Matched Filtering

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LSSTC DATA SCIENCE FELLOWSHIP PROGRAM

Goals for this lesson

- Matching filtering
  - Detection with matched filtering
  - Template banks with matched filtering
  - Matched filtering for GW detection

How do we detect a weak signal in noisy data?

Matched Filtering

Consider noisy data
containing a signal

$$s(x) = n(x) + h(x)$$

$n(x)$: Instrumental/environment/other-type noise

$h(x)$: Signal with relatively well-understood morphology
eg: binary black hole GWs, pulsar pulse profiles, eco-planet transits

In general $|n(x)| > |h(x)|$

x : some data coordinate. Time, frequency, photo detector pixel position

What is the optimal way
to dig out the signal?

Stationary noise

- A random process with time-independent **mean** and **autocorrelation function** is stationary in the weak sense

$$\mu_n = E\left[n(t)\right] \qquad\qquad R_n(\tau) = E\left[(n(t)\,n(t+\tau)\right]$$

- Weiner-Khintchin Relation: We define the (one-sided) power spectral density (PSD) of a stationary process as the Fourier Transform of its autocorrelation function.

$$\langle \tilde{n}(f)\tilde{n}^*(f)\rangle = \frac{1}{2}S_n(f) = \int_{-\infty}^{\infty} d\tau\, R_n(\tau)\, \exp\left(2\pi i f \tau\right)$$

- Cross terms vanish for stationary noise† i.e. $\langle \tilde{n}(f)\tilde{n}^*(f')\rangle = \frac{1}{2}S_n(f)\,\delta(f-f')$

- Alternate definition of stationarity : The PSD* does not change with time.

† , * Ask me about the fine prints!

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Stationary Gaussian Noise

- The statistical properties of the noise in time domain are entirely characterized by mean and autocorrelation function

(or)

- The distribution of the noise is a complex Gaussian in frequency domain with the PSD as the variance.

- We also assume the mean to be zero.

- Under the stationary Gaussian approximation we model it as

$$n(f) \sim \frac{2}{\pi T S(f)} \exp\left(-\frac{2}{T}\frac{|n(f)|}{\pi S(f)}\right)$$

Matched Filtering

Consider: $\hat{s} = \int dx\, s(x) K(x)$

$s(x) = n(x) + h(x)$

What filter function $K(x)$ maximizes the S/N ratio of $\hat{s}$?

$$S = \langle \hat{s} \rangle = \int dx\, \langle s(x) \rangle\, K(x)$$

$$= \int dx\, h(x)\, K(x) + \int dx\, \cancel{\langle n(x) \rangle}\, K(x)$$

$$N^2 = \left(\langle \hat{s}^2 \rangle - \cancel{\langle \hat{s} \rangle^2} \right) \Big|_{h=0}$$

$$N^2 = \int dx\, dx'\, \underbrace{\langle n(x) n(x') \rangle}\, K(x) K(x')$$

Autocorrelation Function: $R_n(x - x')$

Matched Filtering as a dot product

$$\hat{s} = \int dx\, s(x) K(x)$$

What filter function maximizes the S/N ratio of $\hat{s}$?

$$\frac{S}{N} = \frac{\int dx\, h(x)K(x)}{\sqrt{\int dx\, dx' \,\langle n(x) n(x')\rangle\, K(t)K(t')}}$$

Take Fourier transforms

$$\frac{S}{N} = \frac{\int df\, h(f)K^*(f)}{\sqrt{\int df\, \frac{1}{2}S(f)\rangle\, |K(f)|^2}}$$

$$\langle \tilde{n}(f)\tilde{n}^*(f')\rangle = \frac{1}{2}S_n(f)\,\delta(f-f')$$

Matched Filtering as a dot product

$$\frac{S}{N} = \frac{\int df\, h(f) K^*(f)}{\sqrt{\int dx\, \frac{1}{2} S(f) \rangle\, |K(f)|^2}}$$

