Supplementary Appendix
"Heterogeneous beliefs and short selling taxes: A note"

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This appendix provides (i) derivations of the optimal demand schedules, (ii) a proof of Proposition 1, and (iii) some additional supporting material (algorithm + extra simulations).

1 Baseline results

In this section we derive the optimal demands in the main text and prove Proposition 1.

1.1 Derivation of demands

At each date $t \geq 1$, every type $h \in \mathcal{H}$ solves the problem:[1]

$$\max_{z_{t,h}} \tilde{E}_{t,h}[w_{t+1,h}] - \frac{a}{2}\tilde{V}_{t,h}[w_{t+1,h}] \tag{1}$$

where future wealth is $w_{t+1,h} = (1+r)w_{t,h} + (p_{t+1} + d_{t+1} - (1+r)p_t)z_{t,h} - (1+r)T|z_{t,h}|\mathbb{1}_{\{z_{t,h}<0\}}$, $\mathbb{1}_{\{z_{t,h}<0\}}$ equals 1 if $z_{t,h} < 0$ and 0 otherwise, and $\tilde{V}_{t,h}[w_{t+1,h}] = \sigma^2 z_{t,h}^2$ with $\sigma^2 > 0$.

Formulating the above problem as a Lagrangean:

$$\max_{z_{t,h},\lambda_{t,h}} \mathcal{L}_{t,h} = \tilde{E}_{t,h}\left[w_{t+1,h}\right] - \frac{a}{2}\tilde{V}_{t,h}\left[w_{t+1,h}\right] + \lambda_{t,h}\mathbb{1}_{\{p_t^h<p_t\leq\tilde{p}_t^h\}}z_{t,h} \tag{2}$$

where $\lambda_{t,h} \geq 0$ is a Lagrange multiplier on non-participation and $p_t^h, \tilde{p}_t^h$ are 'kink' prices.

The first-order conditions are

$$z_{t,h}: \quad \tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+r)(p_t - \mathbb{1}_{\{z_{t,h}<0\}}T) - a\sigma^2 z_{t,h} + \lambda_{t,h}\mathbb{1}_{\{p_t^h<p_t\leq\tilde{p}_t^h\}} = 0 \tag{3}$$

$$\lambda_{t,h}: \quad \mathbb{1}_{\{p_t^h<p_t\leq\tilde{p}_t^h\}}z_{t,h} = 0 \tag{4}$$

and the complementary slackness condition is:

$$\lambda_{t,h}\mathbb{1}_{\{p_t^h<p_t\leq\tilde{p}_t^h\}}z_{t,h} = 0. \tag{5}$$

If $p_t \leq p_t^h$ or $p_t > \tilde{p}_t^h$, then $\mathbb{1}_{\{p_t^h<p_t\leq\tilde{p}_t^h\}} = 0$, $\lambda_{t,h} = 0$, so by guess-verify on (3):

$$z_{t,h} = \begin{cases} \frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]-(1+r)p_t}{a\sigma^2} \geq 0 & \text{if } p_t \leq p_t^h = \frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+r} \\ \frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]-(1+r)(p_t-T)}{a\sigma^2} < 0 & \text{if } p_t > \tilde{p}_t^h = \frac{\tilde{E}_{t,h}[p_{t+1}]+\tilde{E}_{t,h}[d_{t+1}]}{1+r} + T. \end{cases} \tag{6}$$

Else, if $p_t^h < p_t \leq \tilde{p}_t^h$, then $\mathbb{1}_{\{p_t^h<p_t\leq\tilde{p}_t^h\}} = \mathbb{1}_{\{p_t>p_t^h\}} = 1$, $z_{t,h} = 0$ by (4), and by (3) we have

$$\lambda_{t,h} = -(\tilde{E}_{t,h}\left[p_{t+1}\right] + \tilde{E}_{t,h}\left[d_{t+1}\right] - (1+r)p_t) > 0 \quad (\text{since } z_{t,h} = 0,\ p_t > p_t^h) \tag{7}$$

such that non-participation is binding. Equations (6)–(7) give the demand schedules in Equation (2) of the main text, which match those in Anufriev and Tuinstra (2013).

