COL100: Introduction to Computer Science

3.2: Variations of induction

Using the previous approach, it's not easy to analyze correctness of *fastPower*:

$$
\mathit{fastPower}(x, n) = \begin{cases} 1 & \text{if } n = 0, \\ \mathit{fastPower}(x^2, \lfloor n/2 \rfloor) & \text{if } n \text{ is even}, \\ x \cdot \mathit{fastPower}(x^2, \lfloor n/2 \rfloor) & \text{if } n \text{ is odd}. \end{cases}
$$

The principle of mathematical induction

Version 2

A property P holds for all natural numbers $n \geq k$, provided

1. (**base case:**) P holds for k, and
2. (**induction step:**) if P holds for an arbitrary $n \geq k$, then it also holds for $n + 1$.

Suppose we have stamps of two different denominations, Rs 3 and Rs 5.

Show that it is possible to make up exactly any postage of Rs 8 or more using stamps of these two denominations.

To prove: Any $n \geq 8$ can be expressed as $3i + 5j$ for some $i, j \geq 0$.

Base case: $8 = 3 + 5$, so the claim is true with $i = 1$, $j = 1$.

Induction hypothesis: $n = 3i + 5j$ for some $n \geq 8$, $i, j \geq 0$.

Now what?

Hint: There are two cases: either $j > 0$, or $j = 0$.

To prove: Any $n \geq 8$ can be expressed as $3i + 5j$ for some $i, j \geq 0$.

Base case: $8 = 3 + 5$, so the claim is true with $i = 1$, $j = 1$.

Induction hypothesis: $n = 3i + 5j$ for some $n \geq 8$, $i, j \geq 0$.

Induction step:

If $j > 0$, we have at least one Rs 5 stamp. We can remove it and add two Rs 3 stamps:

$$3(i + 2) + 5(j - 1) = 3i + 5j + 6 - 5 = n + 1.$$

If $j = 0$, n is a multiple of 3 and so we have at least three Rs 3 stamps. Similarly,

$$3(i - 3) + 5(j + 2) = 3i + 5j - 9 + 10 = n + 1.$$

The principle of mathematical induction

Version 3 a.k.a. “strong induction”

A property P holds for all natural numbers, provided

1. (**base case:**) P holds for 0, and
2. (**induction step:**) if P holds for *all* $m \leq n$ for an arbitrary $n \in \mathbb{N}$, then it also holds for $n + 1$.

Equivalent version of induction step:
If P holds for *all* $m < n$ for an arbitrary $n \geq 1$, then it also holds for n.

Consider the Fibonacci numbers,

$$F_0 = 0,$$
$$F_1 = 1,$$
$$F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 2.$$

Let $\phi = (1 + \sqrt{5})/2$. This number has the property that $\phi^2 = \phi + 1$.

Show that $F_n \leq \phi^{n-1}$ for all $n \geq 0$.

$F_0 = 0$
$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$
$F_5 = 5$
$F_6 = 8$
$F_7 = 13$
$\vdots$

Base case: $F_0 = 0 \leq \phi^{-1}$.

Induction hypothesis: For some $n \geq 1$, $F_m \leq \phi^{m-1}$ for all m with $0 \leq m < n$.

Induction step: We need to show that $F_n \leq \phi^{n-1}$.

$$\begin{gathered} F_n \\ = F_{n-1} + F_{n-2} \\ \leq \phi^{n-2} + \phi^{n-3} \\ = \phi^{n-3}\,(\phi + 1) \\ = \phi^{n-3} \cdot \phi^2 \\ = \phi^{n-1}. \end{gathered}$$

Is this valid?

Base case: $F_0 = 0 \leq \phi^{-1}$.

Induction hypothesis: For some $n \geq 1$, $F_m \leq \phi^{m-1}$ for all m with $0 \leq m < n$.

Induction step: We need to show that $F_n \leq \phi^{n-1}$.

$$\begin{gathered} F_n \\ = F_{n-1} + F_{n-2} \\ \leq \phi^{n-2} + \phi^{n-3} \\ = \phi^{n-3}(\phi + 1) \\ = \phi^{n-3} \cdot \phi^2 \\ = \phi^{n-1}. \end{gathered}$$

Is this valid?

Not if $n = 1$! F_{n-2} is not covered by the I.H. (and not even defined).

Base case: $F_0 = 0 \leq \phi^{-1}$.

Induction hypothesis: For some $n \geq 1$, $F_m \leq \phi^{m-1}$ for all m with $0 \leq m < n$.

Induction step: We need to show that $F_n \leq \phi^{n-1}$.

If $n = 1$, $F_n = 1 = \phi^0$.

Otherwise,

$$
\begin{gathered}
F_n \\
= F_{n-1} + F_{n-2} \\
\leq \phi^{n-2} + \phi^{n-3} \\
= \phi^{n-3}\,(\phi + 1) \\
= \phi^{n-3} \cdot \phi^2 \\
= \phi^{n-1}.
\end{gathered}
$$

Correctness of *fastPower*

$$fastPower(x, n) = \begin{cases} 1 & \text{if } n = 0, \\ fastPower(x^2, \lfloor n/2 \rfloor) & \text{if } n \text{ is even}, \\ x \cdot fastPower(x^2, \lfloor n/2 \rfloor) & \text{if } n \text{ is odd}. \end{cases}$$

We want to prove that $fastPower(x, n) = x^n$ for all nonzero $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Proof by induction version 3 (strong induction) on n.

Correctness of *fastPower*

To prove: $\mathit{fastPower}(x, n) = x^n$ for all $x \neq 0$ and $n \in \mathbb{N}$.

Base case: $\mathit{fastPower}(x, 0) = 1 = x^0$ for all $x \neq 0$.

Induction hypothesis: $\mathit{fastPower}(x, m) = x^m$ for all m with $0 \leq m < n$, all $x \neq 0$.

If n is even, then $n = 2k$ with $k = \lfloor n/2 \rfloor < n$. So

$$\begin{gathered} \mathit{fastPower}(x, n) \\ = \mathit{fastPower}(x^2, \lfloor n/2 \rfloor) \\ = \mathit{fastPower}(x^2, k) \\ = (x^2)^k = x^{2k} = x^n. \end{gathered}$$

If n is odd, then…

Afterwards

- Implement *power* and *fastPower* as SML functions of type `real * int -> real` and test them on some example inputs.
- Implement an SML function of type `int -> int` to compute the *n*th Fibonacci number.
- Read Sections 2.0.3 and 3.6.1 of the notes.
- Do problems 3 to 9 of Chapter 2 of the notes.