Define $u(f) \propto \frac{1}{2} S(f)\, K(f)$

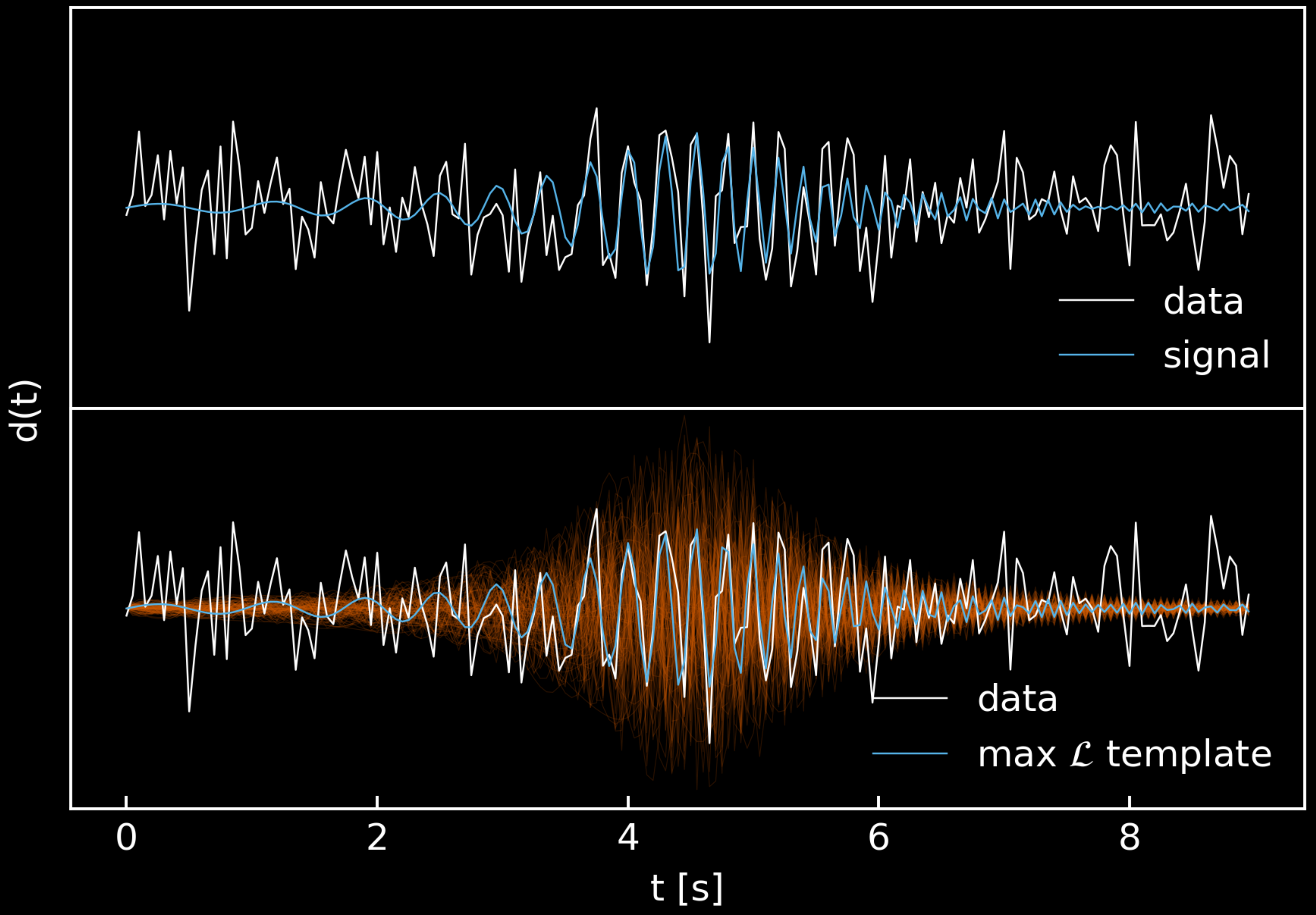
We can define the S/N as a inner product: $\dfrac{S}{N} = \dfrac{(u\,|\,h)}{(u\,|\,h)^{1/2}}$

