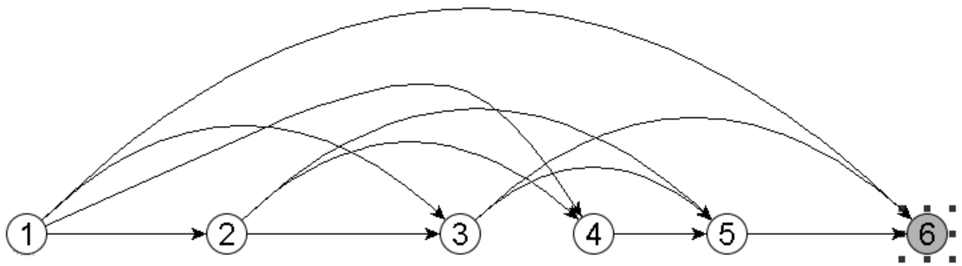
$G = (V, E)$ is a biconnected graph, the nodes of G are st-numbered $V_k = \{1, 2, \ldots, k\}$,

$G_k = (V_k, E_k)$, graph induced by V_k

B_k extends G_k: For each $(v, w) \in E$ with $v \leq k$ и $w \geq k$ there is a node and an edge in B_k

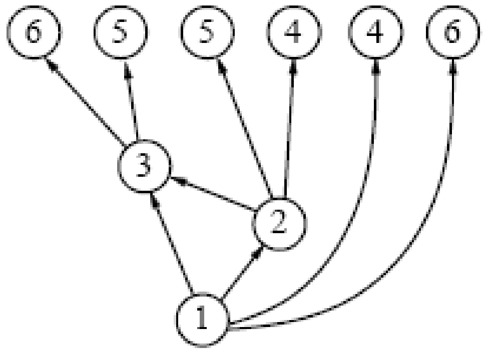
They are called virtual nodes and virtual edges respectively.

We label every virtual node with its counterpart in G.



G_3 = induced by {1,2,3}

Bush form B_3 extends G_3



Bush form

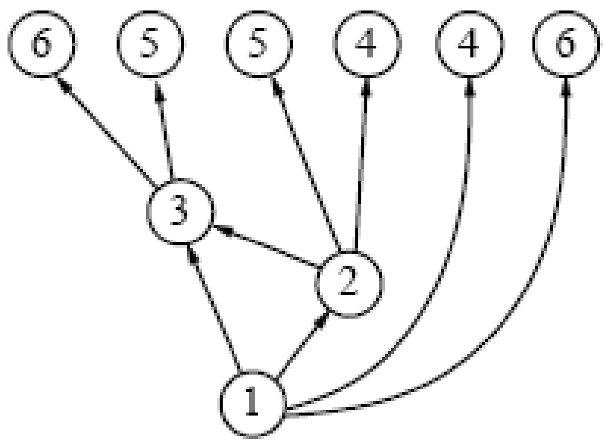
If G is planar, B_k can be drawn in a way which resembles a bush:

Node v, $1 \leq v \leq k$ is drawn at height v

All virtual nodes are put on a horizontal line at height $k+1$

All edges are drawn as a y-monotone curves

We call the horizontal line at height $k+1$ the *horizon*



Bush form

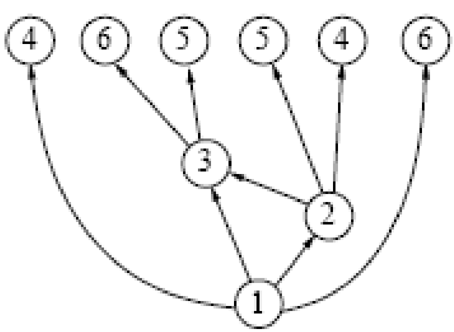
Let v be a cut vertex of $B_{k,}$ k >1.

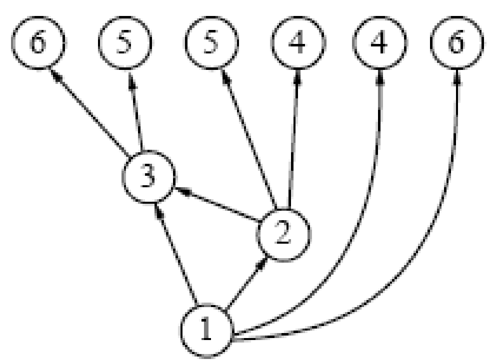
Root component of v = unique component containing lower bounded nodes and all other components (does not exist if v = 1)

Non-root components of v = all other components

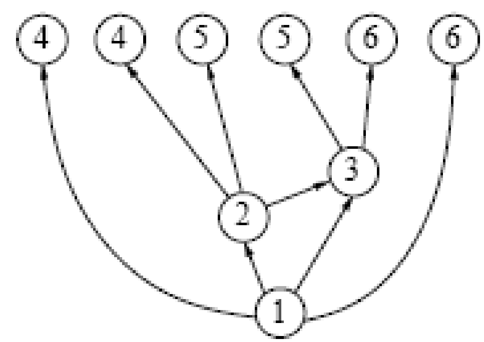
A permutation operation permutes the non-root components with respect to a cut vertex B_k.

