Lecture 8

(36)

Last time: ① Indicator Random Variables

$$X = \begin{cases} 1 & \text{an event occurs} \\ 0 & \text{o.w.} \end{cases}$$

$$E[X] = \underbrace{E\left[\sum\sum x_{ij}\right] = \sum\sum E[X_{ij}]}_{\text{linearity of expectation}} = \sum\sum P[X_{ij}]$$

② Solving recurrence equations that we encounter in Divide & Conquer algorithms

2.1 – open and iterate the recurrence

→ 2.2 – Master theorem. (M.T.)

Today: M.T. proof & Selection & Sorting Alg. that use divide & Conq.

Side bar: • Fermat's Little theorem:

$$a^{n-1} \equiv 1 \mod n \qquad \text{if } n \text{ is prime, } \gcd(a,n)=1 \text{ or } a<n$$

– what is $\boxed{a^{-1}}$ mod n?

multiplicative inverse

$$a \cdot X \equiv 1 \mod n.$$

$$a \cdot \boxed{a^{n-2}} \equiv 1 \mod n. \quad \text{due to Fermat's}$$

• Euler's theorem

$$a^{\phi(n)} \equiv 1 \mod n \qquad \gcd(a,n)=1$$

$\phi(n)$ euler's ~~totient~~ fn.

set of integer for mod n : $\{0, \dots\dots n-1\} = R$

$$Q = \{x \mid x \in R \text{ and } \gcd(x,n)=1\}$$

$Q \subseteq R$

$|Q| = \phi(n)$ $\qquad n = p \cdot q$ (prime p, q)

Factorials

$$n! \leq n^n$$

Stirling's Approximation

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

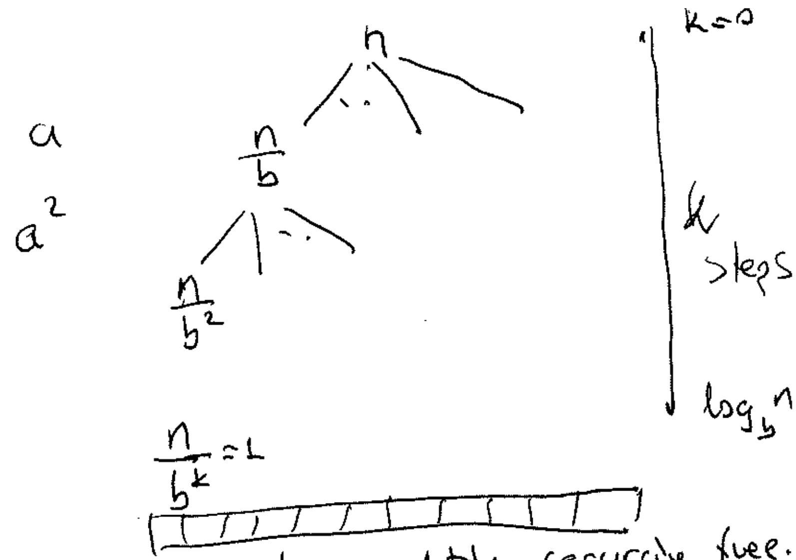
$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n} \qquad \frac{1}{(12n)+1} < \alpha_n < \frac{1}{12n}$$

$$\log(n!) = \Theta(n \log n)$$

end of side bar

Recurrence eq'

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + \Theta(n^d)$$



$$a^k\, \Theta\left(\frac{n}{b^k}\right)^d = \Theta(n^d)\cdot \underbrace{\left(\frac{a}{b^d}\right)^k}_{\text{geometric series.}}$$

this ratio determines if the series is

1. increasing
2. decreasing
3. same.

M.T: $T(n) = \Theta(n^d) \cdot \sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k$ is the solution to recursive equatiation
$T(n) = a \cdot T\left(\frac{n}{b}\right) + \Theta(n^d)$

Proof:

Uses a lemma: if C is a positive real # then $g(n) = 1 + C + C^2 + \dots + C^n$ i.e. sum of the geometric series

a) $\Theta(1)$ if $c < 1$

b) $\Theta(n)$ if $c = 1$

c) $\Theta(c^n)$ if $c > 1$ ☒

Case I: $a = b^d \Rightarrow \log_b a = d$

$$T(n) = \Theta(n^d) \sum_{k=0}^{\log_b n} (1)^k = \Theta(n^d \log_b n)$$

$\Rightarrow$ i.e. $T(n) = \Theta(n^d \log_b n)$ using (b) of lemma

Case II: $a > b^d \Rightarrow \log_b a > d \Rightarrow \left(\frac{a}{b^d}\right) > 1$ increasing

$$T(n) = \Theta(n^d) \cdot \left[\frac{\left(\frac{a}{b^d}\right)^{\log_b n+1} - 1}{\frac{a}{b^d} - 1}\right]$$

notice here $r = \frac{a}{b^d}$

then $\sum_{k=0}^{N} r^k = \frac{r^{N+1} - 1}{r - 1}$

$$T(n) = \Theta(n^d)\left[\frac{a^{\log_b n}}{b^{d \log_b n}}\right] = \Theta(n^d) \cdot \frac{n^{\log_b a}}{n^d}$$

$$T(n) = n^{\log_b a} = \Theta(n^{\log_b a})$$

Case III: $a < b^d \Rightarrow \log_b a < d$

$T(n) = \Theta(n^d) \sum_{k=0} \left(\frac{a}{b^d}\right)^k$ $\quad \left(\frac{a}{b^d}\right) < 1$ decreasing

