EDPs y series de Fourier, 3º Matemáticas/4º PES **Test 2, 2/4/2020**

Nombre:

. . . .

1. Considera la función

$$f(x) = \begin{cases} \cos(\pi x), & x \in (0, 1/2) \\ -\cos(\pi x), & x \in (-1/2, 0) \end{cases}$$

a) Demuestra que $f(x) \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \operatorname{sen}(2\pi n x)$

b) Esboza la gráfica de f y determina la convergencia de la serie en cada $x \in [-\frac{1}{2}, \frac{1}{2}]$

c) Demuestra que $\frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{10^2-1} - \frac{7}{14^2-1} + \ldots = \frac{\pi\sqrt{2}}{16}$.

d) Calcula la suma $S = \sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2}$.

Nota: Para facilitar los cálculos de integrales en a) puedes usar fórmulas del tipo

$$\operatorname{sen}(A)\cos(B) = \frac{\operatorname{sen}(A+B) + \operatorname{sen}(A-B)}{2},$$

o bien, si prefieres las exponenciales complejas, usar $\cos(\pi x) = (e^{i\pi x} + e^{-i\pi x})/2$.

$$f(x) = \begin{cases} \cos(\pi x), & x \in (0, 1/2) \\ -\cos(\pi x), & x \in (-1/2, 0) \end{cases}$$

a) Demuestra que $f(x) \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} \operatorname{sen}(2\pi n x)$

$$f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{2\pi i n x} \qquad \hat{f}(n) = \int_{-1/2}^{1/2} f(x) e^{-2\pi i n x}\, dx$$

$$\hat{f}(n) = -\int_{-1/2}^{0} \cos(\pi x) e^{-2\pi i n x}\, dx + \int_{0}^{1/2} \cos(\pi x) e^{-2\pi i n x}\, dx = ⊛$$

$$\int \cos(\pi x) e^{-2\pi i n x}\, dx = \int \frac{e^{i\pi x} + e^{-i\pi x}}{2} e^{-2\pi i n x}\, dx =$$

$$= \frac{1}{2} \int e^{-i\pi x(2n-1)} + e^{-i\pi x(2n+1)}\, dx =$$

$$= \frac{1}{2} \left[-\frac{e^{-i\pi x(2n-1)}}{i\pi(2n-1)} - \frac{e^{-i\pi x(2n+1)}}{i\pi(1+2n)} \right]$$

$$⊛ = -\frac{1}{2} \left[-\frac{1}{i\pi(2n-1)} - \frac{1}{i\pi(2n+1)} + \frac{\cancelto{(-1)^{n+1}}{e^{i\frac{\pi}{2}(2n-1)}}}{i\pi(2n-1)} + \frac{\cancelto{(-1)^{n}}{e^{i\frac{\pi}{2}(2n+1)}}}{i\pi(2n+1)} \right] +$$

$$+ \frac{1}{2} \left[-\frac{\cancelto{(-1)^{n}}{e^{-i\frac{\pi}{2}(2n-1)}}}{i\pi(2n-1)} - \frac{\cancelto{(-1)^{n+1}}{e^{-i\frac{\pi}{2}(2n+1)}}}{i\pi(2n+1)} + \frac{1}{i\pi(2n-1)} + \frac{1}{i\pi(2n+1)} \right] =$$

$$\circledast = -\frac{1}{2}\left[-\frac{1}{i\pi(2n-1)} - \frac{1}{i\pi(2n+1)} + \frac{\cancelto{(-1)^{n+1}}{e^{i\frac{\pi}{2}(2n-1)}}}{i\pi(2n-1)} + \frac{\cancelto{(-1)^{n}}{e^{i\frac{\pi}{2}(2n+1)}}}{i\pi(2n+1)}\right] +$$

$$+\frac{1}{2}\left[-\frac{\cancelto{(-1)^{n}}{e^{-i\frac{\pi}{2}(2n-1)}}}{i\pi(2n-1)} - \frac{\cancelto{(-1)^{n+1}}{e^{-i\frac{\pi}{2}(2n+1)}}}{i\pi(2n+1)} + \frac{1}{i\pi(2n-1)} + \frac{1}{i\pi(2n+1)}\right] =$$