[1] We assume (as is standard) that $\tilde{E}_{t,h}[y_t] = y_t$ and $\tilde{V}_{t,h}[y_t] = 0$ for any variable y_t that is determined at date t; $\tilde{E}_{t,h}[x_{t+1} + y_{t+1}] = \tilde{E}_{t,h}[x_{t+1}] + \tilde{E}_{t,h}[y_{t+1}]$ for any variables x and y; and $\tilde{V}_{t,h}[x_t y_{t+1}] = x_t^2\tilde{V}_{t,h}[y_{t+1}]$.

1.2 Proposition 1 and proof

A proof of Proposition 1 (repeated below) is provided in the Appendix to this document.

Proposition 1 *Let x_t be the market-clearing price at date $t \in \mathbb{N}_+$. Let $\mathcal{B}_t \subseteq \mathcal{H}$ be the non-empty set of buyers at date t, $\mathcal{S}_{1,t} \subset \mathcal{H} \setminus \mathcal{B}_t$ ($\mathcal{S}_{2,t} = \mathcal{H} \setminus (\mathcal{B}_t \cup \mathcal{S}_{1,t})$) be the sets of zero-position types (taxed short-sellers), and $\tilde{T}_t := (1+r)T \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}} n_{t,h}$. Then the following holds:*

1. If $\sum_{h \in \mathcal{H}} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2 \overline{Z}$, all types are unconstrained buyers ($\mathcal{B}_t^* = \mathcal{H}$, $\mathcal{S}_{1,t}^* = \mathcal{S}_{2,t}^* = \emptyset$), demands are $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \ \forall h \in \mathcal{H}$, and

$$x_t = \frac{\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}}{1+r} := x_t^*. \tag{8}$$

2. If $\sum_{h \in \mathcal{H}} n_{t,h} (f_{t,h} - \min_{h \in \mathcal{H}}\{f_{t,h}\}) > a\sigma^2 \overline{Z}$, one or more types are non-buyers at date t (i.e. $\mathcal{B}_t^* \subset \mathcal{H}$, $\mathcal{S}_{1,t}^* \cup \mathcal{S}_{2,t}^* = \mathcal{H} \setminus \mathcal{B}_t^* \neq \emptyset$) and we have the following:

(i) If $\exists\ \mathcal{B}_t^*, \mathcal{S}_{1,t}^* = \mathcal{H} \setminus \mathcal{B}_t^*$ s.t. $\max\{d_{\mathcal{B}_t^*}, d_{\mathcal{S}_{1,t}^*}\} \leq a\sigma^2\overline{Z} < \sum_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h \in S_{1,t}^*}\{f_{t,h}\})$, then $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_{1,t}^*$, price is

$$x_t = \frac{\sum_{h \in \mathcal{B}_t^*} n_{t,h} f_{t,h} - (1 - \sum_{h \in \mathcal{B}_t^*} n_{t,h}) a\sigma^2 \overline{Z}}{(1+r)\sum_{h \in \mathcal{B}_t^*} n_{t,h}} := \tilde{x}_t > x_t^* \tag{9}$$

where $d_{\mathcal{B}_t^*} := \sum\limits_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min\limits_{h \in \mathcal{B}_t^*}\{f_{t,h}\})$, $d_{\mathcal{S}_{1,t}^*} := \sum\limits_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min\limits_{h \in \mathcal{S}_{1,t}^*}\{f_{t,h}\}) - (1+r)T \sum\limits_{h \in \mathcal{B}_t^*} n_{t,h}$.

