General DP Remarks

Optimal Substructure

- Create optimal solution to problem using optimal solutions to subproblems.
- Can't use DP if optimal solution to a problem does not require subproblem solutions to be optimal.
  → Often happens when subproblems are *not independent* of each other.

Overlapping Subproblems

- For DP to be useful, recursive algorithm should require us to compute optimal solutions to the *same subproblems* over and over again.
- In total, there should be a small number of distinct subproblems (i.e. polynomial in input size).

LCS

$$LCS[i,j] = \begin{cases} 1 + LCS[i-1, j-1] & \text{if } x_i = y_i \\ \max(LCS[i-1,j], LCS[i,j-1]) & \text{otherwise} \end{cases}$$

		j=0	j=1	j=2	j=3	j=4	j=5	j=6	j=7	j=8
			p	r	i	n	t	i	n	g
i=0		0	0	0	0	0	0	0	0	0
i=1	s	0	0	0	0	0	0	0	0	0
i=2	p	0	1↖	1←	1←	1←	1←	1←	1←	1←
i=3	r	0	1↑	2↖	2←	2←	2←	2←	2←	2←
i=4	i	0	1↑	2↑	3↖	3←	3←	3←	3←	3←
i=5	n	0	1↑	2↑	3↑	4↖	4←	4←	4←	4←
i=6	g	0	1↑	2↑	3↑	4↑	4↑	4↑	4↑	5↖
i=7	t	0	1↑	2↑	3↑	4↑	5↖	5←	5←	5←
i=8	i	0	1↑	2↑	3↖	4↑	5↑	6↖	6←	6←
i=9	m	0	1↑	2↑	3↑	4↑	5↑	6↑	6↑	6↑
i=10	e	0	1↑	2↑	3↑	4↑	5↑	6↑	6↑	6↑

OBST

$$e[i,j] = \begin{cases} 0 & \text{if } i = j-1 \\ \min_{i \le r \le j}\{e[i,r-1] + e[r+1,j] + w(i,j)\} & \text{if } i \le j \end{cases}$$

$$root[i,j] = \text{root of subtree with keys } k_i, \ldots, k_j \text{ for } 1 \le i \le j \le n$$

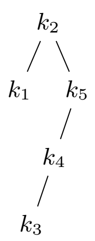
$$w[1,\ldots,n+1, 0, \ldots, n] = \text{ sum of probabilities}$$
$$w[i,i-1] = 0 \text{ for } 1 \le i \le n$$
$$w[i,j] = w[i,j-1] + p_j \text{ for } 1 \le i \le j \le n$$

Consider 5 keys with search probabilities $p_1 = 0.25, p_2 = 0.2, p_3 = 0.05, p_4 = 0.2, p_5 = 0.3$

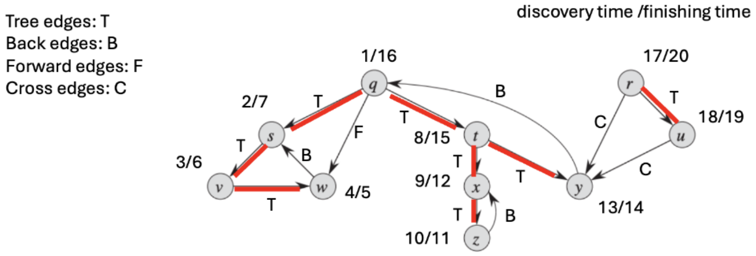
w	0	1	2	3	4	5
1	0	0.25	0.45	0.5	0.7	1.0
2		0	0.2	0.25	0.45	0.55
3			0	0.05	0.25	0.55
4				0	0.2	0.5
5					0	0.3
6						0

e	0	1	2	3	4	5
1	0	0.25	0.65	0.8	1.25	2.1
2		0	0.2	0.3	0.75	1.35
3			0	0.05	0.3	0.85
4				0	0.2	0.7
5					0	0.3
6						0

r	0	1	2	3	4	5
1		1	1	1	2	2
2			2	2	2	4
3				3	4	5
4					4	5
5						5
6						



DFS



$$(q\,[s\{v(ww)v\}s]\;[t\{x(zz)x\}yyt]q)\;(r[uu]r)$$

Tree Edges Are edges in depth-first forest G_π. Edge (u,v) is a tree edge if v was first discovered by exploring edge (u,v)
Back Edges Are edges (u,v) connecting a vertex u to an ancestor in a depth-first tree. We consider self-loops, which may occur in directed graphs, to be back edges.
Forward Edges Are nontree edges (u,v) connecting a vertex u to a descendant v in a depth-first tree.
Cross Edges Are all other edges. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other, or they can go between verticies in different depth-first trees.