$$= +\frac{1}{2}\left[+\frac{1}{i\pi(2n-1)} + \frac{1}{i\pi(2n+1)} + \cancel{\frac{(-1)^{n+1}}{i\pi(2n-1)}} + \cancel{\frac{(-1)^{n}}{i\pi(2n+1)}}\right] + \frac{1}{2}\left[\cancel{\frac{(-1)^{n+1}}{i\pi(2n-1)}} + \cancel{\frac{(-1)^{n}}{i\pi(2n+1)}} +\right.$$

$$\left. + \frac{1}{i\pi(2n-1)} + \frac{1}{i\pi(2n+1)}\right] = \frac{1}{i\pi(2n-1)} + \frac{1}{i\pi(2n+1)} = \frac{2n+1+2n-1}{i\pi(4n^2-1)} = \frac{4n}{i\pi(4n^2-1)}$$

Θ see

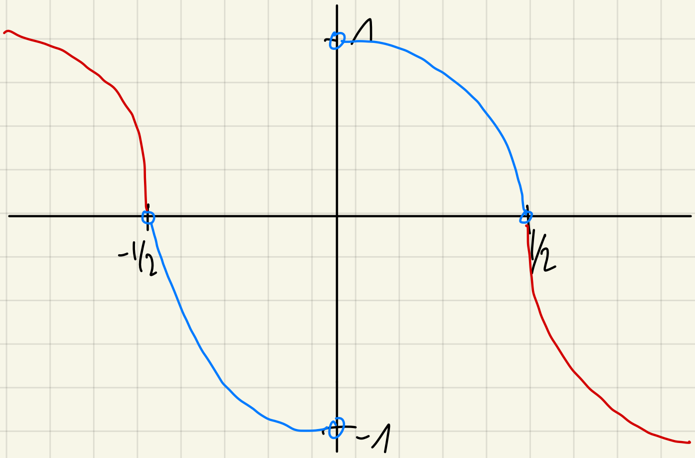
$$f(x) \sim \sum_{n\in\mathbb{Z}} \frac{4n}{i\pi(4n^2-1)} e^{2\pi i n x} = \sum_{n>0} \frac{4n}{i\pi(4n^2-1)} e^{2\pi i n x} + \sum_{n<0} \frac{4n}{i\pi(4n^2-1)} e^{2\pi i n x} =$$

$$= \sum_{n>0} \frac{4n}{i\pi(4n^2-1)} e^{2\pi i n x} - \sum_{n>0} \frac{4n}{i\pi(4n^2-1)} e^{-2\pi i n x} = \sum_{n>0} \frac{4n}{i\pi(4n^2-1)}\left(e^{2\pi i n x} - e^{-2\pi i n x}\right) =$$

$$= \sum_{n>0} \frac{8n}{\pi(4n^2-1)} \operatorname{sen}(2\pi n x)$$

//

b) Esboza la gráfica de f y determina la convergencia de la serie en cada $x \in [-\frac{1}{2}, \frac{1}{2}]$



DINI 1: converge en $(-1/2, 0) \cup (0, 1/2)$

DINI 2: en $x=0$ converge a $\frac{-1+1}{2} = 0$ //

en $x = \pm 1/2$ converge a 0 //

c) Demuestra que $\frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{10^2-1} - \frac{7}{14^2-1} + \ldots = \frac{\pi\sqrt{2}}{16}$.

$$f(x) \overset{\text{excepto en } 0}{=} \sum_{n>0} \frac{8n}{\pi(4n^2-1)} \operatorname{sen}(2\pi n x) \qquad f(x) = \begin{cases} \cos(\pi x) & x \in (0, 1/2) \\ -\cos(\pi x) & x \in (-1/2, 0) \end{cases}$$

$$\frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right) = \sum_{n>0} \frac{8n}{\pi(4n^2-1)} \underbrace{\operatorname{sen}\left(\frac{\pi}{2} n\right)}_{\begin{cases} 0 & n \text{ par} \\ 1 & n = 1, 5, 9, \ldots \\ -1 & n = 3, 7, 11, \ldots \end{cases}} = \frac{8}{\pi}\left(\frac{1}{2^2-1} - \frac{3}{\underbrace{36}_{4n^2 = (2n)^2} - 1} + \ldots\right)$$

