



Universidad de Guanajuato - CIMAT

OWO

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ICPC World Finals 2023

2024-09-09

1 Contest

2 Mathematics

3 Data structures

4 Numerical

5 Number theory

6 Combinatorial

7 Graph

8 Geometry

9 Strings

10 Various

Contest (1)

```
template.cpp
34 lines

#include <bits/stdc++.h>
#define fi first
#define se second
#define mp make_pair
#define pb push_back
#define all(v) v.begin(), v.end()
#define sz(x) ((int) (x).size())
using namespace std;
using ll = long long;
using pii = pair<int,int>;
using pll = pair<ll,ll>;
using vi = vector<int>;
using vp = vector<pii>;
using vl = vector<ll>;
using vvi = vector<vi>;
using vvl = vector<vl>;
using vb = vector<bool>;

template<typename A, typename B> ostream& operator<<(ostream &
os, const pair<A, B> &p){return os << ' (' << p.fi << ", "
<< p.se << ')';}
template<typename C, typename T = typename enable_if<!is_same<C
, string>::value, typename C::value_type>::type>
ostream& operator<<(ostream &os, const C &v){string sep; for(
const T &x : v) os << sep << x, sep = " "; return os;}
#define deb(...) logger(#__VA_ARGS__, __VA_ARGS__)
template<typename ...Args>
void logger(string vars, Args&&... values){
cout << "[Debug]\n\t" << vars << " = ";
string d = "[";
(..., (cout << d << values, d = " ]");
cout << "\n";
}
int main()
{
ios_base::sync_with_stdio(0); cin.tie(0);
return 0;
}
```

Mathematics (2)

In general, given an equation  $Ax = b$ , the solution to a variable  $x_i$  is given by

$$x_i = \frac{\det A'_i}{\det A}$$

where  $A'_i$  is  $A$  with the  $i$ 'th column replaced by  $b$ .

2.1 Recurrences

If  $a_n = c_1a_{n-1} + \dots + c_ka_{n-k}$ , and  $r_1, \dots, r_k$  are distinct roots of  $x^k - c_1x^{k-1} - \dots - c_k$ , there are  $d_1, \dots, d_k$  s.t.

$$a_n = d_1r_1^n + \dots + d_kr_k^n.$$

Non-distinct roots  $r$  become polynomial factors, e.g.

$$a_n = (d_1n + d_2)r^n.$$

2.2 Trigonometry

$$\sin(v+w) = \sin v \cos w + \cos v \sin w$$

$$\cos(v+w) = \cos v \cos w - \sin v \sin w$$

$$\tan(v+w) = \frac{\tan v + \tan w}{1 - \tan v \tan w}$$

$$\sin v + \sin w = 2 \sin \frac{v+w}{2} \cos \frac{v-w}{2}$$

$$\cos v + \cos w = 2 \cos \frac{v+w}{2} \cos \frac{v-w}{2}$$

$$(V+W) \tan(v-w)/2 = (V-W) \tan(v+w)/2$$

where  $V, W$  are lengths of sides opposite angles  $v, w$ .

$$a \cos x + b \sin x = r \cos(x - \phi)$$

$$a \sin x + b \cos x = r \sin(x + \phi)$$

where  $r = \sqrt{a^2 + b^2}, \phi = \text{atan2}(b, a)$ .

2.3 Geometry

2.3.1 Triangles

Side lengths:  $a, b, c$

Semiperimeter:  $p = \frac{a+b+c}{2}$

Area:  $A = \sqrt{p(p-a)(p-b)(p-c)}$

Circumradius:  $R = \frac{abc}{4A}$

Inradius:  $r = \frac{A}{p}$

Length of median (divides triangle into two equal-area triangles):

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

Length of bisector (divides angles in two):

$$s_a = \sqrt{bc \left[ 1 - \left( \frac{a}{b+c} \right)^2 \right]}$$

Law of sines:  $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R}$

Law of cosines:  $a^2 = b^2 + c^2 - 2bc \cos \alpha$

Law of tangents:  $\frac{a+b}{a-b} = \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}$

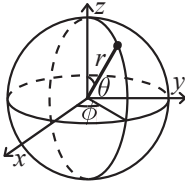
2.3.2 Quadrilaterals

With side lengths  $a, b, c, d$ , diagonals  $e, f$ , diagonals angle  $\theta$ , area  $A$  and magic flux  $F = b^2 + d^2 - a^2 - c^2$ :

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is  $180^\circ$ ,  $ef = ac + bd$ , and  $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$ .

2.3.3 Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z/\sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \text{atan2}(y, x) \end{aligned}$$

2.4 Derivatives/Integrals

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan x = 1 + \tan^2 x \quad \frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\int \tan ax = -\frac{\ln |\cos ax|}{a} \quad \int x \sin ax = \frac{\sin ax - ax \cos ax}{a^2}$$

$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2} \text{erf}(x) \quad \int x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1)$$

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

2.5 Sums

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$$

$$1^4 + 2^4 + 3^4 + \dots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

## 2.6 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, \, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \, (-\infty < x < \infty)$$

## 2.7 Markov chains

$\pi$  is a stationary distribution if  $\pi = \pi \mathbf{P}$ . If the Markov chain is *irreducible* (it is possible to get to any state from any state), then  $\pi_i = \frac{1}{\mathbb{E}(T_i)}$  where  $\mathbb{E}(T_i)$  is the expected time between two visits in state  $i$ .  $\pi_j/\pi_i$  is the expected number of visits in state  $j$  between two visits in state  $i$ .

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors,  $\pi_i$  is proportional to node  $i$ ’s degree.

# Data structures (3)

### OrderStatisticTree.h

**Description:** A set (not multiset!) with support for finding the n’tth element, and finding the index of an element. To get a map, change null\_type.  
**Usage:** Same functions as set + find\_by\_order(k): Returns an iterator to the k-th element (0 indexed)  
order\_of\_key(x): Returns how many elements go strictly before x  
t.join(t1): Assuming T < T2 or T > T2, merge t2 into t  
Declare as Tree<int>, Tree<pii>, ...  
**Time:**  $\mathcal{O}(\log N)$

```
#include <bits/extc++.h>
using namespace __gnu_pbds;
```

```
template<class T>
using Tree = tree<T, null_type, less<T>, rb_tree_tag,
tree_order_statistics_node_update>;
```

### HashMap.h

**Description:** Hash map with mostly the same API as unordered\_map, but ~3x faster. Uses 1.5x memory. Initial capacity must be a power of 2 (if provided).

```
#include <bits/extc++.h>
// To use most bits rather than just the lowest ones:
struct chash { // large odd number for C
    const uint64_t C = 11(4e18 * acos(0)) | 71;
    ll operator()(ll x) const { return __builtin_bswap64(x*C); }
};
__gnu_pbds::gp_hash_table<ll,int,chash> h({},{},{},{},{1<16});
```

### SegmentTree.h

**Description:** Zero-indexed max-tree. Bounds are inclusive to the left and exclusive to the right. Can be changed by modifying T, f and unit.  
**Time:**  $\mathcal{O}(\log N)$

```
struct Tree {
    typedef int T;
    static constexpr T unit = INT_MIN;
    T f(T a, T b) { return max(a, b); } // (any associative fn)
    vector<T> s; int n;
    Tree(int n = 0, T def = unit) : s(2*n, def), n(n) {}
    void update(int pos, T val) {
        for (s[pos += n] = val; pos /= 2;)
            s[pos] = f(s[pos * 2], s[pos * 2 + 1]);
    }
    T query(int b, int e) { // query [b, e)
        T ra = unit, rb = unit;
        for (b += n, e += n; b < e; b /= 2, e /= 2) {
            if (b % 2) ra = f(ra, s[b++]);
            if (e % 2) rb = f(s[--e], rb);
        }
        return f(ra, rb);
    }
};
```

### UnionFindRollback.h

**Description:** Disjoint-set data structure with undo. If undo is not needed, skip st, time() and rollback().  
**Usage:** int t = uf.time(); ...; uf.rollback(t);  
**Time:**  $\mathcal{O}(\log(N))$

```
struct RollbackUF {
    vi e; vector<pii> st;
    RollbackUF(int n) : e(n, -1) {}
    int size(int x) { return -e[find(x)]; }
    int find(int x) { return e[x] < 0 ? x : find(e[x]); }
    int time() { return sz(st); }
    void rollback(int t) {
        for (int i = time(); i --> t;)
            e[st[i].first] = st[i].second;
        st.resize(t);
    }
    bool join(int a, int b) {
        a = find(a), b = find(b);
        if (a == b) return false;
        if (e[a] > e[b]) swap(a, b);
        st.push_back({a, e[a]});
        st.push_back({b, e[b]});
        e[a] += e[b]; e[b] = a;
        return true;
    }
};
```

### LineContainer.h

**Description:** Container where you can add lines of the form  $kx+m$ , and query maximum values at points  $x$ . Useful for dynamic programming (“convex hull trick”).  
**Time:**  $\mathcal{O}(\log N)$

```
struct Line {
    mutable ll k, m, p;
    bool operator<(const Line& o) const { return k < o.k; }
    bool operator<(ll x) const { return p < x; }
};
```

```
struct LineContainer : multiset<Line, less<>> {
    // (for doubles, use inf = 1/.0, div(a,b) = a/b)
    static const ll inf = LLONG_MAX;
    ll div(ll a, ll b) { // floored division
        return a / b - ((a ^ b) < 0 && a % b); }
    bool isect(iterator x, iterator y) {
```

```
        if (y == end()) return x->p = inf, 0;
        if (x->k == y->k) x->p = x->m > y->m ? inf : -inf;
        else x->p = div(y->m - x->m, x->k - y->k);
        return x->p >= y->p;
    }
    void add(ll k, ll m) {
        auto z = insert({k, m, 0}), y = z++, x = y;
        while (isect(y, z)) z = erase(z);
        if (x != begin() && isect(--x, y)) isect(x, y = erase(y));
        while ((y = x) != begin() && (--x->p >= y->p)
            isect(x, erase(y)));
    }
    ll query(ll x) {
        assert(!empty());
        auto l = *lower_bound(x);
        return l.k * x + l.m;
    }
};
```

### LiChaoTree.h

**Description:** min Li Chao Tree for linear functions.  
**Usage:** Functions can be changed, just make sure they cross at most once.

```
struct func {
    ll m, b;
    func(ll m = 0, ll b = 1e18) : m(m), b(b) {}
    ll eval(int x) { return m * x + b; }
} st[4 * N];
void add(int u, int l, int r, func f) {
    int m = (l + r) / 2;
    bool lef = f.eval(l) < st[u].eval(l);
    bool mid = f.eval(m) < st[u].eval(m);
    if(mid) swap(f, st[u]);
    if(l == r) return;
    if(lef != mid) add(2*u+1, l, m, f);
    else add(2*u+2, m+1, r, f);
}
ll get(int u, int l, int r, int x) {
    if(l == r) return st[u].eval(x);
    int m = (l + r) / 2;
    if(x <= m) return min(st[u].eval(x), get(2*u+1, l, m, x));
    return min(st[u].eval(x), get(2*u+2, m+1, r, x));
}
```

### Treap.h

**Description:** A short self-balancing tree. It acts as a sequential container with log-time splits/joins, and is easy to augment with additional data.  
**Time:**  $\mathcal{O}(\log N)$

```
struct Node {
    Node *l = 0, *r = 0;
    int val, y, c = 1;
    Node(int val) : val(val), y(rand()) {}
    void recalc();
};
```

```
int cnt(Node* n) { return n ? n->c : 0; }
void Node::recalc() { c = cnt(l) + cnt(r) + 1; }
```

```
template<class F> void each(Node* n, F f) {
    if (!n) { each(n->l, f); f(n->val); each(n->r, f); }
}
```

```
pair<Node*, Node*> split(Node* n, int k) {
    if (!n) return {};
    if (cnt(n->l) >= k) { // "n->val >= k" for lower_bound(k)
        auto pa = split(n->l, k);
        n->l = pa.second;
```

```
n->recalc();
return {pa.first, n};
} else {
    auto pa = split(n->r, k - cnt(n->l) - 1); // and just "k"
    n->r = pa.first;
    n->recalc();
    return {n, pa.second};
}
}

Node* merge(Node* l, Node* r) {
    if (!l) return r;
    if (!r) return l;
    if (l->y > r->y) {
        l->r = merge(l->r, r);
        l->recalc();
        return l;
    } else {
        r->l = merge(l, r->l);
        r->recalc();
        return r;
    }
}

Node* ins(Node* t, Node* n, int pos) {
    auto pa = split(t, pos);
    return merge(merge(pa.first, n), pa.second);
}
```

```
// Example application: move the range [l, r) to index k
void move(Node*& t, int l, int r, int k) {
    Node *a, *b, *c;
    tie(a,b) = split(t, l); tie(b,c) = split(b, r - 1);
    if (k <= l) t = merge(ins(a, b, k), c);
    else t = merge(a, ins(c, b, k - r));
}
```

### FenwickTree.h

**Description:** Computes partial sums  $a[0] + a[1] + \dots + a[pos - 1]$ , and updates single elements  $a[i]$ , taking the difference between the old and new value.

**Time:** Both operations are  $\mathcal{O}(\log N)$ .

```
struct FT {
    vector<ll> s;
    FT(int n) : s(n) {}
    void update(int pos, ll dif) { // a[pos] += dif
        for (; pos < sz(s); pos |= pos + 1) s[pos] += dif;
    }
    ll query(int pos) { // sum of values in [0, pos)
        ll res = 0;
        for (; pos > 0; pos &= pos - 1) res += s[pos-1];
        return res;
    }
    int lower_bound(ll sum) { // min pos st sum of [0, pos] >= sum
        // Returns n if no sum is >= sum, or -1 if empty sum is.
        if (sum <= 0) return -1;
        int pos = 0;
        for (int pw = 1 << 25; pw; pw >= 1) {
            if (pos + pw <= sz(s) && s[pos + pw-1] < sum)
                pos += pw, sum -= s[pos-1];
        }
        return pos;
    }
};
```

### FenwickTree2d.h

**Description:** Computes sums  $a[i,j]$  for all  $i < I, j < J$ , and increases single elements  $a[i,j]$ . Requires that the elements to be updated are known in advance (call `fakeUpdate()` before `init()`).

**Time:**  $\mathcal{O}(\log^2 N)$ . (Use persistent segment trees for  $\mathcal{O}(\log N)$ .)

```
"FenwickTree.h"
22 lines

struct FT2 {
    vector<vi> ys; vector<FT> ft;
    FT2(int limx) : ys(limx) {}
    void fakeUpdate(int x, int y) {
        for (; x < sz(ys); x |= x + 1) ys[x].push_back(y);
    }
    void init() {
        for (vi& v : ys) sort(all(v)), ft.emplace_back(sz(v));
    }
    int ind(int x, int y) {
        return (int)(lower_bound(all(ys[x]), y) - ys[x].begin());
    }
    void update(int x, int y, ll dif) {
        for (; x < sz(ys); x |= x + 1)
            ft[x].update(ind(x, y), dif);
    }
    ll query(int x, int y) {
        ll sum = 0;
        for (; x; x &= x - 1)
            sum += ft[x-1].query(ind(x-1, y));
        return sum;
    }
};
```

### RMQ.h

**Description:** Range Minimum Queries on an array. Returns  $\min(V[a], V[a + 1], \dots, V[b - 1])$  in constant time.

**Usage:** `RMQ rmq(values);`  
`rmq.query(inclusive, exclusive);`

**Time:**  $\mathcal{O}(|V| \log |V| + Q)$

```
template<class T>
struct RMQ {
    vector<vector<T>> jmp;
    RMQ(const vector<T>& V) : jmp(1, V) {
        for (int pw = 1, k = 1; pw * 2 <= sz(V); pw *= 2, ++k) {
            jmp.emplace_back(sz(V) - pw * 2 + 1);
            rep(j, 0, sz(jmp[k]))
                jmp[k][j] = min(jmp[k - 1][j], jmp[k - 1][j + pw]);
        }
    }
    T query(int a, int b) {
        assert(a < b); // or return inf if a == b
        int dep = 31 - __builtin_clz(b - a);
        return min(jmp[dep][a], jmp[dep][b - (1 << dep)]);
    }
};
```

### MoQueries.h

**Description:** Answer interval or tree path queries by finding an approximate TSP through the queries, and moving from one query to the next by adding/removing points at the ends. If values are on tree edges, change step to add/remove the edge  $(a, c)$  and remove the initial add call (but keep in).

**Time:**  $\mathcal{O}(N\sqrt{Q})$

```
void add(int ind, int end) { ... } // add a[ind] (end = 0 or 1)
void del(int ind, int end) { ... } // remove a[ind]
int calc() { ... } // compute current answer

vi mo(vector<pii> Q) {
    int L = 0, R = 0, blk = 350; // ~N/sqrt(Q)
    vi s(sz(Q)), res = s;
    #define K(x) pii(x.first/blk, x.second ^ -(x.first/blk & 1))
    iota(all(s), 0);
```

```
sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
for (int qi : s) {
    pii q = Q[qi];
    while (L > q.first) add(--L, 0);
    while (R < q.second) add(R++, 1);
    while (L < q.first) del(L++, 0);
    while (R > q.second) del(--R, 1);
    res[qi] = calc();
}
return res;
}
```

```
vi moTree(vector<array<int, 2>> Q, vector<vi>& ed, int root=0){
    int N = sz(ed), pos[2] = {}, blk = 350; // ~N/sqrt(Q)
    vi s(sz(Q)), res = s, I(N), L(N), R(N), in(N), par(N);
    add(0, 0), in[0] = 1;
    auto dfs = [&](int x, int p, int dep, auto& f) -> void {
        par[x] = p;
        L[x] = N;
        if (dep) I[x] = N++;
        for (int y : ed[x]) if (y != p) f(y, x, !dep, f);
        if (!dep) I[x] = N++;
        R[x] = N;
    };
    dfs(root, -1, 0, dfs);
    #define K(x) pii(I[x[0]] / blk, I[x[1]] ^ -(I[x[0]] / blk & 1))
    iota(all(s), 0);
    sort(all(s), [&](int s, int t){ return K(Q[s]) < K(Q[t]); });
    for (int qi : s) rep(end, 0, 2) {
        int &a = pos[end], b = Q[qi][end], i = 0;
        #define step(c) { if (in[c]) { del(a, end); in[a] = 0; } \
            else { add(c, end); in[c] = 1; } a = c; }
        while (!L[b] <= L[a] && R[a] <= R[b])
            I[i++] = b, b = par[b];
        while (a != b) step(par[a]);
        while (i--) step(I[i]);
        if (end) res[qi] = calc();
    }
    return res;
}
```

### IntervalContainer.h

**Description:** Add and remove intervals from a set of disjoint intervals, maintaining some info. for them. To add  $[l, r]$  first it removes intersections with it, then it adds it as its own interval. If 'maximal' is on and endpoint touches other interval of same info, they will merge.

**Time:**  $\mathcal{O}(\log N)$  amortized

```
template <class T = int, class Info = int>
struct IntervalContainer {
    bool maximal;
    IntervalContainer(bool f = false) : maximal(f) {}
    struct Inter {
        T l, r;
        Info u;
        Inter(T r_ = 0) : r(r_) {}
        Inter(T l_, T r_, Info u_) : l(l_), r(r_), u(u_) {}
        bool operator < (const Inter &o) const { return r < o.r; }
    };
    set <Inter> S;
    //returns deleted intervals
    vector<Inter> del(T l, T r) { if(l > r) return {};
        vector <Inter> updated;
        for(T pos : {l, r}) {
            auto it = S.lower_bound(pos);
            if(it != S.end() && it->l <= pos) {
                auto old = *it;
                S.erase(it);
```

```
        if(old.l != pos) S.insert(Inter(old.l, pos - 1,
            old.u));
        if(old.r != pos) S.insert(Inter(pos + 1, old.r,
            old.u));
    }
}
for(auto it = S.lower_bound(l); it != S.upper_bound(r);
    it++) {
    updated.pb(*it);
}
S.erase(S.lower_bound(l), S.upper_bound(r));
return updated;
}
//adds [l, r] with info u
vector<Inter> upd(T l, T r, Info u) { if(l > r) return {};
    auto updated = del(l, r);
    if(maximal) {
        auto it_l = S.lower_bound(l);
        if(it_l != S.begin()) {
            it_l--;
            if(it_l->u == u && it_l->r == l - 1) {
                l = it_l->l;
                S.erase(it_l);
            }
        }
        auto it_r = S.lower_bound(r);
        if(it_r != S.end() && it_r->u == u && it_r->l == r
            + 1) {
            r = it_r->r;
            S.erase(it_r);
        }
    }
    S.insert(Inter(l, r, u));
    return updated;
}
bool elem(T pos) {
    auto it = S.lower_bound(pos);
    return it != S.end() && it->l <= pos;
}
Inter cover(T pos) { //assert(elem(pos));
    return *S.lower_bound(pos);
}
};
```

## Numerical (4)

### 4.1 Polynomials and recurrences

Polynomial.h17 lines

```
struct Poly {
    vector<double> a;
    double operator()(double x) const {
        double val = 0;
        for (int i = sz(a); i--;) (val += x) += a[i];
        return val;
    }
    void diff() {
        rep(i,1,sz(a)) a[i-1] = i*a[i];
        a.pop_back();
    }
    void divroot(double x0) {
        double b = a.back(), c; a.back() = 0;
        for(int i=sz(a)-1; i--;) c = a[i], a[i] = a[i+1]*x0+b, b=c;
        a.pop_back();
    }
};
```

PolyRoots.h23 lines

```
vector<double> polyRoots(Poly p, double xmin, double xmax) {
    if (sz(p.a) == 2) { return {-p.a[0]/p.a[1]}; }
    vector<double> ret;
    Poly der = p;
    der.diff();
    auto dr = polyRoots(der, xmin, xmax);
    dr.push_back(xmin-1);
    dr.push_back(xmax+1);
    sort(all(dr));
    rep(i,0,sz(dr)-1) {
        double l = dr[i], h = dr[i+1];
        bool sign = p(l) > 0;
        if (sign ^ (p(h) > 0)) {
            rep(it,0,60) { // while (h - l > 1e-8)
                double m = (l + h) / 2, f = p(m);
                if ((f <= 0 ^ sign) l = m;
                    else h = m;
                }
                ret.push_back((l + h) / 2);
            }
        }
    }
    return ret;
}
```

PolyInterpolate.h13 lines

```
typedef vector<double> vd;
vd interpolate(vd x, vd y, int n) {
    vd res(n), temp(n);
    rep(k,0,n-1) rep(i,k+1,n)
        y[i] = (y[i] - y[k]) / (x[i] - x[k]);
    double last = 0; temp[0] = 1;
    rep(k,0,n) rep(i,0,n) {
        res[i] += y[k] * temp[i];
        swap(last, temp[i]);
        temp[i] -= last * x[k];
    }
    return res;
}
```

BerlekampMassey.h20 lines

```
vector<ll> berlekampMassey(vector<ll> s) {
    int n = sz(s), L = 0, m = 0;
    vector<ll> C(n), B(n), T;
    C[0] = B[0] = 1;

    ll b = 1;
    rep(i,0,n) { ++m;
        ll d = s[i] % mod;
        rep(j,1,L+1) d = (d + C[j] * s[i - j]) % mod;
        if (!d) continue;
        T = C; ll coef = d * modpow(b, mod-2) % mod;
```

```
        rep(j,m,n) C[j] = (C[j] - coef * B[j - m]) % mod;
        if (2 * L > i) continue;
        L = i + 1 - L; B = T; b = d; m = 0;
    }

    C.resize(L + 1); C.erase(C.begin());
    for (ll& x : C) x = (mod - x) % mod;
    return C;
}
```

LinearRecurrence.h26 lines

```
typedef vector<ll> Poly;
ll linearRec(Poly S, Poly tr, ll k) {
    int n = sz(tr);

    auto combine = [&](Poly a, Poly b) {
        Poly res(n * 2 + 1);
        rep(i,0,n+1) rep(j,0,n+1)
            res[i + j] = (res[i + j] + a[i] * b[j]) % mod;
        for (int i = 2 * n; i > n; --i) rep(j,0,n)
            res[i - 1 - j] = (res[i - 1 - j] + res[i] * tr[j]) % mod;
        res.resize(n + 1);
        return res;
    };

    Poly pol(n + 1), e(pol);
    pol[0] = e[1] = 1;

    for (++k; k; k /= 2) {
        if (k % 2) pol = combine(pol, e);
        e = combine(e, e);
    }

    ll res = 0;
    rep(i,0,n) res = (res + pol[i + 1] * S[i]) % mod;
    return res;
}
```

### 4.2 Optimization

GoldenSectionSearch.h14 lines

```
double gss(double a, double b, double (*f)(double)) {
    double r = (sqrt(5)-1)/2, eps = 1e-7;
    double x1 = b - r*(b-a), x2 = a + r*(b-a);
    double f1 = f(x1), f2 = f(x2);
    while (b-a > eps)
        if (f1 < f2) { //change to > to find maximum
            b = x2; x2 = x1; f2 = f1;
            x1 = b - r*(b-a); f1 = f(x1);
        } else {
            a = x1; x1 = x2; f1 = f2;
            x2 = a + r*(b-a); f2 = f(x2);
        }
    return a;
}
```

HillClimbing.h

Description: Poor man’s optimization for unimodal functions.

