Modelling Laboratory

Modelling species interactions and its applications to FOREX market prediction

March 14, 2022

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Academic Year 2021/22

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1 Introduction

The purpose of this essay is twofold:

- The first and foremost is to explore mathematical models for interaction between species. We generalise the Lotka-Volterra equations to an arbitrary (finite) number of populations and propose three difference equations that also model these interactions, as fitting their parameters is much easier.
- Secondly, we test whether Foreign Exchange market currencies can be accurately predicted using similar models with real world data. The idea was presented by supervisor José Juan López Espín, although we later found a precedent in [1], where Lee *et al.* tries to forecast two competing markets at the Korean stock market using this precise idea.

The motivation behind the second objective is the following observation: market products (currencies in the FOREX case) can be considered as populations competing for the available resources (the buyer's attention and money, investors in the FOREX case). We expect an analogous cyclic behaviour to that of the classic Lotka-Volterra model, as when a currency gets weaker its price drops, making it more attractive to investors.

2 Models for species interactions

In nature, animal species can interact with one another in several ways. According to National Geographic [2], the relationship between species can be generally classified as:

- **Mutualism**: both organisms benefit from the interaction.
- **Commensalism**: one organism benefits from the interacion, which is harmless to the other organism.
- **Predation**: one organism, the predator, feeds by eating the other, the prey.
- **Parasitism**: one organism benefits by causing damage to the other one.
- **Competition**: two organisms compete for the same resources.

This section will explore different mathematical models for populations that relate in these ways, extending the material covered in [3]. The accuracy of these models will be assessed in the next section.

2.1 The original Lotka–Volterra equations

During World War I and the consequent decrease in fishing, biologists found a greater increase in predator species' populations in the Mediterranean Sea than that of the prey's. This counter intuitive phenomenom was studied by mathematician Vito Volterra [3].

Volterra developed a model for a Predator-Prey interaction based on the following hypothesis [3]:

- No contention for resources is considered for preyed on species, which results in exponential growth according to a ratio a in the absence of predators.
- In the absence of preys to feed upon, predators die according to a constant ratio c.
- The number of contacts between predators and preys is proportional to both populations, and has a negative effect on the preys (b) and a positive one on the predators (d).

Denoting $x(t)$ and $y(t)$ as the amount of preys and predators, respectively, the resulting ODE system is the following:

$$\begin{cases} \frac{dx}{dt} = & ax - bxy \\ \frac{dy}{dt} = & -cy + dxy. \end{cases}$$

Chemical physicist Alfred J. Lotka reached the same model independently while studying a chemistry problem, resulting in its well-known name.

2.2 Generalised model for species interaction

The previous model presents two caveats. Firstly, it is restricted to two species only, a very unrealistic situation in a stock market type of ecosystem. Secondly, it only models one of the five main types of interactions listed previously, although only minor tweaks are required to solve this problem. The solution of both can be found in [4], albeit they only extend the model to three species.

The generalisation of the original model is simple. For each population the growth rate in isolation remains the same, whereas we now have to consider interactions with every other population. Let $P(t) = (P_1(t), \ldots, P_n(t))$, where $P_i(t)$ denotes the population of the i-th species at instant t. For $i = 1, \ldots, n$ the resulting equation is

$$P_i' = g_i P_i + \sum_{j=1, j\neq i}^{n} d_{ij} P_i P_j = \left(g_i + \sum_{j=1, j\neq i}^{n} d_{ij} P_j \right) \cdot P_i \tag{1}$$

for suitable real constants g_i, d_{ij}. Here g_i is the growth rate in isolation of the i-th species, and d_{ij} is the effect of the interaction between the i-th and j-th species on i-th's population. For a predation relationship we shall assume that $d_{ij} \cdot d_{ji} < 0$. We note now that whether d_{ij} is greater, equal or less than 0 determines if the i-th species benefits, is indifferent to or is damaged by the interaction with the j-th one, and thus unrestricted $d_{ij} \in \mathbb{R}$ model every type of interaction described at the beginning of the section.

Equation 1 can be expressed in matrix form as

$$P' = (g + DP) \odot P, \tag{2}$$

where $\odot$ denotes the element-wise product of same-dimensional matrices, also called the Hadamard Product. Note that

$$g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}, D = (d_{ij}) \text{ with } d_{ii} = 0 \text{ for each } i = 1, \ldots, n.$$

2.3 Logistical growth in isolation

The previous model assumes exponential growth in isolation, as in the Malthus model. This is not realistic, as Verhulst pointed out, since it does not take into account that resources in an ecosystem are limited. This is true as well for the FOREX market, since wealth is finite in a determined period of time. This means that since money is only a representation of wealth used to easily exchange goods and services, increasing the monetary supply only redounds in a decrease of the value of the money, unless a significant increase in the capacity to create wealth has been made. The resulting model can be expressed through the same equation, except now the diagonal elements of D are negative.

