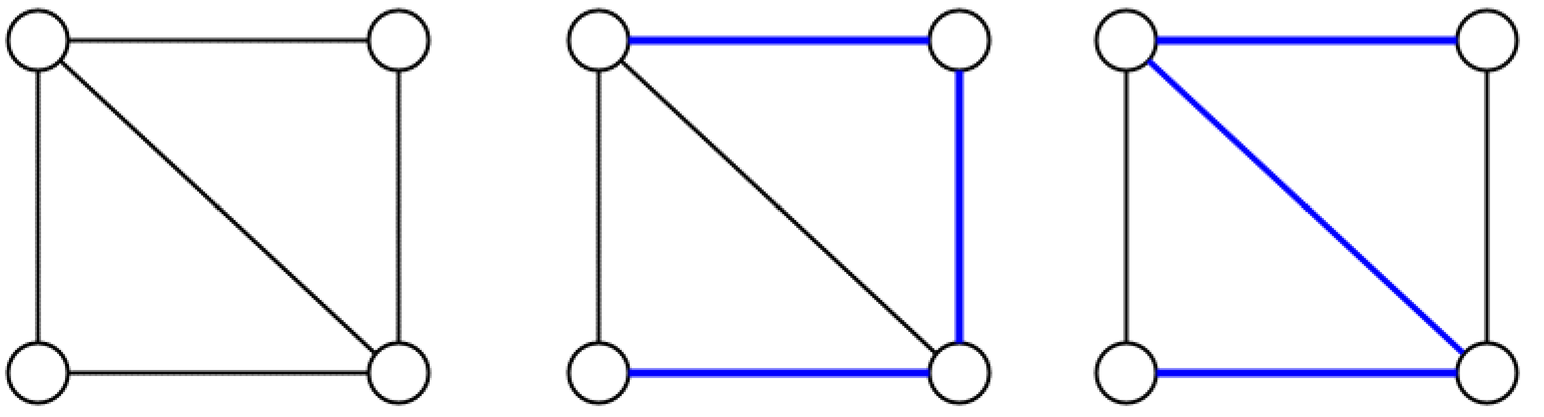
The number of spanning trees in a graph

Apanovich Z.V.

IIS SB RAS, NSU

Example



This graph has 8 distinct spanning trees = 4+4.

4 spanning trees use the diagonal edge and 4 spanning trees do not use the diagonal edge.

A recursive formula for the number of spanning trees in a graph

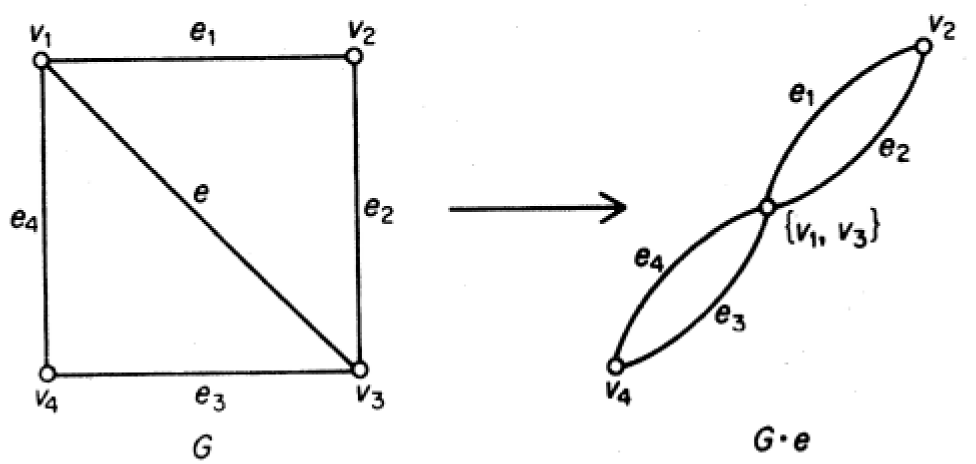
There is a simple and elegant recursive formula for the number of spanning trees in a graph.

