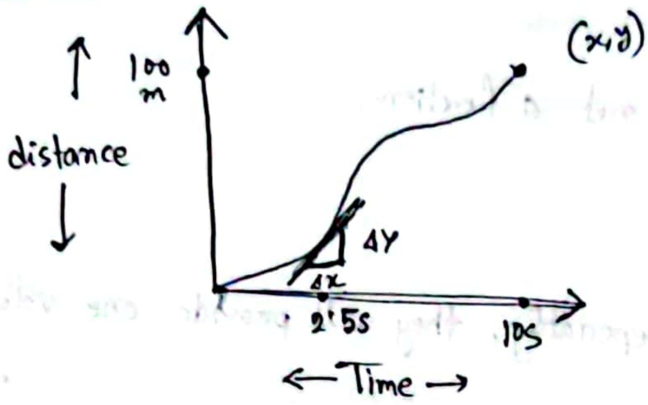
Math Connection (Calculus)

Definition:



$$\lim_{x\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$ (Differential calculus)

So, here for a particular time period, the study of (change in Y axis/change in x axis) is called differential calculus. Like Here we calculating for 2.5s (during that time) what was the distance covered.

Function:

$x \rightarrow \boxed{f} \rightarrow f(x) \Rightarrow y$

input ↲ ↳ output

Function is some expression where each input (x) has exactly one output (y)

→ Is $x^2+y^2=25$ is a function?

$x^2+y^2=25$

$\Rightarrow y^2 = 25-x^2$

$\Rightarrow y = \pm\sqrt{25-x^2}$

at $x=4$, $y=+3$, $y=-3$

As for one input more than one output is coming, so it is not a function.

The previous equation was an equation of circle.

As, $y = \pm\sqrt{25-x^2}$

Gives us two outputs, So It is not a function,

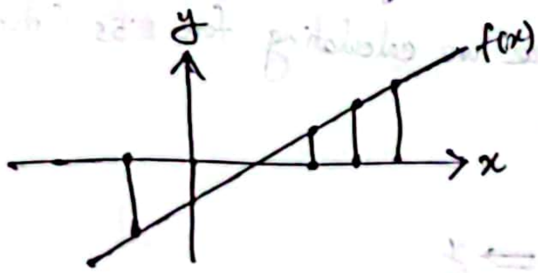
But, if I take $y = +\sqrt{25-x^2}$

or, $y = -\sqrt{25-x^2}$ Seperatly, they will provide one value.

So, individually they are function.

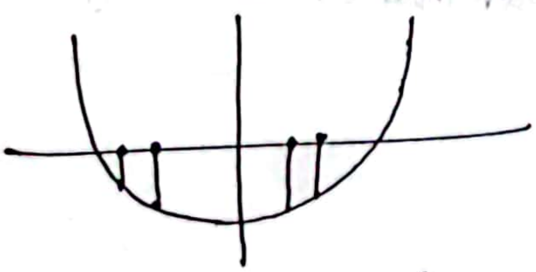
Checking if a graph is representing a function or not?

(i)



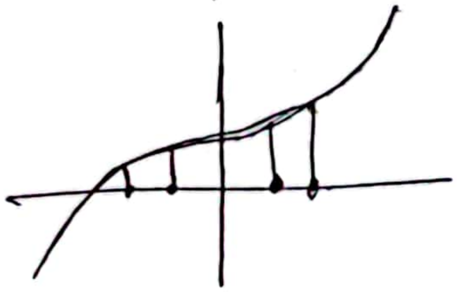
Here for each x value, there is only 1 value in y. So, it is a function

(ii)



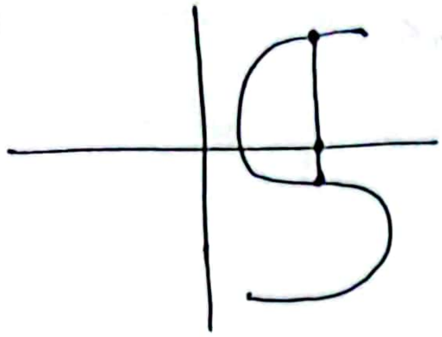
→ This is also a function

(3)



→ This is also a function

(4)



Here, we can see there are more than one output for some value of x. So, it is not a function.