2.4 Discrete models

Lee *et al.* [1] transformed the equations into a discrete model to better fit discrete time data. Even though our model is meant to predict prices in short time spans, this is also the case for us. We also note that difference equations are typically easier to fit than differential equations, as we will see. We explore three models:

(MD-A) Lee *et al.* [1] proposed a Beverton-Holt-like model, resulting in the following equations for two species:

$$\begin{cases} x_{n+1} = \frac{\alpha_1 x_n}{1+\beta_1 x_n + \gamma_1 y_n} \\ y_{n+1} = \frac{\alpha_2 y_n}{1+\beta_2 y_n + \gamma_2 x_n} \end{cases}.$$

The simplest interpretation goes as follows. The coefficient α_i represents the ratio of growth of the i-th species, β_i represents the limitation of resources for each population in isolation (as in the logistical model), and $\gamma_i \geq 0$ the *negative* effect of the other species. This limits the type of interaction to competition or commensalism. The system is naturally extended to n populations, which takes the following matrix form:

$$P_{n+1} = (\alpha \odot P_n) \oslash (\mathbf{1}_n + BP_n),$$

where $\alpha \in \mathbb{R}^n$, $B \in \mathcal{M}_{n\times n}(\mathbb{R}_{\geq 0})$ and $\mathbf{1}_n = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^T$ of dimension n. Here $\oslash$ represent the element-wise (or Hadamard) division.

(MD-B) We may consider the approximation $\frac{dx}{dt} \approx \frac{x(t+h)-x(t)}{h}$, which transforms Equation 2 into

$$\begin{aligned} P(t+h) &\approx P(t) + h(g + DP(t)) \odot P(t) \\ &= (\mathbf{1}_n + hg + hDP(t)) \odot P(t). \end{aligned} \tag{3}$$

As $\mathbf{1}_n$ and $h > 0$ can be included in g and D, this leads to the following difference equation:

$$P_{n+1} = (g + DP_n) \odot P_n.$$

(MD-C) The last model presents the same problem as the logistic model seen in [5]: it is well defined in $\mathbb{R}^n$, but it only makes sense biologically in $[0,\infty)^n$, where it can lead to negative values. We try an analogous solution to that of Ricker:

$$P_{n+1} = \exp(g + DP_n) \odot P_n,$$

where the exponential is applied element-wise. The same remarks can be made here. Firstly, for small values of $g + DP_n$ the equation approximates the linear one. Secondly, if no species benefit from the interaction with others, the same probabilistic interpretation is valid. For each $i = 1, \dots, n$ the i-th species in the $(n+1)$-th generation has a potential population of $s_i P_n^i$, where $s_i > 0$. It has to survive competition with the j-th species, with probability q_{ij} (possibly 1 if the interaction does not have any effect) to survive each match. Assuming independence the probability of survival is $q_{ij}^{s_j P_n^j}$ against the j-th species, and in general $\prod_{j=1}^n q_{ij}^{s_j P_n^j}$. We reach

$$\begin{aligned} P_{n+1}^i &= s_i P_n^i \prod_{j=1}^n q_{ij}^{s_j P_n^j} \\ &= s_i P_n^i \prod_{j=1}^n \exp\left(s_j \ln(q_{ij}) \cdot P_n^j\right). \end{aligned} \tag{4}$$

By taking $g_i = \ln s_i$ and $d_{ij} = s_j \ln q_{ij}$ we reach the original system. We note that $d_{ij} > 0$ implies $q_{ij} > 1$, which cannot be interpreted in the same probabilistic terms. In this case the i-th species benefits from the interaction with the j-th one, and we may regard q_{ij} as the factor by which the encounter with each individual of the j-th kind improves the potential i-th population in the next generation.

2.5 Equilibrium Points

In this section we are going to study the equilibrium points of the models previously stated.

In the case of the continuous model (Equation 2), it is easy to see that for non null populations (otherwise we can reduce the dimension of the problem) the equilibrium points are the solutions of the system

$$Dx = -g.$$

Necessary and sufficient conditions for the stability of the system are given in [6], but it is out of the scope of this essay due to its complexity.

