Monge–Ampère Geometry and Vortices

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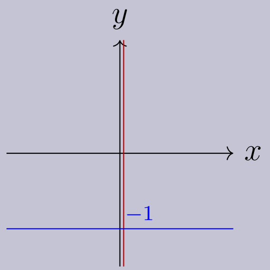
Outline of Talk

1. Geometry of Monge–Ampère PDEs
2. Monge–Ampère Geometry on 2D Background
3. Geometry of 2D Incompressible Fluid Flows
4. Extensions to Curved Background and Higher Dimension
5. Conclusions and Outlook

1. Geometry of Monge–Ampère PDEs

PDEs as Manifolds

- *k-th Jet Bundle* $J^k(M, N)$ is space of all possible values of $x, y, D^1y, \cdots D^ky$ [Ehresmann 1951, Bryant et al. 1991]
- k-th order PDE $F(x, y, D^1y, \cdots D^ky) = 0$ can be seen as the space $\mathcal{E} \subset J^k(M, N)$ of points satisfying equation.
- Solutions $\psi : M \to N$ are submanifolds $L \subset \mathcal{E}$, e.g. $F(x, \psi(x), D^1\psi, \cdots D^k\psi) = 0$.
- Properties of geometry tell us about properties of equation and solutions.



$3x^2(1 + y) = 0$ consists of two lines, each of which is a solution.

What are Monge–Ampère Equations?

- *MAE*: non-linear, second-order PDE, given by quasi-linear combinations of the minor determinants of the *Hessian* of ψ:

$$\mathrm{Hess}(\psi) = \begin{pmatrix} \psi_{x^1x^1} & \psi_{x^1x^2} & \cdots & \psi_{x^1x^n} \\ \psi_{x^2x^1} & \psi_{x^2x^1} & \cdots & \psi_{x^2x^n} \\ \vdots & \vdots & \ldots & \vdots \\ \psi_{x^nx^1} & \psi_{x^nx^2} & \cdots & \psi_{x^nx^n} \end{pmatrix}$$

- *Quasi-Linear*: coefficients can depend on x, ψ and $D^1\psi$ non-linearly.
- *k-th Minor Determinant*: determinant of the $k \times k$ sub-matrix with entries given by intersections of k rows and columns.

MAEs in Two Dimensions

- In two dimensions, MAEs take the form

$$A\psi_{x^1x^1} + 2B\psi_{x^1x^2} + C\psi_{x^2x^2} + D\left(\psi_{x^1x^1}\psi_{x^2x^2} - \psi^2_{x^1x^2}\right) + E = 0\,.$$

  where $A, B, \ldots E$ can depend on $x^1, x^2, \psi, \psi_{x^1}, \psi_{x^2}$ non-linearly.
- If $A, B, \ldots E$ do not depend on ψ, we have a *Symplectic* MAE.
- Symplectic MAEs can be encoded in T^*M rather than $J^2(M, N)$. We call M the *Configuration Space/Background*.

Some Examples You May Know

- 2D Reaction-Diffusion: $\psi^\alpha \psi_{xx} + [\alpha\psi^{\alpha-1}\psi_x - \psi_t + F(\psi)] = 0.$
- 3D Chynoweth–Sewell: $[\psi_{xx}\psi_{yy} - (\psi_{xy})^2] + \psi_{zz} = 0.$
- 4D Khokhlov–Zabolotskaya: $\psi_{tt} + \psi_{yy} + \psi_{zz} - \psi_{xt} + (\psi_t)^2 = 0.$
- Laplace: $\Delta\psi := \psi_{x^1x^1} + \psi_{x^2x^2} + \cdots + \psi_{x^nx^n} = 0.$
- Wave: $\Box\psi := \psi_{tt} - \psi_{x^1x^1} - \psi_{x^2x^2} - \cdots - \psi_{x^nx^n} = 0$.

