Partial Derivative

Multivariate Function: A Function that contains multiple variables.

A simple regression such as $y = mx + b$ is a multivariate function (y) where there are two variables (m and b)

Partial derivative enable us to calculate multivariate equations.

Problem: $z = x^2 - y^2$

Determine:

1. ~~x=3, y=0~~ The value of z
2. The slope of z with respect to x.
3. The slope of z with respect to y.

at the points where

1. $x = 3,\ y = 0$
2. $x = 2,\ y = 3$
3. $x = -2,\ y = -3$

(1) For, $x = 3$ and $y = 0$, (i) $z = x^2 - y^2 = 9 - 0 = 9$

ii) $\frac{\partial z}{\partial x} = 2x = 2 \times 3 = 6$

iii) $\frac{\partial z}{\partial y} = -2y = -2 \times 0 = 0$

Ans

2 For $x=2, y=3$, $z=x^2-y^2 = 2^2-3^2 = -5$

ii) $\frac{\partial z}{\partial x} = 2x = 2\times2 = 4$

iii) $\frac{\partial z}{\partial y} = -2y = -2\times3 = -6$ Ans

3 For, $x=-2, y=-3$, $z=x^2-y^2 = 4-9 = -5$

ii) $\frac{\partial z}{\partial x} = 2x = 2(-2) = -4$

iii) $\frac{\partial z}{\partial y} = -2y = -2(-3) = 6$ Ans

Advanced Partial Derivative Exercises:

1 $z = y^3 + 5xy$

Treating y as a constant

$$\frac{\partial z}{\partial y} = 3y^2 + 5x(1)$$
$$= 3y^2 + 5x$$

Treating x as a constant

$$\frac{\partial z}{\partial x} = 0 + 5y(1)$$
$$= 5y$$

2 $z = 2\pi r^2 + 2\pi r h$

$$\frac{\partial z}{\partial r} = 4\pi r + 2\pi h$$

[Treating h as constant]

$$\frac{\partial z}{\partial h} = 0 + 2\pi r(1)$$
$$= 2\pi r$$

[Treating r as constant].

(3) $v = x^2y - z^3$

- Treating y and z as constant

$$\frac{\partial v}{\partial x} = 2xy - 0$$
$$= 2xy$$

Treating x and z as constant

$$\frac{\partial v}{\partial y} = x^2$$

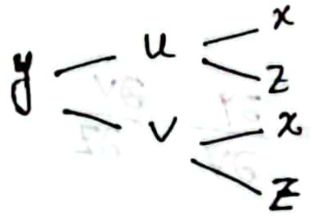
Treating x and y as constant

$$\frac{\partial v}{\partial z} = 0 - 3z^2$$
$$= -3z^2$$

For Partial Derivative Chain Rule:

$$\frac{dy}{dx} = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x}$$

if, $z = f(u,v)$, $u = g(x,z)$, $v = h(x,z)$



$$\therefore \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\therefore \frac{\partial y}{\partial z} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial z}$$

Generalized Formula:

$$z = f(u_1, u_2, u_3, \ldots u_m) \quad u_j = g(x_1, x_2, x_3, \ldots x_n)$$

for, $i = 1, 2, 3, \ldots n;$

$$\frac{\partial z}{\partial x_i} = \frac{\partial z}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_i} + \frac{\partial z}{\partial u_2} \frac{\partial u_2}{\partial x_i} + \cdots + \frac{\partial z}{\partial u_m} \cdot \frac{\partial u_m}{\partial x_i}$$

Problem: Find all the partial derivatives of y, where :

① $y = f(u,v)$, $u = g(x)$, $v = h(z)$

② $y = f(u,v)$, $u = g(x)$, $v = h(x,y)$

③ $y = f(u,v,w)$, $u = g(x)$, $v = h(x)$, $w = J(x)$

Problem 1: $y = f(u,v)$, $u = g(x)$, $v = h(z)$

$$y \begin{cases} u \to x \\ v \to z \end{cases}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} \quad , \quad \frac{\partial y}{\partial z} = \frac{\partial y}{\partial v} \frac{\partial v}{\partial z}$$

Problem 2:

$$y = f(u,v), \quad u = g(x), \quad v = h(x,z)$$

$$\begin{array}{l} y \to u \to x \\ \;\;\, \searrow v \to x \\ \qquad\quad \searrow z \end{array}$$

$$\frac{\partial y}{\partial x} = \left(\frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x}\right) + \left(\frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial x}\right)$$

$$\frac{\partial y}{\partial z} = \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial z}$$

Problem 3:

$$y = f(u,v,w), \quad u = g(x), \quad v = h(x), \quad w = J(x)$$

$$\begin{array}{l} y \to u \to x \\ \;\;\, \searrow v \to x \\ \;\;\, \searrow w \to x \end{array}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial y}{\partial w} \cdot \frac{\partial w}{\partial x}$$

Gradients:

if $f(x,y) = x^2 + y^2$, then $\frac{\partial}{\partial x} f(x,y) = 2x$ and $\frac{\partial}{\partial y} f(x,y) = 2y$

So, the gradiant of $f(x,y)$ is : $\nabla f = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

Problem: Find the gradient of $f(x,y)$ at $(2,3)$

The gradient at $f(2,3)$ is $\rightarrow$ $\Delta f = \begin{bmatrix} 2 \cdot 2 \\ 2 \cdot 3 \end{bmatrix}$

$$= \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

An

~~Find~~

Problem: Minimize the cost functions $\rightarrow$

$$E(m,b) = 14m^2 + 3b^2 + 38 + 12mb - 42m - 20b$$

$$\frac{\partial E}{\partial m} = 28m + 12b - 42$$

$$\frac{\partial E}{\partial b} = 6b + 12m - 20 = 6b - \cancel{8} 12m - 20 \cancel{20}$$

Now, to find the minima/maxima both $\frac{\partial E}{\partial m}$ and $\frac{\partial E}{\partial b}$ has to be 0.

$\therefore \frac{\partial E}{\partial m} = 0$ $\qquad$ $\therefore \frac{\partial E}{\partial b} = 0$

$\therefore 28m + 12b - 42 = 0$ $\qquad$ $\therefore 6b - \cancel{8\,20}\ 12m - 20 = 0$

$$(m,b) = \cancel{\left(\frac{5}{6}, \frac{4}{3}\right)} \left(\frac{1}{2}, \frac{7}{3}\right)$$

An

Optimization using Gradient Descent:

The concept is to take any random value in the graph. Check it's right and left. Where the value is smaller, shift the pointer in that area. There will be a scenario, when the left side value and the right side value both would be greater than the point. So, that can be called the minima or very close to minima.

Algorithm of Gradient Descent: (For 1 variable)

Function: $F(x)$ Goal: find minimum of $f(x)$

Step 1: → Define a learning rate α
→ Choose a starting point x_0

Step 2: Update: $x_K = x_{K-1} - \alpha f''(x_{K-1})$
α → [Learning rate]

Step 3: Repeat Step 2 untill you are close enough to the true minimum

Algorithm of Gradient Descent: (For 2 variables)

Function: $F(x, y)$ Goal: find minimum of $f(x, y)$

Step 1: Define a learning rate α
choose a starting point x_0

Step 2: Update $\begin{bmatrix} x_K \\ y_K \end{bmatrix} = \begin{bmatrix} x_{K-1} \\ y_{K-1} \end{bmatrix} - \alpha \nabla f(x_{K-1}, y_{K-1})$

Step 3: Repeat step 2, until you reach close enough to the true minimum.