A flipping operation flips over a sub-bush





permutation

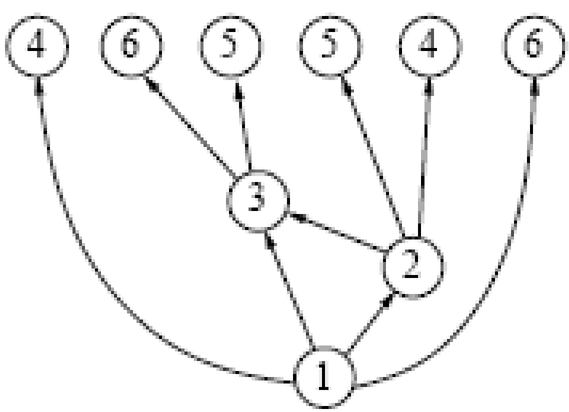


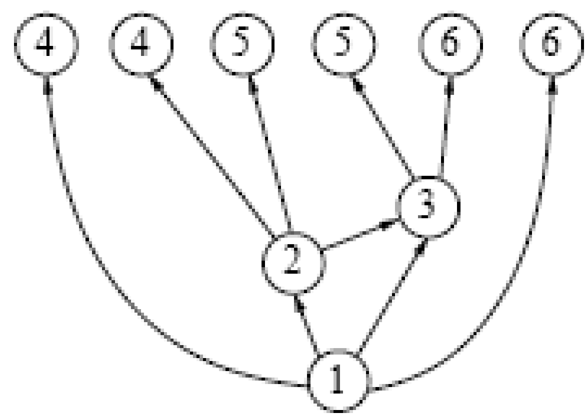
flipping

Flipping (отражение) operation

When we want to draw B_4 we must first transform B_3 so that all virtual vertices with label 4 form a consecutive sequence on level 4.

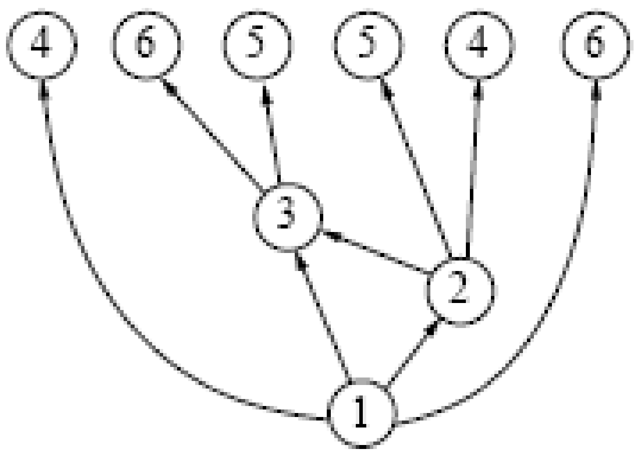
This can be done by flipping (отражение) around vertex 1 epy the component which includes the vertices 2 and 3, so that the virtual vertices labeled 5 and 4 in the split component swap their positions.

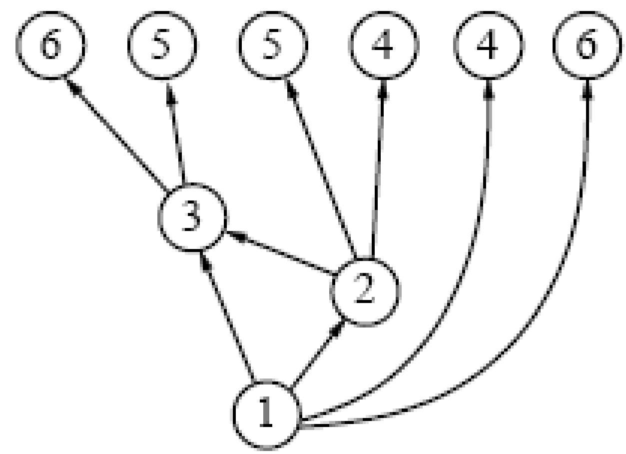




Permutation (перестановка) operation

We also can to move the left virtual vertex labeled 4 to the right and put it next to another virtual vertex labeled 4





Reduction of a bush form

The goal of these transformations is to find such an order where all the virtual vertices with the same label are placed consequtively.

This kind of transformation is called reduction (редукция).

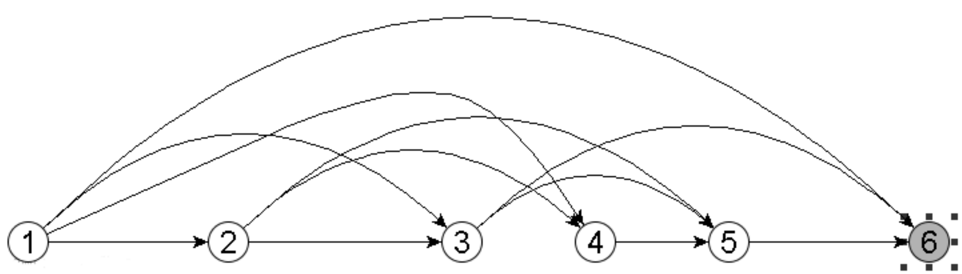
If it is not possible to make the vertices labeled $k+1$ consequtive, we know that the graph is not planar.