(ii) If $\exists\ \mathcal{B}_t^*, \mathcal{S}_{2,t}^* = \mathcal{H} \setminus \mathcal{B}_t^*$ s.t. $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} - (1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} < \sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \max_{h \in \mathcal{S}_{2,t}^*}\{f_{t,h}\}) - (1+r)T$, then $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)(x_t - T)) < 0 \ \forall h \in \mathcal{S}_{2,t}^*$, and price is

$$x_t = \frac{\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} + (1+r)T \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}}{1+r} := \hat{x}_t > x_t^*. \tag{10}$$

(iii) Else, $\exists\ \mathcal{B}_t^*, \mathcal{S}_{1,t}^*, \mathcal{S}_{2,t}^* \neq \emptyset$ s.t. $\max\{d_{1,t}, \tilde{d}_{1,t}\} \leq a\sigma^2\overline{Z} - (1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} < \min\{d_{2,t}, \tilde{d}_{2,t}\}$, then $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_{1,t}^*$, $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)(x_t - T)) < 0 \ \forall h \in \mathcal{S}_{2,t}^*$ and price is

$$x_t = \frac{\sum\limits_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h} + (1+r)T \sum\limits_{h \in \mathcal{S}_{2,t}^*} n_{t,h} - (\sum\limits_{h \in \mathcal{S}_{1,t}^*} n_{t.h}) a\sigma^2 \overline{Z}}{(1+r)\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}} := \overline{x}_t > x_t^* \tag{11}$$

where $d_{1,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{B}_t^*}\{f_{t,h}\})$, $d_{2,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \max_{h \in \mathcal{S}_{1,t}^*}\{f_{t,h}\})$, $\tilde{d}_{1,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{S}_{1,t}^*}\{f_{t,h}\}) - \tilde{T}_t$, $\tilde{d}_{2,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*}(f_{t,h} - \max_{h \in \mathcal{S}_{2,t}^*}\{f_{t,h}\}) - \tilde{T}_t$.

Proof. See next page. ■

Proof of Proposition 1

Existence of a unique equilibrium is shown in Anufriev and Tuinstra (2013, Proposition 2.1).

Case 1: $z_{t,h} \geq 0$ **for all investor types** $h \in \mathcal{H}$

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \; \forall h \in \mathcal{H}$, which implies by the market-clearing condition $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \overline{Z}$ that $x_t = x_t^* := (1+r)^{-1}\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t^* \geq 0 \; \forall h \in \mathcal{H}$, which amounts to $\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} \leq \min\limits_{h\in\mathcal{H}}\{f_{t,h}\} + a\sigma^2\overline{Z}$. Given $\sum_{h\in\mathcal{H}} n_{t,h} = 1$, the above inequality simplifies to $\sum_{h\in\mathcal{H}} n_{t,h}(f_{t,h} - \min\limits_{h\in\mathcal{H}}\{f_{t,h}\}) \leq a\sigma^2\overline{Z}$, as stated in Proposition 1 Part 1.

Case 2(i): $z_{t,h} \geq 0 \; \forall h \in \mathcal{B}_t^*$ **and** $z_{t,h} = 0 \; \forall h \in \mathcal{S}_{1,t}^*$ **for** $\mathcal{B}_t^*$ **= set of types with untaxed long positions and** $\mathcal{S}_{1,t}^* := \mathcal{H} \setminus \mathcal{B}_t^*$ **= set of types with after-tax positions of zero**

