

# Supplementary Appendix

## “Heterogeneous beliefs and short selling taxes: A note”

Michael Hatcher, University of Southampton

This appendix provides (i) derivations of the optimal demand schedules, (ii) a proof of Proposition 1, and (iii) some additional supporting material (algorithm + extra simulations).

## 1 Baseline results

In this section we derive the optimal demands in the main text and prove Proposition 1.

### 1.1 Derivation of demands

At each date  $t \geq 1$ , every type  $h \in \mathcal{H}$  solves the problem:<sup>1</sup>

$$\max_{z_{t,h}} \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2} \tilde{V}_{t,h}[w_{t+1,h}] \quad (1)$$

where future wealth is  $w_{t+1,h} = (1+r)w_{t,h} + (p_{t+1} + d_{t+1} - (1+r)p_t)z_{t,h} - (1+r)T\mathbb{1}_{\{z_{t,h} < 0\}}$ ,  $\mathbb{1}_{\{z_{t,h} < 0\}}$  equals 1 if  $z_{t,h} < 0$  and 0 otherwise, and  $\tilde{V}_{t,h}[w_{t+1,h}] = \sigma^2 z_{t,h}^2$  with  $\sigma^2 > 0$ .

Formulating the above problem as a Lagrangean:

$$\max_{z_{t,h}, \lambda_{t,h}} \mathcal{L}_{t,h} = \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2} \tilde{V}_{t,h}[w_{t+1,h}] + \lambda_{t,h} \mathbb{1}_{\{p_t^h < p_t \leq \tilde{p}_t^h\}} z_{t,h} \quad (2)$$

where  $\lambda_{t,h} \geq 0$  is a Lagrange multiplier on non-participation and  $p_t^h, \tilde{p}_t^h$  are ‘kink’ prices.

The first-order conditions are

$$z_{t,h} : \tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+r)(p_t - \mathbb{1}_{\{z_{t,h} < 0\}}T) - a\sigma^2 z_{t,h} + \lambda_{t,h} \mathbb{1}_{\{p_t^h < p_t \leq \tilde{p}_t^h\}} = 0 \quad (3)$$

$$\lambda_{t,h} : \mathbb{1}_{\{p_t^h < p_t \leq \tilde{p}_t^h\}} z_{t,h} = 0 \quad (4)$$

and the complementary slackness condition is:

$$\lambda_{t,h} \mathbb{1}_{\{p_t^h < p_t \leq \tilde{p}_t^h\}} z_{t,h} = 0. \quad (5)$$

If  $p_t \leq p_t^h$  or  $p_t > \tilde{p}_t^h$ , then  $\mathbb{1}_{\{p_t^h < p_t \leq \tilde{p}_t^h\}} = 0$ ,  $\lambda_{t,h} = 0$ , so by guess-verify on (3):

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+r)p_t}{a\sigma^2} \geq 0 & \text{if } p_t \leq p_t^h = \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}]}{1+r} \\ \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+r)(p_t - T)}{a\sigma^2} < 0 & \text{if } p_t > \tilde{p}_t^h = \frac{\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}]}{1+r} + T. \end{cases} \quad (6)$$

Else, if  $p_t^h < p_t \leq \tilde{p}_t^h$ , then  $\mathbb{1}_{\{p_t^h < p_t \leq \tilde{p}_t^h\}} = 1$ ,  $z_{t,h} = 0$  by (4), and by (3) we have

$$\lambda_{t,h} = -(\tilde{E}_{t,h}[p_{t+1}] + \tilde{E}_{t,h}[d_{t+1}] - (1+r)p_t) > 0 \quad (\text{since } z_{t,h} = 0, p_t > p_t^h) \quad (7)$$

such that non-participation is binding. Equations (6)–(7) give the demand schedules in Equation (2) of the main text, which match those in Anufriev and Tuinstra (2013).