Geometry to Equation: A Quick Example

Consider a 2-form on $T^*\mathbb{R}^2$ (with coordinates x^1, x^2, q_1, q_2):

$$\alpha = \mathrm{d}q_1 \wedge \mathrm{d}x^2 - \mathrm{d}q_2 \wedge \mathrm{d}x^1 \, .$$

Define $L_\psi \coloneqq \{(x^1, x^2, \psi_{x^1}, \psi_{x^2})\} \subset T^*\mathbb{R}^2$ (fix q_1 and q_2 at each x).

$$\begin{aligned} \alpha|_{L_\psi} &= \mathrm{d}(\psi_{x^1}) \wedge \mathrm{d}x^2 - \mathrm{d}(\psi_{x^2}) \wedge \mathrm{d}x^1 \\ &= (\psi_{x^1x^1} + \psi_{x^2x^2})\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \end{aligned}$$

So $\alpha|_{L_\psi} = 0$ if and only if $\Delta\psi = 0$, i.e. ψ solves $\Delta\psi = 0$.

Symplectic Forms and Non-Uniqueness

A *Symplectic* form ω on $T^*\mathbb{R}^m$ is

- a 2-form: skew-symmetric and bilinear,
- *Closed*: $\mathrm{d}\omega \equiv 0$,
- *Non-Degenerate*: $\omega(X, \cdot) \equiv 0$ if and only if $X \equiv 0$.

The canonical choice is

$$\omega = \mathrm{d}q_i \wedge \mathrm{d}x^i = \begin{pmatrix} 0_m & -I_m \\ I_m & 0_m \end{pmatrix}$$

Then $\omega|_{L_\psi} = 0$ is trivial, so $\alpha|_{L_\psi} = 0$ and $(\alpha + F(x,q)\,\omega)|_{L_\psi} = 0$ are the same equation! Which one do we pick?

Effective Forms and Equivalence Classes

- An m-form α on $T^*\mathbb{R}^m$ is called *ω-Effective* if $\alpha \wedge \omega = 0$.
- For symplectic form ω, every m-form β on $T^*\mathbb{R}^m$ decomposes as

$$\beta = \alpha + \omega \wedge \beta_0 \,,$$

  for some unique $(m-2)$-form β_0 and ω-effective m-form α [Hodge–Lepage–Lychagin].
- This defines equivalence classes $[\alpha]$ where the only effective form is α and all forms in class give same equation.

Monge–Ampère Structures

- A *Monge–Ampère Structure* on $T^*\mathbb{R}^m$ is a pair (ω, α), where ω is a symplectic form and α is an ω-effective m-form [Banos 2002].

- Fixing ω canonical, the ω-effective forms in 2D are

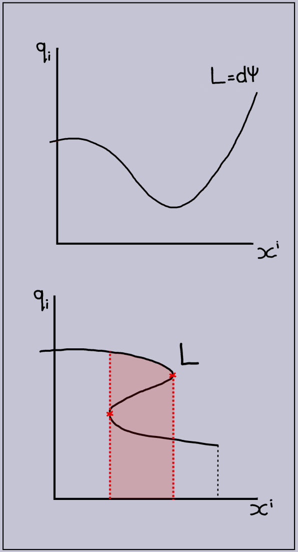
$$\begin{aligned}\alpha = {} & A\,\mathrm{d}q_1 \wedge \mathrm{d}x^2 + B\left(\mathrm{d}x^1 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}x^2\right) \\ & + C\,\mathrm{d}x^1 \wedge \mathrm{d}q_2 + D\,\mathrm{d}q_1 \wedge \mathrm{d}q_2 + E\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\end{aligned}$$

- These α are in bijection with MAEs: $\alpha|_{L_\psi} = 0$ is precisely

$$A\psi_{x^1x^1} + 2B\psi_{x^1x^2} + C\psi_{x^2x^2} + D\left(\psi_{x^1x^1}\psi_{x^2x^2} - \psi_{x^1x^2}^2\right) + E = 0\,.$$

Classical and Generalised Solutions

- A *Classical Solution* $\psi \in \mathcal{C}^\infty(\mathbb{R}^m)$ to a MAE corresponds to $L_\psi = \{x, D^1\psi(x)\} \subset T^*\mathbb{R}^m$.
- A *Generalised Solution* of a MAS is a Lagrangian submanifold $L \subset T^*\mathbb{R}^m$ s.t. $\dim(L) = m$, $\omega|_L = 0$, and $\alpha|_L = 0$.
- If projection $\pi : L \to \mathbb{R}^m$ is not
  – surjective, ψ not defined on whole domain.
  – injective, ψ is multivalued [Vinogradov 1970].
  – immersive, ψ is singular [Arnold 1990].