14 lines

```
typedef array<double, 2> P;

template<class F> pair<double, P> hillClimb(P start, F f) {
    pair<double, P> cur(f(start), start);
    for (double jmp = 1e9; jmp > 1e-20; jmp /= 2) {
        rep(j,0,100) rep(dx,-1,2) rep(dy,-1,2) {
            P p = cur.second;
            p[0] += dx*jmp;
            p[1] += dy*jmp;
            cur = min(cur, make_pair(f(p), p));
        }
    }
    return cur;
}
```

Integrate.h

Description: Simple integration of a function over an interval using Simpson’s rule. The error should be proportional to  $h^4$ , although in practice you will want to verify that the result is stable to desired precision when epsilon changes.

7 lines

```
template<class F>
double quad(double a, double b, F f, const int n = 1000) {
    double h = (b - a) / 2 / n, v = f(a) + f(b);
    rep(i,1,n*2)
        v += f(a + i*h) * (i&1 ? 4 : 2);
    return v * h / 3;
}
```

IntegrateAdaptive.h

Description: Fast integration using an adaptive Simpson’s rule.

Usage: double sphereVolume = quad(-1, 1, [](double x) { return quad(-1, 1, [&](double y) { return quad(-1, 1, [&](double z) { return x\*x + y\*y + z\*z < 1; }}});});

15 lines

```
typedef double d;
#define S(a,b) (f(a) + 4*f((a+b) / 2) + f(b)) * (b-a) / 6

template <class F>
d rec(F& f, d a, d b, d eps, d S) {
    d c = (a + b) / 2;
    d S1 = S(a, c), S2 = S(c, b), T = S1 + S2;
    if (abs(T - S) <= 15 * eps || b - a < 1e-10)
        return T + (T - S) / 15;
    return rec(f, a, c, eps / 2, S1) + rec(f, c, b, eps / 2, S2);
}

template<class F>
d quad(d a, d b, F f, d eps = 1e-8) {
    return rec(f, a, b, eps, S(a, b));
}
```

Simplex.h

Description: Solves a general linear maximization problem: maximize  $c^T x$  subject to  $Ax \leq b, x \geq 0$ . Returns -inf if there is no solution, inf if there are arbitrarily good solutions, or the maximum value of  $c^T x$  otherwise. The input vector is set to an optimal  $x$  (or in the unbounded case, an arbitrary solution fulfilling the constraints). Numerical stability is not guaranteed. For better performance, define variables such that  $x = 0$  is viable.

Usage: vvd A = {{1,-1}, {-1,1}, {-1,-2}}; vd b = {1,1,-4}, c = {-1,-1}, x; T val = LPSolver(A, b, c).solve(x);

Time:  $\mathcal{O}(NM * \#pivots)$ , where a pivot may be e.g. an edge relaxation.  $\mathcal{O}(2^n)$  in the general case.

68 lines

```
typedef double T; // long double, Rational, double + mod<P>...
typedef vector<T> vd;
```

```
typedef vector<vd> vvd;

const T eps = 1e-8, inf = 1/.0;
#define MP make_pair
#define ltj(X) if(s == -1 || MP(X[j],N[j]) < MP(X[s],N[s])) s=j

struct LPSolver {
    int m, n;
    vi N, B;
    vvd D;

    LPSolver(const vvd& A, const vd& b, const vd& c) :
        m(sz(b)), n(sz(c)), N(n+1), B(m), D(m+2, vd(n+2)) {
            rep(i,0,m) rep(j,0,n) D[i][j] = A[i][j];
            rep(i,0,m) { B[i] = n+i; D[i][n] = -1; D[i][n+1] = b[i]; }
            rep(j,0,n) { N[j] = j; D[m][j] = -c[j]; }
            N[n] = -1; D[m+1][n] = 1;
        }

    void pivot(int r, int s) {
        T *a = D[r].data(), inv = 1 / a[s];
        rep(i,0,m+2) if (i != r && abs(D[i][s]) > eps) {
            T *b = D[i].data(), inv2 = b[s] * inv;
            rep(j,0,n+2) b[j] -= a[j] * inv2;
            b[s] = a[s] * inv2;
        }
        rep(j,0,n+2) if (j != s) D[r][j] *= inv;
        rep(i,0,m+2) if (i != r) D[i][s] *= -inv;
        D[r][s] = inv;
        swap(B[r], N[s]);
    }

    bool simplex(int phase) {
        int x = m + phase - 1;
        for (;;) {
            int s = -1;
            rep(j,0,n+1) if (N[j] != -phase) ltj(D[x]);
            if (D[x][s] >= -eps) return true;
            int r = -1;
            rep(i,0,m) {
                if (D[i][s] <= eps) continue;
                if (r == -1 || MP(D[i][n+1] / D[i][s], B[i])
                    < MP(D[r][n+1] / D[r][s], B[r])) r = i;
            }
            if (r == -1) return false;
            pivot(r, s);
        }
    }

    T solve(vd &x) {
        int r = 0;
        rep(i,1,m) if (D[i][n+1] < D[r][n+1]) r = i;
        if (D[r][n+1] < -eps) {
            pivot(r, n);
            if (!simplex(2) || D[m+1][n+1] < -eps) return -inf;
            rep(i,0,m) if (B[i] == -1) {
                int s = 0;
                rep(j,1,n+1) ltj(D[i]);
                pivot(i, s);
            }
        }
        bool ok = simplex(1); x = vd(n);
        rep(i,0,m) if (B[i] < n) x[B[i]] = D[i][n+1];
        return ok ? D[m][n+1] : inf;
    }
};
```

4.3 Matrices

Determinant.h

Description: Calculates determinant of a matrix. Destroys the matrix.

Time:  $\mathcal{O}(N^3)$

15 lines

```
double det(vector<vector<double>>& a) {
    int n = sz(a); double res = 1;
    rep(i,0,n) {
        int b = i;
        rep(j,i+1,n) if (fabs(a[j][i]) > fabs(a[b][i])) b = j;
        if (i != b) swap(a[i], a[b]), res *= -1;
        res *= a[i][i];
        if (res == 0) return 0;
        rep(j,i+1,n) {
            double v = a[j][i] / a[i][i];
            if (v != 0) rep(k,i+1,n) a[j][k] -= v * a[i][k];
        }
    }
    return res;
}
```

IntDeterminant.h

Description: Calculates determinant using modular arithmetics. Modulos can also be removed to get a pure-integer version.

Time:  $\mathcal{O}(N^3)$

18 lines

```
const ll mod = 12345;
ll det(vector<vector<ll>>& a) {
    int n = sz(a); ll ans = 1;
    rep(i,0,n) {
        rep(j,i+1,n) {
            while (a[j][i] != 0) { // gcd step
                ll t = a[i][i] / a[j][i];
                if (t) rep(k,i,n)
                    a[i][k] = (a[i][k] - a[j][k] * t) % mod;
                swap(a[i], a[j]);
                ans *= -1;
            }
        }
        ans = ans * a[i][i] % mod;
        if (!ans) return 0;
    }
    return (ans + mod) % mod;
}
```

SolveLinear.h

Description: Solves  $A * x = b$ . If there are multiple solutions, an arbitrary one is returned. Returns rank, or -1 if no solutions. Data in  $A$  and  $b$  is lost.

Time:  $\mathcal{O}(n^2 m)$

38 lines

```
typedef vector<double> vd;
const double eps = 1e-12;

int solveLinear(vector<vd>& A, vd& b, vd& x) {
    int n = sz(A), m = sz(x), rank = 0, br, bc;
    if (n) assert(sz(A[0]) == m);
    vi col(m); iota(all(col), 0);

    rep(i,0,n) {
        double v, bv = 0;
        rep(r,i,n) rep(c,i,m)
            if ((v = fabs(A[r][c])) > bv)
                br = r, bc = c, bv = v;
        if (bv <= eps) {
            rep(j,i,n) if (fabs(b[j]) > eps) return -1;
            break;
        }
        swap(A[i], A[br]);
        swap(b[i], b[br]);
```

```
swap(col[i], col[bc]);
rep(j,0,n) swap(A[j][i], A[j][bc]);
bv = 1/A[i][i];
rep(j,i+1,n) {
    double fac = A[j][i] * bv;
    b[j] -= fac * b[i];
    rep(k,i+1,m) A[j][k] -= fac*A[i][k];
}
rank++;
}

x.assign(m, 0);
for (int i = rank; i--;) {
    b[i] /= A[i][i];
    x[col[i]] = b[i];
    rep(j,0,i) b[j] -= A[j][i] * b[i];
}
return rank; // (multiple solutions if rank < m)
```

SolveLinear2.h  
**Description:** To get all uniquely determined values of  $x$  back from SolveLinear, make the following changes:

"SolveLinear.h"3 lines

```
rep(j,0,n) if (j != i) // instead of rep(j,i+1,n)
// ... then at the end:
x.assign(m, undefined);
rep(i,0,rank) {
    rep(j,rank,m) if (fabs(A[i][j]) > eps) goto fail;
    x[col[i]] = b[i] / A[i][i];
fail:; }
```

SolveLinearBinary.h  
**Description:** Solves  $Ax = b$  over  $\mathbb{F}_2$ . If there are multiple solutions, one is returned arbitrarily. Returns rank, or -1 if no solutions. Destroys  $A$  and  $b$ .  
**Time:**  $\mathcal{O}(n^2m)$

typedef bitset<1000> bs;34 lines

```
int solveLinear(vector<bs>& A, vi& b, bs& x, int m) {
    int n = sz(A), rank = 0, br;
    assert(m <= sz(x));
    vi col(m); iota(all(col), 0);
    rep(i,0,n) {
        for (br=i; br<n; ++br) if (A[br].any()) break;
        if (br == n) {
            rep(j,i,n) if(b[j]) return -1;
            break;
        }
        int bc = (int)A[br]._Find_next(i-1);
        swap(A[i], A[br]);
        swap(b[i], b[br]);
        swap(col[i], col[bc]);
        rep(j,0,n) if (A[j][i] != A[j][bc]) {
            A[j].flip(i); A[j].flip(bc);
        }
        rep(j,i+1,n) if (A[j][i]) {
            b[j] ^= b[i];
            A[j] ^= A[i];
        }
        rank++;
    }

    x = bs();
    for (int i = rank; i--;) {
        if (!b[i]) continue;
        x[col[i]] = 1;
        rep(j,0,i) b[j] ^= A[j][i];
    }
```

```
return rank; // (multiple solutions if rank < m)
}
```

MatrixInverse.h  
**Description:** Invert matrix  $A$ . Returns rank; result is stored in  $A$  unless singular ( $\text{rank} < n$ ). Can easily be extended to prime moduli; for prime powers, repeatedly set  $A^{-1} = A^{-1}(2I - AA^{-1}) \pmod{p^k}$  where  $A^{-1}$  starts as the inverse of  $A \pmod{p}$ , and  $k$  is doubled in each step.  
**Time:**  $\mathcal{O}(n^3)$

35 lines

```
int matInv(vector<vector<double>>& A) {
    int n = sz(A); vi col(n);
    vector<vector<double>> tmp(n, vector<double>(n));
    rep(i,0,n) tmp[i][i] = 1, col[i] = i;

    rep(i,0,n) {
        int r = i, c = i;
        rep(j,i,n) rep(k,i,n)
            if (fabs(A[j][k]) > fabs(A[r][c]))
                r = j, c = k;
        if (fabs(A[r][c]) < 1e-12) return i;
        A[i].swap(A[r]); tmp[i].swap(tmp[r]);
        rep(j,0,n)
            swap(A[j][i], A[j][c]), swap(tmp[j][i], tmp[j][c]);
        swap(col[i], col[c]);
        double v = A[i][i];
        rep(j,i+1,n) {
            double f = A[j][i] / v;
            A[j][i] = 0;
            rep(k,i+1,n) A[j][k] -= f*A[i][k];
            rep(k,0,n) tmp[j][k] -= f*tmp[i][k];
        }
        rep(j,i+1,n) A[i][j] /= v;
        rep(j,0,n) tmp[i][j] /= v;
        A[i][i] = 1;
    }

    for (int i = n-1; i > 0; --i) rep(j,0,i) {
        double v = A[j][i];
        rep(k,0,n) tmp[j][k] -= v*tmp[i][k];
    }

    rep(i,0,n) rep(j,0,n) A[col[i]][col[j]] = tmp[i][j];
    return n;
}
```

Tridiagonal.h  
**Description:**  $x = \text{tridiagonal}(d, p, q, b)$  solves the equation system

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{n-1} \end{pmatrix} = \begin{pmatrix} d_0 & p_0 & 0 & 0 & \cdots & 0 \\ q_0 & d_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_1 & d_2 & p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{n-3} & d_{n-2} & p_{n-2} \\ 0 & 0 & \cdots & 0 & q_{n-2} & d_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

This is useful for solving problems on the type

$$a_i = b_i a_{i-1} + c_i a_{i+1} + d_i, \, 1 \leq i \leq n,$$

where  $a_0, a_{n+1}, b_i, c_i$  and  $d_i$  are known.  $a$  can then be obtained from

$$\{a_i\} = \text{tridiagonal}(\{1, -1, -1, \dots, -1, 1\}, \{0, c_1, c_2, \dots, c_n\}, \{b_1, b_2, \dots, b_n, 0\}, \{a_0, d_1, d_2, \dots, d_n, a_{n+1}\}).$$

Fails if the solution is not unique.  
If  $|d_i| > |p_i| + |q_{i-1}|$  for all  $i$ , or  $|d_i| > |p_{i-1}| + |q_i|$ , or the matrix is positive definite, the algorithm is numerically stable and neither `tr` nor the check for `diag[i] == 0` is needed.  
**Time:**  $\mathcal{O}(N)$

```
typedef double T;
vector<T> tridiagonal(vector<T> diag, const vector<T>& super,
    const vector<T>& sub, vector<T> b) {
    int n = sz(b); vi tr(n);
    rep(i,0,n-1) {
        if (abs(diag[i]) < 1e-9 * abs(super[i])) { // diag[i] == 0
            b[i+1] -= b[i] * diag[i+1] / super[i];
            if (i+2 < n) b[i+2] -= b[i] * sub[i+1] / super[i];
            diag[i+1] = sub[i]; tr[++i] = 1;
        } else {
            diag[i+1] -= super[i]*sub[i]/diag[i];
            b[i+1] -= b[i]*sub[i]/diag[i];
        }
    }
    for (int i = n; i--;) {
        if (tr[i]) {
            swap(b[i], b[i-1]);
            diag[i-1] = diag[i];
            b[i] /= super[i-1];
        } else {
            b[i] /= diag[i];
            if (i) b[i-1] -= b[i]*super[i-1];
        }
    }
    return b;
}
```

## 4.4 Fourier transforms

FastFourierTransform.h  
**Description:** `fft(a)` computes  $\hat{f}(k) = \sum_x a[x] \exp(2\pi i \cdot kx/N)$  for all  $k$ .  $N$  must be a power of 2. Useful for convolution: `conv(a, b) = c`, where  $c[x] = \sum a[i]b[x-i]$ . For convolution of complex numbers or more than two vectors: FFT, multiply pointwise, divide by  $n$ , reverse(`start+1, end`), FFT back. Rounding is safe if  $(\sum a_i^2 + \sum b_i^2) \log_2 N < 9 \cdot 10^{14}$  (in practice  $10^{16}$ ; higher for random inputs). Otherwise, use NTT/FFTMod.  
**Time:**  $\mathcal{O}(N \log N)$  with  $N = |A| + |B|$  ( $\sim 1s$  for  $N = 2^{22}$ )

35 lines

```
typedef complex<double> C;
typedef vector<double> vd;
void fft(vector<C>& a) {
    int n = sz(a), L = 31 - __builtin_clz(n);
    static vector<complex<long double>> R(2, 1);
    static vector<C> rt(2, 1); // (^ 10% faster if double)
    for (static int k = 2; k < n; k *= 2) {
        R.resize(n); rt.resize(n);
        auto x = polar(1.0L, acos(-1.0L) / k);
        rep(i,k,2*k) rt[i] = R[i] = i&1 ? R[i/2] * x : R[i/2];
    }
    vi rev(n);
    rep(i,0,n) rev[i] = (rev[i / 2] | (i & 1) << L) / 2;
    rep(i,0,n) if (i < rev[i]) swap(a[i], a[rev[i]]);
    for (int k = 1; k < n; k *= 2)
        for (int i = 0; i < n; i += 2 * k) rep(j,0,k) {
            C z = rt[j+k] * a[i+j+k]; // (25% faster if hand-rolled)
            a[i + j + k] = a[i + j] - z;
            a[i + j] += z;
        }
}

vd conv(const vd& a, const vd& b) {
    if (a.empty() || b.empty()) return {};
    vd res(sz(a) + sz(b) - 1);
    int L = 32 - __builtin_clz(sz(res)), n = 1 << L;
    vector<C> in(n), out(n);
    copy(all(a), begin(in));
    rep(i,0,sz(b)) in[i].imag(b[i]);
    fft(in);
    for (C& x : in) x *= x;
    rep(i,0,n) out[i] = in[-i & (n - 1)] - conj(in[i]);
}
```

```
fft(out);
rep(i,0,sz(res)) res[i] = imag(out[i]) / (4 * n);
return res;
}
```

### FastFourierTransformMod.h

**Description:** Higher precision FFT, can be used for convolutions modulo arbitrary integers as long as  $N \log_2 N \cdot \text{mod} < 8.6 \cdot 10^{14}$  (in practice  $10^{16}$  or higher). Inputs must be in  $[0, \text{mod})$ .

**Time:**  $\mathcal{O}(N \log N)$ , where  $N = |A| + |B|$  (twice as slow as NTT or FFT)

"FastFourierTransform.h"	22 lines
<pre>typedef vector&lt;ll&gt; vl; template&lt;int M&gt; vl convMod(const vl &amp;a, const vl &amp;b) {     if (a.empty()    b.empty()) return {};     vl res(sz(a) + sz(b) - 1);     int B=32-__builtin_clz(sz(res)), n=1&lt;&lt;B, cut=int(sqrt(M));     vector&lt;C&gt; L(n), R(n), outs(n), outl(n);     rep(i,0,sz(a)) L[i] = C((int)a[i] / cut, (int)a[i] % cut);     rep(i,0,sz(b)) R[i] = C((int)b[i] / cut, (int)b[i] % cut);     fft(L), fft(R);     rep(i,0,n) {         int j = -i &amp; (n - 1);         outl[j] = (L[i] + conj(L[j])) * R[i] / (2.0 * n);         outs[j] = (L[i] - conj(L[j])) * R[i] / (2.0 * n) / 1i;     }     fft(outl), fft(outs);     rep(i,0,sz(res)) {         ll av = ll(real(outl[i])+.5), cv = ll(imag(outs[i])+.5);         ll bv = ll(imag(outl[i])+.5) + ll(real(outs[i])+.5);         res[i] = ((av % M * cut + bv) % M * cut + cv) % M;     }     return res; }</pre>	

### NumberTheoreticTransform.h

**Description:** ntt(a) computes  $\hat{f}(k) = \sum_x a[x]g^{xk}$  for all  $k$ , where  $g = \text{root}^{(\text{mod}-1)/N}$ .  $N$  must be a power of 2. Useful for convolution modulo specific nice primes of the form  $2^ab + 1$ , where the convolution result has size at most  $2^a$ . For arbitrary modulo, see FFTMod.  $\text{conv}(a, b) = c$ , where  $c[x] = \sum a[i]b[x - i]$ . For manual convolution: NTT the inputs, multiply pointwise, divide by  $n$ , reverse(start+1, end), NTT back. Inputs must be in  $[0, \text{mod})$ .

**Time:**  $\mathcal{O}(N \log N)$

"../number-theory/ModPow.h"	33 lines
<pre>const ll mod = (119 &lt;&lt; 23) + 1, root = 62; // = 998244353 // For p &lt; 2^30 there is also e.g. 5 &lt;&lt; 25, 7 &lt;&lt; 26, 479 &lt;&lt; 21 // and 483 &lt;&lt; 21 (same root). The last two are &gt; 10^9. typedef vector&lt;ll&gt; vl; void ntt(vl &amp;a) {     int n = sz(a), L = 31 - __builtin_clz(n);     static vl rt(2, 1);     for (static int k = 2, s = 2; k &lt; n; k *= 2, s++) {         rt.resize(n);         ll z[] = {1, modpow(root, mod &gt;&gt; s)};         rep(i,k,2*k) rt[i] = rt[i / 2] * z[i &amp; 1] % mod;     }     vi rev(n);     rep(i,0,n) rev[i] = (rev[i / 2]   (i &amp; 1) &lt;&lt; L) / 2;     rep(i,0,n) if (i &lt; rev[i]) swap(a[i], a[rev[i]]);     for (int k = 1; k &lt; n; k *= 2)         for (int i = 0; i &lt; n; i += 2 * k) rep(j,0,k) {             ll z = rt[j + k] * a[i + j + k] % mod, &amp;ai = a[i + j];             a[i + j + k] = ai - z + (z &gt; ai ? mod : 0);             ai += (ai + z &gt;= mod ? z - mod : z);         } } vl conv(const vl &amp;a, const vl &amp;b) {     if (a.empty()    b.empty()) return {};</pre>	

```
int s = sz(a) + sz(b) - 1, B = 32 - __builtin_clz(s), n = 1
    << B;
int inv = modpow(n, mod - 2);
vl L(a), R(b), out(n);
L.resize(n), R.resize(n);
ntt(L), ntt(R);
rep(i,0,n) out[-i & (n - 1)] = (ll)L[i] * R[i] % mod * inv %
    mod;
ntt(out);
return {out.begin(), out.begin() + s};
}
```

### FastSubsetTransform.h

**Description:** Transform to a basis with fast convolutions of the form

$c[z] = \sum_{z=x \oplus y} a[x] \cdot b[y]$ , where  $\oplus$  is one of AND, OR, XOR. The size of  $a$  must be a power of two.

**Time:**  $\mathcal{O}(N \log N)$

	16 lines
<pre>void FST(vi&amp; a, bool inv) {     for (int n = sz(a), step = 1; step &lt; n; step *= 2) {         for (int i = 0; i &lt; n; i += 2 * step) rep(j,i,i+step) {             int &amp;u = a[j], &amp;v = a[j + step]; tie(u, v) =                 inv ? pii(v - u, u) : pii(v, u + v); // AND                 inv ? pii(v, u - v) : pii(u + v, u); // OR                 pii(u + v, u - v); // XOR         }         if (inv) for (int&amp; x : a) x /= sz(a); // XOR only     }     vi conv(vi a, vi b) {         FST(a, 0); FST(b, 0);         rep(i,0,sz(a)) a[i] *= b[i];         FST(a, 1); return a;     } }</pre>	

## Number theory (5)

### 5.1 Modular arithmetic

#### ModInverse.h

**Description:** Pre-computation of modular inverses. Assumes  $\text{LIM} \leq \text{mod}$  and that  $\text{mod}$  is a prime.

	3 lines
<pre>const ll mod = 1000000007, LIM = 200000; ll* inv = new ll[LIM] - 1; inv[1] = 1; rep(i,2,LIM) inv[i] = mod - (mod / i) * inv[mod % i] % mod;</pre>	

#### ModPow.h

	8 lines
<pre>const ll mod = 1000000007; // faster if const</pre>	

```
ll modpow(ll b, ll e) {
    ll ans = 1;
    for (; e; b = b * b % mod, e /= 2)
        if (e & 1) ans = ans * b % mod;
    return ans;
}
```

#### ModLog.h

**Description:** Returns the smallest  $x > 0$  s.t.  $a^x = b \pmod m$ , or  $-1$  if no such  $x$  exists.  $\text{modLog}(a, \text{l}, m)$  can be used to calculate the order of  $a$ .