$T(n) \leq \Theta(n^d) \cdot \frac{1}{1 - \frac{a}{\dots}} = \Theta(n^d)$

M.T. restated:

If $T(n) = a\,T\left(\left\lceil \frac{n}{b} \right\rceil\right) + O(n^d)$

for some constants $a > 0$ $b > 1$, $d > 0$ then

$$T(n) = \begin{cases} O(n^d) & \text{if increasing } d > \log_b a \\ O(n^d \log_b n) & \text{if constant } d = \log_b a \\ O(n^{\log_b a}) & \text{if decreasing } d < \log_b a \end{cases}$$

<u>example</u>: $T(n) = 2T\left(\frac{n}{2}\right) + O(n) \leq cn$

<u>method 1:</u> expand the equation and "see" a pattern emerges

$$\begin{aligned} T(n) &\leq 2\left[2T\left(\frac{n}{4}\right) + c \cdot \frac{n}{2}\right] + c \cdot n = 4T\left(\frac{n}{4}\right) + 2cn \\ &\leq 4\left[2T\left(\frac{n}{8}\right) + c \cdot \frac{n}{4}\right] + 2cn = 8T\left(\frac{n}{8}\right) + 3cn \\ &\leq 8\left[2T\left(\frac{n}{16}\right) + c \cdot \frac{n}{8}\right] + 3cn = 16T\left(\frac{n}{16}\right) + 4cn \\ &\vdots \\ T(n) &\leq 2^k\, T\left(\frac{n}{2^k}\right) + kcn \qquad \text{plug } k = \log_2 n \\ T(n) &\leq n \cdot T(1) + \log_2 n \cdot cn \\ &\leq O(n \log n) \end{aligned}$$

<u>Method 2:</u> Master theorem with $a = 2$, $b = 2$, $d = 1$ $\Rightarrow O(n \log n)$

Example 2 $\quad T(n) = 3\,T(n/2) + \Theta(n)$

Method 1: $T(n) \leq 3T\left(\frac{n}{2}\right) + cn$

$$\leq 3\left[3T\left(\frac{n}{4}\right) + c\frac{n}{2}\right] + cn = 9T\left(\frac{n}{4}\right) + 2cn$$

$$\vdots \quad \vdots$$

$$\leq 3^k\, T\left(\frac{n}{2^k}\right) + cn \sum_{i=1}^{k-1} \left(\frac{3}{2}\right)^i$$

$$\leq 3^k\, T\left(\frac{n}{2^k}\right) + 2cn\left(\left(\frac{3}{2}\right)^k - 1\right)$$

$k = \lg n \quad \Rightarrow \quad T\left(\frac{n}{2^k}\right) = T(1) = \Theta(1)$

$$T(n) = d \cdot n^{\log_2 3} + 2cn\left(\frac{n^{\log_2 3}}{n}\right)$$

$$= \Theta\left(n^{\log_2 3}\right) \quad \square$$

Method 2: $\quad a = 3 \quad b = 2 \quad d = 1$

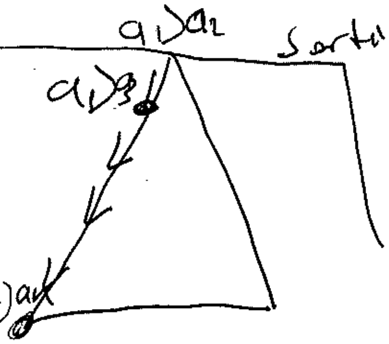
example 3: $\quad T(n) = 9T\left(\frac{n}{3}\right) + n^2 \quad = ?$

$a = 9 \quad b = 3 \quad d = 2 \qquad$ hint: case 2 ?

Next: Applications of Divide and Conq.

"Merge Sort" $\qquad$ what is the complexity of Sorting problem

we need to prove a Lowerbound.



sorting (n) numbers

1		a_i		...		n

If you are lucky

$a_i \geq a_{i-1} \quad i = 2 \dots n$

$i = 1$ | 2 3 4 - - - | n.

$a_n < a_1$

$a_5 < a_4$

$n\ 2\ 3\ 4 \cdots 1$

$n\ 2\ 3\ 5\ 4 \cdots 1$

$\Rightarrow n!$ ways of organizing the indexes.

is $a_1 > a_2$?

yes / no

$a_1 > a_3$

$n! = 2^d$

# of leafs $\quad 2^d$

$\Rightarrow \log(n!) \dots \Theta(n \log n)$