Spherical Pendulum with Python

Arun Leander

2019

1 Introduction

Simulate a spherical pendulum with python.

1.1 References

These are clickable links.

Spherical Pendulum Spherical Pendulum describes how the equations of a spherical pendulum are derived. Starts with the dependencies between cartesian coordinates and the lagrangian coordinates θ and ϕ.

Pendulum code example Case Of the Spherical Pendulum briefly derives the energies, lagrangian and movement equations and hosts a matlab example script.

Python ODE solver nathantyp nski.com/... describes how to solve a second-order ODE with NumPy and SciPy.

2 Equations

The spherical pendulums position can be described with two coordinates. θ is the angle in a plane which includes the Z-axis and ϕ is the angle on the plane perpendicular to the Z-axis. With l as the length of the pendulum, the position in cartesian coordinates is

$$x = l \sin\theta \cos\phi \tag{1}$$

$$y = l \sin\theta \sin\phi \tag{2}$$

$$z = -l \cos\theta \tag{3}$$

The energies.

$$E_{kin} = T = \frac{1}{2}mv^2 \tag{4}$$
$$E_{pot} = V = mgh \tag{5}$$

The Lagrangian.

$$L = T - V \tag{6}$$

To calculate the kinetic energy T, I have to replace v^2.

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \tag{7}$$

2.1 Velocity

v^2 can be written using θ and ϕ only. This requires the derivates in respect to the change of time t.

$$\begin{aligned}
\dot{x}^2 &= (\frac{\partial x}{\partial t})^2 \\
&= (\frac{\partial}{\partial t} l \sin\theta \cos\phi)^2 \\
&= l^2(\sin\theta(-\sin\phi)\dot{\phi} + \cos\theta\cos\phi\dot{\theta})^2 \\
&= l^2[\sin^2\theta \sin^2\phi\dot{\phi}^2 + \cos^2\theta\cos^2\phi\dot{\theta}^2 - 2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi\dot{\phi}]
\end{aligned} \tag{8}$$

$$\begin{aligned}
\dot{y}^2 &= (\frac{\partial y}{\partial t})^2 \\
&= (\frac{\partial}{\partial t} l \sin\theta \sin\phi)^2 \\
&= l^2(\sin\theta\cos\phi\dot{\phi} + \cos\theta\sin\phi\dot{\theta})^2 \\
&= l^2[\sin^2\theta \cos^2\phi\dot{\phi}^2 + \cos^2\theta\sin^2\phi\dot{\theta}^2 + 2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi\dot{\phi}]
\end{aligned} \tag{9}$$

$$\begin{aligned}
\dot{z}^2 &= (\frac{\partial z}{\partial t})^2 \\
&= (\frac{\partial}{\partial t} - l\cos\theta)^2 \\
&= l^2(\sin^2\theta\dot{\theta}^2)
\end{aligned} \tag{10}$$

$$
\begin{aligned}
\frac{v^2}{l^2} &= (\sin^2\theta\sin^2\phi\dot{\phi}^2 + \cos^2\theta\cos^2\phi\dot{\theta}^2 - 2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi\dot{\phi}) \\
&\quad + (\sin^2\theta\cos^2\phi\dot{\phi}^2 + \cos^2\theta\sin^2\phi\dot{\theta}^2 + 2\sin\theta\cos\theta\dot{\theta}\sin\phi\cos\phi\dot{\phi}) \\
&\quad + \sin^2\theta\dot{\theta}^2
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\frac{v^2}{l^2} &= \sin^2\theta\sin^2\phi\dot{\phi}^2 + \sin^2\theta\cos^2\phi\dot{\phi}^2 \\
&\quad + \cos^2\theta\cos^2\phi\dot{\theta}^2 + \cos^2\theta\sin^2\phi\dot{\theta}^2 \\
&\quad + \sin^2\theta\dot{\theta}^2
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\frac{v^2}{l^2} &= \sin^2\theta\dot{\phi}^2(\sin^2\phi + \cos^2\phi) + \cos^2\theta\dot{\theta}^2(\cos^2\phi + \sin^2\phi) \\
&\quad + \sin^2\theta\dot{\theta}^2
\end{aligned} \tag{13}
$$

$$
\frac{v^2}{l^2} = \sin^2\theta\dot{\phi}^2 + (\cos^2\theta + \sin^2\theta)\dot{\theta}^2 \tag{14}
$$

$$
v^2 = l^2(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) \tag{15}
$$

Now the energies and the lagrangian can be written using θ and ϕ.

