ISTA 116: Statistical Foundations for the Information Age

Discrete Random Variables

31 October 2011

Outline

Random Variables

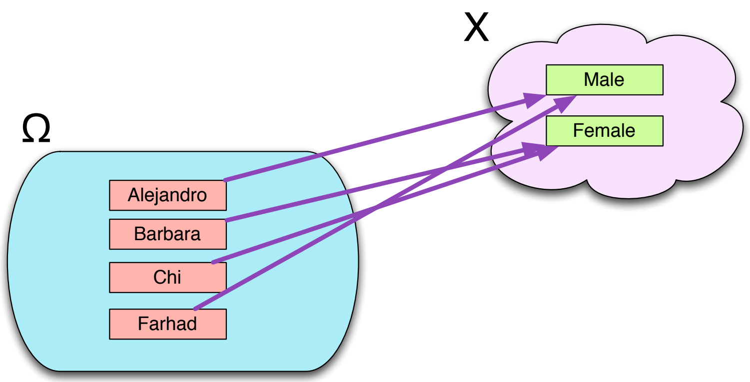
- A single random sample may have more than one characteristic that we can observe (i.e., it may be bi-/multivariate data).
- We can represent each characteristic (e.g., sex, weight, cancer status, etc.) using a **random variable**

(Definition: Random Variable)

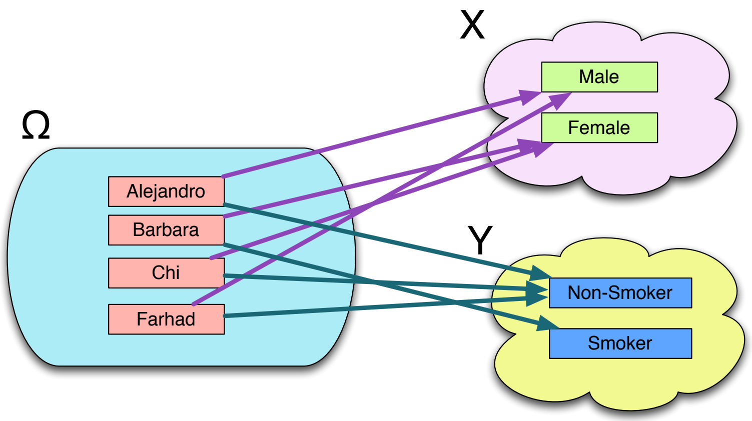
A **random variable** connects each element of the sample space to a value or quantity of interest.

- X: People $\mapsto$ Their Sex
- Y: Sequences of coin flips $\mapsto$ Number of Heads

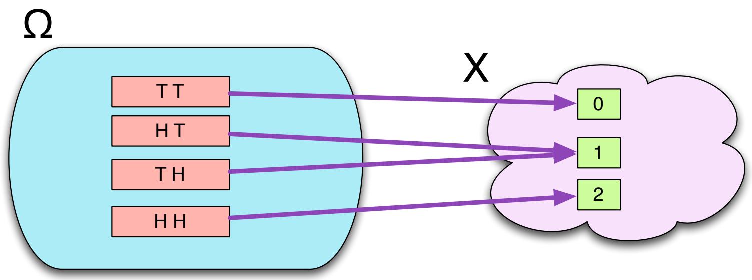
Examples of Random Variables



Examples of Random Variables



Examples of Random Variables



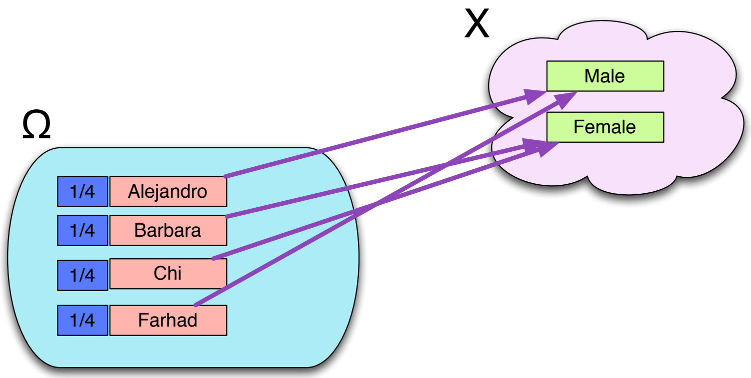
Types of Random Variables

- We can classify random variables based on the types of values they take on.
- **Discrete** random variables take on discrete values (e.g., categories or integers)
- **Continuous** random variables take on continuous values (e.g., real numbers).

