Monge–Ampère PDEs And Their Geometry: A Strange Relationship

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About Me

My Work:

- Final year Ph.D. researcher at the University of Surrey, UK.
- Visiting La Rochelle as part of a French Embassy SSHN fellowship.
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My Current Projects:

- Monge–Ampère Geometry and Fluid Dynamics.
- Lorentzian Length Spaces and Curvature.
- Cellular Automata on Aperiodic Tilings of Surfaces.

Outline of Talk

1. Monge–Ampère Equations
2. Differential Geometry Glossary
3. From Forms To Equations (And Back Again)
4. Some Useful $(2\mathrm{D})$ Tools
5. An Example: The Poisson Equation For Incompressible Fluids

Part 2: Curved Backgrounds and High-Dimensional Systems

Select References

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1. Monge–Ampère Equations

What is a Monge–Ampère Equation?

- $x^1, x^2 \cdots x^n$ and ψ are the independent and dependent variables.
- Non-linear, second-order PDEs, given by quasi-linear combinations of the determinant of the Hessian of ψ and its minors.

$$\mathrm{Hess}(\psi) = \begin{pmatrix} \psi_{x^1x^1} & \psi_{x^1x^2} & \cdots & \psi_{x^1x^n} \\ \psi_{x^2x^1} & \psi_{x^2x^1} & \cdots & \psi_{x^2x^n} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_{x^nx^1} & \psi_{x^n,x^2} & \cdots & \psi_{x^nx^n} \end{pmatrix}$$

- Minors of a matrix M are $k \times k$ sub-matrices with entries given by the intersections of k rows and k columns of M.

General Form in Two Dimensions

- Set $x^1 = x$ and $x^2 = y$.
- In two dimensions, MAEs take the form

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0\,.$$

  where $A, B, \ldots E$ can depend on $x, y, \psi, \psi_x, \psi_y$ non-linearly.
- If $A, B, \ldots E$ do not depend on ψ, we have a symplectic Monge–Ampère equation, e.g. 2D Poisson.

Some Examples You May Know

- 2D Reaction-Diffusion: $\psi^\alpha \psi_{xx} + [\alpha \psi^{\alpha-1}\psi_x - \psi_t + F(\psi)] = 0.$
- 3D Chynoweth–Sewell: $[\psi_{xx}\psi_{yy} - (\psi_{xy})^2] + \psi_{zz} = 0.$
- 4D Khokhlov–Zabolotskaya: $\psi_{tt} + \psi_{yy} + \psi_{zz} - \psi_{xt} + (\psi_t)^2 = 0.$
- Laplace: $\Delta\psi := \psi_{x^1x^1} + \psi_{x^2x^2} + \cdots + \psi_{x^nx^n} = 0.$
- Wave: $\Box\psi := \psi_{tt} - \psi_{x^1x^1} - \psi_{x^2x^2} - \cdots - \psi_{x^nx^n} = 0$.
- 2D Poisson: $\psi_{xx}\psi_{yy} - (\psi_{xy})^2 = F(x, y).$

2. Differential Geometry Glossary

Terminology: Configuration and Phase Space

- Configuration Space/Background – $\mathbb{R}^n$ with coordinates $x^1, x^2, \cdots x^n$ representing independent variables and points $x \in \mathbb{R}^n$ representing positions in space.
- A particle at point $x \in \mathbb{R}^n$ has an m-dimensional space of possible momenta (one in each direction x^i) with coordinates $q_1, q_2, \cdots q_n$. These will be related to the dependent variable.
- Phase space T^*M – $2m$-dimensional space (manifold), with coordinates $x^1 \cdots x^m, q_1 \cdots q_m$, representing all possible combinations of positions and momenta.

Terminology: Tangent Vectors and Forms

Let M be a space (manifold) with coordinates $y^1, y^2, \cdots y^n$.

