ISTA 116: Statistical Foundations for the Information Age

Discrete Random Variables

31 October 2011

Outline

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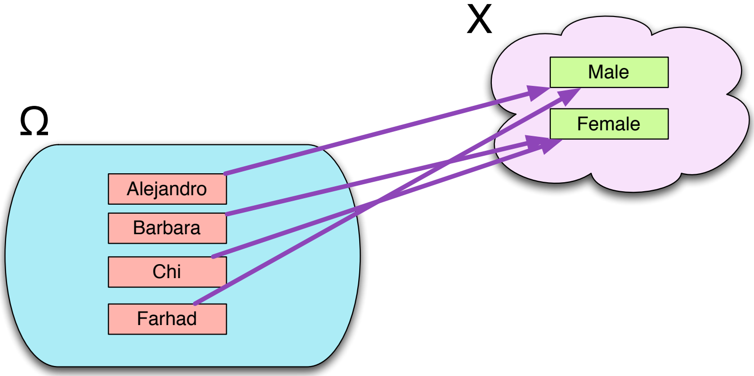
- X: People $\mapsto$ Their Sex
- Y: Sequences of coin flips $\mapsto$ Number of Heads

Examples of Random Variables



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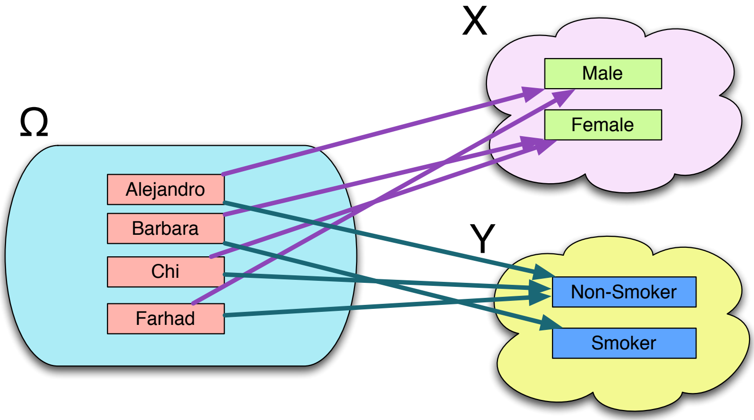

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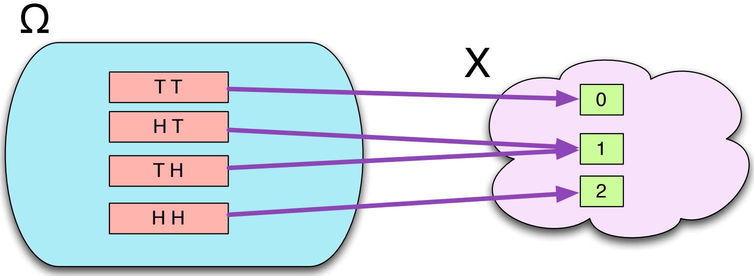

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- **Discrete** random variables take on discrete values (e.g., categories or integers)
- **Continuous** random variables take on continuous values (e.g., real numbers).

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Discrete Distributions

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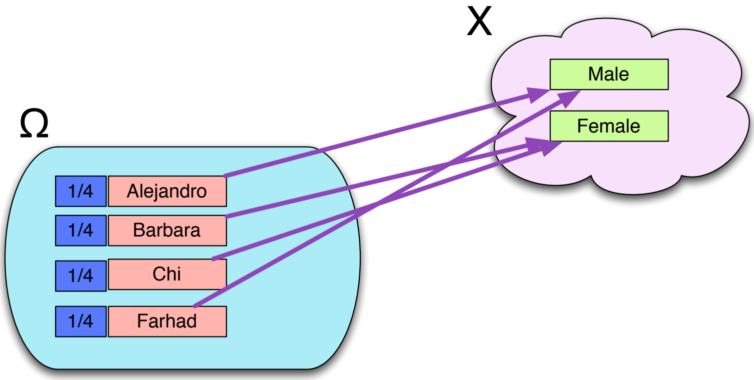
- These probabilities are determined by the probabilities on the sample space itself
- If the sample space is a finite population and we make a simple random draw, then the probability of a value is the proportion of individual outcomes assigned to it.

