UNIVERSIDAD DE MURCIA
Departamento de Matemáticas

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Importante: Justifica detalladamente todas tus respuestas.

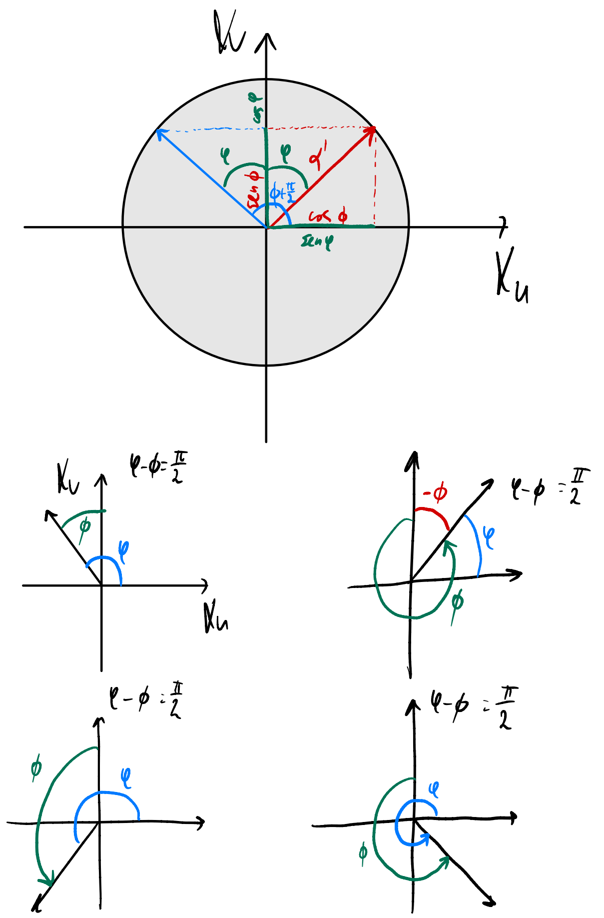
Considera la superficie de revolución S generada al girar la curva parametrizada regular $(f(u), 0, g(u))$ alrededor del eje OZ, dada por la parametrización

$$X(u,v) = (f(u)\cos v, f(u)\operatorname{sen} v, g(u)),$$

con $f(u) > 0$ y $(f'(u))^2 + (g'(u))^2 > 0$.

Sea $\alpha(s) = X(u(s), v(s))$ un geodésica de S parametrizada por el arco y definida en un cierto intervalo $I \subset \mathbb{R}$, $\alpha : I \to S$. Demuestra que la función $f(u(s))\operatorname{sen}\varphi(s)$ es una constante c, donde $\varphi(s)$ es el ángulo que forman los vectores $\frac{\partial X}{\partial u}(u(s), v(s))$ y $\alpha'(s)$.

Indicación: Si denotamos por $\phi(s)$ el ángulo que forman los vectores $\frac{\partial X}{\partial v}(u(s), v(s))$ y $\alpha'(s)$, comprueba primero que $\operatorname{sen}\varphi(s) = \cos\phi(s) = f(u(s))v'(s)$. La ecuación intrínseca de las geodésicas en la parametrización $X(u,v)$ también puede ayudar. ¡Y recuerda en todo momento que α está parametrizada por el arco!



$\alpha + \beta = \frac{\pi}{2} \Rightarrow \cos(\alpha) = \operatorname{sen}(\beta)$

$0 = \cos\left(\frac{\pi}{2}\right) = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta$

$\Rightarrow \cos\alpha\cos\beta = \operatorname{sen}\alpha\operatorname{sen}\beta$

$\cos^2\alpha\cos^2\beta = \operatorname{sen}^2\alpha\operatorname{sen}^2\beta$

$\cos^2\alpha(1-\operatorname{sen}^2\beta) = (1-\cos^2\alpha)\operatorname{sen}^2\beta$

$\cos^2\alpha - \cancel{\cos^2\alpha\operatorname{sen}^2\beta} = \operatorname{sen}^2\beta - \cancel{\cos^2\alpha\operatorname{sen}^2\beta}$

$\alpha - \beta = \frac{\pi}{2} \Rightarrow 0 = \cos(\alpha-\beta) = \cos\alpha\cos\beta + \operatorname{sen}\alpha\operatorname{sen}\beta$

$\longrightarrow \cos^2\alpha\cos^2\beta = \operatorname{sen}^2\alpha\operatorname{sen}^2\beta \longrightarrow \cos^2\alpha = \operatorname{sen}^2\beta$

