Facility Location Using Markov Chains

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January 16, 2018

Abstract

We consider using Markov chains to solve facility location problems. In particular, we find a Markov chain when coupled with realistic assumptions about a consumer-supplier network is able to answer questions such as: Where should a facility be placed to have the most consumer throughput, or so that the most neglected customers can have increased ability to access it? We present an example construction including computational results for finding the best location in the United States for a new temple to be built by a national church.

I. Introduction

Problems in facility location theory focus on finding optimal locations for facilities to meet consumer demand, either in adding new facilities or relocating existing ones. There are a myriad of ways to define an optimal location, but the standard definition is a location that minimizes the average distance traveled by a consumer to fulfill a demand (Owen & Daskin, 1998). Additional definitions of optimal include the location that minimizes the maximum distance traveled to fulfill a demand, the location to maximize the throughput of the suppliers, or the location that minimizes the average time consumers are required to travel to reach suppliers. In this paper, we show that a Markov chain can be used to solve facility location problems based on some of these definitions of optimal, as well as others we introduce in the section on proposed metrics.

A fundamental facility location problem is the P-median problem, which tries to minimize the distance that a consumer must travel to reach their nearest facility (Current, Min, & Schilling, 1990). Typical applications of P-median problems include finding optimal locations for industrial warehouses to minimize shipping distances (Jakubovskis, 2017), stores to maximize availability to customers (Klose, Drexl, 2005), and emergency response public services where response time is imperative (Ghosh, Gosavi, 2017). As a special case of the P-median problem, we will be contrasting our analysis with the Fermat-Weber problem (Bose, Maheshwari, Morin, 2003). We call our approximate solution to the Fermat-Weber problem the Weighted-Distance Method (WDM). This method considers the Euclidean distance between the consumers and the potential facility under a function that we believe models the effect of distance on the likelihood of people to visit the consumer. An example of this model in application can be found in the paper by Ndiaye, Ndiaye and Ly (2012).

In comparison to the WDM, the main approach employed in this paper is to use a Markov chain to model the behavior of consumers as they interact with facilities. An interaction could be a sale made, a visit to a retail location, or any other meaningful experience that a consumer would have with a supplier. Markov chains reveal the effects that simple local behaviors have on more complex global behaviors through modeling the behavior as a network. This is similar to the goal of many facility location problems, as it is often the case that the behavior being optimized is about the aggregate populous and not the specific actions of certain individuals. Markov chains have been used in many applications, including: mathematical ecology (Kirkland, 2014), computational biology (Krogh, Brown, Mian, Sjolander, Haussler, 1994), and network modeling (Newman, 2003), but they have not been commonly used in facility location problems. In this paper, we show that Markov chains provide a flexible and efficient way to model facility location problems that offers additional information not found in the standard analysis of the P-median problem.

This paper proceeds as follows. Section 2 contains a literary review of facility location problems on which Markov chains could potentially be used, as well as literature pertaining to the WDM and the Markov chain method. Section 3 will outline the construction of the Markov transition matrix used to solve facility location problems, give the results of the general testing of the model and expand on insights gained while studying the model. Section 4 details the set up and use of our model through a case study about determining new locations for temples of The Church of Jesus Christ of Latter-day Saints (LDS church). Section 5 will conclude the paper with suggestions for further work.

II. Literature Review

Facility location management is a well-established area within operations research. Current, Min and Schilling report that facility location management has been researched through the perspective of many different academic disciplines and cite a recent bibliography on this subject that had over 1500 titles, showing the vast expanse of research in this area (1990). Owen and Daskin, in their literature review, give an overview of the most-used models among this immense amount of literature. They comment on 15 differing subsets of facility location problems, again demonstrating the vastness of the literature and applicability of the problem (1998). Although the library of facility location management is large, this review focuses primarily on literature pertaining to median problems, with a specific emphasis on the use of Markov chains.

The study of location theory began in 1909 when Alfred Weber was researching possible warehouse location based on the average distance to be traveled by those who would visit the warehouse (Owen & Daskin, 1998). His research created the basis for *median problems*, or problems that determine a facility location by minimizing the average distance traveled by consumers. This distance is often weighted using factors like speed limits, traffic, proximity to competitors, and tolls. As noted by Melo, Nickel and Saldanha-da-Gamma, the P-median problem is considered to be the foundational median problem. They define the P-median problem as the problem of placing P facilities in locations that minimize the total weighted distance (or costs) to satisfy customer demands (2009).

