Module 1: What is DG?

DG Motivation: Why DG?

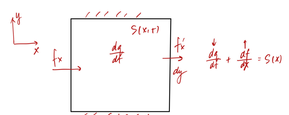
- Overall purpose: PDE models physical system — Solve PDE numerically
- Compare to common techniques FDM, FEM, FVM:
  - Increasing order of solution
    - can be more efficient
    - smooth functions especially
    - Most comm. packages low order (LO)
  - Comparison

FDM	FVM	DG
not explicitly conservative	constant solution of volume	explicitly conservative
extended stencil for high order(HO)	necessitates extended stencil for HO	well-suited for hyperbolic problems
a problem for unstructured girds	ruins flexibility for unstructured grids	able to handle discontinuities
		can still use unstructured grids at HO

- Local nature of solution — Permits good parallelizability
- Can use numerical flux functions — to capture physical behavior

Example PDE: Scalar Conservation Law

- General Approach
  - $$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = S(x)$$
    As worked out, PDE for system in ID is
  - We will need to
    - decide how to solve this system
    - and discretize in space and time
      - 1st term: Forward Euler
      - 2nd term: DG
  - For simplicity
    - $S(x) = 0$ Assume no sources
    - $f(q) = Cq$ A linear flux with an externally prescribed velocity field
- A brief motivating example
  - Imagine some scalar quantity



  ...that is subject to a conservation law

The Weak Form of the PDE

- For now permits the following: $$\int_\Omega \frac{\partial q}{\partial t}\phi dx + \int_\Omega \frac{\partial f(q)}{\partial x}\phi dx = 0$$
  - over the global domain Ω
  - of our solution times some test function Φ

  The weak form solution of the PDE is the integral
- In DG the global domain is split into K elements
  - where the local sol'n is defined for a particular element
  - the global sol'n is then — the direct sum
    - of each of these local polynomials
    - i.e. a piecewise polynomial
- This is similar to a FEM approach — however continuity is not enforced across elements

Domain Decomposition: Global vs Local

- Notice that we include the endpoints of our domain k — Endpoints of neighboring elements are coincident
  - So q is multiply defined — what ramifications does this have?
  - Also, we now have K independent local solutions — how do we recover the global solution?
- To reduce smoothness requirements on the flux $$\int_k \frac{\partial q}{\partial t}\phi dx + [f(q)\phi]\Big|_{x_L}^{x_R} - \int_k f(q)\frac{d\phi}{dx}dx = 0$$
  we integrate the 2nd term by parts to get
- The integration domain is now
  - over the element
  - $\{x | x \in k, x_L \leq x \leq x_R\}$

  instead of the whole domain s.t.

Element Boundaries: Multiply Defined?

- If q is multiply defined at endpoints — what should then our flux be?
- In order to ensure conservation flux between elements — $-\hat{f}_{k-1}(q(x_R)) = \hat{f}_k(q(x_L))$ should be equal i.e. inflow = outflow
- Take a FVM approach — $\hat{f}(q)$ and permit a numerical flux denoted
- The numerical flux function
  - permits communication between elements
  - allowing recovery of global sol'n

[Recall] Flux Functions

$$\hat{f}(q^+, q^-) = \begin{cases} cq^-, c > 0 \\ cq^+, c < 0 \end{cases}$$

For simplicity we will use the upwind flux where

- Many choices
  - Lax-Friedrichs
  - Richtmyer
  - Godunov
  - Osher
  - Roe
- Permits insertion of problem specific knowledge — into an otherwise agnostic PDE
- Flux functions describe the "flow" — of some quantity depending on
  - the quantity itself
  - and other factors (possibly)