Hoja 4: Series de Fourier

1. Hallar la serie de Fourier de las siguientes funciones de $L^1(-\frac{1}{2},\frac{1}{2})$, esbozando sus gráficas y estableciendo en qué puntos se tiene convergencia:

   (a) $f(x) = |x|$

   (b) $f(x) = \operatorname{sen}(\pi x) + \cos^2(2\pi x)$

   (c) $f(x) = |\operatorname{sen}(2\pi x)|$

   (d) $f(x) = e^{ax}$

   (e) $f(x) = \begin{cases} 0 & \text{si } -\frac{1}{2} < x < 0 \\ \operatorname{sen}(2\pi x) & \text{si } 0 < x < \frac{1}{2} \end{cases}$

   (f) $f(x) = \begin{cases} 0 & \text{si } -\frac{1}{2} < x < 0 \\ x & \text{si } 0 < x < \frac{1}{2} \end{cases}$

   (g) $f_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon} & \text{si } |x| \le \varepsilon \\ 0 & \text{si } \varepsilon < |x| < \frac{1}{2} \end{cases}$

   (h) $f_\varepsilon(x) = \begin{cases} 1 - \frac{|x|}{\varepsilon} & \text{si } |x| \le \varepsilon \\ 0 & \text{si } \varepsilon < |x| < \frac{1}{2} \end{cases}$

   Nota: Una tabla de soluciones aparece en el libro de Folland "*Fourier Analysis*", pág. 26.

2. Dado un número $\alpha \notin \mathbb{Z}$: (i) demuestra (justificando el tipo de convergencia) que

   $$e^{-i\alpha x} = c_\alpha \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{n+\alpha} e^{inx}, \quad |x| < \pi,$$

   para una constante c_α apropiada. ¿Qué ocurre cuando $x = \pm\pi$?

   (ii) Dando valores adecuados a x demuestra las fórmulas

   $$\frac{\pi}{\operatorname{sen}(\pi\alpha)} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\alpha+n} \qquad \text{y} \qquad \frac{\pi}{\tan(\pi\alpha)} = \sum_{n\in\mathbb{Z}} \frac{1}{\alpha+n}.$$

   (iii) Utilizando Parseval, demuestra la fórmula

   $$\frac{\pi^2}{\operatorname{sen}^2(\pi\alpha)} = \sum_{n\in\mathbb{Z}} \frac{1}{(\alpha+n)^2}.$$

3. Sabiendo que $x = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{\pi n} \operatorname{sen}(2\pi n x)$, para $|x| < 1/2$

   (i) hallar la serie de Fourier de x^2 en $[-\frac{1}{2},\frac{1}{2}]$, estableciendo el tipo de convergencia

   (ii) calcula las siguientes sumas

   $$\sum_{n=1}^\infty \frac{1}{n^2}, \qquad \sum_{n=1}^\infty \frac{1}{(2n-1)^2}, \qquad \sum_{n=0}^\infty \frac{(-1)^{n+1}}{n^2}.$$

   (iii) aplicando Parseval a la función x^2, calcula las sumas

   $$\sum_{n=1}^\infty \frac{1}{n^4}, \qquad \sum_{n=1}^\infty \frac{1}{(2n-1)^4}, \qquad \sum_{n=0}^\infty \frac{(-1)^{n+1}}{n^4}.$$

4. Sabiendo que $|x| = \frac{1}{4} - \frac{2}{\pi^2}\sum_{k=0}^\infty \frac{\cos[(2k+1)2\pi x]}{(2k+1)^2}$, para $|x| < 1/2$, demuestra que

   $$\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32} \qquad \text{y} \qquad \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^5} = \frac{5\pi^5}{1536}.$$

   Sugerencia: Integra la serie de $|x|$ un número suficiente de veces.

1. Hallar la serie de Fourier de las siguientes funciones de $L^1(-\frac{1}{2},\frac{1}{2})$, esbozando sus gráficas y estableciendo en qué puntos se tiene convergencia:

$$(a)\ f(x) = |x|$$

$\boxed{n=0}$

$$\hat{f}(0) = \int_{-1/2}^{1/2} f(y) e^{-2\pi i n y} dy = \int_{-1/2}^{1/2} |y| dy = 2\int_0^{1/2} y\, dy = \frac{1}{4}$$

$\boxed{n\neq 0}$

$$\hat{f}(n) = \int_{-1/2}^{1/2} |y| e^{-2\pi i n y} dy = \int_{-1/2}^{0} (-y) e^{-2\pi i n y} dy + \int_0^{1/2} y e^{-2\pi i n y} dy =$$

$$\underset{\substack{\uparrow \\ t=-y \\ dt=-dy}}{=} \int_0^{1/2} t e^{2\pi i n t} dt + \int_0^{1/2} y e^{-2\pi i n y} dy = \int_0^{1/2} y\left(e^{2\pi i n y} + e^{-2\pi i n y}\right) dy =$$

$$= \int_0^{1/2} y \cdot 2\cos(2\pi n y)\, dy = \overbrace{2\left[y \cdot \frac{\sin(2\pi n y)}{2\pi n}\right]_0^{1/2}}^{0} - \cancel{2}\int_0^{1/2} \frac{\sin(2\pi n y)}{\cancel{2}\pi n} dy =$$

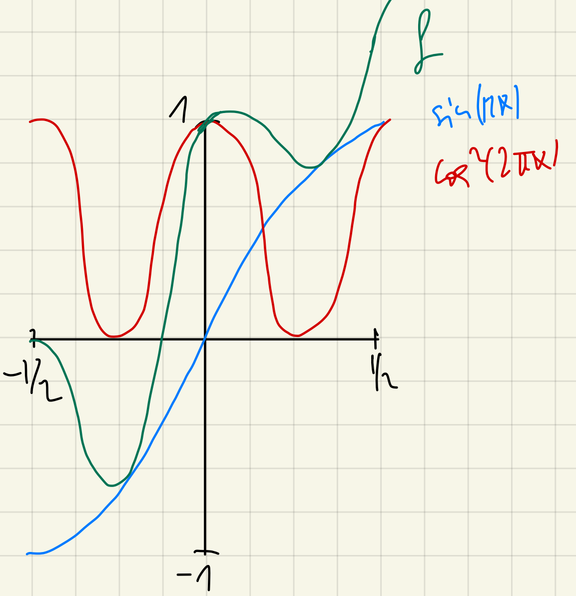
$$= \left.\frac{\cos(2\pi n y)}{2\pi^2 n^2}\right]_0^{1/2} = \frac{\cos(\pi n) - 1}{2\pi^2 n^2} = \frac{(-1)^n - 1}{2\pi^2 n^2} = \begin{cases} 0 & n \text{ par} \\ \frac{-1}{\pi^2 n^2} & n \text{ impar} \end{cases}$$

$$f(x) = \frac{1}{4} + \sum_{n=-\infty}^{\infty} \frac{-1}{\pi^2(2n-1)^2} e^{2\pi i (2n-1)x} = \frac{1}{4} - \sum_{n=1}^{\infty} \frac{2\cos(2\pi(2n-1)x)}{\pi^2(2n-1)^2}, \quad \forall x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

por Dini 1 y 2

(b) $f(x) = \mathrm{sen}(\pi x) + \cos^2(2\pi x)$

$$\hat{f}(n) = \int_{-1/2}^{1/2} f(y)\, e^{-2\pi i n y}\, dy$$



$$\hat{f}(n) = \int_{-1/2}^{1/2} \left(\mathrm{sen}(\pi y) + \cos^2(2\pi y)\right) e^{-2\pi i n y}\, dy =$$