From this rudimentary definition of a P-median problem arises a multitude of slight variations that account for the context of the problem. For example, Berman, Drezner, and Wesolowsky use a model formulation which allows demands to be serviced by facilities other than the one closest to them (2003). Bruni, Beraldi, and Conforti use an undirected graph to model a complex water distribution network (2016).

Paragraph (or two) about distance models, applications, successes, and disadvantages.

Consider, for example, the P-median problem. Using the optimization techniques developed by the literature, a decision maker can introduce a new facility that will give the most access, as measured by the inverse of distance, to the consumers that they want to reach. Even with some additional conditions imposed on the problem, such as access to multiple facilities and geographical obstacles, there are still studied solutions to these questions.Via modeling the same system as a network, a Markov chain can be used to optimize many things, including the ease of access to a facility under the feedback mechanism of busyness, the amount of throughput occurring in each facility, the access of only those consumers who have the least access initially, and so on.

One of the significant tools in this process is the equilibrium state, or steady state, of the Markov chain. This can be calculated in a straightforward manner from the transition matrix used in the Markov chain. This equilibrium state is a vector consisting of what is qualitatively understood as the long-term proportion of consumers that are in a certain state. As the transition matrix can be easily updated, we can easily measure the long-term effects of placing a new facility in a particular location. An example of this is a paper by Faizrahnemoon et al. detailing the access to a hospital by changing the location of bus stops in a transportation network in England (2015). By measuring the long term effects of a change, we can find the optimal location for a new facility or the relocation of a facility. Because it is simple to create a network that models the desirable qualities, properties, and obstacles in the real life situation, we have a flexible and adaptable framework with which to solve facility location problems.

III. Markov Chain Modeling

Using properties of a Markov chain, one can make decisions about the "best" place to have a facility under different definitions of "best". Where the P-median problem seeks to optimize distance, a Markov chain can be used to optimize many different attributes of a supply network. In this section, we develop this idea and demonstrate the construction and analysis of such on real-world networks. Finally, we discuss the qualitative and quantitative results of the models that were created, in which the strengths and weaknesses of each objective function is discussed.

.1 Background

A Markov chain is an example of a stochastic simulation of a closed system with n states in discrete time. We create an $n \times n$ matrix X where $x_{i,j}$ represents the probability of moving from state i to state j in one time step. As this is a closed system, in each time step the probability that a person moves from state i to some state in the system (including possibly remaining at i) is 1. This means that each row in our transition matrix should have sum equal to one, which is known as *row stochastic*. We note that an equivalent construction can be used where the columns sum to one, but we continue with the row stochastic formulation here.

One of the powerful applications of Markov chains is that we can derive a equilibrium solution, or steady state solution, very easily. We

$$\left[\begin{array}{c|c} CC & CS \\ \hline SC & SS \end{array}\right]$$

Figure 1: *Block diagram of transition matrix*

can model any initial condition (i.e. distribution of people) as a vector w_0 and see what happens after one time step by letting $w_1 = w_0 X$. More generally, we say $w_{n+1} = w_n X$. Because we have a row stochastic matrix, we can see that the all-ones vector is a right eigenvector of X. In the subsection Model Setup, we add a "teleport" factor to X, which results in it being an all positive matrix. The Perron-Frobenius Theorem tells us that, in this case, the dominant eigenvalue of X is 1 and that it has an all-positive left eigenvector. This eigenvector is known as the Perron eigenvector and represents the steady state of the model. We use this steady state solution to observe what happens to the distribution of consumers when a new facility is added in a given location.

For a more in-depth background to Markov chain modeling please refer to the "backround" section of the paper by Faizrahnemoon et al. (2015).

.2 Model Setup

The Markov chain method that we use is based on that of Faizrahnemoon et al. (2015). It involves two stages: the initialization of the model transition matrix and the computation of the metrics, including the Perron eigenvector and mean first passage time.

To begin, we initialize the transition matrix X to be an empty $n \times n$ matrix. Each row and column $0 \leq i \leq k$ represents a consumer i and each row and column $k < j \leq n$ represents a supplier j. We use this notation will continue to distinguish consumers and suppliers. However, for individual entries in the matrix, we employ the standard notation, $x_{i,j}$.