---

<sup>1</sup>We assume (as is standard) that  $\tilde{E}_{t,h}[y_t] = y_t$  and  $\tilde{V}_{t,h}[y_t] = 0$  for any variable  $y_t$  that is determined at date  $t$ ;  $\tilde{E}_{t,h}[x_{t+1} + y_{t+1}] = \tilde{E}_{t,h}[x_{t+1}] + \tilde{E}_{t,h}[y_{t+1}]$  for any variables  $x$  and  $y$ ; and  $\tilde{V}_{t,h}[x_t y_{t+1}] = x_t^2 \tilde{V}_{t,h}[y_{t+1}]$ .

## 1.2 Proposition 1 and proof

A proof of Proposition 1 (repeated below) is provided in the Appendix to this document.

**Proposition 1** *Let  $x_t$  be the market-clearing price at date  $t \in \mathbb{N}_+$ . Let  $\mathcal{B}_t \subseteq \mathcal{H}$  be the non-empty set of buyers at date  $t$ ,  $\mathcal{S}_{1,t} \subset \mathcal{H} \setminus \mathcal{B}_t$  ( $\mathcal{S}_{2,t} = \mathcal{H} \setminus (\mathcal{B}_t \cup \mathcal{S}_{1,t})$ ) be the sets of zero-position types (taxed short-sellers), and  $\tilde{T} := (1+r)T \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}} n_{t,h}$ . Then the following holds:*

1. *If  $\sum_{h \in \mathcal{H}} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{H}} \{f_{t,h}\}) \leq a\sigma^2 \bar{Z}$ , all types are unconstrained buyers ( $\mathcal{B}_t^* = \mathcal{H}$ ,  $\mathcal{S}_{1,t}^* = \mathcal{S}_{2,t}^* = \emptyset$ ), demands are  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \bar{Z} - (1+r)x_t) \geq 0 \forall h \in \mathcal{H}$ , and*

$$x_t = \frac{\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}}{1+r} := x_t^*. \quad (8)$$

2. *If  $\sum_{h \in \mathcal{H}} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{H}} \{f_{t,h}\}) > a\sigma^2 \bar{Z}$ , one or more types are non-buyers at date  $t$  (i.e.  $\mathcal{B}_t^* \subset \mathcal{H}$ ,  $\mathcal{S}_{1,t}^* \cup \mathcal{S}_{2,t}^* = \mathcal{H} \setminus \mathcal{B}_t^* \neq \emptyset$ ) and we have the following:*

(i) *If  $\exists \mathcal{B}_t^*, \mathcal{S}_{1,t}^* = \mathcal{H} \setminus \mathcal{B}_t^*$  s.t.  $\max\{d_{\mathcal{B}_t^*}, d_{\mathcal{S}_{1,t}^*}\} \leq a\sigma^2 \bar{Z} < \sum_{h \in \mathcal{B}_t^*} n_{t,h} (f_{t,h} - \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\})$ , then  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \bar{Z} - (1+r)x_t) \geq 0 \forall h \in \mathcal{B}_t^*$ ,  $z_{t,h} = 0 \forall h \in \mathcal{S}_{1,t}^*$ , price is*

$$x_t = \frac{\sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h}) a\sigma^2 \bar{Z}}{(1+r) \sum_{h \in \mathcal{B}_t^*} n_{t,h}} := \tilde{x}_t > x_t^* \quad (9)$$

where  $d_{\mathcal{B}_t^*} := \sum_{h \in \mathcal{B}_t^*} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{B}_t^*} \{f_{t,h}\})$ ,  $d_{\mathcal{S}_{1,t}^*} := \sum_{h \in \mathcal{B}_t^*} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}) - (1+r)T \sum_{h \in \mathcal{B}_t^*} n_{t,h}$ .