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \; \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = 0 \; \forall h \in \mathcal{S}_{1,t}^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ and $\mathcal{S}_t^* := \mathcal{H} \setminus \mathcal{B}_t^*$. Clearly, the above conditions imply $\min_{h\in\mathcal{B}_t^*}\{f_{t,h}\} > \max_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}$. Under the above guess, $\sum_{h\in\mathcal{H}} n_{t,h}z_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h}$ so market-clearing $\sum_{h\in\mathcal{B}_t^*} n_{t,h}z_{t,h} = \overline{Z}$ gives $x_t = [(1+r)\sum_{h\in\mathcal{B}_t^*} n_{t,h}]^{-1}\left(\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z}\right)$ $:= \tilde{x}_t$. The guess is verified if and only if $f_{t,h} + a\sigma^2\overline{Z} - (1+r)\tilde{x}_t \geq 0 \; (<0) \; \forall h \in \mathcal{B}_t^* \; (\forall h \in \mathcal{S}_{1,t}^*)$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)(\tilde{x}_t - T) \geq 0 \; \forall h \in \mathcal{S}_{1,t}^*$, i.e. iff the following inequalities hold: $(f_{t,h} + a\sigma^2\overline{Z})\sum_{h\in\mathcal{B}_t^*} n_{t,h} \geq (<) \sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} \; \forall h \in \mathcal{B}_t^* \; (\forall h \in \mathcal{S}_{1,t}^*)$ and $(f_{t,h} + a\sigma^2\overline{Z} + (1+r)T)\sum_{h\in\mathcal{B}_t^*} n_{t,h} \geq \sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - (1-\sum_{h\in\mathcal{B}_t^*} n_{t,h})a\sigma^2\overline{Z} \; \forall h \in \mathcal{S}_{1,t}^*$ which simplify to $\max\{d_{\mathcal{B}_t^*}, d_{\mathcal{S}_{1,t}^*}\} \leq a\sigma^2\overline{Z} < \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\})$, where $d_{\mathcal{S}_{1,t}^*} := \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}) - (1+r)T\sum_{h\in\mathcal{B}_t^*} n_{t,h}$, $d_{\mathcal{B}_t^*} := \sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h\in\mathcal{B}_t^*}\{f_{t,h}\})$, as stated in Proposition 1 Part 2(i).

It remains to show $\tilde{x}_t > x_t^* = \frac{\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}}{1+r}$, where x_t^* is the price if short-selling constraints are absent. Note $(1+r)(\tilde{x}_t - x_t^*) = (1 - \frac{1}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}})a\sigma^2\overline{Z} + \frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \sum_{h\in\mathcal{H}} n_{t,h}f_{t,h}$ and $\sum_{h\in\mathcal{B}_t^*} n_{t,h} = 1 - \sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}$. Using $\sum_{h\in\mathcal{H}} n_{t,h}f_{t,h} = \sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} + \sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}f_{t,h}$, we get

$$(1+r)(\tilde{x}_t - x_t^*) = \sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}\left[\frac{\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - a\sigma^2\overline{Z}}{\sum_{h\in\mathcal{B}_t^*} n_{t,h}} - \frac{\sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}f_{t,h}}{\sum_{h\in\mathcal{S}_{1,t^*}} n_{t,h}}\right] > 0$$

since $\sum\limits_{h\in\mathcal{S}_{1,t}^*} \frac{n_{t,h}}{\sum_{h\in\mathcal{S}_{1,t}^*} n_{t,h}} f_{t,h} \leq \max\limits_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}$ and $[\sum\limits_{h\in\mathcal{B}_t^*} n_{t,h}]^{-1}\left(\sum_{h\in\mathcal{B}_t^*} n_{t,h}f_{t,h} - a\sigma^2\overline{Z}\right) > \max\limits_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}$ by the condition $\sum_{h\in\mathcal{B}_t^*} n_{t,h}(f_{t,h} - \max_{h\in\mathcal{S}_{1,t}^*}\{f_{t,h}\}) > a\sigma^2\overline{Z}$ above.

Case 2(ii): $z_{t,h} \geq 0 \ \forall h \in \mathcal{B}_t^*$ **and** $z_{t,h} < 0 \ \forall h \in \mathcal{S}_{2,t}^*$ **for** $\mathcal{B}_t^*$ **= set of types with untaxed long positions and** $\mathcal{S}_{2,t}^* := \mathcal{H} \setminus \mathcal{B}_t^*$ **= set of types with negative after-tax positions**