**Time:**  $\mathcal{O}(\sqrt{m})$

	11 lines
<pre>ll modLog(ll a, ll b, ll m) {     ll n = (ll) sqrt(m) + 1, e = 1, f = 1, j = 1;     unordered_map&lt;ll, ll&gt; A;     while (j &lt;= n &amp;&amp; (e = f = e * a % m) != b % m)         A[e * b % m] = j++;</pre>	

```
if (e == b % m) return j;
if (__gcd(m, e) == __gcd(m, b))
    rep(i,2,n+2) if (A.count(e = e * f % m))
        return n * i - A[e];
return -1;
}
```

### ModSum.h

**Description:** Sums of mod'ed arithmetic progressions.

$\text{modsum}(\text{to}, c, k, m) = \sum_{i=0}^{\text{to}-1} (ki + c) \% m$ .  $\text{divsum}$  is similar but for floored division.

**Time:**  $\log(m)$ , with a large constant.

	16 lines
<pre>typedef unsigned long long ull; ull sumsq(ull to) { return to / 2 * ((to-1)   1); }</pre>	

```
ull divsum(ull to, ull c, ull k, ull m) {
    ull res = k / m * sumsq(to) + c / m * to;
    k %= m; c %= m;
    if (!k) return res;
    ull to2 = (to * k + c) / m;
    return res + (to - 1) * to2 - divsum(to2, m-1 - c, m, k);
}
```

```
ll modsum(ull to, ll c, ll k, ll m) {
    c = ((c % m) + m) % m;
    k = ((k % m) + m) % m;
    return to * c + k * sumsq(to) - m * divsum(to, c, k, m);
}
```

### ModMulLL.h

**Description:** Calculate  $a \cdot b \pmod c$  (or  $a^b \pmod c$ ) for  $0 \leq a, b \leq c \leq 7.2 \cdot 10^{18}$ .

**Time:**  $\mathcal{O}(1)$  for  $\text{modmul}$ ,  $\mathcal{O}(\log b)$  for  $\text{modpow}$

	11 lines
<pre>typedef unsigned long long ull; ull modmul(ull a, ull b, ull M) {     ll ret = a * b - M * ull(1.L / M * a * b);     return ret + M * (ret &lt; 0) - M * (ret &gt;= (ll)M); } ull modpow(ull b, ull e, ull mod) {     ull ans = 1;     for (; e; b = modmul(b, b, mod), e /= 2)         if (e &amp; 1) ans = modmul(ans, b, mod);     return ans; }</pre>	

### ModSqrt.h

**Description:** Tonelli-Shanks algorithm for modular square roots. Finds  $x$  s.t.  $x^2 = a \pmod p$  ( $-x$  gives the other solution).

**Time:**  $\mathcal{O}(\log^2 p)$  worst case,  $\mathcal{O}(\log p)$  for most  $p$

"ModPow.h"	24 lines
<pre>ll sqrt(ll a, ll p) {     a %= p; if (a &lt; 0) a += p;     if (a == 0) return 0;     assert(modpow(a, (p-1)/2, p) == 1); // else no solution     if (p % 4 == 3) return modpow(a, (p+1)/4, p);     // a^(n+3)/8 or 2^(n+3)/8 * 2^(n-1)/4 works if p % 8 == 5     ll s = p - 1, n = 2;     int r = 0, m;     while (s % 2 == 0)         ++r, s /= 2;     while (modpow(n, (p - 1) / 2, p) != p - 1) ++n;     ll x = modpow(a, (s + 1) / 2, p);     ll b = modpow(a, s, p), g = modpow(n, s, p);     for (; r = m) {         ll t = b;         for (m = 0; m &lt; r &amp;&amp; t != 1; ++m)             t = t * t % p;         if (m == 0) return x;</pre>	



```
ll gs = modpow(g, 1LL << (r - m - 1), p);
g = gs * gs % p;
x = x * gs % p;
b = b * g % p;
}
}
```

## 5.2 Primality

### FastEratosthenes.h

**Description:** Prime sieve for generating all primes smaller than LIM.  
**Time:** LIM=1e9  $\approx$  1.5s

<pre>const int LIM = 1e6; bitset&lt;LIM&gt; isPrime; vi eratosthenes() {     const int S = (int)round(sqrt(LIM)), R = LIM / 2;     vi pr = {2}, sieve(S+1); pr.reserve((int)(LIM/log(LIM)*1.1));     vector&lt;pii&gt; cp;     for (int i = 3; i &lt;= S; i += 2) if (!sieve[i]) {         cp.push_back({i, i * i / 2});         for (int j = i * i; j &lt;= S; j += 2 * i) sieve[j] = 1;     }     for (int L = 1; L &lt;= R; L += S) {         array&lt;bool, S&gt; block{};         for (auto &amp;[p, idx] : cp)             for (int i=idx; i &lt; S+L; idx = (i+=p)) block[i-L] = 1;         rep(i,0,min(S, R - L))             if (!block[i]) pr.push_back((L + i) * 2 + 1);     }     for (int i : pr) isPrime[i] = 1;     return pr; }</pre>	20 lines
---	----------

### MillerRabin.h

**Description:** Deterministic Miller-Rabin primality test. Guaranteed to work for numbers up to  $7 \cdot 10^{18}$ ; for larger numbers, use Python and extend A randomly.  
**Time:** 7 times the complexity of  $a^b \bmod c$ .

<pre>"ModMulLL.h" bool isPrime(u1ll n) {     if (n &lt; 2    n % 6 % 4 != 1) return (n   1) == 3;     u1l A[] = {2, 325, 9375, 28178, 450775, 9780504, 1795265022},         s = __builtin_ctzll(n-1), d = n &gt;&gt; s;     for (u1l a : A) { // ^ count trailing zeroes         u1l p = modpow(a%n, d, n), i = s;         while (p != 1 &amp;&amp; p != n - 1 &amp;&amp; a % n &amp;&amp; i--)             p = modmul(p, p, n);         if (p != n-1 &amp;&amp; i != s) return 0;     }     return 1; }</pre>	12 lines
--	----------

### Factor.h

**Description:** Pollard-rho randomized factorization algorithm. Returns prime factors of a number, in arbitrary order (e.g. 2299 -> {11, 19, 11}).  
**Time:**  $\mathcal{O}\left(n^{1/4}\right)$ , less for numbers with small factors.

<pre>"ModMulLL.h", "MillerRabin.h" u1l pollard(u1l n) {     auto f = [n](u1l x) { return modmul(x, x, n) + 1; };     u1l x = 0, y = 0, t = 30, prd = 2, i = 1, q;     while (t++ % 40    __gcd(prd, n) == 1) {         if (x == y) x = ++i, y = f(x);         if ((q = modmul(prd, max(x,y) - min(x,y), n))) prd = q;         x = f(x), y = f(f(y));     }     return __gcd(prd, n); }</pre>	18 lines
--	----------

```
vector<u1l> factor(u1l n) {
    if (n == 1) return {};
    if (isPrime(n)) return {n};
    u1l x = pollard(n);
    auto l = factor(x), r = factor(n / x);
    l.insert(l.end(), all(r));
    return l;
}
```

## 5.3 Divisibility

### euclid.h

**Description:** Finds two integers  $x$  and  $y$ , such that  $ax + by = \gcd(a, b)$ . If you just need gcd, use the built in `__gcd` instead. If  $a$  and  $b$  are coprime, then  $x$  is the inverse of  $a \pmod b$ .

<pre>ll euclid(ll a, ll b, ll &amp;x, ll &amp;y) {     if (!b) return x = 1, y = 0, a;     ll d = euclid(b, a % b, y, x);     return y -= a/b * x, d; }</pre>	5 lines
---	---------

### CRT.h

**Description:** Chinese Remainder Theorem.  
`crt(a, m, b, n)` computes  $x$  such that  $x \equiv a \pmod m, x \equiv b \pmod n$ . If  $|a| < m$  and  $|b| < n$ ,  $x$  will obey  $0 \leq x < \text{lcm}(m, n)$ . Assumes  $mn < 2^{62}$ .  
**Time:**  $\log(n)$

<pre>"euclid.h" ll crt(ll a, ll m, ll b, ll n) {     if (n &gt; m) swap(a, b), swap(m, n);     ll x, y, g = euclid(m, n, x, y);     assert((a - b) % g == 0); // else no solution     x = (b - a) % n * x % n / g * m + a;     return x &lt; 0 ? x + m*n/g : x; }</pre>	7 lines
---	---------

### 5.3.1 Bézout’s identity

For  $a \neq 0, b \neq 0$ , then  $d = \gcd(a, b)$  is the smallest positive integer for which there are integer solutions to

$$ax + by = d$$

If  $(x, y)$  is one solution, then all solutions are given by

$$\left(x + \frac{kb}{\gcd(a,b)}, y - \frac{ka}{\gcd(a,b)}\right), \quad k \in \mathbb{Z}$$

### phiFunction.h

**Description:** Euler’s  $\phi$  function is defined as  $\phi(n) := \#$  of positive integers  $\leq n$  that are coprime with  $n$ .  $\phi(1) = 1$ ,  $p$  prime  $\Rightarrow \phi(p^k) = (p - 1)p^{k-1}$ ,  $m, n$  coprime  $\Rightarrow \phi(mn) = \phi(m)\phi(n)$ . If  $n = p_1^{k_1}p_2^{k_2}...p_r^{k_r}$  then  $\phi(n) = (p_1 - 1)p_1^{k_1-1}...(p_r - 1)p_r^{k_r-1}$ .  $\phi(n) = n \cdot \prod_{p|n} (1 - 1/p)$ .  $\sum_{d|n} \phi(d) = n$ ,  $\sum_{1 \leq k \leq n, \gcd(k,n)=1} k = n\phi(n)/2, n > 1$ .  
**Euler’s thm:**  $a, n$  coprime  $\Rightarrow a^{\phi(n)} \equiv 1 \pmod n$ .  
**Fermat’s little thm:**  $p$  prime  $\Rightarrow a^{p-1} \equiv 1 \pmod p \ \forall a$ .

<pre>const int LIM = 5000000; int phi[LIM];  void calculatePhi() {     rep(i,0,LIM) phi[i] = i&amp;1 ? i : i/2;     for (int i = 3; i &lt; LIM; i += 2) if(phi[i] == i)         for (int j = i; j &lt; LIM; j += i) phi[j] -= phi[j] / i; }</pre>	8 lines
---	---------

## 5.4 Fractions

### FracBinarySearch.h

**Description:** Given  $f$  and  $N$ , finds the smallest fraction  $p/q \in [0, 1]$  such that  $f(p/q)$  is true, and  $p, q \leq N$ . You may want to throw an exception from  $f$  if it finds an exact solution, in which case  $N$  can be removed.  
**Usage:** `fracBS([])(Frac f) { return f.p>=3*f.q; }, 10);` // {1,3}  
**Time:**  $\mathcal{O}(\log(N))$

<pre>struct Frac { ll p, q; };  template&lt;class F&gt; Frac fracBS(F f, ll N) {     bool dir = 1, A = 1, B = 1;     Frac lo{0, 1}, hi{1, 1}; // Set hi to 1/0 to search (0, N]     if (f(lo)) return lo;     assert(f(hi));     while (A    B) {         ll adv = 0, step = 1; // move hi if dir, else lo         for (int si = 0; step; (step *= 2) &gt;= si) {             adv += step;             Frac mid{lo.p * adv + hi.p, lo.q * adv + hi.q};             if (abs(mid.p) &gt; N    mid.q &gt; N    dir == !f(mid)) {                 adv -= step; si = 2;             }         }         hi.p += lo.p * adv;         hi.q += lo.q * adv;         dir = !dir;         swap(lo, hi);         A = B; B = !adv;     }     return dir ? hi : lo; }</pre>	25 lines
---	----------

## 5.5 Pythagorean Triples

The Pythagorean triples are uniquely generated by

$$a = k \cdot (m^2 - n^2), \quad b = k \cdot (2mn), \quad c = k \cdot (m^2 + n^2),$$

with  $m > n > 0, k > 0, m \perp n$ , and either  $m$  or  $n$  even.

## 5.6 Primes

$p = 962592769$  is such that  $2^{21} \mid p - 1$ , which may be useful. For hashing use 970592641 (31-bit number), 31443539979727 (45-bit), 3006703054056749 (52-bit). There are 78498 primes less than 1 000 000.

Primitive roots exist modulo any prime power  $p^a$ , except for  $p = 2, a > 2$ , and there are  $\phi(\phi(p^a))$  many. For  $p = 2, a > 2$ , the group  $\mathbb{Z}_{2^a}^\times$  is instead isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_{2^{a-2}}$ .

## 5.7 Estimates

$$\sum_{d|n} d = \mathcal{O}(n \log \log n).$$

The number of divisors of  $n$  is at most around 100 for  $n < 5e4$ , 500 for  $n < 1e7$ , 2000 for  $n < 1e10$ , 200 000 for  $n < 1e19$ .

## 5.8 Mobius Function

$$\mu(n) = \begin{cases} 0 & n \text{ is not square free} \\ 1 & n \text{ has even number of prime factors} \\ -1 & n \text{ has odd number of prime factors} \end{cases}$$

Mobius Inversion:

$$g(n) = \sum_{d|n} f(d) \Leftrightarrow f(n) = \sum_{d|n} \mu(d)g(n/d)$$

Other useful formulas/forms:

$$\sum_{d|n} \mu(d) = [n = 1] \text{ (very useful)}$$

$$g(n) = \sum_{n|d} f(d) \Leftrightarrow f(n) = \sum_{n|d} \mu(d/n)g(d)$$

$$g(n) = \sum_{1 \leq m \leq n} f(\lfloor \frac{n}{m} \rfloor) \Leftrightarrow f(n) = \sum_{1 \leq m \leq n} \mu(m)g(\lfloor \frac{n}{m} \rfloor)$$

muFunction.h

**Description:** Sieve that computes  $\mu$

4 lines

```
mu[1] = 1;
for(int i = 1; i < LIM; i++)
    for(int j = i+i; j < LIM; j += i)
        mu[j] -= mu[i];
```

5.8.1 Mobius example:

Coprime pairs:

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n [\gcd(i, j) = 1] &= \sum_{i=1}^n \sum_{j=1}^n \sum_{d|\gcd(i, j)} \mu(d) \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{d=1}^n [d|i][d|j] \mu(d) \\ &= \sum_{d=1}^n \mu(d) \left( \sum_{j=1}^n [d|j] \right)^2 \\ &= \sum_{d=1}^n \mu(d) \left\lfloor \frac{n}{d} \right\rfloor^2. \end{aligned}$$

Gcd sum:

$$\begin{aligned} \sum_{k=1}^n k \sum_{i=1}^n \sum_{j=1}^n [\gcd(i, j) = k] &= \sum_{k=1}^n k \sum_{a=1}^{\lfloor n/k \rfloor} \sum_{b=1}^{\lfloor n/k \rfloor} [\gcd(a, b) = 1] \\ &= \sum_{k=1}^n k \sum_{d=1}^{\lfloor n/k \rfloor} \mu(d) \left\lfloor \frac{n}{kd} \right\rfloor^2. \end{aligned}$$

doing  $k = ld$  we get

$$\sum_{l=1}^n \left\lfloor \frac{n}{l} \right\rfloor^2 \sum_{k|l} k \mu\left(\frac{k}{l}\right) = \sum_{l=1}^n \left\lfloor \frac{n}{l} \right\rfloor^2 \phi(l).$$

5.9 Interpolation

Interpolation.h

**Description:** Interpolates polynomial P at x assuming sample = P(1)...P(N) and P has deg < n

**Time:**  $\mathcal{O}(N)$ . Change mints to ints and care for modulo when using  $\mathfrak{z}_{11}$

21 lines

```
mint interpolate(vector<mint> &sample, mint x) {
    int n = sz(sample);
    mint fac = fact[n - 1];
    vector<mint> pref(n), suf(n);
    pref[0] = mint(1);
    for(int i = 2; i <= n; i++) {
        pref[i - 1] = pref[i - 2] * (x - (i - 1));
    }
    suf[n - 1] = mint(1);
    for(int i = n - 2; i >= 0; i--)
        suf[i] = suf[i + 1] * (x - (i + 2));
    mint ans = 0;
    for(int i = 1; i <= n; i++) {
        mint term = invfact[i - 1] * invfact[n - i];
        if((n - i) % 2 == 1)
            term = term * mint(-1);
        term *= pref[i - 1] * suf[i - 1] * sample[i - 1];
        ans += term;
    }
    return ans;
}
```

Combinatorial (6)

6.1 Permutations

6.1.1 Factorial

IntPerm.h

**Description:** Permutation -> integer conversion. (Not order preserving.)

**Time:**  $\mathcal{O}(n)$

6 lines

```
int permToInt(vi& v) {
    int use = 0, i = 0, r = 0;
    for(int x:v) r = r * ++i + __builtin_popcount(use & -(1<<x)),
        use |= 1 << x;
    return r;
}
```

6.1.2 Cycles

Let  $g_S(n)$  be the number of  $n$ -permutations whose cycle lengths all belong to the set  $S$ . Then

$$\sum_{n=0}^\infty g_S(n) \frac{x^n}{n!} = \exp\left(\sum_{n \in S} \frac{x^n}{n}\right)$$

6.1.3 Derangements

Permutations of a set such that none of the elements appear in their original position.

$$D(n) = (n-1)(D(n-1) + D(n-2)) = nD(n-1) + (-1)^n = \left\lfloor \frac{n!}{e} \right\rfloor$$

6.1.4 Burnside’s lemma

Given a group  $G$  of symmetries and a set  $X$ , the number of elements of  $X$  up to *symmetry* equals

$$\frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where  $X^g$  are the elements fixed by  $g$  ( $g.x = x$ ).

If  $f(n)$  counts “configurations” (of some sort) of length  $n$ , we can ignore rotational symmetry using  $G = \mathbb{Z}_n$  to get

$$g(n) = \frac{1}{n} \sum_{k=0}^{n-1} f(\gcd(n, k)) = \frac{1}{n} \sum_{k|n} f(k) \phi(n/k).$$

6.2 Partitions and subsets

6.2.1 Partition function

Number of ways of writing  $n$  as a sum of positive integers, disregarding the order of the summands.

$$p(0) = 1, \quad p(n) = \sum_{k \in \mathbb{Z} \setminus \{0\}} (-1)^{k+1} p(n - k(3k - 1)/2)$$

$$p(n) \sim 0.145/n \cdot \exp(2.56\sqrt{n})$$

$n$	0	1	2	3	4	5	6	7	8	9	20	50	100
$p(n)$	1	1	2	3	5	7	11	15	22	30	627	$\sim 2\text{e}5$	$\sim 2\text{e}8$

6.2.2 Lucas’ Theorem

Let  $n, m$  be non-negative integers and  $p$  a prime. Write  $n = n_k p^k + \dots + n_1 p + n_0$  and  $m = m_k p^k + \dots + m_1 p + m_0$ . Then  $\binom{n}{m} \equiv \prod_{i=0}^k \binom{n_i}{m_i} \pmod{p}$ .

6.2.3 Binomials

multinomial.h

**Description:** Computes  $\binom{k_1 + \dots + k_n}{k_1, k_2, \dots, k_n} = \frac{(\sum k_i)!}{k_1! k_2! \dots k_n!}$ .

6 lines

```
ll multinomial(vi& v) {
    ll c = 1, m = v.empty() ? 1 : v[0];
    rep(i, 1, sz(v)) rep(j, 0, v[i])
        c = c * ++m / (j+1);
    return c;
}
```

6.3 General purpose numbers

6.3.1 Bernoulli numbers

EGF of Bernoulli numbers is  $B(t) = \frac{t}{e^t - 1}$  (FFT-able).  
 $B[0, \dots] = [1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots]$

Sums of powers:

$$\sum_{i=1}^n n^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k \cdot (n+1)^{m+1-k}$$

Euler-Maclaurin formula for infinite sums:

$$\begin{aligned} \sum_{i=m}^\infty f(i) &= \int_m^\infty f(x) dx - \sum_{k=1}^\infty \frac{B_k}{k!} f^{(k-1)}(m) \\ &\approx \int_m^\infty f(x) dx + \frac{f(m)}{2} - \frac{f'(m)}{12} + \frac{f'''(m)}{720} + O(f^{(5)}(m)) \end{aligned}$$

### 6.3.2 Stirling numbers of the first kind

Number of permutations on  $n$  items with  $k$  cycles.

$$c(n,k)=c(n-1,k-1)+(n-1)c(n-1,k),\; c(0,0)=1$$
$$\sum_{k=0}^nc(n,k)x^k=x(x+1)\dots(x+n-1)$$

$$c(8,k)=8,0,5040,13068,13132,6769,1960,322,28,1$$
$$c(n,2)=0,0,1,3,11,50,274,1764,13068,109584,\dots$$

### 6.3.3 Eulerian numbers

Number of permutations  $\pi\in S_n$  in which exactly  $k$  elements are greater than the previous element.  $k\;j$ :s s.t.  $\pi(j)>\pi(j+1)$ ,  $k+1\;j$ :s s.t.  $\pi(j)\geq j$ ,  $k\;j$ :s s.t.  $\pi(j)>j$ .

$$E(n,k)=(n-k)E(n-1,k-1)+(k+1)E(n-1,k)$$
$$E(n,0)=E(n,n-1)=1$$
$$E(n,k)=\sum_{j=0}^k(-1)^j\binom{n+1}{j}(k+1-j)^n$$

### 6.3.4 Stirling numbers of the second kind

Partitions of  $n$  distinct elements into exactly  $k$  groups.

$$S(n,k)=S(n-1,k-1)+kS(n-1,k)$$
$$S(n,1)=S(n,n)=1$$
$$S(n,k)=\frac{1}{k!}\sum_{j=0}^k(-1)^{k-j}\binom{k}{j}j^n$$

### 6.3.5 Bell numbers

Total number of partitions of  $n$  distinct elements.  $B(n)=1,1,2,5,15,52,203,877,4140,21147,\dots$ . For  $p$  prime,

$$B(p^m+n)\equiv mB(n)+B(n+1)\pmod p$$

### 6.3.6 Labeled unrooted trees

# on  $n$  vertices:  $n^{n-2}$   
# on  $k$  existing trees of size  $n_i$ :  $n_1n_2\cdots n_kn^{k-2}$   
# with degrees  $d_i$ :  $(n-2)!/((d_1-1)!\cdots(d_n-1)!)$

### 6.3.7 Catalan numbers

$$C_n=\frac{1}{n+1}\binom{2n}{n}=\binom{2n}{n}-\binom{2n}{n+1}=\frac{(2n)!}{(n+1)!n!}$$
$$C_0=1,\; C_{n+1}=\frac{2(2n+1)}{n+2}C_n,\; C_{n+1}=\sum C_iC_{n-i}$$
$$C_n=1,1,2,5,14,42,132,429,1430,4862,16796,58786,\dots$$

- sub-diagonal monotone paths in an  $n\times n$  grid.
- strings with  $n$  pairs of parenthesis, correctly nested.
- binary trees with with  $n+1$  leaves (0 or 2 children).

## xorbasis PushRelabelMinCostMaxFlow2

- ordered trees with  $n+1$  vertices.
- ways a convex polygon with  $n+2$  sides can be cut into triangles by connecting vertices with straight lines.
- permutations of  $[n]$  with no 3-term increasing subseq.