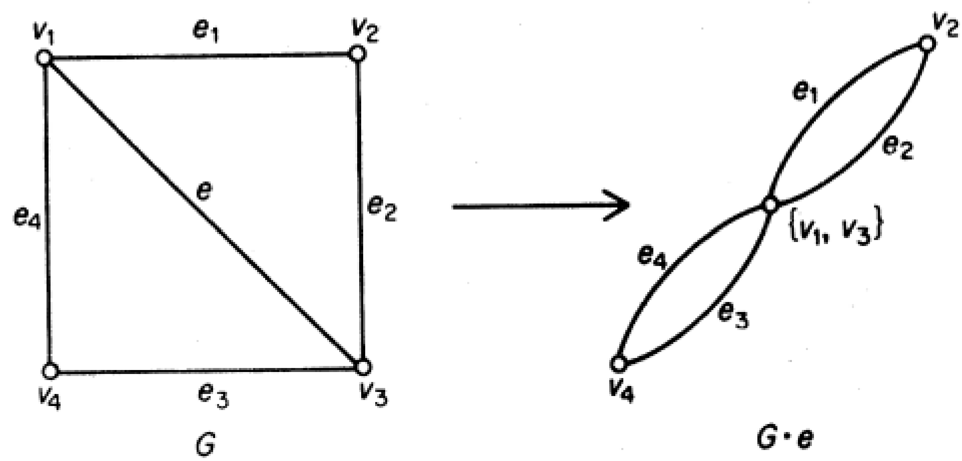
It involves the operation of contraction (стягивание ребра) of an edge, which we now introduce.

An edge e of G is said to be *contracted* if it is deleted and its ends are identified;

the resulting graph is denoted by $G \cdot e$.

Figure bellow illustrates the effect of contracting an edge.





It is clear that if e has distinct ends in G, then $v(G \cdot e) = v(G) - 1$ and $e(G \cdot e) = e(G) - 1$ and the number of components of G
$w(G \cdot e) = w(G)$.

Therefore, if T is a tree, so too is $T \cdot e$.
We denote the number of spanning trees of G by $T(G)$

***Theorem* 1** If e has distinct ends in G (e is not a loop) , then $T(G)=T(G-e)+T(G\cdot e)$.

Proof Since every spanning tree of G that does not contain e is also a spanning tree of $G - e$, and conversely, $T(G - e)$ is the number of spanning trees of G that do not contain e.

Now to each spanning tree T of G that contains e, there corresponds a spanning tree

$T\cdot e$ of $G\cdot e$.

This correspondence is clearly a bijection (see figure bellow).

Therefore $T(G \cdot e)$ is precisely the number of spanning trees of G that contain e.

