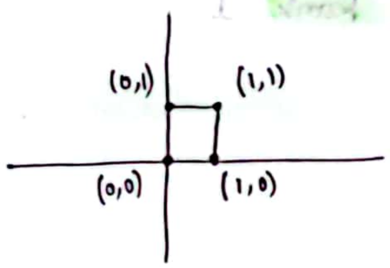
Determinants in Depth

Singularity and rank of linear transformations:

- Linear transformation can also be singular or non singular.

Non singular transformation:

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow$$

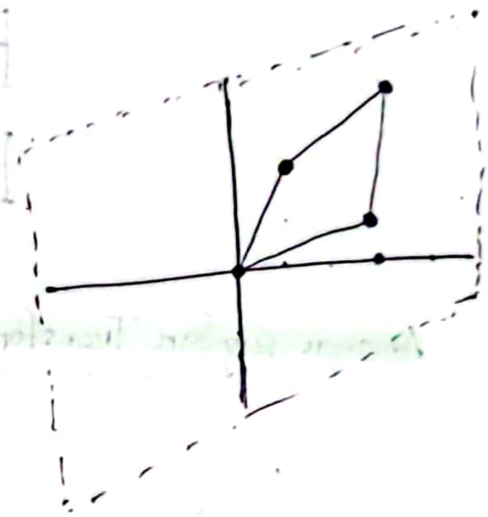


$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (0,0)$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (3,1)$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (1,2)$$

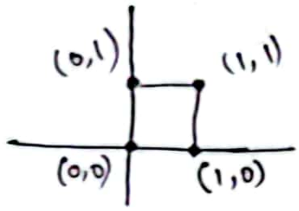
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (4,3)$$



Dimension = 2

Rank = 2

Singular Transformation: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

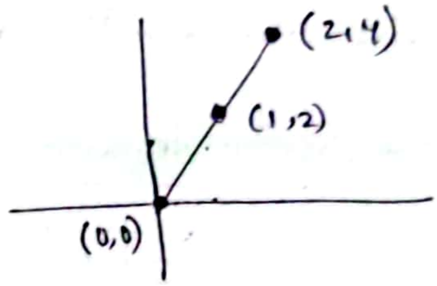


$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (0,0)$

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (1,2)$

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (1,2)$

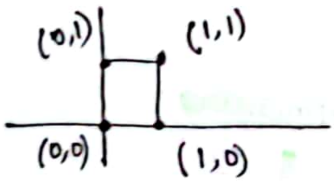
$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (2,4)$.



Dimension: 1D

Rank = 1

Another singular Transformation: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

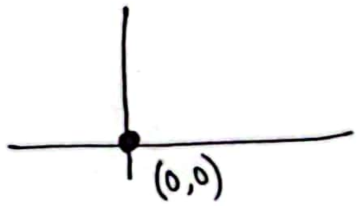


$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = (0,0)$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (0,0)$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (0,0)$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (0,0)$



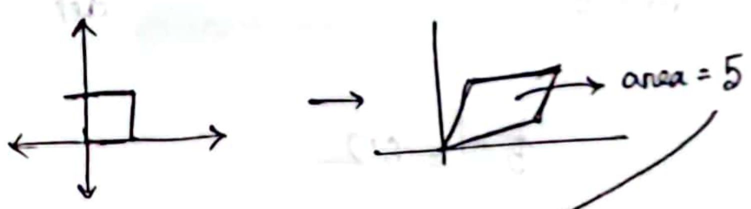
Dimension = 0

Rank = 0

Determinant as an area:

Deteremimant of any matrix is same, as the area of the transformed matrix ~~after~~ from unit matrix

Example → $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$



det = 6 - 1
= 5

Determinant of product of matrix:

Suppos Det of matrix A = a,
Det of matrix B = b

if we multiply 2 matrices A×B, suppose new matrix is C
and it's determinant = c

So, $a \times b = c$

Means the multiplication of the determinants of two matrix A and B will be equal to the determinant of matrix C.

