Module 2: A Simple 1D DG Solver

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Linear Solution Approximation: Definition

- $M = 1$ We must have two basis functions because
  - We shall choose two ramp functions, let's see what they look like...
- V_h We will choose in this simple example
  - to be the space of all polynomials
    - of order 1 and less
- We can think of the approximate solution
  - $\tilde{q} \in V_h$ ...belonging to some finite vector space
  - This vector space must
    - $\tilde{q}(x) = \sum_{i=0}^{M} a_i \psi_i(x)$ have a set of basis vectors ψ s.t.
- $q(x)$ The exact sol'n is not usually obtainable
  - $\tilde{q}(x)$ The best we can do is an approximate sol'n

Linear Solution Approximation: Bases and Resultant Approximation

- Basis funs
  - $\psi_0(x) = 1 - x$ Blue
  - $\psi_1(x) = x$ Red
- $\tilde{q} = a_0(1-x) + a_1(x)$ Resultant approximation is linear

Test Function Choice (Galerkin)

- In order to solve for the approximation
  - Our weak form sol'n
  - $\int \frac{\partial q}{\partial t}\phi_j + \int \frac{\partial f}{\partial x}\phi_j = 0, \forall j$ must be satisfied for each weighting function
- This means our weighting functions
  - are ramp functions as well
    - $\phi_0(x) = \psi_0(x)$
    - $\phi_1(x) = \psi_1(x)$
- We will discuss this in greater detail
  - but as in a FEM...
  - ...DG chooses the space of weighting function to be the same
    - $\phi \in V_h$ as the approximation space
- We have now defined our approximation space
  - But what about the weighting function in the weak form ϕ?

Upwind Flux

- We must have...everywhere
  - A constant flow with velocity c
  - if
    - Our flow is incompressible
    - and the global domain is 1D
- $c > 1$ Let's assume that
  - $c\tilde{q}^-$ the numerical flux is then...at all boundaries
- Note that there are
  - two distinct numerical fluxes
  - K for element
    - $cq^-(x_L)$ on the left it is
    - $cq^-(x_R)$ on the right is
    - $q^-(x_L) = q^-(x_R)$ where not necessarily

Mass Matrix

- If the time derivative...
  - of each of the approximation coefficients
  - $\sum_{i=0}^{M} \frac{da_i(t)}{dt} \int_k \psi_i(x)\phi_j(x)dx = \mathbf{M}\mathbf{a}'$ is a vector a' then
- Let's substitute in our expressions
  - for approximation & the 1st weighting basis & analytically integrate
    - $\int_k \psi_0(x)\phi_0(x)dx = (1-x)(1-x)\Big|_0^1 = \frac{1}{3}$ ψ - approximation
    - $\int_k \psi_1(x)\phi_0(x)dx = x(1-x)\Big|_0^1 = \frac{1}{6}$ ϕ - weighting basis
  - $\mathbf{M}_{j,i} = \int \begin{bmatrix} \psi_0\phi_0 & \psi_1\phi_0 \\ \psi_0\phi_1 & \psi_1\phi_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}$ Do this for each weighting basis to get 2*2 matrix:
- We now have explicit formula...
  - $\int_k \frac{\partial \tilde{q}}{\partial t}\phi_j dx = \sum_{i=0}^{M} \frac{da_i(t)}{dt} \int_k \psi_i(x)\phi_j(x)dx$ for \tilde{q} and ϕ , substituting
- From convention for dynamic systems
  - the left integral in the weak form is called the:
  - $\int_k \frac{\partial \tilde{q}}{\partial t}\phi_j dx + [f(\tilde{q})\phi_j]\Big|_{x_L}^{x_R} - \int_k f(\tilde{q})\frac{d\phi}{dx}dx = 0, \ \forall j \le N$ *Mass Matrix*