$$= \int_{-1/2}^{1/2} \underbrace{\mathrm{sen}(\pi y)}_{\frac{e^{i\pi y} - e^{-i\pi y}}{2i}} e^{-2\pi i n y}\, dy + \int_{-1/2}^{1/2} \underbrace{\cos^2(2\pi y)}_{\left(\frac{e^{i2\pi y} + e^{-i2\pi y}}{2}\right)^2} e^{-2\pi i n y}\, dy =$$

$$= \frac{1}{2i} \int_{-1/2}^{1/2} e^{i\pi y(1-2n)} - e^{-i\pi y(1+2n)}\, dy + \frac{1}{4} \int_{-1/2}^{1/2} \left(e^{i4\pi y} + 2 + e^{-i4\pi y}\right) e^{-2\pi i n y}\, dy =$$

$$= \frac{1}{2} \left[-\frac{e^{i\pi y(1-2n)}}{\pi(1-2n)} - \frac{e^{-i\pi y(1+2n)}}{\pi(1+2n)} \right]_{-1/2}^{1/2} + \frac{1}{4} \int_{-1/2}^{1/2} \left(e^{i2\pi y(2-n)} + 2e^{-2\pi i n y} + e^{-i2\pi y(2+n)} \right) dy =$$

$$= \frac{1}{2} \left(-\frac{e^{i\frac{\pi}{2}(1-2n)}}{\pi(1-2n)} - \frac{e^{-i\frac{\pi}{2}(1+2n)}}{\pi(1+2n)} + \frac{e^{-i\frac{\pi}{2}(1-2n)}}{\pi(1-2n)} + \frac{e^{i\frac{\pi}{2}(1+2n)}}{\pi(1+2n)} \right) + \frac{1}{4} \left[\frac{e^{i2\pi y(2-n)}}{i2\pi(2-n)} - \cancel{2} \frac{e^{-2\pi i n y}}{\cancel{2}\pi i n} - \frac{e^{-i2\pi y(2+n)}}{i2\pi(2+n)} \right]_{-1/2}^{1/2} =$$

$$= \frac{1}{2} \left(\frac{-2i\,\mathrm{sen}\left(\frac{\pi}{2}(1-2n)\right)}{\pi(1-2n)} + \frac{2i\,\mathrm{sen}\left(\frac{\pi}{2}(2n+1)\right)}{\pi(1+2n)} \right) + \frac{1}{4} \left(\frac{e^{i\pi(2-n)}}{i2\pi(2-n)} - \frac{e^{-i\pi(2-n)}}{i2\pi(2-n)} - \frac{e^{-\pi i n}}{\pi i n} + \frac{e^{\pi i n}}{\pi i n} - \frac{e^{-i\pi(2+n)}}{i2\pi(2+n)} + \frac{e^{i\pi(2+n)}}{i2\pi(2+n)} \right)$$

$\mathrm{sen}\left(\frac{\pi}{2}(1-2n)\right) = (-1)^{|n|}$, $\mathrm{sen}\left(\frac{\pi}{2}(2n+1)\right) = (-1)^{|n|}$

$$\frac{-i\cdot(-1)^{|n|}}{\pi(1-2n)} + \frac{i\cdot(-1)^{|n|}}{\pi(1+2n)} = \frac{-i\cdot(-1)^{|n|}\cdot(1+2n) + i(-1)^{|n|}\cdot(1-2n)}{\pi(1-4n^2)} = \frac{i(-1)^{|n|}(1-2n-1-2n)}{\pi(1-4n^2)} = i\,\frac{2n(-1)^{|n|}}{\pi(4n^2-1)}$$

$$\frac{1}{4}\left(\underbrace{\frac{e^{i\pi(2-n)}}{i2\pi(2-n)}-\frac{e^{-i\pi(2-n)}}{i2\pi(2-n)}}_{\frac{\cancel{\operatorname{sen}(\pi(2-n))}\to 0}{\pi(2-n)}}-\underbrace{\frac{e^{-\pi i n}}{\pi i n}+\frac{e^{\pi i n}}{\pi i n}}_{\frac{2\cancel{\operatorname{sen}(\pi n)}\to 0}{\pi n}}-\frac{e^{-i\pi(2+n)}}{i2\pi(2+n)}+\frac{e^{i\pi(2+n)}}{i2\pi(2+n)}\right)$$

$$\frac{\cancel{\operatorname{sen}(\pi(2+n))}\to 0}{\pi(2+n)}$$

$\boxed{n=0}$

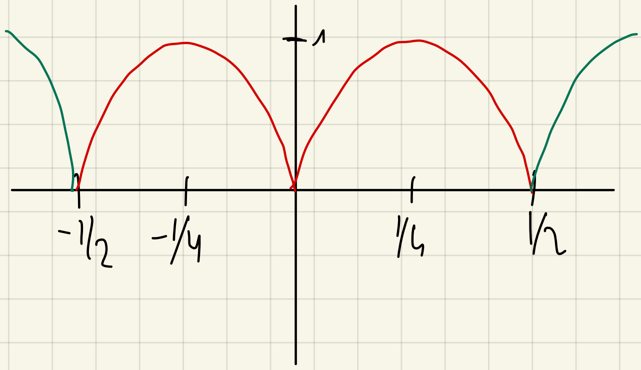
$$\hat{f}(0)=\int_{-1/2}^{1/2}\underbrace{\operatorname{sen}(\pi y)}_{\text{impar}}+\cos^2(2\pi y)\,dy=\int_{-1/2}^{1/2}\cos^2(2\pi y)\,dy=\int_{-1/2}^{1/2}\frac{1+\cos(4\pi y)}{2}\,dy=$$

$$=\frac{1}{2}+\left.\frac{\operatorname{sen}(4\pi y)}{2\pi}\right]_{-1/2}^{1/2}=\frac{1}{2}$$

Así, $f\underset{q}{=}\frac{1}{2}+\sum_{n\neq 0}$

BIEN 1

(c) $f(x) = |\operatorname{sen}(2\pi x)|$



$$\hat{f}(n) = \int_{-1/2}^{1/2} |\operatorname{sen}(2\pi y)|\, e^{-i2\pi n y}\, dy = \int_{-1/2}^{0} -\operatorname{sen}(2\pi y)\, e^{-i2\pi n y}\, dy + \int_{0}^{1/2} \operatorname{sen}(2\pi y)\, e^{-i2\pi n y}\, dy = \circledast$$

$$\int_0^{1/2} s(2\pi y)\, e^{-i2\pi y}\, dy = \int_0^{1/2} \left(\frac{e^{i2\pi y} - e^{-i2\pi y}}{2i}\right) e^{-i2\pi n y}\, dy = \frac{1}{2i}\int_0^{1/2} e^{i2\pi y(1-n)} - e^{-i2\pi y(1+n)}\, dy$$

$$= \frac{1}{2i}\left(\frac{e^{i2\pi y(1-n)}}{i2\pi(1-n)} - \frac{e^{-i2\pi y(1+n)}}{i2\pi(1+n)}\right)\Bigg]_0^{1/2} = -\frac{e^{i2\pi y(1-n)}}{4\pi(1-n)} + \frac{e^{-i2\pi y(1+n)}}{4\pi(1+n)}\Bigg]_0^{1/2} =$$