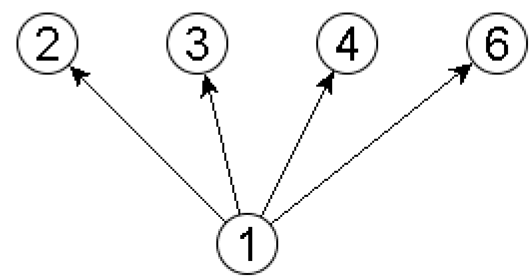
Otherwise, the algorithm will produce a planar drawing of the graph.

Theorem 10 Graph G is planar $\Leftrightarrow$ for every vertex k it admits a bush form B_k

An example of planarity testing

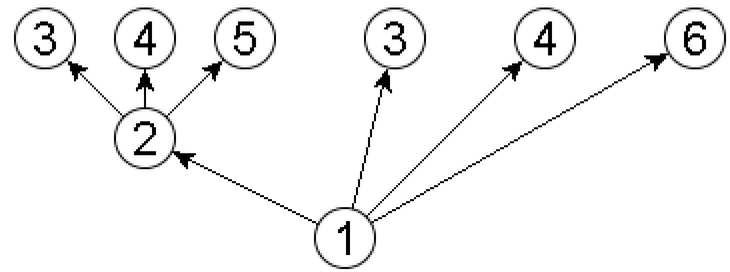
Graph G with *st*- numbering and a bush form B_1