Knapsack

$$c[i,w] = \begin{cases} 0 & \text{if } i = 0 \text{ or } w = 0 \\ c[i-1,w] & \text{if } w_i > w \\ \max(c[i-1,w], c[i-1, w-w_i] + v_i) & \text{if } i > 0 \text{ and } w \ge w_i \end{cases}$$

Graphs

Handshaking Lemma

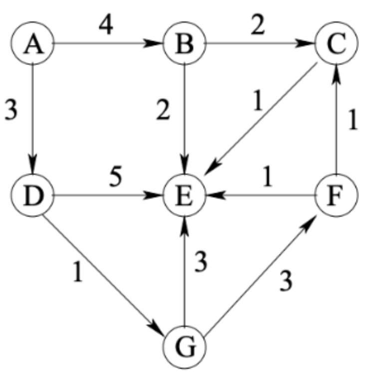
$$\sum_{v \in V} \deg(v) = 2|E|$$

BFS vs DFS

- DFS is usually for finding relationship among vertices.
- BFS Is usually for finding shortest path from a given source.

Djikstra

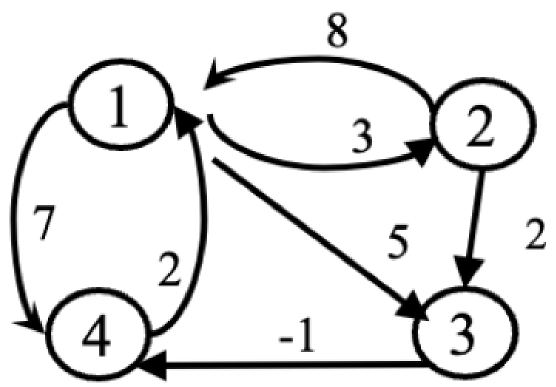
Execute Djikstra's algorithm on the graph below starting at A. If there are ties, the vertex with the lowest letter comes first.



A: 0	D: 3	B: 4	G: 4	C: 6	E:
B: ∞	B: 4	G: 4	C: 6	E: 6	F:
C: ∞	C: ∞	E: 8	E:6	F: 7	
D: ∞	E: ∞	C: ∞	F: ∞		
E: ∞	F: ∞	F: ∞			
F: ∞	G: ∞				
G: ∞					

Floyd-Warshall

Execute the Floyd-Warshall algorithm on the graph below, provide the D^k and P matrixes on each step.



$$D^0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 3 & 5 & 7 \\ 2 & 8 & 0 & 2 & \infty \\ 3 & \infty & \infty & 0 & -1 \\ 4 & 2 & \infty & \infty & 0 \end{array} \qquad P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$D^1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 3 & 5 & 7 \\ 2 & 8 & 0 & 2 & 15 \\ 3 & \infty & \infty & 0 & -1 \\ 4 & 2 & 5 & 7 & 0 \end{array} \qquad P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

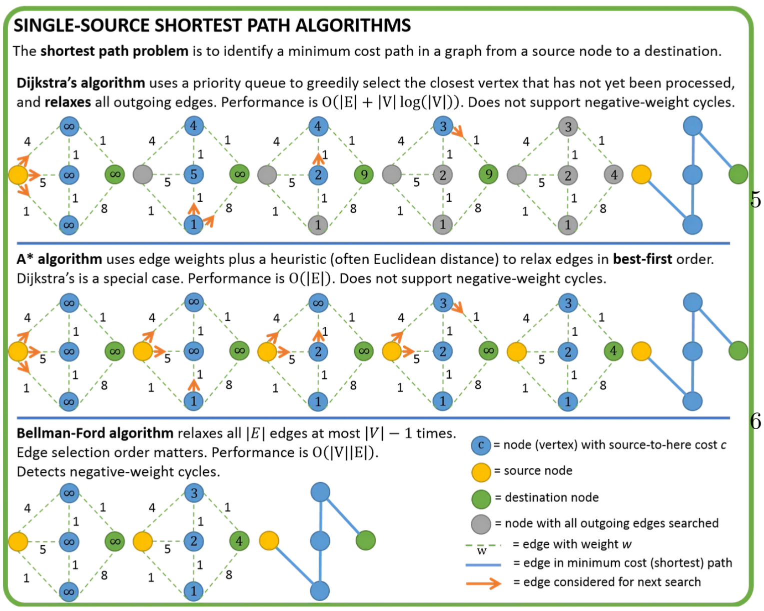
$$D^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 3 & 5 & 7 \\ 2 & 8 & 0 & 2 & 15 \\ 3 & \infty & \infty & 0 & -1 \\ 4 & 2 & 5 & 7 & 0 \end{array} \qquad P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

$$D^3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 3 & 5 & 4 \\ 2 & 8 & 0 & 2 & 1 \\ 3 & \infty & \infty & 0 & -1 \\ 4 & 2 & 5 & 7 & 0 \end{array} \qquad P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

$$D^4 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 3 & 5 & 4 \\ 2 & 3 & 0 & 2 & 1 \\ 3 & 1 & 4 & 0 & -1 \\ 4 & 2 & 5 & 7 & 0 \end{array} \qquad P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 3 \\ 2 & 4 & 0 & 0 & 3 \\ 3 & 4 & 4 & 0 & 0 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