(ii) *If  $\exists \mathcal{B}_t^*, \mathcal{S}_{2,t}^* = \mathcal{H} \setminus \mathcal{B}_t^*$  s.t.  $\sum_{h \in \mathcal{H}} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{B}_t^*} \{f_{t,h}\}) \leq a\sigma^2 \bar{Z} - (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} < \sum_{h \in \mathcal{H}} n_{t,h} (f_{t,h} - \max_{h \in \mathcal{S}_{2,t}^*} \{f_{t,h}\}) - (1+r)T$ , then  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \bar{Z} - (1+r)x_t) \geq 0 \forall h \in \mathcal{B}_t^*$ ,  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \bar{Z} - (1+r)(x_t - T)) < 0 \forall h \in \mathcal{S}_{2,t}^*$ , and price is*

$$x_t = \frac{\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} + (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}}{1+r} := \hat{x}_t > x_t^*. \quad (10)$$

(iii) *Else,  $\exists \mathcal{B}_t^*, \mathcal{S}_{1,t}^*, \mathcal{S}_{2,t}^* \neq \emptyset$  s.t.  $\max\{d_{1,t}, \tilde{d}_{1,t}\} \leq a\sigma^2 \bar{Z} - (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} < \min\{d_{2,t}, \tilde{d}_{2,t}\}$ ,  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \bar{Z} - (1+r)x_t) \geq 0 \forall h \in \mathcal{B}_t^*$ ,  $z_{t,h} = 0 \forall h \in \mathcal{S}_{1,t}^*$ ,  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2 \bar{Z} - (1+r)(x_t - T)) < 0 \forall h \in \mathcal{S}_{2,t}^*$  and price is*

$$x_t = \frac{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h} + (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} - (\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}) a\sigma^2 \bar{Z}}{(1+r) \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}} := \bar{x}_t > x_t^* \quad (11)$$

where  $d_{1,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{B}_t^*} \{f_{t,h}\})$ ,  $d_{2,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} (f_{t,h} - \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\})$ ,  $\tilde{d}_{1,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}) - \tilde{T}$ ,  $\tilde{d}_{2,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} (f_{t,h} - \max_{h \in \mathcal{S}_{2,t}^*} \{f_{t,h}\}) - \tilde{T}$ .

**Proof.** See next page. ■

## Proof of Proposition 1

Existence of a unique equilibrium is shown in Anufriev and Tuinstra (2013, Proposition 2.1).

### Case 1: $z_{t,h} \geq 0$ for all investor types $h \in \mathcal{H}$

Let us guess that  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\bar{Z} - (1+r)x_t) \geq 0 \forall h \in \mathcal{H}$ , which implies by the market-clearing condition  $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \bar{Z}$  that  $x_t = x_t^* := (1+r)^{-1} \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}$ . The guess is verified if and only if  $f_{t,h} + a\sigma^2\bar{Z} - (1+r)x_t^* \geq 0 \forall h \in \mathcal{H}$ , which amounts to  $\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} \leq \min_{h \in \mathcal{H}} \{f_{t,h}\} + a\sigma^2\bar{Z}$ . Given  $\sum_{h \in \mathcal{H}} n_{t,h} = 1$ , the above inequality simplifies to  $\sum_{h \in \mathcal{H}} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{H}} \{f_{t,h}\}) \leq a\sigma^2\bar{Z}$ , as stated in Proposition 1 Part 1.

### Case 2(i): $z_{t,h} \geq 0 \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0 \forall h \in \mathcal{S}_{1,t}^*$ for $\mathcal{B}_t^* =$ set of types with untaxed long positions and $\mathcal{S}_{1,t}^* := \mathcal{H} \setminus \mathcal{B}_t^* =$ set of types with after-tax positions of zero