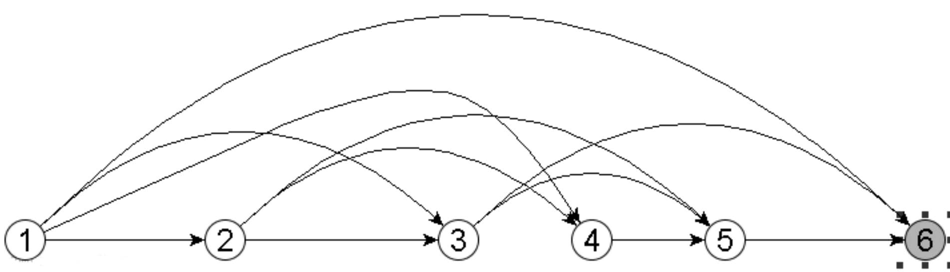


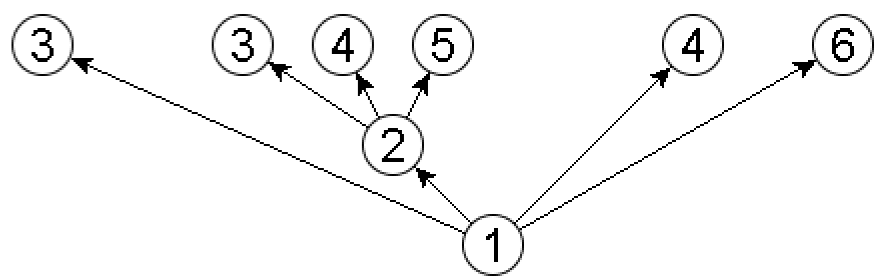


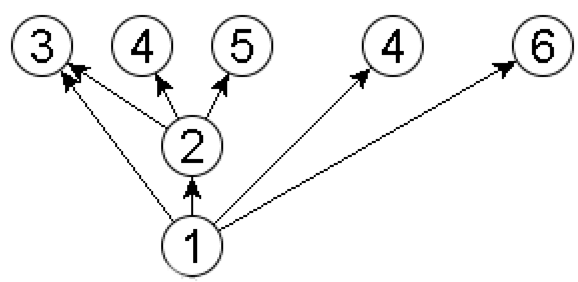
An example of planarity testing

Bush form B_2



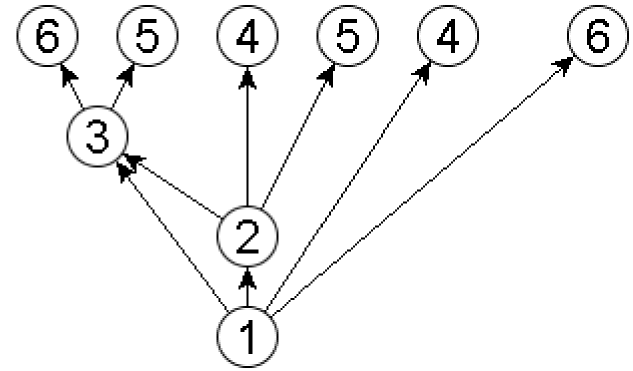


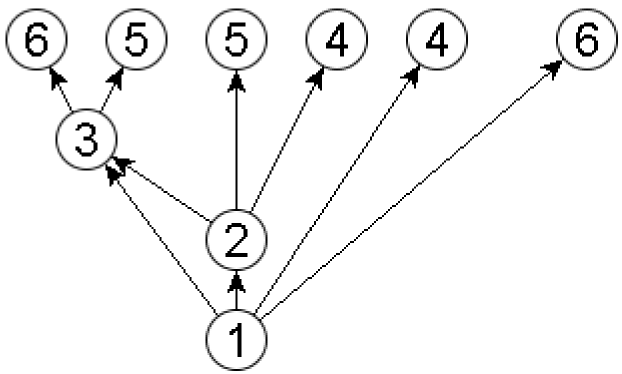


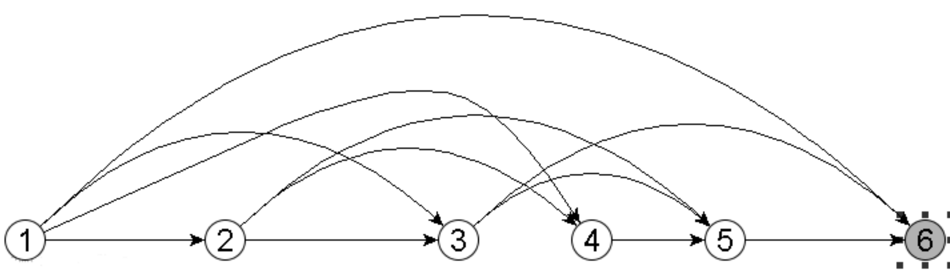


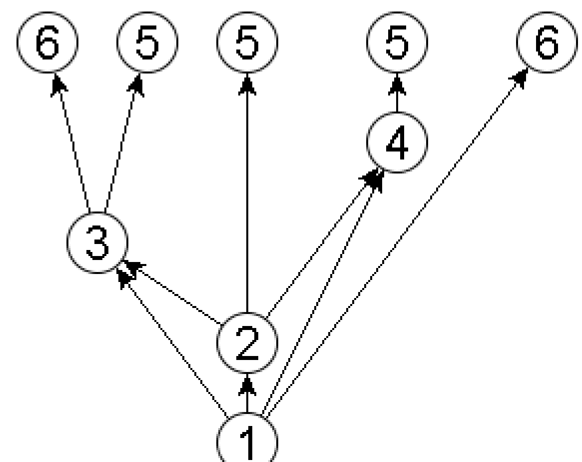
An example of planarity testing

Bush form B_3



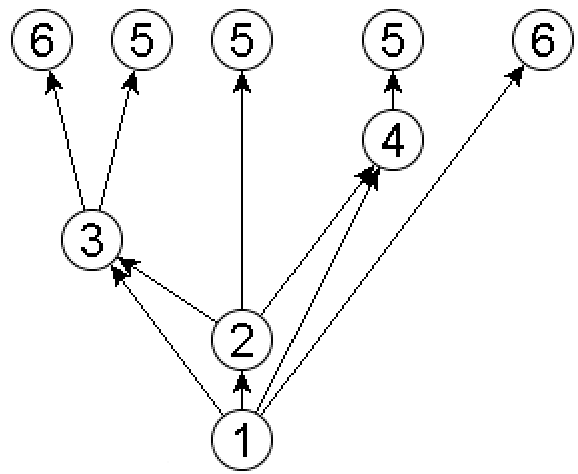


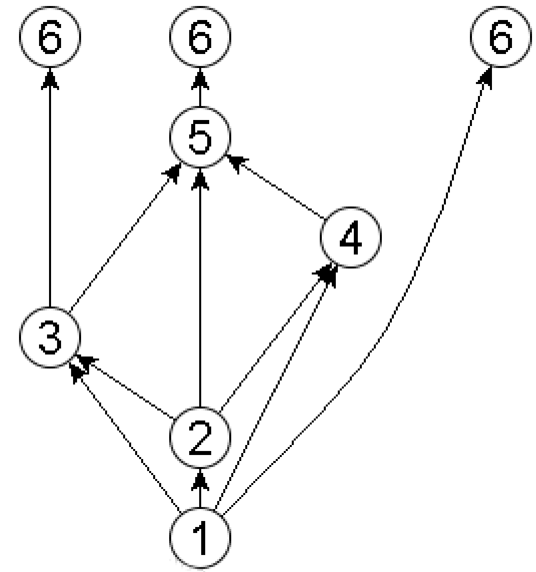


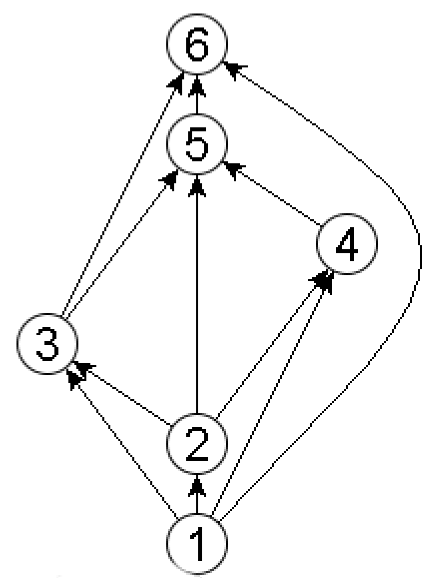


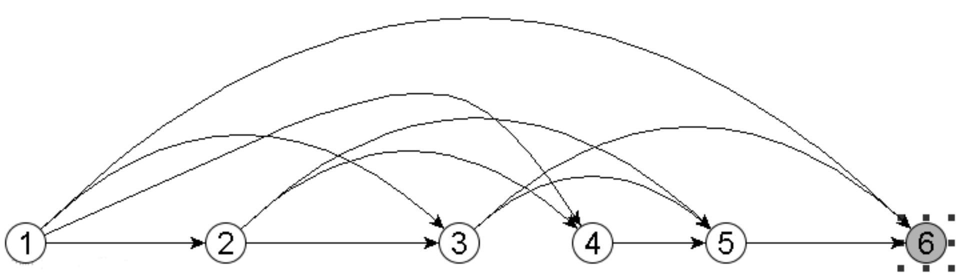
An example of planarity testing

Bush form B_4









What happens if the graph G is not planar???