$$
T = \frac{1}{2}ml^2(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) \tag{16}
$$

$$
V_g = -mgl\cos\theta \tag{17}
$$

$$
V_f = ml(f_z\cos\theta - (f_x\cos\phi + f_y sin(\phi))\sin(\theta)) \tag{18}
$$

$$
L = \frac{1}{2}ml^2(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) + mgl\cos\theta \tag{19}
$$

$$
L_f = \frac{1}{2}ml^2(\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2) - ml(f_z\cos\theta - (f_x\cos\phi + f_y sin(\phi))\sin(\theta)) \tag{20}
$$

I inserted a generalized force f. For gravity use $f = \left(\begin{smallmatrix} 0 \\ 0 \\ -10 \end{smallmatrix}\right)$.

2.2 Relative Velocity

The velocity relative to a moving system can be expressed with $v_w = v - w$ where w is the velocity of the system.

$$
{v_w}^2 = (\dot{x} - w_x)^2 + (\dot{y} - w_y)^2 + (\dot{z} - w_z)^2 \tag{21}
$$

$$
\begin{aligned}
{v_w}^2 &= (-l\sin\theta\sin\phi\dot{\phi} + l\cos\theta\cos\phi\dot{\theta} - w_x)^2 \\
&\quad + (l\sin\theta\cos\phi\dot{\phi} + l\cos\theta\sin\phi\dot{\theta} - w_y)^2 + (l\sin\theta\dot{\theta} - w_z)^2
\end{aligned} \tag{22}
$$

2.3 Second Lagrange Equation

The second Lagrange equation for each coordinate q_i is

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{23}$$

Insert the friction.

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial C}{\partial \dot{q}_i} = 0 \tag{24}$$

Solve it for $\ddot{\theta}$ and $\ddot{\phi}$.

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} + \frac{\partial C}{\partial \dot{q}_i} \\
&= \frac{\partial}{\partial t}(\frac{1}{2}ml^2 2\dot{\theta}) - [\frac{1}{2}ml^2 2\sin\theta\cos\theta\dot{\phi}^2 - ml(-f_z\sin\theta + (f_x\cos\phi - f_y\sin(-\phi))(-\cos(\theta)))] + \frac{\partial C}{\partial \dot{q}_i} \\
&= ml^2\ddot{\theta} - ml^2\sin\theta\cos\theta\dot{\phi}^2 - mlf_z\sin\theta - ml(f_x\cos\phi - f_y\sin(-\phi))\cos\theta + \frac{\partial C}{\partial \dot{q}_i} \\
&= \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 - \frac{\sin\theta}{l}f_z - \frac{\cos\theta}{l}(f_x\cos\phi - f_y\sin(-\phi)) + \frac{1}{ml^2}\frac{\partial C}{\partial \dot{q}_i} \\
\ddot{\theta} &= \frac{\sin(2\theta)}{2}\dot{\phi}^2 + \frac{\sin\theta}{l}f_z + \frac{\cos\theta}{l}(f_x\cos\phi - f_y\sin(-\phi)) - \frac{1}{ml^2}\frac{\partial C}{\partial \dot{q}_i}
\end{aligned} \tag{25}$$

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} + \frac{\partial C}{\partial \dot{\phi}} \\
&= \frac{\partial}{\partial t}(ml^2\sin^2\theta\dot{\phi}) - (-1)ml(-f_x\sin\phi + f_y\cos(\phi))\sin(-\theta) + \frac{\partial C}{\partial \dot{\phi}} \\
&= ml^2(\sin^2\theta\ddot{\phi} + 2\sin\theta\cos\theta\dot{\theta}\dot{\phi}) + ml(-f_x\sin\phi + f_y\cos(\phi))\sin(-\theta) + \frac{\partial C}{\partial \dot{\phi}} \\
&= \ddot{\phi} + 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi} + \frac{\sin(-\theta)}{l}(-f_x\sin\phi + f_y\cos\phi) + \frac{1}{ml^2\sin^2\theta}\frac{\partial C}{\partial \dot{\phi}} \\
\ddot{\phi} &= -2\cot\theta\dot{\theta}\dot{\phi} + \frac{\sin(\theta)}{l}(-f_x\sin\phi + f_y\cos\phi) - \frac{1}{ml^2\sin^2\theta}\frac{\partial C}{\partial \dot{\phi}}
\end{aligned} \tag{26}$$