- At each $p \in M$, there is a vector space of Tangent Vectors to M. Space has a basis $\partial_{y^1}, \cdots \partial_{y^n}$.
- At each $p \in M$ there is a dual vector space of 1-Forms on M. Space has a basis $\mathrm{d}y^1, \cdots \mathrm{d}y^n$, satisfying $\mathrm{d}y^i(\partial_{y^j}) = \delta^i_j$.
- Coefficients are in $\mathscr{C}^\infty(M)$, so a 1-form on $T^*\mathbb{R}^2$ looks like

$$\alpha = \alpha_i(x^1, x^2, q_1, q_2)\,\mathrm{d}x^i + \tilde{\alpha}^j(x^1, x^2, q_1, q_2)\,\mathrm{d}q_j$$

Terminology: k-Forms and Operators

- <u>Wedge Product $\wedge$</u> – A skew-symmetric bilinear operator on forms. For 1-forms α, β, we have

$$\alpha \wedge \beta = -\beta \wedge \alpha\,,$$
$$(\alpha + \tilde{\alpha}) \wedge \beta = \alpha \wedge \beta + \tilde{\alpha} \wedge \beta\,.$$

- <u>k-Forms</u> – totally skew-symmetric, k-linear operator on tangent vectors $\gamma = \gamma_{i_1,\cdots i_k} \mathrm{d}y^{i_1} \wedge \mathrm{d}y^{i_2} \wedge \cdots \wedge \mathrm{d}y^{i_k}$.

- The wedge product of a k-form α with an ℓ-form β is a $(k+\ell)$-form satisfying $\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha$:

$$\alpha \wedge \beta = (\alpha_i \mathrm{d}y^i) \wedge (\beta_j \mathrm{d}y^j) = (\alpha_i \beta_j) \mathrm{d}y^i \wedge \mathrm{d}y^j\,.$$

Exterior Derivative

- Exterior Derivative – Operator d taking k-forms to $(k+1)$-forms.
- Exterior derivative of a function (think total derivative): $\mathrm{d}(f) = (\partial_{y^i} f)\mathrm{d}y^i = (f_{y^i})\mathrm{d}y^i$.
- Exterior derivative of a 1-form:

$$\begin{aligned} \mathrm{d}(\alpha_i \mathrm{d}y^i) &= (\mathrm{d}\alpha_i) \wedge \mathrm{d}y^i + \alpha_i \mathrm{d}(\mathrm{d}y^i) \\ &= (\partial_{y_j} \alpha_i)\mathrm{d}y^j \wedge \mathrm{d}y^i \,. \end{aligned}$$

- Note: $\mathrm{d}^2 = 0$

3. From Forms To Equations (And Back Again)

Example: Equation from Form

Consider a 2-form on $T^*\mathbb{R}^2$, where x^1, x^2 are our independent variables,

$$\alpha = \mathsf{d}q_1 \wedge \mathsf{d}x^2 - \mathsf{d}q_2 \wedge \mathsf{d}x^1 \,.$$

Define $L := \{(x^1, x^2, \psi_{x^1}, \psi_{x^2})\} \subset T^*\mathbb{R}^2$ (fix q_1 and q_2 at each x).

$$\begin{aligned} \alpha|_L &= \mathsf{d}(\psi_{x^1}) \wedge \mathsf{d}x^2 - \mathsf{d}(\psi_{x^2}) \wedge \mathsf{d}x^1 \\ &= (\psi_{x^1x^1} + \psi_{x^2x^2})\, \mathsf{d}x^1 \wedge \mathsf{d}x^2 \end{aligned}$$

So $\alpha|_L = 0$ if and only if $\Delta\psi = 0$, that is, ψ solves $\Delta\psi = 0$ when L solves $\alpha|_L = 0$.

Symplectic Forms and Equivalent Equations

A symplectic form ω on $T^*\mathbb{R}^n$ is

- a 2-form: skew-symmetric and bilinear,
- closed: $d\omega \equiv 0$,
- non-degenerate: $\omega(X, \cdot) \equiv 0$ if and only if $X \equiv 0$.

The canonical choice is

$$\omega = dq_i \wedge dx^i = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$$

Then $\omega|_L = 0$ is trivial, so $\alpha|_L = 0$ and $(\alpha + \omega)|_L = 0$ are the same equation! Which one do we pick?