Let us guess that  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\bar{Z} - (1+r)x_t) \geq 0 \forall h \in \mathcal{B}_t^*$  and  $z_{t,h} = 0 \forall h \in \mathcal{S}_{1,t}^*$ , where  $\mathcal{B}_t^* \subset \mathcal{H}$  and  $\mathcal{S}_{1,t}^* := \mathcal{H} \setminus \mathcal{B}_t^*$ . Clearly, the above conditions imply  $\min_{h \in \mathcal{B}_t^*} \{f_{t,h}\} > \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}$ . Under the above guess,  $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \sum_{h \in \mathcal{B}_t^*} n_{t,h} z_{t,h}$  so market-clearing  $\sum_{h \in \mathcal{B}_t^*} n_{t,h} z_{t,h} = \bar{Z}$  gives  $x_t = [(1+r) \sum_{h \in \mathcal{B}_t^*} n_{t,h}]^{-1} \left( \sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h}) a\sigma^2\bar{Z} \right) := \tilde{x}_t$ . The guess is verified if and only if  $f_{t,h} + a\sigma^2\bar{Z} - (1+r)\tilde{x}_t \geq 0 (< 0) \forall h \in \mathcal{B}_t^* (\forall h \in \mathcal{S}_{1,t}^*)$  and  $f_{t,h} + a\sigma^2\bar{Z} - (1+r)(\tilde{x}_t - T) \geq 0 \forall h \in \mathcal{S}_{1,t}^*$ , i.e. iff the following inequalities hold:  $(f_{t,h} + a\sigma^2\bar{Z}) \sum_{h \in \mathcal{B}_t^*} n_{t,h} \geq (<) \sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h}) a\sigma^2\bar{Z} \forall h \in \mathcal{B}_t^* (\forall h \in \mathcal{S}_{1,t}^*)$  and  $(f_{t,h} + a\sigma^2\bar{Z} + (1+r)T) \sum_{h \in \mathcal{B}_t^*} n_{t,h} \geq \sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h}) a\sigma^2\bar{Z} \forall h \in \mathcal{S}_{1,t}^*$ , which simplify to  $\max\{d_{\mathcal{B}_t^*}, d_{\mathcal{S}_{1,t}^*}\} \leq a\sigma^2\bar{Z} < \sum_{h \in \mathcal{B}_t^*} n_{t,h} (f_{t,h} - \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\})$ , where  $d_{\mathcal{S}_{1,t}^*} := \sum_{h \in \mathcal{B}_t^*} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}) - (1+r)T \sum_{h \in \mathcal{B}_t^*} n_{t,h}$ ,  $d_{\mathcal{B}_t^*} := \sum_{h \in \mathcal{B}_t^*} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{B}_t^*} \{f_{t,h}\})$ , as stated in Proposition 1 Part 2(i).

It remains to show  $\tilde{x}_t > x_t^* = \frac{\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}}{1+r}$ , where  $x_t^*$  is the price if short-selling constraints are absent. Note  $(1+r)(\tilde{x}_t - x_t^*) = (1 - \frac{1}{\sum_{h \in \mathcal{B}_t^*} n_{t,h}}) a\sigma^2\bar{Z} + \frac{\sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h}}{\sum_{h \in \mathcal{B}_t^*} n_{t,h}} - \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}$  and  $\sum_{h \in \mathcal{B}_t^*} n_{t,h} = 1 - \sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}$ . Using  $\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} = \sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} + \sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h}$ , we get

$$(1+r)(\tilde{x}_t - x_t^*) = \sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h} \left[ \frac{\sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - a\sigma^2\bar{Z}}{\sum_{h \in \mathcal{B}_t^*} n_{t,h}} - \frac{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h}}{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}} \right] > 0$$

since  $\sum_{h \in \mathcal{S}_{1,t}^*} \frac{n_{t,h}}{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}} f_{t,h} \leq \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}$  and  $[\sum_{h \in \mathcal{B}_t^*} n_{t,h}]^{-1} \left( \sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - a\sigma^2\bar{Z} \right) > \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}$  by the condition  $\sum_{h \in \mathcal{B}_t^*} n_{t,h} (f_{t,h} - \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}) > a\sigma^2\bar{Z}$  above.