The optimal filter that maximizes the S/N

$$K_{\rm opt}(f) \propto \frac{h(f)}{S(f)}$$

$$\left.\frac{S}{N}\right|_{opt} = (h\,|\,h)^{1/2} = 4 \int df \frac{|h(f)|^2}{S(f)}$$

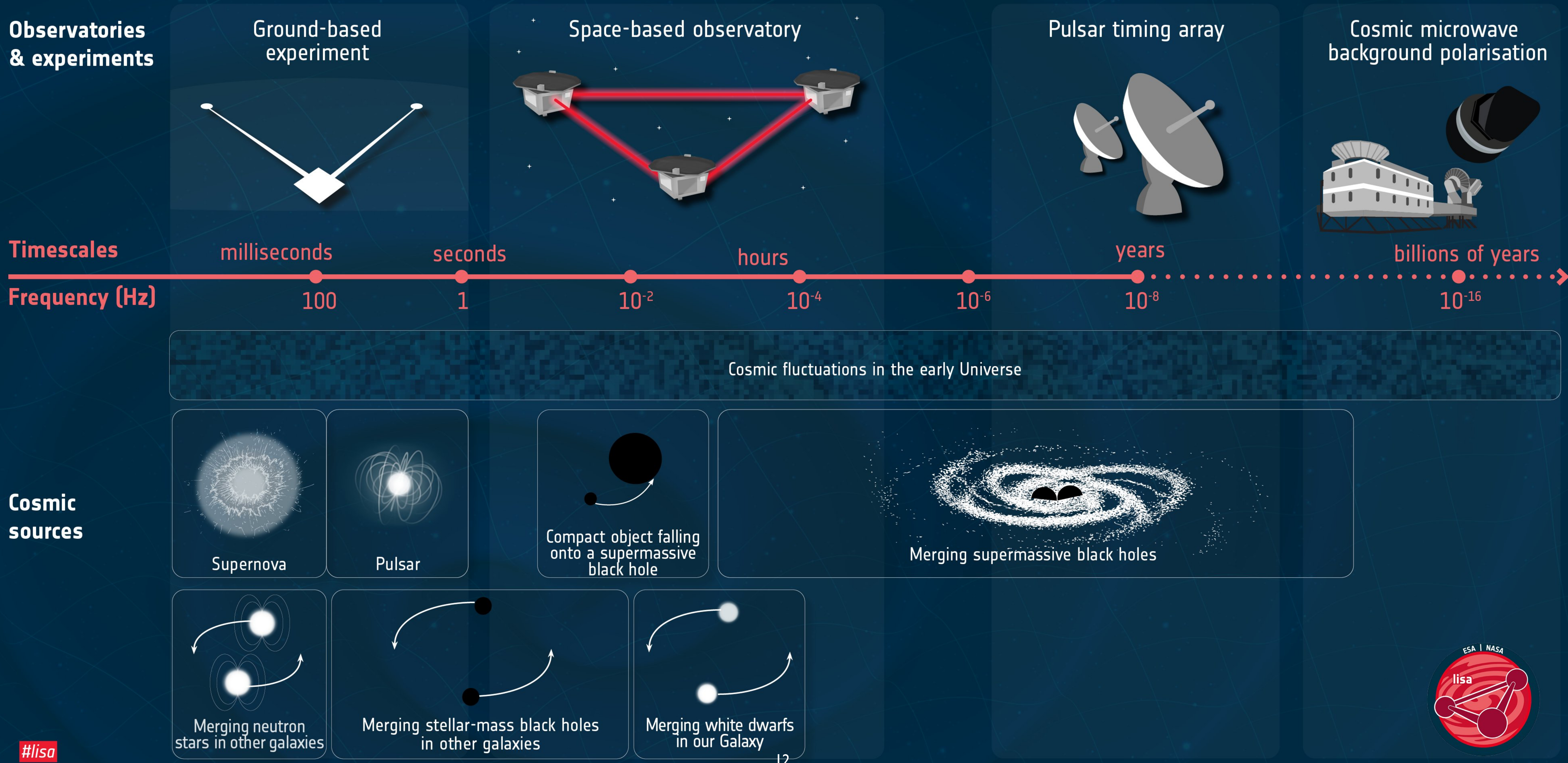
Matched Filtering detection



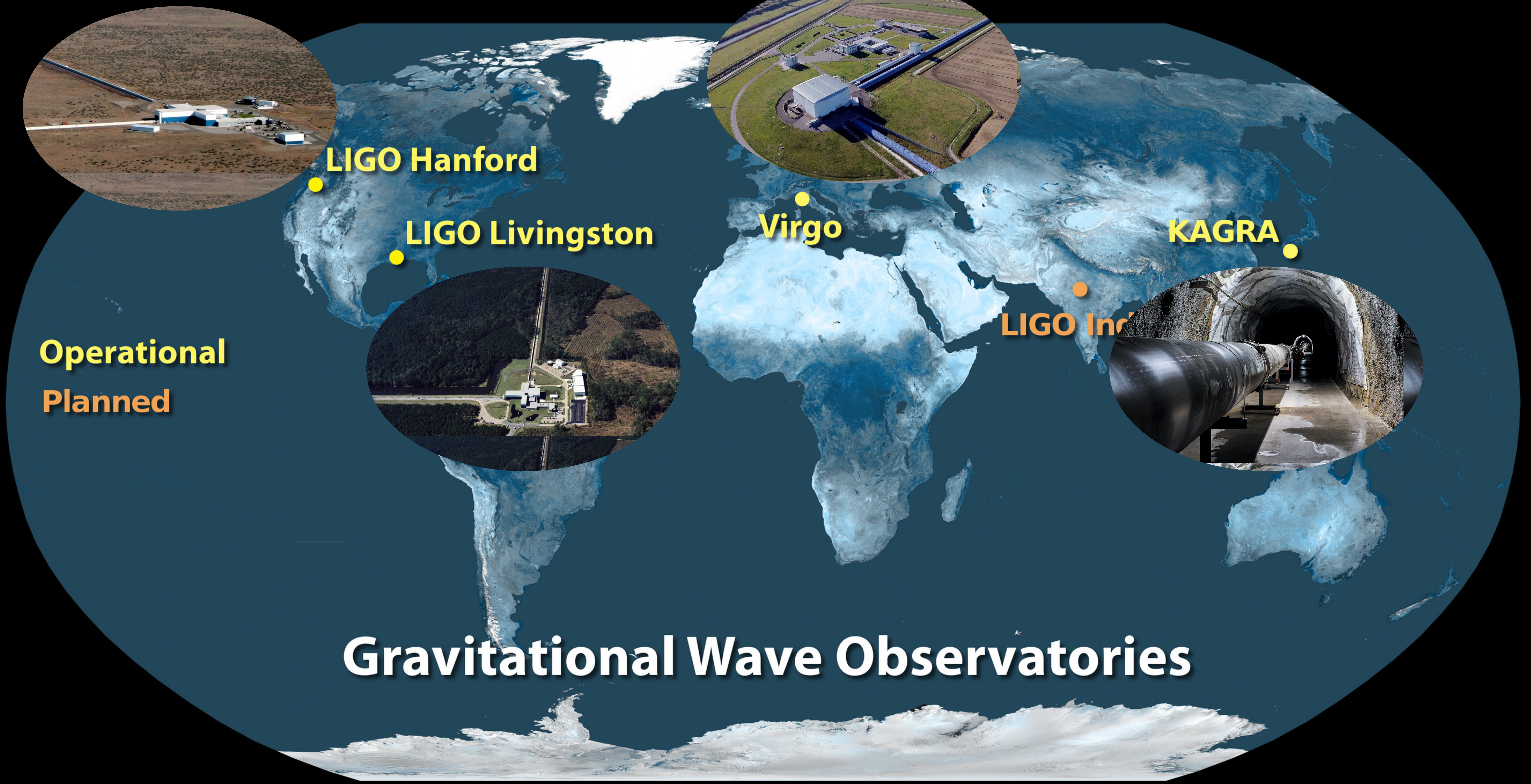
- Unknown signal of a known type
- $n(t) \sim h(t)$
- Detect with matched filtering
  - The S/N is optimized by the true waveform.
  - The true waveform is the one that maximizes the S/N

Gravitational-waves : A Crash Course

THE SPECTRUM OF GRAVITATIONAL WAVES



#lisa



Time domain data!