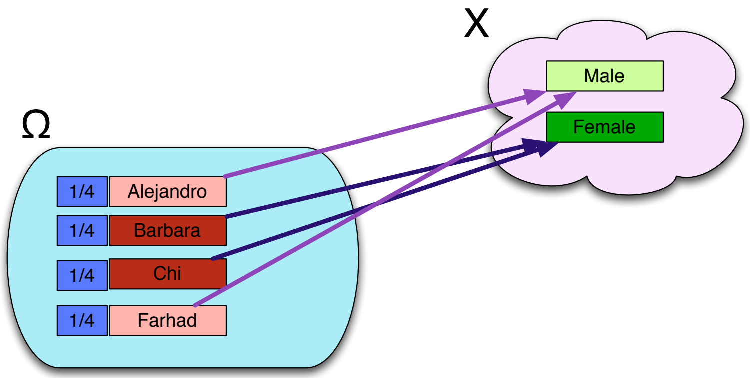
Discrete Distributions

- When a random variable is discrete, its **distribution** is characterized by the probabilities assigned to each distinct value.
- These probabilities are determined by the probabilities on the sample space itself
- If the sample space is a finite population and we make a simple random draw, then the probability of a value is the proportion of individual outcomes assigned to it.

The Distribution of a Random Variable



The Distribution of a Random Variable



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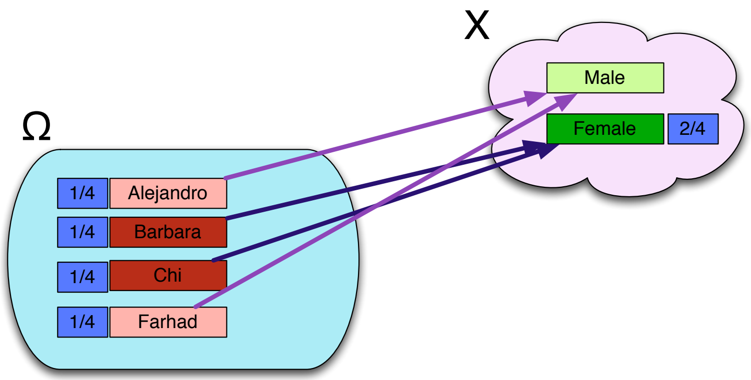
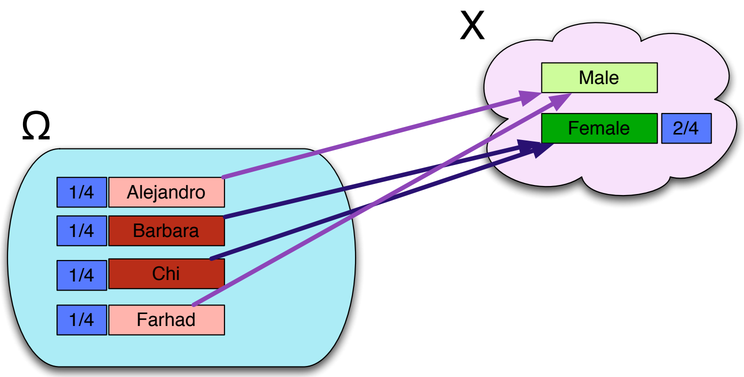
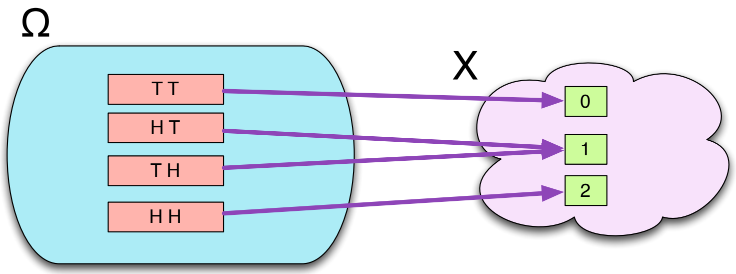


The Distribution of a Random Variable



x	Male	Female
$P(X = x)$	$1/2$	$1/2$

The Distribution of a Random Variable



x	0	1	2
$P(X = x)$			

Properties of Discrete Distributions

- Note that each value of a discrete random variable corresponds to an event in the original sample space.
- The probability associated with the value is the probability associated with the event.
- Moreover, every outcome in the sample space is associated with exactly one value of the random variable.
- Therefore, the values of a discrete random variable give us a set of **disjoint** events whose **union** is the entire sample space.