Effective Forms in Two Dimensions

- An n-form α on $T^*\mathbb{R}^n$ is called <u>ω-effective</u> if it is skew-orthogonal to ω, i.e. $\alpha \wedge \omega = 0$.
- (Hodge–Lepage–Lychagin) For any 2-form β and symplectic form ω,

$$\beta = \alpha + F\,\omega\,,$$

  for unique ω-effective 2-form α and $F \in \mathscr{C}^\infty(T^*\mathbb{R}^2)$. $\quad (n=2)$
- If $\omega|_L = 0$, all β corresponding to a given α correspond to the same equation. So we only need to consider the ω-effective forms. More formally...

Monge–Ampère Structures

- A <u>Monge–Ampère Structure</u> on $T^*\mathbb{R}^n$ is a pair (ω, α), where ω is a symplectic form and α is an ω-effective n-form [Banos 2002].
- Fixing canonical ω, then $\omega|_L = 0$ trivially and $\alpha|_L = 0$ is a Monge–Ampère equation.
- <u>Classical Solutions</u> $\psi \in \mathscr{C}^\infty(\mathbb{R}^n)$ of the Monge–Ampère equations correspond to choices of $L = \{(x^1, \cdots x^n, \psi_{x^1}, \cdots \psi_{x^n})\}$.
- <u>Generalised Solutions</u> are n-dimensional spaces (submanifolds) $L \subset T^*\mathbb{R}^n$ such that $\omega|_L = 0$ and $\alpha|_L = 0$.

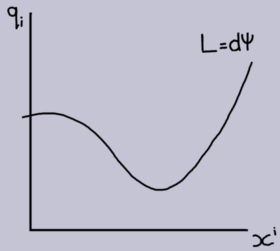
Classical and Generalised Solutions

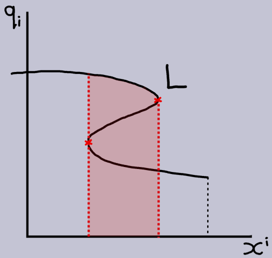
Classical $L = \{(x^1, \cdots x^n, \psi_{x^1}, \cdots \psi_{x^n})\}$:

- $\pi : L \to \mathbb{R}^n$ is bijective (diffeomorphic).

Generalised L:

- When π is not surjective, ψ is not defined on the whole domain.
- When π is not injective, ψ is a multivalued solution. [Vinogradov 1973]
- When π is not immersive, we have Arnold's singularities.





Monge–Ampère Forms in Two Dimensions

For canonical ω on $T^*\mathbb{R}^2$, the ω-effective 2-forms are

$$\begin{aligned}\alpha = {} & A\,\mathrm{d}q_1 \wedge \mathrm{d}x^2 + B\left(\mathrm{d}x^1 \wedge \mathrm{d}q_1 + \mathrm{d}q_2 \wedge \mathrm{d}x^2\right) \\ & + C\,\mathrm{d}x^1 \wedge \mathrm{d}q_2 + D\,\mathrm{d}q_1 \wedge \mathrm{d}q_2 + E\,\mathrm{d}x^1 \wedge \mathrm{d}x^2\end{aligned}$$

.

For classical solutions $L = \{(x, y, \psi_x, \psi_y)\}$, the constraint $\alpha|_{\mathrm{d}\psi} = 0$ gives the Monge–Ampère equation

$$A\psi_{xx} + 2B\psi_{xy} + C\psi_{yy} + D\left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right) + E = 0\,.$$

This is a bijection between Monge–Ampère equations and ω-effective forms.

4. Some Useful (2D) Tools

The Pfaffian in Two Dimensions

- The Pfaffian is defined by $\alpha \wedge \alpha =: f_\alpha \omega \wedge \omega$ where $f_\alpha = AC - B^2 - DE$.
- Note that f_α on L is the determinant of the linearisation matrix of the equation $\alpha|_L = 0$ (Rellich invariant).
- Hence, the Monge–Ampère equation $\alpha|_L = 0$ is

  elliptic $\Leftrightarrow f_\alpha > 0.$
  hyperbolic $\Leftrightarrow f_\alpha < 0.$
  parabolic $\Leftrightarrow f_\alpha = 0.$