We explore now the discrete models.

(MD-A) We need to solve

$$P = (\alpha \odot P) \oslash (\mathbf{1_n} + BP) \iff (\mathbf{1_n} + BP) \odot P = \alpha \odot P,$$

and by cancelling P we get

$$\mathbf{1_n} + BP = \alpha \iff BP = \alpha - \mathbf{1_n}.$$

(MD-B) The equilibrium points are the solutions of

$$P = (g + DP) \odot P \iff \mathbf{1_n} - g = DP.$$

(MD-C) It is necessary and sufficient that

$$P = \exp(g + DP) \odot P \iff \mathbf{1_n} = \exp(g + DP)$$

for P to be an equilibrium point, or, equivalently, that P is a solution of

$$g + DP = 0 \iff DP = -g.$$

3 Fitting the models

We now focus our interest in the estimation of the parameters of our models given enough experimental data. It is important to note that we aim to predict the market in short intervals and speed of calculation of the parameters can be key.

The case of all discrete models can be reduced to fitting an affine transformation to the data set, this is, finding $a \in \mathbb{R}^n, B \in \mathcal{M}_{n\times n}$ that best aproximate $a + BX_i = Y_i$ for every X_i, Y_i calculated from of our data points. By putting —as in [7]—

$$\tilde{B} = \begin{pmatrix} B \mid a \end{pmatrix}; \tilde{X}_i = \begin{pmatrix} X_i \\ \hline 1 \end{pmatrix}$$

we now have to fit $\tilde{B}\tilde{X} = Y$, where $\tilde{X}$ is the matrix with columns $\tilde{X}_i$ and Y the corresponding matrix with columns Y_i. This is a linear system with (probably) much more equations than variables, and it is well know that the least square fit can be computed as the solution of

$$\tilde{B}\tilde{X}\tilde{X}^T = Y\tilde{X}^T.$$

Let now P_n for $n = 1, ..., N$ be our data points. We will show how to get to an affine system for each discrete model to apply the previous approach.

(MD-A) For the first model, we can make the following transformation

$$P_n \oslash P_{n+1} = (\mathbf{1}_n + BP_n) \oslash \alpha,$$

and by including alpha into the other parameters, we finally obtain

$$R_n = P_n \oslash P_{n+1} = g + DP_n.$$

(MD-B) For the second difference equation, the transformation is as follows:

$$R_n = P_{n+1} \oslash P_n = g + DP_n.$$

(MD-C) The transformation is very similar to that of (MD-B):

$$R_n = \log P_{n+1} - \log P_n = g + DP_n.$$

All discrete models present one caveat: they require certain restrictions on the values their parameters can take, but this type of fitting does not account for that. For clearly unreasonable values we will simply discard the model, as fixing this situation is out of the scope of the essay. For values that are very close to being acceptable we may regard them as numerical errors and ignore or truncate them (checking if the model still fits the data).

Fitting the generalised continuous model is not a trivial problem, specially if a fast algorithm is needed. We define a loss function as the square error of a numerically calculated solution, and minimise it. We observe that the Jacobian is not easy to calculate analytically, and that the loss function is very expensive (we calculate a solution of the ODE in each call), making numerical estimation of the Jacobian unfeasible. We thus use a solver that does not require derivatives, such as the function *fminsearch* in Matlab. The community[1] made a variation that allows bounding of the variables, which solves the problem mentioned earlier for discrete models. Since most algorithms require an initial estimation, we use the one provided by (MD-B), which is meant to approximate the continuous model. Note that the parameters are not the same, a transformation with h and $\mathbf{1}_n$ is required (see Equation 3).

3.1 Fitting example

We adjust the parameters to fit the classical example of the American hares and the Canadian lynxes from 1900 to 1930 [3] by applying the methods explained earlier.

We will start with (MD-A), even though we should not expect good results, as this model is only suitable for a situation in which the species relate via competition or commensalism. Figure 1 shows both empirical data and the model's approximation.

As expected, it does not model the situation accurately, and this should be taken into account in the election of the model. It will be, in many situations, preferable to use a more flexible model, in which different types of interactions could be detected. We do however expect relationships between most currencies to be of competition.

(MD-B) does not present this problem, and a more accurate model is obtained. The results can be consulted in Figure 2. Both (MD-B) and (MD-C) (Figure 3) present the same delay-like type of inaccuracy.

[1] Visit [8] for further details.