**Case 2(ii):**  $z_{t,h} \geq 0 \forall h \in \mathcal{B}_t^*$  and  $z_{t,h} < 0 \forall h \in \mathcal{S}_{2,t}^*$  for  $\mathcal{B}_t^* =$  set of types with untaxed long positions and  $\mathcal{S}_{2,t}^* := \mathcal{H} \setminus \mathcal{B}_t^* =$  set of types with negative after-tax positions

Let us guess that  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\bar{Z} - (1+r)x_t) \geq 0 \forall h \in \mathcal{B}_t^*$  and  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\bar{Z} - (1+r)(x_t - T)) < 0 \forall h \in \mathcal{S}_{2,t}^*$ , where  $\mathcal{B}_t^* \subset \mathcal{H}$  and  $\mathcal{S}_{2,t}^* := \mathcal{H} \setminus \mathcal{B}_t^*$ . Clearly, the above conditions imply that  $\min_{h \in \mathcal{B}_t^*} \{f_{t,h}\} > \max_{h \in \mathcal{S}_{2,t}^*} \{f_{t,h}\}$ . Under the above guess, market-clearing  $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \bar{Z}$  gives  $x_t = (1+r)^{-1} \left[ \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} + \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} (1+r)T \right] := \hat{x}_t$ . The guess is verified iff  $f_{t,h} + a\sigma^2\bar{Z} - (1+r)\hat{x}_t \geq 0 \forall h \in \mathcal{B}_t^*$  and  $f_{t,h} + a\sigma^2\bar{Z} - (1+r)(\hat{x}_t - T) < 0 \forall h \in \mathcal{S}_{2,t}^*$ , i.e.  $f_{t,h} + a\sigma^2\bar{Z} \geq \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} + (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} \forall h \in \mathcal{B}_t^*$  and  $f_{t,h} + a\sigma^2\bar{Z} + (1+r)T \sum_{h \in \mathcal{B}_t^*} n_{t,h} < \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} \forall h \in \mathcal{S}_{2,t}^*$ , which simplify to  $\sum_{h \in \mathcal{H}} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{B}_t^*} \{f_{t,h}\}) \leq a\sigma^2\bar{Z} - (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} < \sum_{h \in \mathcal{B}_t^*} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{S}_{2,t}^*} \{f_{t,h}\}) - (1+r)T$ , as stated in Proposition 1 Part 2(ii). Finally, note that  $\hat{x}_t > x_t^* := (1+r)^{-1} \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}$  since  $(1+r)(\hat{x}_t - x_t^*) = \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} (1+r)T > 0$ .

**Case 2(iii):**  $z_{t,h} \geq 0 \forall h \in \mathcal{B}_t^*$ ,  $z_{t,h} = 0 \forall h \in \mathcal{S}_{1,t}^*$  and  $z_{t,h} < 0 \forall h \in \mathcal{S}_{2,t}^* := \mathcal{H} \setminus (\mathcal{B}_t^* \cup \mathcal{S}_{1,t}^*)$  for  $\mathcal{B}_t^*$  buyers and  $\mathcal{S}_{1,t}^* = (\mathcal{S}_{2,t}^*)^c$  set of types with zero (negative) after-tax positions