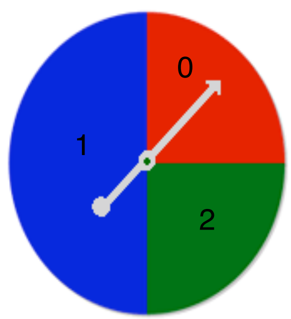
Properties of Discrete Distributions

- What must be true of the set of probabilities then?

(Properties of Discrete Distributions)

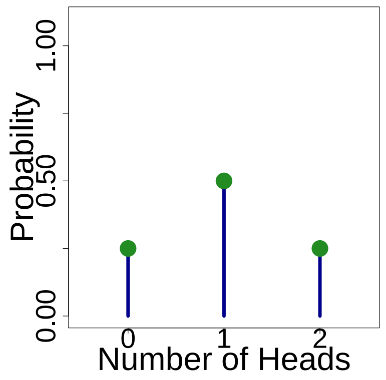
1. For every x in the range of X, $P(X = x) \geq 0$.
2. $$\sum_{x \in \text{Range}(X)} P(X = x) = 1$$

Visualizing Discrete Distributions



- Very simple distributions can be visualized with a pie chart.
- Can imagine a spinner mechanism that lands on a slice according to its probability.
- But, like pie charts, this is limited in its ability to convey information.

The Spike Plot



- An alternative is the **spike plot**
- Like a bar plot, but with probabilities, instead of frequencies or proportions, on the y-axis.

Probability Mass Function

- The **distribution** of a random variable is characterized by the set of values and their probabilities.
- For finite sets of values, can think of a table.
- One way to use such a table is to start with a value and *look up* its probability.
- This process is characterized by the **probability mass function**, which takes a value and returns its probability.

Probability Mass Function

(Definition: The Probability Mass Function)

A discrete random variable, X, can be characterized by its **probability mass function**, f_X, which takes values and returns probabilities:

$$f_X(x) = P(X = x)$$

- In the most general case, we just have to consult a table.
- However, we will see examples later of PMFs that have algebraic expressions.

The Cumulative Distribution Function

- Often times we are interested in the probability of falling in some *range* of values.
- For this purpose, we can use the **cumulative distribution function** (or CDF), which gives the "accumulated probability" up to a particular value.

(Definition: The Cumulative Distribution Function)

A random variable, X, can be characterized by its **cumulative distribution function**, F_X, which takes values and returns *cumulative* probabilities:

$$F_X(x) = P(X \leq x)$$

The Cumulative Distribution Function

- How can we calculate $F_X(x)$ from the distribution table?
- How would we calculate $P(X > x)$?
- How about $P(X \geq x)$?
- How would we calculate $P(a < X \leq b)$?
- How about $P(a \leq X \leq b)$?
- $P(a \leq X < b)$?

Summarizing Random Variables

- As with data, it is useful to characterize the center and spread of a probability distribution.
- Most of the measures we've seen can be computed; but the most common are the **mean** and **variance**.

Expected Value

- The mean of a random variable is also called its **expected value**.
- As with a sample mean, it represents an average over the possible values; but it is **weighted** by the probabilities.

Example: Mean Number of Heads

- To compute the mean number of heads in two coin tosses:

$$\begin{aligned}
\mu_X = E(X) &= 0 \times P(X = 0) + 1 \times P(X = 1) \\
&\quad +2 \times P(X = 2) \\
&= 0 \times 1/4 + 1 \times 1/2 + 2 \times 1/4 \\
&= 1/2 + 2/4 = 1
\end{aligned}$$

Expected Value of a Discrete Random Variable

- In general, we have:

(Definition: Expected Value of a Discrete Random Variable)

$$\mu_X = E(X) = \sum_{x \in \text{Range}(X)} x P(X = x)$$

- If X is a numeric variable with values from 0 to some number n, we have

(Expected Value of Finite, Integer-Valued Random Variable)

$$\mu_X = E(X) = \sum_{x=0}^{n} x P(X = x)$$

Variance of a Discrete Random Variable

- The variance is the expected squared deviation:

(Definition: Variance of a Discrete Random Variable)

$$\sigma_X^2 = E((X - \mu_X)^2) = \sum_{x \in \text{Range}(X)} (x - \mu_X)^2 P(X = x)$$