3 Friction and Drag

The non generalized *angular* friction power C_A and *drag* friction power C_D are as follows. For the drag, I will pretend the the surface A and the density p (through volume) is proportional to l. The extra components are included in the drag coefficient c_d. While the velocity for C_A can be described only through θ and ϕ, C_D depends on the velocity of the system and has to include the wind speed.

$$
\begin{aligned}
C_A &= \frac{1}{2} c_a v^2 \\
C_A &= \frac{1}{2} c_a l^2 (\sin^2\theta \dot{\phi}^2 + \dot{\theta}^2)
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
C_D &= \frac{1}{2} c_{d0} p A (v - w)^3 \\
C_D &= \frac{1}{2} c_d \frac{m}{l} l (|v - w|)^3 \\
C_D &= \frac{1}{2} c_d m l^3 (\frac{{v_w}^2}{l^2})^{\frac{3}{2}}
\end{aligned}
\tag{28}
$$

3.1 Derivates of the frictions for the main equation

3.1.1 Angular Friction

$$
\frac{\partial C_A}{\partial \dot{\theta}} = c_a l^2 \dot{\theta} \tag{29}
$$

$$
\frac{\partial C_A}{\partial \dot{\phi}} = c_a l^2 \sin^2\theta \dot{\phi} \tag{30}
$$

$$
\tag{31}
$$

3.1.2 Drag

${v_w}^2$ has to be inserted, see this previous section (click).

$$
\begin{aligned}
\frac{\partial C_D}{\partial \dot{\theta}} &= \frac{\partial}{\partial \dot{\theta}} \frac{1}{2} c_d m ({v_w}^2)^{\frac{3}{2}} \\
&= \frac{3}{4} c_d m (\frac{{v_w}^2}{l^2})^{\frac{1}{2}} (2l \cos\theta \cos\phi (l\dot{\theta} \cos\theta \cos\phi - l\dot{\phi} \sin\theta \sin\phi - x_w) \\
&\quad + 2l \cos\theta \sin\phi (l\dot{\theta} \cos\theta \sin\phi + l\dot{\phi} \sin\theta \cos\phi - y_w) + 2l \sin\theta (l\dot{\theta} \sin\theta - z_w))
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\frac{\partial C_D}{\partial \dot{\phi}} &= \frac{1}{2} c_d m \frac{\partial (\frac{{v_w}^2}{l^2})^{\frac{3}{2}}}{\partial \dot{\phi}} \\
&= \frac{3}{4} c_d m (\frac{{v_w}^2}{l^2})^{\frac{1}{2}} \\
&\quad (2l \sin\theta (l\dot{\phi} \sin\theta (\sin^2\phi + \cos^2\phi + x_w \sin\phi - y_w \cos\phi)))
\end{aligned}
\tag{33}
$$

4 Multiple Segments

Here is an attempt to generalize the equation to compute the motion of multiple pendulums concatenated.

$$
\begin{aligned}
x &= l \sin\theta \cos\phi & \dot{x} &= -l \sin\theta \sin\phi \dot{\phi} + l \cos\theta \cos\phi \dot{\theta} \\
y &= l \sin\theta \sin\phi & \dot{y} &= l \sin\theta \cos\phi \dot{\phi} + l \cos\theta \sin\phi \dot{\theta} \\
z &= -l \cos\theta & \dot{z} &= l \sin\theta \dot{\theta}
\end{aligned} \tag{34}
$$

Some partial derivates.

$$
\begin{aligned}
\frac{\partial x}{\partial \theta} &= x_{n\theta} = l \cos\theta \cos\phi \\
\frac{\partial y}{\partial \theta} &= y_{n\theta} = l \cos\theta \sin\phi \\
\frac{\partial z}{\partial \theta} &= z_{n\theta} = l \sin\theta
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\frac{\partial \dot{x}}{\partial t} &= \dot{x}_t = -l \cos\theta \sin\phi \dot{\theta}\dot{\phi} - l \sin\theta \cos\phi \dot{\phi}^2 - l \sin\theta \sin\phi \ddot{\phi} \\
&\qquad - l \cos\theta \sin\phi \dot{\theta}\dot{\phi} - l \sin\theta \cos\phi \dot{\theta}^2 + l \cos\theta \cos\phi \ddot{\theta} \\
\frac{\partial \dot{y}}{\partial t} &= \dot{y}_t = l \cos\theta \cos\phi \dot{\theta}\dot{\phi} - l \sin\theta \sin\phi \dot{\phi}^2 + l \sin\theta \cos\phi \ddot{\phi} \\
&\qquad + l \cos\theta \cos\phi \dot{\theta}\dot{\phi} - l \sin\theta \sin\phi \dot{\theta}^2 - l \cos\theta \sin\phi \ddot{\theta} \\
\frac{\partial \dot{z}}{\partial t} &= \dot{z}_t = l \cos\theta \dot{\theta}^2 + l \sin\theta \ddot{\theta}
\end{aligned} \tag{36}
$$