Determinants of the inverses:

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5   det = 0.2

$$5^{-1} = 0.2$$

So, the determinant of the resultant matrix is also the the inverse of the determinant of the main matrix.

Here is the proof,

We know from the determinant of product of Matrices →

$$\det(AB) = \det(A) \times \det(B)$$

$$\Rightarrow \det(AA^{-1}) = \det(A) \times \det(A^{-1})$$

$$\Rightarrow \det(I) = \det(A) \times \det(A^{-1}) \qquad \left[AA^{-1} = I \ (I = \text{Identity Matrix}) \right]$$

Now, $\Rightarrow \det(I) = ~~det~~ 1$ $\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \det = 1 \right]$

$$\therefore \det(A) \times \det(A^{-1}) = 1$$

$$\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$

Bases :

Examples: 

Any form of two vectors called basis. They can reach to any point from the origin (different directions)

Not Bases:

Examples → 

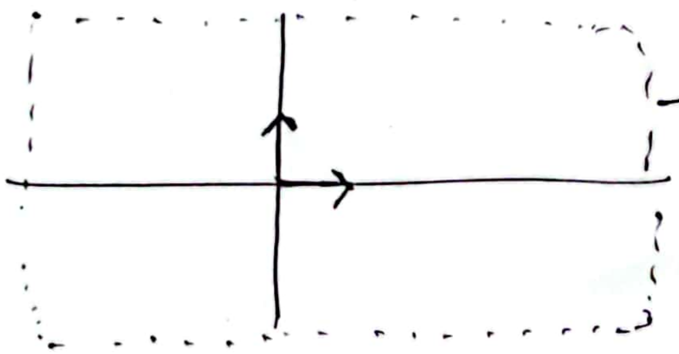
Any form of two vectors (same direction/opossite) is not bases. They can't reach to many particular points from the origin

But a single vector can be a bases →

Example: 

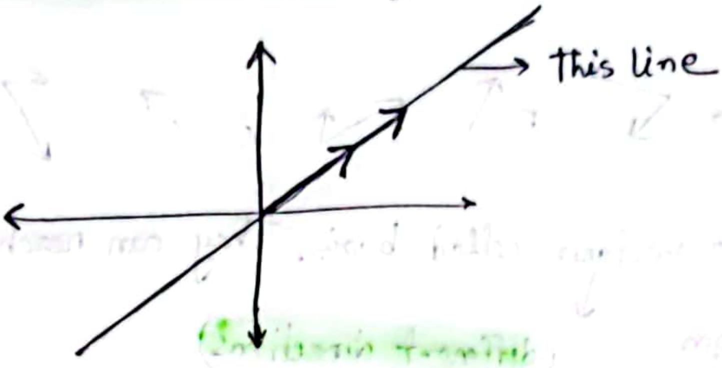
Span :