How do gravitational waves couple?

Two Polarizations (+ , ×)

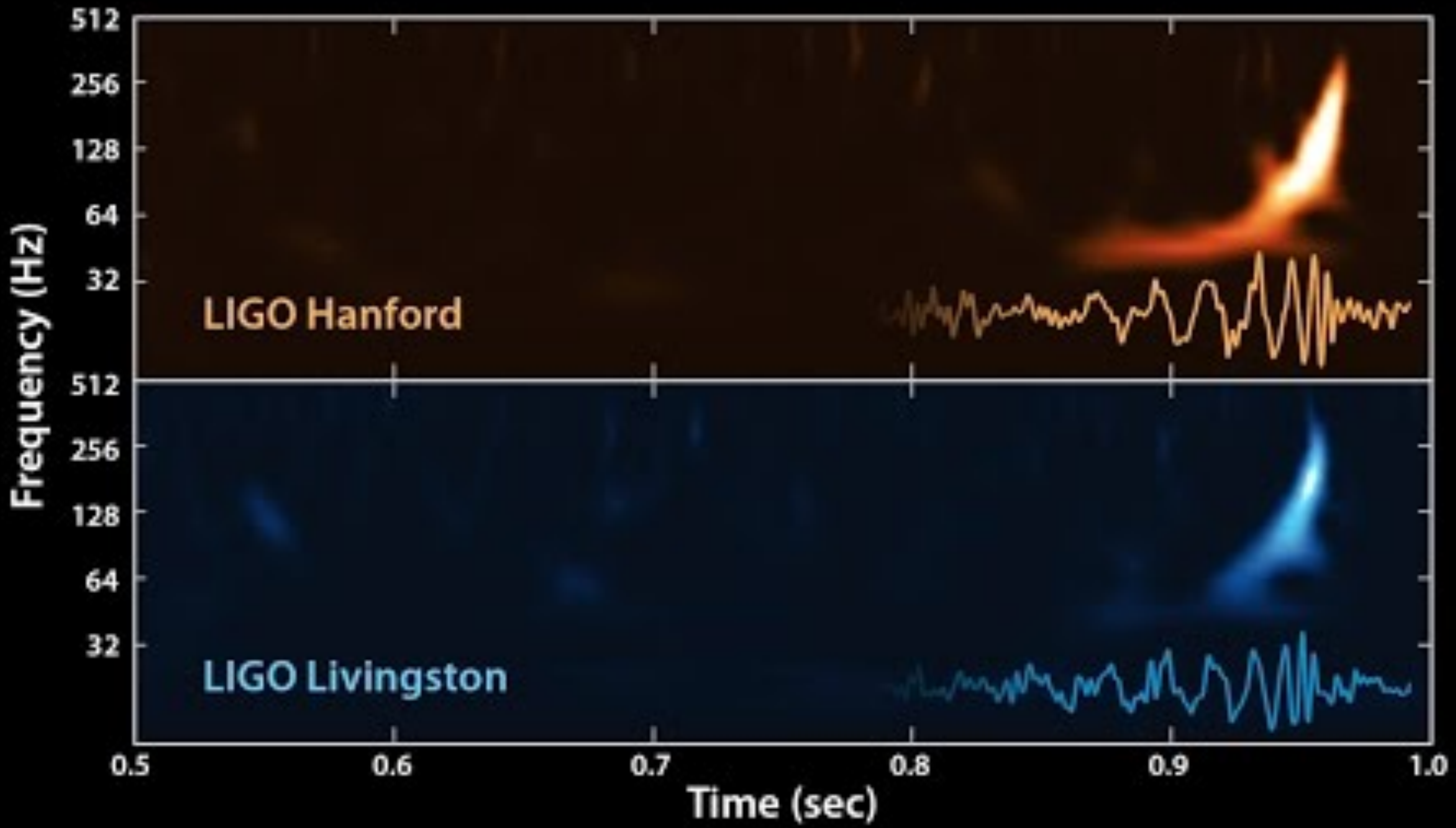
$$h(t) = F_+ h_+(t\,|\,\theta) + F_\times h_\times(t\,|\,\theta)$$

Antenna/Detector Patterns
Response functions

Depends on (ra, dec) & t

Also depends on f and arm length for LISA

GW150914 : The first detection



What do GW detectors output?