Leaf Words and Extendible Bush Form

Leaf word (листовое слово) is a sequence $\{N, E\}^*$, where

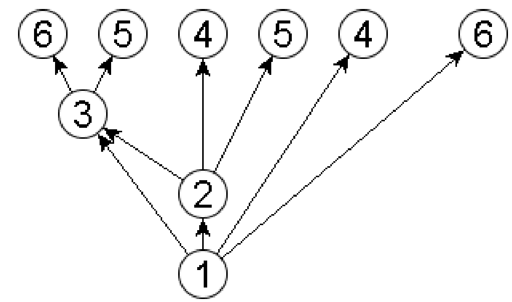
E represents $k+1$,

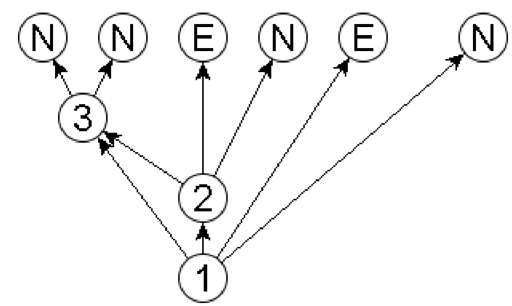
N stands for $\geq k+2$

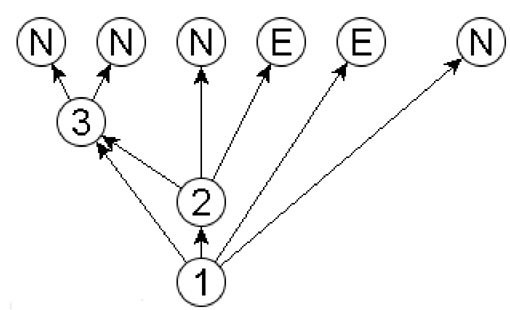
NNENEN and NNNEEN

Extendible bush form (расширяемая кустовая форма): leaf word in $N^*E^*N^*$

An extendible bush form $B_k^\wedge$ is readily extended to a bush form B_{k+1}







Leaf Words and Extendible Bush Form

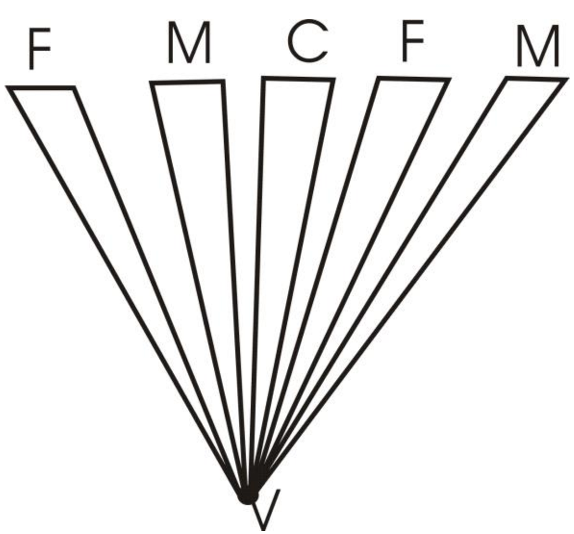
Theorem 11(Extendibility) If B_k has no obstruction (to extedibility) then any bush form B_k can be transformed into an extendible form of B_k^ by permutations and flippings