Catalan convolution ( $k=0$  is regular Catalan):

$$C_n^{(k)}=\frac{k+1}{n+k+1}\binom{2n+k}{n}=\frac{(2n+k-1)(2n+k)}{n(n+k+1)}C_{n-1}^k.$$

### 6.3.8 Stars and bars

- La cantidad de formas de sumar  $N$  con  $K$  sumandos positivos (no necesariamente distintos y ordenados  $-(1,2)\neq(2,1)$ ) es  $\binom{N-1}{K-1}$
- La cantidad de formas de sumar  $N$  con  $K$  sumandos no negativos (no necesariamente distintos y ordenados) es  $\binom{N+K-1}{N}$

### 6.3.9 XOR basis

```
xorbasis.h
Description: Computes de basis for a set of elements in  $[0,2^D)$ . Every element in the basis has a different most significant bit. The element with i-th bit on is saved on basis[i] (if it exists).
Time:  $\mathcal{O}(D)$ 


---


const int D = 60;
ll basis[D];
int dim;

//To check if an element is in the span, do the same
//but, if !basis[i] return false, otherwise true.
void insert_vector(ll k) {
    for(int i = D-1; i >= 0; i--) {
        if(!(k >> i & 1)) continue;
        if(!basis[i]) {
            basis[i] = k;
            dim++;
            return;
        }
        k ^= basis[i];
    }
}

//k-th smallest in linspace
ll get_kth(ll k) {
    ll x = 0;
    int pref = dim;
    for(ll i = D-1; i >= 0; i--) {
        if(!basis[i]) continue;
        ll p2 = 1ll << (pref - 1);
        ll mini_leq = 1;
        ll maxi_leq = p2;
        if(!(x >> i & 1))
            mini_leq += p2, maxi_leq += p2;
        if(mini_leq <= k && maxi_leq >= k) {
            if(!(x >> i & 1)) k -= p2;
            x ^= basis[i];
        }
        else if(x >> i & 1) k -= p2;
        pref--;
    }
    return x;
}
```

## Graph (7)

### 7.1 Network flow

```
PushRelabel.h
Description: Push-relabel using the highest label selection rule and the gap heuristic. Quite fast in practice. To obtain the actual flow, look at positive values only.
Time:  $\mathcal{O}\left(V^2\sqrt{E}\right)$ 


---


struct PushRelabel {
    struct Edge {
        int dest, back;
        ll f, c;
    };
    vector<vector<Edge>> g;
    vector<ll> ec;
    vector<Edge*> cur;
    vector<vi> hs; vi H;
    PushRelabel(int n) : g(n), ec(n), cur(n), hs(2*n), H(n) {}

    void addEdge(int s, int t, ll cap, ll rcap=0) {
        if (s == t) return;
        g[s].push_back({t, sz(g[t]), 0, cap});
        g[t].push_back({s, sz(g[s])-1, 0, rcap});
    }

    void addFlow(Edge& e, ll f) {
        Edge &back = g[e.dest][e.back];
        if (!ec[e.dest] && f) hs[H[e.dest]].push_back(e.dest);
        e.f += f; e.c -= f; ec[e.dest] += f;
        back.f -= f; back.c += f; ec[back.dest] -= f;
    }

    ll calc(int s, int t) {
        int v = sz(g); H[s] = v; ec[t] = 1;
        vi co(2*v); co[0] = v-1;
        rep(i,0,v) cur[i] = g[i].data();
        for (Edge& e : g[s]) addFlow(e, e.c);

        for (int hi = 0;;) {
            while (hs[hi].empty()) if (!hi--) return -ec[s];
            int u = hs[hi].back(); hs[hi].pop_back();
            while (ec[u] > 0) // discharge u
                if (cur[u] == g[u].data() + sz(g[u])) {
                    H[u] = 1e9;
                    for (Edge& e : g[u]) if (e.c && H[u] > H[e.dest]+1)
                        H[u] = H[e.dest]+1, cur[u] = &e;
                    if (++co[H[u]], !--co[hi] && hi < v)
                        rep(i,0,v) if (hi < H[i] && H[i] < v)
                            --co[H[i]], H[i] = v + 1;
                    hi = H[u];
                }
            else if (cur[u]->c && H[u] == H[cur[u]->dest]+1)
                addFlow(*cur[u], min(ec[u], cur[u]->c));
            else ++cur[u];
        }
    }

    bool leftOfMinCut(int a) { return H[a] >= sz(g); }
};

MinCostMaxFlow2.h
Description: MCMF using Dijkstra and Potentials.  $\mathcal{O}(nm)$  in potentials calc. +  $\mathcal{O}(F\;m\;\log\;m)$  where  $F$  = max flow (faster in practice).


---


template <class T = int>
struct MCF {
    int n, s, t;
    static constexpr T T_inf = numeric_limits<T>::max();
    T bound;
    struct Edge {
```

```
int u, v;
T cap, cost;
Edge(int u_, int v_, T c, T w) : u(u_), v(v_), cap(c),
    cost(w) {}

};
vector<Edge> edges;
vector<vi> adj;
vector<T> pot, dist;
bool any_flow;
MCF(int n_, int s_, int t_, T b = T_inf, bool af = false)
: n(n_), s(s_), t(t_), bound(b), adj(n + 1), pot(n + 1),
    dist(n + 1), any_flow(af) {}
void add_edge(int u, int v, T c, T w) {
    edges.pb(Edge(u, v, c, w));
    edges.pb(Edge(v, u, 0, -w));
    adj[u].pb(sz(edges) - 2);
    adj[v].pb(sz(edges) - 1);
}
pair<T, T> mincost_flow() {
    //call get_potentials if needed!
    T flow = 0;
    T cost = 0;
    vector<bool> vis(n + 1, false);
    vi comes_from(n + 1);
    while(bound) {
        fill(all(vis), false);
        fill(all(dist), T_inf);
        priority_queue<pair<T, int>, vector<pair<T, int>>,
            greater<>> pq;
        pq.push({0, s});
        dist[s] = 0;
        while(!pq.empty()) {
            auto [d, u] = pq.top();
            pq.pop();
            if(vis[u]) continue;
            vis[u] = true;
            for(int i : adj[u]) {
                int v = edges[i].v;
                T c = edges[i].cap;
                T d = dist[u] + edges[i].cost + pot[u] -
                    pot[v];
                if(c > 0 && !vis[v] && d < dist[v]) {
                    dist[v] = d;
                    comes_from[v] = i;
                    pq.push({dist[v], v});
                }
            }
        }
        if(!vis[t]) break;
        T path_cost = 0;
        T path_flow = bound;
        for(int v = t; v != s; v = edges[comes_from[v]].u)
            {
                path_flow = min(path_flow, edges[comes_from[v]
                    ]].cap);
                path_cost += edges[comes_from[v]].cost;
            }
        path_cost *= path_flow;
        if(any_flow && path_cost >= 0) break;
        bound -= path_flow;
        for(int v = t; v != s; v = edges[comes_from[v]].u)
            {
                int i = comes_from[v];
                edges[i].cap -= path_flow;
                edges[i ^ 1].cap += path_flow;
            }
        flow += path_flow;
        cost += path_cost;
    }
}
```

```
for(int i = 1; i <= n; i++) {
    dist[i] += pot[i] - pot[s];
}
pot = dist;
}
return {flow, cost};
}
void get_potentials() {
    for(int i = 1; i <= n; i++) {
        for(auto e : edges) {
            if(e.cap == 0) continue;
            if(pot[e.u] + e.cost < pot[e.v]) {
                pot[e.v] = pot[e.u] + e.cost;
                assert(i != n);
            }
        }
    }
}
};
```

Dinic.h

**Description:** Flow algorithm with complexity  $O(VE \log U)$  where  $U = \max|cap|$ .  $O(\min(E^{1/2}, V^{2/3})E)$  if  $U = 1$ ;  $O(\sqrt{VE})$  for bipartite matching.

```
struct Dinic {
    struct Edge {
        int to, rev;
        ll c, oc;
        ll flow() { return max(oc - c, 0LL); } // if you need flows
    };
    vi lvl, ptr, q;
    vector<vector<Edge>> adj;
    Dinic(int n) : lvl(n), ptr(n), q(n), adj(n) {}
    void addEdge(int a, int b, ll c, ll rcap = 0) {
        adj[a].push_back({b, sz(adj[b]), c, c});
        adj[b].push_back({a, sz(adj[a]) - 1, rcap, rcap});
    }
    ll dfs(int v, int t, ll f) {
        if (v == t || !f) return f;
        for (int& i = ptr[v]; i < sz(adj[v]); i++) {
            Edge& e = adj[v][i];
            if (lvl[e.to] == lvl[v] + 1)
                if (ll p = dfs(e.to, t, min(f, e.c))) {
                    e.c -= p, adj[e.to][e.rev].c += p;
                    return p;
                }
        }
        return 0;
    }
    ll calc(int s, int t) {
        ll flow = 0; q[0] = s;
        rep(L, 0, 31) do { // 'int L=30' maybe faster for random data
            lvl = ptr = vi(sz(q));
            int qi = 0, qe = lvl[s] = 1;
            while (qi < qe && !lvl[t]) {
                int v = q[qi++];
                for (Edge e : adj[v])
                    if (!lvl[e.to] && e.c >> (30 - L))
                        q[qi++] = e.to, lvl[e.to] = lvl[v] + 1;
            }
            while (ll p = dfs(s, t, LLONG_MAX)) flow += p;
        } while (lvl[t]);
        return flow;
    }
    bool leftOfMinCut(int a) { return lvl[a] != 0; }
};
```

MinCut.h

**Description:** After running max-flow, the left side of a min-cut from  $s$  to  $t$  is given by all vertices reachable from  $s$ , only traversing edges with positive residual capacity.

GlobalMinCut.h

**Description:** Find a global minimum cut in an undirected graph, as represented by an adjacency matrix.

**Time:**  $O(V^3)$  21 lines

```
pair<int, vi> globalMinCut(vector<vi> mat) {
    pair<int, vi> best = {INT_MAX, {}};
    int n = sz(mat);
    vector<vi> co(n);
    rep(i, 0, n) co[i] = {i};
    rep(ph, 1, n) {
        vi w = mat[0];
        size_t s = 0, t = 0;
        rep(it, 0, n-ph) { //  $O(V^2) \rightarrow O(E \log V)$  with prio. queue
            w[t] = INT_MIN;
            s = t, t = max_element(all(w)) - w.begin();
            rep(i, 0, n) w[i] += mat[t][i];
        }
        best = min(best, {w[t] - mat[t][t], co[t]});
        co[s].insert(co[s].end(), all(co[t]));
        rep(i, 0, n) mat[s][i] += mat[t][i];
        rep(i, 0, n) mat[i][s] = mat[s][i];
        mat[0][t] = INT_MIN;
    }
    return best;
}
```

GomoryHu.h

**Description:** Given a list of edges representing an undirected flow graph, returns edges of the Gomory-Hu tree. The max flow between any pair of vertices is given by minimum edge weight along the Gomory-Hu tree path.

**Time:**  $O(V)$  Flow Computations

"PushRelabel.h" 13 lines

```
typedef array<ll, 3> Edge;
vector<Edge> gomoryHu(int N, vector<Edge> ed) {
    vector<Edge> tree;
    vi par(N);
    rep(i, 1, N) {
        PushRelabel D(N); // Dinic also works
        for (Edge t : ed) D.addEdge(t[0], t[1], t[2], t[2]);
        tree.push_back({i, par[i], D.calc(i, par[i])});
        rep(j, i+1, N)
            if (par[j] == par[i] && D.leftOfMinCut(j)) par[j] = i;
    }
    return tree;
}
```

7.2 Matching

hopcroftKarp.h

**Description:** Fast bipartite matching algorithm. Graph  $g$  should be a list of neighbors of the left partition, and  $btoa$  should be a vector full of -1's of the same size as the right partition. Returns the size of the matching.  $btoa[i]$  will be the match for vertex  $i$  on the right side, or  $-1$  if it's not matched.

**Usage:** vi btoa(m, -1); hopcroftKarp(g, btoa);

**Time:**  $O(\sqrt{VE})$  42 lines

```
bool dfs(int a, int L, vector<vi>& g, vi& btoa, vi& A, vi& B) {
    if (A[a] != L) return 0;
    A[a] = -1;
    for (int b : g[a]) if (B[b] == L + 1) {
        B[b] = 0;
        if (btoa[b] == -1 || dfs(btoa[b], L + 1, g, btoa, A, B))
```

```
        return btoa[b] = a, 1;
    }
    return 0;
}

int hopcroftKarp(vector<vi>& g, vi& btoa) {
    int res = 0;
    vi A(g.size()), B(btoa.size()), cur, next;
    for (;;) {
        fill(all(A), 0);
        fill(all(B), 0);
        cur.clear();
        for (int a : btoa) if(a != -1) A[a] = -1;
        rep(a,0,sz(g)) if(A[a] == 0) cur.push_back(a);
        for (int lay = 1;; lay++) {
            bool islast = 0;
            next.clear();
            for (int a : cur) for (int b : g[a]) {
                if (btoa[b] == -1) {
                    B[b] = lay;
                    islast = 1;
                }
                else if (btoa[b] != a && !B[b]) {
                    B[b] = lay;
                    next.push_back(btoa[b]);
                }
            }
            if (islast) break;
            if (next.empty()) return res;
            for (int a : next) A[a] = lay;
            cur.swap(next);
        }
        rep(a,0,sz(g))
            res += dfs(a, 0, g, btoa, A, B);
    }
}
```

DFSMatching.h

**Description:** Simple bipartite matching algorithm. Graph  $g$  should be a list of neighbors of the left partition, and  $btoa$  should be a vector full of -1's of the same size as the right partition. Returns the size of the matching.  $btoa[i]$  will be the match for vertex  $i$  on the right side, or  $-1$  if it's not matched.

**Usage:** vi btoa(m, -1); dfsMatching(g, btoa);

**Time:**  $\mathcal{O}(VE)$

22 lines

```
bool find(int j, vector<vi>& g, vi& btoa, vi& vis) {
    if (btoa[j] == -1) return 1;
    vis[j] = 1; int di = btoa[j];
    for (int e : g[di])
        if (!vis[e] && find(e, g, btoa, vis)) {
            btoa[e] = di;
            return 1;
        }
    return 0;
}

int dfsMatching(vector<vi>& g, vi& btoa) {
    vi vis;
    rep(i,0,sz(g)) {
        vis.assign(sz(btoa), 0);
        for (int j : g[i])
            if (find(j, g, btoa, vis)) {
                btoa[j] = i;
                break;
            }
    }
    return sz(btoa) - (int)count(all(btoa), -1);
}
```

MinimumVertexCover.h

**Description:** Finds a minimum vertex cover in a bipartite graph. The size is the same as the size of a maximum matching, and the complement is a maximum independent set.

20 lines

```
vi cover(vector<vi>& g, int n, int m) {
    vi match(m, -1);
    int res = dfsMatching(g, match);
    vector<bool> lfound(n, true), seen(m);
    for (int it : match) if (it != -1) lfound[it] = false;
    vi q, cover;
    rep(i,0,n) if (lfound[i]) q.push_back(i);
    while (!q.empty()) {
        int i = q.back(); q.pop_back();
        lfound[i] = 1;
        for (int e : g[i]) if (!seen[e] && match[e] != -1) {
            seen[e] = true;
            q.push_back(match[e]);
        }
    }
    rep(i,0,n) if (!lfound[i]) cover.push_back(i);
    rep(i,0,m) if (seen[i]) cover.push_back(n+i);
    assert(sz(cover) == res);
    return cover;
}
```

Hungarian.h

**Description:** Maximum matching minimum cost. 1-indexed. For non complete bipartite graph add edges (u, v) with cost inf. For Maximum matching maximum cost multiply the weights by -1. Matching (u, v) is stored in M. Make sure  $n = \text{rows} \leq \text{columns} = m$ .

**Time:**  $\mathcal{O}(n^3) - \mathcal{O}(n^2m)$  for non-square matrix-

53 lines

```
const ll inf = 1e18;

struct Hungarian{
public:
    int n,m;
    vvl w;
    ll min_cost;
    vp M;
    Hungarian(vvl &w){
        this->w = w;
        n = w.size()-1;
        m = w[0].size()-1;
        run();
        return;
    }
    void run(){
        vl u(n+1), v(m+1), p(m+1);
        vi path(m+1);
        for (int i=1; i<=n; i++){
            p[0] = i;
            int j0 = 0;
            vl minv(m+1, inf);
            vb used(m+1, false);
            do{
                used[j0] = true;
                int i0 = p[j0], j1;
                ll delta = inf;
                for (int j=1; j<=m; j++)
                    if (!used[j]){
                        ll cur = w[i0][j]-u[i0]-v[j];
                        if (cur < minv[j])
                            minv[j] = cur, path[j] = j0;
                        if (minv[j] < delta)
                            delta = minv[j], j1 = j;
                    }
                for (int j=0; j<=m; ++j)
```

```
                    if (used[j])
                        u[p[j]] += delta, v[j] -= delta;
                    else
                        minv[j] -= delta;

                j0 = j1;
            }while (p[j0] != 0);
            do{
                int j1 = path[j0];
                p[j0] = p[j1];
                j0 = j1;
            }while (j0);
        }
        for(int i=1; i<=m; i++) M.pb({p[i], i});
        min_cost = -v[0];
        return;
    }
};
```

GeneralMatching.h

**Description:** Matching for general graphs. Fails with probability  $N/mod$ .

**Time:**  $\mathcal{O}(N^3)$

40 lines

```
vector<pii> generalMatching(int N, vector<pii>& ed) {
    vector<vector<ll>> mat(N, vector<ll>(N)), A;
    for (pii pa : ed) {
        int a = pa.first, b = pa.second, r = rand() % mod;
        mat[a][b] = r, mat[b][a] = (mod - r) % mod;
    }

    int r = matInv(A = mat), M = 2*N - r, fi, fj;
    assert(r % 2 == 0);

    if (M != N) do {
        mat.resize(M, vector<ll>(M));
        rep(i,0,N) {
            mat[i].resize(M);
            rep(j,N,M) {
                int r = rand() % mod;
                mat[i][j] = r, mat[j][i] = (mod - r) % mod;
            }
        }
    } while (matInv(A = mat) != M);

    vi has(M, 1); vector<pii> ret;
    rep(it,0,M/2) {
        rep(i,0,M) if (has[i])
            rep(j,i+1,M) if (A[i][j] && mat[i][j]) {
                fi = i; fj = j; goto done;
            }
        assert(0); done:
        if (fj < N) ret.emplace_back(fi, fj);
        has[fi] = has[fj] = 0;
        rep(sw,0,2) {
            ll a = modpow(A[fi][fj], mod-2);
            rep(i,0,M) if (has[i] && A[i][fj]) {
                ll b = A[i][fj] * a % mod;
                rep(j,0,M) A[i][j] = (A[i][j] - A[fi][j] * b) % mod;
            }
            swap(fi,fj);
        }
    }
    return ret;
}
```

### 7.3 DFS algorithms

SCC.h

**Description:** Finds strongly connected components in a directed graph. If vertices  $u, v$  belong to the same component, we can reach  $u$  from  $v$  and vice versa.

**Usage:** scc(graph, [&](vi& v) { ... }) visits all components in reverse topological order. comp[i] holds the component index of a node (a component only has edges to components with lower index). ncomps will contain the number of components.  
**Time:**  $\mathcal{O}(E + V)$

24 lines

```
vi val, comp, z, cont;
int Time, ncomps;
template<class G, class F> int dfs(int j, G& g, F& f) {
    int low = val[j] = ++Time, x; z.push_back(j);
    for (auto e : g[j]) if (comp[e] < 0)
        low = min(low, val[e] ?: dfs(e,g,f));

    if (low == val[j]) {
        do {
            x = z.back(); z.pop_back();
            comp[x] = ncomps;
            cont.push_back(x);
        } while (x != j);
        f(cont); cont.clear();
        ncomps++;
    }
    return val[j] = low;
}
template<class G, class F> void scc(G& g, F f) {
    int n = sz(g);
    val.assign(n, 0); comp.assign(n, -1);
    Time = ncomps = 0;
    rep(i,0,n) if (comp[i] < 0) dfs(i, g, f);
}
```

BiconnectedComponents.h

**Description:** Finds all biconnected components in an undirected graph, and runs a callback for the edges in each. In a biconnected component there are at least two distinct paths between any two nodes. Note that a node can be in several components. An edge which is not in a component is a bridge, i.e., not part of any cycle.  
**Usage:** int eid = 0; ed.resize(N);  
for each edge (a,b) {  
ed[a].emplace\_back(b, eid);  
ed[b].emplace\_back(a, eid++); }  
bicomps([&](const vi& edgelist) {...});  
**Time:**  $\mathcal{O}(E + V)$

33 lines

```
vi num, st;
vector<vector<pii>> ed;
int Time;
template<class F>
int dfs(int at, int par, F& f) {
    int me = num[at] = ++Time, e, y, top = me;
    for (auto pa : ed[at]) if (pa.second != par) {
        tie(y, e) = pa;
        if (num[y]) {
            top = min(top, num[y]);
            if (num[y] < me)
                st.push_back(e);
        } else {
            int si = sz(st);
            int up = dfs(y, e, f);
            top = min(top, up);
            if (up == me) {
                st.push_back(e);
                f(vi(st.begin() + si, st.end()));
                st.resize(si);
            }
            else if (up < me) st.push_back(e);
            else { /* e is a bridge */ }
        }
    }
    return top;
}
```

```
}

template<class F>
void bicomps(F f) {
    num.assign(sz(ed), 0);
    rep(i,0,sz(ed)) if (!num[i]) dfs(i, -1, f);
}
```

2sat.h

**Description:** Calculates a valid assignment to boolean variables a, b, c,... to a 2-SAT problem, so that an expression of the type  $(a \vee b) \wedge (\neg a \vee c) \wedge (d \vee \neg b) \wedge \dots$  becomes true, or reports that it is unsatisfiable. Negated variables are represented by bit-inversions ( $\sim x$ ).  
**Usage:** TwoSat ts(number of boolean variables);  
ts.either(0, ~3); // Var 0 is true or var 3 is false  
ts.setValue(2); // Var 2 is true  
ts.atMostOne({0,~1,2}); //  $\leq 1$  of vars 0, ~1 and 2 are true  
ts.solve(); // Returns true iff it is solvable  
ts.values[0..N-1] holds the assigned values to the vars  
**Time:**  $\mathcal{O}(N + E)$ , where N is the number of boolean variables, and E is the number of clauses.

50 lines

```
struct TwoSat {
    int N;
    vector<vi> gr;
    vi values; // 0 = false, 1 = true
    TwoSat(int n = 0) : N(n), gr(2*n) {}
    int addVar() { // (optional)
        gr.emplace_back();
        gr.emplace_back();
        return N++;
    }
    void either(int f, int j) {
        f = max(2*f, -1-2*f);
        j = max(2*j, -1-2*j);
        gr[f].push_back(j^1);
        gr[j].push_back(f^1);
    }
    void setValue(int x) { either(x, x); }
    void atMostOne(const vi& li) { // (optional)
        if (sz(li) <= 1) return;
        int cur = ~li[0];
        rep(i,2,sz(li)) {
            int next = addVar();
            either(cur, ~li[i]);
            either(cur, next);
            either(~li[i], next);
            cur = ~next;
        }
        either(cur, ~li[1]);
    }
}
vi val, comp, z; int time = 0;
int dfs(int i) {
    int low = val[i] = ++time, x; z.push_back(i);
    for(int e : gr[i]) if (!comp[e])
        low = min(low, val[e] ?: dfs(e));
    if (low == val[i]) do {
        x = z.back(); z.pop_back();
        comp[x] = low;
        if (values[x>>1] == -1)
            values[x>>1] = x&1;
    } while (x != i);
    return val[i] = low;
}
bool solve() {
    values.assign(N, -1);
    val.assign(2*N, 0); comp = val;
    rep(i,0,2*N) if (!comp[i]) dfs(i);
    rep(i,0,N) if (comp[2*i] == comp[2*i+1]) return 0;
}
```

```
    return 1;
}
};
```

BridgesAndArticulation.h

**Description:** Tarjan's algorithm for finding bridges and articulation points in undirected graph. Can be used for building the bridge tree.  
**Time:** construction  $\mathcal{O}(N \log M)$  since briges and points are kept in sets, but easy to make  $\mathcal{O}(N + M)$  keeping some bool vector.