It follows that $T(G)=T(G-e)+T(G-e)$

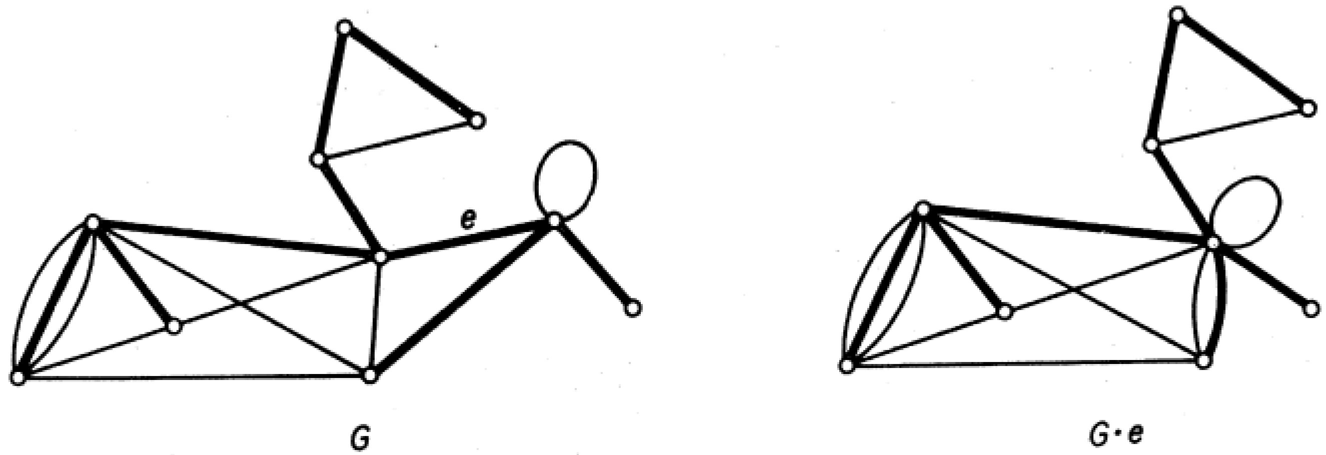
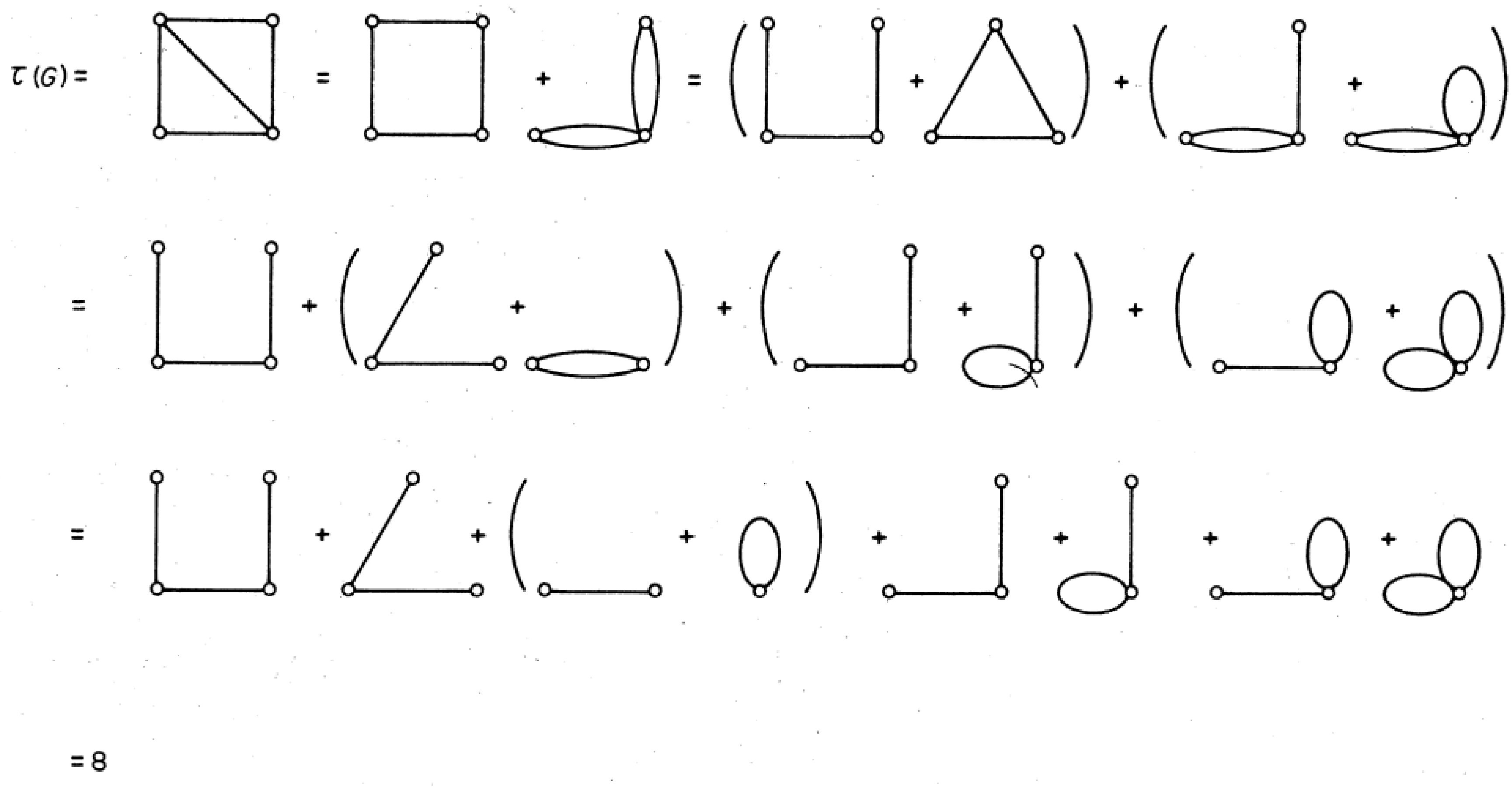


Figure bellow illustrates the recursive calculation of $T(G)$ by means of theorem 1; the number of spanning trees in a graph is represented symbolically by the graph itself.



Although theorem 1 provides a method of calculating the number of spanning trees in a graph, this method is not suitable for large graphs.
Fortunately, and rather surprisingly, there is a formula for $T(G)$ which expresses $T(G)$ as a determinant;
In the special case when G is complete, a simple formula for $T(G)$ was discovered by Cayley (1889).

The proof we give is due to Prufer (1918).

Theorem 2 $T(K_n) = n^{n-2}$.

Proof Let the vertex set of K_n be $N = \{1, 2, \ldots, n\}$.

We note that n^{n-2} is the number of sequences of length $n - 2$ that can be formed from N.