Intervals:

~~$x \to [1,5]$~~
~~$x \to (1,5)$~~
~~$x \to$~~

$x \in [1,5] \Rightarrow x = 1$ to $x = 5$, including 1 and 5 $(1 \leq x \leq 5)$

$x \in (1,5) \Rightarrow x = 1$ to $x = 5$, excluding 1 and 5 $(1 < x < 5)$

$x \in [1,5) \Rightarrow x = 1$ to $x = 5$, including 1 and excluding 5 $(1 \leq x < 5)$

$x \in (1,5] \Rightarrow x = 1$ to $x = 5$, excluding 1 and including 5 $(1 < x \leq 5)$

Domain and Range:

Domain: All input (x) values

Range: All output (y) values

Some domains of some functions:

$f(x) = x^3 \longrightarrow$ Domain: $x \in \mathbb{R}$

$f(x) = \frac{1}{x} \longrightarrow$ Domain: $x \in \mathbb{R}$, except $x = 0$

$f(x) = \sqrt{x} \longrightarrow$ Domain: $x \geq 0$

$f(x) = \frac{1}{(x-1)(x-5)} \longrightarrow$ Domain: $x \in \mathbb{R}$, except $x = 1$ and $x = 5$

Problem: Find the domain of $y = \sqrt{x^2 - 5x + 6}$.

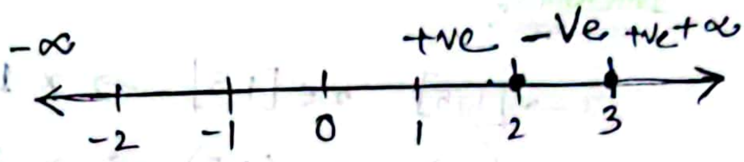
Condition for $\sqrt{x}$ is, $x \geq 0$, So

$$x^2 - 5x + 6 \geq 0$$

$$\Rightarrow x^2 - 3x - 2x + 6 \geq 0$$

$$\Rightarrow x(x-3) - 2(x-3) \geq 0$$

$$\Rightarrow (x-3)(x-2) \geq 0$$



Our point came 2 and 3,

Let's take a value between them and put it on the expression.

Suppose value = 2.5

$$(2.5)^2 - 5(2.5) + 6$$

$$= 6.25 - 12.5 + 6$$

$$\Rightarrow 12.25 - 12.5$$

$$= -ve \text{ (Negative value)}$$

As 2 to 3 there we found -v, So before 2 would be Alternate (+ve) and after 3 will be also alternate (+ve)

So, Domain = $(-\infty, 2] \cup [3, +\infty)$

Ans

For finding domains of a function we have to remember some things →

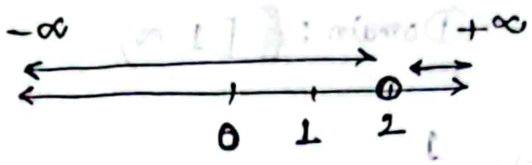
① Denominator can't be zero

② $\sqrt{+ve}$

$F(x) = \dfrac{1}{x-2}$; find the domain

$\dfrac{1}{x-2} \Rightarrow x-2 \neq 0$

$\Rightarrow x \neq 2$



Domain : $[-\infty, 2) \cup (2, \infty)$

Problem: $F(x) = \dfrac{1}{x^2-x-6}$

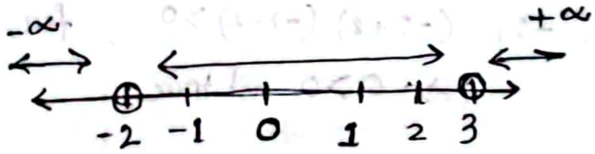
$x^2 - x - 6 \neq 0$

$\Rightarrow x^2 - 3x + 2x - 6 \neq 0$

$\Rightarrow x(x-3) + 2(x-3) \neq 0$

$\Rightarrow (x-3)(x+2) \neq 0$

$\therefore x \neq 3$, $x \neq -2$



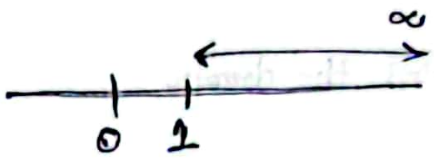
Domain: $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

Problem 3 $f(x) = \dfrac{\sqrt{x-1}}{x^2+4}$

(The denominator will always be positive. So don't need to be checked)