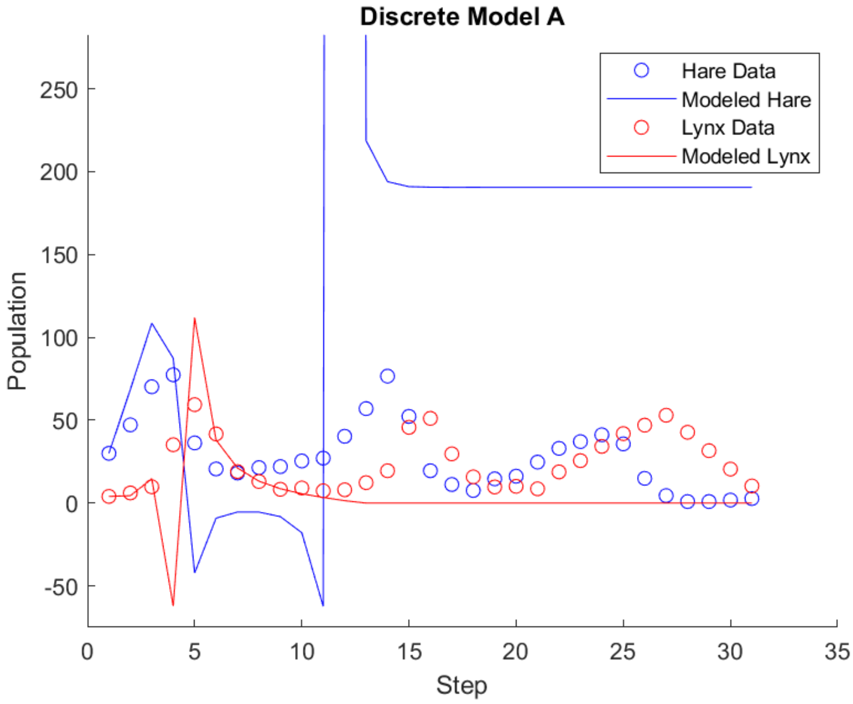

Figure 1: MD-A fitted to data

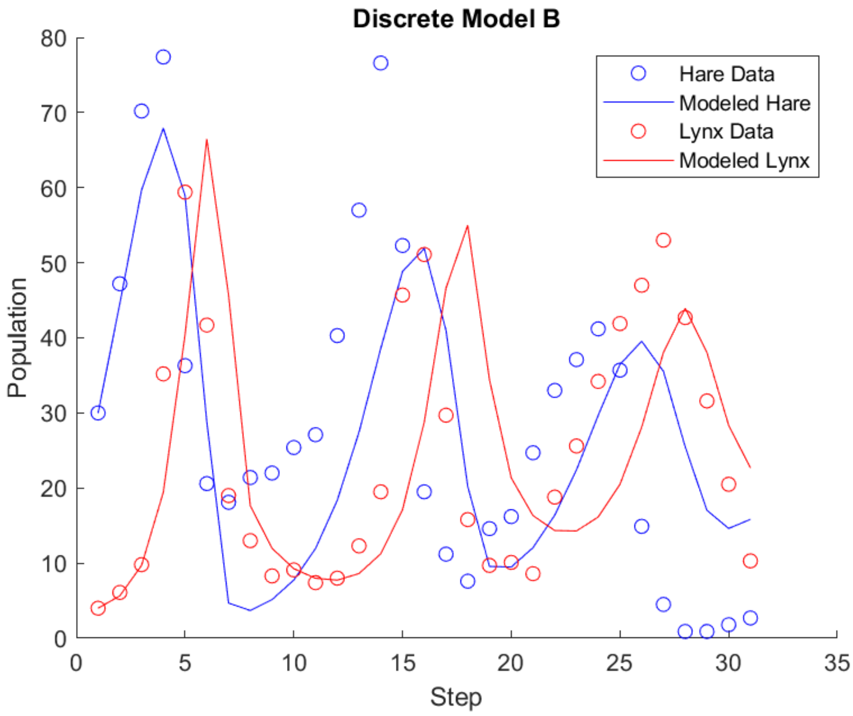

Figure 2: MD-B fitted to data

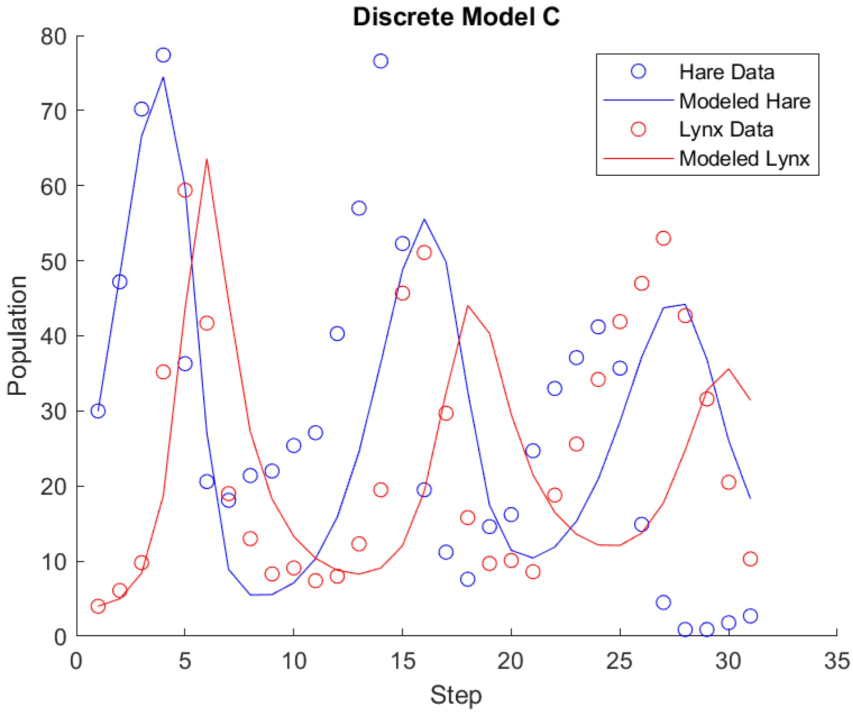

Figure 3: MD-C fitted to data

A better fit is obtained using the continuous models, both with logistic growth in isolation or without it. The results of the former are presented in Figure 4. We can observe the cycles are accurately modelled and the estimations for populations are fairly precise. For the latter of the models, the results can be seen in Figure 5. The cycles are also well described by the model, but the amplitude does not adapt, resulting in a worse fit. This model seems to find difficulties in representing the variation in the population peak sizes of the different species.