Thus, to prove the theorem, it suffices to establish a bijection between the set of spanning trees of K_n and the set of such sequences.

With each spanning tree T of K_n, we associate a unique sequence $(t_1, t_2, .., t_{n-2})$ as follows.

Regarding N as an ordered set, let s_1 be the first vertex of degree 1 in T;

the vertex adjacent to s_1 is taken as t_1.

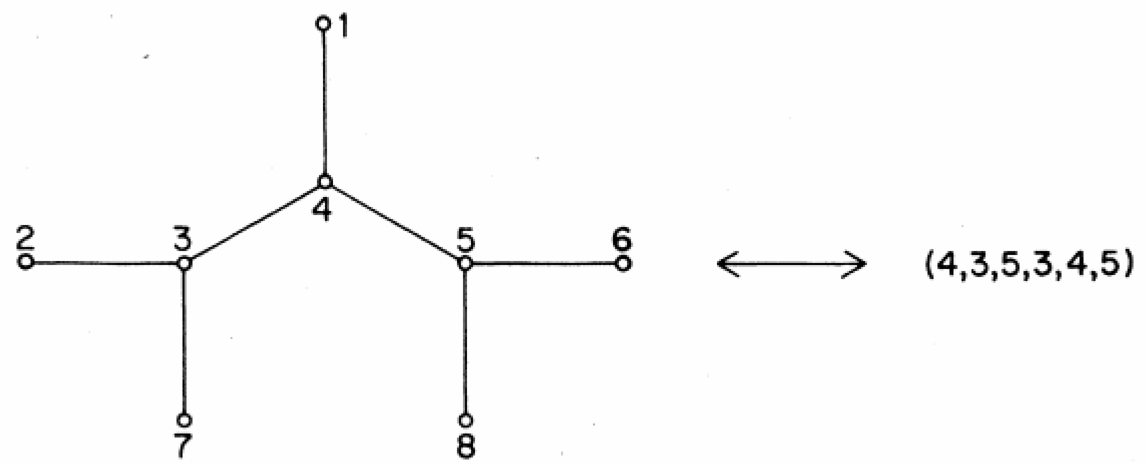
We now delete s_1 from T, denote by s_2 the first vertex of degree 1 in $T - s_1$, and take the vertex adjacent to s_2 as t_2.

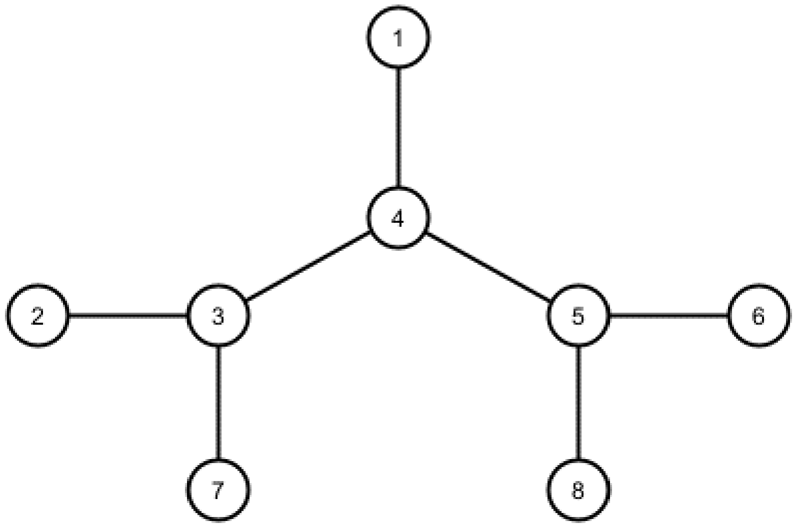
This operation is repeated until t_{n-2} has been defined and a tree with just 2 vertices remains;

Example

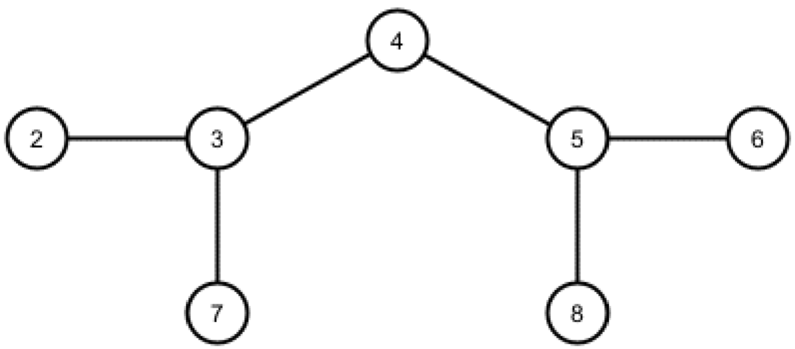
The tree in figure bellow, for instance, gives rise to the sequence (4, 3, 5, 3, 4, 5).