Almost (Para-)Complex Structure

- Set $\tilde{\alpha} = \frac{1}{\sqrt{|f_\alpha|}}\alpha$, so that $f_{\tilde{\alpha}} = \mathrm{sign}(f_\alpha)$.
  Multiples of α have the same $\tilde{\alpha}$ (removes choice).
- Define endomorphism of vector fields $J : \mathfrak{X}(T^*\mathbb{R}^2) \to \mathfrak{X}(T^*\mathbb{R}^2)$ by
$$\tilde{\alpha}(\cdot\,,\cdot) =: \omega(J\cdot\,,\cdot) \quad \left(J = \omega^{-1}\tilde{\alpha} \text{ as matrices}\right)\,,$$
- $f_\alpha \lesseqgtr 0 \Leftrightarrow J^2 = \pm I_4$. [Lychagin et al. 1993]

The Lychagin–Rubtsov Theorem and Equivalence

Two Monge–Ampère equations are locally equivalent if there exists a bijection (diffeomorphism) $F : (T^*\mathbb{R}^2, \omega, \alpha_1) \to (T^*\mathbb{R}^2, \omega, \alpha_2)$

$$\omega(F_* \cdot, F_* \cdot) = \omega(\cdot, \cdot) \text{ and } \alpha_2(F_* \cdot, F_* \cdot) = \alpha_1(\cdot, \cdot)$$

[Lychagin–Rubtsov] The following conditions are equivalent:

- $\alpha|_L = 0$ is locally equivalent to $\Box\psi = 0$ or $\Delta\psi = 0$.
- $\mathrm{d}(\tilde{\alpha}) = 0$ ($f_\alpha \lesseqgtr 0$).
- J is integrable ($J^2 = \pm I_2$).

These criteria do not always hold.

The Lychagin–Rubtsov Metric

- Picking a non-degenerate, ω-effective, and α-effective 2-form K, we can define a symmetric, bilinear form

$$\hat{g}(\cdot\,,\cdot) := -K(J\cdot\,,\cdot)$$

called a Lychagin–Rubtsov metric. [Roulstone et al. 2001]

- There exists a choice of K s.t. the metric in (x^i, q_i) coordinates is

$$\hat{g} = \begin{pmatrix} f_\alpha I_2 & 0 \\ 0 & I_2 \end{pmatrix}$$

with signature dictated by the sign of f_α.

5. An Example: The Poisson Equation For Incompressible Fluids

Pressure, Vorticity, and Strain

- Homogeneous, Incompressible Navier–Stokes on $\mathbb{R}^m$

$$\partial_t v^j = -v^i \nabla_i v^j - \nabla_j p + \nu \Delta v^j \; (-c_j) \; .$$

$$\nabla_i v^i = 0$$

Here $\nabla_i := \partial_{x^i}$, time t is a parameter, ν is viscosity and v^i is the x^i-component of velocity.

Pressure, Vorticity, and Strain

- Homogeneous, Incompressible Navier–Stokes on $\mathbb{R}^m$

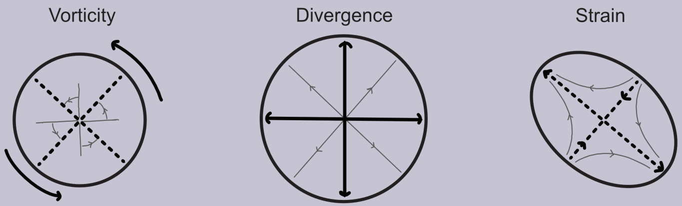
$$\partial_t v^j = -v^i \nabla_i v^j - \nabla_j p + \nu \Delta v^j \; (-c_j)\,.$$

- Taking the divergence and applying $\nabla_i v^i = 0$ one finds

$$\zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} = \Delta p \; (+\nabla_i c^i)\,.$$

where $\zeta_{ij} = \frac{1}{2}(\nabla_j v_i - \nabla_i v_j)$ is the vorticity form and $S_{ij} = \frac{1}{2}(\nabla_j v_i + \nabla_i v_j)$ is the strain-rate tensor.