$$= \frac{e^{-i\pi(1+n)}}{4\pi(1+n)} - \frac{e^{i\pi(1-n)}}{4\pi(1-n)} - \frac{1}{4\pi(1+n)} + \frac{1}{4\pi(1-n)}$$

$$\frac{e^{-i2\pi y(1+n)}}{4\pi(1+n)} - \frac{e^{i2\pi y(1-n)}}{4\pi(1-n)}\Bigg]_{-1/2}^{0} = \frac{1}{4\pi(1+n)} - \frac{1}{4\pi(1-n)} - \frac{e^{i\pi(1+n)}}{4\pi(1+n)} + \frac{e^{-i\pi(1-n)}}{4\pi(1-n)}$$

$$\circledast = \frac{e^{-i\pi(1+n)} + e^{i\pi(1+n)}}{4\pi(1+n)} - \frac{e^{i\pi(1-n)} + e^{-i\pi(1-n)}}{4\pi(1-n)} - \frac{1}{2\pi(1+n)} + \frac{1}{2\pi(1-n)}$$

$$= \frac{\cos(\pi(1+n))}{2\pi(1+n)} - \frac{\cos(\pi(1-n))}{2\pi(1-n)} + \frac{1+n-1+n}{2\pi(1-n^2)} = \frac{(-1)^{n+1}}{2\pi(1+n)} - \frac{(-1)^{n+1}}{2\pi(1-n)} + \frac{n}{\pi(1-n^2)} =$$

$$= \frac{(-1)^{n+1}(1-n) - (-1)^{n+1}(1+n) + 2n}{2\pi(1-n^2)} = \frac{(-1)^{n+1}(1-n-1-n)+2n}{2\pi(1-n^2)} = \frac{(-1)^{n+1}\cdot(-2n)+2n}{2\pi(1-n^2)} = \frac{(-1)^n\cdot 2n+2n}{2\pi(1-n^2)} = \frac{n((-1)^n+1)}{\pi(1-n^2)} = \begin{cases} 0 & n \text{ impar} \\ \frac{2n}{\pi(1-n^2)} & n \text{ par} \end{cases}$$

$\boxed{n=0}$

$$f(0) = \int_{-1/2}^{1/2} |\cos(2\pi y)| \, dy = \frac{2}{\pi}$$

DINI 1 + DINI 2:

$$f(x) = |\cos(2\pi x)| = \frac{2}{\pi} + \sum_{n \neq 0} \frac{4n}{\pi(1-4n^2)} e^{2\pi i n x}$$

(d) $f(x) = e^{ax}$

$$\hat{f}(n) = \int_{-1/2}^{1/2} e^{ax} e^{-2\pi i n x} dx = \frac{e^{(a-2\pi i n)x}}{a-2\pi i n}\Big]_{-1/2}^{1/2} = \frac{e^{a/2}(-1)^n - e^{-a/2}(-1)^n}{a-2\pi i n} =$$

$$= (-1)^n \frac{2\operatorname{senh}(\frac{a}{2})}{a-2\pi i n}$$

$$f(x) \sim 2\operatorname{senh}\left(\frac{a}{2}\right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a-2\pi i n} e^{2\pi i n x}$$

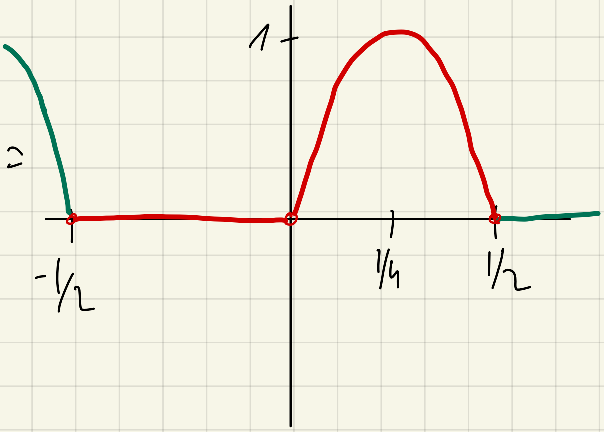
Por DNI 1, $e^{ax} = 2\operatorname{sen} h\left(\frac{a}{2}\right) \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a-2\pi i n} e^{2\pi i n x}$

Por DINI 2, $2\operatorname{senh}\left(\frac{a}{2}\right) \sum \frac{(-1)^n}{a-2\pi i n} (-1)^n = 2\operatorname{senh}\left(\frac{a}{2}\right) \sum \frac{1}{a-2\pi i n} = \frac{f(1/2^+) + f(1/2^-)}{2} =$

$$= \frac{e^{-a/2} + e^{a/2}}{2} = \cosh\left(\frac{a}{2}\right)$$

ídem para $x = -\frac{1}{2}$

(e) $f(x) = \begin{cases} 0 & \text{si } -\frac{1}{2} < x < 0 \\ \operatorname{sen}(2\pi x) & \text{si } 0 < x < \frac{1}{2} \end{cases}$



$$\hat{f}(n) = \int_{-1/2}^{1/2} f(x)\, e^{-2i\pi n x}\, dx = \int_0^{1/2} \operatorname{sen}(2\pi x)\, e^{-2i\pi n x}\, dx =$$

(en ©)

$$= \frac{e^{-i\pi(1+n)}}{4\pi(1+n)} - \frac{e^{i\pi(1-n)}}{4\pi(1-n)} - \frac{1}{4\pi(1+n)} + \frac{1}{4\pi(1-n)} =$$

$$= \frac{e^{-i\pi(1+n)}(1-n) - e^{i\pi(1-n)}(1+n) - (1-n) + (1+n)}{4\pi(1-n^2)} =$$

$$= \frac{e^{-i\pi} \cdot e^{-i\pi n}(1-n) - e^{i\pi} e^{-i\pi n}(1+n) + 2n}{4\pi(1-n^2)} = \frac{-e^{-i\pi n}(1-n) + e^{-i\pi n}(1+n) + 2n}{4\pi(1-n^2)} =$$

$$= \frac{e^{-i\pi n}(1+n-1+n) + 2n}{4\pi(1-n^2)} = \frac{n e^{-i\pi n} + n}{2\pi(1-n^2)} = \frac{n(\cos(n\pi) - i\cancel{\operatorname{sen}(n\pi)}) + n}{2\pi(1-n^2)} = \frac{n(\cos(n\pi)+1)}{2\pi(1-n^2)} =$$

$$= \frac{n((-1)^n+1)}{2\pi(1-n^2)} = \begin{cases} 0 & n \text{ impar} \\ \frac{n}{\pi(1-n^2)} & n \text{ par} \end{cases}$$

$$\hat{f}(0) = \int_{-1/2}^{1/2} f(x)\, dx = \int_0^{1/2} \operatorname{sen}(2\pi x)\, dx = \frac{1}{\pi}$$

y es

$$f(x) \underset{\substack{\text{DINI 1} \\ (-\frac{1}{2},0)\cup(0,\frac{1}{2})}}{=} \frac{1}{\pi} + \sum_{n \neq 0} \frac{2n}{\pi(1-4n^2)} e^{2\pi i n x}$$

$$(f)\ f(x) = \begin{cases} 0 & \text{si } -\frac{1}{2} < x < 0 \\ x & \text{si } 0 < x < \frac{1}{2} \end{cases}$$

$$\hat{f}(n) = \int_{-1/2}^{1/2} f(y)\, e^{-2\pi i n y}\, dy = \int_0^{1/2} y\, e^{-2\pi i n y}\, dy = -\frac{y e^{-2\pi i n y}}{2\pi i n}\Big]_0^{1/2} + \frac{1}{2\pi i n}\int_0^{1/2} e^{-2\pi i n y}\, dy =$$