It can be seen that different spanning trees of K_n determine different sequences.

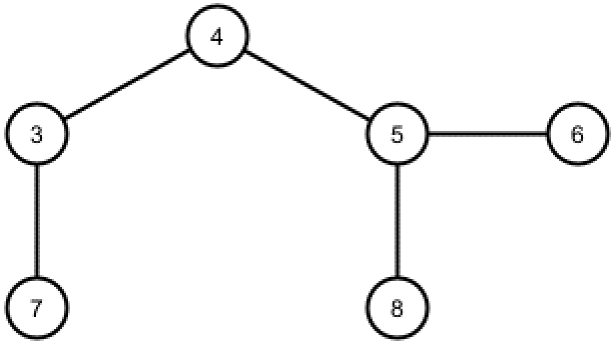




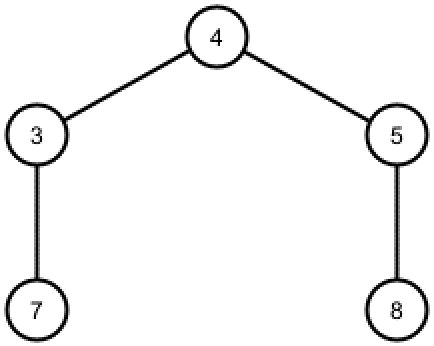
	1	2	3	4	5	6	7	8	s	t	Prufer
D(v)	1	1	3	3	3	1	1	1			
	0	1	3	2	3	1	1	1	1	4	4,



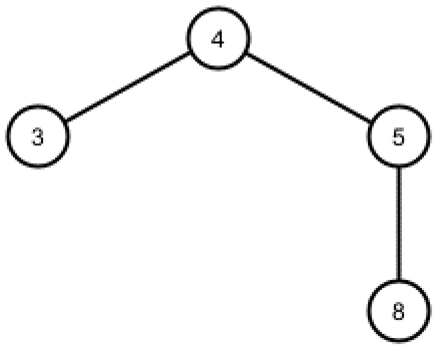
	1	2	3	4	5	6	7	8			Prufer
D(v)	1	1	3	3	3	1	1	1			
	0	1	3	2	3	1	1	1	1	4	4
	0	0	2	2	3	1	1	1	2	3	4,3



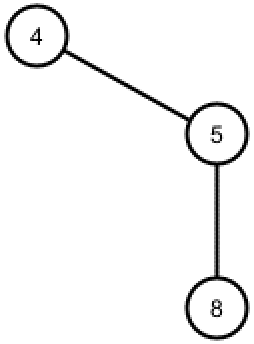
	1	2	3	4	5	6	7	8	s	t	Prufer
D(v)	1	1	3	3	3	1	1	1			
	0	1	3	2	3	1	1	1	1	4	4
	0	0	2	2	3	1	1	1	2	3	4,3
	0	0	2	2	2	0	1	1	6	5	4,3,5



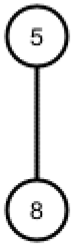
	1	2	3	4	5	6	7	8			Prufer
D(v)	1	1	3	3	3	1	1	1			
	0	1	3	2	3	1	1	1	1	4	4
	0	0	2	2	3	1	1	1	2	3	4,3
	0	0	2	2	2	0	1	1	6	5	4,3,5
	0	0	1	2	2	0	0	1	7	3	4,3,5,3



	1	2	3	4	5	6	7	8	s	t	Prufer
D(v)	1	1	3	3	3	1	1	1			
	0	1	3	2	3	1	1	1	1	4	4
	0	0	2	2	3	1	1	1	2	3	4,3
	0	0	2	2	2	0	1	1	6	5	4,3,5
	0	0	1	2	2	0	0	1	7	3	4,3,5,3
	0	0	0	1	2	0	0	1	3	4	4,3,5,3,4



	1	2	3	4	5	6	7	8	s	t	Prufer
D(v)	1	1	3	3	3	1	1	1			
	0	1	3	2	3	1	1	1	1	4	4
	0	0	2	2	3	1	1	1	2	3	4,3
	0	0	2	2	2	0	1	1	6	5	4,3,5
	0	0	1	2	2	0	0	1	7	3	4,3,5,3
	0	0	0	1	2	0	0	1	3	4	4,3,5,3,4
	0	0	0	0	1	0	0	1	4	5	4,3,5,3,4,5