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$ and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)(x_t - T)) < 0 \ \forall h \in \mathcal{S}_{2,t}^*$, where $\mathcal{B}_t^* \subset \mathcal{H}$ and $\mathcal{S}_{2,t}^* := \mathcal{H} \setminus \mathcal{B}_t^*$. Clearly, the above conditions imply that $\min_{h \in \mathcal{B}_t^*}\{f_{t,h}\} > \max_{h \in \mathcal{S}_{2,t^*}}\{f_{t,h}\}$. Under the above guess, market-clearing $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \overline{Z}$ gives $x_t = (1+r)^{-1}\left[\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} + \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}(1+r)T\right] := \hat{x}_t$. The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r)\hat{x}_t \geq 0 \ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)(\hat{x}_t - T) < 0 \ \forall h \in \mathcal{S}_{2,t}^*$, i.e. $f_{t,h} + a\sigma^2\overline{Z} \geq \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} + (1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} \ \forall h \in \mathcal{B}_t^*$ and $f_{t,h} + a\sigma^2\overline{Z} + (1+r)T\sum_{h \in \mathcal{B}_t^*} n_{t,h} < \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} \ \forall h \in \mathcal{S}_{2,t}^*$, which simplify to $\sum_{h \in \mathcal{H}} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{B}_t^*}\{f_{t,h}\}) \leq a\sigma^2\overline{Z} - (1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} < \sum_{h \in \mathcal{B}_t^*} n_{t,h}(f_{t,h} - \min_{h \in \mathcal{S}_{2,t}^*}\{f_{t,h}\}) - (1+r)T$, as stated in Proposition 1 Part 2(ii). Finally, note that $\hat{x}_t > x_t^* := (1+r)^{-1}\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}$ since $(1+r)(\hat{x}_t - x_t^*) = \sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}(1+r)T > 0$.

Case 2(iii): $z_{t,h} \geq 0 \ \forall h \in \mathcal{B}_t^*$**,** $z_{t,h} = 0 \ \forall h \in \mathcal{S}_{1,t}^*$ **and** $z_{t,h} < 0 \ \forall h \in \mathcal{S}_{2,t}^* := \mathcal{H} \setminus (\mathcal{B}_t^* \cup \mathcal{S}_{1,t}^*)$ **for** $\mathcal{B}_t^*$ **buyers and** $\mathcal{S}_{1,t}^* = (\mathcal{S}_{2,t}^* =)$ **set of types with zero (negative) after-tax positions**

Let us guess that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t) \geq 0 \ \forall h \in \mathcal{B}_t^*$, $z_{t,h} = 0 \ \forall h \in \mathcal{S}_{1,t}^*$, and $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)(x_t - T)) < 0 \ \forall h \in \mathcal{S}_{2,t}^*$, where $\mathcal{B}_t^*, \mathcal{S}_{1,t}^*, \mathcal{S}_{2,t}^* \subset \mathcal{H}$, $\mathcal{B}_t^* \cup \mathcal{S}_{1,t}^* \cup \mathcal{S}_{2,t}^* = \mathcal{H}$ and $\mathcal{B}_t^*, \mathcal{S}_{1,t}^*, \mathcal{S}_{2,t}^* \neq \emptyset$. Clearly, the above conditions imply that $\min_{h \in \mathcal{B}_t^*}\{f_{t,h}\} > \max_{h \in \mathcal{S}_{1,t^*}}\{f_{t,h}\} > \max_{h \in \mathcal{S}_{2,t^*}}\{f_{t,h}\}$. Under the above guess, $\sum_{h \in \mathcal{H}} n_{t,h} z_{t,h} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} z_{t,h}$ (where $\mathcal{H} \setminus \mathcal{S}_{1,t}^* = \mathcal{B}_t^* \cup \mathcal{S}_{2,t}^*$) and market-clearing $\sum_{h \in \mathcal{B}_t^* \cup \mathcal{S}_{2,t}} n_{t,h} z_{t,h} = \overline{Z}$ gives $x_t = [(1+r)\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}]^{-1}(\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}[f_{t,h} + a\sigma^2\overline{Z}] + (1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} - a\sigma^2\overline{Z}) := \overline{x}_t$. The guess is verified iff $f_{t,h} + a\sigma^2\overline{Z} - (1+r)\overline{x}_t \geq 0 \ (< 0) \ \forall h \in \mathcal{B}_t^* \ (\forall h \in \mathcal{S}_{1,t}^*)$ and $f_{t,h} + a\sigma^2\overline{Z} - (1+r)(\overline{x}_t - T) \geq 0 \ (< 0) \ \forall h \in \mathcal{S}_{1,t}^* \ (\forall h \in \mathcal{S}_{2,t}^*)$, i.e. iff $\max\{d_{1,t}, \tilde{d}_{1,t}\} \leq a\sigma^2\overline{Z} - (1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} < \min\{d_{2,t}, \tilde{d}_{2,t}\}$ for $d_{1,t}, \tilde{d}_{1,t}, d_{2,t}, \tilde{d}_{2,t}$ as in Proposition 1, 2(iii).

