Spherical Pendulum with Python

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2019

1 Introduction

Simulate a spherical pendulum with python.

1.1 References

These are clickable links.

Spherical Pendulum Spherical Pendulum describes how the equations of a spherical pendulum are derived. Starts with the dependencies between cartesian coordinates and the lagrangian coordinates θ and ϕ.

Pendulum code example Case Of the Spherical Pendulum briefly derives the energies, lagrangian and movement equations and hosts a matlab example script.

Python ODE solver nathantypanski.com/... describes how to solve a second-order ODE with NumPy and SciPy.

2 Equations

The spherical pendulums position can be described with two coordinates. θ is the angle in a plane which includes the Z-axis and ϕ is the angle on the plane perpendicular to the Z-axis. With l as the length of the pendulum, the position in cartesian coordinates is

$$x = l \sin\theta \cos\phi \tag{1}$$
$$y = l \sin\theta \sin\phi \tag{2}$$
$$z = -l \cos\theta \tag{3}$$

The energies.

$$E_{kin} = T = \frac{1}{2}mv^2 \tag{4}$$
$$E_{pot} = V = mgh \tag{5}$$

The Lagrangian.

$$L = T - V \tag{6}$$

To calculate the kinetic energy T, we have to replace v^2.

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{7}$$

v^2 can be written using θ and ϕ only. This requires the derivates in respect to the change of time t.

$$
\begin{aligned}
\dot{x}^2 &= (\frac{\partial x}{\partial t})^2 \\
&= (\frac{\partial}{\partial t} l \sin\theta\cos\phi)^2 \\
&= l^2(\sin\theta(-\sin\phi)\dot{\phi} + \cos\theta\cos\phi\dot{\theta})^2 \\
&= l^2[\sin^2\theta(-1)^2\sin^2\phi\dot{\phi}^2 + \cos^2\theta\cos^2\phi\dot{\theta}^2 + 2\sin\theta(-\sin\phi)\dot{\phi}\cos\theta\cos\phi\dot{\theta}] \\
&= l^2[\sin^2\theta\sin^2\phi\dot{\phi}^2 + \cos^2\theta\cos^2\phi\dot{\theta}^2 - 2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi\dot{\phi}]
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\dot{y}^2 &= (\frac{\partial y}{\partial t})^2 \\
&= (\frac{\partial}{\partial t} l \sin\theta\sin\phi)^2 \\
&= l^2(\sin\theta\cos\phi\dot{\phi} + \cos\theta\sin\phi\dot{\theta})^2 \\
&= l^2[\sin^2\theta\cos^2\phi\dot{\phi}^2 + \cos^2\theta\sin^2\phi\dot{\theta}^2 + 2\sin\theta\cos\phi\dot{\phi}\cos\theta\sin\phi\dot{\theta}] \\
&= l^2[\sin^2\theta\cos^2\phi\dot{\phi}^2 + \cos^2\theta\sin^2\phi\dot{\theta}^2 + 2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi\dot{\phi}]
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\dot{z}^2 &= (\frac{\partial z}{\partial t})^2 \\
&= (\frac{\partial}{\partial t} - l\cos\theta)^2 \\
&= l^2(-\sin\theta\dot{\theta})^2 \\
&= l^2[\sin^2\theta\dot{\theta}^2]
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\frac{v^2}{l^2} &= (\sin^2\theta\sin^2\phi\dot{\phi}^2 + \cos^2\theta\cos^2\phi\dot{\theta}^2 - 2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi\dot{\phi}) \\
&\quad + (\sin^2\theta\cos^2\phi\dot{\phi}^2 + \cos^2\theta\sin^2\phi\dot{\theta}^2 + 2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi\dot{\phi}) \\
&\quad + \sin^2\theta\dot{\theta}^2
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\frac{v^2}{l^2} &= \sin^2\theta\sin^2\phi\dot{\phi}^2 + \sin^2\theta\cos^2\phi\dot{\phi}^2 \\
&\quad + \cos^2\theta\cos^2\phi\dot{\theta}^2 + \cos^2\theta\sin^2\phi\dot{\theta}^2 \\
&\quad + \sin^2\theta\dot{\theta}^2
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\frac{v^2}{l^2} &= \sin^2\theta\dot{\phi}^2(\sin^2\phi + \cos^2\phi) + \cos^2\theta\dot{\theta}^2(\cos^2\phi + \sin^2\phi) \\
&\quad + \sin^2\theta\dot{\theta}^2
\end{aligned} \tag{13}
$$

