Monge–Ampère PDEs And Their Geometry: A Strange Relationship

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Slides available at https://lewisn3142.github.io

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Select References

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Part 1 Recap

Monge–Ampère Equations

- A Monge–Ampère equation is a non-linear, second order PDE, given by quasi-linear combinations of the determinant of the Hessian of function ψ and its minors.
- In $2\mathrm{D}$, they look like

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0\,.$$

  where $A, B, \ldots E$ can depend on $x, y, \psi, \psi_x, \psi_y$ non-linearly.
- We call the equation symplectic if $A, B, \ldots E$ do not depend on ψ.

Monge–Ampère Structures

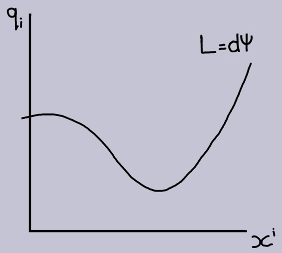
- A Monge–Ampère Structure on $T^*\mathbb{R}^n$ is a pair of forms (ω, α):
  - ω is a symplectic 2-form (ω is non-degenerate and closed).
  - α is an ω-effective n-form ($\alpha \wedge \omega = 0$).
- In 2D, if ω is canonical, we have

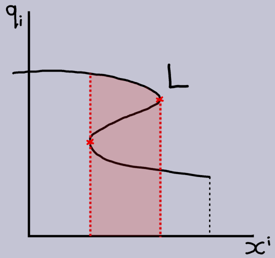
$$\begin{aligned}\alpha = {} & A\, \mathrm{d}q_1 \wedge \mathrm{d}x^2 + B\, (\mathrm{d}x^1 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}x^2) \\ & + C\, \mathrm{d}x^1 \wedge \mathrm{d}q_2 + D\, \mathrm{d}q_1 \wedge \mathrm{d}q_2 + E\, \mathrm{d}x^1 \wedge \mathrm{d}x^2\end{aligned}$$

There is a bijection between these forms and Monge–Ampère equations $\alpha|_L = 0$.

Solutions to Monge–Ampère Equations

- A Generalised Solution is an n-dimensional submanifold $L \subset T^*\mathbb{R}^n$ with $\omega|_L = 0$ and $\alpha|_L = 0$.
- A Classical Solution $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ of a Monge–Ampère equation corresponds to $L = \{(x^1, \cdots, x^n, \psi_{x^1}, \cdots \psi_{x^n})\}$, where ω is canonical and $\omega|_L = 0$ is trivial.





Useful Tools in 2D

- The Pfaffian $f_\alpha = AC - B^2 - DE$.
- The almost (para-)complex structure J given by $\alpha(\cdot,\cdot) = \sqrt{|f_\alpha|}\omega(J\cdot,\cdot)$.
- The Lychagin–Rubtsov metrics $\hat{g}(\cdot,\cdot) = -K(J\cdot,\cdot)$ on $T^*\mathbb{R}^2$, for choice of 2-form K.
- The Lychagin–Rubtsov theorem states that $\alpha|_L = 0$ is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$ if and only if $\mathrm{d}(\frac{1}{\sqrt{|f_\alpha|}}\alpha) = 0$.

The 2D Poisson Equation

- In general dimension, $\Delta p = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij}$, while in 2D, there exists a $\psi \in \mathscr{C}^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned}
&\Delta p = 2(\psi_{xx}\psi_{yy} - (\psi_{xy})^2) \\
&\alpha = \mathrm{d}q_1 \wedge \mathrm{d}q_2 - f_\alpha \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&2f_\alpha = \Delta p
\end{aligned}$$

- This is equivalent to either the Laplace or Wave equation when Δp is constant (by the Lychagin–Rubtsov theorem).

Summary of Relationship

Δp	> 0	< 0	$= 0$
Dominance	Vorticity	Strain	None
$\alpha\|_L = 0$	Elliptic	Hyperbolic	Parabolic
f_α	> 0	< 0	$= 0$
J^2	$-I_2$	I_2	Singular
$\hat{g}$	Riemannian $(4, 0)$	Kleinian $(2, 2)$	Degenerate
$\hat{g}\|_L$	Riemannian $(2, 0)$	Kleinian $(1, 1)$*	Degenerate

*Except when $\zeta = 0$, in which case it is degenerate.

6. Modifications in 2D

Some Riemannian Geometry

- Manifold M – space which looks locally like $\mathbb{R}^n$, has some global structure.
- Riemannian manifold (M, g) – equip M with metric g which defines inner product on tangent vectors $X \cdot Y = X^T g Y$.
- Christoffel Symbols ${\Gamma_{ij}}^k$ – measure of how basis vectors change as we move about manifold (circle vs line).
- Ricci tensor R_{ij} – measures how curved a manifold is at each point.