2. Monge–Ampère Geometry on 2D Background

The Pfaffian (2D)

- The *Pfaffian* is defined by $\alpha \wedge \alpha =: f_\alpha \omega \wedge \omega$ where $f_\alpha = AC - B^2 - DE$.
- Here, f_α is the determinant of the coefficient matrix of the linearisation of $\alpha|_{L_\psi} = 0$.
- Hence, the Monge–Ampère equation $\alpha|_{L_\psi} = 0$ is

$$\textit{elliptic} \Leftrightarrow f_\alpha > 0.$$
$$\textit{hyperbolic} \Leftrightarrow f_\alpha < 0.$$
$$\textit{parabolic} \Leftrightarrow f_\alpha = 0.$$

Lychagin–Rubtsov Metrics (2D)

- Set $\tilde{\alpha} = \frac{1}{\sqrt{|f_\alpha|}}\alpha$ and define an endomorphism of vector fields J by

$$\tilde{\alpha}(\cdot\,,\cdot) =: \omega(J\,\cdot\,,\cdot) \quad \left(J = \omega^{-1}\tilde{\alpha} \text{ as matrices}\right)\,,$$

$f_\alpha \lessgtr 0 \;\Leftrightarrow\; J^2 = \pm I_4$ and $\mathrm{tr}(J) = 0$ [Lychagin et al. 1993].

- For a non-degenerate, ω-effective, α-effective 2-form K, the Lychagin–Rubtsov metric is a symmetric bilinear form

$$\hat{g}(\cdot,\cdot) := -K(J\cdot,\cdot)\,.$$

The Lychagin–Rubtsov Theorem (2D)

MAEs are *locally equivalent* if there exists a (local) symplectomorphism $F : (T^*\mathbb{R}^2, \omega, \alpha_1) \to (T^*\mathbb{R}^2, \omega, \alpha_2)$, i.e.

$$F^*\omega = \omega \text{ and } F^*\alpha_2 = \alpha_1 \,.$$

The following conditions are equivalent [Lychagin et al. 1993]:

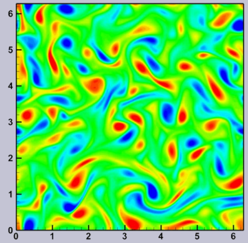
- $\alpha|_{L_\psi} = 0$ is locally equivalent to $\Box\psi = 0$ or $\Delta\psi = 0$.
- $\mathrm{d}(\tilde{\alpha}) = 0$ (with $f_\alpha \lessgtr 0$).
- J is integrable (with $J^2 = \pm I_2$).

These criteria do not always hold.

3. Geometry of 2D Incompressible Fluid Flows

Understanding Turbulence

- Turbulent flows consist of complex interactions of vortex structures – defined qualitatively.
- (2D) Vortices combine forming stable, coherent structures characterised by circulation.
- (3D) Vortices take the form of knotted/linked tubes which accumulate at small scale.
- Want to understand why behaviour so different and provide unified description.



Evolving 2D Flow Vorticity (A. Ovsyannikov - Ecole Centrale de Lyon)

Incompressible Navier–Stokes

- Homogeneous, Incompressible Navier–Stokes on $\mathbb{R}^m$

$$\begin{aligned} \partial_t v^j &= -v^i \nabla_i v^j - \nabla_j p + \nu \Delta v^j \; (-c_j)\,, \\ \nabla_i v^i &= 0\,. \end{aligned}$$

- Taking the divergence of the first and applying the second:

$$\zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} = \Delta p \; (+\nabla_i c^i)\,.$$

where $\zeta_{ij} = \frac{1}{2}(\nabla_j v_i - \nabla_i v_j)$ is the vorticity form
and $S_{ij} = \frac{1}{2}(\nabla_j v_i + \nabla_i v_j)$ is the strain-rate tensor.