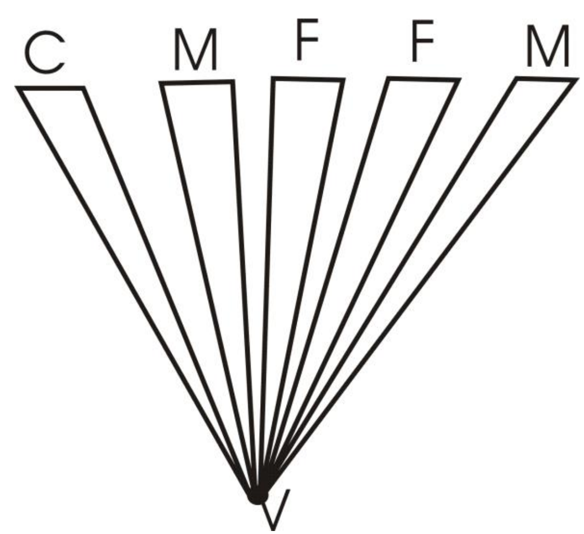
Leaf Words and Extendible Bush Form

Subgraph of B_k is called Clean, if none all of its virtual vertces are labeled k+1

Subgraph of B_k is called Mixed, if some but not all of its virtual vertces are labeled k+1

Subgraph of B_k is called Full if all of its virtual vertces are labeled k+1

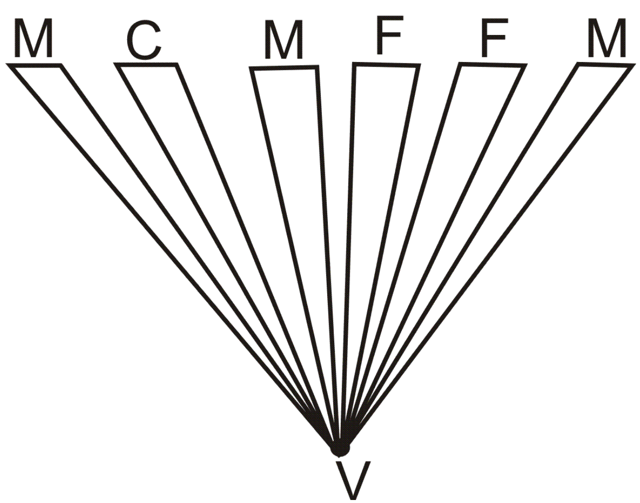




Leaf Words and Extendible Bush Form

A cut vertex v of B_k *is* obstructing if there are ≥ 3 components of $B_k \backslash v$ that are mixed

Without obstruction, we can gather the occurences of *k+1.*



Summary

Lempel –Even- Cederbaum planarity test constructs bush forms $B_0, B_1, \ldots, B_n$

in iteration k+1 the bush form B_k is first transformed into an extendible bush form $B^\wedge_k$, and then extended to a bush form B_{k+1}

Transformation to an extendible bush form uses permutations and flippings and is possible if B_k *contains no obstructions* .

The running time of the Lempel–Even- Cederbaum test is $O(n^2)$ in its original form

Booth and Lueker improved the running time to $O(n+m)$ time by the introduction of the PQ –tree data structure.ложности

Combinatorial embeddings

The definition of a planar graph, while easy to understand in theory, is hard to capture for algorithmic purposes.

In the current definition, we allow drawings with arbitrary curves for edges.

How would we store this in a computer, i.e., represent it in a discrete way?

One idea would be to store vertices as points and declare edges as straight lines, but to be allowed to do this, we would first need to prove that all planar graphs have such a straight-line drawing.
(They do, but this is not at all an easy result)
Instead, we will represent planar graphs in a very different way, via what is called a combinatorial embedding.

Combinatorial embedding

Definition Let G be a graph.

A combinatorial embedding (комбинаторная укладка) of G is a set of orderings π_v for each vertex $v \in V$, where π_v specifies a cyclic ordering of edges incident to v.