$$
\begin{aligned}
\frac{\partial \dot{x}}{\partial \dot{\theta}} &= \dot{x}_{\dot{\theta}} = l \cos\theta \cos\phi \\
\frac{\partial \dot{y}}{\partial \dot{\theta}} &= \dot{y}_{\dot{\theta}} = l \cos\theta \sin\phi \\
\frac{\partial \dot{z}}{\partial \dot{\theta}} &= \dot{z}_{\dot{\theta}} = l \sin\theta
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial \dot{x}}{\partial \dot{\theta}} &= \dot{x}_{\dot{\theta}t} = -l \sin\theta \cos\phi \dot{\theta} - l \cos\theta \sin\phi \dot{\phi} \\
\frac{\partial}{\partial t}\frac{\partial \dot{y}}{\partial \dot{\theta}} &= \dot{y}_{\dot{\theta}t} = -l \sin\theta \cos\phi \dot{\theta} + l \cos\theta \cos\phi \dot{\phi} \\
\frac{\partial}{\partial t}\frac{\partial \dot{z}}{\partial \dot{\theta}} &= \dot{z}_{\dot{\theta}t} = -l \sin\theta \dot{\theta}
\end{aligned} \tag{38}
$$

The updated lagrangian.

$$
\begin{aligned}
T &= \frac{1}{2}mv^2 \\
&= \frac{1}{2}((\sqrt{m}\dot{x})^2 + (\sqrt{m}\dot{y})^2 + (\sqrt{m}\dot{z})^2) \\
&= \frac{1}{2}((\sqrt{m_a}\dot{x}_a + \sqrt{m_b}\dot{x}_b \dots)^2 + (\sqrt{m_a}\dot{y}_a + \sqrt{m_b}\dot{y}_b \dots)^2 + (\sqrt{m_a}\dot{z}_a + \sqrt{m_b}\dot{z}_b \dots)^2)
\end{aligned} \tag{39}
$$

$$
V = -(((m_a + m_b)x_a + m_b x_b)x_f) + ((m_a + m_b)y_a + m_b y_b)y_f) + ((m_a + m_b)z_a + m_b x_b)z_f)) \tag{40}
$$

$$
\begin{aligned}
L = \frac{1}{2}(&m_a\dot{x}_a^2 + 2\sqrt{m_a + m_b}\dot{x}_a\dot{x}_b + m_b\dot{x}_b^2 + m_a\dot{y}_a^2 + 2\sqrt{m_a + m_b}\dot{y}_a\dot{y}_b + m_b\dot{y}_b^2 \\
&+ m_a\dot{z}_a^2 + 2\sqrt{m_a + m_b}\dot{z}_a\dot{z}_b + m_b\dot{z}_b^2) \\
&+ ((m_a + m_b)x_a + m_b x_b)x_f + ((m_a + m_b)y_a + m_b y_b)y_f + ((m_a + m_b)z_a + m_b x_b)z_f
\end{aligned} \tag{41}
$$

For the topmost θ and ϕ, this will result in.

$$
\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{42}
$$

5 Numpy ODE

By entering θ, $\dot{\theta}$, ϕ and $\dot{\phi}$ into an ODE, they motion can be solved numerically.

```python

def pend(y, t, mass, gravity, L, c):
    theta, d_theta, phi, d_phi = y
    dydt = [d_theta,
            np.square(d_phi) * np.sin(theta) * np.cos(theta) - gravity/L * np.sin(theta) - c/mass * d_phi,
            d_phi,
            (-2) * d_theta * d_phi / np.tan(theta) - c/mass * d_phi
            ]
    return dydt

mass = 10
gravity = 10
L = 0.2
friction = 8

# intial values
y0 = [np.pi/2.5, 0.0, 0.0, 4]

# time values
t = np.linspace(0, 20, 400)

# create the solver
sol = scipy.integrate.odeint(pend, y0, t, args=(mass, gravity, L, friction))
```

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