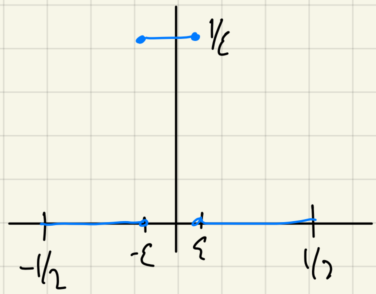
$$u = y,\ du = dy$$
$$dv = e^{-2\pi i n y}\, dy,\ v = \frac{e^{-2\pi i n y}}{-2\pi i n}$$

$$= -\frac{e^{-\pi i n}}{4\pi i n} + \frac{1}{4\pi^2 n}\, e^{-2\pi i n y}\Big]_0^{1/2} = \frac{e^{-\pi i n}}{4\pi i n} + \frac{1}{4\pi^2 n}\left(e^{-\pi i n} - 1\right) =$$

$$e^{-\pi i n} = \cos(\pi n) - i\,\cancel{\sin}(\pi n) = \cos(\pi n) = (-1)^n$$

$$= \frac{(-1)^n}{4\pi i n} + \frac{\left((-1)^n - 1\right)}{4\pi^2 n}$$

(g) $f_\varepsilon(x) = \begin{cases} \frac{1}{2\varepsilon} & \text{si } |x| \le \varepsilon \\ 0 & \text{si } \varepsilon < |x| < \frac{1}{2} \end{cases}$



$$\hat{f}_\varepsilon(n) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} e^{-2\pi i n x}\,dx = \frac{1}{2\varepsilon}\,\frac{e^{-2\pi i n \varepsilon} - e^{2\pi i n \varepsilon}}{-2\pi i n} =$$

$$= \frac{1}{2\varepsilon}\cdot\frac{2i\sin(2\pi n\varepsilon)}{2\pi i n} = \frac{\sin(2\pi n\varepsilon)}{2\pi n\varepsilon} \qquad n \neq 0$$

$$\hat{f}_\varepsilon(0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} dx = 1$$

Así:

$$\frac{1}{2\varepsilon}\mathbb{1}_{(-\varepsilon,\varepsilon)}(x) = 1 + \sum_{n\neq 0}\frac{\sin(2\pi n\varepsilon)}{2\pi n\varepsilon}e^{2\pi i n x} \qquad \forall x \in \left[-\tfrac{1}{2},\tfrac{1}{2}\right]\setminus\{\pm\varepsilon\}$$

por ANJ 1

Si $x = \pm\varepsilon \longrightarrow S_N f(\pm\varepsilon) \xrightarrow{N\to\infty} \frac{1}{4\varepsilon}$

Nota $x=0$

$$\frac{1}{2\varepsilon} = 1 + \sum_{n\neq 0}\frac{\sin(2\pi n\varepsilon)}{2\pi n\varepsilon} \longrightarrow \underbrace{\left(\frac{1}{2\varepsilon}-1\right)2\pi\varepsilon}_{=\,\pi(1-2\varepsilon)} = \sum_{n\neq 0}\frac{\sin(2\pi n\varepsilon)}{n} \qquad \forall \varepsilon < \tfrac{1}{2}$$

Nota

$\varepsilon \to 0$

$$\delta_0(x) \sim 1 + \sum_{n\neq 0} e^{2\pi i n x} = \sum_{n\in\mathbb{Z}} e^{2\pi i n x}$$

2. Dado un número $\alpha \notin \mathbb{Z}$: (i) demuestra (justificando el tipo de convergencia) que

$$e^{-i\alpha x} = c_\alpha \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{n+\alpha} e^{inx}, \quad |x| < \pi,$$

para una constante c_α apropiada. ¿Qué ocurre cuando $x = \pm\pi$?

$$|x|<\pi \longleftrightarrow x\in(-\pi,\pi) \longleftrightarrow \frac{x}{2}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \longleftrightarrow \frac{x}{2\pi}\in\left(-\frac{1}{2},\frac{1}{2}\right)$$

$$f(x) = e^{-i\alpha 2\pi x}$$

$$\hat{f}(n) = \int_{-1/2}^{1/2} e^{-i\alpha 2\pi x} e^{-i2\pi n x}\,dx = \int_{-1/2}^{1/2} e^{-2i\pi x(\alpha+n)}\,dx =$$

$$= \left.\frac{-e^{-2i\pi x(\alpha+n)}}{2i\pi(\alpha+n)}\right]_{-1/2}^{1/2} = \frac{-e^{-i\pi(\alpha+n)}+e^{i\pi(\alpha+n)}}{2i\pi(\alpha+n)} = \frac{\operatorname{sen}(\pi(\alpha+n))}{\pi(\alpha+n)} =$$

$$= \frac{\operatorname{sen}(\pi\alpha+n\pi)}{\pi(\alpha+n)} = \frac{\operatorname{sen}(\pi\alpha)\cos(\pi n)+\cancel{\cos(\pi\alpha)\operatorname{sen}(\pi n)}^{0}}{\pi(\alpha+n)} = \frac{\operatorname{sen}(\pi\alpha)(-1)^n}{\pi(\alpha+n)}$$

$$e^{-i\alpha 2\pi x} \sim \frac{\operatorname{sen}(\pi\alpha)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{n+\alpha}e^{2\pi i n x} \qquad |x|<\frac{1}{2}$$

$$\longrightarrow e^{-i\alpha x} \underset{=}{\sim} \frac{\operatorname{sen}(\pi\alpha)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{n+\alpha}e^{inx}$$

DINI 1, $f\in C^1(-\frac{1}{2},\frac{1}{2})$

$$x=\pm\pi \to e^{-i\alpha\pi} \neq e^{i\alpha\pi} \sim \quad S(\pi) = \frac{e^{i\alpha\pi}+e^{-i\alpha\pi}}{2} = \cos(\alpha\pi)$$

(ii) Dando valores adecuados a x demuestra las fórmulas

$$\frac{\pi}{\operatorname{sen}(\pi\alpha)} = \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\alpha+n} \qquad \text{y} \qquad \frac{\pi}{\tan(\pi\alpha)} = \sum_{n\in\mathbb{Z}} \frac{1}{\alpha+n}.$$

$$1 = e^{-i\alpha 0} = \frac{\operatorname{sen}(\pi\alpha)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{\alpha+n} \rightarrow \frac{\pi}{\operatorname{sen}(\pi\alpha)} = \sum_{n\in\mathbb{Z}}\frac{(-1)^n}{\alpha+n} \quad //$$

$$e^{-i\alpha x} = \frac{\operatorname{sen}(\pi\alpha)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{\alpha+n}e^{inx}$$

$$e^{i\alpha x} = \frac{\operatorname{sen}(\pi\alpha)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{\alpha+n}e^{-inx}$$