The span of these two vectors 

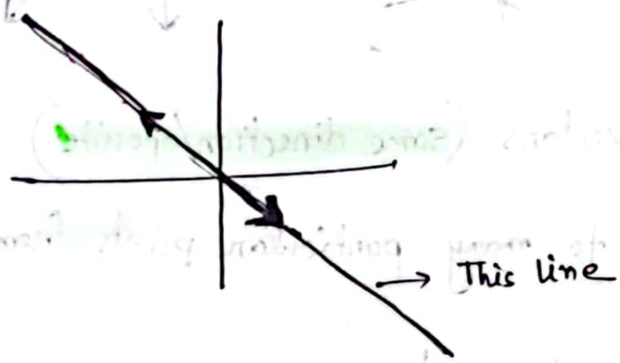


→ This whole area

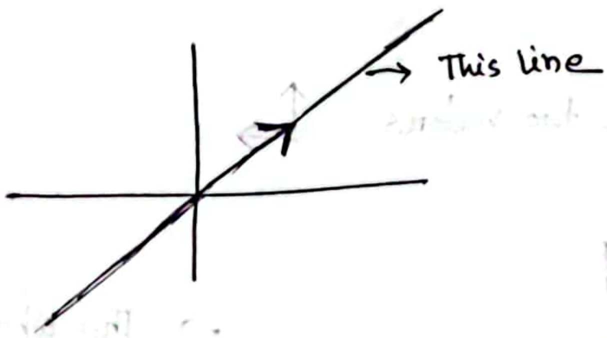
The span of these two vectors →



The span of these two vectors →



Span of a single vector →



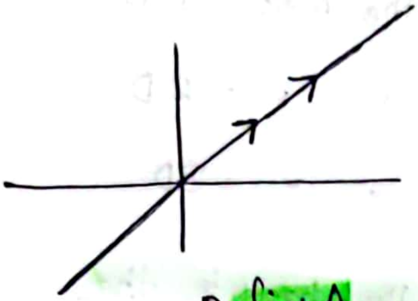


fig: A

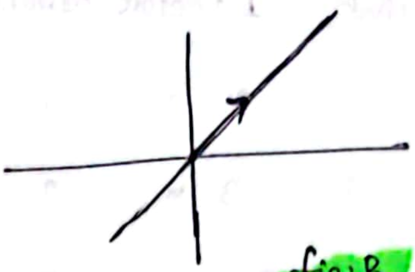


fig: B

Among the above vectors which is a basis?

A basis has to be a minimal spanning set.

In the above context, any single vector can form that span of line.

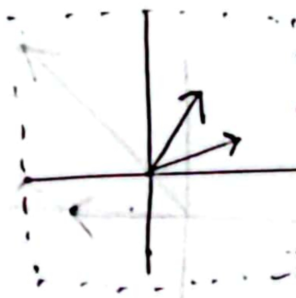
So, the minimum vector needed to create that span is 1.

So, Fig B would be a basis and fig: A would not be a basis.

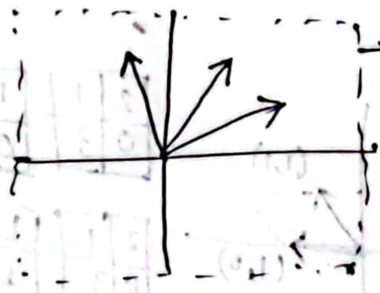
So A basis is a minimal spanning set.

And a spanning set is the minimum numbers of vectors needed to create a span

Another example:



Basis



Not basis

because third vector is redundant

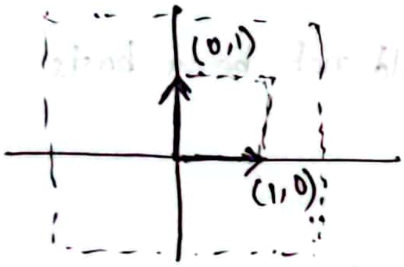
It can be said that, 1 vector element within the basis = 1 Dimension

2 " " " " " = 2D

3 " " " " " = 3D

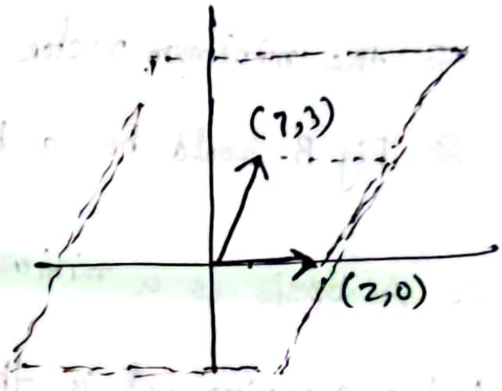
So, Number of vector elements in the basis is the ~~Number of~~ Dimension.