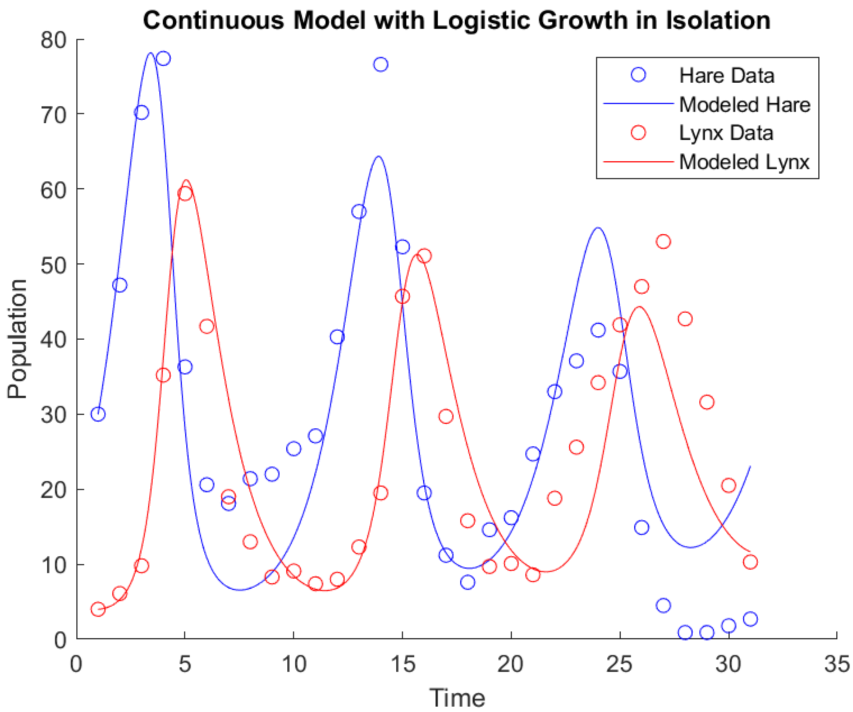

Figure 4: Continuous model with logistic growth in isolation fitted to data

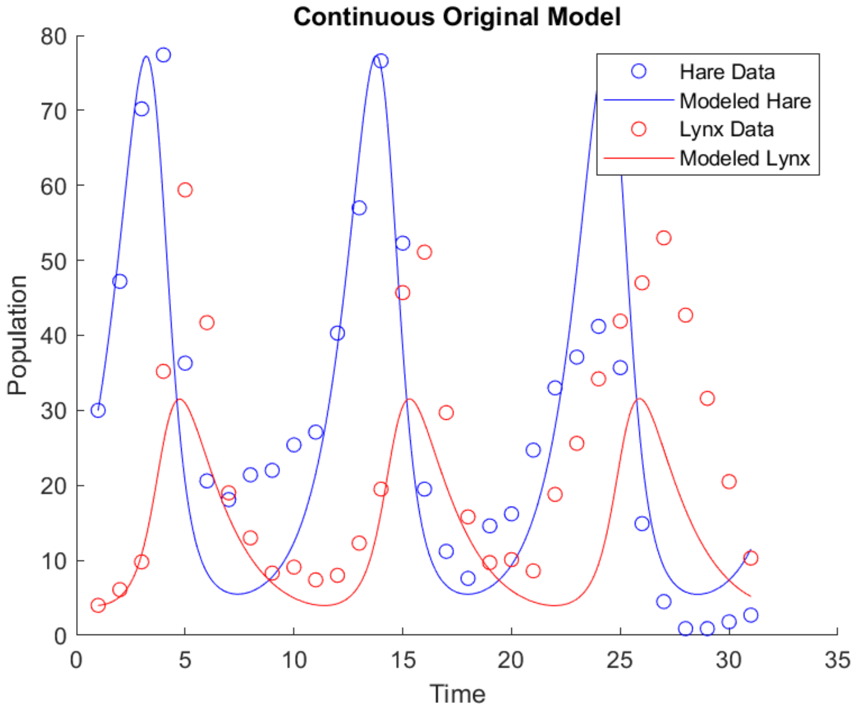

Figure 5: Continuous model without logistic growth in isolation fitted to data

4 Application to the FOREX market prediction

4.1 Introduction to FOREX

The sources for this subsection can be found in [9] and [10].

The foreign exchange market, or FOREX (FX) for short, is a decentralised market place that facilitates the buying and selling of different currencies. This takes place over the counter (OTC) instead of on a centralised exchange.

Forex trading is a term used to describe individuals that are engaged in the active exchange of foreign currencies, often for the purpose of financial benefit or gain. That can take on the form of speculators, who are looking to buy or sell a currency with the goal of profiting from the currency's price movement; or it can be a hedger that's looking to protect their accounts in the event of an adverse move against their own currency positions.

Moreover, one unique aspect of the Forex market is the manner in which prices are quoted. Because currencies are the base of the financial system, the only way to quote a currency is by using other currencies. An explanatory example:

Let's take the Euro and let's say a trader has optimistic projections for the European economy and would thusly like to get long the currency. But, let's say this investor is also bullish for the US economy, but is bearish for the UK economy. Well, in this example, the investor isn't forced to buy the Euro against the US Dollar (which would be a long EUR/USD trade); and they can, instead, buy the Euro against the British Pound (going long EUR/GBP).