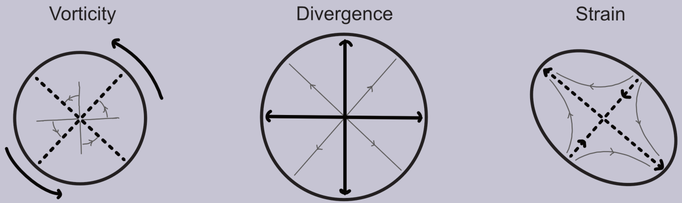
Pressure Equation in Two Dimensions

- In $2\mathrm{D}$, there exists a *stream function* $\psi \in \mathscr{C}^\infty(\mathbb{R}^2)$ such that $v^1 = -\psi_{x^2}$ and $v^2 = \psi_{x^1}$.
- Function dictates the direction of a particle dropped into the flow.
- Substituting this into Navier–Stokes, $\nabla_i v^i = 0$ is trivially satisfied and the pressure equation becomes an MAE for ψ:

$$\Delta p = 2\left(\psi_{x^1x^1}\psi_{x^2x^2} - (\psi_{x^1x^2})^2\right) .$$

Known Result: The Q-Criterion

- $\zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} = \Delta p = 2\left(\psi_{x^1x^1}\psi_{x^2x^2} - (\psi_{x^1x^2})^2\right).$
- Q-criterion [Weiss 1991, Larchevêque 1993]:
  Vorticity dominates $\Leftrightarrow \Delta p > 0 \Leftrightarrow$ *Elliptic equation.*
  Strain dominates $\Leftrightarrow \Delta p < 0 \Leftrightarrow$ *Hyperbolic equation.*
  No dominance $\Leftrightarrow \Delta p = 0 \Leftrightarrow$ *Parabolic equation.*



Based on Figure from Clough et al. 2014

Known Result From Geometry: Q-Criterion

- The pressure equation is given by the 2-form [Roulstone et al. 2009]

$$\alpha = \mathrm{d}q_1 \wedge \mathrm{d}q_2 - \frac{\Delta p}{2}\mathrm{d}x^1 \wedge \mathrm{d}x^2\,.$$

- Pfaffian is $f_\alpha = \frac{1}{2}\Delta p$
- Hence, the Q-criterion is recovered from the geometry:

$$\begin{aligned} \text{elliptic} \; &\Leftrightarrow f_\alpha > 0 \Leftrightarrow \Delta p > 0\,, \\ \text{hyperbolic} \; &\Leftrightarrow f_\alpha < 0 \Leftrightarrow \Delta p < 0\,, \\ \text{parabolic} \; &\Leftrightarrow f_\alpha = 0 \Leftrightarrow \Delta p = 0\,. \end{aligned}$$

Extras From Geometry: Lychagin–Rubtsov Theorem

- For $\tilde{\alpha} = \frac{1}{\sqrt{|f_\alpha|}}\alpha$, we find $\mathrm{d}\tilde{\alpha} = 0$ if and only if Δp is constant.
- Hence, by the Lychagin–Rubtsov Theorem,

$$\frac{\Delta p}{2} = (\psi_{xx}\psi_{yy} - \psi_{xy}^2)$$

  is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$ iff Δp is constant.
- So this equivalence only applies to some (relatively uninteresting) problems.

Extras From Geometry: Lychagin–Rubtsov Metric

- There is a Lychagin–Rubtsov metric on $T^*\mathbb{R}^2$ given by

$$\hat{g} = \begin{pmatrix} \frac{\Delta p}{2} I & 0 \\ 0 & I \end{pmatrix} .$$

- When pulling back to a classical solution L_ψ, we find

$$\hat{g}|_{L_\psi} = \zeta \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

where $\zeta = \Delta\psi$ is the vorticity.