Let us guess that  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\bar{Z} - (1+r)x_t) \geq 0 \forall h \in \mathcal{B}_t^*$ ,  $z_{t,h} = 0 \forall h \in \mathcal{S}_{1,t}^*$ , and  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\bar{Z} - (1+r)(x_t - T)) < 0 \forall h \in \mathcal{S}_{2,t}^*$ , where  $\mathcal{B}_t^*, \mathcal{S}_{1,t}^*, \mathcal{S}_{2,t}^* \subset \mathcal{H}$ ,  $\mathcal{B}_t^* \cup \mathcal{S}_{1,t}^* \cup \mathcal{S}_{2,t}^* = \mathcal{H}$  and  $\mathcal{B}_t^*, \mathcal{S}_{1,t}^*, \mathcal{S}_{2,t}^* \neq \emptyset$ . Clearly, the above conditions imply that  $\min_{h \in \mathcal{B}_t^*} \{f_{t,h}\} > \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\} > \max_{h \in \mathcal{S}_{2,t}^*} \{f_{t,h}\}$ . Under the above guess,  $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} z_{t,h}$  (where  $\mathcal{H} \setminus \mathcal{S}_{1,t}^* = \mathcal{B}_t^* \cup \mathcal{S}_{2,t}^*$ ) and market-clearing  $\sum_{h \in \mathcal{B}_t^* \cup \mathcal{S}_{2,t}^*} n_{t,h} z_{t,h} = \bar{Z}$  gives  $x_t = [(1+r) \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}]^{-1} (\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} [f_{t,h} + a\sigma^2\bar{Z}] + (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} - a\sigma^2\bar{Z}) := \bar{x}_t$ . The guess is verified iff  $f_{t,h} + a\sigma^2\bar{Z} - (1+r)\bar{x}_t \geq 0 (< 0) \forall h \in \mathcal{B}_t^* (\forall h \in \mathcal{S}_{1,t}^*)$  and  $f_{t,h} + a\sigma^2\bar{Z} - (1+r)(\bar{x}_t - T) \geq 0 (< 0) \forall h \in \mathcal{S}_{1,t}^* (\forall h \in \mathcal{S}_{2,t}^*)$ , i.e. iff  $\max\{d_{1,t}, \tilde{d}_{1,t}\} \leq a\sigma^2\bar{Z} - (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} < \min\{d_{2,t}, \tilde{d}_{2,t}\}$  for  $d_{1,t}, \tilde{d}_{1,t}, d_{2,t}, \tilde{d}_{2,t}$  as in Proposition 1, 2(iii).

To show  $\bar{x}_t > x_t^*$ :  $(1+r)(\bar{x}_t - x_t^*) = \frac{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} [f_{t,h} + a\sigma^2\bar{Z}] + (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} - a\sigma^2\bar{Z}}{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}} - \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}$ .

Using  $\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h} + \sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h}$  and simplifying, we have:

$$(1+r)(\bar{x}_t - x_t^*) = \sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h} \left[ \frac{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h} + \frac{(1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}}{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}} - a\sigma^2\bar{Z}}{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}} - \frac{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h}}{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}} \right] > 0$$

where  $\sum_{h \in \mathcal{S}_{1,t}^*} \frac{n_{t,h}}{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}} f_{t,h} \leq \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}$ , and  $\frac{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h} + \frac{(1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}}{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}} - a\sigma^2\bar{Z}}{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}} > \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}$  since  $d_{2,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} (f_{t,h} - \max_{h \in \mathcal{S}_{1,t}^*} \{f_{t,h}\}) > a\sigma^2\bar{Z} - (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}$  (see above). ■

## 2 Supporting material

In this section we provide some supporting material for the ‘fast’ version of our algorithm and some additional numerical results for computation speed and accuracy.

### 2.1 Foundations for the algorithm

For easy reference, the ‘fast’ version of our algorithm is repeated below.

#### Algorithm 2 (fast)

1. Find the set  $\tilde{\mathcal{H}}_t$  and the population shares  $n_{t,h}$  for  $h = 1, \dots, \tilde{H}_t$ . Compute  $disp_{t,1}$ . If  $disp_{t,1} \leq a\sigma^2\bar{Z}$ , then  $x_t = x_t^*$  is the date  $t$  price, compute the demands  $z_{t,h} \geq 0$  for  $h = 1, \dots, \tilde{H}_t$  and move to period  $t + 1$  and repeat. If  $disp_{t,1} > a\sigma^2\bar{Z}$ , move to Step 2.
2. Find the largest  $h$  such that  $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\bar{Z} - (1+r)x_t^*) < 0$ , say  $k_0$ , where  $x_t^*$  is the price if a short-sales tax were absent (see Proposition 1 Part 1). If desired,  $k_0$  may be updated in an iterative manner by updating the price and then updating  $k_0$ .
3. Run Steps 3–5 of Algorithm 1, starting from  $k = k_0$  (see Step 2). Continue until a solution is found, then move to period  $t + 1$  and repeat.