Vorticity, Divergence, and Strain



Based on Figure from Clough et al. 2014

$$(\zeta_{ij})_{2D} = \frac{1}{2}\begin{pmatrix} 0 & \zeta \\ -\zeta & 0 \end{pmatrix} \qquad (\zeta_{ij})_{3D} = \frac{1}{2}\begin{pmatrix} 0 & \zeta_3 & -\zeta_2 \\ -\zeta_3 & 0 & \zeta_1 \\ \zeta_2 & -\zeta_1 & 0 \end{pmatrix}$$

Pressure Equation in Two Dimensions

- In 2D, there exists a stream function $\psi \in \mathscr{C}^\infty(\mathbb{R}^2)$ such that $v^1 = -\psi_y$ and $v^2 = \psi_x$.
- The constraint $\nabla_i v^i = 0$ is trivially satisfied and the pressure equation is a Monge–Ampère equation for ψ

$$\Delta p = 2\left(\psi_{xx}\psi_{yy} - (\psi_{xy})^2\right) .$$

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$$\zeta_{ij}\zeta^{ij} - S_{ij}S^{ij} = \Delta p = 2\left(\psi_{xx}\psi_{yy} - (\psi_{xy})^2\right) .$$

- *Vorticity dominates* $\Leftrightarrow \ \Delta p > 0 \ \Leftrightarrow$ *Elliptic equation.*
  Strain dominates $\Leftrightarrow \ \Delta p < 0 \ \Leftrightarrow$ *Hyperbolic equation.*
  No dominance $\Leftrightarrow \ \Delta p = 0 \ \Leftrightarrow$ *Parabolic equation.*
  [Weiss 1991, Larchevêque 1993]

What Can the Geometry Tell Us?

- Which differential form α corresponds to the equation?
- Does the geometry see vorticity/strain dominating regions?
- Can we apply the LR theorem to get (local) solutions of the Poisson equation from the Laplace or Wave equation?
- Does the Lychagin–Rubtsov metric pick up on any interesting properties of solutions?
- What does this tell us about solutions to Euler/Navier–Stokes?

Question 1: What Differential Form?

One can recover the pressure equation

$$\frac{\Delta p}{2} = \left(\psi_{xx}\psi_{yy} - \psi_{xy}^2\right)$$

by choosing the Monge–Ampère form [Roulstone et al. 2009]

$$\alpha = \mathsf{d}q_1 \wedge \mathsf{d}q_2 - f_\alpha \mathsf{d}x^1 \wedge \mathsf{d}x^2\,,$$

Question 2: Does Geometry See Existing Results?

- The Pfaffian of α is given by

$$f_\alpha = \frac{\Delta p(x,y)}{2}\,.$$

  Recall that the sign of f_α tells us if a Monge–Ampère equation is elliptic/hyperbolic/parabolic.

- Hence, we have a geometric justification for the Poisson equation being:

$$\begin{aligned} \text{elliptic} &\Leftrightarrow \Delta p > 0\,, \\ \text{hyperbolic} &\Leftrightarrow \Delta p < 0\,, \\ \text{parabolic} &\Leftrightarrow \Delta p = 0\,. \end{aligned}$$

Question 3: Do We Have Equivalence?

- For $\tilde{\alpha} = \frac{1}{\sqrt{|f_\alpha|}}\alpha$, we find $\mathrm{d}\tilde{\alpha} = 0$ if and only if Δp is constant.
- Hence, by the Lychagin–Rubtsov Theorem,

$$\frac{\Delta p}{2} = (\psi_{xx}\psi_{yy} - \psi_{xy}^2)$$

  is locally equivalent to $\Delta\psi = 0$ or $\Box\psi = 0$ if and only if Δp is constant.
- So this equivalence cannot be applied to some physical problems.

Question 4: What Does the Metric on $T^*\mathbb{R}^2$ Say?

The Lychagin–Rubtsov metric on $T^*\mathbb{R}^2$ given by

$$\hat{g} = \begin{pmatrix} \frac{\Delta p}{2} I & 0 \\ 0 & I \end{pmatrix}$$

is

$$\text{Riemannian} \Leftrightarrow \Delta p > 0.$$
$$\text{Kleinian} \Leftrightarrow \Delta p < 0.$$
$$\text{Degenerate} \Leftrightarrow \Delta p = 0.$$

These degeneracies are where the scalar curvature of $\hat{g}$ blows up (curvature singularities) — they persist under coordinate changes.

Question 4: What Does the Metric on L Say?

