Tarea 3: El aparato de Frenet

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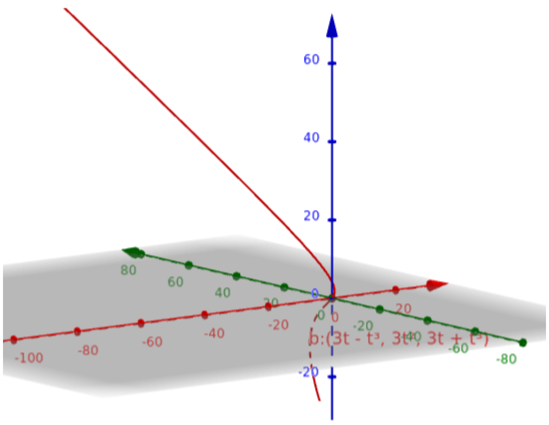
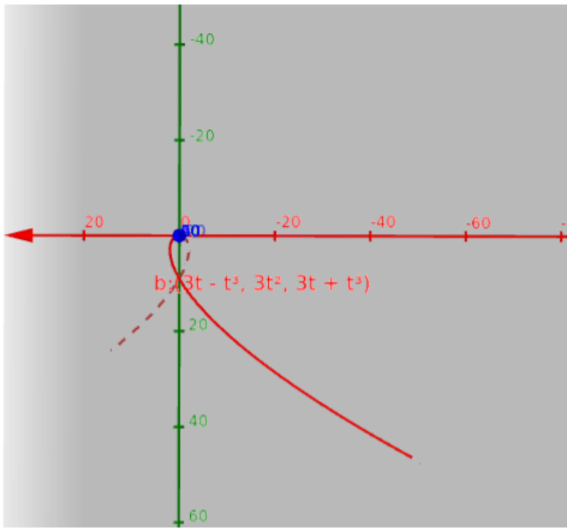
Dada una curva regular en $\mathbb{R}^3$, $\boldsymbol{\alpha : I \subset \mathbb{R} \to \mathbb{R}^3}$, su aparato de Frenet es la familia $\{\boldsymbol{k, \tau, T, N, B}\}$ formada por su curvatura, su torsión y su triedro de Frenet. Calcule el aparato de Frenet de la siguiente curva:

$$\boldsymbol{\alpha(s) = \left(3s - s^3, 3s^2, 3s + s^3\right)}$$

La curva es así:





Debemos tener cuidado al hacer esto, pues las definiciones del libro están dadas para curvas p.p.a..

Tenemos, primero, que

$$\alpha' = \left(3 - 3s^2, 6s, 3 + 3s^2\right)$$
$$\alpha'' = (-6s, 6, 6s)$$
$$\alpha''' = (-6, 0, 6)$$

Por lo que

$$\alpha' \wedge \alpha'' = \begin{vmatrix} i & j & k \\ 3 - 3s^2 & 6s & 3 + 3s^2 \\ -6s & 6 & 6s \end{vmatrix} = \left(36s^2 - 18 - 18s^2, -18s - 18s^2 - 18s + 18s^2, 18 - 18s^2 + 36s^2\right) =$$

$$= \left(18s^2 - 18, -36s, 18s^2 + 18\right) = 18\left(s^2 - 1, -2s, s^2 + 1\right)$$

Usando la proposición 1.3.5, podemos ya calcular la curvatura y la torsión:

$$k = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3} = \frac{18\sqrt{s^4 + 1 - 2s^2 + 4s^2 + s^4 + 1 + 2s^2}}{\left(9 + 9s^4 - 18s^2 + 36s^2 + 9 + 9s^4 + 18s^2\right)^{\frac{3}{2}}} = \frac{18\sqrt{2s^4 + 2 + 4s^2}}{\left(18 + 18s^4 + 36s^2\right)^{\frac{3}{2}}} = \frac{18\sqrt{2}\sqrt{s^4 + 2s^2 + 1}}{\sqrt{18}^3\sqrt{1 + 2s^2 + s^4}^3} =$$

$$= \frac{\sqrt{2}}{\sqrt{18}\left(s^2 + 1\right)^2} = \frac{1}{\left(s^2 + 1\right)^2}\sqrt{\frac{2}{18}} = \frac{1}{\left(s^2 + 1\right)^2}\sqrt{\frac{1}{9}} = \frac{1}{3\left(s^2 + 1\right)^2}$$

Por otro lado,

$$det\left(\alpha',\alpha'',\alpha'''\right)=\begin{vmatrix} 3-3s^2 & 6s & 3+3s^2 \\ -6s & 6 & 6s \\ -6 & 0 & 6 \end{vmatrix}=216$$