This affords the investor or trader that extra bit of flexibility, allowing them to avoid 'going short' the US Dollar to buy the Euro and, instead, allowing them to buy the Euro while going short the British Pound.

One important distinction of a Forex quote is the convention: The first currency listed in the quote is known as the 'base' currency of the pair, and this is the asset that's being quoted. The second currency in the pair is known as the 'counter' currency, and this is the convention of the quote, or the currency that's being used to define the value of the first currency in the pair.

Let's take EUR/USD as an example:

The Euro is the first currency in the quote, so the Euro would be the base currency in the EUR/USD currency pair. The US Dollar is the second currency in the quote, and this is the currency that the EUR/USD quote is using to define the value of the Euro.

So, let's say that the EUR/USD quote is 1.3000. That would mean that 1 Euro is worth \$1.30. If the price moves up to \$1.35 – then the Euro would have increased in value and, on a relative basis, the US Dollar would've decreased in value.

In a nutshell, the foreign exchange market works like many other markets in that it's driven by supply and demand. Using a very basic example, if there is a strong demand for the US Dollar from European citizens holding Euros, they will exchange their Euros into Dollars. The value of the US Dollar will rise while the value of the Euro will fall. Keep in mind that this transaction only affects the EUR/USD currency pair and will not necessarily cause the USD to depreciate against the Japanese Yen, for example.

In reality, the above example is only one of many factors that can move the FX market. Others include broad macro-economic events like the election of a new president, or country specific factors such as the prevailing interest rate, GDP, unemployment, inflation and the debt to GDP ratio, to name a few.