$\boxed{x=\pi}$ Hemos visto que $S(\pi) = \cos(\alpha\pi)$

Por tanto

$$\cos(\alpha\pi) = \frac{\operatorname{sen}(\pi\alpha)}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{\alpha+n}e^{in\pi} = \frac{\operatorname{sen}(\pi\alpha)}{\pi}\sum\frac{(-1)^n}{\alpha+n}(-1)^n$$

$$\rightarrow \frac{\pi}{\tan(\pi\alpha)} = \sum_{n\in\mathbb{Z}}\frac{1}{\alpha+n} \quad //$$

(iii) Utilizando Parseval, demuestra la fórmula

$$\frac{\pi^2}{\operatorname{sen}^2(\pi\alpha)} = \sum_{n\in\mathbb{Z}} \frac{1}{(\alpha+n)^2}.$$

Parseval

$$f \in L^2(\mathbb{T}) \Rightarrow \int_{\mathbb{T}} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

$$e^{-i\alpha x} = \frac{\operatorname{sen}(\pi\alpha)}{\pi} \sum_{n\in\mathbb{Z}} \frac{(-1)^n}{\alpha+n} e^{inx}$$

$$\int_{-1/2}^{1/2} |e^{-i\alpha x}|^2 dx = \int_{-1/2}^{1/2} 1\, dx = 1$$

''

$$\int_{\mathbb{T}} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2 = \frac{\operatorname{sen}^2(\pi\alpha)}{\pi^2} \sum_{n\in\mathbb{Z}} \frac{1}{(\alpha+n)^2}$$

3. Sabiendo que $x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \operatorname{sen}(2\pi n x)$, para $|x| < 1/2$

(i) hallar la serie de Fourier de x^2 en $[-\frac{1}{2}, \frac{1}{2}]$, estableciendo el tipo de convergencia

$$x^2 = 2\int_0^x t\,dt$$

$$f \in L^1(\mathbb{T}) \quad , \quad F(x) = \int_0^x f(t)\,dt$$

$$\rightarrow \hat{F}(n) = \frac{\hat{f}(n) - \hat{f}(0)}{2\pi i n}$$

$$x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} \operatorname{sen}(2\pi n x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n 2i}\left(e^{2\pi n x i} - e^{-2\pi n x i}\right) =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n 2i} e^{2\pi n x i} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n 2i} e^{-2\pi n x i} =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n 2i} e^{2\pi n x i} + \sum_{n=1}^{-\infty} \frac{(-1)^{n+1}}{\pi n 2i} e^{2\pi n x i} = \sum_{n\neq 0} \frac{(-1)^{n+1}}{2i\pi n} e^{2\pi n x i}$$

$$\frac{x^2}{2} \sim \sum_{n\neq 0} \frac{\hat{f}(n)}{2\pi i n} e^{2\pi n x i} + C_0 = \sum_{n\neq 0} \frac{(-1)^n}{4\pi^2 n^2} e^{2\pi n x i} + C_0$$

$$\uparrow \int_{\mathbb{T}} x = 0$$

$$C_0 = \hat{F}(0) = \int_{-1/2}^{1/2} \frac{t^2}{2}\,dt = \left.\frac{t^3}{6}\right]_{-1/2}^{1/2} = \frac{1/8}{6} + \frac{1/8}{6} = \frac{1/4}{6} = \frac{1}{24}$$

$$x^2 = \frac{1}{12} + \sum_{n\neq 0} \frac{(-1)^n}{2\pi^2 n^2} e^{2\pi n x i}$$

$\uparrow$ DINI 1

Por DINI 2, tmb en los extremos

(ii) calcula las siguientes sumas

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

$$x^2 = \frac{1}{12} + \sum_{n\neq 0} \frac{(-1)^n}{2\pi^2 n^2} e^{2\pi n x i}$$

$$\frac{1}{4} = \frac{1}{12} + \sum_{n\neq 0} \frac{(-1)^n}{2\pi^2 n^2} (-1)^n = \frac{1}{12} + \sum_{n\neq 0} \frac{1}{2\pi^2 n^2}$$

$$\to \sum_{n\neq 0} \frac{1}{2\pi^2 n^2} = \frac{1}{4} - \frac{1}{12} = \frac{3-1}{12} = \frac{2}{12} = \frac{1}{6}$$

$$\to \sum_{n\neq 0} \frac{1}{n^2} = \frac{\pi^2}{3} \to \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\parallel$$

$$2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = S_{par} + S_{impar}$$

$$S_{par} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{4} \frac{\pi^2}{6}$$

$$\to S_{impar} = \frac{3}{4} \frac{\pi^2}{6} = \frac{\pi^2}{8}$$

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{-1}{(2n)^2} + \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} = -\frac{\pi^2}{24} + \frac{\pi^2}{8} = \frac{\pi^2}{12}$$

(iii) aplicando Parseval a la función x^2, calcula las sumas

$$\sum_{n=1}^{\infty} \frac{1}{n^4}, \qquad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n^4}.$$

$$f(x) = x^2 = \frac{1}{12} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{(-1)^n}{2\pi^2 n^2} e^{2\pi i n x}$$

$$f(x) \in L^2(\mathbb{T}) \Longrightarrow \int_{\mathbb{T}} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$$

$$\Longrightarrow \int_{-1/2}^{1/2} |x^2|^2 dx = \frac{x^5}{5}\Big|_{-1/2}^{1/2} = \frac{2}{160} = \frac{1}{80}$$

$$\parallel$$

$$\sum |\hat{f}(n)|^2 = \frac{1}{144} + \sum_{n\neq 0} \left|\frac{(-1)^n}{2\pi^2 n^2}\right|^2 = \frac{1}{144} + \frac{1}{4\pi^4} \sum \frac{1}{n^4}$$

$$\rightarrow \sum_{n\neq 0} \frac{1}{n^4} = 4\pi^4 \left(\frac{1}{80} - \frac{1}{144}\right) = \frac{4\pi^4}{180} = \frac{\pi^4}{45}$$

$$\rightarrow \sum_{n\neq 0} \frac{1}{n^4} = \sum_{n<0} \frac{1}{n^4} + \sum_{n>0} \frac{1}{n^4} = 2\sum_{n>0} \frac{1}{n^4} \rightarrow \boxed{\sum_{n>0} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$$

$$S = \sum_{n=1}^{\infty} \frac{1}{n^4}, \quad S = S_{par} + S_{impar} = \underbrace{\sum_{k=1}^{\infty} \frac{1}{(2k)^4}}_{= \frac{1}{2^4} \cdot S} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4}$$

Así, $S\left(1-\frac{1}{2^4}\right) = S_{impar}$

$$\frac{\pi^4}{90}\left(1-\frac{1}{2^4}\right) = \frac{\pi^4}{90} \frac{16-1}{16} = \frac{\pi^4 \cdot 15}{16 \cdot 90} = \boxed{\frac{\pi^4}{96} = \sum_{n>0} \frac{1}{(2n-1)^4}}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = S^*$$

$$S^* = \sum_{k=1}^{\infty} \frac{1}{(2k)^4} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^4} = -\frac{1}{2^4} S + S_{impar} = \frac{\pi^4}{2^4\ 90} + \frac{\pi^4}{96} =$$

$$= \frac{7\pi^4}{720}$$

4. Sabiendo que $|x| = \frac{1}{4} - \frac{2}{\pi^2}\sum_{k=0}^{\infty} \frac{\cos[(2k+1)2\pi x]}{(2k+1)^2}$, para $|x| < 1/2$, demuestra que