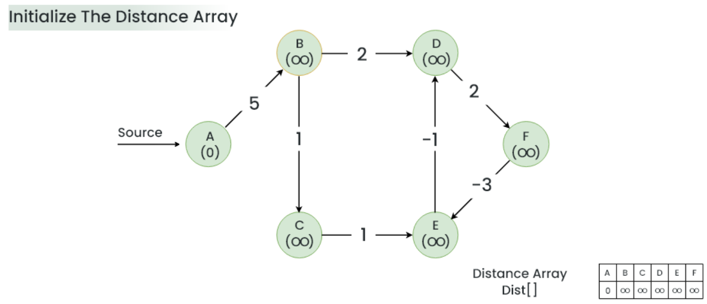
Knapsack

$$c[i,w] = \begin{cases} 0 & \text{if } i = 0 \text{ or } w = 0 \\ c[i-1,w] & \text{if } w_i > w \\ \max(c[i-1,w], c[i-1,w-w_i]+v_i) & \text{otherwise} \end{cases}$$



Bellman-Ford

Bellman-Ford is a single source shortest path algorithm that determines the shortest path between a given source vertex and every other vertex in a graph. This algorithm can be used on both weighted and unweighted graphs.



Bellman-Ford To Detect A Negative Cycle In A Graph

1. Initialize distance array to store shortest dist for each vertex. Initialize source as 0, and all others to be ∞.
2. Relax all edges $|V| - 1$ times.
3. Relax all edges one more time to detect negative cycles.

Case 1 For any edge(u, v, weight), if dist[u] + weight ¡ dist[v], then there is a negative cycle.

Case 2 Case 1 fails for all edges

A	B	C	D	E	F
0	∞	∞	∞	∞	∞
0	5	∞	∞	∞	∞
0	5	6	7	∞	∞
0	5	6	7	7	9
0	5	6	6	6	9
0	5	6	5	6	8

1. Start relaxing edges, 1st relaxation:

$$Dist(B) > Dist(A) + w(A,B)$$
$$\infty > 0 + 5 \Rightarrow Dist(B) = 5$$

2. 2nd relaxation:

$$Dist(D) > Dist(B) + w(B,D)$$
$$\infty > 5 + 2 \Rightarrow Dist(D) = 7$$

$$Dist(C) > Dist(B) + w(B,C)$$
$$\infty > 5 + 1 \Rightarrow Dist(C) = 6$$

3. 3rd relaxation:

$$Dist(F) > Dist(D) + w(D,F)$$
$$\infty > 7 + 2 \Rightarrow Dist(F) = 9$$

$$Dist(E) > Dist(C) + w(C,E)$$
$$\infty > 6 + 1 \Rightarrow Dist(E) = 7$$

4. 4th relaxation:

$$Dist(D) > Dist(E) + w(E,D)$$
$$7 > 7 + (-1) \Rightarrow Dist(D) = 6$$

$$Dist(E) > Dist(F) + w(F,E)$$
$$7 > 9 + (-3) \Rightarrow Dist(E) = 6$$

5. 5th relaxation:

$$Dist(F) > Dist(D) + w(D,F)$$
$$9 > 6 + 2 \Rightarrow Dist(F) = 8$$

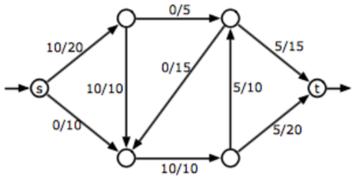
$$Dist(D) > Dist(E) + w(E,D)$$
$$6 > 6 + (-1) \Rightarrow Dist(D) = 6$$

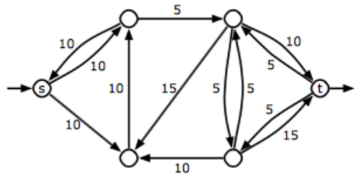
6. 6th relaxation (final):

$$Dist(E) > Dist(F) + w(F,E)$$
$$6 > 8 + (-3) \Rightarrow Dist(E) = 5$$

$$Dist(F) > Dist(D) + w(D,F)$$
$$8 > 5 + 2 \Rightarrow Dist(F) = 7$$

Network Flow





Flow Amount of material that can be transported between two nodes.
Capacity Maximum amount of material that can be transported between two nodes.
Residual Capacity Amount of material that can still be transported between two nodes.
Augmenting Path Path from source to sink where all edges have residual capacity.
Residual Graph Graph where edges have residual capacity.