To show $\overline{x}_t > x_t^*$: $(1+r)(\overline{x}_t - x_t^*) = \frac{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}[f_{t,h} + a\sigma^2\overline{Z}] + (1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h} - a\sigma^2\overline{Z}}{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}} - \sum_{h \in \mathcal{H}} n_{t,h} f_{t,h}$. Using $\sum_{h \in \mathcal{H}} n_{t,h} f_{t,h} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h} + \sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h}$ and simplifying, we have:

$$(1+r)(\overline{x}_t - x_t^*) = \sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}\left[\frac{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h} + \frac{(1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}}{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}} - a\sigma^2\overline{Z}}{\sum\limits_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}} - \frac{\sum\limits_{h \in \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h}}{\sum\limits_{h \in \mathcal{S}_{1,t}^*} n_{t,h}}\right] > 0$$

where $\sum\limits_{h \in \mathcal{S}_{1,t}^*} \frac{n_{t,h}}{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}} f_{t,h} \leq \max\limits_{h \in \mathcal{S}_{1,t}^*}\{f_{t,h}\}$, and $\frac{\sum\limits_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h} f_{t,h} + \frac{(1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}}{\sum_{h \in \mathcal{S}_{1,t}^*} n_{t,h}} - a\sigma^2\overline{Z}}{\sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}} > \max\limits_{h \in \mathcal{S}_{1,t}^*}\{f_{t,h}\}$ since $d_{2,t} = \sum_{h \in \mathcal{H} \setminus \mathcal{S}_{1,t}^*} n_{t,h}(f_{t,h} - \max_{h \in \mathcal{S}_{1,t}^*}\{f_{t,h}\}) > a\sigma^2\overline{Z} - (1+r)T\sum_{h \in \mathcal{S}_{2,t}^*} n_{t,h}$ (see above). ■

2 Supporting material

In this section we provide some supporting material for the 'fast' version of our algorithm and some additional numerical results for computation speed and accuracy.

2.1 Foundations for the algorithm

For easy reference, the 'fast' version of our algorithm is repeated below.

Algorithm 2 (fast)

1. Find the set $\tilde{\mathcal{H}}_t$ and the population shares $n_{t,h}$ for $h = 1, ..., \tilde{H}_t$. Compute $disp_{t,1}$. If $disp_{t,1} \leq a\sigma^2\overline{Z}$, then $x_t = x_t^*$ is the date t price, compute the demands $z_{t,h} \geq 0$ for $h = 1, \ldots, \tilde{H}_t$ and move to period $t+1$ and repeat. If $disp_{t,1} > a\sigma^2\overline{Z}$, move to Step 2.

2. Find the largest h such that $z_{t,h} = (a\sigma^2)^{-1}(f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t^*) < 0$, say k_0, where x_t^* is the hypothetical price if a short-sales tax were absent (see Proposition 1 Part 1).

3. Run Steps 3–5 of Algorithm 1, starting from $k = k_0$ (see Step 2). Continue until a solution is found, then move to period $t+1$ and repeat.