Navier–Stokes equations in spherical geometry describe ocean/atmosphere dynamics (Joshua Stevens - NASA Earth Observatory)

Extension to Riemannian Manifolds

- On a Riemannian manifold (M, g), the approach is similar:

$$\Delta p + R_{ij} v^i v^j \ (+\nabla_i c^i) = \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij}\ .$$

- Schematically take

$$\mathrm{d}q_i \to \nabla q_i := \mathrm{d}q_i - \mathrm{d}x^j {\Gamma_{ij}}^k q_k.$$
$$I \to g.$$
$$f_\alpha = \tfrac{1}{2}\Delta p \to f_\alpha = \tfrac{1}{2}(\Delta p + R^{ij} q_i q_j).$$

- Vorticity/strain dominance $\Leftrightarrow$ $\mathrm{sign}(f_\alpha) \Leftrightarrow$ $\mathrm{type}(\alpha|_L = 0)$ holds on manifolds (e.g. $\mathbb{S}^2$) with Pfaffian as geometric justification.

Velocity and Divergence in 2D

- Rather than ψ with $v_1 = -\psi_y$, $v_2 = \psi_x$, work with velocity directly and consider solutions $L = \{(x, y, v_1(x, y), v_2(x, y))\}$.
- $\alpha|_L = 0$ gives Poission equation for pressure in terms of vorticity and strain, but now $\omega|_L = 0$ requires vanishing vorticity (bad!).
- Use a different symplectic form:

$$\begin{aligned} \varpi &= \nabla q_i \wedge \star \mathrm{d}x^i \\ &= \mathrm{d}q_1 \wedge \mathrm{d}x^2 - \mathrm{d}q_2 \wedge \mathrm{d}x^1 \end{aligned}$$

where $\varpi|_L = 0$ gives $\nabla_i v^i = 0$.

Jacobi Systems in 2D

- Having $\alpha|_L = 0$ and $\varpi|_L = 0$ simultaneously is equivalent to

$$\nabla \cdot v = 0$$

$$\det \begin{pmatrix} \partial_x v_1 & \partial_y v_1 \\ \partial_x v_2 & \partial_y v_2 \end{pmatrix} = \frac{1}{2}\Delta p$$

- This is a Jacobi System – first order system of PDEs with nonlinearity given by determinants of Jacobian and its minors.

- Extension of Monge–Ampère equations to multiple dependent variables and coupled equations.

7. Poisson Equation in 3D

Setup for Higher Dimensional Fluids

The 2-forms ϖ, α from 2D generalise in n dimensions to n-forms

$$\begin{aligned}\varpi &= \nabla q_i \wedge \star \mathrm{d}x^i \\ \alpha &= \tfrac{1}{2}\nabla q_i \wedge \nabla q_j \wedge \star(\mathrm{d}x^i \wedge \mathrm{d}x^j) - f_\alpha \,\mathrm{vol}_M\end{aligned}$$

With $L = \{(x^i, v_i(x))\}$, the equations $\varpi|_L = 0$ and $\alpha|_L = 0$ are

$$\begin{aligned}\nabla_i v^i &= 0 \\ \Delta p &= \zeta_{ij}\zeta^{ij} - S_{ij}S^{ij}\end{aligned}$$

the divergence free equation and Poisson equation respectively.

Lychagin–Rubtsov Metric on T^*M

- As in 2D, we can find an endomorphism of vector fields J such that, for chosen 2-form K, we have the Lychagin–Rubtsov metric on T^*M

$$\hat{g}(\cdot,\cdot) := -K(J\cdot,\cdot)$$

- The analogous choice of K to 2D again gives

$$\hat{g} = \begin{pmatrix} f_\alpha g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

in (x^i, q_i) coordinates, with signature dictated by the sign of f_α (dominance of vorticity and strain).

Lychagin–Rubtsov Metric on L

- Let $A_{ij} = \nabla_j v_i$ be the velocity gradient tensor.
- The Lychagin–Rubtsov metric on solutions $L = \{(x^i, v_i(x))\}$ is then

$$(\hat{g}|_L)_{ij} = A^k{}_i A_{kj} - \frac{1}{2} g_{ij} A_{kl} A^{lk} \, .$$

- In general, signature change of $\hat{g}|_L$ does not coincide with sign change in f — more complicated relationship.

  $f > 0 \Rightarrow \hat{g}|L$ is Riemannian.
  $\hat{g}|_L$ is Kleinian $\Rightarrow f < 0$.

Almost (Para-)Complex Structure in 3D

- But how do we construct J? Beyond 2D,

$$\alpha(\cdot, \cdots, \cdot) = \varpi(J\cdot, \cdots, \cdot)$$

does not define an endomorphism J in general (more later).