Let d_i be the distance from consumer i to their nearest supplier. These distances are transformed by a sigmoid function $g(d_i) : \mathbb{R} \to [0, 1]$ of the form:

$$g(d_i) = \frac{1}{1 + e^{\gamma(d_i - \alpha)}} - \beta. \quad (1)$$

This transformation allows the modeler to define a region of significance. As an example of the use of this function, if a potential customer of a retail chain lives several hundreds of miles away from the nearest location, then cutting that distance by twenty percent will not be a significance change for that consumer. Equally useful is that this function allows the modeler to determine a minimum distance of significance. For example, if a potential customer lives three hundred yards away from the nearest supplier, then cutting that will have a negligible effect on the customer. Thus, an annulus of significance can be defined. The parameters γ, α, and β represent, respectively, a stretch factor, a domain shift, and a range shift. These will vary with the desired region of significance. If this region of significance does not hold significance for the modeler, a different transformation can best suit the situation.

It is assumed that no consumer relocates to a different area. Even though people often move, Markov chains use and report information about aggregate behavior, so this is a reasonable assumption to make. So, for $0 \leq i, m \leq k$, $x_{i,m} = x_{m,i} = 0$ if $i \neq m$.

Let p be the probability that a consumer in the immediate vicinity of a supplier visits that supplier on any given time step. The values on the diagonal of X represent the probability of a consumer not visiting a supplier in one time period. These are calculated as follows:

$$x_{i,i} = p + (1-p) \cdot (g(d_i))$$

where $0 \leq i \leq k$. We observe here that the greater the distance a given consumer has to its nearest supplier, the less likely that consumer is to visit a supplier. This is desirable behavior in our model because it represents tendencies that people have.

To consider movement from a consumer to a supplier, we have $d(i,j)$ as the Euclidean distance from consumer i to supplier j and:

$$D_i = \sum_{j=k}^{n} \frac{1}{e^{d(i,j)}}$$

Then,

$$x_{i,j} = x_{i,i} \times \frac{\frac{1}{e^{d(i,j)}}}{D_i}$$

The movement from suppliers to other suppliers is then set to 0. So,

$$y_{j,j} = 0.$$

And, for $k < l \leq n$,

$$y_{j,l} = 0.$$

Or, equivalently, the SS block is the appropriately-sized matrix of zeroes. This models the assumption that once a consumer has visited a supplier, they have no intention of waiting around at that supplier or immediately visiting another supplier. Once again, although there may be individual exceptions to this assumption, the overall behavior being modeled fits the assumption.

The next consideration is movement from suppliers to consumers, which is calculated as follows:

$$y_{j,i} = \frac{(1 - x_{i,i}) \times x_{j,i}}{E_j}$$

where,

$$E_j = \sum_{m=0}^{k} (1 - x_{m,m}) \times x_{k,j}.$$

This construction results in X being a row stochastic matrix. To make X a positive matrix, we add a "teleport" factor distributed across each entry of each row. While this does not model true behavior, it ensures that the matrix satisfies the hypotheses of the analysis and speeds convergence.

This is done as follows: Given e, a small number greater than zero, all elements of X are multiplied by $(1-e)$ and

$$x_{cd} = x_{cd} + \frac{e}{n}$$

This results in a matrix that is both row stochastic and ergodic.e should be chosen to be significantly smaller than any values in the transition matrix so the results are not influenced too greatly by the presence of the teleport factor.

Variable	Description
i	A given consumer
j	A given supplier
n	Number of states
k	Number of states corresponding to consumers
X	Transition matrix of the Markov Chain
γ, α, β	Parameters for weighted distance function
p	Probability a consumer visits the supplier in one time step
d_i	Distance from consumer i to the nearest supplier
e	Teleport factor

Table 1: *Generic variables and parameters used in the setup of the Markov chain method.*

i. Proposed Metrics

The following metrics are ways to measure the effectiveness of a network. Each metric measures the network differently and each can be optimized in the network. We will explain each in detail including what each metric measures. We will also give justification in the form of an empirical test case. We will refer to the transition matrix in blocks, as outlined in figure 1.