- If X is a numeric variable with values from 0 to some number n, we have

(Variance of Finite, Integer-Valued Random Variable)

$$\sigma_X^2 = E((X - \mu_X)^2) = \sum_{x=0}^{n} (x - \mu_x)^2 P(X = x)$$

Example: Variance of Number of Heads

- To compute the variance in the number of heads in two coin tosses:

$$\begin{aligned}
\sigma_X^2 &= (0 - \mu_X)^2 \times P(X = 0) + (1 - \mu_X)^2 \times P(X = 1) \\
&\quad + (2 - \mu_X)^2 \times P(X = 2) \\
&= (0 - 1)^2 \times 1/4 + (1 - 1)^2 \times 1/2 \\
&\quad + (2 - 1)^2 \times 1/4 \\
&= 1/4 + 1/4 = 1/2
\end{aligned}$$

Standard Deviation

- The standard deviation is then ______________

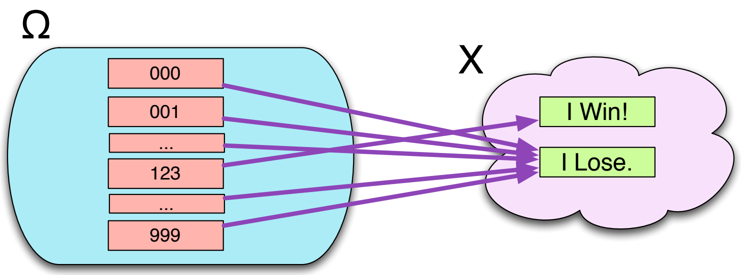
Named Distributions

- Any discrete set of values, along with associated probabilities, forms a discrete probability distribution
  - as long as ________
  - and ________
- However, some sets show up so often, they are given names.
- We will look at three of these:
  - The Bernoulli Distribution
  - The Discrete Uniform Distribution
  - The Binomial Distribution

The Bernoulli Distribution

- The simplest possible probability distribution is the **Bernoulli Distribution**, named after Jakob Bernoulli, who is also credited with discovering the constant e.
- Any random variable with only two possible outcomes has a Bernoulli distribution with "success" probability p (where we choose one of the outcomes to call a "success").
- Examples:
  - Flipping a single coin
  - Playing the lottery
  - Getting or not getting cancer
  - Winning or losing a baseball game
  - Graduating or not graduating from college
  - etc. etc.

Example: Playing the Lottery



x	Lose	Win
$P(X = x)$		

The Bernoulli Distribution

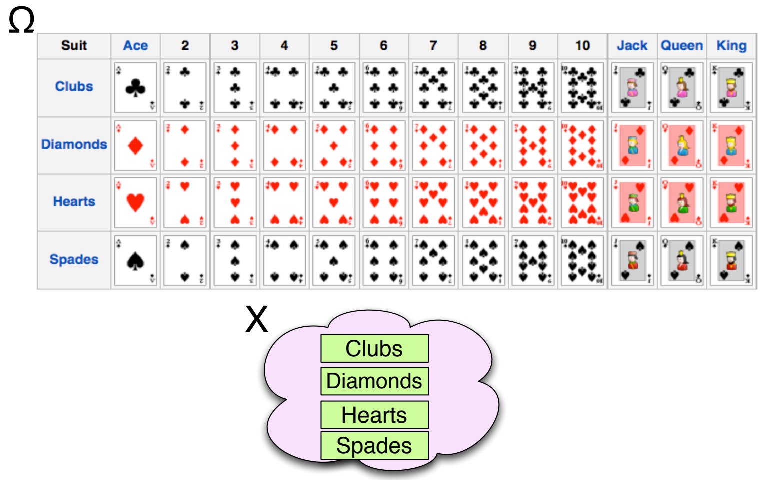
- If a "success" has probability p, then a failure has probability ________
- Typically, we label the success outcome as "1", and the failure outcome as "0".
- We can then compute a mean and variance.
- In terms of the **parameter** p, the mean of a Bernoulli distribution is ________
- The variance is ________

$$\begin{aligned}
\sigma^2 &= (0-p)^2(1-p) + (1-p)^2 p \\
&= p^2(1-p) + (1-2p+p^2)p \\
&= p^2 - p^3 + p - 2p^2 + p^3 \\
&= p - p^2 = p(1-p)
\end{aligned}$$

The Discrete Uniform Distribution

- Another common distribution is the **discrete uniform distribution**.
- Here, we have n possible outcomes (labeled 1 through n) which are all equally likely.
- In terms of the **parameter** n, $f(k) = ?$ for $k = 1, \ldots, n$
- What are some examples?
- What is $F(k)$ equal to?
- How would we find the mean and variance?