$\varphi - \phi = \frac{3}{2}\pi$

$$\left(f(u(s))\,\text{sen}\,\varphi(s)\right)' = f'(u(s))\,u'(s)\,\text{sen}\,\varphi(s) + f(u(s))\cos\varphi(s)$$

$$\varphi(s) = \measuredangle\left(\frac{\partial X}{\partial u}, \gamma'\right)$$

$$X(u,v) = (f(u)\cos v, f(u)\,\text{sen}\,v, g(u)),$$

$$\phi(s) = \measuredangle\left(\frac{\partial X}{\partial v}, \gamma'\right)$$

$$\frac{\partial X}{\partial u} = (f'(u)\cos v, f'(u)\,\text{sen}\,v, g'(u)) \qquad E = f'(u)^2 + g'(u)^2 \qquad E_u = 2(f'f'' + g'g'')$$

$$F = 0 \qquad E_v = 0 \qquad G_v = 0$$

$$\frac{\partial X}{\partial v} = (-f(u)\,\text{sen}\,v, f(u)\cos v, 0) \qquad G = f^2(u) \qquad G_u = 2f'f$$

$$\underset{\text{sen}\,\varphi(s)}{\underset{\shortparallel}{\cos\phi(s)}} = \frac{\langle \frac{\partial X}{\partial v}, \gamma' \rangle}{\|\frac{\partial X}{\partial v}\| \underset{1}{\underset{\shortparallel}{\|\gamma'\|}}} = \frac{v'(s)\cdot\langle \frac{\partial X}{\partial v}, \frac{\partial X}{\partial v}\rangle}{\|\frac{\partial X}{\partial v}\|} = \frac{v'(s)\cdot\|\frac{\partial X}{\partial v}\|^{\cancel{2}}}{\cancel{\|\frac{\partial X}{\partial v}\|}} = f(u)v' \;/\!/$$

$$\gamma(s) = X(u(s), v(s)) \longrightarrow \gamma'(s) = \frac{\partial X}{\partial u}(u(s),v(s))\cdot u'(s) + \frac{\partial X}{\partial v}(u(s),v(s))\cdot v'(s)$$

$$\cos\varphi(s) = \frac{\langle \frac{\partial X}{\partial u}, \gamma' \rangle}{\|\frac{\partial X}{\partial u}\|\,\|\gamma'\|} = \frac{u'(s)\,\|\frac{\partial X}{\partial u}\|^{\cancel{2}}}{\cancel{\|\frac{\partial X}{\partial u}\|}} = u'(s)\sqrt{f'^2 + g'^2}$$

$$\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} \frac{E_u}{2} & \frac{E_v}{2} & F_v - \frac{G_u}{2} \\ F_u - \frac{E_v}{2} & \frac{G_u}{2} & \frac{G_v}{2} \end{pmatrix}$$

$$EG - F^2 = (f'(u)^2 + g'(u)^2)\,f^2(u)$$

$$\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \frac{1}{(f'^2+g'^2)f^2}\begin{pmatrix} f^2 & 0 \\ 0 & f'^2+g'^2 \end{pmatrix}\begin{pmatrix} f'f''+g'g'' & 0 & -f'f \\ 0 & f'f & 0 \end{pmatrix}$$

$$\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \frac{1}{(f'^2+g'^2)f^2} \begin{pmatrix} f^2 & 0 \\ 0 & f'^2+g'^2 \end{pmatrix} \begin{pmatrix} f'f''+g'g'' & 0 & -f'f \\ 0 & f'f & 0 \end{pmatrix}$$

$$= \frac{1}{(f'^2+g'^2)f^2} \begin{pmatrix} f^2(f'f''+g'g'') & 0 & -f'f^3 \\ 0 & f'f(f'^2+g'^2) & 0 \end{pmatrix}$$

$$\Gamma^1_{11} = \frac{f'f''+g'g''}{(f'^2+g'^2)} \qquad \Gamma^1_{12} = 0 \qquad \Gamma^1_{22} = \frac{-f'f}{(f'^2+g'^2)}$$

$$\Gamma^2_{11} = 0 \qquad \Gamma^2_{12} = \frac{f'}{f} \qquad \Gamma^2_{22} = 0$$

$$\begin{cases} u'' + (u')^2\Gamma^1_{11}(u,v) + 2u'v'\Gamma^1_{12}(u,v) + (v')^2\Gamma^1_{22}(u,v) = 0 \\ v'' + (u')^2\Gamma^2_{11}(u,v) + 2u'v'\Gamma^2_{12}(u,v) + (v')^2\Gamma^2_{22}(u,v) = 0 \end{cases}$$

$$\begin{cases} u'' + u'^2 \dfrac{f'f''+g'g''}{(f'^2+g'^2)} - v'^2 \dfrac{f'f}{(f'^2+g'^2)} = 0 \\ v'' + 2u'v' \dfrac{f'}{f} = 0 \end{cases}$$

$$\left(f(u(s)) \cdot \operatorname{sen} \varphi(s) \right)' = \left(f(u(s)) \cdot f(u(s))\, v'(s) \right)' = \left(f^2(u(s))\, v'(s) \right)' =$$

$$= 2 f'(u(s)) \cdot f(u(s)) \cdot v'(s)\, u'(s) + f^2(u(s))\, v''(s)$$

$\cdot f^2$ //