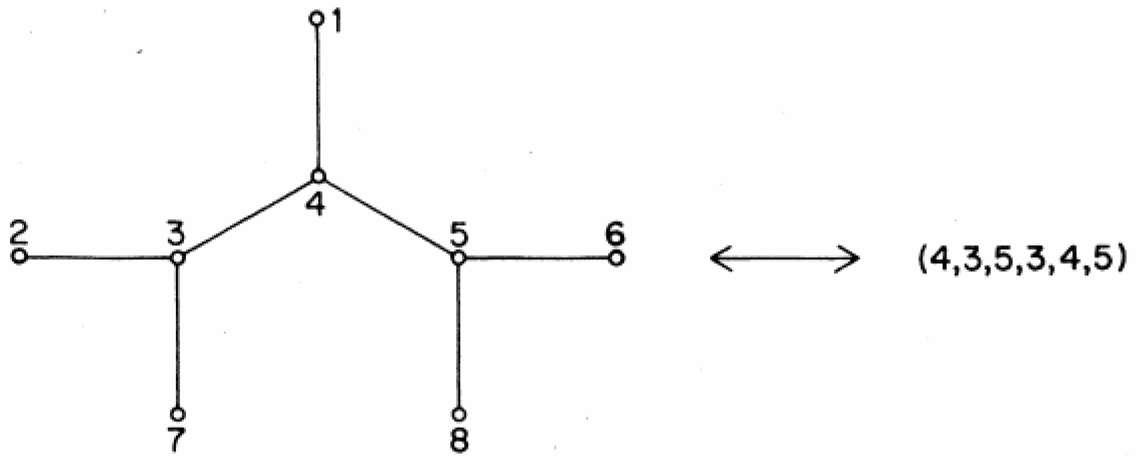
Prufer code: 4,3,5,3,4,5

Reverse procedure

The reverse procedure is equally straightforward.

Observe, first, that any vertex v of T occurs $d_T(v) - 1$ times in $(t_1, t_2, \ldots, t_{n-2})$.

Thus the vertices of degree 1 in T are precisely those that do not appear in this sequence.



Example

Prufer code: 4,3,5,3,4,5

Hence, the tree has 8 nodes

Absent nodes are: 1, 2, 6, 7, 8

They are the leaves of the tree.

Other vertices have the following degrees: deg(4) = 2+1, deg(3) = 2+1, deg(5) = 2+1

The degree sequence of the tree is:

1	2	3	4	5	6	7	8
1	1	3	3	3	1	1	1

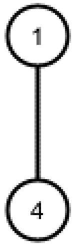
Prufer code: 4,3,5,3,4,5

$V(T)$ not in $(t_1, t_2, \ldots, t_{n-2})$ = 1, 2, 6, 7, 8

To reconstruct T from $(t_1, t_2, \ldots, t_{n-2})$, we therefore proceed as follows.

	1	2	3	4	5	6	7	8	s	t
D(v)	1	1	3	3	3	1	1	1		
	0	1	3	2	3	1	1	1	1	4

Let s_1 be the first vertex of $V(G)$ not in Prufer code$(t_1, t_2, \ldots, t_{n-2})$; join s_1 to t_1.



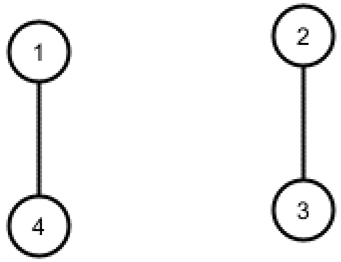
Next, let s_2 be the first vertex of $V(T)\backslash\{s_1\}$ not in $(t_2, \ldots, t_{n-2})$,

and join s_2 to t_2.

Continue in this way until the n - 2 edges $s_1t_1, s_2t_2, \ldots, s_{n-2}\, t_{n-2}$ have been determined.