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32} \qquad \text{y} \qquad \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^5} = \frac{5\pi^5}{1536}.$$

$$\xrightarrow{\int_0^x} \frac{x^2}{2} = \frac{x}{4} - \frac{\cancel{2}}{\pi^2}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \underbrace{\int_0^x \cos(2\pi(2k+1)t)\,dt}_{\frac{\operatorname{sen}(2\pi(2k+1)t)}{\cancel{2}\pi(2k+1)}}$$

$$\frac{x^2}{2} - \frac{x}{4} = -\frac{1}{\pi^3}\sum_{k=0}^{\infty} \frac{\operatorname{sen}(2\pi(2k+1)x)}{(2k+1)^3}$$

$$\xrightarrow{x=1/4} \frac{1}{32} - \frac{1}{16} = \frac{-1}{\pi^3}\sum_{k=0}^{\infty} \frac{\operatorname{sen}\left(\frac{\pi}{2}(2k+1)\right)}{(2k+1)^3} = \frac{-1}{\pi^3}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

$$\frac{1-2}{32} = -\frac{1}{32}$$

$$\longrightarrow \frac{\pi^3}{32} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$$

$$\frac{x^2}{2}-\frac{x}{4}=-\frac{1}{\pi^3}\sum_{k=0}^{\infty}\frac{\cos(2\pi(2k+1)x)}{(2k+1)^3}$$

$$\xrightarrow{\int_0^x}\quad \frac{x^3}{6}-\frac{x^2}{8}=-\frac{1}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}\underbrace{\int_0^x\cos(2\pi(2k+1)t)\,dt}_{\left.\frac{-\cos(2\pi(2k+1)t)}{2\pi(2k+1)}\right]_0^x =}$$

$$=\frac{-\cos(2\pi(2k+1)x)+1}{2\pi(2k+1)}$$

$$\frac{x^3}{6}-\frac{x^2}{8}=-\frac{1}{2\pi^4}\sum_{k=0}^{\infty}\frac{1-\cos(2\pi(2k+1)x)}{(2k+1)^4}$$

$$\xrightarrow{x=1/2}\quad \frac{1/8}{6}-\frac{1/4}{8}=-\frac{1}{2\pi^4}\left(\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}+\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}\right)=-\frac{1}{\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}$$

$$\frac{1}{48}-\frac{1}{32}=\frac{32-48}{1536}=\frac{-16}{1536}=\frac{-1}{96}$$

$$\rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}=\frac{\pi^4}{96}\quad\sim\quad -\frac{1}{2\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}=-\frac{1}{2\pi^4}\frac{\pi^4}{96}=-\frac{1}{192}$$

$$\rightarrow \frac{x^3}{6}-\frac{x^2}{8}=-\frac{1}{192}+\frac{1}{2\pi^4}\sum_{k=0}^{\infty}\frac{\cos(2\pi(2k+1)x)}{(2k+1)^4}$$

$$\xrightarrow{\int_0^x}\quad \frac{x^4}{24}-\frac{x^3}{24}=-\frac{x}{192}+\frac{1}{2\pi^4}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}\underbrace{\int_0^x\cos(2\pi(2k+1)t)\,dt}_{\frac{\sin(2\pi(2k+1)x)}{2\pi(2k+1)}}$$

$$\rightarrow \frac{x^4}{24}-\frac{x^3}{24}+\frac{x}{192}=\frac{1}{4\pi^5}\sum_{k=0}^{\infty}\frac{\sin(2\pi(2k+1)x)}{(2k+1)^5}$$

$$\frac{x^4}{24} - \frac{x^3}{24} + \frac{x}{192} = \frac{1}{4\pi^5} \sum_{k=0}^{\infty} \frac{\sin(2\pi(2k+1)x)}{(2k+1)^5}$$

$$\xrightarrow{x=1/4} \quad \frac{1/256}{24} - \frac{1/64}{24} + \frac{1/4}{192} = \frac{1}{4\pi^5} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^5}$$

$$\frac{1}{6144} - \frac{1}{1536} + \frac{1}{768} = \frac{5}{6144}$$

$$\longrightarrow \quad \frac{5\pi^5}{1536} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^5}$$

5. *Series de Fourier reales:* Si $f \in L^1[0,1)$, podemos formalmente escribir

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx} \sim \tfrac{a_0}{2} + \sum_{n=1}^{\infty} \big[a_n \cos(2\pi nx) + b_n \sin(2\pi nx)\big].$$

a) Expresa a_n y b_n en términos de $\hat{f}(n)$, y viceversa

b) Demuestra que $b_n = 2\int_0^1 f(x)\sin(2\pi nx)\,dx$ y encuentra una fórmula similar para a_n.

6. *Series de funciones reales, pares e impares.* Sea $f \in L^1[-\frac{1}{2}, \frac{1}{2})$.

(i) Demuestra que f es real si y sólo si $\hat{f}(n) = \overline{\hat{f}(-n)}$, $\forall\, n \in \mathbb{Z}$.

(ii) Demuestra que f es real si y sólo si los coeficientes a_n, b_n del ejercicio 5 son reales.

(iii) Caracteriza los coeficientes de Fourier $\hat{f}(n), a_n, b_n$ para las funciones pares.

(iv) Ídem para las funciones impares.

7. *Suavidad de f implica decaimiento de $\hat{f}$.* Demostrar que si $f \in C^k_{\rm per}(\mathbb{R})$, entonces

$$\hat{f}(n) = o(1/|n|^k), \quad \text{para } |n| \to \infty.$$

Sugerencia: Utiliza integración por partes y el lema de Riemann-Lebesgue.

8. *Decaimiento de $\hat{f}$ implica suavidad de f:* Demuestra que si $f \in L^1[0,1)$ es tal que $\widehat{f}(n) = O(1/|n|^{k+2})$ para $|n| \to \infty$, entonces $f \in C^k_{\rm per}(\mathbb{R})$.

Nota: Necesitarás citar algún resultado sobre funciones de una variable, que garantice que $\frac{d}{dx}(\sum_{n=1}^{\infty} f_n(x)) = \sum_{n=1}^{\infty} f_n'(x)$; ver el libro de Spivak.

9. (i) *Fórmula de sumación por partes*: si $S_k = s_0 + s_1 + \ldots + s_k$, demuestra que

$$\sum_{k=0}^{N} a_k s_k = a_N S_N - \sum_{k=0}^{N-1} (a_{k+1} - a_k) S_k.$$

(ii) *Criterio de Dirichlet*: demuestra que la serie numérica $\sum_{n=0}^{\infty} a_n s_n$ es convergente cuando

$$a_n \searrow 0, \quad \text{y} \quad \{s_n\} \quad \text{tiene sumas parciales acotadas}$$

(iii) Demostrar la convergencia de $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \ldots$ y $\operatorname{sen} x + \frac{\operatorname{sen} 2x}{2} + \frac{\operatorname{sen} 3x}{3} + \ldots$.

Nota: En (iii) hay que justificar que las sumas parciales $|\operatorname{sen} x + \operatorname{sen}(2x) + \ldots + \operatorname{sen}(nx)|$ están acotadas en n (puedes usar series geométricas para calcular esta expresión).

10. (i) Demostrar que la serie $\sum_{n=2}^{\infty} \frac{1}{\log n} \sin(2\pi nx)$ es convergente $\forall\, x \in \mathbb{R}$.