In Step 1, we check whether the usual price $x_t = x_t^*$ is a solution. This will be the case if belief dispersion is small enough (no type wants an unconstrained negative position); otherwise, there must at least one non-buyer in equilibrium at date t (and at most $\tilde{H}_t - 1$), and we proceed to Step 2. In *Step 2*, we count the number k_0 of negative demands at the price x_t^*, because the equilibrium price must satisfy $x_t > x_t^*$ (shown in Proposition 1), such that any type with an unrestricted non-positive position at price x_t^* must also have a non-positive position at the equilibrium price x_t (i.e. be a non-buyer):

$$\frac{f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t^*}{a\sigma^2} \leq 0 \implies \frac{f_{t,h} + a\sigma^2\overline{Z} - (1+r)x_t}{a\sigma^2} < 0 \implies z_{t,h} \leq 0, \quad h \in \tilde{H}_t$$

by Equation (5) of the main text.

Hence, the equilibrium number of non-buyers at date t, say k^*, must satisfy $k^* \geq k_0$, implying that k_0 is a lower bound for k^*; thus we give our algorithm initial guess $k = k_0$.

As noted in the main text (see Section 2.1), an updated guess k_0' may be obtained by finding k_0 and then checking among types $1, \ldots, k_0$ which are short-sellers after tax at price x_t^* and which (if any) have after-tax positions of zero. Let k_0^s (k_0^0) be the number of after-tax short-selling types (zero-position types) at price x_t^*. If $k_0^s = k_0$ ($k_0^s < k_0$), an updated price can be computed via Proposition 1 Part 2(ii) (Proposition 1 Part 2(iii)). Our update k_0' is then the number of unrestricted demands ≤ 0 at that price, analogous to Step 2 above.[2]

[2]Our algorithm has a section to perform the update described above, which can improve computation speeds non-trivially when there is a large number of types or beliefs are highly concentrated (i.e. very similar).

2.2 Additional numerical examples

This section reports computation times and accuracy measures for a longer simulation and other scenarios in Figure 2 of the main text. As in Table 1 (main text), dividends are stochastic. The results tell a similar story to Table 1 (main text) and highlight the importance of periods with coexistence of short and zero-position types in increasing computation times (see columns 3–4 in Tables 3,4 below). Computer software and hardware as in the main text.

Table 1: Computation times and accuracy in Scenario 3: $T_{sim} = 250$ periods

No. of types	Regime	Time (s)	Freq. 1 (2)	$\max(Error_t)$
	No tax: $T = 0$	0.03	-	6.9e-17
$H = 100$	Scenario 2: $T = 0.10$	0.08	250 (190)	7.6e-16
	Scenario 2: $T = 1/8$	0.06	250 (145)	1.1e-15
	No tax: $T = 0$	0.04	-	8.3e-17
$H = 1,000$	Scenario 2: $T = 0.10$	0.64	250 (195)	1.1e-15
	Scenario 2: $T = 1/8$	0.36	250 (135)	1.3e-15
	No tax: $T = 0$	0.06	-	1.4e-16
$H = 2,500$	Scenario 2: $T = 0.10$	2.20	250 (196)	1.5e-15
	Scenario 2: $T = 1/8$	1.23	250 (134)	1.5e-15

Notes: $\max(Error_t) := \max\{Error_1, ..., Error_{T_{sim}}\}$, where we define the date t simulation error as $Error_t = |\sum_{h\in\mathcal{H}} n_{t,h} z_{t,h} - \overline{Z}|$. Demands $z_{t,h}$ depend on the computed market-clearing price. Freq. 1 = number of periods with $\mathcal{S}^*_{1,t} \cup \mathcal{S}^*_{2,t} \neq \emptyset$ (at least one short or zero position at date t), and Freq. 2 = number of periods with $\mathcal{S}^*_{1,t}, \mathcal{S}^*_{2,t} \neq \emptyset$ (both short and zero positions at date t).