Prufer code: 4,3,5, 3,4,5

$V(T)$ not in $(t_1, t_2, \ldots, t_{n-2})$ = 1, 2, 6, 7, 8

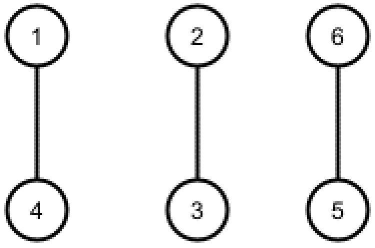


	1	2	3	4	5	6	7	8		
D(v)	1	1	3	3	3	1	1	1		
	0	1	3	2	3	1	1	1	1	4
	0	0	2	2	3	1	1	1	2	3

Prufer code: 4,3,5, 3,4,5

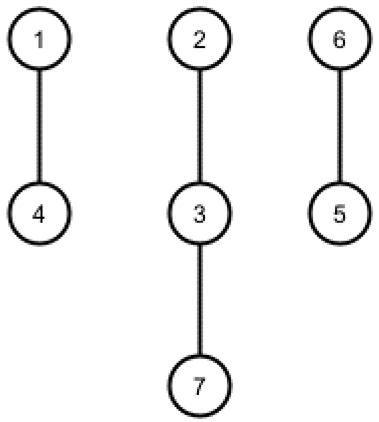
$V(T)$ not in $(t_1, t_2, \ldots, t_{n-2})$ = 1, 2, 6, 7, 8

	1	2	3	4	5	6	7	8	s	t
D(v)	1	1	3	3	3	1	1	1		
	0	1	3	2	3	1	1	1	1	4
	0	0	2	2	3	1	1	1	2	3
	0	0	2	2	2	0	1	1	6	5



Prufer code: 4,3,5,3,4,5

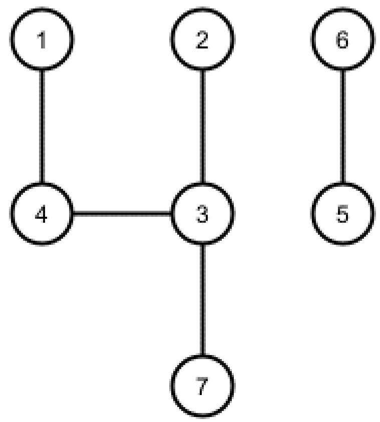
$V(T)$ not in $(t_1, t_2, \ldots, t_{n-2})$ = 1, 2, 6, 7,



	1	2	3	4	5	6	7	8		
D(v)	1	1	3	3	3	1	1	1		
	0	1	3	2	3	1	1	1	1	4
	0	0	2	2	3	1	1	1	2	3
	0	0	2	2	2	0	1	1	6	5
	0	0	1	2	2	0	0	1	7	3

Prufer code: 4,3,5,3,4,5

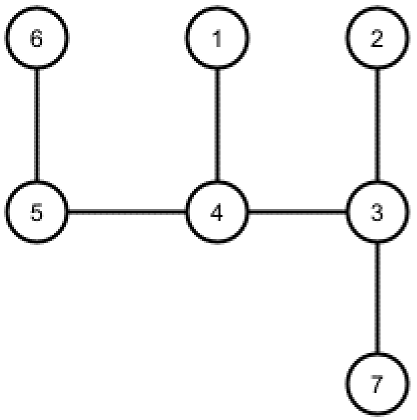
$V(T)$ not in $(t_1, t_2, \ldots, t_{n-2})$ = 1, 2, 6, 7,



	1	2	3	4	5	6	7	8	s	t	Prufer
D(v)	1	1	3	3	3	1	1	1			
	0	1	3	2	3	1	1	1	1	4	4
	0	0	2	2	3	1	1	1	2	3	4,3
	0	0	2	2	2	0	1	1	6	5	4,3,5
	0	0	1	2	2	0	0	1	7	3	4,3,5,3
	0	0	0	1	2	0	0	1	3	4	4,3,5,3,4

Prufer code: 4,3,5,3,4,5

$V(T)$ not in $(t_1, t_2, \ldots, t_{n-2})$ = 1, 2, 6, 7, 4



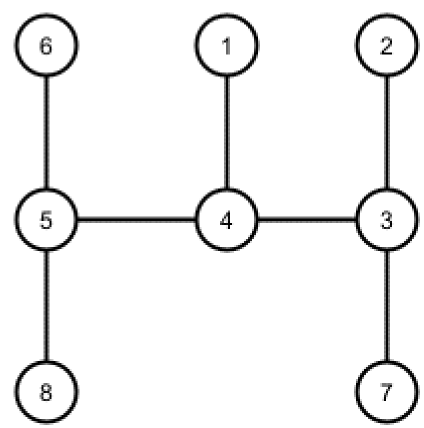
	1	2	3	4	5	6	7	8	s	t	Prufer
D(v)	1	1	3	3	3	1	1	1			
	0	1	3	2	3	1	1	1	1	4	4
	0	0	2	2	3	1	1	1	2	3	4,3
	0	0	2	2	2	0	1	1	6	5	4,3,5
	0	0	1	2	2	0	0	1	7	3	4,3,5,3
	0	0	0	1	2	0	0	1	3	4	4,3,5,3,4
	0	0	0	0	1	0	0	1	4	5	4,3,5,3,4,5

T is now obtained by adding the edge joining the 2 remaining vertices of $V \backslash \{s_1, s_2, \ldots, s_{n-2}\}$.