(ii) Demostrar que la serie en (i) no es la serie de Fourier de ninguna función integrable.

Sugerencia: En (ii) utiliza el teorema de integración término a término de series de Fourier.

11. *La desigualdad de Wirtinger:* Sea f una función T-periódica y de clase C^1.

(i) Si la media $\int_0^T f = 0$, demuestra que

$$\|f\|_{L^2[0,T)} \;\le\; \tfrac{T}{2\pi}\, \|f'\|_{L^2[0,T)}.$$

(ii) Si $f(0) = f(T) = 0$, demuestra que

$$\|f\|_{L^2[0,T)} \;\le\; \tfrac{T}{\pi}\, \|f'\|_{L^2[0,T)}.$$

(iii) ¿Sabrías decir para qué funciones se tiene la igualdad en cada caso?

Sugerencia: En (i) utiliza Parseval. En (ii), extiende f de modo impar a $[-T, T]$ y utiliza (i), apropiadamente adaptado a funciones $2T$-periódicas.

5. *Series de Fourier reales:* Si $f \in L^1[0,1)$, podemos formalmente escribir

$$\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi nx) + b_n \sin(2\pi nx)].$$

a) Expresa a_n y b_n en términos de $\hat{f}(n)$, y viceversa

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi inx} = \hat{f}(0) + \sum_{n>0} \hat{f}(n) e^{2\pi inx} + \sum_{n<0} \hat{f}(n) e^{2\pi inx} =$$

$$= \underbrace{\hat{f}(0)}_{2\cdot a_0} + \sum_{n>0} \left[\hat{f}(n) e^{2\pi inx} + \hat{f}(-n) e^{-2\pi inx} \right]$$

$$\hat{f}(n) e^{2\pi inx} + \hat{f}(-n) e^{-2\pi inx} = \hat{f}(n) \left[\cos(2\pi nx) + i\,\text{sen}(2\pi nx) \right]$$
$$+ \hat{f}(-n) \left[\cos(2\pi nx) - i\,\text{sen}(2\pi nx) \right]$$

$$= \underbrace{\left(\hat{f}(n) + \hat{f}(-n)\right)}_{a_n} \cos(2\pi nx) + \underbrace{\left(\hat{f}(n) - \hat{f}(-n)\right) i}_{b_n}\, \text{sen}(2\pi nx)$$

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(2\pi nx) + b_n\, \text{sen}(2\pi nx)\right) =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \frac{e^{2\pi inx} + e^{-2\pi inx}}{2} \cdots$$

$$\left.\begin{array}{l} \hat{f}(n) + \hat{f}(-n) = a_n \\ i\left(\hat{f}(n) - \hat{f}(-n)\right) = b_n \end{array}\right\} \begin{array}{l} 2\hat{f}(n) = a_n - ib_n \to \hat{f}(n) = \dfrac{a_n - ib_n}{2} \\ 2\hat{f}(-n) = a_n - b_n \to \hat{f}(-n) = \dfrac{a_n - b_n/i}{2} = \dfrac{a_n + ib_n}{2} \end{array}$$

b) Demuestra que $b_n = 2\int_0^1 f(x)\sin(2\pi n x)\,dx$ y encuentra una fórmula similar para a_n.

$$b_n = \left(\hat{f}(n) - \hat{f}(-n)\right) i = i \cdot \left(\int_0^1 f(t) \cdot e^{-2\pi i n t}\, dt - \int_0^1 f(t) \cdot e^{2\pi i n t}\, dt \right) =$$

$$= i \cdot \int_0^1 -f(t) \cdot 2i \,\text{sen}(2\pi n t)\, dt = 2\int_0^1 f(t)\, \text{sen}(2\pi n t)\, dt$$

$$a_n = \hat{f}(n) + \hat{f}(-n) = \int_0^1 f(t) e^{-2\pi i n t}\, dt + \int_0^1 f(t) e^{2\pi i n t}\, dt =$$

$$= \int_0^1 f(t) \cdot 2\cos(2\pi n t)\, dt = 2\int_0^1 f(t) \cos(2\pi n t)\, dt$$

6. *Series de funciones reales, pares e impares.* Sea $f \in L^1[-\frac{1}{2}, \frac{1}{2})$.

(i) Demuestra que f es real si y sólo si $\hat{f}(n) = \overline{\hat{f}(-n)}$, $\forall\, n \in \mathbb{Z}$.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (\hat{f}(n) + \hat{f}(-n)) \cos(2\pi n x) + i \sum_{n=1}^{\infty} (\hat{f}(n) - \hat{f}(-n)) \operatorname{sen}(2\pi n x)$$

$\boxed{\Leftarrow}$

$$\int_{-1/2}^{1/2} f(x) e^{-2\pi i n x} dx = \overline{\int_{-1/2}^{1/2} f(x) e^{2\pi i n x} dx} = \int_{-1/2}^{1/2} \overline{f(x) e^{2\pi i n x}}\, dx =$$

$$\overline{(a+bi)(c+di)} = ac + adi + bci - bd = \overline{(ac-bd) + (ad+bc)i} =$$

$$= (ac-bd) - (ad+bc)i$$

$$\overline{(a+bi)}\;\overline{(c+di)} = (a-bi)(c-di) = ac - adi - bci - bd = (ac-bd) - (ad+bc)i$$

$$= \int_{-1/2}^{1/2} \overline{f(x)}\, e^{-2\pi i n x} dx$$

Y entonces $f(x) = \overline{f(x)} \to f$ es real

$\boxed{\Rightarrow}$ $f(x) = \overline{f(x)} \Rightarrow \underbrace{\int_{-1/2}^{1/2} f(x) e^{-2\pi i n x} dx}_{=\hat{f}(n)} = \int_{-1/2}^{1/2} \overline{f(x)}\, e^{-2\pi i n x} dx =$

$$= \int_{-1/2}^{1/2} \overline{f(x)}\, \overline{e^{2\pi i n x}} dx = \overline{\int_{-1/2}^{1/2} f(x) e^{2\pi i n x} dx} = \overline{\hat{f}(-n)}$$

(ii) Demuestra que f es real si y sólo si los coeficientes a_n, b_n del ejercicio 5 son reales.

f real $\iff \hat{f}(n) = \overline{\hat{f}(-n)}$

$\boxed{\Rightarrow}$

$$a_n = \hat{f}(n) + \hat{f}(-n) = \overline{\hat{f}(-n)} + \hat{f}(-n) = 2\operatorname{Re}(\hat{f}(-n)) \in \mathbb{R}$$

$$b_n = (\hat{f}(n) - \hat{f}(-n))\cdot i = (\overline{\hat{f}(-n)} - \hat{f}(-n))\, i = (-2i\operatorname{Im}(\hat{f}(-n)))\, i =$$

$$= 2\operatorname{Im}(\hat{f}(-n)) \in \mathbb{R} \quad //$$

$\boxed{\Leftarrow}$

$$\hat{f}(n) = \frac{a_n - i b_n}{2} = \overline{\hat{f}(-n)} \quad //$$

(iii) Caracteriza los coeficientes de Fourier $\hat{f}(n), a_n, b_n$ para las funciones pares.