Figure: The test cases - ellipse, hyperbola, circles, with the results from each metric plotted on top.

i.1 Mean First Passage Time

Instead of minimizing the median Euclidean distance or the transformed distance from equation (1), we consider minimizing the mean first passage time. The mean first passage time from a to b represents the average amount of time taken for Markov chain to reach state b given that it started in state a. Using the formulation of our transition matrix, the mean first passage time from i to j would represent the average time it takes for consumer i to visit supplier j. Thus, by minimizing the mean first passage time, we can minimize the average amount of time it takes for a consumer to visit a supplier. However, the mean first passage time is a measure from one specific state to another specific state, so to use it to measure the network, we must use a function of the entries in the mean first passage time matrix.

Following the notation and exposition in Kirkland et al. (2008), the mean first passage time is given by

$$M = [I - X^{\#} + J_n X_d^{\#}]\Pi^{-1}$$

Where I is the $n \times n$ identity matrix, $X^{\#}$ is the *group inverse* of the transition matrix X, J_n is the $n \times n$ matrix of all ones, $X_d^{\#}$ is the $n \times n$ diagonal matrix whose diagonal entries are the corresponding diagonal entries of $X_{\#}$, and where Π is the diagonal matrix whose diagonal entries are the corresponding entries of the equilibrium state vector of the transition matrix.The group inverse is the unique matrix that satisfies $X^{\#}XX^{\#} = X^{\#}$, $XX^{\#}X = X$, and $XX^{\#} = X^{\#}X$. It is a matrix the same size as the transition matrix and the i, jth entry gives the average number of time steps required for a person starting in state i to get to state j. The block format and meaning of the transition matrix carries over into the matrix of mean first passage times.

We are not concerned with the values in the CC, SC, or SS block because it is only the CS block that represents the amount of steps it takes for consumers to get to a supplier. However, under a different problem formulation, these blocks could give meaningful information. In fact, we are not even interested in all of the values in the CS section, only the smallest entry in each row. In a large network, a person in Alabama is not likely to drive to Montana to do their shopping, so while there is a mean first passage time from a consumer in Alabama to a center in Montana, it is very large, and this time should have little effect on our decision because it is not typical behavior. Thus, we sum the minimum of each row in the CS block of the mean first passage time matrix.

We want to minimize this sum, as this represents a small average travel time for a consumer to arrive at a supplier, and having consumers closer to suppliers is valuable to a supplier. By optimizing on the mean first passage time, we are reducing the overall difficulty of getting to a supplier, which can be a desirable behavior in the network.

To minimize the relevant portion of the mean first passage time, we choose a set of locations where we may desire new facilities. For each location in the set, we add the location to the existing network, create the transition matrix for the modified network, and compute the mean first passage time matrix. Following this, we sum the smallest entries in each row. We accept the location with the smallest sum as the most optimal location in the set of proposed locations.

i.2 Throughput

In order to increase revenues, a reasonable thing to maximize is the number of people that visit suppliers during each time step. For example, a restaurant wants the most people to go through each location. If there are more people at the store daily, then revenues will increase, which is likely a desirable behavior in the network.

Referencing the first Perron eigenvector again, we can see the steady state of the network. Each entry in the eigenvector gives the proportion of people in the in that state in the long term or equilibrium. By this construction, the last $n-k$ places in the eigenvector correspond to the number of people who visit a supplier in a given day. To maximize the number of people who visit the supplier, we maximize the sum of these last $n-k$ places in the eigenvector. To carry this out, we take the proposed location for the new facility, calculate the transition matrix for the network with the proposed facility, and compute the first Perron eigenvector. Then, we sum over the last $n-k$ entries in the vector. In comparison to the mean first passage time, the optimization problem induced by this metric is throughput-centric.

i.3 Top-tier

It also may be of interest to enlarge a consumer base by focusing on increasing accessibility for those who are currently having the most difficulty getting to suppliers. This would represent situations such as hospitals or fire stations that desire to minimize the difficulty for outliers to get to a supplier's location. By doing this, the worst-case response time of the fire station will be decreased because the ability of outlier consumers to visit suppliers is increased.

