Module 3A: To Higher-Orders - Solution Approximation

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L2 Projection: Minimize norm

- Where does the weak form come from?
  - $$\frac{\partial \hat{q}}{\partial t} + \frac{\partial f(\hat{q})}{\partial x} = R(x)$$
    - Approximate sol'n doesn't satisfy PDE exactly...
    - left with some residual
    - substitute R(x) into
- A Projection
  - finds the "closest" fit
    - on a vector space
    - An orthogonal projection does just that
    - Compared to the original
- L^2 Minimizes the...norm
  - w.r.t. the space projected upon
    - $< f, g >= \int f(x)g(x) = 0$ — *in general*
    - $< R, \phi >= \int R(x)\phi_j = 0$ — *in our case*
      - same formatting
      - $$= \int \left(\frac{\partial \hat{q}}{\partial t} + \frac{\partial f(\hat{q})}{\partial x} \right) \phi_j = 0$$
      - recovering weak form

Arbitrary Order Basis: Monomials?

- The choice of polynomials
  - $$P(x) = a_1 x + a_2 x^2$$
    - for our basis functions, e.g.
    - Let's continue with it!
  - LET fail
- The easy & obvious choice is monomials
  - $$\psi_1(x) = a_i x^i$$
    - substitute in
    - Monomials
  - $$\hat{q}(x) = \sum_{i=0}^{M} a_i \psi_i(x) = \sum_{i=0}^{M} a_i x^i$$
    - So our solution approximation is
- Can you think of why...
  - this is a poor choice for a vector basis?
  - Remember, a basis needs to be
    - Complete
    - Linearly-independent
  - Let's see two separate ways why this is a poor choice...

The ill-conditioned Hilbert Matrix

- The alternate way to consider this
  - ...is to look at the Mass Matrix for a monomial basis
    - $$\mathbf{M} = \int \psi_i \psi_j$$
    - Recall Mass Matrix
- When we integrate from 0 to 1
  - $$\int_0^1 \begin{bmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & \\ x^2 & x^3 & & \\ x^3 & \cdots & & \end{bmatrix}$$
    - it's easy to see the pattern of values
  - $$\rightarrow \int x^i dx = \frac{x^{i+1}}{i+1} = \frac{1}{i+1}$$
    - $\rightarrow$ this is a Hilbert matrix!
  - $$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \\ \frac{1}{3} & \frac{1}{4} & & \\ \frac{1}{4} & \cdots & & \end{bmatrix}$$
    - *Hilbert Matrix*
- It's ill-conditioned
  - Let's see why this is in a problem in Matlab

Runge's Phenomenon

- The divergence of the interpolation is a result of
  - *Runge's phenomenon*
- Why this fails, is explainable
  - In terms of two relations
    - *Bernstein*
    - *Markov*
    - Inequalities
  - $$P^N \in [-1, 1] \rightarrow |P'(x)| \leq \frac{N}{\sqrt{1-x^2}} \|P\|_\infty$$
    - For a polynomial
    - satisfy
  - $$\|P'\|_\infty \leq N^2 \|P\|_\infty$$
    - In the Bernstein inequality
    - can be bounded with the Markov inequality
- The singularity
  - $$|P'(x)| \leq \min\left(\frac{N}{\sqrt{1-x^2}}, N^2\right) \|P'\|_\infty$$
    - with THE OVERALL being

Weak Form- Revisited: Approximation Space

- We glossed over...
  - $$\hat{q}(x) = \sum_{i=0}^{M} \alpha_i \psi_i(x)$$
    - *the construction of the bases*
    - we can think in terms of
      - *nodal*
      - *modal*
- To get a flexible method...
  - we'd like to be able to generate a solution of
    - arbitrary order
  - HOW do we do this in code?
- Let's revisit the weak form
  - in greater detail
  - $$\int_\Omega \frac{\partial q}{\partial t} \phi_j dx + \int_\Omega \frac{\partial f(q)}{\partial x} \phi_j dx = 0$$
    - we glossed over some important details

The Galerkin Condition

- The projected space
  - doesn't have to be the same space as the solution approx, i.e. choose 2 different spaces
    - *Petrov-Galerkin*
  - but if do choose them to be the same:
    - *Bubnov-Galerkin*
  - 2 types of approach
- An orthogonal projection means that
  - the difference between
    - *the orthogonal*
    - *the projection*
    - diff should be orthogonal to the space projected upon
  - $$\phi \in V_h \rightarrow \int (f - f_h)\phi dx = 0 = \int R(x)\phi dx$$
    - so for each basis
    - Recall, inner product = 0
- $$R(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$
  - e.g. A weighted space of delta functions yields
  - a collocation method
  - Restating

Linear Dependence

- VERY nearly dependent basis vectors
  - means small perturbations -> can lead to trouble finding the correct solution
- Linear independent vectors are orthogonal
  - *i.e. their inner inner product is zero*
  - But for our monomials
    - Even moderate values of *i*, give an inner product of close to !!
- Normalize the basis
  - $$|g| = \sqrt{\int g^2} \qquad \hat{\psi}_i = \sqrt{2i+1}\, x^i$$
    - w.r.t. the L^2 norm
    - so...
  - $$\int \hat{\psi}_{i-1} \hat{\psi}_i dx = \sqrt{1 - \frac{1}{4i^2}}$$
    - Consider the inner product of two adjacent basis functions:

[Recall] Lagrange Interpolation

- We can compactly represent this as
  - ```
    m = length(T_N); N=T.N(T.N);
    Lag = ones(m);
    for(j0;j<=N;j++){for(k;k<=m;k++){Lag(k,j)*=(x-T.x(j))/(T.x(k)-T.x(j))}}
    ```
  - $$L_q(x) = \prod_{p=1, p\neq q}^{N_p} \frac{x - x_p}{x_q - x_p}$$
    - It has the form
    - Recall, the Lagrange basis
      - 0 at all interp nodes
      - 1 except for the basis node
        - which is 1 at
    - Instead of monomial basis
      - Commonly used for interpolation
      - what if we were to use a Lagrange basis?

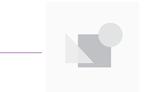
Lagrange Spatial Approximation

- We can now represent...
  - $$\hat{q}(x) = \sum_{i=0}^{M} \alpha_i \psi_i(x) = \sum_{i=0}^{M} q(x_i) L_i(x)$$
    - our arbitrary order sol'n approximation
- We must choose...
  - a collection of points
    - to use as our interpolation nodes

Interpolation Points: Equispaced?

- Let's see how it behaves at large Np!
  - No
  - N_p
  - so one would expect the accuracy to ↑ as Np ↑
  - works well for low Np
- Indeed, these points
- It would seem logical that
  - Equispaced Points are a good choice
    - they evenly cover the domain
    - are simple to calculate positions for

Interpolation Points: Legendre Roots



Legendre Roots

- The overall inequality implies
  - we'd like quadratic spacing near the element boundaries
- Though it may seem arbitrary
  - the roots of the Legendre polynomials have a favorable spacing that matches our needs
- They avoid Runge's phenomenon
  - & have several other very nice features
    ```