From the investor's point of view, there are two possible investing approaches:

- **Long**: when the investment is made in the prospect of future gains in the long term. Usually, long term decisions in economy are those which are though to earn profit in a period longer than 6 months. Although today's rapid computerised economy is shortening the time which is to be considered a long term investment.
- **Short**: short term decisions are those whose objective is to earn profit in a short period of time. Nowadays, a short term decision could be to buy and sell a share in a period of 5 minutes, even several times.

The scope in which we believe our model to be useful is in the short term Forex investment. Using available data sources to estimate the amount of currencies that are being exchanged in a certain period of time. This prediction is useful for the purpose of predicting Forex rates, as it is a reflection of the current supply and demand of the different currencies involved.

4.2 Interpretation of the FOREX market as an ecosystem

As stated in the introduction, we interpret currencies as populations competing for the the investor's attention and money. Cyclic behaviour is expected, as drops in prices might restart the investments in certain currencies and make its price rebound.

Now, let's say we are an investment fund and we have historical data of two competing companies with stock of different currencies. For simplicity, suppose we are only interested in EUR and USD. We have access to their data of EUR and USD currencies, and their transactions, which have been made according to a certain decision methodology. This information could be useful to predict (added to many other variables) the exchange rate EUR/USD. We are thus interested in predicting their next currency movement, so we can better estimate the future exchange rate and improve our investment decision.

4.3 Evaluation of the models

We do not have access to FOREX market offer data, as the company involved requires validation of the models before handing it. Every year, the European Central Bank (ECB), the Federal Reserve Economic

Data (FRED) and the Swiss Central Bank (SCB) release data on how many EUR, USD and CHF, respectively, are in circulation. We believe this data is suitable to obtain a first validation for our model. The mentioned data has been obtained from [11], [12] and [13][2].

The fitting methodology is as described in 3. All discrete models seem to accurately fit the experimental data. Results for (MD-A) are presented in Figure 6. Note that the amounts in circulation are presented in a logarithmic scale for better visualisation. For both (MD-B) and (MD-C) very similar results are obtained, they can be consulted in Figure 7 and Figure 8, respectively.

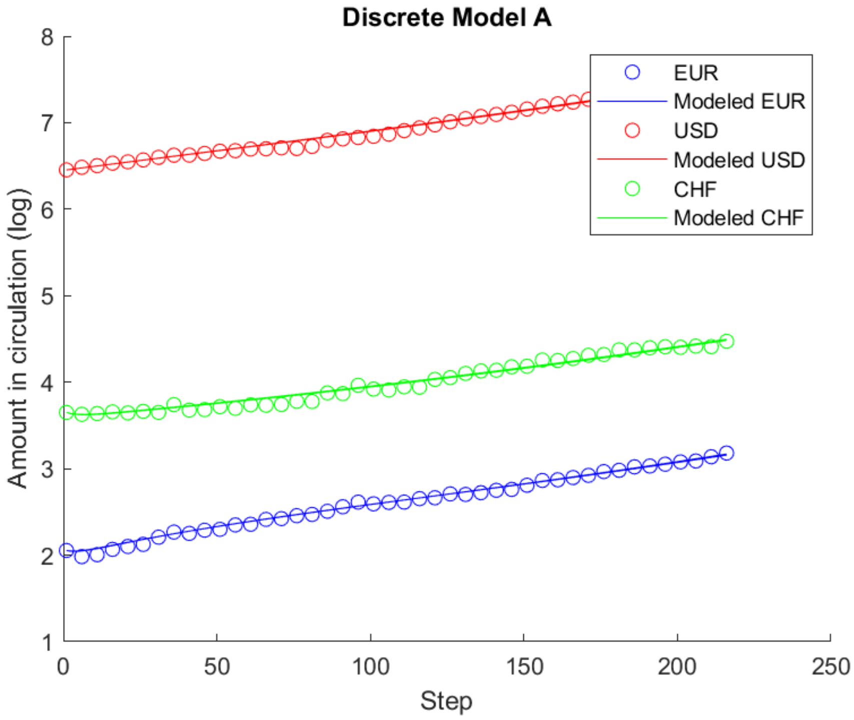

Figure 6: MD-A fitted to currencies data

[2]USD data has been slightly modified, as some data points were clearly mistaken. For example, there were cases in which at month N, the USD amount in circulation was X; in month N+1 it was $\sim X/10$; and in month N+2 it was $\sim X$ again. We've multiplied those values by 10.