Stiffness Matrix

- $\int_k \frac{\partial q}{\partial t}\phi dx + [f(q)\phi]\Big|_{x_L}^{x_R} - \int_k f(q)\frac{d\phi}{dx}dx = 0$ Similar to mass matrix...
  - $\int_k f(q)\frac{d\phi}{dx}dx = \sum_{i=0}^{M} ca_i(t)\int_k \psi_i(x)\phi_j'(x)dx$ Substitute our approximation & bases into the 3rd term
- The integral for all weighting bases
  - Combine to form
    - $\mathbf{K}$ *Stiffness Matrix*
  - we can analytically integrate for this simple case
- Similar to the mass matrix...
  - $\sum_{i=0}^{M} ca_i(t)\int_k \psi_i(x)\phi_j'(x)dx = c\mathbf{K}\mathbf{a}$ we have a vector of our element approximation coefficients s.t.
- Let's substitute in our expressions
  - for approximation & the 1st weighting basis & analytically integrate
    - $\int_k \psi_i(x)\phi_j'(x)dx = []\Big|_0^1$
    - $\mathbf{K}_{j,i} = \int \begin{bmatrix} \psi_0\phi_0' & \psi_1\phi_0' \\ \psi_0\phi_1' & \psi_1\phi_1' \end{bmatrix} = \int \begin{bmatrix} (1-x)\cdot(-1) & x(-1) \\ (1-x)\cdot 1 & x \cdot 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ Again, the 2*2 matrix

Putting it all Together

- So what do we get?
  - $\hat{f} = \begin{bmatrix} -cq_{k-1}(x_R) \\ cq_k(x_R) \end{bmatrix} = c\begin{bmatrix} -q_{k-1}(x_R) \\ q_k(x_R) \end{bmatrix}$ numerical flux
- ϕ_j ...will be zero sometimes
  - depending on j
- Just need to calculate now the...
  - $[\hat{f}(q)\phi]\Big|_{x_L}^{x_R} = cq_k(x_R)\phi_j(x_R) - cq_{k-1}(x_R)\phi_j(x_L)$ *numerical flux vector*

Semi-discrete system

- We can now express the semi-discrete system...
  - $\mathbf{M}\mathbf{a}' - \hat{\mathbf{f}} - c\mathbf{K}\mathbf{a} = 0$ compactly in matrix/vector form
- $\mathbf{a}$ For each time we have
  - $\mathbf{a}_0$ starting with...from ICs
- We'd like to calculate
  - the time rate of change of our coefficients
    - to use in our time discretization
    - $\mathbf{a}' = \mathbf{M}^{-1}[c\mathbf{K}\mathbf{a} - \hat{\mathbf{f}}]$ so solve for a'

Time Discretization: Forward Euler

- We must...
  - choose Δt carefully
    - Can you predict why
- We may want to...
  - periodically save for plotting
- We can overwrite...
  - our previous timestep's coefficients
  - with the new ones to save memory
- Let's use a Forward Euler approach!
  - $a_{t+1} = a_t + a_t'\Delta t$ "Very simple" explicit ODE solver
- Now we need to complete the method
  - by discretizing in time, the semi-discrete system

Investigation- h-Convergence

- What do we expect...
  - for h-convergence
- How coarse can our elements be and...
  - still get a "good enough" answer
- What effects does
  - *the shape/smoothness*
    - of the solution have on convergence

Investigation- t-Convergence

- How do...
  - h convergence
  - t convergence
  - relate?
- Do we expect
  - *periodic solutions*
    - to stay steady over time
- How big can we make Δt

Investigation- Stability(CFL)

- As was seen...
  - too large of a time step lead to issue
    - why?
- What happens
  - if a parcel of quantity
  - ...moves fast enough to skip elements between timesteps?
- Methods
  - *Explicit methods*
    - $CFL = \frac{c\Delta t}{\Delta x} \le C_{max} = 1$ need to ensure
  - *Implicit methods*
    - relax this restriction
      - to permit higher values