It is easily verified that different sequences give rise to different spanning trees of K_n.

We have thus established the desired bijection

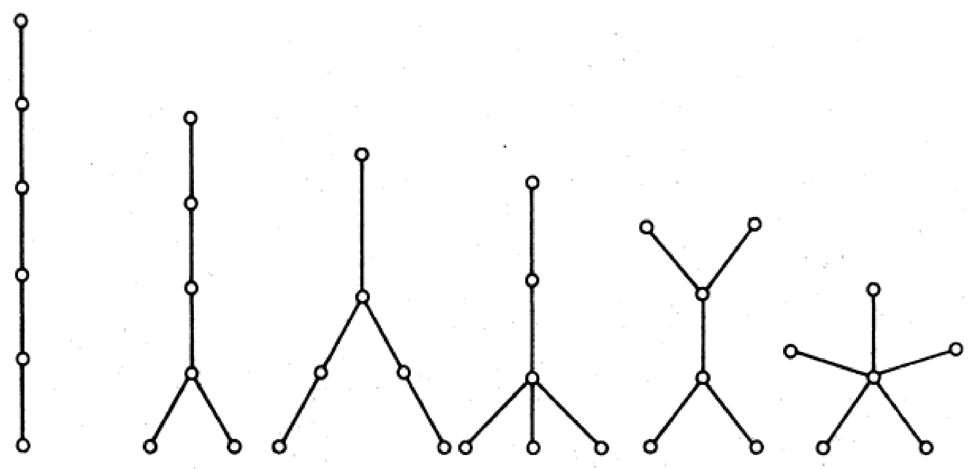
The last added edge is (8, 5)



Note that n^{n-2} is not the number of nonisomorphic spanning trees of K_n , but the number of distinct spanning trees of K_n ;

there are just 6 nonisomorphic spanning trees of K_6 (see figure bellow),

whereas there are
6^4 = 1296 **distinct** spanning trees of K_6



Laplacian matrix for the number of spanning trees

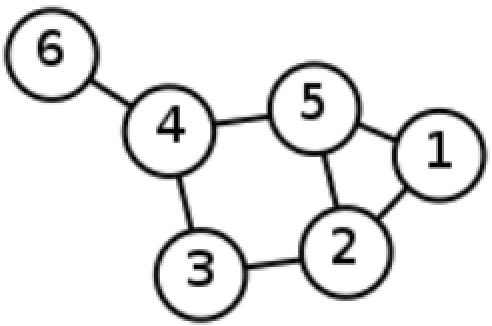
Given a simple graph with vertices , its Laplacian matrix is defined element-wise as
Laplacian matrix

$$L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently by the matrix **L** = **D-A**,
where **D** is the degree matrix and **A** is the adjacency matrix of the graph.

Example

Labelled graph



Degree matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Adjacency matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Laplacian matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

What is the spectrum of L_G?

We observe that **e** (all 1s vector) is an eigenvector of eigenvalue 0 for L_G,

Fact 1 $\lambda_1 = 0$.

Fact 2 $\lambda_2 = 0$ ⇔ *G is disconnected.*

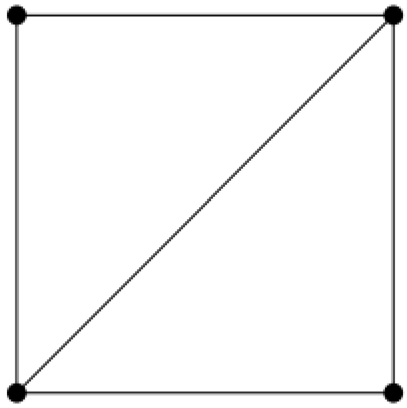
Fact 3 $\lambda_k = 0$ ⇔ *G has at least k components.*

The Matrix-Tree Theorem

Let $A[i]$ be the matrix A with its ith row and column removed.

Theorem 3 (Kirchhoff's Matrix-Tree Theorem) *The number of spanning trees in a graph G is given by* $\det(LG[i])$, *for any i.*

Example



$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}.$$

$$L^* = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix},$$

Det(**L***) = 8.