COL100: Introduction to Computer Science

3.1: Recursion and induction

Problem: Given a nonzero real $x \in \mathbb{R}$ and a natural number $n \in \mathbb{N}$, compute x^n.

$$power(x, n) = \begin{cases} 1 & \text{if } n = 0, \\ x \cdot power(x, n-1) & \text{otherwise.} \end{cases}$$

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fun power(x, n) =
  if n = 0
  then 1.0
  else x * power(x, n - 1)
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$$\begin{aligned} & power(0.2, 3) \\ &= 0.2 \cdot power(0.2, 2) \\ &= 0.2 \cdot (0.2 \cdot power(0.2, 1)) \\ &= 0.2 \cdot (0.2 \cdot (0.2 \cdot power(0.2, 0))) \\ &= 0.2 \cdot (0.2 \cdot (0.2 \cdot 1)) \\ &= 0.2 \cdot (0.2 \cdot 0.2) \\ &= 0.2 \cdot 0.04 \\ &= 0.008 \end{aligned}$$

$$power(x, n) = \begin{cases} 1 & \text{if } n = 0, \\ x \cdot power(x, n-1) & \text{otherwise.} \end{cases}$$

1. Does this algorithm always terminate?
   - Yes, *n* decreases by 1 on each recursive step.
2. Does it give the correct result for all *x* and *n*?
   - …

$$
\mathit{fastPower}(x, n) = \begin{cases} 1 & \text{if } n = 0, \\ \mathit{fastPower}(x^2, \lfloor n/2 \rfloor) & \text{if } n \text{ is even}, \\ x \cdot \mathit{fastPower}(x^2, \lfloor n/2 \rfloor) & \text{if } n \text{ is odd}. \end{cases}
$$

What about this version? Does *it* always give the correct result?

How can we prove that?

The principle of mathematical induction

How to prove a statement about infinitely many numbers?

e.g.

"Prove that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$."

Write an infinitely long proof? Prove that $0^3 + 2 \cdot 0$ is divisible by 3, and that $1^3 + 2 \cdot 1$ is divisible by 3, and $2^3 + 2 \cdot 2$ is divisible by 3, and…

The principle of mathematical induction

Version 1

A property P holds for all natural numbers, provided

1. (**base case:**) P holds for 0, and
2. (**induction step:**) if P holds for an arbitrary $n \in \mathbb{N}$, then it also holds for $n + 1$.