Calibrated
data

$$h(t) = F_+ h_+(\theta) + F_\times h_\times(\theta)$$

$$s(t) \quad = \quad n(t) \quad + \quad h(t)$$

$$\langle f(n) \rangle = E_T \left[f(n(t)) \right]$$

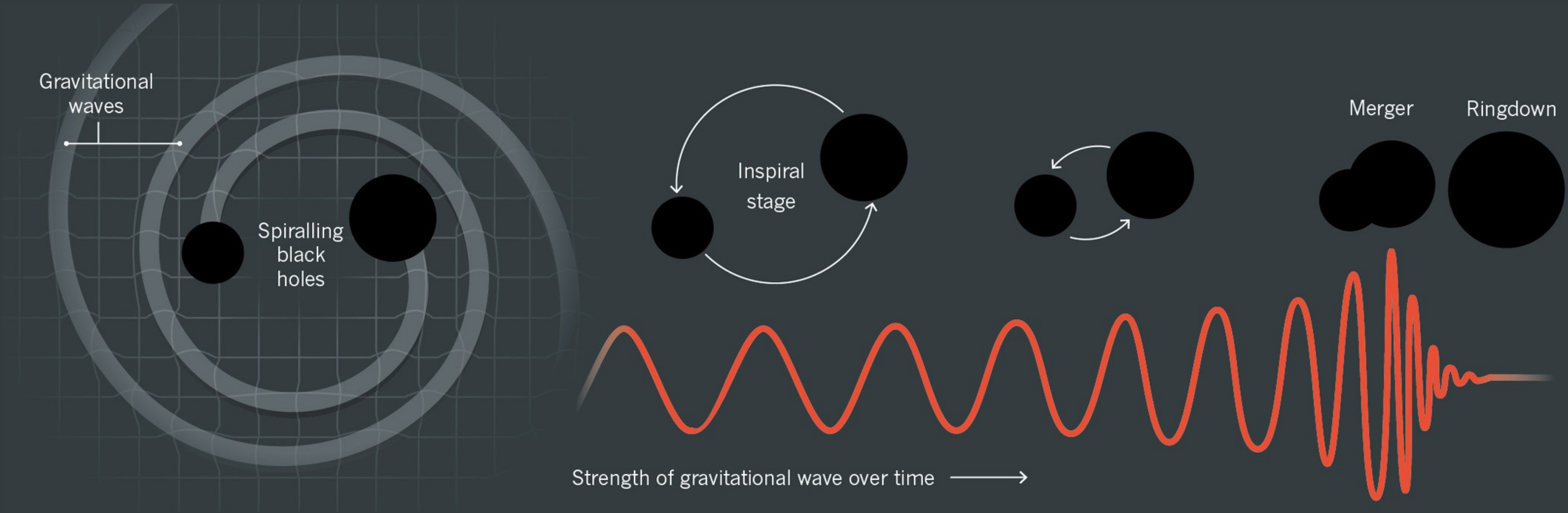
<u>Instrumental noise</u>

Ergodic

Quasi-stationary with $\mu = 0$


[1253923218-1254009618, state: Locked]
GEO-LIGO-Virgo gravitational-wave strain
GW amplitude spectral density [strain/√Hz]
Frequency [Hz]
GEO600
Hanford
Livingston
Virgo
LIGO-Virgo ASD