- In 3D, we use the Hitchin endomorphism [Hitchin. 2000]

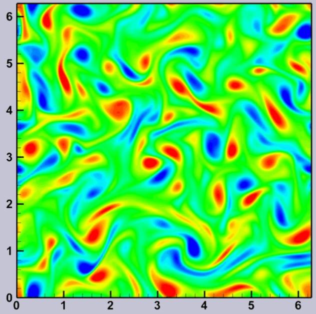
$$\tilde{A}(X)\mathrm{vol}_{M_3} = \alpha(X, \cdot, \cdot) \wedge \alpha$$

- Then $f_\alpha = \frac{1}{6}\mathrm{tr}\left(\tilde{A}^2\right)$ is the Hitchin Pfaffian and $\sqrt{|f_\alpha|}J = \tilde{A}$. Note that J does not depend on ϖ here...

8. Interlude: On the Topology of Vortices

The Importance of Vortices

- Turbulent flows consist of complex interactions of vortex structures.
- In 2D, they combine as they evolve, forming stable structures with circulation/elliptic motion.
- In 3D, one finds knotted/linked tubes which accumulate at small scale.



Vorticity of evolving 2d turbulence at early time (Andrey Ovsyannikov - Ecole Centrale de Lyon)

Topology of 2D Vortices

- Let Σ be a simply connected region (no holes) in 2D, with $\Delta p > 0$ and boundary given by closed streamline.
  All streamlines within Σ are closed [Larchevêque 1993]

- Σ is topologically a disc (can be deformed to one)
  So the Gauß–Bonnet theorem on classical solution L gives

$$\int_{\partial\Sigma} \mathrm{d}s\ \kappa(x(s)) = 2\pi - \int_{\Sigma} \mathrm{vol}_{\Sigma} \hat{R}|_L$$

- $\hat{R}|_L$ is curvature of L, which is given in terms of gradients of vorticity and strain — these dictate the mean curvature of the boundary of the 'vortex.'

Topology of 3D Vortices

- No Gauss–Bonnet Theorem in odd dimensions...
- Let $\theta = q_i \mathrm{d}x^i$. The helicity density is

$$(\theta \wedge \omega)|_L = v_i \zeta^i \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$$

- In ideal conditions, helicity is invariant under volume preserving diffeomorphisms on M and vorticity is conserved.
- Can relate helicity to topological quantities from knot theory (vortex lines) e.g. Jones Polynomial [Liu and Ricca 2012].

3D Flows with Symmetry

- 2.5D Euclidean flows flows have velocity [Ohkitani et al. 2000]

$$v := \left(v_1(x^1, x^2, t), v_2(x^1, x^2, t), z\gamma(x^1, x^2, t) + W(x^1, y^2, t)\right)$$

- We have a 1D symmetry generated by
  – $\partial_{x^3} \in \mathfrak{X}(\mathbb{R}^3)$ when $\gamma \equiv 0$.
  – $\partial_{x^3} + \gamma\partial_{q_3} \in \mathfrak{X}(T^*\mathbb{R}^3)$ when $W = c\gamma$ for some $c \in \mathbb{R}$.

- Shown that for Burgers' Vortex ($W \equiv 0$, $\gamma = \gamma(t)$), dimensional reduction reproduces Lundgren's Transformation and yields a 2D (compressible) flow [Banos et al. 2016].

- Explicitly extended to $\gamma \equiv 0$ in [Napper et al. 2023]

9. Higher Monge–Ampère Structures?

Some Questions... (Ongoing Work)

- Why does $\alpha(\cdot, \cdots, \cdot) = \varpi(A\cdot, \cdots, \cdot)$ not define an endomorphism A for our choice of (ϖ, α)?
- Is there an equivalent pair (linear combination) where A is defined?
- The pair (ϖ, α) is not a Monge–Ampère structure, but what is it?
- Can we classify these new structures and the equations they correspond to?

Compatible Forms and Endomorphisms

- Let $\dim(M) = m$. The pair (ϖ, A), where ϖ is an m-form on T^*M and A is an endomorphism on $\mathfrak{X}(T^*M)$, is <u>compatible</u> if

$$\begin{aligned}\varpi(AX_1, X_2, \cdots X_m) &= \varpi(X_1, AX_2, \cdots X_m) = \cdots \\ &= \varpi(X_1, X_2, \cdots X_{m-1}, AX_m)\,.\end{aligned}$$

- Define $\alpha := \varpi(A\cdot, \cdots)$, then:

  (ϖ, A) compatible $\Leftrightarrow$ α a differential form.

  (ϖ, A) compatible $\Rightarrow$ (α, A) compatible.