In order to have the most benefit for the consumers that have the most difficulty getting to the suppliers, we again consider the first Perron eigenvector. The consumer entries with the highest values in the equilibrium vector will have the highest proportion of people in that location at any given day. In order to get their business, these consumers must be able to visit suppliers more easily, which corresponds to a reduction in their entry in the Perron eigenvector. To do this, we compute the Perron eigenvector for the network before adding a new facility and note the states that have the highest values in the eigenvector. We choose the top $L\%$ to minimize, where L varies based on the context of the problem being considered. If L is large, this metric turns into the throughput metric. If L is very small, then a few locations will dominate the decision-making process. For example, if we choose only one location to reduce, then the logical thing to do would be to place a supplier right on that consumer's location. By optimizing on this metric, we ensure that the people that have the most difficulty getting to a supplier will be benefited the most.

i.4 Kemeny Constant

The Kemeny constant is defined to be

$$K = \sum_{\substack{\lambda \in \sigma(X) \\ \lambda \neq 1}} \frac{1}{1-\lambda}$$

Where $\sigma(X)$ is the spectrum of X (get reference for cho and meyer 2001). It measures the connectedness of a network. By maximizing the Kemeny constant, the network becomes more connected; that is, it is easier to reach any state from any other state. This would be useful in situations like warehouses and distribution centers, where it is important for there to be many links from warehouses to distribution centers to allow for flexible and optimized scheduling of transport of goods.

IV. Case Study

In this section we present an example of a facility location problem, in which some, but not all, of the metrics discussed previously give valuable information. The example given here is that of The Church of Jesus Christ of Latter-Day Saints (the LDS church) building new temples.

For members of the LDS Church, temples are a facility of worship and visiting them is believed to be imperative in the process of gaining salvation. In this problem, we assume that the LDS Church is attempting to satisfy one of two goals, either increasing the access that its members have to temples (Top-tier metric) or generating the highest attendance to its temples overall (Throughput metric).

As a control case, we use a P-median method to minimize the distance of each church member to their closest temple. The distance is measured as a transformation of the geophysical distance, which is in turn calculated via Vincenty's Formulae. We will refer to the Vincenty distance by d_v. The transformation is given by:

$$f(d_v) = \frac{1}{1 + e^{-0.053*(d_v - 70)}} - 0.024. \tag{2}$$

This is a special case of 1 with the constants γ, α, and β chosen to mimic the effect of a temple a certain distance away. Here, the assumption is that the difference between having a temple 5 miles away and 10 miles away is marginal, as is the difference between having a temple 200 miles away and 400 miles away. The assumption implies that the biggest impact will come from people who were previously about 90 miles away from a temple and are now 20 miles away.

In the LDS church, church members are grouped by geographical area. Each grouping of members is called a stake, and a building of worship, called a stake center, is located within each stake. For our model, instead of the residence of each individual member, we use the location of the stake center to calculate d_v. Let each stake center be represented as s_i, each temple as t_j, the set of all the stake centers as S, the set of original temples as T, and the set of new temples as T_{new}. An example representation of the setup of this problem with each stake being connected to its two closest temples can be found in Figure 1. The average distance a given member of an LDS stake has to travel to get to their closest temple is

$$h(S,T) = \frac{1}{n} \sum_i \min_j \{d(s_i, t_j)\}. \tag{3}$$

Thus, the unconstrained optimization problem can be written as

$$\underset{T^*}{\text{minimize}}\, h(S, T^*), \text{ where } T^* = T \cup T_{\text{new}}. \tag{4}$$

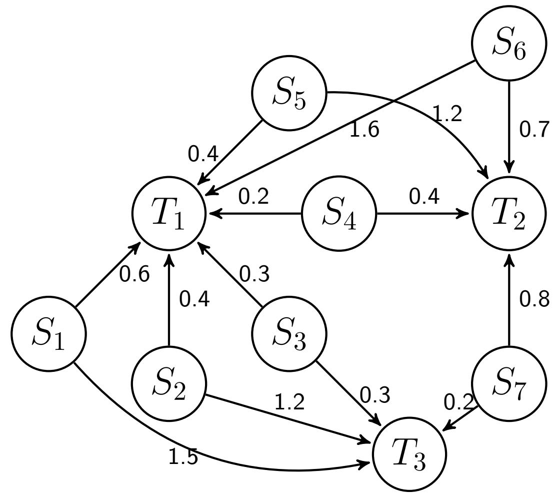

Figure 2: *A sample graph including 7 stakes and 3 temples. **Note:** Edges in this figure are a measure of distance, not a measure of probability. Not to scale.*

To determine the cardinality of T_{new} to use in our experiments, we performed hill climbing to see if building multiple temples at once would change where the new temple locations would be. We found that the US is spread out enough that determining the location of multiple temples at once is not significantly different than determining the temples' locations one at a time. Thus, for ease of computation, we only considered T_{new} with carnality 1; That is, we assume only one temple is to be added at a time.