Por lo que

$$\tau=-\frac{det\left(\alpha',\alpha'',\alpha'''\right)}{|\alpha'\wedge\alpha''|^2}=-\frac{216}{648\left(s^4+2s^2+1\right)}=-\frac{1}{3\left(s^2+1\right)^2}$$

Ahora, como queremos formar el triedro de Frenet, pero la curva no es p.p.a., tenemos que normalizar para obtener T.

$$T=t\left(s\right)=\frac{\alpha'}{|\alpha'|}=\frac{\left(3-3s^2,6s,3+3s^2\right)}{\sqrt{18}\left(s^2+1\right)}=\frac{1}{\sqrt{18}}\left(\frac{3-3s^2}{s^2+1},\frac{6s}{s^2+1},3\right)=\frac{1}{\sqrt{2}}\left(\frac{1-s^2}{s^2+1},\frac{2s}{s^2+1},1\right)$$

Una vez tenemos el vector tangente unitario, calculamos el vector normal unitario (estas cuentas las realicé antes de hacer la última simplificación en el vector tangente, pero sale bien y no merece la pena volver a hacerlas):

$$N=n\left(s\right)=\frac{t'}{|t'|}=\frac{\frac{1}{\sqrt{18}}\left(\frac{-6s\left(s^2+1\right)-\left(3-3s^2\right)2s}{\left(s^2+1\right)^2},\frac{6\left(s^2+1\right)-12s^2}{\left(s^2+1\right)^2},0\right)}{|t'|}=$$

$$=\frac{\frac{1}{\sqrt{18}}\left(\frac{-12s}{\left(s^2+1\right)^2},\frac{-6\left(s^2-1\right)}{\left(s^2+1\right)^2},0\right)}{\frac{1}{\sqrt{18}}\sqrt{\frac{144s^2+36\left(s^2-1\right)^2}{\left(s^2+1\right)^4}}}=\frac{\left(-12s,-6\left(s^2-1\right),0\right)}{\sqrt{144s^2+36\left(s^4+1-2s^2\right)}}=\frac{\left(-12s,-6\left(s^2-1\right),0\right)}{\sqrt{36s^4+72s^2+36}}=$$

$$=\frac{\left(-12s,-6\left(s^2-1\right),0\right)}{6s^2+6}=\left(\frac{-2s}{s^2+1},\frac{1-s^2}{s^2+1},0\right)$$

Y, por último, el binormal unitario:

$$B=b\left(s\right)=t\left(s\right)\wedge n\left(s\right)=\begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{2}}\frac{1-s^2}{s^2+1} & \frac{1}{\sqrt{2}}\frac{2s}{s^2+1} & \frac{1}{\sqrt{2}} \\ \frac{-2s}{s^2+1} & \frac{1-s^2}{s^2+1} & 0 \end{vmatrix}=\left(\frac{1}{\sqrt{2}}\frac{s^2-1}{s^2+1},\frac{-\sqrt{2}s}{s^2+1},\frac{\left(1-1s^2\right)\left(1-s^2\right)+4s^2}{\sqrt{2}\left(s^2+1\right)^2}\right)=$$

$$=\frac{1}{\sqrt{2}}\left(\frac{s^2-1}{s^2+1},\frac{-2s}{s^2+1},\frac{1-s^2-s^2+s^4+4s^2}{\left(s^2+1\right)^2}\right)=\frac{1}{\sqrt{2}}\left(\frac{s^2-1}{s^2+1},\frac{-2s}{s^2+1},\frac{1+2s^2+1s^4}{\left(s^2+1\right)^2}\right)=$$

$$=\frac{1}{\sqrt{2}}\left(\frac{s^2-1}{s^2+1},\frac{-2s}{s^2+1},1\right)$$

Y ahora podemos hacer las comprobaciones oportunas:

- $|t|=1$:

$$|t|=\frac{1}{\sqrt{2}}\sqrt{\left(\frac{1-s^2}{s^2+1}\right)^2+\left(\frac{2s}{s^2+1}\right)^2+1}=\frac{1}{\sqrt{2}}\sqrt{\frac{1+s^4-2s^2+4s^2}{\left(s^2+1\right)^2}+1}=$$

$$=\frac{1}{\sqrt{2}}\sqrt{\frac{1+2s^2+1s^4}{\left(s^2+1\right)^2}+1}=\frac{1}{\sqrt{2}}\sqrt{\frac{\left(s^2+1\right)^2}{\left(s^2+1\right)^2}+1}=\frac{1}{\sqrt{2}}\sqrt{1+1}=1$$