Example: Card Suit



The Binomial Distribution

- Many times we have some basic process with two outcomes (i.e., a Bernoulli process), which is repeated some number of times.
- We may be interested in the *number* of "successes".
- If both
    - (a) the success probability stays constant
    - (b) success events are mutually independent
- then the number of successes has a **Binomial Distribution** with **parameters** n (number of trials) and p (individual success probability)

The Binomial Distribution

- We've seen an example already. What was it?
- The number of heads seen in 2 tosses is Binomial with $n = ?$ and $p = ?$
- Q: How do we find the probability of seeing k heads in 2 tosses?
- A: Find all the individual sequences with k heads, and add their probabilities (since individual sequences are disjoint events).
- Q: What is the probability of any particular sequence?
- A: In two independent tosses, it's $P(\text{First Outcome}) \times P(\text{Second Outcome})$

The Binomial Distribution

- Q: In general, for a sequence of n **independent** trials, each with success probability p, how would we find the probability of a specific sequence of successes and failures?

$$\begin{aligned} P(\text{Sequence}) &= P(\text{First Outcome} \times P(\text{Second Outcome}) \\ &\quad \times \cdots \times P(\text{Last Outcome}) \\ &= p^{\#\text{ successes}} \times (1-p)^{\#\text{ failures}} \end{aligned}$$

- If there were k successes, how many failures were there?

The Binomial Distribution

- Therefore, every sequence of n trials with k successes has probability :

$$P(\text{Each sequence with } k \text{ successes}) = p^k \times (1-p)^{n-k}$$

- So, is this the probability of k successes in n (identical, independent) tries?
- **No!** This is the probability of *each sequence*! What else do we need to know?

Binomial Coefficients

- How many different sequences of length n with k successes are there?
- Equivalently: how many arrangements are there of k pegs in n holes?
- Consider, say, $n = 5$ and $k = 2$.
- The first peg can go in 5 different places...
- For each of those, the second peg can go in 4 places, for a total of 5×4.
- But now we've counted $\{1, 2\}$ and $\{2, 1\}$ separately, when really, both represent the same sequence: $(1, 1, 0, 0, 0)$.
- So, divide by 2, to get $(5 \times 4)/2$.

Binomial Coefficients

- How about $n = 5$ and $k = 3$?
- Now we have $5 \times 4 \times 3$...
- But $\{1, 2, 3\}$, $\{1, 3, 2\}$, etc. give us the same sequence. How many times are we counting each sequence now?
- How many ways are there to order 3 things?
- Answer: $3 \times 2 \times 1$

Binomial Coefficients

- So, in general, the number of subsets of size k you can draw from a big set of size n is:

$$\begin{aligned} \binom{n}{k} &= \frac{n \times (n-1) \times \cdots \times (n-k+1)}{k \times (k-1) \times (k-2) \times \cdots \times 2 \times 1} \\ &= \frac{(n)_k}{k!} \\ &= \frac{n!}{(n-k)!k!} \end{aligned}$$

Binomial Coefficients

(Binomial Coefficients)

The **binomial coefficients** give the number of sequences of length n that contain k successes:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

This is read "n choose k"

Binomial PMF

- Combining this with the probability of each of these sequences gives us the overall probability of k successes.

(Binomial Probability Mass Function)

For a binomial random variable X with parameters n and p, the probability of k successes is

$$f_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{(n-k)}$$

Binomial CDF

- Unfortunately, there's no closed form expression for the binomial CDF; we just have to take a sum:

(Binomial CDF)

For a binomial random variable X with parameters n and p, the probability of $\leq k$ successes is

$$F_X(k) = P(X \leq k) = \sum_{j=0}^{k} f_x(k) = \sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{(n-j)}$$

Binomial Mean and Variance

- The mean and variance of the binomial distribution follow from two general facts about sums of random variables (remember, the Binomial is the sum of n identical and independent Bernoullis)

Sums of Random Variables

1. If X_1, X_2, $\ldots X_n$ are n random variables, and we define Y to be their sum, then

$$E(Y) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$$

2. If in addition X_1 through X_n are **independent**, then

$$Var(Y) = Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$$

Binomial Mean and Variance

- Remember that the Bernoulli mean is p, and the Bernoulli variance is $p(1-p)$.
- The Binomial mean is then ____________
- The Binomial variance is ____________
- The Binomial standard deviation is ____________

Applying the Binomial Distribution

- 40% of the population support a particular proposition. If the entire population voted, the proposition would be defeated. What is the probability that a simple random sample of 3 people would vote to pass the proposition?