To minimize the objective function, we sampled the value of h in the United States at one million evenly spaced points. For the test case, we used the WDM as a comparison of what the P-median problem would yield.

i. Weighted Distance Model Setup

To fully understand the model, it is crucial to know what factors were considered relevant. Chief among these factors is the distance $d(s_i, t_j)$ from a stake center to a temple. This distance is measured using the Vincenty Algorithm in order to account for the shape of the earth and give accurate measurements based on latitude and longitude. This is a naive Euclidean distance not based upon actual driving or walking distances. Using an API to find all of these actual distances could be considered for increased accuracy in future work.

The only other relevant factor to the WDM was the weight associated with each distance, ω_i. For convenience, we set these weights to 1 to get our initial result. As this is a standard P-median problem, the weights can be adjusted to account for additional factors, as desired.

The WDM is designed to minimize the average weighted distance a given member of a LDS stake has to travel to get to their closest temple. As such, this model does not take into account the capacity of the temples, nor the relative accessibility of other temples, even if they are a similar distance away. The only assumption made about the behavior of church members is that they desire to have a temple close to their stake center, and that they will only use the closest temple to them.

ii. Markov Model Setup

To solve this problem, we used the Markov chain model that can be found in Section 3 using the variables in Table 1. To simplify this model a number of assumptions were made. First, we assumed that every LDS stake in the United States has the same number of active, temple-going members contained within its boundaries. This approximation, combined with the assumption that every stake center is located at the center of the stake, allowed us to estimate the member density of each stake, ρ_i. The purpose of ρ_i is to relate how much traffic each stake would provide to its nearest five temples.

Next, for each stake, we assumed the number of members that attend the temple on an average day given the stake center's Euclidean distance (again determined by the Vincenty Algorithm) to and the busyness of its 5 closest temples. We determined the busyness of a temple, τ_i, based on the number of stakes that listed that temple as one of its five closest temples.

In order to create this model, we initialize a square probability matrix P in which the first

Variable	Description
$d(s_i, t_j)$	Distance from stake s_i to temple t_j
ρ_i	Density of stake s_i
S	Set of all stakes
s_i	Stake number i
T	Set of all temples
T_{new}	The proposed new temple locations
t_i	Temple number i
τ_i	Temple score of temple t_i

Table 2: *Important variables for the Markov model used to solve the LDS location problem.*

k rows and columns correspond to the stakes in the United States and the next m rows and columns correspond to the temples in the United States. Each $P_{i,j}$ in the matrix represents the probability that a person in row i will travel to column j in the current state. The diagonal entries $P_{i,i}$ represent the probability that a person will remain in their current state (i.e. not visit a temple on a given day). The time, in this case, is in days, so each multiplication of the transition matrix onto a distribution vector will simulate one day passing.

We begin populating the matrix by allowing the diagonals $P_{i,i} = p + f(d)$ with $p = 29/30$ which numerically represents the assumption that in each state, 1/30 of a stake's temple attendees will attend the temple in a given day (using f from (1)). We follow the procedures outlined in section 3.2 to populate the remainder of the matrix.

iii. Results

This will be filled in soon, once we've run the final model on the case study data. We will present the results of the case study, compare them to the results from the P-median problem, and discuss the advantages and disadvantages of using the Markov chain.

V. Conclusion

Discrete mathematics is already in wide use in facility location theory. The development and discussion of methods based on Markov chains is another example of using tools from discrete mathematics to give researchers and decision makers more information. A system modeled as a network via a Markov chain can be analyzed to optimize its throughput, connectedness, and many other mathematical properties, each with possible applications in the area of facility location theory. According to the arguments given here and the computational evidence, there is strong reason to believe that the simplified qualitative interpretation of the metrics used to optimize a network indeed work as presented.

In the conclusion, we will also cover ideas for application and further work.

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