$$f(x) = f(-x) \to b_n = 2\int_{-1/2}^{1/2} \underbrace{\underbrace{f(x)}_{\text{par}}\ \underbrace{\sin(2\pi n x)}_{\text{impar}}}_{\text{impar}}\, dx = 0 \quad //$$

$$\hat{f}(n) = \frac{a_n - i b_n}{2} = \frac{a_n}{2} \qquad \hat{f}(n) = \hat{f}(-n)$$

$$\hat{f}(-n) = \frac{a_n + i b_n}{2} = \frac{a_n}{2} \qquad a_n = 2\hat{f}(n) \quad //$$

y es

$$f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x} = \hat{f}(0) + \sum_{n>0} \hat{f}(n) e^{2\pi i n x} + \sum_{n<0} \hat{f}(n) e^{2\pi i n x} =$$

$$= \hat{f}(0) + \sum_{n>0} \hat{f}(n) e^{2\pi i n x} + \sum_{n>0} \underbrace{\hat{f}(-n)}_{= \hat{f}(n)} e^{2\pi i n x} = \hat{f}(0) + \sum_{n>0} \hat{f}(n)\cdot 2\cos(2\pi n x)$$

$$= \frac{a_0}{2} + \sum_{n>0} a_n \cos(2\pi n x)$$

(iv) Ídem para las funciones impares.

$$f(x) = -f(-x) \to a_n = \int_{-1/2}^{1/2} \underbrace{\underbrace{f(x)}_{\text{impar}} \underbrace{\cos(2\pi n x)}_{\text{par}}}_{\text{impar}} dx = 0$$

$$\hat{f}(n) = \frac{-i b_n}{2}$$

$$\hat{f}(-n) = \frac{i b_n}{2} = -\hat{f}(n)$$

$$b_n = 2i\hat{f}(n)$$

7. *Suavidad de f implica decaimiento de $\hat{f}$*. Demostrar que si $f \in C^k_{\text{per}}(\mathbb{R})$, entonces

$$\hat{f}(n) = o(1/|n|^k), \quad \text{para } |n| \to \infty.$$

Sugerencia: Utiliza integración por partes y el lema de Riemann-Lebesgue.

8. *Decaimiento de $\hat{f}$ implica suavidad de f:* Demuestra que si $f \in L^1[0,1)$ es tal que $\widehat{f}(n) = O(1/|n|^{k+2})$ para $|n| \to \infty$, entonces $f \in C^k_{\mathrm{per}}(\mathbb{R})$.

Nota: Necesitarás citar algún resultado sobre funciones de una variable, que garantice que $\frac{d}{dx}(\sum_{n=1}^{\infty} f_n(x)) = \sum_{n=1}^{\infty} f_n'(x)$; ver el libro de Spivak.

9. (i) *Fórmula de sumación por partes*: si $S_k = s_0 + s_1 + \ldots + s_k$, demuestra que

$$\sum_{k=0}^{N} a_k s_k = a_N S_N - \sum_{k=0}^{N-1} (a_{k+1} - a_k) S_k.$$

$s_k = S_k - S_{k-1} \quad k = 0, 1, \ldots \quad S_{-1} = 0$

$$\sum_{k=0}^{N} a_k s_k = \sum_{k=0}^{N} a_k (S_k - S_{k-1}) = \sum_{k=0}^{N} a_k S_k - \sum_{k=0}^{N} a_k S_{k-1} =$$

$$= a_N S_N + \sum_{k=0}^{N-1} a_k S_k - \cancel{a_0 \underbrace{S_{-1}}_{0}} - \sum_{j=0}^{N-1} a_{j+1} S_j =$$

$$= a_N S_N + \sum_{k=0}^{N-1} (a_k - a_{k+1}) S_k = a_N S_N - \sum_{k=0}^{N-1} (a_{k+1} - a_k) S_k$$

(ii) *Criterio de Dirichlet*: demuestra que la serie numérica $\sum_{n=0}^{\infty} a_n s_n$ es convergente cuando

$$a_n \searrow 0, \quad \text{y} \quad \{s_n\} \quad \text{tiene sumas parciales acotadas}$$

$a_n \geq a_{n+1} \geq \dots \geq 0$

$$\sum_{n=0}^{\infty} a_n s_n = \lim_{N\to\infty} \sum_{k=0}^{N} a_k s_k = \lim_{N\to\infty} a_N S_N - \sum_{k=0}^{N-1} (a_{k+1} - a_k) S_k =$$

$\{s_n\} \rightsquigarrow |S_k| < C \ , \ \forall k$

$$= \lim_{N} a_N S_N + \sum_{k=0}^{N-1} \underset{\substack{\vee\!\vee \\ 0}}{(a_k - a_{k+1})} S_k \leq \lim_{N} a_N C + \sum_{k=0}^{N-1} (a_k - a_{k+1}) C =$$

$$= \lim_{N} a_N C + (a_0 - a_N) C = \lim_{N} a_0 C = a_0 C$$

Está acotada. ¿Es convergente?

$$\lim_{M,N\to\infty} \left| \sum_{k=M}^{N} a_k s_k \right| = \lim_{M,N\to\infty} \left| a_N S_N - a_M S_{M-1} + \sum_{n=M}^{N-1} (a_n - a_{n+1}) S_n \right| \leq$$

$$\leq \lim_{M,N\to\infty} |a_N| C + |a_M| C + \sum_{n=M}^{N-1} |a_n - a_{n+1}| |S_n| \leq$$

$$\leq \lim_{M,N\to\infty} C\Big(|a_N| + |a_M| + \sum_{n=M}^{N-1} \underset{\substack{\vee\!\vee \\ 0}}{(a_n - a_{n+1})}\Big) = \lim_{M,N\to\infty} C(\cancel{a_N} + a_M + a_M - \cancel{a_N}) =$$

$$= \lim_{M,N\to\infty} 2 \cdot C \cdot a_M = 0$$

(iii) Demostrar la convergencia de $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \ldots$ y $\operatorname{sen} x + \frac{\operatorname{sen} 2x}{2} + \frac{\operatorname{sen} 3x}{3} + \ldots$.

$a_n = \frac{1}{n} \quad n \geq 1 \qquad s_n = (-1)^n \qquad \sum_{n=1}^{N} s_n = \begin{cases} -1 & n \text{ impar} \\ 0 & n \text{ par} \end{cases}$ acotado

$b_n = \frac{1}{n} \qquad s_n =$

10. (i) Demostrar que la serie $\sum_{n=2}^{\infty} \frac{1}{\log n} \sin(2\pi n x)$ es convergente $\forall\, x \in \mathbb{R}$.

(ii) Demostrar que la serie en (i) no es la serie de Fourier de ninguna función integrable.

(i) Por el ej 9, $a_n = \frac{1}{\log n}$, $s_n = \sin(2\pi n x)$

(ii) De serlo, entonces

$$g \in L^1(\mathbb{T}) \quad b_n(g) = \frac{1}{\log n}, \quad n \geq 2$$

$$a_n(g) = b_1(n) = 0, \ \forall n$$

y la función primitiva sería $G(x) = \sum_{n=2}^{\infty} \frac{-\cos(2\pi n x)}{2\pi n \log n}$

converge $\forall x \in \mathbb{T}$

pero $G(0) = \frac{-1}{2\pi} \sum \frac{1}{n \log n} = -\infty$ $\#$

11. *La desigualdad de Wirtinger:* Sea f una función T-periódica y de clase C^1.

(i) Si la media $\int_0^T f = 0$, demuestra que

$$\|f\|_{L^2[0,T)} \leq \frac{T}{2\pi} \|f'\|_{L^2[0,T)}.$$