Simulation times are higher, as expected, when we simulate Scenario 3 for $T_{sim} = 250$ periods (rather than 100) and our measure of accuracy (final column) has similar values.

Table 2: Computation times and accuracy in Scenario 1: $T_{sim} = 100$ periods

No. of types	Regime	Time (s)	Freq. 1 (2)	$\max(Error_t)$
	No tax: $T = 0$	0.02	-	6.9e-17
$H = 100$	Scenario 2: $T = 0.10$	0.03	100 (98)	1.1e-16
	Scenario 2: $T = 1/8$	0.03	100 (99)	1.3e-16
	No tax: $T = 0$	0.03	-	5.6e-17
$H = 1,000$	Scenario 2: $T = 0.10$	0.13	100 (98)	1.4e-16
	Scenario 2: $T = 1/8$	0.14	100 (99)	1.5e-16
	No tax: $T = 0$	0.03	-	9.7e-17
$H = 2,500$	Scenario 2: $T = 0.10$	0.43	100 (98)	2.2e-16
	Scenario 2: $T = 1/8$	0.46	100 (99)	3.6e-16

Notes: Please see Table 1 above for a full description of the column entries.

In Scenario 2, computation times and accuracy are similar to Scenario 3 (Table 1, paper).

Table 3: Computation times and accuracy in Scenario 2: $T_{sim} = 100$ periods

No. of types	Regime	Time (s)	Freq. 1 (2)	max($Error_t$)
	No tax: $T = 0$	0.01	-	5.6e-17
$H = 100$	Scenario 2: $T = 0.10$	0.03	100 (98)	1.1e-16
	Scenario 2: $T = 1/8$	0.04	100 (99)	1.9e-16
	No tax: $T = 0$	0.02	-	5.6e-17
$H = 1,000$	Scenario 2: $T = 0.10$	0.22	100 (98)	1.1e-16
	Scenario 2: $T = 0.1/8$	0.27	100 (99)	1.7e-16
	No tax: $T = 0$	0.03	-	8.3e-17
$H = 2,500$	Scenario 2: $T = 0.10$	0.72	100 (98)	2.2e-16
	Scenario 2: $T = 1/8$	0.93	100 (99)	3.6e-16

Notes: Please see Table 1 above for a full description of the column entries.

Computation times are a bit higher in Scenario 2 due to the increased number of periods in which both zero-position types and short-sellers coexist relative to Scenario 3 (see Col. 4).

Table 4: Computation times and accuracy in Scenario 4: $T_{sim} = 50$ periods

No. of types	Regime	Time (s)	Freq. 1 (2)	max($Error_t$)
	No tax: $T = 0$	0.01	-	1.9e-16
$H = 100$	Scenario 2: $T = 0.10$	0.02	50 (33)	1.4e-15
	Scenario 2: $T = 1/8$	0.02	50 (37)	9.2e-15
	No tax: $T = 0$	0.01	-	3.6e-16
$H = 1,000$	Scenario 2: $T = 0.10$	0.14	50 (33)	1.4e-15
	Scenario 2: $T = 1/8$	0.15	50 (37)	8.2e-15
	No tax: $T = 0$	0.02	-	5.6e-16
$H = 2,500$	Scenario 2: $T = 0.10$	0.44	50 (33)	2.8e-15
	Scenario 2: $T = 1/8$	0.57	50 (37)	4.2e-15

Notes: Please see Table 1 above for a full description of the column entries.

Computation times are higher than in Scenario 3, taking into account the smaller number of simulated periods. In this scenario, there is an fast-exploding price path that diverges to $+\infty$ (see Figure 2, main text); for this reason we simulated $T_{sim} = 50$ periods in this case.

References

Anufriev, M. and Tuinstra, J. (2013). The impact of short-selling constraints on financial market stability in a heterogeneous agents model. *Journal of Economic Dynamics and Control*, 37(8):1523–1543.