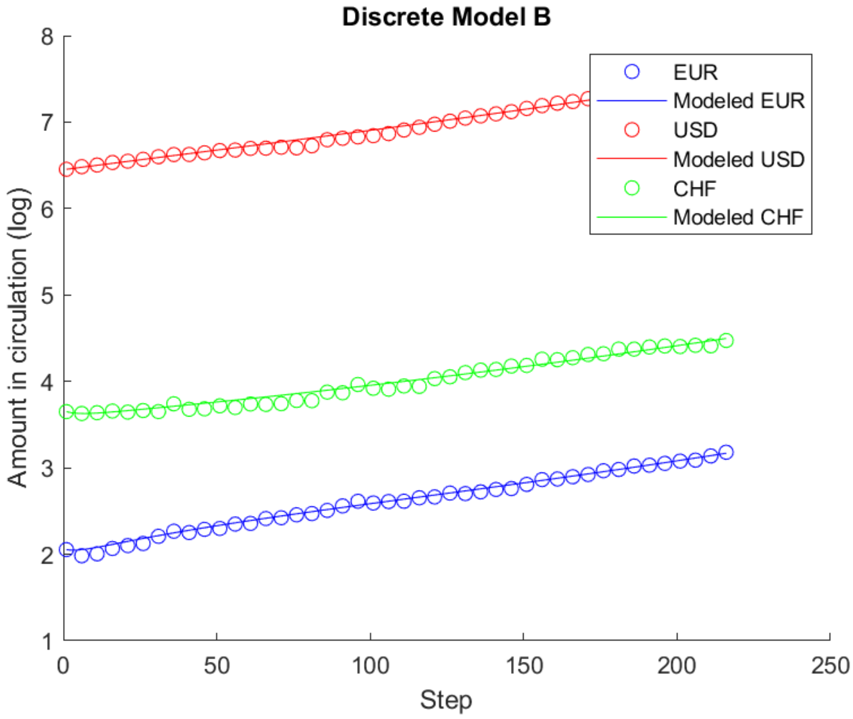

Figure 7: MD-B fitted to currencies data

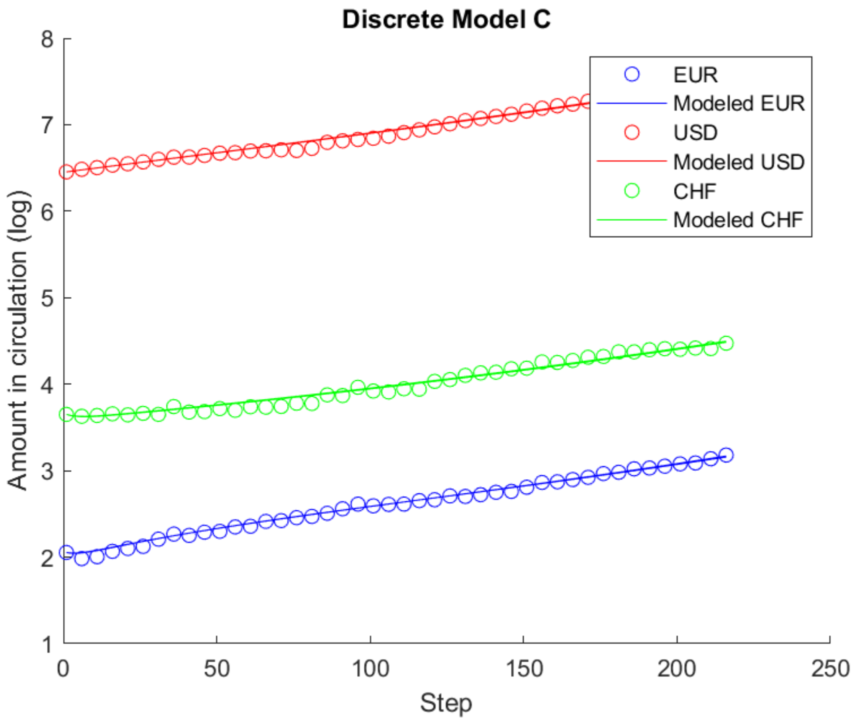

Figure 8: MD-C fitted to currencies data

5 Conclusions

We reach two major conclusions in this essay.

1. Even though discrete models are easier to fit, they appear worse for modelling interactions between species. It is only fair to note that a very small data set was used, and the species did not follow any seasonal behaviour, which can benefit difference equations.
2. Evaluation with preliminary data shows that these models might work in predicting FOREX market currency offer. It is clear that further evaluation using proper data is necessary.

We believe the essay also successfully aided our learning about mathematically modelling interaction between species, which was the main objective.

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