40 lines

```
struct Tarjan {
    int t, n;
    vi mt, dt, in_stack; //mt = min time, dt = discovery time
    vector<vi> adj;
    set<pii> bridges;
    set<int> artic_points;
    Tarjan(int n_) : n(n_), t(1), adj(n+1), mt(n+1), dt(n+1),
        in_stack(n+1) {}
    void add_edge(int u, int v) { adj[u].pb(v); adj[v].pb(u); }
    void tarjan_dfs(int u, int par) {
        if(dt[u]) return;
        int child_cnt = 0;
        dt[u] = mt[u] = t++;
        in_stack[u] = 1;
        for(int v : adj[u]) {
            if(v == par) continue;
            if(!dt[v]) {
                tarjan_dfs(v, u);
                child_cnt++;
                mt[u] = min(mt[u], mt[v]);
                if(mt[v] >= dt[u] && par != -1) {
                    artic_points.insert(u);
                }
                if(mt[v] > dt[u]) {
                    bridges.insert({min(v, u), max(u, v)});
                }
                else {
                    //can connect (u, v) a bridge tree
                }
            }
            else if(in_stack[v]) {
                mt[u] = min(mt[u], dt[v]);
            }
        }
        in_stack[u] = 0;
        if(par == -1 && child_cnt > 1) artic_points.insert(u);
    }
    void find_points_and_bridges() {
        for(int i = 1; i <= n; i++) tarjan_dfs(i, -1);
    }
};
```

EulerWalk.h

**Description:** Eulerian undirected/directed path/cycle algorithm. Input should be a vector of (dest, global edge index), where for undirected graphs, forward/backward edges have the same index. Returns a list of nodes in the Eulerian path/cycle with src at both start and end, or empty list if no cycle/path exists. To get edge indices back, add .second to s and ret.  
**Time:**  $\mathcal{O}(V + E)$

15 lines

```
vi eulerWalk(vector<vector<pii>>& gr, int nedges, int src=0) {
    int n = sz(gr);
    vi D(n), its(n), eu(nedges), ret, s = {src};
    D[src]++; // to allow Euler paths, not just cycles
    while (!s.empty()) {
        int x = s.back(), y, e, &it = its[x], end = sz(gr[x]);
        if (it == end){ ret.push_back(x); s.pop_back(); continue; }
        tie(y, e) = gr[x][it++];
        if (!eu[e]) {
```

```
D[x]--; D[y]++;
eu[e] = 1; s.push_back(y);
}
}
for (int x : D) if (x < 0 || sz(ret) != nedges+1) return {};
return {ret.rbegin(), ret.rend()};
}
```

## 7.4 Coloring

### EdgeColoring.h

**Description:** Given a simple, undirected graph with max degree  $D$ , computes a  $(D + 1)$ -coloring of the edges such that no neighboring edges share a color. ( $D$ -coloring is NP-hard, but can be done for bipartite graphs by repeated matchings of max-degree nodes.)

**Time:**  $\mathcal{O}(NM)$

31 lines

```
vi edgeColoring(int N, vector<pii> eds) {
    vi cc(N + 1), ret(sz(eds)), fan(N), free(N), loc;
    for (pii e : eds) ++cc[e.first], ++cc[e.second];
    int u, v, ncols = *max_element(all(cc)) + 1;
    vector<vi> adj(N, vi(ncols, -1));
    for (pii e : eds) {
        tie(u, v) = e;
        fan[0] = v;
        loc.assign(ncols, 0);
        int at = u, end = u, d, c = free[u], ind = 0, i = 0;
        while (d = free[v], !loc[d] && (v = adj[u][d]) != -1)
            loc[d] = ++ind, cc[ind] = d, fan[ind] = v;
        cc[loc[d]] = c;
        for (int cd = d; at != -1; cd ^= c ^ d, at = adj[at][cd])
            swap(adj[at][cd], adj[end = at][cd ^ c ^ d]);
        while (adj[fan[i]][d] != -1) {
            int left = fan[i], right = fan[++i], e = cc[i];
            adj[u][e] = left;
            adj[left][e] = u;
            adj[right][e] = -1;
            free[right] = e;
        }
        adj[u][d] = fan[i];
        adj[fan[i]][d] = u;
        for (int y : {fan[0], u, end})
            for (int& z = free[y] = 0; adj[y][z] != -1; z++);
    }
    rep(i, 0, sz(eds))
        for (tie(u, v) = eds[i]; adj[u][ret[i]] != v;) ++ret[i];
    return ret;
}
```

## 7.5 Heuristics

### MaximalCliques.h

**Description:** Runs a callback for all maximal cliques in a graph (given as a symmetric bitset matrix; self-edges not allowed). Callback is given a bitset representing the maximal clique.

**Time:**  $\mathcal{O}\left(3^{n/3}\right)$ , much faster for sparse graphs

12 lines

```
typedef bitset<128> B;
template<class F>
void cliques(vector<B>& eds, F f, B P = ~B(), B X={}, B R={}) {
    if (!P.any()) { if (!X.any()) f(R); return; }
    auto q = (P | X)._Find_first();
    auto cands = P & ~eds[q];
    rep(i, 0, sz(eds)) if (cands[i]) {
        R[i] = 1;
        cliques(eds, f, P & eds[i], X & eds[i], R);
        R[i] = P[i] = 0; X[i] = 1;
    }
}
```

### MaximumClique.h

**Description:** Quickly finds a maximum clique of a graph (given as symmetric bitset matrix; self-edges not allowed). Can be used to find a maximum independent set by finding a clique of the complement graph.

**Time:** Runs in about 1s for n=155 and worst case random graphs (p=.90). Runs faster for sparse graphs.

49 lines

```
typedef vector<bitset<200>> vb;
struct Maxclique {
    double limit=0.025, pk=0;
    struct Vertex { int i, d=0; };
    typedef vector<Vertex> vv;
    vb e;
    vv V;
    vector<vi> C;
    vi qmax, q, S, old;
    void init(vv& r) {
        for (auto& v : r) v.d = 0;
        for (auto& v : r) for (auto j : r) v.d += e[v.i][j.i];
        sort(all(r), [](auto a, auto b) { return a.d > b.d; });
        int mxD = r[0].d;
        rep(i, 0, sz(r)) r[i].d = min(i, mxD) + 1;
    }
    void expand(vv& R, int lev = 1) {
        S[lev] += S[lev - 1] - old[lev];
        old[lev] = S[lev - 1];
        while (sz(R)) {
            if (sz(q) + R.back().d <= sz(qmax)) return;
            q.push_back(R.back().i);
            vv T;
            for(auto v:R) if (e[R.back().i][v.i]) T.push_back({v.i});
            if (sz(T)) {
                if (S[lev]++ / ++pk < limit) init(T);
                int j = 0, mxk = 1, mnk = max(sz(qmax) - sz(q) + 1, 1);
                C[1].clear(), C[2].clear();
                for (auto v : T) {
                    int k = 1;
                    auto f = [&](int i) { return e[v.i][i]; };
                    while (any_of(all(C[k]), f)) k++;
                    if (k > mxk) mxk = k, C[mxk + 1].clear();
                    if (k < mnk) T[j++] .i = v.i;
                    C[k].push_back(v.i);
                }
                if (j > 0) T[j - 1].d = 0;
                rep(k, mnk, mxk + 1) for (int i : C[k])
                    T[j].i = i, T[j++].d = k;
                expand(T, lev + 1);
            } else if (sz(q) > sz(qmax)) qmax = q;
            q.pop_back(), R.pop_back();
        }
    }
    vi maxClique() { init(V), expand(V); return qmax; }
    Maxclique(vb conn) : e(conn), C(sz(e)+1), S(sz(C)), old(S) {
        rep(i, 0, sz(e)) V.push_back({i});
    }
};
```

### MaximumIndependentSet.h

**Description:** To obtain a maximum independent set of a graph, find a max clique of the complement. If the graph is bipartite, see MinimumVertex-Cover.

## 7.6 Trees

### LCA.h

**Description:** Data structure for computing lowest common ancestors in a tree (with 0 as root). C should be an adjacency list of the tree, either directed or undirected.

**Time:**  $\mathcal{O}(N \log N + Q)$

../data-structures/RMQ.h"21 lines

```
struct LCA {
    int T = 0;
    vi time, path, ret;
    RMQ<int> rmq;

    LCA(vector<vi>& C) : time(sz(C)), rmq((dfs(C, 0, -1), ret)) {}
    void dfs(vector<vi>& C, int v, int par) {
        time[v] = T++;
        for (int y : C[v]) if (y != par) {
            path.push_back(v), ret.push_back(time[v]);
            dfs(C, y, v);
        }
    }

    int lca(int a, int b) {
        if (a == b) return a;
        tie(a, b) = minmax(time[a], time[b]);
        return path[rmq.query(a, b)];
    }
    //dist(a,b){return depth[a] + depth[b] - 2*depth[lca(a,b)];}
};
```

### CompressTree.h

**Description:** Given a rooted tree and a subset S of nodes, compute the minimal subtree that contains all the nodes by adding all (at most  $|S| - 1$ ) pairwise LCA's and compressing edges. Returns a list of (par, orig\_index) representing a tree rooted at 0. The root points to itself.

**Time:**  $\mathcal{O}(|S| \log |S|)$

"LCA.h"21 lines

```
typedef vector<pair<int, int>> vpi;
vpi compressTree(LCA& lca, const vi& subset) {
    static vi rev; rev.resize(sz(lca.time));
    vi li = subset, &T = lca.time;
    auto cmp = [&](int a, int b) { return T[a] < T[b]; };
    sort(all(li), cmp);
    int m = sz(li)-1;
    rep(i, 0, m) {
        int a = li[i], b = li[i+1];
        li.push_back(lca.lca(a, b));
    }
    sort(all(li), cmp);
    li.erase(unique(all(li)), li.end());
    rep(i, 0, sz(li)) rev[li[i]] = i;
    vpi ret = {pi(0, li[0])};
    rep(i, 0, sz(li)-1) {
        int a = li[i], b = li[i+1];
        ret.emplace_back(rev[lca.lca(a, b)], b);
    }
    return ret;
}
```

### HLD.h

**Description:** Decomposes a tree into vertex disjoint heavy paths and light edges such that the path from any leaf to the root contains at most  $\log(n)$  light edges. Code does additive modifications and max queries, but can support commutative segtree modifications/queries on paths and subtrees. Takes as input the full adjacency list. VALS\_EDGES being true means that values are stored in the edges, as opposed to the nodes. All values initialized to the segtree default. Root must be 0.

**Time:**  $\mathcal{O}\left((\log N)^2\right)$

../data-structures/LazySegmentTree.h"46 lines

```
template<bool VALS_EDGES> struct HLD {
    int N, tim = 0;
    vector<vi> adj;
    vi par, siz, depth, rt, pos;
    Node *tree;
```

```

HLD(vector<vi> adj_)
: N(sz(adj_)), adj(adj_), par(N, -1), siz(N, 1), depth(N),
  rt(N), pos(N), tree(new Node(0, N)) { dfsSz(0); dfsHld(0); }
void dfsSz(int v) {
  if (par[v] != -1) adj[v].erase(find(all(adj[v]), par[v]));
  for (int& u : adj[v]) {
    par[u] = v, depth[u] = depth[v] + 1;
    dfsSz(u);
    siz[v] += siz[u];
    if (siz[u] > siz[adj[v][0]]) swap(u, adj[v][0]);
  }
}
void dfsHld(int v) {
  pos[v] = tim++;
  for (int u : adj[v]) {
    rt[u] = (u == adj[v][0] ? rt[v] : u);
    dfsHld(u);
  }
}
template <class B> void process(int u, int v, B op) {
  for (; rt[u] != rt[v]; v = par[rt[v]]) {
    if (depth[rt[u]] > depth[rt[v]]) swap(u, v);
    op(pos[rt[v]], pos[v] + 1);
  }
  if (depth[u] > depth[v]) swap(u, v);
  op(pos[u] + VALS_EDGES, pos[v] + 1);
}
void modifyPath(int u, int v, int val) {
  process(u, v, [&](int l, int r) { tree->add(l, r, val); });
}
int queryPath(int u, int v) { // Modify depending on problem
  int res = -1e9;
  process(u, v, [&](int l, int r) {
    res = max(res, tree->query(l, r));
  });
  return res;
}
int querySubtree(int v) { // modifySubtree is similar
  return tree->query(pos[v] + VALS_EDGES, pos[v] + siz[v]);
}
};

```

## LinkCutTree.h

**Description:** Represents a forest of unrooted trees. You can add and remove edges (as long as the result is still a forest), and check whether two nodes are in the same tree.

**Time:** All operations take amortized  $\mathcal{O}(\log N)$ .

90 lines

```

struct Node { // Splay tree. Root's pp contains tree's parent.
  Node *p = 0, *pp = 0, *c[2];
  bool flip = 0;
  Node() { c[0] = c[1] = 0; fix(); }
  void fix() {
    if (c[0]) c[0]->p = this;
    if (c[1]) c[1]->p = this;
    // (+ update sum of subtree elements etc. if wanted)
  }
  void pushFlip() {
    if (!flip) return;
    flip = 0; swap(c[0], c[1]);
    if (c[0]) c[0]->flip ^= 1;
    if (c[1]) c[1]->flip ^= 1;
  }
  int up() { return p ? p->c[1] == this : -1; }
  void rot(int i, int b) {
    int h = i ^ b;
    Node *x = c[i], *y = b == 2 ? x : x->c[h], *z = b ? y : x;
    if ((y->p = p) p->c[up()] = y;
    c[i] = z->c[i ^ 1];

```

```

    if (b < 2) {
      x->c[h] = y->c[h ^ 1];
      z->c[h ^ 1] = b ? x : this;
    }
    y->c[i ^ 1] = b ? this : x;
    fix(); x->fix(); y->fix();
    if (p) p->fix();
    swap(pp, y->pp);
  }
  void splay() {
    for (pushFlip(); p; ) {
      if (p->p) p->p->pushFlip();
      p->pushFlip(); pushFlip();
      int c1 = up(), c2 = p->up();
      if (c2 == -1) p->rot(c1, 2);
      else p->p->rot(c2, c1 != c2);
    }
  }
  Node* first() {
    pushFlip();
    return c[0] ? c[0]->first() : (splay(), this);
  }
};

struct LinkCut {
  vector<Node> node;
  LinkCut(int N) : node(N) {}

  void link(int u, int v) { // add an edge (u, v)
    assert(!connected(u, v));
    makeRoot(&node[u]);
    node[u].pp = &node[v];
  }
  void cut(int u, int v) { // remove an edge (u, v)
    Node *x = &node[u], *top = &node[v];
    makeRoot(top); x->splay();
    assert(top == (x->pp ? x->c[0]));
    if (x->pp) x->pp = 0;
    else {
      x->c[0] = top->p = 0;
      x->fix();
    }
  }
  bool connected(int u, int v) { // are u, v in the same tree?
    Node* nu = access(&node[u])->first();
    return nu == access(&node[v])->first();
  }
  void makeRoot(Node* u) {
    access(u);
    u->splay();
    if (u->c[0]) {
      u->c[0]->p = 0;
      u->c[0]->flip ^= 1;
      u->c[0]->pp = u;
      u->c[0] = 0;
      u->fix();
    }
  }
  Node* access(Node* u) {
    u->splay();
    while (Node* pp = u->pp) {
      pp->splay(); u->pp = 0;
      if (pp->c[1]) {
        pp->c[1]->p = 0; pp->c[1]->pp = pp; }
      pp->c[1] = u; pp->fix(); u = pp;
    }
    return u;
  }
};

```

## DirectedMST.h

**Description:** Finds a minimum spanning tree/arborescence of a directed graph, given a root node. If no MST exists, returns -1.

**Time:**  $\mathcal{O}(E \log V)$

../../data-structures/UnionFindRollback.h

60 lines

```

struct Edge { int a, b; ll w; };
struct Node {
  Edge key;
  Node *l, *r;
  ll delta;
  void prop() {
    key.w += delta;
    if (l) l->delta += delta;
    if (r) r->delta += delta;
    delta = 0;
  }
  Edge top() { prop(); return key; }
};
Node *merge(Node *a, Node *b) {
  if (!a || !b) return a ? b;
  a->prop(), b->prop();
  if (a->key.w > b->key.w) swap(a, b);
  swap(a->l, (a->r = merge(b, a->r)));
  return a;
}
void pop(Node& a) { a->prop(); a = merge(a->l, a->r); }

pair<ll, vi> dmst(int n, int r, vector<Edge>& g) {
  RollbackUF uf(n);
  vector<Node*> heap(n);
  for (Edge e : g) heap[e.b] = merge(heap[e.b], new Node(e));
  ll res = 0;
  vi seen(n, -1), path(n), par(n);
  seen[r] = r;
  vector<Edge> Q(n), in(n, {-1, -1}), comp;
  deque<tuple<int, int, vector<Edge>>> cys;
  rep(s, 0, n) {
    int u = s, qi = 0, w;
    while (seen[u] < 0) {
      if (!heap[u]) return {-1, {}};
      Edge e = heap[u]->top();
      heap[u]->delta -= e.w, pop(heap[u]);
      Q[qi] = e, path[qi++] = u, seen[u] = s;
      res += e.w, u = uf.find(e.a);
      if (seen[u] == s) {
        Node* cyc = 0;
        int end = qi, time = uf.time();
        do cyc = merge(cyc, heap[w = path[--qi]]);
        while (uf.join(u, w));
        u = uf.find(u), heap[u] = cyc, seen[u] = -1;
        cys.push_front({u, time, {Q[qi], Q[end]}});
      }
    }
    rep(i, 0, qi) in[uf.find(Q[i].b)] = Q[i];
  }

  for (auto& [u, t, comp] : cys) { // restore sol (optional)
    uf.rollback(t);
    Edge inEdge = in[u];
    for (auto& e : comp) in[uf.find(e.b)] = e;
    in[uf.find(inEdge.b)] = inEdge;
  }
  rep(i, 0, n) par[i] = in[i].a;
  return {res, par};
}

```



DominatorTree.h

**Description:** 0-indexed. Computes dominator tree of a directed graph. run(*s*) returns a vector *p*.  $p[i] = -1$  if  $i = s$  or if  $i$  is not reachable from  $s$ , otherwise is equal to the parent.  $s$  is the root.  $u$  is ancestor of  $v$  if  $u$  appears in every path from  $s$  to  $v$ .

**Usage:** add\_edge(*u*, *v*) - run(*s*)

**Time:**  $\mathcal{O}(|E|\log|V|)$  50 lines

```
struct DominatorTree{
    vector< basic_string<int> > g, rg, bucket;
    basic_string<int> arr, par, rev, sdom, dom, dsu, label;
    int n, t;

    explicit DominatorTree(int n) : g(n), rg(n), bucket(n), arr
        (n, -1), par(n, -1), rev(n, -1), sdom(n, -1), dom(n,
        -1), dsu(n, 0), label(n, 0), n(n), t(0) {}

    void add_edge(int u, int v){
        g[u] += v;
    }

    void dfs(int u){
        arr[u] = t;
        rev[t] = u;
        label[t] = sdom[t] = dsu[t] = t;
        t++;
        for(int w : g[u]){
            if(arr[w] == -1){
                dfs(w);
                par[arr[w]] = arr[u];
            }
            rg[arr[w]] += arr[u];
        }
    }

    int find(int u, int x=0){
        if(u == dsu[u]) return x ? -1 : u;
        int v = find(dsu[u], x + 1);
        if(v < 0) return u;
        if(sdom[label[dsu[u]]] < sdom[label[u]])
            label[u] = label[dsu[u]];
        dsu[u] = v;
        return x ? v : label[u];
    }

    vi run(int root){
        dfs(root);
        iota(all(dom), 0);
        for(int i=t-1; i>=0; i--){
            for(int w : rg[i]) sdom[i] = std::min(sdom[i], sdom
                [find(w)]);
            if(i) bucket[sdom[i]] += i;
            for(int w : bucket[i]){
                int v = find(w);
                if (sdom[v] == sdom[w]) dom[w] = sdom[w];
                else dom[w] = v;
            }
            if(i > 1) dsu[i] = par[i];
        }
        for(int i=1; i<t; i++) if(dom[i] != sdom[i]) dom[i] =
            dom[dom[i]];
        vi outside_dom(n, -1);
        for(int i=1; i<t; i++) outside_dom[rev[i]] = rev[dom[i]
            ];
        return outside_dom;
    }
};
```

ReachabilityTree.h

**Description:** Computes the Reachability/KruskalReconstruction tree  $T$  of a graph  $G$  when inserting the edges in the given order. Nodes of  $G$  correspond to leaves of  $T$  and internal nodes of  $T$  corresponds to edges that connects the set of leaves of its subtrees. Nodes of  $G$  should be 1-indexed. edges should be  $\{u, v, w\}$  (endpoints and weight), this vector should be 0-indexed.  $T$  has at most  $2N - 1$  vertices. If  $u$  is an internal node of  $T$  then  $w[u]$  corresponds to weight of the associated edge.

**Time:**  $\mathcal{O}(N \log N)$  but can be improved to  $\mathcal{O}(N)$ ,  $\log N$  comes from DSU.

```
struct DSU{
    int n, comp;
    vi dsu;
    DSU(int n) : n(n), dsu(n + 1), comp(n){iota(all(dsu), 0);}
    int getF(int u){
        if(dsu[u] == u) return u;
        return dsu[u] = getF( dsu[u] );
    }

    bool join(int u, int v){
        int Fu = getF(u), Fv = getF(v);
        if(Fu == Fv) return false;
        comp--;
        dsu[ Fu ] = Fv;
        return true;
    }
};

struct ReachabilityTree{
    int n, idx, root;
    DSU dsu = DSU(0);
    vvi T, p;
    vi parent, depth, sz_subtree, sz_leaves, w;
    vector< array<int, 3> > edges;
    ReachabilityTree(int n, vector< array<int, 3> > edges){
        this->n = n;
        this->idx = n+1;
        this->edges = edges;

        T.resize(n + n);
        parent.resize(n + n);
        dsu = DSU(n + n - 1); // DSU indexa en 1 solito
        depth.resize(n + n);
        sz_subtree.resize(n + n);
        sz_leaves.resize(n + n);
        w.resize(n + n);

        __build();
        __dfs(root, 0);
    }

    void __build(){
        int u, v;
        for(auto e:edges){
            u = dsu.getF(e[0]);
            v = dsu.getF(e[1]);
            if(u != v){
                T[idx].pb(u);
                T[idx].pb(v);
                parent[u] = idx;
                parent[v] = idx;
                // Warning: Join debe asignar como nuevo
                // representante a idx
                dsu.join(u, idx);
                dsu.join(v, idx);
                w[idx] = e[2];
                if(dsu.comp == 1) root = idx;
                idx++;
                continue;
            }
        }
        return;
    }
};
```

```
void __dfs(int u, int d){
    depth[u] = d;
    sz_subtree[u] = 1;
    sz_leaves[u] = !T[u].size();
    for(auto v:T[u]){
        __dfs(v, d+1);
        sz_subtree[u] += sz_subtree[v];
        sz_leaves[u] += sz_leaves[v];
    }
    return;
};
```

7.7 Math

7.7.1 Number of Spanning Trees

Create an  $N \times N$  matrix *mat*, and for each edge  $a \rightarrow b \in G$ , do *mat* [*a*] [*b*]--, *mat* [*b*] [*b*]++ (and *mat* [*b*] [*a*]--, *mat* [*a*] [*a*]++ if  $G$  is undirected). Remove the *i*th row and column and take the determinant; this yields the number of directed spanning trees rooted at  $i$  (if  $G$  is undirected, remove any row/column).

7.7.2 Erdős–Gallai theorem

A simple graph with node degrees  $d_1 \geq \dots \geq d_n$  exists iff  $d_1 + \dots + d_n$  is even and for every  $k = 1 \dots n$ ,

$$\sum_{i=1}^k d_i \leq k(k-1) + \sum_{i=k+1}^n \min(d_i, k).$$

Geometry (8)

8.1 Geometric primitives

Point.h

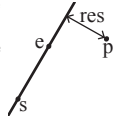
**Description:** Class to handle points in the plane. T can be e.g. double or long long. (Avoid int.) 28 lines

```
template <class T> int sgn(T x) { return (x > 0) - (x < 0); }
template<class T>
struct Point {
    typedef Point P;
    T x, y;
    explicit Point(T x=0, T y=0) : x(x), y(y) {}
    bool operator<(P p) const { return tie(x,y) < tie(p.x,p.y); }
    bool operator==(P p) const { return tie(x,y)==tie(p.x,p.y); }
    P operator+(P p) const { return P(x+p.x, y+p.y); }
    P operator-(P p) const { return P(x-p.x, y-p.y); }
    P operator*(T d) const { return P(x*d, y*d); }
    P operator/(T d) const { return P(x/d, y/d); }
    T dot(P p) const { return x*p.x + y*p.y; }
    T cross(P p) const { return x*p.y - y*p.x; }
    T cross(P a, P b) const { return (a-*this).cross(b-*this); }
    T dist2() const { return x*x + y*y; }
    // angle to x-axis in interval [-pi, pi]
    double angle() const { return atan2(y, x); }
    P unit() const { return *this/dist(); } // makes dist()==1
    P perp() const { return P(-y, x); } // rotates +90 degrees
    P normal() const { return perp().unit(); }
    // returns point rotated 'a' radians ccw around the origin
```

```
P rotate(double a) const {
    return P(x*cos(a)-y*sin(a),x*sin(a)+y*cos(a)); }
friend ostream& operator<<(ostream& os, P p) {
    return os << "(" << p.x << ", " << p.y << ")"; }
};
```

lineDistance.h

**Description:**  
Returns the signed distance between point p and the line containing points a and b. Positive value on left side and negative on right as seen from a towards b. a==b gives nan. P is supposed to be Point<T> or Point3D<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long. Using Point3D will always give a non-negative distance. For Point3D, call .dist on the result of the cross product.