The Distribution of a Random Variable



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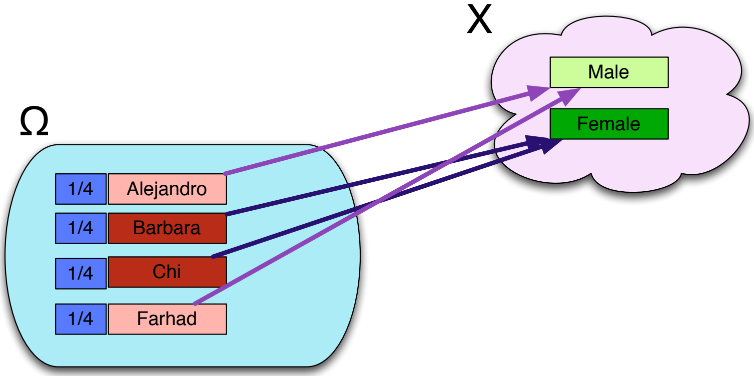

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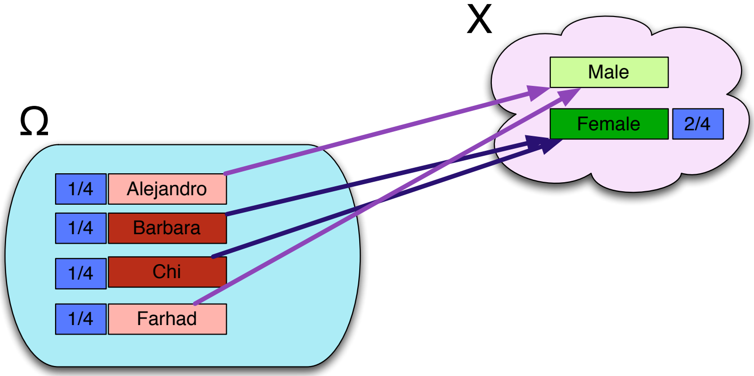

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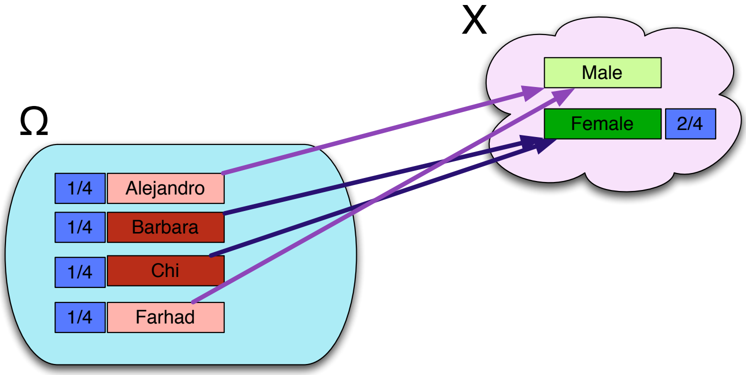
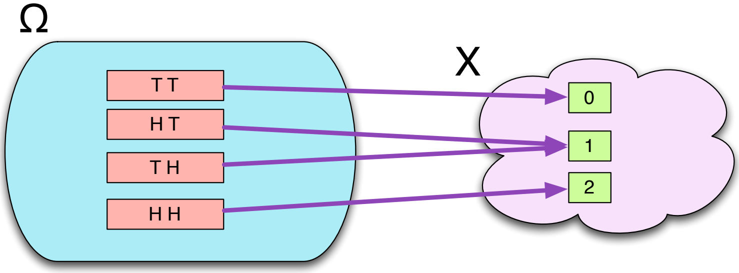


x	Male	Female
$P(X = x)$	$1/2$	$1/2$

The Distribution of a Random Variable



x	0	1	2
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- Note that each value of a discrete random variable corresponds to an event in the original sample space.
- The probability associated with the value is the probability associated with the event.
- Moreover, every outcome in the sample space is associated with exactly one value of the random variable.
- Therefore, the values of a discrete random variable give us a set of **disjoint** events whose **union** is the entire sample space.

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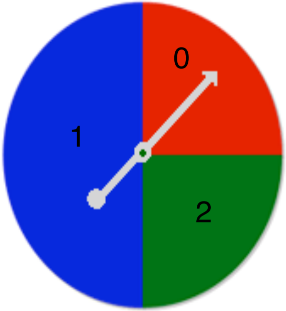
1. For every x in the range of X, $P(X = x) \geq 0$.
2. $$\sum_{x \in \text{Range}(X)} P(X = x) = 1$$

Outline

Visualizing Discrete Distributions



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- Can imagine a spinner mechanism that lands on a slice according to its probability.

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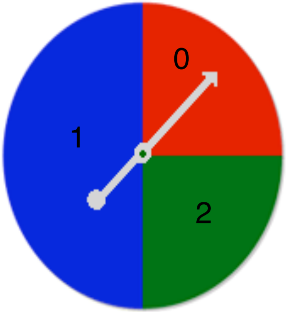

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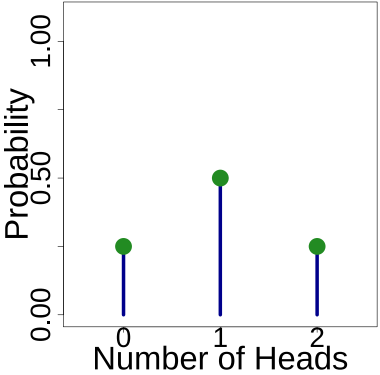
- Can imagine a spinner mechanism that lands on a slice according to its probability.
- But, like pie charts, this is limited in its ability to convey information.