In Step 1, we check whether the usual price  $x_t = x_t^*$  is a solution. This will be the case if (and only if) belief dispersion is small enough (no type wants an unconstrained negative position); otherwise, there must at least one non-buying type in equilibrium at date  $t$  (and at most  $\tilde{H}_t - 1$ ), and we proceed to Step 2. In *Step 2*, we count the number  $k_0$  of negative demands at the price  $x_t^*$ , because the equilibrium price must satisfy  $x_t > x_t^*$  (shown in Proposition 1), such that any type  $h \in \tilde{H}_t$  with an unrestricted non-positive position at price  $x_t^*$  must also have a non-positive position at the equilibrium price  $x_t$  (i.e. be a non-buyer):

$$\frac{f_{t,h} + a\sigma^2\bar{Z} - (1+r)x_t^*}{a\sigma^2} \leq 0 \implies \frac{f_{t,h} + a\sigma^2\bar{Z} - (1+r)x_t}{a\sigma^2} < 0 \implies z_{t,h} \leq 0$$

by Equation (5) of the main text.

Hence, the equilibrium number of non-buyers at date  $t$ , say  $k^*$ , must satisfy  $k^* \geq k_0$ , implying that  $k_0$  is a lower bound for  $k^*$ ; thus we give our algorithm initial guess  $k = k_0$ .

As noted in the main text (see Section 2.1), an updated guess  $k'_0$  may be obtained by finding  $k_0$  and then checking among types  $1, \dots, k_0$  which are short-sellers after tax at price  $x_t^*$  and which (if any) have after-tax positions of zero. Let  $k_0^s$  ( $k_0^0$ ) be the number of after-tax short-selling types (zero-position types) at price  $x_t^*$ . If  $k_0^s = k_0$  ( $k_0^s < k_0$ ), an updated price can be computed via Proposition 1 Part 2(ii) (Proposition 1 Part 2(iii)). Our update  $k'_0$  is then the number of unrestricted demands  $\leq 0$  at that price, analogous to Step 2 above.<sup>2</sup>

---

<sup>2</sup>Our algorithm has a section to perform the update described above, which can improve computation speeds non-trivially when there is a large number of types or beliefs are highly concentrated (i.e. very similar).

## 2.2 Additional numerical examples

This section reports computation times and accuracy measures for a longer simulation and other scenarios in Figure 2 of the main text. As in Table 1 (main text), dividends are stochastic. The results tell a similar story to Table 1 (main text) and highlight the importance of periods with coexistence of short and zero-position types in increasing computation times (see columns 3–4 in Tables 1,4 below). Computer software and hardware as in the main text.

Table 1: Computation times and accuracy in Scenario 3:  $T_{sim} = 250$  periods

No. of types	Regime	Time (s)	Freq. 1 (2)	$\max(Error_t)$
$H = 100$	No tax: $T = 0$	0.03	-	6.9e-17
	Scenario 2: $T = 0.10$	0.08	250 (190)	7.6e-16
	Scenario 2: $T = 1/8$	0.06	250 (145)	1.1e-15
$H = 1,000$	No tax: $T = 0$	0.04	-	8.3e-17
	Scenario 2: $T = 0.10$	0.64	250 (195)	1.1e-15
	Scenario 2: $T = 1/8$	0.36	250 (135)	1.3e-15
$H = 2,500$	No tax: $T = 0$	0.06	-	1.4e-16
	Scenario 2: $T = 0.10$	2.20	250 (196)	1.5e-15
	Scenario 2: $T = 1/8$	1.23	250 (134)	1.5e-15

**Notes:**  $\max(Error_t) := \max\{Error_1, \dots, Error_{T_{sim}}\}$ , where we define the date  $t$  simulation error as  $Error_t = |\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} - \bar{Z}|$ . Demands  $z_{t,h}$  depend on the computed market-clearing price. Freq. 1 = number of periods with  $\mathcal{S}_{1,t}^* \cup \mathcal{S}_{2,t}^* \neq \emptyset$  (at least one short or zero position at date  $t$ ), and Freq. 2 = number of periods with  $\mathcal{S}_{1,t}^*, \mathcal{S}_{2,t}^* \neq \emptyset$  (both short and zero positions at date  $t$ ).