- If we restrict $\hat{g}$ to the classical solution $L = \{(x, y, \psi_x, \psi_y)\}$, we get

$$\hat{g}|_L = \zeta \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix}$$

  where $\zeta = \Delta\psi$ is the vorticity.

- Degenerate when $\zeta = 0$ or $\Delta p = 0$.
  Riemannian when $\Delta p > 0$.
  Kleinian when $\Delta p < 0$.

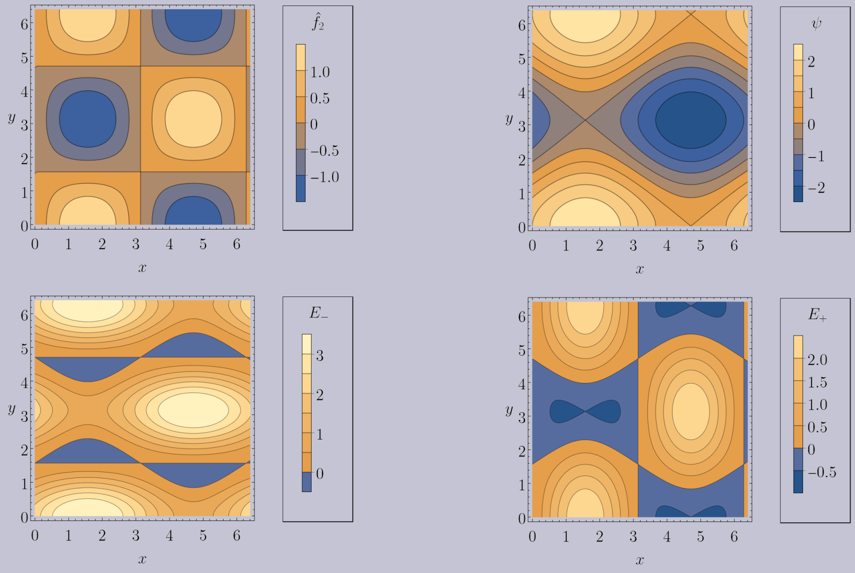
- Degeneracy when $\zeta = 0$ not always curvature singularity.

Summary of Relationship

Δp	> 0	< 0	$= 0$
Dominance	Vorticity	Strain	None
$\alpha\|_L = 0$	Elliptic	Hyperbolic	Parabolic
f_α	> 0	< 0	$= 0$
J^2	$-I_2$	I_2	Singular
$\hat{g}$	Riemannian $(4, 0)$	Kleinian $(2, 2)$	Degenerate
$\hat{g}\|_L$	Riemannian $(2, 0)$	Kleinian $(1, 1)$*	Degenerate

*Except when $\zeta = 0$, in which case it is degenerate.

2D ABC Flow: $\psi(x,y) = \frac{3}{2}\cos(y) + \sin(x) = -\zeta$



Summary and Outlook

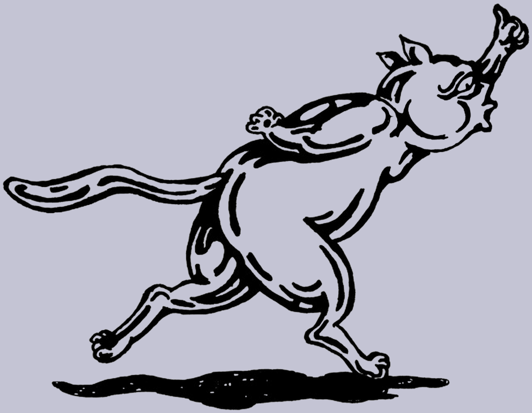
This time we:

- Introduced MA equations and their geometry as a tool for studying solutions and discussed the LR theorem and metric.
- Applied this to the Poisson equation in 2D to recover existing results and pick up interesting features of solutions.

Next time we will:

- show how our fluid dynamical results can be extended to curved backgrounds, such as the sphere, and what they tell us about the topology of vortices.
- discuss what happens when we take a non-canonical symplectic form and how to consider the Poisson equation in three dimensions, when it is not Monge–Ampère.

Thank you!



Any questions?

(Image Credit [Kushner, Lychagin, Rubtsov. 2007])

Part 2: Curved Backgrounds and High-Dimensional Systems