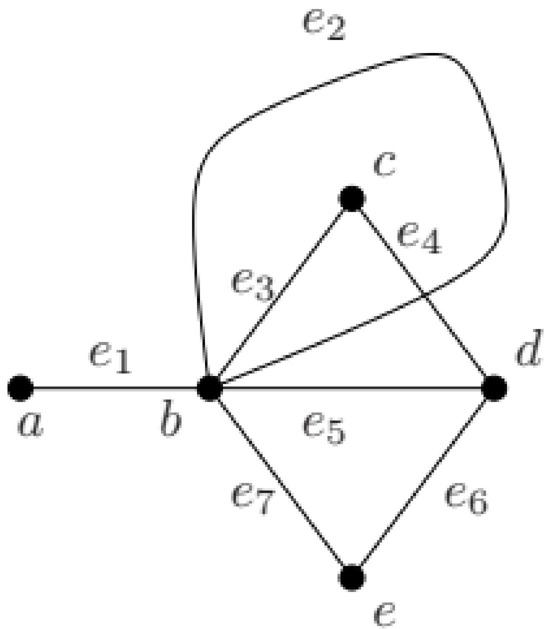
π_a: $\{e_1\}$

π_b: $\{e_1, e_2, e_3, e_2, e_5, e_7\}$

π_c: $\{e_3, e_4\}$

π_d :$\{e_4, e_6, e_5\}$

π_e :$\{e_6, e_7\}$



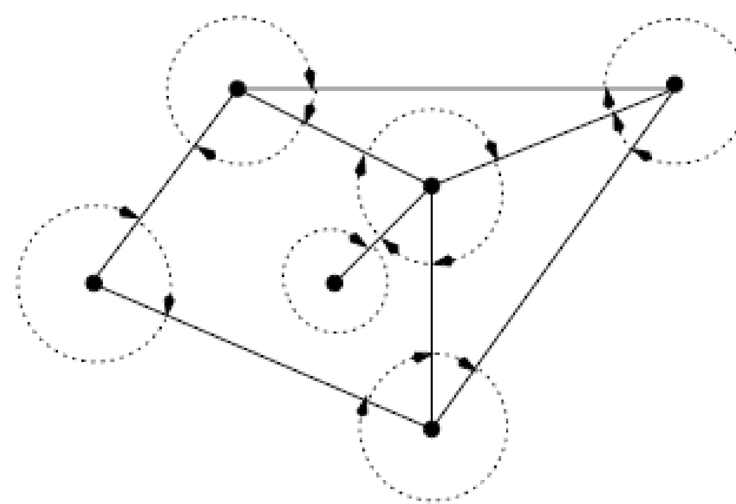
Combinatorial embeddings of planar graphs

While the above definitions holds for arbitrary combinatorial embeddings, the concepts are easier to visualize for a planar graph.

Assume for a moment that someone gives us a drawing of a planar graph *G* that has no crossing.

Then use as combinatorial embedding the one induced by this drawing, i.e., order the edges as they appear in clockwise order around each vertex.

The problem of constructing a planar drawing that corresponds to a given combinatorial embedding is much more difficult…



Combinatorial embeddings

If we are given a drawing of a graph $G = (V, E)$ (with crossings or without), then this always implies a combinatorial embedding, by taking the clockwise order of the edges incident to each vertex.

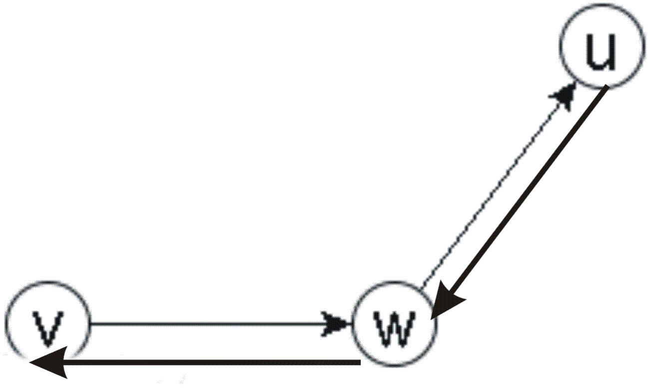
On the other hand, if we are given a combinatorial embedding of a graph $G = (V, E)$, then it is easy to create a drawing (with crossings, possibly) such that the combinatorial embedding exactly corresponds to the clockwise order of edges at each vertex.

Next edge

Given a combinatorial embedding, we can define for each edge a next edge by following the combinatorial embedding.

To make this precise, we will for a little while replace each edge (*v, w*) by 2 directed edges *v-> w* and *w-> v*, and replace the entry of (*v, w*) in the list of *v* by *v ->w*, followed by *w-> v*.

Then for edge *v -> w*, the next edge after *v -> w* is the edge *w -> u* that immediately follows *v-> w* in the cyclic order of edges around vertex *w*.



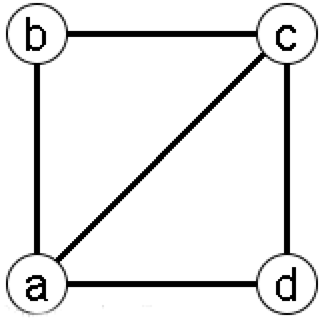
Face of a planar graph, combinatorial definition

a:{(a, b), (a, c), (a, d)}

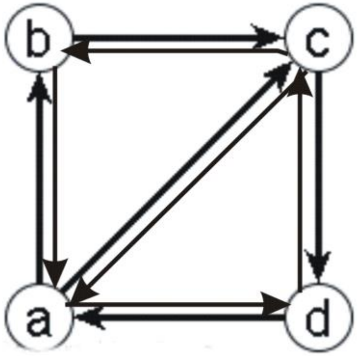
b:{(b, c), (b, a)}

c:{(c, b), (c, a), (c, d)}

d:{(d, c), (d, a)}



a:{(a, b), (b, a), (a, c), (c, a), (a, d), (d, a)};
b:{(b, c),(c, b), (b, a)(a, b)};
c:{(c, b), (c, a), (c, d), (d, c)};
d:{(d, c), (c, d), (d, a), (a, d)};



Using the notion of the next edge, we can define a face as the equivalence classes of edges that can reach each other via the next operation