Simulation times are higher, as expected, when we simulate Scenario 3 for  $T_{sim} = 250$  periods (rather than 100) and our measure of accuracy (final column) has similar values.

Table 2: Computation times and accuracy in Scenario 1:  $T_{sim} = 100$  periods

No. of types	Regime	Time (s)	Freq. 1 (2)	$\max(Error_t)$
$H = 100$	No tax: $T = 0$	0.02	-	6.9e-17
	Scenario 2: $T = 0.10$	0.03	100 (98)	1.1e-16
	Scenario 2: $T = 1/8$	0.03	100 (99)	1.3e-16
$H = 1,000$	No tax: $T = 0$	0.03	-	5.6e-17
	Scenario 2: $T = 0.10$	0.13	100 (98)	1.4e-16
	Scenario 2: $T = 1/8$	0.14	100 (99)	1.5e-16
$H = 2,500$	No tax: $T = 0$	0.03	-	9.7e-17
	Scenario 2: $T = 0.10$	0.43	100 (98)	2.2e-16
	Scenario 2: $T = 1/8$	0.46	100 (99)	3.6e-16

**Notes:** Please see Table 1 above for a full description of the column entries.

In Scenario 2, computation times and accuracy are similar to Scenario 3 (Table 1, paper).

Table 3: Computation times and accuracy in Scenario 2:  $T_{sim} = 100$  periods

No. of types	Regime	Time (s)	Freq. 1 (2)	$\max(Error_t)$
$H = 100$	No tax: $T = 0$	0.01	-	5.6e-17
	Scenario 2: $T = 0.10$	0.03	100 (98)	1.1e-16
	Scenario 2: $T = 1/8$	0.04	100 (99)	1.9e-16
$H = 1,000$	No tax: $T = 0$	0.02	-	5.6e-17
	Scenario 2: $T = 0.10$	0.22	100 (98)	1.1e-16
	Scenario 2: $T = 0.1/8$	0.27	100 (99)	1.7e-16
$H = 2,500$	No tax: $T = 0$	0.03	-	8.3e-17
	Scenario 2: $T = 0.10$	0.72	100 (98)	2.2e-16
	Scenario 2: $T = 1/8$	0.93	100 (99)	3.6e-16

**Notes:** Please see Table 1 above for a full description of the column entries.

Computation times are a bit higher in Scenario 2 due to the increased number of periods in which both zero-position types and short-sellers coexist relative to Scenario 3 (see Col. 4).

Table 4: Computation times and accuracy in Scenario 4:  $T_{sim} = 50$  periods

No. of types	Regime	Time (s)	Freq. 1 (2)	$\max(Error_t)$
$H = 100$	No tax: $T = 0$	0.01	-	1.9e-16
	Scenario 2: $T = 0.10$	0.02	50 (33)	1.4e-15
	Scenario 2: $T = 1/8$	0.02	50 (37)	9.2e-15
$H = 1,000$	No tax: $T = 0$	0.01	-	3.6e-16
	Scenario 2: $T = 0.10$	0.14	50 (33)	1.4e-15
	Scenario 2: $T = 1/8$	0.15	50 (37)	8.2e-15
$H = 2,500$	No tax: $T = 0$	0.02	-	5.6e-16
	Scenario 2: $T = 0.10$	0.44	50 (33)	2.8e-15
	Scenario 2: $T = 1/8$	0.57	50 (37)	4.2e-15

**Notes:** Please see Table 1 above for a full description of the column entries.

Computation times are higher than in Scenario 3, taking into account the smaller number of simulated periods. In this scenario, there is an fast-exploding price path that diverges to  $+\infty$  (see Figure 2, main text); for this reason we simulated  $T_{sim} = 50$  periods in this case.

## References

Anufriev, M. and Tuinstra, J. (2013). The impact of short-selling constraints on financial market stability in a heterogeneous agents model. *Journal of Economic Dynamics and Control*, 37(8):1523–1543.