- Given 2 non-degenerate m-forms, (ϖ, α), then A given by $\alpha =: \varpi(A\cdot, \cdots)$ is an automorphism if it exists. When does it exist?

Question 1: Endomorphisms For $m = 3$

Non-degenerate 3-forms in 6 dimensions (on T^*M) look like one of the following in some basis $\{e^i\}_{i=1}^6$ [Bryant. 2006]

$$\begin{array}{lll} (1) & e^{123} + e^{456} & f > 0 \\ (2) & e^{136} + e^{426} + e^{235} + e^{145} & f < 0 \\ (3) & e^{135} + e^{416} + e^{326} & f = 0 \end{array}$$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$.

No (ϖ, α) from different classes $(1), (2), (3)$ define an endomorphism A. It is necessary for the signs of their Pfaffians f to match. [N. In Progress]

Questions 1-3: A Framework

- All compatible (ϖ, A) define an $\alpha = \varpi(A\cdot, \cdots)$ of the same class.
  Do we get all forms in the class this way?
  Does a pair of forms in the same class always define an A?
- Given a pair (ϖ, α) in different classes, does there exist a function F such that $(\varpi, \alpha + F\varpi)$ are in the same class?
- If correct, then higher Monge–Ampère structures are (a subset of) pairs of non-degenerate forms from a class of our choice, plus some effectiveness condition.

Some Questions... (Ongoing Work)

- Why does $\alpha(\cdot, \cdots, \cdot) = \varpi(A\cdot, \cdots, \cdot)$ not define an endomorphism A for our choice of (ϖ, α)?
- Is there an equivalent pair (linear combination) where A is defined?
- The pair (ϖ, α) is not a Monge–Ampère structure, but what is it?
- Can we classify these new structures and the equations they correspond to?

Question 4: Classification of Structures

- ϖ from our example was an $(m-1)$-plectic form:
  A closed and non-degenerate m-form. [Cantrijn et al. 2009]
- Given compatible (ϖ, J), where ϖ is $(m-1)$-plectic and J is an almost (para-)complex structure. Set $\alpha := \varpi(J\cdot, \cdots)$.
  Then J is integrable if and only if α is closed. [N. et al, In Progress]
- This is part of the Lychagin–Rubtsov theorem for higher structures.
  - What effectiveness gives almost (para-)complex J?
  - When do we have an $(m-1)$-plectic form in our pair?
  - What Pfaffian tells us the equations we're equivalent to?

10. Summary and Outlook

Summary

- We demonstrated schematically that the results from part 1 could be applied to curved backgrounds and that the Weiss criterion extend to manifolds.
- We showed how a sensible non-canonical choice of symplectic form can be used to couple two equations for n dependent variables.
- The MA structure for the Poisson equation generalises to n dimensions and we can again define a LR metric.
- This generalisation is no longer a MA structure but some multisymplectic generalisation. We took first steps to precisely define these structures.

Outlook

- Does Bryant's classification of 3-forms in 6 dimensions extend to m-forms in $2m$-dimensions?
- Can we find a corresponding notion of 'effectiveness' and 'Pfaffian' for higher structures that ensure A is almost (para-)complex?
- Can we complete the Lychagin–Rubtsov theorem for these higher structures? What equations are we locally equivalent to?
- What do $\varpi, \alpha, \hat{g}$ look like for the Poisson equation if this is accounted for? Does this formulation better describe solutions?

Outlook

- Is it possible to encode dynamics as well as kinematics? Could the vorticity equation

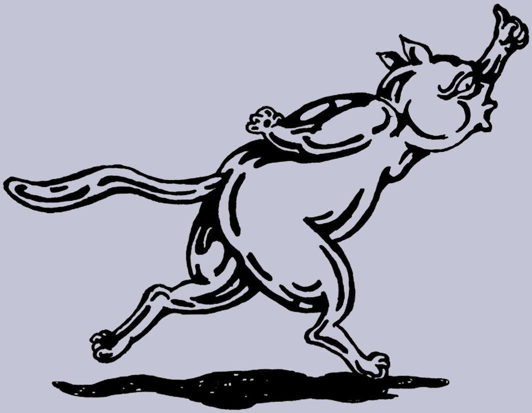
$$\partial_t \zeta + \nabla(\zeta \cdot v) - \nu \Delta \zeta = 0$$

be used as a flow equation over time t for solutions L?

- We studied (locally) classical solutions – what happens when we consider generalised solutions to the Poisson equation which have non-immersive projections?

- In semi-geostrophic theory, these produce degeneracy and type change in $\hat{g}|_L$, representing weather fronts [D'Onofrio et al. 2023] Maybe for us they model vortex sheets?

Thank you!



Any questions?

(Image Credit [Kushner, Lychagin, Rubtsov. 2007])