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② considering the function
$f(x) = 1 - \cos x$

ⓐ relative condition number at $x=0$

i.e. $\frac{\|x\|}{\|f(x)\|} \sup_{\delta x} \frac{\|\delta y\|}{\|\delta x\|}$ where $y = f(x)$

Given $f(x) = 1 - \cos x$
$f : \mathbb{R} \to \mathbb{R}$

hence $K = \frac{|x|}{|f(x)|} \cdot \sup_{\delta x} \left|\frac{\delta f}{\delta x}\right|$

$K = \frac{|x|}{|f(x)|} \cdot |f'(x)|$

$f'(x) = \frac{d}{dx}(1 - \cos x) = -(-\sin x) = \sin x$

At $x = 0$, $K = \lim_{x \to 0} \frac{|x|}{|1 - \cos x|} \cdot |\sin x|$

$K = \lim_{x \to 0} \frac{|x \sin x|}{|1 - \cos x|}$

For both $x \to 0^-$ and $x \to 0^+$

$\frac{(x\sin x)}{(1-\cos x)} = \frac{x \sin x}{1 - \cos x}$ as it converges to +ve

so, $K = \lim_{x \to 0} \frac{x \sin x}{1 - \cos x}$

Applying L Hospital rule

$\lim_{x \to 0} \frac{|x \sin x|}{|1 - \cos x|} = \lim_{x \to 0} \frac{x \, 2 \sin x/2 \cos x/2}{2 \sin^2 x/2}$

$\lim_{x \to 0} \left(\lim_{x \to 0} \frac{x}{\sin x/2}\right)\left(\lim_{x \to 0} \cos \frac{x}{2}\right)$

$2 \left(\lim_{x/2 \to 0} \frac{x/2}{\sin x/2}\right)\left(\lim_{x \to 0} \cos x/2\right)$

$= 2 \times 1 \times 1$

So, answer is $\underline{\underline{2}}$

(5) function $f: \mathbb{C}^m \to \mathbb{R}$

$$f(x) = \frac{1}{2}\left(\min\{|x_1|, \dots |x_m|\} + \max\{|x_1|, \dots |x_m|\}\right) \to (1)$$

(a) To show:-

f is a norm when $m=2$ ie satisfies all three norm conditions. for all vectors x and y & for all

Scalars $\alpha \in \mathbb{C}$

1) $\|x\| \geq 0$ and $\|x\| = 0$ only if $x = 0$

2) $\|x+y\| \leq \|x\| + \|y\|$

3) $\|\alpha x\| = |\alpha| \|x\|$

from equation (1) $\min(a,b) + \max(a,b) = a+b$

hence $f(x) = \frac{1}{2}[|x_1| + |x_2|] \longrightarrow (2)$

As we stated above, to be a norm it should satisfy 3 conditions

i) $\|x\| = f(x) > 0$ if $x \neq 0$

$f(x) = \frac{1}{2}[|x_1| + |x_2|] > 0$

so, it is true and condition satisfies

ii) $\|x+y\| \leq \|x\| + \|y\|$. i.e. $f(x) + f(y) \geq f(x+y)$

$f(x+y) = \frac{1}{2}[|x_1+y_1| + |x_2+y_2|]$

$\leq \frac{1}{2}[|x_1| + |y_1| + |x_2| + |y_2|]$

$\leq \frac{1}{2}[|x_1| + |x_2|] + \frac{1}{2}[|y_1| + |y_2|]$

$f(x+y) \leq f(x) + f(y)$ is true, condition satisfied

iii) $\|\alpha x\| = f(\alpha x) = |\alpha| \|x\| = |\alpha| f(x)$

$f(\alpha x) = \frac{1}{2}[|\alpha x_1| + |\alpha x_2|] = \frac{|\alpha|}{2}[|x_1| + |x_2|]$

$= |\alpha| . f(x)$

$f(x)$ satisfied the condition

So, f is norm for $m = 2$

To show :-

5(b) $m \geq 3$, f is not a norm

to show that f is not a norm, we have following generic example for each m.

$x = (0, c, c, c \ldots\ldots c) \geq 0$

$y = (c, 0, c, c, c \ldots\ldots c) \geq 0$

$f(x+y) = \frac{1}{2}[\min\{|x_1+y_1|, |x_2+y_2| \ldots\} + \max\{|x_1+y_1|, |x_2+y_2| \ldots\}]$

i.e. $x+y = (c, c, 2c, 2c \ldots\ldots 2c) \geq 0$ i.e. $x_1, x_2 = c$

$x_3, x_4 \ldots x_m = 2c$

$\therefore \min\{|x_1+y_1|, |x_2+y_2| \ldots\} = c$

$\max\{|x_1+y_1|, |x_2+y_2| \ldots\} = 2c$

$f(x+y) = \frac{1}{2}[c+2c] = \frac{3c}{2}$

$f(x) = \frac{1}{2}[\min\{|x_1|, |x_2| \ldots\ldots |x_m|\} + \max\{|x_1|, |x_2| \ldots\ldots\}]$

