Module 3A: To Higher-Orders - Solution Approximation

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Weak Form- Revisited: Approximation Space

- We glossed over...
  - $\tilde{q}(x) = \sum_{i=0}^{M} \alpha_i \psi_i(x)$
    - *the construction of the bases*
  - we can think in terms of:
    - *nodal*
    - *modal*
- To get a flexible method...
  - we'd like to be able to generate a solution of
    - arbitrary order
  - HOW do we do this in code?
- Let's revisit the weak form
  - in greater detail
  - $\int_\Omega \frac{\partial q}{\partial t}\phi_j dx + \int_\Omega \frac{\partial f(q)}{\partial x}\phi_j dx = 0$
    - we glossed over some important details

L2 Projection: Minimize norm

- Where does the weak form come from?
  - Approximate sol'n doesn't satisfy PDE exactly...
  - $\frac{\partial \tilde{q}}{\partial t} + \frac{\partial f(\tilde{q})}{\partial x} = R(x)$
    - left with some residual
- *A Projection*
  - finds the "closest" fit
    - Compared to the original
    - on a vector space
    - An orthogonal projection does just that
    - substitute R(x) into
- L^2 Minimizes the...norm
  - w.r.t. the space projected upon
    - $< f, g >= \int f(x)g(x) = 0$
      - *in general*
      - same formatting
    - $< R, \phi >= \int R(x)\phi_j = 0$
      - *in our case*
    - $= \int \left(\frac{\partial \tilde{q}}{\partial t} + \frac{\partial f(\tilde{q})}{\partial x} \right) \phi_j = 0$
      - *recovering weak form*

The Galerkin Condition

- The projected space
  - doesn't have to be the same space as the solution approx, i.e. choose 2 different spaces
    - *Petrov–Galerkin*
  - but if do choose them to be the same:
    - *Bubnov–Galerkin*
  - 2 types of approach
- $R(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$
  - e.g. A weighted space of delta functions yields
  - *a collocation method*
  - Restating !
- An orthogonal projection means that:
  - the difference between
    - *the orthogonal*
    - *the projection*
    - diff should be orthogonal to the space projected upon
  - $\phi \in V_h$ so for each basis:
    - $\int (f - f_h)\phi dx = 0 = \int R(x)\phi dx$
      - Recall, inner product = 0

Arbitrary Order Basis: Monomials?

- The choice of polynomials
  - Let's continue with it!
  - $P(x) = a_1 x + a_2 x^2$
    - for our basis functions, e.g.
  - *LET'S!*
- The easy & obvious choice is monomials
  - $\psi_i(x) = a_i x^i$ → substitute in
    - Monomials
  - $\tilde{q}(x) = \sum_{i=0}^{M} a_i \psi_i(x) = \sum_{i=0}^{M} a_i x^i$
    - So our solution approximation is
- Can you think of why...
  - this is a poor choice for a vector basis?
    - *Complete*
    - *Linearly-independent*
  - Remember, a basis needs to be
  - Let's see two separate ways why this is a poor choice...

Linear Dependence

- *VERY nearly dependent basis vectors*
  - means small perturbations -> can lead to trouble finding the correct solution
- Linear independent vectors are orthogonal
  - *i.e. their inner product is zero*
  - But for our monomials
    - Even moderate values of i, give an inner product of close to 1!
- Normalize the basis
  - $|g| = \sqrt{\int g^2}$ → $\hat{\psi}_i = \sqrt{2i+1}x^i$
    - w.r.t. the L^2 norm
    - so...
  - $\int \hat{\psi}_i, \hat{\psi}_j dx = \sqrt{1 - \frac{1}{4i^2}}$
    - Consider the inner product of two adjacent basis functions

The ill-conditioned Hilbert Matrix

- The alternate way to consider this
  - ...is to look at the Mass Matrix for a monomial basis
  - $M = \int \psi_i \psi_j$
    - Recall Mass Matrix
- When we integrate from 0 to 1
  - $\int_0^1 \begin{bmatrix} 1 & x & x^2 & x^3 \\ x & x^2 & x^3 & \\ x^2 & x^3 & & \\ x^3 & \cdots & & \end{bmatrix}$
    - It's easy to see the pattern of values
  - $= x \int x^i dx = \frac{x^{i+1}}{i+1}\Big|_0^1 = \frac{1}{i}$
    - → this is a Hilbert matrix
  - $= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \\ \frac{1}{3} & \frac{1}{4} & & \\ \frac{1}{4} & \cdots & & \end{bmatrix}$
    - *Hilbert Matrix*
- It's ill-conditioned
  - Let's see why this is in a problem in Matlab

[Recall] Lagrange Interpolation

- We can compactly represent this as
  - $L_q(x) = \prod_{p=1, p\neq q}^{N} \frac{x - x_p}{x_q - x_p}$
  - It has the form
  - Recall, the Lagrange basis
    - 0 at all interp nodes
    - 1 except for the basis node
      - which is 1 at
  - Instead of monomial basis
    - Commonly used for interpolation
    - what if we were to use a Lagrange basis?

Lagrange Spatial Approximation

- We can now represent...
  - $\tilde{q}(x) = \sum_{i=0}^{M} a_i \psi_i(x) = \sum_{i=0}^{M} q(x_i) L_i(x)$
    - our arbitrary order sol'n approximation
- We must choose...
  - *a collection of points*
  - to use as our interpolation nodes

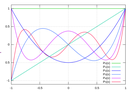
Interpolation Points: Equispaced?

- Let's see how it behaves at large Np !
  - No
  - N_p works well for low Np
  - so one would expect the accuracy to ↑ as Np ↑
- Indeed, these points
- It would seem logical that
  - Equispaced Points are a good choice
    - they evenly cover the domain
    - are simple to calculate positions for

Runge's Phenomenon

- The divergence of the interpolation is a result of
  - *Runge's phenomenon*
- Why this fails, is explainable
  - in terms of two relations
    - *Bernstein*
    - *Markov*
    - *Inequalities*
  - $P^N \in [-1, 1] \rightarrow |P'(x)| \leq \frac{N}{\sqrt{1 - x^2}} \|P\|_\infty$
    - For a polynomial
    - satisfy
  - The singularity
    - in the Bernstein inequality
    - $\|P'\|_\infty \leq N^2 \|P\|_\infty$
      - can be bounded with the Markov inequality
    - $|P'(x)| \leq \min\left(\frac{N}{\sqrt{1 - x^2}}, N^2\right)\|P'\|_\infty$
      - with THE OVERALL being

Interpolation Points: Legendre Roots



Legendre Roots

- The overall inequality implies
  - we'd like quadratic spacing near the element boundaries
- Though it may seem arbitrary
  - the roots of the Legendre polynomials have a favorable spacing that matches our needs
- They avoid Runge's phenomenon
  - & have several other very nice features