$$
\frac{v^2}{l^2} = \sin^2\theta\dot{\phi}^2 + (\cos^2\theta + \sin^2\theta)\dot{\theta}^2 \tag{14}
$$

$$
v^2 = l^2(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) \tag{15}
$$

Now the energies and the lagrangian can be written using θ and ϕ.

$$T = \frac{1}{2}ml^2(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) \tag{16}$$
$$V = -mgl\cos\theta \tag{17}$$
$$L = \frac{1}{2}ml^2(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) + mgl\cos\theta \tag{18}$$

2.1 Second Lagrange Equation

The second Lagrange equation for each coordinate q_i is

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{19}$$

The non generalized friction force C is

$$\begin{aligned} C &= \frac{1}{2}cv^2 \\ C &= \frac{1}{2}cl^2(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) \end{aligned} \tag{20}$$

which we insert into the Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial C}{\partial \dot{q}_i} = 0 \tag{21}$$

Solve it for θ and ϕ.

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \frac{\partial C}{\partial \dot{\theta}} \\ &= \frac{\partial}{\partial t}(\frac{1}{2}ml^2 2\dot{\theta}) - (\frac{1}{2}ml^2 2\sin\theta\cos\theta\dot{\phi}^2 + mgl(-\sin\theta)) + \frac{1}{2}cl^2 2\dot{\theta} \\ &= ml^2\ddot{\theta} - ml^2\sin\theta\cos\theta\dot{\phi}^2 + mgl\sin\theta + cl^2\dot{\theta} \\ &= \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 + \frac{g}{l}\sin\theta + \frac{c}{m}\dot{\theta} \\ \ddot{\theta} &= \frac{1}{2}\sin(2\theta)\dot{\phi}^2 - \frac{g}{l}\sin\theta - \frac{c}{m}\dot{\theta} \end{aligned} \tag{22}$$

$$
\begin{aligned}
0 &= \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} + \frac{\partial C}{\partial \dot{\phi}} \\
&= \frac{\partial}{\partial t}(ml^2 \sin^2\theta\dot{\phi}) - 0 + cl^2\sin^2\theta\dot{\phi} \\
&= ml^2(\sin^2\theta\ddot{\phi} + 2\sin\theta\cos\theta\dot{\theta}\dot{\phi}) + cl^2\sin^2\theta\dot{\phi} \\
&= \ddot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} + \frac{c}{m}\dot{\phi} \\
\ddot{\phi} &= -2\cot\theta\dot{\theta}\dot{\phi} - \frac{c}{m}\sin^2\theta\dot{\phi}
\end{aligned}
\tag{23}
$$

3 Numpy ODE

By entering θ, $\dot{\theta}$, ϕ and $\dot{\phi}$ into an ODE, they motion can be solved numerically.

```

def pend(y, t, mass, gravity, L, c):
    theta, d_theta, phi, d_phi = y
    dydt = [d_theta,
            np.square(d_phi) * np.sin(theta) * np.cos(theta) - gravity/L * np.sin(theta) - c/
     mass * d_phi,
            d_phi,
            (-2) * d_theta * d_phi / np.tan(theta) - c/mass * d_phi
            ]
    return dydt

mass = 10
gravity = 10
L = 0.2
friction = 8

# intial values
y0 = [np.pi/2.5, 0.0, 0.0, 4]

# time values
t = np.linspace(0, 20, 400)

# create the solver
sol = scipy.integrate.odeint(pend, y0, t, args=(mass, gravity, L, friction))
```

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