Summary of Relationship

Δp	> 0	< 0	$= 0$
Dominance	Vorticity	Strain	None
$\alpha\|_{L_\psi} = 0$	Elliptic	Hyperbolic	Parabolic
f_α	> 0	< 0	$= 0$
J^2	$-I_2$	I_2	Singular
$\hat{g}$	Riemannian $(4,0)$	Kleinian $(2,2)$	Degenerate*
$\hat{g}\|_{L_\psi}$	Riemannian $(2,0)$	Kleinian $(1,1)$**	Degenerate*

*These degeneracies are curvature singularities.

**The $\zeta = 0$ degeneracy may occur here and be removable.

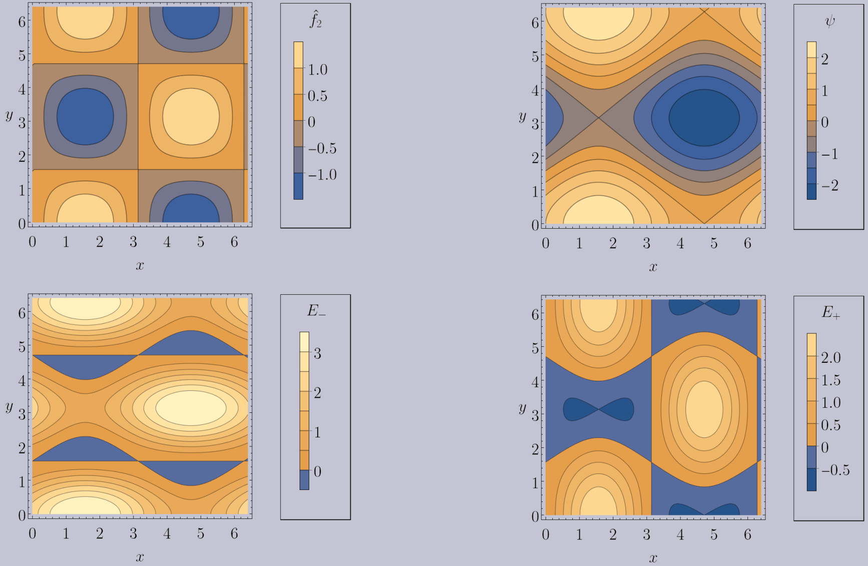
Topology of 2D Vortices

- Let $\Sigma \subset \mathbb{R}^2$ be a simply connected region with $\Delta p > 0$ and boundary given by a closed streamline.
- $\hat{g}_{L_\psi}$ is Riemannian over Σ, which is topologically a disc, so the Gauß–Bonnet theorem yields

$$\int_{\partial L_\psi(\Sigma)} \mathrm{d}s\, \kappa(x(s)) = 2\pi - \int_{L_\psi(\Sigma)} \mathrm{vol}_{L_\psi} \tilde{R}\,.$$

- The mean curvature of the boundary of a "vortex" is described by gradients of vorticity and strain.

2D ABC Flow: $\psi(x,y) = \frac{3}{2}\cos(y) + \sin(x) = -\zeta$



4. Extensions to Curved Background and Higher Dimension

Extension to Riemannian Manifolds

- On a Riemannian manifold (M, g), the approach is similar:

$$\Delta p + \frac{1}{2} R|v|^2 \ (+\nabla_i c^i) = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} \ .$$

- Schematically take

$$\mathrm{d}q_i \to \nabla q_i := \mathrm{d}q_i - \mathrm{d}x^j \Gamma_{ij}{}^k q_k.$$
$$I \to g.$$
$$f_\alpha = \tfrac{1}{2}\Delta p \to f_\alpha = \tfrac{1}{2}\Delta p + \tfrac{1}{4} R|v|^2.$$

- Vorticity/strain dominance $\Leftrightarrow$ $\mathrm{sign}(f_\alpha) \Leftrightarrow$ $\mathrm{type}(\alpha|_L = 0)$

  Pfaffian justifies Q-criterion hold on manifolds (e.g. $\mathbb{S}^2$, basin).