Eigenbases:



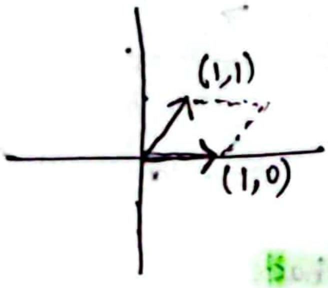
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (1,3)$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (2,0)$$



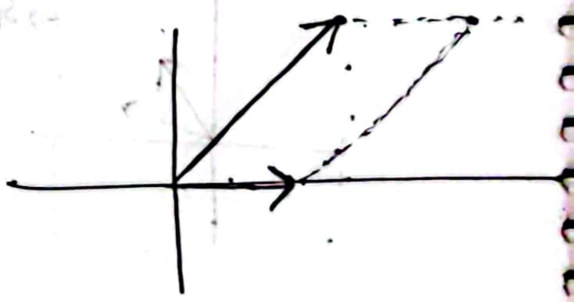
From squared coordintes to parallelogram coordinates.

Another figure

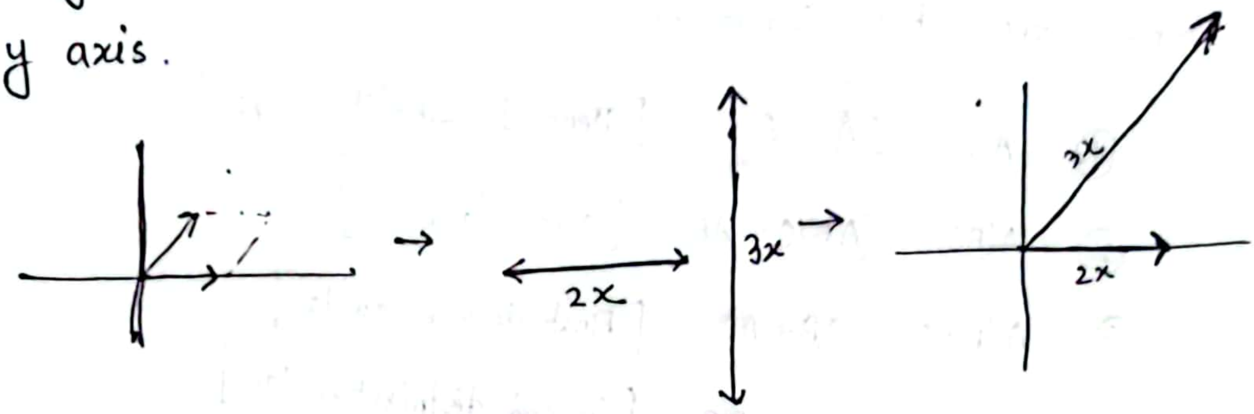


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = (2,0)$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (3,3)$$



By multiplying with another matrix what we are doing is we are stretching the left vector by 2x time in x axis and 3x time in y axis.



Here, two vectors in the basis will be call Eigen vectors.

and the stretching factor 2 and 3 will called eigen values

Eigen values and Eigen vectors:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

determinant,

$$(2-\lambda)(3-\lambda) - 0 = 0$$

$$\Rightarrow 6 - 2\lambda - 3\lambda + \lambda^2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

$$\therefore \lambda = (2,3)$$ → These are eigen values

Let's they the eigen values to solve the equations.

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x + \cancel{0}y \\ 0x + 3y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

⇒

$2x + \cancel{0}y = 2x \rightarrow$ (I)

$3y = 2y \rightarrow$ (II)

$2x + y = 2x$

$x + \frac{y}{2} = x$

$2x + y = 2x$

$\Rightarrow y = 0 \rightarrow$ (III)

$y = 0$, on (I) →

$2x = 2x$

⇒ x can be any real number

let $x = 1$

So, the eigen vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Now, for $\lambda_2 = 3$,

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x+y \\ 3y \end{bmatrix} = \begin{bmatrix} 3x \\ 3y \end{bmatrix}$$

$\therefore \quad 2x + y = 3x \longrightarrow (i)$

$\qquad 3y = 3y \longrightarrow (ii)$

from, (i) $\rightarrow$ $2x - 3x + y = 0$

$\Rightarrow -x = -y$

$\Rightarrow x = y$

from, (ii) $\rightarrow$ $3y = 3y$

$\Rightarrow y = y$

$\Rightarrow$ y can be any real number, let's take $y = 1$

$\therefore x = 1,$

the second eigen vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ans