Gravitational-wave waveforms

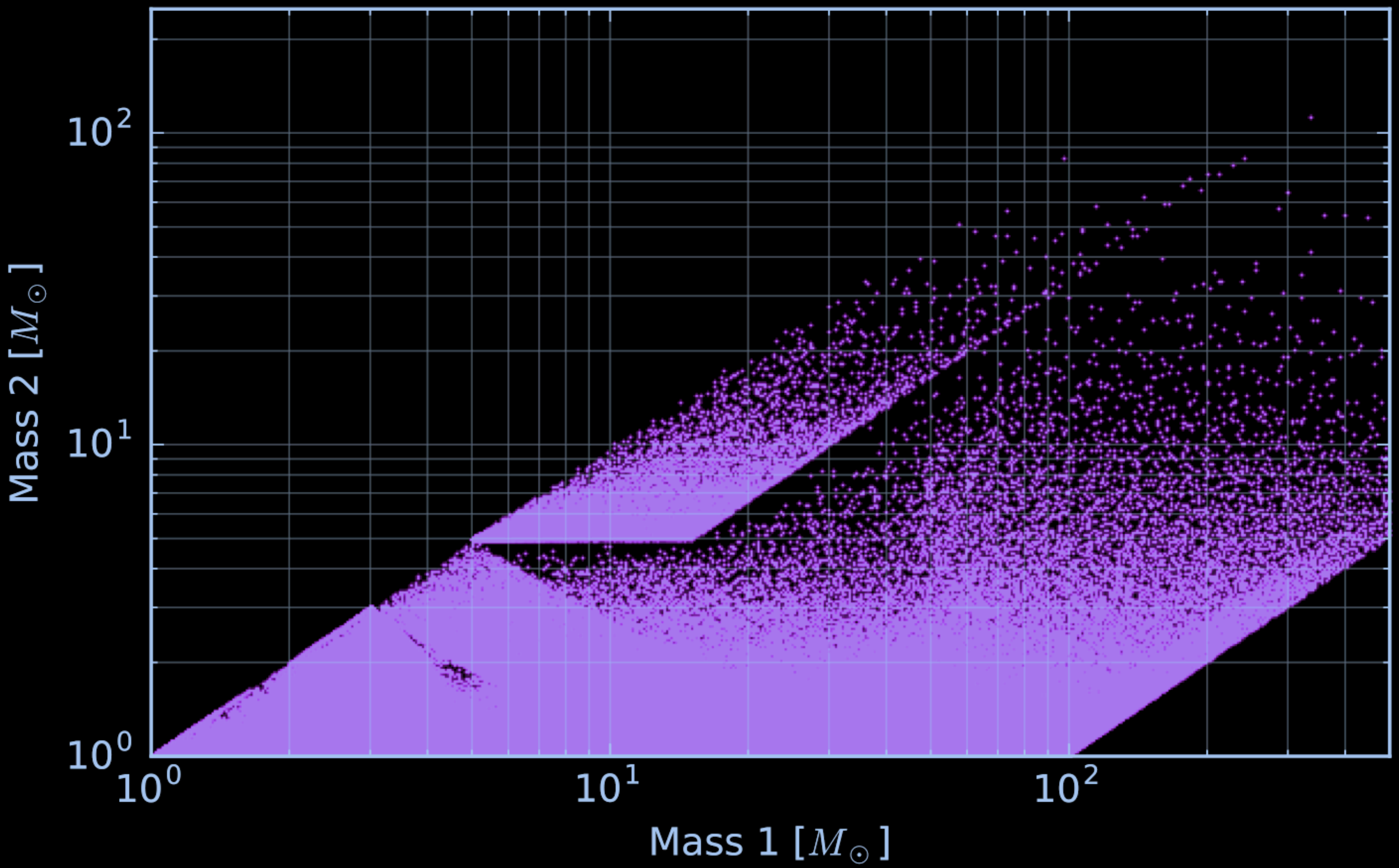


General Relativity is solved in a variety of ways to develop waveforms

$$m_1, m_2, \overrightarrow{\chi_1}, \overrightarrow{\chi_2}, \iota, d, t_{\rm inp}, \phi_0$$

How is Matched Filtering performed in gravitational-wave analyses?