$= \frac{1}{2}[0+c] = c/2$

Since, $\min\{|x_1|, |x_2| \ldots\ldots |x_m|\} = 0$

$\max\{|x_1|, |x_2| \ldots\ldots |x_m|\} = c$

Similarly $f(y) = c/2$

It is clear that $f(x+y) > f(x) + f(y)$

$\therefore$ f is not a norm for $m \geq 3$

③ Using lagrange polynomial (interpolation)

ⓐ $P(z) = az^2+bz+c = P(0)*\frac{(z-2)(z-1)}{(2-0)(1-0)} + P(1)*\frac{(z-0)(z-2)}{(1-0)(1-2)}$

$+ P(2)*\frac{(z-0)(z-1)}{(2-0)(2-1)} \longrightarrow eq ①$

so, $P(z) = x_1 e_1(z) + x_2 e_2(z) + x_3 e_3(z) \longrightarrow eq ②$

from eq ① & eq ②

$e_1(z) = \frac{(z-2)(z-1)}{2}, \; e_2(z) = \frac{z(z-2)}{-1}, \; e_3 = \frac{z(z-1)}{2}$

$P(0) = a(0)^2 + b(0) + c = c$

$P(1) = a(1)^2 + b(1) + c = a+b+c$

$p(2) = a(2)^2 + b(2) + c = 4a+2b+c$

now, we have

$e_1(z) = \frac{(z-2)(z-1)}{2}, \; e_2(z) = -z(z-2) \Rightarrow z(2-z)$

$e_3(z) = \frac{(z)(z-1)}{2}$

We can see that e_1, e_2, e_3 are linearly independent and they span quadratic polynomials therefore e_1, e_2, e_3 forms a basis.

3 (b) let $P_1(x) = a_n x^n + a_{n-1} x^{n-1} + \dots\dots + a_0$

$P_2(x) = b_m x^m + b_{m-1} x^{m-1} + \dots\dots + b_0$

$$\frac{\partial (P_1(x))}{\partial x} = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + a_1$$

$$\frac{\partial (P_2(x))}{\partial x} = m b_n x^{m-1} + (m-1) b_{m-1} x^{m-2} + \dots b_2$$

$$\frac{\partial (P_1(x) + P_2(x))}{\partial x} = \_\_\_\_ + \_\_\_\_ \dots$$

now, Let $c \in \mathbb{F}$.

Then

$$\frac{\partial (c P_1(n))}{\partial x} = \partial/\partial x \,(c a_n x^n + c a_{n-1} x^{n-1} + \dots c a_1 x$$

$$= c n a_n x^{n-1} + c(n-1) a_{n-1} x^{n-2} + \dots c a_1$$

$$= c (n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots a_1)$$

$$= \left(\frac{\partial (P_1(x))}{\partial x} \right.$$

therefore from the above differentiation is a linear transformation of polynomials

we have $e_1(z) = \frac{z^2}{2} - \frac{3^2}{2} + 1$

$e_2(z) = 2z - z^2$

$e_3(z) = \frac{z^2}{2} - z/2$

So, $e_1(z) + e_2(z) + e_3(z) = 1 \longrightarrow$ eq (1)

$e_3(z) - e_1(z) = z - 1$

$\Rightarrow 2e_3 + e_2 = z \longrightarrow$ eq (2)

$$\frac{\partial(e_1(z))}{\partial(z)} = z - \frac{3}{2} \longrightarrow (3')$$

$$= (2e_3 + e_2) - \frac{3}{2}(e_1 + e_2 + e_3)$$

$$= -\frac{3}{2} e_1(z) - \frac{1}{2} e_2 + \frac{1}{2} e_3$$

Similarly from eq (3)

$$\frac{\partial(e_2(z))}{\partial(z)} = 2e_1(z) - 2e_3(z)$$

$$\frac{\partial(e_3(z))}{\partial(z)} = -\frac{1}{2} e_1(z) + \frac{1}{2} e_2(z) + \frac{3}{2} e_3(z)$$

matrix of D that represents it in this basis is

$$D = \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -2 & \frac{3}{2} \end{bmatrix}$$

3 (C) By rank nullity theorem, for a transformation

$D : \mathbb{P}_n(R) \longrightarrow \mathbb{P}_{(n-1)}(F)$

$D : A \rightarrow B$

$\text{Rank}(D) + \text{Nullity}(D) = \dim(A)$

since all constant in $\mathbb{P}_n(F)$ goes to 0, we can say that nullity $(D) = 1$