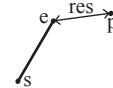


"Point.h" 4 lines

```
template<class P>
double lineDist(const P& a, const P& b, const P& p) {
    return (double) (b-a).cross(p-a)/(b-a).dist();
}
```

SegmentDistance.h

**Description:**  
Returns the shortest distance between point p and the line segment from point s to e.  
**Usage:** Point<double> a, b(2,2), p(1,1);  
bool onSegment = segDist(a,b,p) < 1e-10;

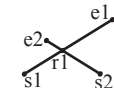


"Point.h" 6 lines

```
typedef Point<double> P;
double segDist(P& s, P& e, P& p) {
    if (s==e) return (p-s).dist();
    auto d = (e-s).dist2(), t = min(d,max(.0, (p-s).dot(e-s)));
    return ((p-s)*d-(e-s)*t).dist()/d;
}
```

SegmentIntersection.h

**Description:**  
If a unique intersection point between the line segments going from s1 to e1 and from s2 to e2 exists then it is returned. If no intersection point exists an empty vector is returned. If infinitely many exist a vector with 2 elements is returned, containing the endpoints of the common line segment. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or long long.  
**Usage:** vector<P> inter = segInter(s1,e1,s2,e2);  
if (sz(inter)==1)  
cout << "segments intersect at " << inter[0] << endl;

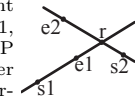


"Point.h", "OnSegment.h" 13 lines

```
template<class P> vector<P> segInter(P a, P b, P c, P d) {
    auto oa = c.cross(d, a), ob = c.cross(d, b),
        oc = a.cross(b, c), od = a.cross(b, d);
    // Checks if intersection is single non-endpoint point.
    if (sgn(oa) * sgn(ob) < 0 && sgn(oc) * sgn(od) < 0)
        return {(a * ob - b * oa) / (ob - oa)};
    set<P> s;
    if (onSegment(c, d, a)) s.insert(a);
    if (onSegment(c, d, b)) s.insert(b);
    if (onSegment(a, b, c)) s.insert(c);
    if (onSegment(a, b, d)) s.insert(d);
    return {all(s)};
}
```

lineIntersection.h

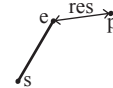
**Description:**  
If a unique intersection point of the lines going through s1,e1 and s2,e2 exists {1, point} is returned. If no intersection point exists {0, (0,0)} is returned and if infinitely many exists {-1, (0,0)} is returned. The wrong position will be returned if P is Point<ll> and the intersection point does not have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow if using int or ll.  
**Usage:** auto res = lineInter(s1,e1,s2,e2);  
if (res.first == 1)  
cout << "intersection point at " << res.second << endl;



"Point.h" 8 lines

sideOf.h

**Description:** Returns where p is as seen from s towards e. 1/0/-1 ⇔ left/on line/right. If the optional argument eps is given 0 is returned if p is within distance eps from the line. P is supposed to be Point<T> where T is e.g. double or long long. It uses products in intermediate steps so watch out for overflow if using int or long long.  
**Usage:** bool left = sideOf(p1,p2,q)==1;



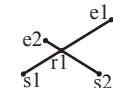
"Point.h" 9 lines

```
template<class P>
int sideOf(P s, P e, P p) { return sgn(s.cross(e, p)); }
```

```
template<class P>
int sideOf(const P& s, const P& e, const P& p, double eps) {
    auto a = (e-s).cross(p-s);
    double l = (e-s).dist()*eps;
    return (a > l) - (a < -l);
}
```

OnSegment.h

**Description:** Returns true iff p lies on the line segment from s to e. Use (segDist(s,e,p)<=epsilon) instead when using Point<double>.

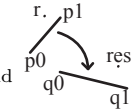


"Point.h" 3 lines

```
template<class P> bool onSegment(P s, P e, P p) {
    return p.cross(s, e) == 0 && (s - p).dot(e - p) <= 0;
}
```

linearTransformation.h

**Description:**  
Apply the linear transformation (translation, rotation and scaling) which takes line p0-p1 to line q0-q1 to point r.

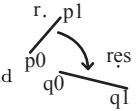


"Point.h" 6 lines

```
typedef Point<double> P;
P linearTransformation(const P& p0, const P& p1,
    const P& q0, const P& q1, const P& r) {
    P dp = p1-p0, dq = q1-q0, num(dp.cross(dq), dp.dot(dq));
    return q0 + P((r-p0).cross(num), (r-p0).dot(num))/dp.dist2();
}
```

LineProjectionReflection.h

**Description:** Projects point p onto line ab. Set refl=true to get reflection of point p across line ab instead. The wrong point will be returned if P is an integer point and the desired point doesn't have integer coordinates. Products of three coordinates are used in intermediate steps so watch out for overflow.

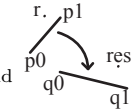


"Point.h" 5 lines

```
template<class P>
P lineProj(P a, P b, P p, bool refl=false) {
    P v = b - a;
    return p - v.perp()*(1+refl)*v.cross(p-a)/v.dist2();
}
```

Angle.h

**Description:** A class for ordering angles (as represented by int points and a number of rotations around the origin). Useful for rotational sweeping. Sometimes also represents points or vectors.  
**Usage:** vector<Angle> v = {w[0], w[0].t360() ...}; // sorted  
int j = 0; rep(i,0,n) { while (v[j] < v[i].t180()) ++j; }  
// sweeps j such that (j-i) represents the number of positively oriented triangles with vertices at 0 and i



35 lines

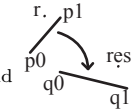
```
struct Angle {
    int x, y;
    int t;
    Angle(int x, int y, int t=0) : x(x), y(y), t(t) {}
    Angle operator-(Angle b) const { return {x-b.x, y-b.y, t}; }
    int half() const {
        assert(x || y);
        return y < 0 || (y == 0 && x < 0);
    }
    Angle t90() const { return {-y, x, t + (half() && x >= 0)}; }
    Angle t180() const { return {-x, -y, t + half()}; }
    Angle t360() const { return {x, y, t + 1}; }
};
bool operator<(Angle a, Angle b) {
    // add a.dist2() and b.dist2() to also compare distances
    return make_tuple(a.t, a.half(), a.y * (ll)b.x <
        make_tuple(b.t, b.half(), a.x * (ll)b.y);
}
```

```
// Given two points, this calculates the smallest angle between
// them, i.e., the angle that covers the defined line segment.
pair<Angle, Angle> segmentAngles(Angle a, Angle b) {
    if (b < a) swap(a, b);
    return (b < a.t180() ?
        make_pair(a, b) : make_pair(b, a.t360()));
}
Angle operator+(Angle a, Angle b) { // point a + vector b
    Angle r(a.x + b.x, a.y + b.y, a.t);
    if (a.t180() < r) r.t--;
    return r.t180() < a ? r.t360() : r;
}
Angle angleDiff(Angle a, Angle b) { // angle b - angle a
    int tu = b.t - a.t; a.t = b.t;
    return {a.x*b.x + a.y*b.y, a.x*b.y - a.y*b.x, tu - (b < a)};
}
```

8.2 Circles

CircleIntersection.h

**Description:** Computes the pair of points at which two circles intersect. Returns false in case of no intersection.



"Point.h" 11 lines

```
typedef Point<double> P;
bool circleInter(P a,P b,double r1,double r2,pair<P, P>* out) {
    if (a == b) { assert(r1 != r2); return false; }
    P vec = b - a;
    double d2 = vec.dist2(), sum = r1+r2, dif = r1-r2,
        p = (d2 + r1*r1 - r2*r2)/(d2*2), h2 = r1*r1 - p*p*d2;
    if (sum*sum < d2 || dif*dif > d2) return false;
    P mid = a + vec*p, per = vec.perp() * sqrt(fmax(0, h2) / d2);
    *out = {mid + per, mid - per};
    return true;
}
```

### CircleTangents.h

**Description:** Finds the external tangents of two circles, or internal if r2 is negated. Can return 0, 1, or 2 tangents – 0 if one circle contains the other (or overlaps it, in the internal case, or if the circles are the same); 1 if the circles are tangent to each other (in which case .first = .second and the tangent line is perpendicular to the line between the centers). .first and .second give the tangency points at circle 1 and 2 respectively. To find the tangents of a circle with a point set r2 to 0.

"Point.h"	13 lines
<pre>template&lt;class P&gt; vector&lt;pair&lt;P, P&gt;&gt; tangents(P c1, double r1, P c2, double r2) {     P d = c2 - c1;     double dr = r1 - r2, d2 = d.dist2(), h2 = d2 - dr * dr;     if (d2 == 0    h2 &lt; 0)    return {};     vector&lt;pair&lt;P, P&gt;&gt; out;     for (double sign : {-1, 1}) {         P v = (d * dr + d.perp() * sqrt(h2) * sign) / d2;         out.push_back({c1 + v * r1, c2 + v * r2});     }     if (h2 == 0) out.pop_back();     return out; }</pre>	

### CircleLine.h

**Description:** Finds the intersection between a circle and a line. Returns a vector of either 0, 1, or 2 intersection points. P is intended to be Point<double>.

"Point.h"	9 lines
<pre>template&lt;class P&gt; vector&lt;P&gt; circleLine(P c, double r, P a, P b) {     P ab = b - a, p = a + ab * (c-a).dot(ab) / ab.dist2();     double s = a.cross(b, c), h2 = r*r - s*s / ab.dist2();     if (h2 &lt; 0) return {};     if (h2 == 0) return {p};     P h = ab.unit() * sqrt(h2);     return {p - h, p + h}; }</pre>	

### CirclePolygonIntersection.h

**Description:** Returns the area of the intersection of a circle with a ccw polygon.

"Time: $\mathcal{O}(n)$ "	19 lines
<pre>..../content/geometry/Point.h" typedef Point&lt;double&gt; P; #define arg(p, q) atan2(p.cross(q), p.dot(q)) double circlePoly(P c, double r, vector&lt;P&gt; ps) {     auto tri = [&amp;](P p, P q) {         auto r2 = r * r / 2;         P d = q - p;         auto a = d.dot(p)/d.dist2(), b = (p.dist2()-r*r)/d.dist2();         auto det = a * a - b;         if (det &lt;= 0) return arg(p, q) * r2;         auto s = max(0., -a-sqrt(det)), t = min(1., -a+sqrt(det));         if (t &lt; 0    1 &lt;= s) return arg(p, q) * r2;         P u = p + d * s, v = p + d * t;         return arg(p,u) * r2 + u.cross(v)/2 + arg(v,q) * r2;     };     auto sum = 0.0;     rep(i,0,sz(ps))         sum += tri(ps[i] - c, ps[(i + 1) % sz(ps)] - c);     return sum; }</pre>	

### circumcircle.h

**Description:** The circumcircle of a triangle is the circle intersecting all three vertices. ccRadius returns the radius of the circle going through points A, B and C and ccCenter returns the center of the same circle.

"Point.h"	9 lines
<pre>typedef Point&lt;double&gt; P; double ccRadius(const P&amp; A, const P&amp; B, const P&amp; C) {     return (B-A).dist()*(C-B).dist()*(A-C).dist()/         abs((B-A).cross(C-A))/2; } P ccCenter(const P&amp; A, const P&amp; B, const P&amp; C) {     P b = C-A, c = B-A;     return A + (b*c.dist2()-c*b.dist2()).perp()/b.cross(c)/2; }</pre>	

### MinimumEnclosingCircle.h

**Description:** Computes the minimum circle that encloses a set of points.  
**Time:** expected  $\mathcal{O}(n)$

"circumcircle.h"	17 lines
<pre>pair&lt;P, double&gt; mec(vector&lt;P&gt; ps) {     shuffle(all(ps), mt19937(time(0)));     P o = ps[0];     double r = 0, EPS = 1 + 1e-8;     rep(i,0,sz(ps)) if ((o - ps[i]).dist() &gt; r * EPS) {         o = ps[i], r = 0;         rep(j,0,i) if ((o - ps[j]).dist() &gt; r * EPS) {             o = (ps[i] + ps[j]) / 2;             r = (o - ps[i]).dist();             rep(k,0,j) if ((o - ps[k]).dist() &gt; r * EPS) {                 o = ccCenter(ps[i], ps[j], ps[k]);                 r = (o - ps[i]).dist();             }         }     }     return {o, r}; }</pre>	

## 8.3 Polygons

### InsidePolygon.h

**Description:** Returns true if p lies within the polygon. If strict is true, it returns false for points on the boundary. The algorithm uses products in intermediate steps so watch out for overflow.

<p><b>Usage:</b> vector&lt;P&gt; v = {P{4,4}, P{1,2}, P{2,1}};</p> <p>bool in = inPolygon(v, P{3, 3}, false);</p> <p><b>Time:</b> <math>\mathcal{O}(n)</math></p>	11 lines
<pre>"Point.h", "OnSegment.h", "SegmentDistance.h" template&lt;class P&gt; bool inPolygon(vector&lt;P&gt; &amp;p, P a, bool strict = true) {     int cnt = 0, n = sz(p);     rep(i,0,n) {         P q = p[(i + 1) % n];         if (onSegment(p[i], q, a)) return !strict;         //or: if (segDist(p[i], q, a) &lt;= eps) return !strict;         cnt ^= ((a.y&lt;p[i].y) - (a.y&lt;q.y)) * a.cross(p[i], q) &gt; 0;     }     return cnt; }</pre>	

### PolygonArea.h

**Description:** Returns twice the signed area of a polygon. Clockwise enumeration gives negative area. Watch out for overflow if using int as T!

"Point.h"	6 lines
<pre>template&lt;class T&gt; T polygonArea2(vector&lt;Point&lt;T&gt;&gt;&amp; v) {</pre>	

T a = v.back().cross(v[0]); rep(i,0,sz(v)-1) a += v[i].cross(v[i+1]); return a; }	
--	--

### PolygonCenter.h

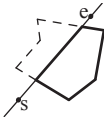
**Description:** Returns the center of mass for a polygon.  
**Time:**  $\mathcal{O}(n)$

"Point.h"	9 lines
<pre>typedef Point&lt;double&gt; P; P polygonCenter(const vector&lt;P&gt;&amp; v) {     P res(0, 0); double A = 0;     for (int i = 0, j = sz(v) - 1; i &lt; sz(v); j = i++) {         res = res + (v[i] + v[j]) * v[j].cross(v[i]);         A += v[j].cross(v[i]);     }     return res / A / 3; }</pre>	

### PolygonCut.h

**Description:** Returns a vector with the vertices of a polygon with everything to the left of the line going from s to e cut away.  
**Usage:** vector<P> p = ...;  
p = polygonCut(p, P(0,0), P(1,0));

"Point.h", "lineIntersection.h"	13 lines
<pre>typedef Point&lt;double&gt; P; vector&lt;P&gt; polygonCut(const vector&lt;P&gt;&amp; poly, P s, P e) {     vector&lt;P&gt; res;     rep(i,0,sz(poly)) {         P cur = poly[i], prev = i ? poly[i-1] : poly.back();         bool side = s.cross(e, cur) &lt; 0;         if (side != (s.cross(e, prev) &lt; 0))             res.push_back(lineInter(s, e, cur, prev).second);         if (side)             res.push_back(cur);     }     return res; }</pre>	



### PolygonUnion.h

**Description:** Calculates the area of the union of  $n$  polygons (not necessarily convex). The points within each polygon must be given in CCW order. (Epsilon checks may optionally be added to sideOf/sgn, but shouldn't be needed.)

<p><b>Time:</b> <math>\mathcal{O}(N^2)</math>, where <math>N</math> is the total number of points</p>	33 lines
<pre>"Point.h", "sideOf.h" typedef Point&lt;double&gt; P; double rat(P a, P b) { return sgn(b.x) ? a.x/b.x : a.y/b.y; } double polyUnion(vector&lt;vector&lt;P&gt;&gt;&amp; poly) {     double ret = 0;     rep(i,0,sz(poly)) rep(v,0,sz(poly[i])) {         P A = poly[i][v], B = poly[i][(v + 1) % sz(poly[i])];         vector&lt;pair&lt;double, int&gt;&gt; segs = {{0, 0}, {1, 0}};         rep(j,0,sz(poly)) if (i != j) {             rep(u,0,sz(poly[j])) {                 P C = poly[j][u], D = poly[j][(u + 1) % sz(poly[j])];                 int sc = sideOf(A, B, C), sd = sideOf(A, B, D);                 if (sc != sd) {                     double sa = C.cross(D, A), sb = C.cross(D, B);                     if (min(sc, sd) &lt; 0)                         segs.emplace_back(sa / (sa - sb), sgn(sc - sd));                 } else if (!sc &amp;&amp; !sd &amp;&amp; j &lt; i &amp;&amp; sgn((B-A).dot(D-C))&gt;0) {                     segs.emplace_back(rat(C - A, B - A), 1);                     segs.emplace_back(rat(D - A, B - A), -1);                 }             }         }     }</pre>	

```
    }
    sort(all(segs));
    for (auto& s : segs) s.first = min(max(s.first, 0.0), 1.0);
    double sum = 0;
    int cnt = segs[0].second;
    rep(j,1,sz(segs)) {
        if (!cnt) sum += segs[j].first - segs[j - 1].first;
        cnt += segs[j].second;
    }
    ret += A.cross(B) * sum;
}
return ret / 2;
}
```

ConvexHull.h

**Description:** Returns a vector of the points of the convex hull in counter-clockwise order. Points on the edge of the hull between two other points are not considered part of the hull.  
**Time:**  $\mathcal{O}(n \log n)$



```
"Point.h" 13 lines
typedef Point<ll> P;
vector<P> convexHull(vector<P> pts) {
    if (sz(pts) <= 1) return pts;
    sort(all(pts));
    vector<P> h(sz(pts)+1);
    int s = 0, t = 0;
    for (int it = 2; it--; s = --t, reverse(all(pts)))
        for (P p : pts) {
            while (t >= s + 2 && h[t-2].cross(h[t-1], p) <= 0) t--;
            h[t++] = p;
        }
    return {h.begin(), h.begin() + t - (t == 2 && h[0] == h[1])};
}
```

Minkowski.h

**Description:** Returns vertices of the Minkowski sum of two convex polygons (also a convex polygon).

```
25 lines
template <class T> //lowest vertex first
void reorder_polygon(vector<Point<T>> &P) {
    int pos = 0;
    for(int i = 1; i < sz(P); i++){
        if(P[i].y < P[pos].y || (P[i].y == P[pos].y && P[i].x < P[pos].x))
            pos = i;
    }
    rotate(P.begin(), P.begin() + pos, P.end());
}

template <class T> //assumes P and Q are ccw
vector<Point<T>> minkowski(vector<Point<T>> P, vector<Point<T>> Q) {
    reorder_polygon(P);
    reorder_polygon(Q);
    P.pb(P[0]); P.pb(P[1]); Q.pb(Q[0]); Q.pb(Q[1]);
    vector<Point<T>> result;
    int i = 0, j = 0;
    while(i < sz(P) - 2 || j < sz(Q) - 2) {
        result.pb(P[i] + Q[j]);
        auto cross = (P[i + 1] - P[i]).cross(Q[j + 1] - Q[j]);
        if(cross >= 0 && i < sz(P) - 2) ++i;
        if(cross <= 0 && j < sz(Q) - 2) ++j;
    }
    return result;
}
```

HullDiameter.h

**Description:** Returns the two points with max distance on a convex hull (ccw, no duplicate/collinear points).

```
Time: O(n)
"Point.h" 12 lines
typedef Point<ll> P;
array<P, 2> hullDiameter(vector<P> S) {
    int n = sz(S), j = n < 2 ? 0 : 1;
    pair<ll, array<P, 2>> res({0, {S[0], S[0]}});
    rep(i,0,j)
        for (; j = (j + 1) % n) {
            res = max(res, {(S[i] - S[j]).dist2(), {S[i], S[j]}});
            if ((S[(j + 1) % n] - S[j]).cross(S[i + 1] - S[i]) >= 0)
                break;
        }
    return res.second;
}
```

PointInsideHull.h

**Description:** Determine whether a point t lies inside a convex hull (CCW order, with no collinear points). Returns true if point lies within the hull. If strict is true, points on the boundary aren't included.  
**Time:**  $\mathcal{O}(\log N)$

```
"Point.h", "sideOf.h", "OnSegment.h" 14 lines
typedef Point<ll> P;

bool inHull(const vector<P>& l, P p, bool strict = true) {
    int a = 1, b = sz(l) - 1, r = !strict;
    if (sz(l) < 3) return r && onSegment(l[0], l.back(), p);
    if (sideOf(l[0], l[a], l[b]) > 0) swap(a, b);
    if (sideOf(l[0], l[a], p) >= r || sideOf(l[0], l[b], p) <= -r)
        return false;
    while (abs(a - b) > 1) {
        int c = (a + b) / 2;
        (sideOf(l[0], l[c], p) > 0 ? b : a) = c;
    }
    return sgn(l[a].cross(l[b], p)) < r;
}
```

LineHullIntersection.h

**Description:** Line-convex polygon intersection. The polygon must be ccw and have no collinear points. lineHull(line, poly) returns a pair describing the intersection of a line with the polygon:  $\bullet(-1, -1)$  if no collision,  $\bullet(i, -1)$  if touching the corner  $i$ ,  $\bullet(i, i)$  if along side  $(i, i + 1)$ ,  $\bullet(i, j)$  if crossing sides  $(i, i + 1)$  and  $(j, j + 1)$ . In the last case, if a corner  $i$  is crossed, this is treated as happening on side  $(i, i + 1)$ . The points are returned in the same order as the line hits the polygon. extrVertex returns the point of a hull with the max projection onto a line.

```
Time: O(log n)
"Point.h" 39 lines
#define cmp(i, j) sgn(dir.perp().cross(poly[(i)%n]-poly[(j)%n]))
#define extr(i) cmp(i + 1, i) >= 0 && cmp(i, i - 1 + n) < 0
template <class P> int extrVertex(vector<P>& poly, P dir) {
    int n = sz(poly), lo = 0, hi = n;
    if (extr(0)) return 0;
    while (lo + 1 < hi) {
        int m = (lo + hi) / 2;
        if (extr(m)) return m;
        int ls = cmp(lo + 1, lo), ms = cmp(m + 1, m);
        (ls < ms || (ls == ms && ls == cmp(lo, m)) ? hi : lo) = m;
    }
    return lo;
}

#define cmpl(i) sgn(a.cross(poly[i], b))
template <class P>
array<int, 2> lineHull(P a, P b, vector<P>& poly) {
    int endA = extrVertex(poly, (a - b).perp());
```

```
int endB = extrVertex(poly, (b - a).perp());
if (cmpl(endA) < 0 || cmpl(endB) > 0)
    return {-1, -1};
array<int, 2> res;
rep(i,0,2) {
    int lo = endB, hi = endA, n = sz(poly);
    while ((lo + 1) % n != hi) {
        int m = ((lo + hi + (lo < hi ? 0 : n)) / 2) % n;
        (cmpl(m) == cmpl(endB) ? lo : hi) = m;
    }
    res[i] = (lo + !cmpl(hi)) % n;
    swap(endA, endB);
}
if (res[0] == res[1]) return {res[0], -1};
if (!cmpl(res[0]) && !cmpl(res[1]))
    switch ((res[0] - res[1] + sz(poly) + 1) % sz(poly)) {
        case 0: return {res[0], res[0]};
        case 2: return {res[1], res[1]};
    }
return res;
}
```

8.4 Misc. Point Set Problems

ClosestPair.h

**Description:** Finds the closest pair of points.  
**Time:**  $\mathcal{O}(n \log n)$

```
"Point.h" 17 lines
typedef Point<ll> P;
pair<P, P> closest(vector<P> v) {
    assert(sz(v) > 1);
    set<P> S;
    sort(all(v), [](P a, P b) { return a.y < b.y; });
    pair<ll, pair<P, P>> ret{LLONG_MAX, {P(), P()}};
    int j = 0;
    for (P p : v) {
        P d{1 + (ll)sqrt(ret.first), 0};
        while (v[j].y <= p.y - d.x) S.erase(v[j++]);
        auto lo = S.lower_bound(p - d), hi = S.upper_bound(p + d);
        for (; lo != hi; ++lo)
            ret = min(ret, {(lo - p).dist2(), {lo, p}});
        S.insert(p);
    }
    return ret.second;
}
```