- $|n| = 1$:

$$|n| = \sqrt{\left(\frac{-2s}{s^2+1}\right)^2 + \left(\frac{1-s^2}{s^2+1}\right)^2} = \sqrt{\frac{4s^2+1+s^4-2s^2}{\left(s^2+1\right)^2}} = \sqrt{\frac{s^4+2s^2+1}{\left(s^2+1\right)^2}} = \sqrt{\frac{\left(s^2+1\right)^2}{\left(s^2+1\right)^2}} = 1$$

- $\langle t, n\rangle = 0$:

$$\langle t, n\rangle = \frac{\left(3-3s^2\right)(-2s) + 6s\left(1-s^2\right)}{\left(s^2+1\right)^2} = \frac{-6s+6s^3+6s-6s^3}{\left(s^2+1\right)^2} = 0$$

- $|b| = 1$:

$$|b| = \frac{1}{\sqrt{2}}\sqrt{\left(\frac{s^2-1}{s^2+1}\right)^2 + \left(\frac{-2s}{s^2+1}\right)^2 + 1} = \frac{1}{\sqrt{2}}\sqrt{\frac{1+s^4-2s^2+4s^2}{\left(s^2+1\right)^2} + 1} = \frac{1}{\sqrt{2}}\sqrt{\frac{1+2s^2+s^4}{\left(s^2+1\right)^2} + 1} = 1$$

- $\langle t, b\rangle = 0$:

$$\langle t, b\rangle = \left\langle \frac{1}{\sqrt{2}}\left(\frac{1-s^2}{s^2+1}, \frac{2s}{s^2+1}, 1\right), \frac{1}{\sqrt{2}}\left(\frac{s^2-1}{s^2+1}, \frac{-2s}{s^2+1}, 1\right)\right\rangle =$$

$$= \frac{1}{2}\left(\frac{s^2-1-s^4+s^2-4s^2}{\left(s^2+1\right)^2} + 1\right) = \frac{1}{2}\left(\frac{-s^4-2s^2-1}{\left(s^2+1\right)^2} + 1\right) = \frac{1}{2}\left(\frac{-\left(s^2+1\right)^2}{\left(s^2+1\right)} + 1\right) =$$

$$= \frac{1}{2}(-1+1) = 0$$

- $\langle n, b\rangle = 0$:

$$\langle n, b\rangle = \left\langle \left(\frac{-2s}{s^2+1}, \frac{1-s^2}{s^2+1}, 0\right), \frac{1}{\sqrt{2}}\left(\frac{s^2-1}{s^2+1}, \frac{-2s}{s^2+1}, 1\right)\right\rangle = \frac{1}{\sqrt{2}}\left(\frac{-2\left(s^2-1\right)+2\left(s^2-1\right)}{\left(s^2+1\right)^2}\right) = 0$$

- El triedro de Frenet es una base positivamente orientada, $det\,(T, N, B) = 1$:

$$\begin{vmatrix} \frac{1}{\sqrt{2}}\frac{1-s^2}{s^2+1} & \frac{1}{\sqrt{2}}\frac{2s}{s^2+1} & \frac{1}{\sqrt{2}} \\ \frac{-2s}{s^2+1} & \frac{1-s^2}{s^2+1} & 0 \\ \frac{1}{\sqrt{2}}\frac{s^2-1}{s^2+1}, & \frac{1}{\sqrt{2}}\frac{-2s}{s^2+1}, & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{2}\frac{\left(1-s^2\right)^2}{\left(s^2+1\right)^2} + \frac{1}{2}\cdot\frac{4s^2}{\left(s^2+1\right)^2} - \left(\frac{1}{2}\frac{-\left(1-s^2\right)^2}{\left(s^2+1\right)^2} + \frac{1}{2}\frac{-4s^2}{\left(s^2+1\right)^2}\right) =$$

$$= \frac{1}{2}\frac{1-2s^2+s^4+4s^2}{\left(s^2+1\right)^2}\cdot 2 = \frac{\left(s^2+1\right)^2}{\left(s^2+1\right)^2} = 1$$

Y vemos como todo funciona como queremos.

Recapitulando, tenemos el aparato de Frenet dado por:

$$k = \frac{1}{3\left(s^2+1\right)^2}$$

$$\tau = -\frac{1}{3\left(s^2+1\right)^2}$$

$$T = \frac{1}{\sqrt{2}}\left(\frac{1-s^2}{s^2+1}, \frac{2s}{s^2+1}, 1\right)$$

$$N = \left(\frac{-2s}{s^2+1}, \frac{1-s^2}{s^2+1}, 0\right)$$

$$B = \frac{1}{\sqrt{2}}\left(\frac{s^2-1}{s^2+1}, \frac{-2s}{s^2+1}, 1\right)$$