Matched Filter Template Banks

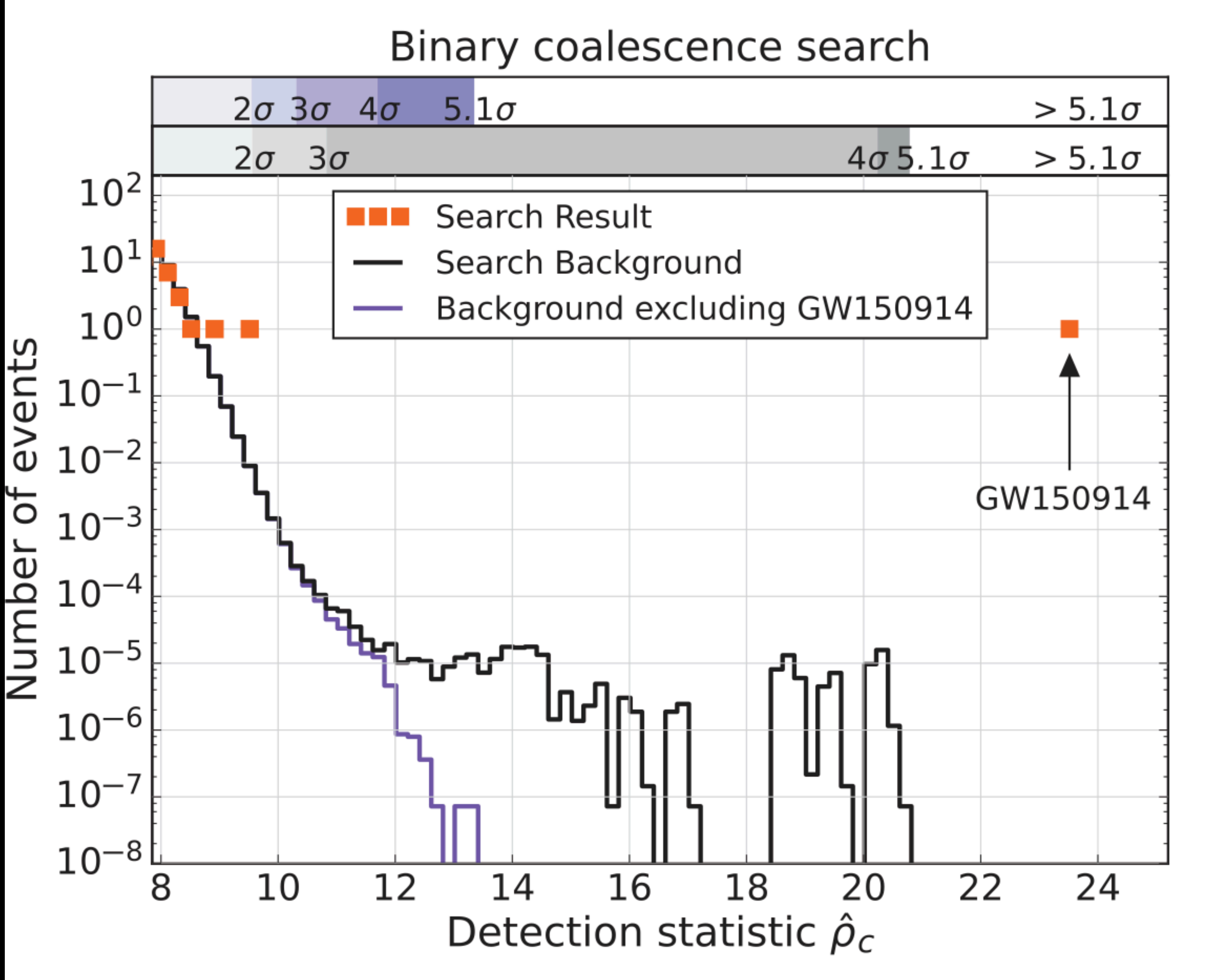


Template banks use by all recent PyCBC searches

Dal Canton+ 1705.01845

Identifying triggers

- Calculate the $\left.\frac{S}{N}\right|_{opt}$ as a function of time
- Local maxima are the triggers of interest.
- Record triggers above some threshold
- Special detection statistics and algorithms to handle non-stationary & non-Gaussian noise
- Compare against background



Background for GW150914