Por tanto, es

$$\frac{\pi\sqrt{2}}{16} = \frac{1}{2^2-1} - \frac{3}{6^2-1} + \frac{5}{10^2-1} - \frac{7}{14^2-1} + \ldots$$

d) Calcula la suma $S = \sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2}$.

excepto en 0

$$f(x) \overset{\downarrow}{=} \sum_{n>0} \frac{8n}{\pi(4n^2-1)} \operatorname{sen}(2\pi n x)$$

Por Parseval: $\int_{\mathbb{T}} |f(x)|^2 dx = \sum_{n\in\mathbb{Z}} |\hat{f}(n)|^2$

$$f(x) = \sum_{n\in\mathbb{Z}} \underbrace{\frac{4n}{i\pi(4n^2-1)}}_{\hat{f}(n)} e^{2\pi i n x}$$

$$\hat{f}(n) \to |\hat{f}(n)|^2 = \frac{16n^2}{\pi^2(4n^2-1)^2}$$

$$\sum_{n\in\mathbb{Z}} \frac{16n^2}{\pi^2(4n^2-1)^2} = \frac{16}{\pi^2} \sum_{n\in\mathbb{Z}} \frac{n^2}{(4n^2-1)^2} = \frac{32}{\pi^2} \sum_{n\geq 1} \frac{n^2}{(4n^2-1)^2}$$

$$\| $$

$$\int_{-1/2}^{1/2} |f(x)|^2 dx = \int_{-1/2}^{1/2} \cos^2(\pi x) dx = \int_{-1/2}^{1/2} \frac{\cos(2\pi x)+1}{2} dx = \frac{\cos(2\pi x)}{4\pi} + \frac{x}{2}\Bigg]_{-1/2}^{1/2} = \cancel{\frac{\cos(\pi)}{4\pi}} + \frac{1}{4} -$$

$$\cos(2\alpha) = \cos^2\alpha - \operatorname{sen}^2\alpha = 2\cos^2\alpha - 1 \to \cos^2\alpha = \frac{\cos(2\alpha)+1}{2}$$

$$- \cancel{\frac{\cos(-\pi)}{4\pi}} + \frac{1}{4} = \frac{1}{2}$$

Por tanto $\frac{32}{\pi^2} \sum_{n\geq 1} \frac{n^2}{(4n^2-1)^2} = \frac{1}{2} \longrightarrow \sum_{n\geq 1} \frac{n^2}{(4n^2-1)^2} = \frac{\pi^2}{64}$

Cosicas

$f\in L^1(\mathbb{T})$:

$$f(x)\sim\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{2\pi i n x}$$

$$\hat{f}(n)=\int_{\mathbb{T}} f(x)e^{-2\pi i n x}\,dx$$

DINI 1:

f derivable en en x_0 $\longrightarrow$ $\sim$ es $=$ en x_0

DINI 2

$\exists\, f(x_0^{\pm})$ y $f'(x_0^{\pm})$ entonces

$$\sum_{n\in\mathbb{Z}}\hat{f}(n)e^{2\pi i n x_0}=\frac{f(x_0^+)+f(x_0^-)}{2}$$

Formula Parceval

$f\in L^2(\mathbb{T})\longrightarrow$ $$\int_{\mathbb{T}}|f(x)|^2\,dx=\sum_{n\in\mathbb{Z}}|\hat{f}(n)|^2$$

Derivación SF

$f\in C^k(\mathbb{T})\longrightarrow$ $$f^{(k)}(x)\sim\sum_{n\in\mathbb{Z}}(2\pi i n)^k\hat{f}(n)e^{2\pi i n x}$$

Integración SF

$f\in L^1(\mathbb{T})$

$F(x)=\int_0^x f(t)\,dt$

$$\longrightarrow \left.\hat{F}(n)=\frac{\hat{f}(n)-\hat{f}(0)}{2\pi i n}\quad n\neq 0\right\}\longrightarrow F(x)-\hat{f}(0)x\sim\sum_{n\in\mathbb{Z}^*}\hat{F}(n)e^{2\pi i n x}+C_0$$