The Spike Plot



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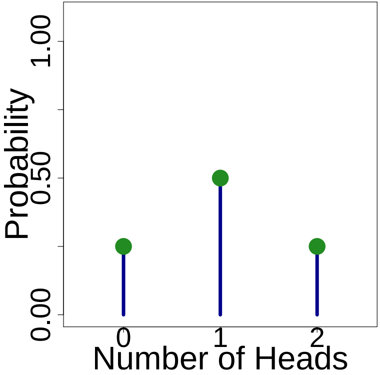

- An alternative is the **spike plot**
- Like a bar plot, but with probabilities, instead of frequencies or proportions, on the y-axis.

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- One way to use such a table is to start with a value and *look up* its probability.
- This process is characterized by the **probability mass function**, which takes a value and returns its probability.

Probability Mass Function

(Definition: The Probability Mass Function)

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- However, we will see examples later of PMFs that have algebraic expressions.

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A random variable, X, can be characterized by its **cumulative distribution function**, F_X, which takes values and returns *cumulative* probabilities:

$$F_X(x) = P(X \leq x)$$

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- Most of the measures we've seen can be computed; but the most common are the **mean** and **variance**.

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Expected Value

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- As with a sample mean, it represents an average over the possible values; but it is **weighted** by the probabilities.

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Variance of a Discrete Random Variable

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$$\sigma_X^2 = E((X - \mu_X)^2) = \sum_{x \in \text{Range}(X)} (x - \mu_X)^2 P(X = x)$$

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Standard Deviation

- The standard deviation is then ____________________

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- However, some sets show up so often, they are given names.
- We will look at three of these:
  - The Bernoulli Distribution
  - The Discrete Uniform Distribution
  - The Binomial Distribution

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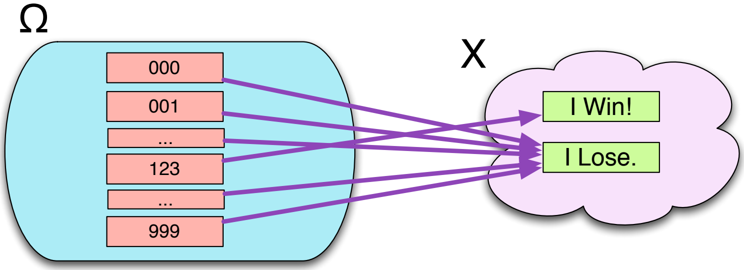
- Examples:
  - Flipping a single coin
  - Playing the lottery
  - Getting or not getting cancer
  - Winning or losing a baseball game
  - Graduating or not graduating from college
  - etc. etc.

Example: Playing the Lottery



x	Lose	Win
$P(X = x)$		

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&= p - p^2 = p(1-p)
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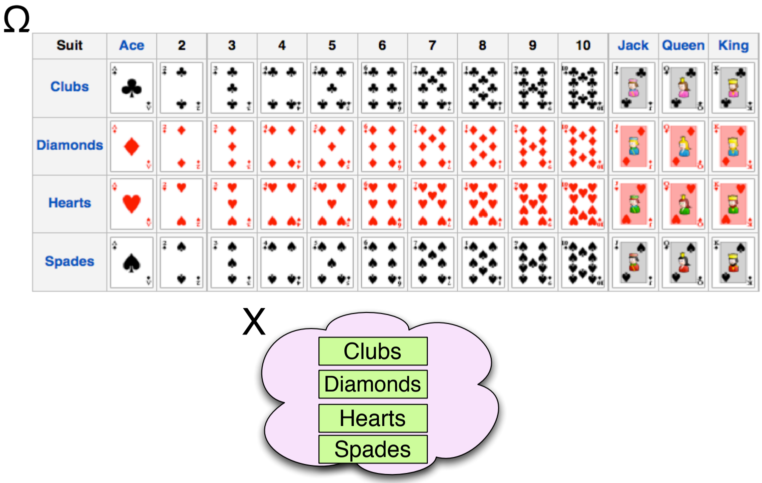
- What is $F(k)$ equal to?
- How would we find the mean and variance?

Example: Card Suit



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- We may be interested in the *number* of "successes".
- If both
  - (a) the success probability stays constant
  - (b) success events are mutually independent
- then the number of successes has a **Binomial Distribution** with **parameters** n (number of trials) and p (individual success probability)

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- A: In two independent tosses, it's $P(\text{First Outcome}) \times P(\text{Second Outcome})$

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- If there were k successes, how many failures were there?

The Binomial Distribution

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- So, is this the probability of k successes in n (identical, independent) tries?
- **No!** This is the probability of *each sequence*! What else do we need to know?