ManhattanMST.h

**Description:** Given  $N$  points, returns up to  $4 \cdot N$  edges, which are guaranteed to contain a minimum spanning tree for the graph with edge weights  $w(p, q) = -p.x - q.x - + -p.y - q.y -$ . Edges are in the form (distance, src, dst). Use a standard MST algorithm on the result to find the final MST.  
**Time:**  $\mathcal{O}(N \log N)$

```
"Point.h" 23 lines
typedef Point<int> P;
vector<array<int, 3>> manhattanMST(vector<P> ps) {
    vi id(sz(ps));
    iota(all(id), 0);
    vector<array<int, 3>> edges;
    rep(k,0,4) {
        sort(all(id), [&](int i, int j) {
            return (ps[i]-ps[j]).x < (ps[j]-ps[i]).y;});
        map<int, int> sweep;
        for (int i : id) {
            for (auto it = sweep.lower_bound(-ps[i].y);
                 it != sweep.end(); sweep.erase(it++)) {
                int j = it->second;
                P d = ps[i] - ps[j];
                if (d.y > d.x) break;
                edges.push_back({d.y + d.x, i, j});
            }
        }
    }
```

```
    }
    sweep[-ps[i].y] = i;
}
for (P& p : ps) if (k & 1) p.x = -p.x; else swap(p.x, p.y);
}
return edges;
}
```

kdTree.h

Description: KD-tree (2d, can be extended to 3d)

"Point.h"63 lines

```
typedef long long T;
typedef Point<T> P;
const T INF = numeric_limits<T>::max();

bool on_x(const P& a, const P& b) { return a.x < b.x; }
bool on_y(const P& a, const P& b) { return a.y < b.y; }

struct Node {
    P pt; // if this is a leaf, the single point in it
    T x0 = INF, x1 = -INF, y0 = INF, y1 = -INF; // bounds
    Node *first = 0, *second = 0;

    T distance(const P& p) { // min squared distance to a point
        T x = (p.x < x0 ? x0 : p.x > x1 ? x1 : p.x);
        T y = (p.y < y0 ? y0 : p.y > y1 ? y1 : p.y);
        return (P(x,y) - p).dist2();
    }

    Node(vector<P>&& vp) : pt(vp[0]) {
        for (P p : vp) {
            x0 = min(x0, p.x); x1 = max(x1, p.x);
            y0 = min(y0, p.y); y1 = max(y1, p.y);
        }
        if (vp.size() > 1) {
            // split on x if width >= height (not ideal...)
            sort(all(vp), x1 - x0 >= y1 - y0 ? on_x : on_y);
            // divide by taking half the array for each child (not
            // best performance with many duplicates in the middle)
            int half = sz(vp)/2;
            first = new Node({vp.begin(), vp.begin() + half});
            second = new Node({vp.begin() + half, vp.end()});
        }
    }
};

struct KDTree {
    Node* root;
    KDTree(const vector<P>& vp) : root(new Node({all(vp)})) {}

    pair<T, P> search(Node *node, const P& p) {
        if (!node->first) {
            // uncomment if we should not find the point itself:
            // if (p == node->pt) return {INF, P()};
            return make_pair((p - node->pt).dist2(), node->pt);
        }

        Node *f = node->first, *s = node->second;
        T bfirst = f->distance(p), bsec = s->distance(p);
        if (bfirst > bsec) swap(bsec, bfirst), swap(f, s);

        // search closest side first, other side if needed
        auto best = search(f, p);
        if (bsec < best.first)
            best = min(best, search(s, p));
        return best;
    }

    // find nearest point to a point, and its squared distance
```

```
// (requires an arbitrary operator< for Point)
pair<T, P> nearest(const P& p) {
    return search(root, p);
};

HalfplaneIntersection.h
Description: Computes convex polygon which represents intersection of half
planes (each half plane is to the left of the segment) Careful with precision!
40 lines

const double EPS = 1e-9;
const double DINF = 1e100;
template <typename T> struct HalfPlane {
    Point<T> p, pq;
    T angle;
    HalfPlane() {}
    HalfPlane(Point<T> _p, Point<T> _q): p(_p), pq(_q - _p),
        angle(atan2(pq.y, pq.x)) {}
    bool operator<(HalfPlane &b) const { return angle < b.angle
        ; }
    bool out(Point<T> q) { return pq.cross(q - p) < EPS; }
    Point<T> intersect(HalfPlane<T> l) {
        if (abs(pq.cross(l.pq)) < EPS) return Point<T>(DINF,
            DINF);
        return l.p + l.pq * ((p - l.p).cross(pq) / l.pq.cross(
            pq));
    }
}; // Halfplane to the left of line

template <typename T> vector<Point<T>> intersect(vector<
    HalfPlane<T>> b) {
    vector<Point<T>> bx = {Point<T>(DINF, DINF), Point<T>(-DINF
        , DINF), Point<T>(-DINF, -DINF), Point<T>(DINF, -DINF)
        };
    for (int i = 0; i < 4; ++i) b.emplace_back(bx[i], bx[(i +
        1) % 4]);
    sort(b.begin(), b.end());
    int n = b.size(), q = 1, h = 0;
    vector<HalfPlane<T>> c(b.size() + 10);
    for (int i = 0; i < n; ++i) {
        while (q < h && b[i].out(c[h].intersect(c[h - 1]))) h
            --;
        while (q < h && b[i].out(c[q].intersect(c[q + 1]))) q
            ++;
        c[++h] = b[i];
        if (q < h && abs(c[h].pq.cross(c[h - 1].pq)) < EPS) {
            if (c[h].pq.dot(c[h - 1].pq) <= 0) return {};
            h--;
            if (b[i].out(c[h].p))c[h] = b[i];
        }
    }
    while (q < h - 1 && c[q].out(c[h].intersect(c[h - 1]))) h
        --;
    while (q < h - 1 && c[h].out(c[q].intersect(c[q + 1]))) q
        ++;
    if (h - q <= 1) return {};
    c[h + 1] = c[q];
    vector<Point<T>> s;
    for (int i = q; i < h + 1; ++i) s.push_back(c[i].intersect(
        c[i + 1]));
    return s;
}

DelaunayTriangulation.h
Description: Computes the Delaunay triangulation of a set of points. Each
circumcircle contains none of the input points. If any three points are
collinear or any four are on the same circle, behavior is undefined.
Time: O(n^2)
10 lines

"Point.h", "3dHull.h"
```

```
template<class P, class F>
void delaunay(vector<P>& ps, F trifun) {
    if (sz(ps) == 3) { int d = (ps[0].cross(ps[1], ps[2]) < 0);
        trifun(0,1+d,2-d); }
    vector<P> p3;
    for (P p : ps) p3.emplace_back(p.x, p.y, p.dist2());
    if (sz(ps) > 3) for(auto t:hull3d(p3)) if ((p3[t.b]-p3[t.a]).
        cross(p3[t.c]-p3[t.a]).dot(P3(0,0,1)) < 0)
        trifun(t.a, t.c, t.b);
}

FastDelaunay.h
Description: Fast Delaunay triangulation. Each circumcircle contains none
of the input points. There must be no duplicate points. If all points are on a
line, no triangles will be returned. Should work for doubles as well, though
there may be precision issues in 'circ'. Returns triangles in order {t[0][0],
t[0][1], t[0][2], t[1][0], ... }, all counter-clockwise.
Time: O(n log n)
88 lines

"Point.h"

typedef Point<ll> P;
typedef struct Quad* Q;
typedef __int128_t ll1; // (can be ll if coords are < 2e4)
P arb(LLONG_MAX,LLONG_MAX); // not equal to any other point

struct Quad {
    Q rot, o; P p = arb; bool mark;
    P& F() { return r()->p; }
    Q& r() { return rot->rot; }
    Q prev() { return rot->o->rot; }
    Q next() { return r()->prev(); }
} *H;

bool circ(P p, P a, P b, P c) { // is p in the circumcircle?
    ll1 p2 = p.dist2(), A = a.dist2()-p2,
        B = b.dist2()-p2, C = c.dist2()-p2;
    return p.cross(a,b)*C + p.cross(b,c)*A + p.cross(c,a)*B > 0;
}
Q makeEdge(P orig, P dest) {
    Q r = H ? H : new Quad{new Quad{new Quad{0}}};
    H = r->o; r->r()->r() = r;
    rep(i,0,4) r = r->rot, r->p = arb, r->o = i & 1 ? r : r->r();
    r->p = orig; r->F() = dest;
    return r;
}

void splice(Q a, Q b) {
    swap(a->o->rot->o, b->o->rot->o); swap(a->o, b->o);
}

Q connect(Q a, Q b) {
    Q q = makeEdge(a->F(), b->p);
    splice(q, a->next());
    splice(q->r(), b);
    return q;
}

pair<Q,Q> rec(const vector<P>& s) {
    if (sz(s) <= 3) {
        Q a = makeEdge(s[0], s[1]), b = makeEdge(s[1], s.back());
        if (sz(s) == 2) return { a, a->r() };
        splice(a->r(), b);
        auto side = s[0].cross(s[1], s[2]);
        Q c = side ? connect(b, a) : 0;
        return {side < 0 ? c->r() : a, side < 0 ? c : b->r() };
    }

#define H(e) e->F(), e->p
#define valid(e) (e->F()).cross(H(base)) > 0)
    Q A, B, ra, rb;
    int half = sz(s) / 2;
```

```
tie(ra, A) = rec({all(s) - half});
tie(B, rb) = rec({sz(s) - half + all(s)});
while ((B->p.cross(H(A)) < 0 && (A = A->next())) ||
        (A->p.cross(H(B)) > 0 && (B = B->r()->o)));
Q base = connect(B->r(), A);
if (A->p == ra->p) ra = base->r();
if (B->p == rb->p) rb = base;
```

```
#define DEL(e, init, dir) Q e = init->dir; if (valid(e)) \
    while (circ(e->dir->F(), H(base), e->F())) { \
        Q t = e->dir; \
        splice(e, e->prev()); \
        splice(e->r(), e->r()->prev()); \
        e->o = H; H = e; e = t; \
    }
for (;;) {
    DEL(LC, base->r(), o); DEL(RC, base, prev());
    if (!valid(LC) && !valid(RC)) break;
    if (!valid(LC) || (valid(RC) && circ(H(RC), H(LC))))
        base = connect(RC, base->r());
    else
        base = connect(base->r(), LC->r());
}
return { ra, rb };
```

```
vector<P> triangulate(vector<P> pts) {
    sort(all(pts)); assert(unique(all(pts)) == pts.end());
    if (sz(pts) < 2) return {};
    Q e = rec(pts).first;
    vector<Q> q = {e};
    int qi = 0;
    while (e->o->F().cross(e->F(), e->p) < 0) e = e->o;
#define ADD { Q c = e; do { c->mark = 1; pts.push_back(c->p); \
    q.push_back(c->r()); c = c->next(); } while (c != e); }
    ADD; pts.clear();
    while (qi < sz(q)) if (!(e = q[qi++]->mark) ADD;
    return pts;
}
```

## 8.5 3D

### PolyhedronVolume.h

**Description:** Magic formula for the volume of a polyhedron. Faces should point outwards.

```
template<class V, class L>
double signedPolyVolume(const V& p, const L& trilst) {
    double v = 0;
    for (auto i : trilst) v += p[i.a].cross(p[i.b]).dot(p[i.c]);
    return v / 6;
}
```

### Point3D.h

**Description:** Class to handle points in 3D space. T can be e.g. double or long long.

```
template<class T> struct Point3D {
    typedef Point3D P;
    typedef const P& R;
    T x, y, z;
    explicit Point3D(T x=0, T y=0, T z=0) : x(x), y(y), z(z) {}
    bool operator<(R p) const {
        return tie(x, y, z) < tie(p.x, p.y, p.z); }
    bool operator==(R p) const {
        return tie(x, y, z) == tie(p.x, p.y, p.z); }
    P operator+(R p) const { return P(x+p.x, y+p.y, z+p.z); }
    P operator-(R p) const { return P(x-p.x, y-p.y, z-p.z); }
    P operator*(T d) const { return P(x*d, y*d, z*d); }
    P operator/(T d) const { return P(x/d, y/d, z/d); }
```

```
T dot(R p) const { return x*p.x + y*p.y + z*p.z; }
P cross(R p) const {
    return P(y*p.z - z*p.y, z*p.x - x*p.z, x*p.y - y*p.x);
}
T dist2() const { return x*x + y*y + z*z; }
double dist() const { return sqrt((double)dist2()); }
//Azimuthal angle (longitude) to x-axis in interval [-pi, pi]
double phi() const { return atan2(y, x); }
//Zenith angle (latitude) to the z-axis in interval [0, pi]
double theta() const { return atan2(sqrt(x*x+y*y),z); }
P unit() const { return *this/(T)dist(); } //makes dist()==1
//returns unit vector normal to *this and p
P normal(P p) const { return cross(p).unit(); }
//returns point rotated 'angle' radians ccw around axis
P rotate(double angle, P axis) const {
    double s = sin(angle), c = cos(angle); P u = axis.unit();
    return u*dot(u)*(1-c) + (*this)*c - cross(u)*s;
}
}
```

### 3dHull.h

**Description:** Computes all faces of the 3-dimension hull of a point set. \*No four points must be coplanar\*, or else random results will be returned. All faces will point outwards.

```
Time:  $\mathcal{O}(n^2)$ 
"Point3D.h" 49 lines
typedef Point3D<double> P3;

struct PR {
    void ins(int x) { (a == -1 ? a : b) = x; }
    void rem(int x) { (a == x ? a : b) = -1; }
    int cnt() { return (a != -1) + (b != -1); }
    int a, b;
};
```

```
struct F { P3 q; int a, b, c; };
```

```
vector<F> hull3d(const vector<P3>& A) {
    assert(sz(A) >= 4);
    vector<vector<PR>> E(sz(A), vector<PR>(sz(A), {-1, -1}));
#define E(x,y) E[f.x][f.y]
    vector<F> FS;
    auto mf = [&](int i, int j, int k, int l) {
        P3 q = (A[j] - A[i]).cross((A[k] - A[i]));
        if (q.dot(A[l]) > q.dot(A[i]))
            q = q * -1;
        F f{q, i, j, k};
        E(a,b).ins(k); E(a,c).ins(j); E(b,c).ins(i);
        FS.push_back(f);
    };
    rep(i,0,4) rep(j,i+1,4) rep(k,j+1,4)
        mf(i, j, k, 6 - i - j - k);
```

```
    rep(i,4,sz(A)) {
        rep(j,0,sz(FS)) {
            F f = FS[j];
            if(f.q.dot(A[i]) > f.q.dot(A[f.a])) {
                E(a,b).rem(f.c);
                E(a,c).rem(f.b);
                E(b,c).rem(f.a);
                swap(FS[j--], FS.back());
                FS.pop_back();
            }
        }
        int nw = sz(FS);
        rep(j,0,nw) {
            F f = FS[j];
#define C(a, b, c) if (E(a,b).cnt() != 2) mf(f.a, f.b, i, f.c);
            C(a, b, c); C(a, c, b); C(b, c, a);
```

```
        }
    }
    for (F& it : FS) if ((A[it.b] - A[it.a]).cross(
        A[it.c] - A[it.a]).dot(it.q) <= 0) swap(it.c, it.b);
    return FS;
};
```

### sphericalDistance.h

**Description:** Returns the shortest distance on the sphere with radius radius between the points with azimuthal angles (longitude) f1 ( $\phi_1$ ) and f2 ( $\phi_2$ ) from x axis and zenith angles (latitude) t1 ( $\theta_1$ ) and t2 ( $\theta_2$ ) from z axis (0 = north pole). All angles measured in radians. The algorithm starts by converting the spherical coordinates to cartesian coordinates so if that is what you have you can use only the two last rows. dx\*radius is then the difference between the two points in the x direction and d\*radius is the total distance between the points.

```
double sphericalDistance(double f1, double t1,
    double f2, double t2, double radius) {
    double dx = sin(t2)*cos(f2) - sin(t1)*cos(f1);
    double dy = sin(t2)*sin(f2) - sin(t1)*sin(f1);
    double dz = cos(t2) - cos(t1);
    double d = sqrt(dx*dx + dy*dy + dz*dz);
    return radius*2*asin(d/2);
}
```

# Strings (9)

### KMP.h

**Description:** pi[x] computes the length of the longest prefix of s that ends at x, other than s[0...x] itself (abacaba -> 0010123). Can be used to find all occurrences of a string.

```
Time:  $\mathcal{O}(n)$  16 lines
vi pi(const string& s) {
    vi p(sz(s));
    rep(i,1,sz(s)) {
        int g = p[i-1];
        while (g && s[i] != s[g]) g = p[g-1];
        p[i] = g + (s[i] == s[g]);
    }
    return p;
}
```

```
vi match(const string& s, const string& pat) {
    vi p = pi(pat + '\0' + s), res;
    rep(i,sz(p)-sz(s),sz(p))
        if (p[i] == sz(pat)) res.push_back(i - 2 * sz(pat));
    return res;
}
```

### Zfunc.h

**Description:** z[x] computes the length of the longest common prefix of s[i:] and s, except z[0] = 0. (abacaba -> 0010301)

```
Time:  $\mathcal{O}(n)$  12 lines
vi Z(const string& S) {
    vi z(sz(S));
    int l = -1, r = -1;
    rep(i,1,sz(S)) {
        z[i] = i >= r ? 0 : min(r - i, z[i - l]);
        while (i + z[i] < sz(S) && S[i + z[i]] == S[z[i]])
            z[i]++;
        if (i + z[i] > r)
            l = i, r = i + z[i];
    }
    return z;
}
```

Manacher.h

**Description:** For each position in a string, computes  $p[0][i]$  = half length of longest even palindrome around pos  $i$ ,  $p[1][i]$  = longest odd (half rounded down).

**Time:**  $\mathcal{O}(N)$

```
array<vi, 2> manacher(const string& s) {
    int n = sz(s);
    array<vi,2> p = {vi(n+1), vi(n)};
    rep(z,0,2) for (int i=0,l=0,r=0; i < n; i++) {
        int t = r-i+!z;
        if (i<r) p[z][i] = min(t, p[z][l+t]);
        int L = i-p[z][i], R = i+p[z][i]-!z;
        while (L>=1 && R+1<n && s[L-1] == s[R+1])
            p[z][i]++, L--, R++;
        if (R>r) l=L, r=R;
    }
    return p;
}
```

13 lines

MinRotation.h

**Description:** Finds the lexicographically smallest rotation of a string.

**Usage:** rotate(v.begin(), v.begin()+minRotation(v), v.end());

**Time:**  $\mathcal{O}(N)$

```
int minRotation(string s) {
    int a=0, N=sz(s); s += s;
    rep(b,0,N) rep(k,0,N) {
        if (a+k == b || s[a+k] < s[b+k]) {b += max(0, k-1); break;}
        if (s[a+k] > s[b+k]) {a = b; break;}
    }
    return a;
}
```

8 lines

SuffixArray.h

**Description:** Builds suffix array for a string.  $sa[i]$  is the starting index of the suffix which is  $i$ 'th in the sorted suffix array. The returned vector is of size  $n+1$ , and  $sa[0] = n$ . The lcp array contains longest common prefixes for neighbouring strings in the suffix array:  $lcp[i] = lcp(sa[i], sa[i-1])$ ,  $lcp[0] = 0$ . The input string must not contain any zero bytes.

**Time:**  $\mathcal{O}(n \log n)$

```
struct SuffixArray {
    vi sa, lcp;
    SuffixArray(string& s, int lim=256) { // or basic_string<int>
        int n = sz(s) + 1, k = 0, a, b;
        vi x(all(s)+1), y(n), ws(max(n, lim)), rank(n);
        sa = lcp = y, iota(all(sa), 0);
        for (int j = 0, p = 0; p < n; j = max(1, j * 2), lim = p) {
            p = j, iota(all(y), n - j);
            rep(i,0,n) if (sa[i] >= j) y[p++] = sa[i] - j;
            fill(all(ws), 0);
            rep(i,0,n) ws[x[i]]++;
            rep(i,1,lim) ws[i] += ws[i - 1];
            for (int i = n; i--;) sa[--ws[x[y[i]]]] = y[i];
            swap(x, y), p = 1, x[sa[0]] = 0;
            rep(i,1,n) a = sa[i - 1], b = sa[i], x[b] =
                (y[a] == y[b] && y[a + j] == y[b + j]) ? p - 1 : p++;
        }
        rep(i,1,n) rank[sa[i]] = i;
        for (int i = 0, j; i < n - 1; lcp[rank[i++]] = k)
            for (k && k--, j = sa[rank[i] - 1];
                s[i + k] == s[j + k]; k++);
    }
};
```

23 lines

**Description:** Ukkonen's algorithm for online suffix tree construction. Each node contains indices  $[l, r]$  into the string, and a list of child nodes. Suffixes are given by traversals of this tree, joining  $[l, r]$  substrings. The root is 0 (has  $l = -1, r = 0$ ), non-existent children are -1. To get a complete tree, append a dummy symbol – otherwise it may contain an incomplete path (still useful for substring matching, though).

**Time:**  $\mathcal{O}(26N)$

```
struct SuffixTree {
    enum { N = 200010, ALPHA = 26 }; // N ~ 2*maxlen+10
    int toi(char c) { return c - 'a'; }
    string a; // v = cur node, q = cur position
    int t[N][ALPHA], l[N], r[N], p[N], s[N], v=0, q=0, m=2;

    void ukkadd(int i, int c) { suff:
        if (r[v]<=q) {
            if (t[v][c]==-1) { t[v][c]=m; l[m]=i;
                p[m++]=v; v=s[v]; q=r[v]; goto suff; }
            v=t[v][c]; q=l[v];
        }
        if (q==-1 || c==toi(a[q])) q++; else {
            l[m+1]=i; p[m+1]=m; l[m]=l[v]; r[m]=q;
            p[m]=p[v]; t[m][c]=m+1; t[m][toi(a[q])]=v;
            l[v]=q; p[v]=m; t[p[m]][toi(a[l[m]])]=m;
            v=s[p[m]]; q=l[m];
            while (q<r[m]) { v=t[v][toi(a[q])]; q+=r[v]-l[v]; }
            if (q==r[m]) s[m]=v; else s[m]=m+2;
            q=r[v]-(q-r[m]); m+=2; goto suff;
        }
    }
}
```

```
SuffixTree(string a) : a(a) {
    fill(r,r+N,sz(a));
    memset(s, 0, sizeof s);
    memset(t, -1, sizeof t);
    fill(t[1],t[1]+ALPHA,0);
    s[0] = 1; l[0] = l[1] = -1; r[0] = r[1] = p[0] = p[1] = 0;
    rep(i,0,sz(a)) ukkadd(i, toi(a[i]));
}
```

```
// example: find longest common substring (uses ALPHA = 28)
pii best;
int lcs(int node, int i1, int i2, int olen) {
    if (l[node] <= i1 && i1 < r[node]) return 1;
    if (l[node] <= i2 && i2 < r[node]) return 2;
    int mask = 0, len = node ? olen + (r[node] - l[node]) : 0;
    rep(c,0,ALPHA) if (t[node][c] != -1)
        mask |= lcs(t[node][c], i1, i2, len);
    if (mask == 3)
        best = max(best, {len, r[node] - len});
    return mask;
}

static pii LCS(string s, string t) {
    SuffixTree st(s + (char)('z' + 1) + t + (char)('z' + 2));
    st.lcs(0, sz(s), sz(s) + 1 + sz(t), 0);
    return st.best;
}
};
```

Hashing.h

**Description:** 0-indexed. Self-explanatory methods for string hashing.

```
// Arithmetic mod 2^64-1. 2x slower than mod 2^64 and more
// code, but works on evil test data (e.g. Thue-Morse, where
// ABBA... and BAAB... of length 2^10 hash the same mod 2^64).
// "typedef ull H;" instead if you think test data is random,
// or work mod 10^9+7 if the Birthday paradox is not a problem.
typedef uint64_t ull;
struct H {
    ull x; H(ull x=0) : x(x) {}
```

50 lines

```
H operator+(H o) { return x + o.x + (x + o.x < x); }
H operator-(H o) { return *this + ~o.x; }
H operator*(H o) { auto m = (__uint128_t)x * o.x;
    return H((ull)m + (ull)(m >> 64)); }
ull get() const { return x + !~x; }
bool operator==(H o) const { return get() == o.get(); }
bool operator<(H o) const { return get() < o.get(); }
};

static const H C = (1ll)1e11+3; // (order ~ 3e9; random also ok)

struct HashInterval {
    vector<H> ha, pw;
    HashInterval(string& str) : ha(sz(str)+1), pw(ha) {
        pw[0] = 1;
        rep(i,0,sz(str))
            ha[i+1] = ha[i] * C + str[i],
            pw[i+1] = pw[i] * C;
    }
    H hashInterval(int a, int b) { // hash [a, b]
        return ha[b] - ha[a] * pw[b - a];
    }
};
```

```
vector<H> getHashes(string& str, int length) {
    if (sz(str) < length) return {};
    H h = 0, pw = 1;
    rep(i,0,length)
        h = h * C + str[i], pw = pw * C;
    vector<H> ret = {h};
    rep(i,length,sz(str)) {
        ret.push_back(h = h * C + str[i] - pw * str[i-length]);
    }
    return ret;
}
```

```
H hashString(string& s){H h{}; for(char c:s) h=h*C+c;return h;}
```

AhoCorasick.h

**Description:** Aho-Corasick automaton, used for multiple pattern matching. Initialize with AhoCorasick ac(patterns); the automaton start node will be at index 0. find(word) returns for each position the index of the longest word that ends there, or -1 if none. findAll(–, word) finds all words (up to  $N\sqrt{N}$  many if no duplicate patterns) that start at each position (shortest first). Duplicate patterns are allowed; empty patterns are not. To find the longest words that start at each position, reverse all input. For large alphabets, split each symbol into chunks, with sentinel bits for symbol boundaries. **Time:** construction takes  $\mathcal{O}(26N)$ , where  $N$  = sum of length of patterns. find(x) is  $\mathcal{O}(N)$ , where  $N$  = length of x. findAll is  $\mathcal{O}(NM)$ .