An Alternative Approach in 2D

- Rather than stream function ψ, work with velocity directly and consider solutions $L_v = \{(x, y, v_1(x, y), v_2(x, y))\}$.
- $\alpha|_{L_v} = 0$ gives Poission equation for pressure in terms of vorticity and strain, but now $\omega|_{L_v} = 0$ requires vanishing vorticity (bad!).
- Use a different symplectic form:

$$\begin{aligned}\varpi &= \nabla q_i \wedge \star_g \mathrm{d}x^i \\ &= \mathrm{d}q_1 \wedge \mathrm{d}x^2 - \mathrm{d}q_2 \wedge \mathrm{d}x^1\end{aligned}$$

where $\varpi|_{L_v} = 0$ gives $\nabla_i v^i = 0$.

Higher MA Structure For Fluids

- (ϖ, α) generalise in m dimensions to $(m-1)$-plectic m-forms

$$\varpi = \nabla q_i \wedge \star_g \, \mathrm{d}x^i$$
$$\alpha = \tfrac{1}{2}\nabla q_i \wedge \nabla q_j \wedge \star_g \left(\mathrm{d}x^i \wedge \mathrm{d}x^j\right) - f_\alpha \, \mathrm{vol}_M$$

- This is not a MAS, but can be studied in the same way, with solutions $L_v = \{(x^i, v_i(x))\} \subset T^*M$ and Lychagin–Rubtsov metrics.
- The velocity gradient and background metric alone describe

$$(\hat{g}|_{L_v})_{ij} = A^k{}_i A_{kj} - \tfrac{1}{2} g_{ij} A_{kl} A^{lk} \quad \text{with} \quad A_{ij} = \nabla_j v_i \,.$$

Hamiltonian Reduction (3D)

- 2.5D Euclidean flows have velocity [Ohkitani et al. 2000]

$$v := \left(v_1(x^1, x^2, t), v_2(x^1, x^2, t), x^3\gamma(x^1, x^2, t) + W(x^1, x^2, t)\right) .$$

- They have 1D symmetry generated by
  - $\partial_{x^3} \in \mathfrak{X}(\mathbb{R}^3)$ when $\gamma \equiv 0$.
  - $\partial_{x^3} + \gamma\partial_{q_3} \in \mathfrak{X}(T^*\mathbb{R}^3)$ when $W = c\gamma$ for $c \in \mathbb{R}$.

- The Marsden–Weinstein(–Blacker) reduction principles to obtain the MAS and LR metric for a (potentially compressible) 2D flow.

Hamiltonian Reduction (3D)

- This can also be applied on manifolds with metric of form $g = g_2 + \mathrm{e}^{-2\varphi(x^1,x^2)}\mathrm{d}x^3 \otimes \mathrm{d}x^3$, where we find

$$\nabla_i v^i = -v^i \nabla_i \varphi \quad \text{and} \quad v_i = -\epsilon_{ij}\mathrm{e}^{-\varphi}\nabla^j \psi\,.$$

- This extends Lundgren's transformation and can be studied as before, to relate vortices in 3D to those in 2D.

- While we cannot use Gauß–Bonnet in 3D, we can access topology using helicity density and knot theory

$$(q_i \mathrm{d}x^i \wedge \omega)|_L = v_i \zeta^i \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,.$$

5. Conclusions and Outlook

Summary

- Introduced Monge–Ampère geometry as a tool for studying PDEs, particularly in 2D.
- Discussed application of this geometry to two-dimensional incompressible fluids, replicating and extending the Q-criterion for dominance of vorticity and strain.
- Extended to curved backgrounds and higher dimensions, highlighting the existence of reduction principles to obtain compressible flows.
- Indicated how we could obtain topological information about vortices from our framework.

Open Problems and Ongoing Work

- Can we classify structures like (ϖ, α) in dimension $m > 2$ and what equations are they equivalent to?
- Generalised solutions of Chynoweth–Sewell equation represent weather fronts in [D'Onofrio et al. 2023].
  What about here and can we detect singular behaviour using TDA?
- Can we encode dynamics using the vorticity equation

$$\partial_t \zeta + \nabla(\zeta \cdot v) - \nu \Delta \zeta = 0$$

as a flow equation for solutions L over time t?

Thank you!

Any questions?