Face of a planar graph, combinatorial definition

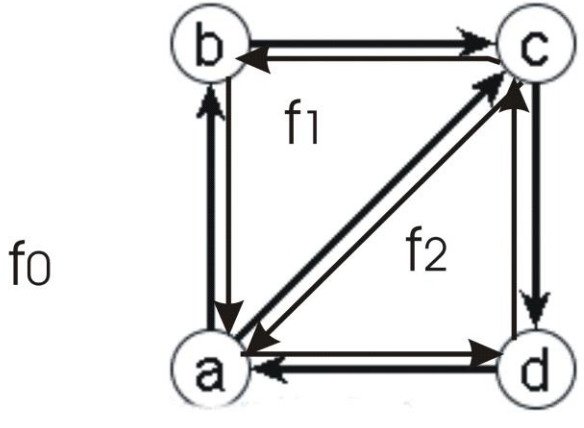
More precisely, this is defined as follows.
Start at an arbitrary edge v_1 ->v_2.

Let v_2 -> v_3 be the next edge after v_1->v_2.

Iterate, i.e. for $i > 0$, let v_i ->v_{i+1} be the next edge after v_{i-1} -> v_i.

Continue this process until at some point we repeat an edge (which must happen since the graph is finite.)

This repeated edge actually must be v_1 -> v_2, since every edge has only one edge for which it is the next.



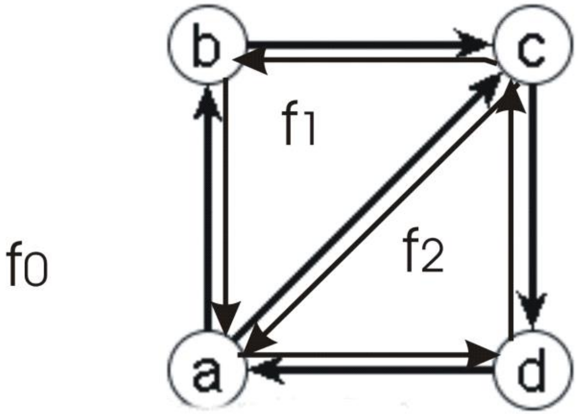
$f_0 = \{(a, b), (b, c), (c, d), (d, a)\}$
$f_1 = \{(a, c), (c, b), (b, a)\};$
$f_2 = \{(a, d), (d, c), (c, a)\};$

Face of a plane graph, combinatorial definition

The resulting circuit

<v_1, v_2, ..., v_k, v_1> is called a facial circuit, or face.

The number k is the degree of the face, and is also denoted deg(F) for face F.



f_1= {(a, c), (c, b), (b, a)};
f_2= {(a, d), (d, c), (c, a)};
f_0= {(a, b), (b, c), (c, d), (d, a)}

Face of a planar graph, combinatorial definition

A plane graph G is a planar graph with a fixed combinatorial embedding.

A graph with fixed combinatorial embedding naturally splits in a number of faces.

Each direction of each edge belongs exactly to 1 face.

Though the 2 directions might belong to the same face twice.

In particular, therefore,

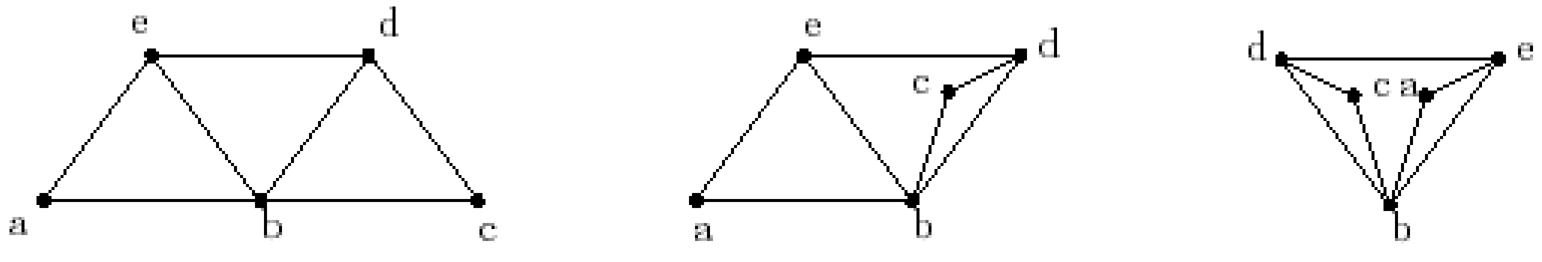
$\sum_F \deg(F) = 2m$.

Different drawings of the same planar graph

Figure bellow provides an example of different planar drawings of the same graph.

Here, the leftmost drawing is truly different from the middle drawing: the leftmost drawing has a face of degree 5 (the outer-face), while the middle drawing has no such face.

The difference between these 2 drawings results from having “flipped” the subgraph at the cutting pair {*a, c*}.



Theorem (Whitney)

Theorem 12 (Whitney) If G is a 3-connected planar graph, then G has a unique planar embedding.