66 lines

```
struct AhoCorasick {
    enum {alpha = 26, first = 'A'}; // change this!
    struct Node {
        // (nmatches is optional)
        int back, next[alpha], start = -1, end = -1, nmatches = 0;
        Node(int v) { memset(next, v, sizeof(next)); }
    };
    vector<Node> N;
    vi backp;
    void insert(string& s, int j) {
        assert(!s.empty());
        int n = 0;
        for (char c : s) {
            int& m = N[n].next[c - first];
            if (m == -1) { n = m = sz(N); N.emplace_back(-1); }
            else n = m;
        }
        if (N[n].end == -1) N[n].start = j;
        backp.push_back(N[n].end);
        N[n].end = j;
    }
};
```



```

    N[n].nmatches++;
}
AhoCorasick(vector<string>& pat) : N(1, -1) {
    rep(i,0,sz(pat)) insert(pat[i], i);
    N[0].back = sz(N);
    N.emplace_back(0);

    queue<int> q;
    for (q.push(0); !q.empty(); q.pop()) {
        int n = q.front(), prev = N[n].back;
        rep(i,0,alpha) {
            int &ed = N[n].next[i], y = N[prev].next[i];
            if (ed == -1) ed = y;
            else {
                N[ed].back = y;
                (N[ed].end == -1 ? N[ed].end : backp[N[ed].start])
                    = N[y].end;
                N[ed].nmatches += N[y].nmatches;
                q.push(ed);
            }
        }
    }
}

vi find(string word) {
    int n = 0;
    vi res; // ll count = 0;
    for (char c : word) {
        n = N[n].next[c - first];
        res.push_back(N[n].end);
        // count += N[n].nmatches;
    }
    return res;
}

vector<vi> findAll(vector<string>& pat, string word) {
    vi r = find(word);
    vector<vi> res(sz(word));
    rep(i,0,sz(word)) {
        int ind = r[i];
        while (ind != -1) {
            res[i - sz(pat[ind]) + 1].push_back(ind);
            ind = backp[ind];
        }
    }
    return res;
}
};
```

SuffixAutomaton.h

**Description:** Computes suffix automaton for a given string. Letters have to be added one by one from left to right. After adding all the letters maybe call markTerminalNodes().

**Usage:** For a string  $s$ .  
SuffixAutomaton sa(sz(s));  
for(auto ch: s) sa.add(ch);  
sa.markTerminalNodes()

**Time:**  $\mathcal{O}(N)$  time and  $\mathcal{O}(N\alpha)$  memory. If array is replaced with map  $\mathcal{O}(N \log \alpha)$  time and  $\mathcal{O}(N)$  memory.

---

```
constexpr short alpha = 26;
constexpr char offset = 'a';
struct state{
    int len, link;
    bool is_terminal;
    array<int,alpha> next;
    state(){
        len = 0; link = -1; is_terminal = false;
        next.fill(0);
    }
};
```

```

struct SuffixAutomaton{
    int n;
    vector<state> sa;
    int sz = 1, last = 0;
    SuffixAutomaton(int n) : n(n), sa(2*n + 1){}
    void add(char ch){
        short ch = ch_offset;
        int curr = sz++;
        sa[curr].len = sa[last].len + 1;
        int p = last;
        while(p!=-1 && !sa[p].next[ch]){
            sa[p].next[ch] = curr;
            p = sa[p].link;
        }
        if(p == -1){
            sa[curr].link = 0;
        }else{
            int q = sa[p].next[ch];
            if(sa[p].len+1 == sa[q].len){
                sa[curr].link = q;
            }else{
                int clone = sz++;
                sa[clone].len = sa[p].len + 1;
                sa[clone].next = sa[q].next;
                sa[clone].link = sa[q].link;
                while(p!=-1 && sa[p].next[ch] == q){
                    sa[p].next[ch] = clone;
                    p = sa[p].link;
                }
                sa[q].link = sa[curr].link = clone;
            }
        }
        last = curr;
        return;
    }
    void markTerminalNodes(){
        int curr = last;
        while(curr){
            sa[curr].is_terminal = true;
            curr = sa[curr].link;
        }
        return;
    }
};
```

Eertree.h

**Description:** Computes eertree for a given string. Letters have to be added one by one from left to right. After adding all the letters maybe call computeFrequency().

**Usage:** Let  $s$  be the original string and  $p$  be the palindrome associated with node  $u$   
 $et[u].l$  is the length of  $p$   
 $et[u].cnt$  is the number of times  $p$  appears as the longest palindrome suffix (not necessarily proper) of a prefix  $s[0 : i]$ . After calling computeFrequency() is the number of times  $p$  appears in  $s$ .  
 $et[u].link$  is the node corresponding to the longest proper palindrome of  $p$

**Time:**  $\mathcal{O}(N)$  time and  $\mathcal{O}(N\alpha)$  memory. If array is replaced with map  $\mathcal{O}(N \log \alpha)$  time and  $\mathcal{O}(N)$  memory.

---

```
constexpr short alpha = 26;
constexpr char offset = 'a';
struct state{
    int l, link, cnt;
    array<int,alpha> go;
    state(){
        l = cnt = 0;
        go.fill(0);
    }
};
```

```

    }
};
struct eertree{
    int n = 2;
    int last = 1;
    vector<state> et;
    string s;
    eertree(){
        et.resize(2);
        et[0].link = et[1].link = 1;
        et[1].l = -1;
    }
    int palSuff(int x){
        while(s[sz(s) - 2 - et[x].l] != s.back()) x = et[x].link;
        return x;
    }
    int add(char ch){
        s.pb(ch);
        last = palSuff(last);
        bool new_pal = !et[last].go[ch_offset];
        if(new_pal){
            et.pb(state());
            et[last].go[ch_offset] = n++;
            et.back().link = et[palSuff(et[last].link)].go[ch_offset];
            et.back().l = et[last].l + 2;
            if(et.back().l == 1) et.back().link = 0;
        }
        last = et[last].go[ch_offset];
        // Do something with last, maybe if new_pal
        et[last].cnt++;
        if(et[last].l == sz(s)) last = et[last].link;
        return new_pal;
    }
    void computeFrequency(){
        for(int i=n-1; i>1; i--){
            et[ et[i].link ].cnt += et[i].cnt;
        }
    }
};
```

Various (10)

10.1 Bitset

- Find\_first() regresa el indice del primer bit encendido. Manejar explícitamente cuando no existe
- Find\_next(idx) regresa el menor indice mayor a idx con el bit encendido. Manejar explícitamente cuando no existe
- count() Regresa la cantidad de 1's
- set() Asigna todos los bits a 1
- reset() Asigna todos los bits a 0
- flip()

10.2 Intervals

IntervalCover.h

**Description:** Compute indices of smallest set of intervals covering another interval. Intervals should be [inclusive, exclusive). To support [inclusive, inclusive], change (A) to add || R.empty(). Returns empty set on failure (or if G is empty).



<b>Time:</b> $\mathcal{O}(N \log N)$	19 lines
<pre>template&lt;class T&gt; vi cover(pair&lt;T, T&gt; G, vector&lt;pair&lt;T, T&gt;&gt; I) {     vi S(sz(I)), R;     iota(all(S), 0);     sort(all(S), [&amp;](int a, int b) { return I[a] &lt; I[b]; });     T cur = G.first;     int at = 0;     while (cur &lt; G.second) { // (A)         pair&lt;T, int&gt; mx = make_pair(cur, -1);         while (at &lt; sz(I) &amp;&amp; I[S[at]].first &lt;= cur) {             mx = max(mx, make_pair(I[S[at]].second, S[at]));             at++;         }         if (mx.second == -1) return {};         cur = mx.first;         R.push_back(mx.second);     }     return R; }</pre>	

### ConstantIntervals.h

**Description:** Split a monotone function on  $[from, to]$  into a minimal set of half-open intervals on which it has the same value. Runs a callback  $g$  for each such interval.

**Usage:** `constantIntervals(0, sz(v), [&](int x){return v[x];}, [&](int lo, int hi, T val){...});`

**Time:**  $\mathcal{O}(k \log \frac{n}{k})$

```
template<class F, class G, class T>
void rec(int from, int to, F& f, G& g, int& i, T& p, T q) {
    if (p == q) return;
    if (from == to) {
        g(i, to, p);
        i = to; p = q;
    } else {
        int mid = (from + to) >> 1;
        rec(from, mid, f, g, i, p, f(mid));
        rec(mid+1, to, f, g, i, p, q);
    }
}

template<class F, class G>
void constantIntervals(int from, int to, F f, G g) {
    if (to <= from) return;
    int i = from; auto p = f(i), q = f(to-1);
    rec(from, to-1, f, g, i, p, q);
    g(i, to, q);
}
```

## 10.3 Misc. algorithms

### TernarySearch.h

**Description:** Find the smallest  $i$  in  $[a, b]$  that maximizes  $f(i)$ , assuming that  $f(a) < \dots < f(i) \geq \dots \geq f(b)$ . To reverse which of the sides allows non-strict inequalities, change the  $<$  marked with (A) to  $<=$ , and reverse the loop at (B). To minimize  $f$ , change it to  $>$ , also at (B).

**Usage:** `int ind = ternSearch(0,n-1,[&](int i){return a[i];});`

**Time:**  $\mathcal{O}(\log(b - a))$

```
template<class F>
int ternSearch(int a, int b, F f) {
    assert(a <= b);
    while (b - a >= 5) {
        int mid = (a + b) / 2;
        if (f(mid) < f(mid+1)) a = mid; // (A)
        else b = mid+1;
    }
    rep(i,a+1,b+1) if (f(a) < f(i)) a = i; // (B)
    return a;
}
```

	17 lines
<pre> }</pre>	
<h3>LIS.h</h3> <p><b>Description:</b> Compute indices for the longest increasing subsequence.</p> <p><b>Time:</b> <math>\mathcal{O}(N \log N)</math></p>	
<pre>template&lt;class I&gt; vi lis(const vector&lt;I&gt;&amp; S) {     if (S.empty()) return {};     vi prev(sz(S));     typedef pair&lt;I, int&gt; p;     vector&lt;p&gt; res;     rep(i,0,sz(S)) {         // change 0 -&gt; i for longest non-decreasing subsequence         auto it = lower_bound(all(res), p{S[i], 0});         if (it == res.end()) res.emplace_back(), it = res.end()-1;         *it = {S[i], i};         prev[i] = it == res.begin() ? 0 : (it-1)-&gt;second;     }     int L = sz(res), cur = res.back().second;     vi ans(L);     while (L--) ans[L] = cur, cur = prev[cur];     return ans; }</pre>	

### FastKnapsack.h

**Description:** Given  $N$  non-negative integer weights  $w$  and a non-negative target  $t$ , computes the maximum  $S \leq t$  such that  $S$  is the sum of some subset of the weights.

**Time:**  $\mathcal{O}(N \max(w_i))$

```
int knapsack(vi w, int t) {
    int a = 0, b = 0, x;
    while (b < sz(w) && a + w[b] <= t) a += w[b++];
    if (b == sz(w)) return a;
    int m = *max_element(all(w));
    vi u, v(2*m, -1);
    v[a+m-t] = b;
    rep(i,b,sz(w)) {
        u = v;
        rep(x,0,m) v[x+w[i]] = max(v[x+w[i]], u[x]);
        for (x = 2*m; --x > m;) rep(j, max(0,u[x]), v[x])
            v[x-w[j]] = max(v[x-w[j]], j);
    }
    for (a = t; v[a+m-t] < 0; a--);
    return a;
}
```

## 10.4 Dynamic programming

### KnuthDP.h

**Description:** When doing DP on intervals:  $a[i][j] = \min_{i < k < j} (a[i][k] + a[k][j]) + f(i, j)$ , where the (minimal) optimal  $k$  increases with both  $i$  and  $j$ , one can solve intervals in increasing order of length, and search  $k = p[i][j]$  for  $a[i][j]$  only between  $p[i][j - 1]$  and  $p[i + 1][j]$ . This is known as Knuth DP. Sufficient criteria for this are if  $f(b, c) \leq f(a, d)$  and  $f(a, c) + f(b, d) \leq f(a, d) + f(b, c)$  for all  $a \leq b \leq c \leq d$ . Consider also: LineContainer (ch. Data structures), monotone queues, ternary search.

**Time:**  $\mathcal{O}(N^2)$

### DivideAndConquerDP.h

**Description:** Given  $a[i] = \min_{l_0(i) \leq k < h_i(i)} (f(i, k))$  where the (minimal) optimal  $k$  increases with  $i$ , computes  $a[i]$  for  $i = L..R - 1$ .

**Time:**  $\mathcal{O}((N + (hi - lo)) \log N)$

```
struct DP { // Modify at will:
    int lo(int ind) { return 0; }
    int hi(int ind) { return ind; }
    ll f(int ind, int k) { return dp[ind][k]; }
    void store(int ind, int k, ll v) { res[ind] = pii(k, v); }
```

<pre>void rec(int L, int R, int LO, int HI) {     if (L &gt;= R) return;     int mid = (L + R) &gt;&gt; 1;     pair&lt;ll, int&gt; best(LLONG_MAX, LO);     rep(k, max(LO, lo(mid)), min(HI, hi(mid)))         best = min(best, make_pair(f(mid, k), k));     store(mid, best.second, best.first);     rec(L, mid, LO, best.second+1);     rec(mid+1, R, best.second, HI); }  void solve(int L, int R) { rec(L, R, INT_MIN, INT_MAX); } };</pre>	
--	--

## 10.5 Optimization tricks

### 10.5.1 Bit hacks

- $x \& -x$  is the least bit in  $x$ .
- `for (int x = m; x; ) { --x &= m; ... }` loops over all subset masks of  $m$  (except  $m$  itself).
- $c = x \& -x, r = x + c; (((r^x) >> 2) / c) \mid r$  is the next number after  $x$  with the same number of bits set.
- `rep(b,0,K) rep(i,0,(1 << K))`  
if  $(i \& 1 << b) D[i] += D[i^ (1 << b)]$ ;  
computes all sums of subsets.

### 10.5.2 Pragmas

- `#pragma GCC optimize ("Ofast")` will make GCC auto-vectorize loops and optimizes floating points better.
- `#pragma GCC target ("avx2")` can double performance of vectorized code, but causes crashes on old machines.
- `#pragma GCC optimize ("trapv")` kills the program on integer overflows (but is really slow).

### FastMod.h

**Description:** Compute  $a \% b$  about 5 times faster than usual, where  $b$  is constant but not known at compile time. Returns a value congruent to  $a \pmod b$  in the range  $[0, 2b)$ .

```
typedef unsigned long long ull;
struct FastMod {
    ull b, m;
    FastMod(ull b) : b(b), m(-1ULL / b) {}
    ull reduce(ull a) { // a % b + (0 or b)
        return a - (ull)((__uint128_t(m) * a) >> 64) * b;
    }
};
```

### FastInput.h

**Description:** Read an integer from stdin. Usage requires your program to pipe in input from file.

**Usage:** `./a.out < input.txt`

**Time:** About 5x as fast as `cin/scanf`.

```
inline char gc() { // like getchar()
    static char buf[1 << 16];
    static size_t bc, be;
    if (bc >= be) {
        buf[0] = 0, bc = 0;
        be = fread(buf, 1, sizeof(buf), stdin);
    }
}
```

```
}
    return buf[bc++]; // returns 0 on EOF
}

int readInt() {
    int a, c;
    while ((a = gc()) < 40);
    if (a == '-') return -readInt();
    while ((c = gc()) >= 48) a = a * 10 + c - 48;
    return a - 48;
}
```

10.6 Theory

10.6.1 Problemas de Flujo

Mínimo corte: El valor del mínimo corte es igual al del máximo flujo. Corriendo máximo flujo recupera un mínimo corte como el conjunto de los nodos alcanzables desde  $s$  en la red residual, y su complemento.

Mínimo costo, cualquier flujo: Correr mincost-flow, terminar cuando el camino aumentante sea no negativo.

Flujo con cotas inferiores/demandas: Sea  $D = \sum_{(u \rightarrow v) \in E} d(u \rightarrow v)$  la suma de todas las demandas en  $G = (V, E)$ . Construir nueva red  $G'$ , con nuevos source y sink  $s'$  y  $t'$ . Para cada  $v \in V$  hacer capacidad  $c'(s' \rightarrow v) = \sum_{u \in V} d(u \rightarrow v)$  (demandas que le entran) y  $c'(v \rightarrow t') = \sum_{u \in v} d(v \rightarrow u)$  (demandas que le salen). Para cada  $(u \rightarrow v) \in E$ ,  $c'(u \rightarrow v) = c(u \rightarrow v) - d(u \rightarrow v)$ . Además,  $c'(t \rightarrow s) = \infty$ . Existe un flujo válido ssi el maximo flujo en  $G'$  es  $D$ .

Para máximo flujo valido, hacer otra red; este es el flujo ya mandado por las demandas, sumado al flujo en otra red  $G''$ . Si  $G'$  mandó  $f'(u \rightarrow v)$  de flujo con  $(u \rightarrow v) \in E$ , poner  $c''(u \rightarrow v) = c(u \rightarrow v) - f'(u \rightarrow v)$ , y en el reverso  $c''(v \rightarrow u) = f'(u \rightarrow v) - d(u \rightarrow v)$ , y resolver en esta red. Para mínimo costo, se debió correr  $G'$  con mincost-flow, para preservar que no haya ciclos negativos, y también hay que agregarle costo infinito a la arista auxiliar  $(t \rightarrow s)$ .

Max-weight closure: Dado  $G$  con pesos  $w$  en nodos encontrar un subconjunto  $V' \subseteq V$  cerrado (si algo de  $V'$  alcanza un nodo, este también está en  $V'$ ) de máximo costo. Si  $w(u) > 0$  se agrega  $s \rightarrow u$  con capacidad  $w(u)$ , de otra manera,  $u \rightarrow t$  con  $-w(u)$ . Para todo  $(u \rightarrow v) \in E$  agregar  $c(u \rightarrow v) = \infty$ . El  $w(V')$  óptimo es la suma de todo  $w(u) > 0$  menos el minimo corte. Recuperar  $V'$  como los nodos del lado de  $s$  en el mincut.

Dilworth (máxima antcadena, cubrir poset con min. num. cadenas): Dado un poset, una cadena es un conjunto en el que cualquier par es comparable, y una antcadena es tal que ninguno lo es. El tamaño de la máxima antcadena es igual al mínimo número de cadenas para particionar al poset, y este valor se define como la longitud del poset.

Duplicar nodos. Si  $u < v$  agregar  $u \rightarrow n + v$ . Esto hace un grafo bipartito, y la longitud del poset es  $n$  menos el máximo matching. Las aristas tomadas son aristas en las cadenas.

Si  $C$  es un vertex cover de cardinalidad máxima en este bipartito (veáse Konig), los nodos que **no** están presentes en  $C$  (en ninguna de sus dos versiones) son una antcadena máxima.

Cubrir DAG en min. num. caminos: Hacer un grafo bipartito como el del punto anterior y correr matching, si los caminos son disjuntos en vértices. Si son disjuntos en aristas, hacer la cerradura transitiva (es un poset) antes de calcular el matching.

Teorema de Mirsky: Dual de Dilworth. La cardinalidad máxima de una cadena es igual al mínimo número de antcadenas para cubrir a un poset.

Grafos bipartitos

Konig’s theorem: En un grafo bipartito la cardinalidad del maximum matching es igual a la cardinalidad del minimum vertex cover (conjunto de vertices que incluye al menos un extremo de cada arista).

- **Construcción con flujo:** Sean  $A, B$  las dos partes. Para  $a \in A$ , agregar  $c(s \rightarrow a) = 1$ , para  $b \in B$ ,  $c(b \rightarrow t) = 1$ . Si  $(a \rightarrow b) \in E$  agregar  $c(a \rightarrow b) = \infty$ . El matching máximo, de tamaño  $|M|$  corresponde a las aristas  $a \rightarrow b$  tomadas por el máximo flujo. Sea  $(S, T)$  un mínimo corte y  $A = A_S \cup A_T$ ,  $B = B_S \cup B_T$  de acuerdo a que lado están del corte. Entonces el corte tiene tamaño  $|A_T| + |B_S| = |M|$  y  $A_T \cup B_S$  es un vertex cover mínimo.

- **Construcción con Kuhn:** En el arreglo  $M$  queda almacenado el matching.  $M[i]$  es el indice del conjunto  $L$  con el que el nodo  $i$  de  $R$  esta matcheado o  $-1$  si no esta con ninguno.

- **Max. Indep. Set:** Tomar los nodos no tomados en el Máx. Vertex Cover.

- **Max Clique:** Construir la red del complemento de  $G$  y tomar Max. Indep. Set.

- **Min. Edge Cover:** Aristas del matching, más para los nodos no tomados en el matching, tomar cualquier arista.

10.6.2 Strings

- Una strings  $S$  tiene a lo más  $N$  palíndromos distintos
- En una string aleatoria  $S$  la cantidad esperada de palíndromos distintos es  $O(\sqrt{N\sigma})$

- Sea  $S$  una string de longitud  $N$  y  $T$  una string de longitud minima tal que  $S = T + T + \dots + T$ . Si consideramos  $k = N - \pi[N]$  –donde  $\pi[N]$  se obtiene con KMP– tenemos lo siguiente, si  $k$  divide a  $N$  se cumple que  $T = S[1 : k]$  si no necesariamente  $T = S$ .

- Pensar el suffix tree como un Aho Corasick comprimido de todos los sufijos de una string y crear en la template

Suffix Automaton

- Es un DAG
- Los caminos del nodo inicial a uno terminal corresponden a sufijos
- Cada nodo representa una clase de equivalencia que contiene las strings que tienen el mismo conjunto **endpos** (el conjunto de posiciones donde terminan). Para el nodo  $u$ , su conjunto **endpos** contiene strings de longitudes consecutivas siendo  $u.len$  la mas grande y  $u.link.len + 1$  la mas corta. Lo anterior se puede pensar como que el link lleva al siguiente conjunto de strings mas cortas consecutivas, además, existe un camino del nodo inicial a  $u$  para toda longitud en  $[u.link.len + 1, u.len]$ .
- Los suffix links inducen un árbol
- El orden topologico se debe obtener como en un DAG general

Eertree

- Recorrer de  $N$  a 2 da un orden topologico, tener cuidado con los nodos 0 y 1 porque son strings vacias
- Los nodos de strings de longitud par inducen un arbol con los links, tambien los impares
- Podemos pensarlo como un DAG donde  $u \rightarrow v$  si  $v$  se obtiene como  $c + u + c$  para algun caracter  $c$ ;  $aba \rightarrow cabac$ .