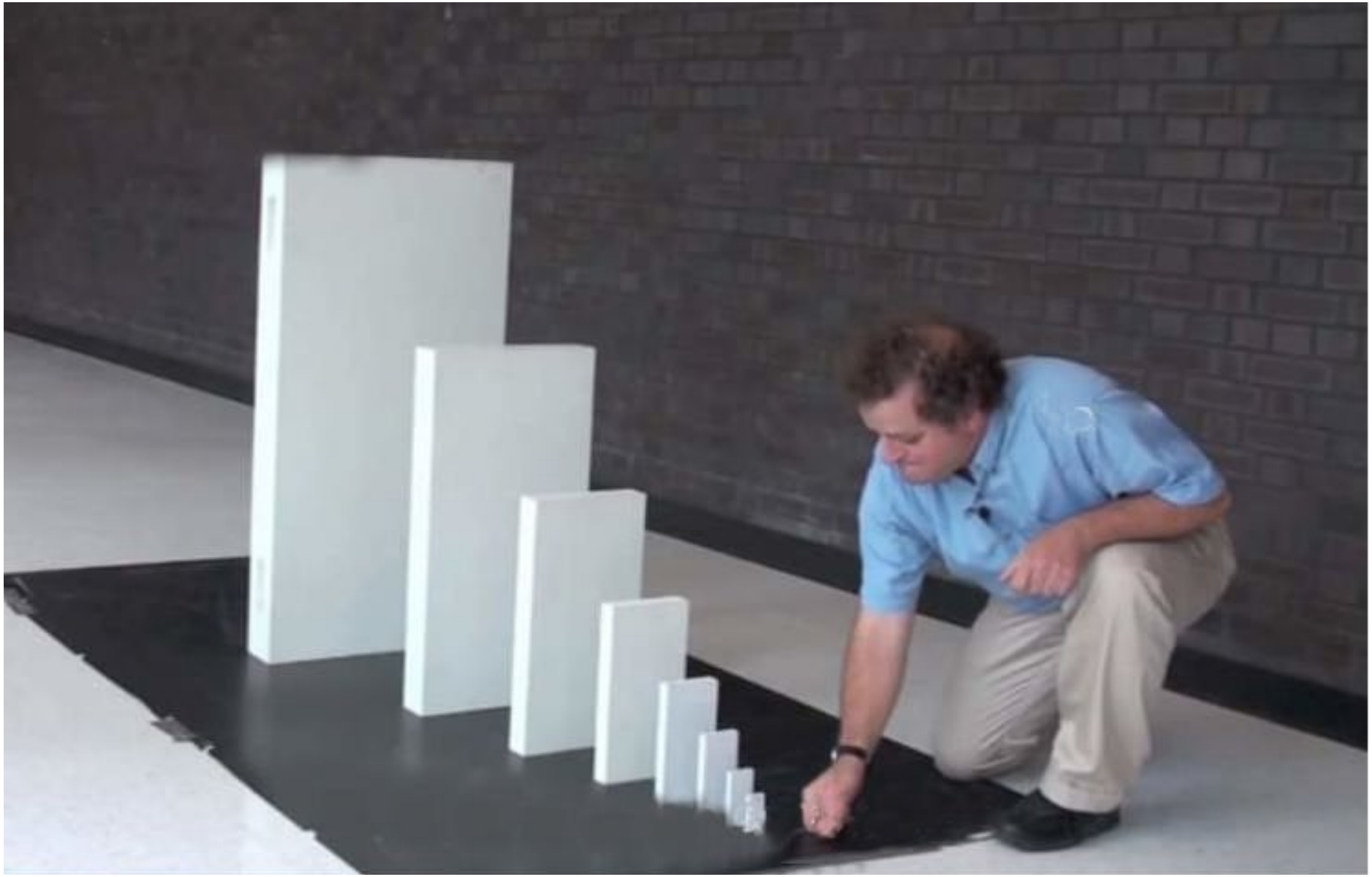
Example: Prove that $1 + 2 + \cdots + n = n\,(n + 1)/2$.

Base case: For $n = 0$, we have 0 on the left and $0 \cdot 1/2 = 0$ on the right.

Induction step: Assume $1 + 2 + \cdots + n = n\,(n + 1)/2$ (**induction hypothesis**).
We have to show that $1 + 2 + \cdots + n + (n + 1) = (n + 1)\,(n + 2)/2$.

$$
\begin{aligned}
&1 + 2 + \cdots + n + (n + 1) \\
&= n\,(n + 1)/2 + (n + 1) \\
&= (n + 1)\,(n + 2)/2.
\end{aligned}
$$

By induction, the result follows.



Example: Prove that $n^3 + 2n$ is divisible by 3 for all $n \in \mathbb{N}$.

Base case: Prove that $0^3 + 2 \cdot 0$ is divisible by 3.

Induction step: Assume that $n^3 + 2n$ is divisible by 3. Prove that $(n + 1)^3 + 2(n + 1)$ is divisible by 3.

Complete this proof yourself.

Correctness of *power*

We want to prove that $power(x, n) = x^n$ for all nonzero $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Here x^n is the correct mathematical definition,

$$x^n = \underbrace{x \cdot x \cdots x}_{n \text{ times}}$$

$power(x, n)$ is the function we have defined,

$$power(x, n) = \begin{cases} 1 & \text{if } n = 0, \\ x \cdot power(x, n-1) & \text{otherwise.} \end{cases}$$

To prove: $power(x, n) = x^n$ for all nonzero $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Proof by induction on n.

Base case: $power(x, 0) = 1 = x^0$ for all nonzero $x \in \mathbb{R}$.

Induction step: Assume $power(x, n) = x^n$.

$$
\begin{aligned}
& power(x, n+1) \\
&= x \cdot power(x, n) \\
&= x \cdot x^n \\
&= x^{n+1}.
\end{aligned}
$$

Exercise: Prove using induction that if a line of unit length is given, then a line of length $\sqrt{n}$ can be constructed using a *straightedge and compass* for any positive integer n.

Exercise: Prove that the factorial function defined in the last class is correct: $factorial(n) = n!$ for all $n \in \mathbb{N}$.