$\therefore x-1 \geq 0$

$\Rightarrow x \geq 1$



Domain : $[1, \infty)$

Problem 4: $f(x) = \dfrac{1}{\sqrt{x^2-4}}$

$\therefore x^2-4 > 0$

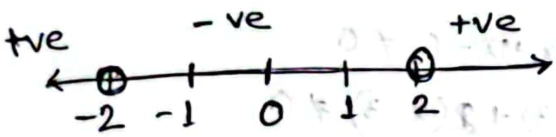
[$x^2-4 \geq 0$, this won't work because, if $x=0$, then it becomes $\dfrac{1}{\sqrt{-4}}$ which is not acceptable to find function Domain]

$\Rightarrow x^2-2^2 > 0$

$\Rightarrow (x+2)(x-2) > 0$

$\therefore$ ~~[illegible]~~

So, it can



for, -2, $(-2+2)(-2-2) > 0$ $\Rightarrow 0 > 0$ not true

for, $2 \rightarrow$ $(2+2)(2-2) > 0$ $\Rightarrow 0 > 0$ not true

So, they won't be included

Let's try a value between them, say $1 \rightarrow$ $(1+2)(1-2) > 0$

$\Rightarrow 3(-1) > 0$

$\Rightarrow -3 > 0$

$\Rightarrow -ve$

Domain: $(-\infty, -2) \cup (2, \infty)$

(Because the domain will be the positive ranged value)

Problem 5: $f(x) = \sin^{-1}(x^2 - 3)$

for, $\sin^{-1}(x) \rightarrow -1 \le x \le 1$

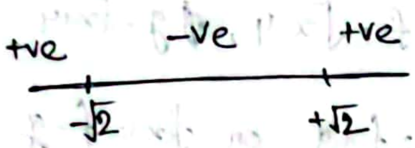
$\therefore -1 \le x^2 - 3 \le 1$

Now, $-1 \le x^2 - 3$

$\Rightarrow x^2 - 3 \ge -1$

~~$\Rightarrow x^2 \ge 2$~~

$\Rightarrow x^2 - 2 \ge 0$



$\rightarrow$ [We want the positive ranges]

Let's put 2 in that, $2^2 - 2 \ge 0$,

$\Rightarrow 2 \ge 0$ (+ve)

As that is (+ve), alternate range will be negative $[-\sqrt{2}, \sqrt{2}]$, then again alternate range will be positive.

Domain : $(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty)$

For the other part,

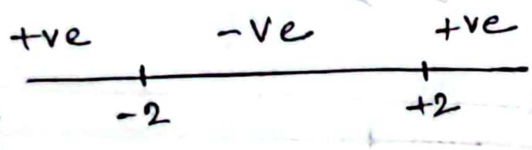
$x^2 - 3 \le 1$

$\Rightarrow x^2 - 4 \le 0$

Let's put 2 in the expression $\rightarrow$

$2^2 - 4 \le 0$

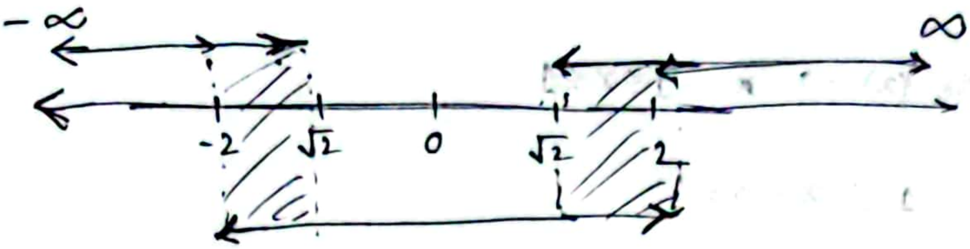
$\Rightarrow 0 \le 0$ (+ve)



[We want the negative range here]

Domain: $[-2, 2]$

Domain Overall:



Domain $[-2, \sqrt{2}] \cup [\sqrt{2}, 2]$

Problem 6: $f(x) = \sqrt{x-4}\,\sqrt{x+4}$

Here we can see 2 functions ; say, $f = \sqrt{x-4}$, & $g = \sqrt{x+4}$

In these case when, $f(x) = f \cdot g$ or $f(x) = f + g$, or $f(x) = f - g$

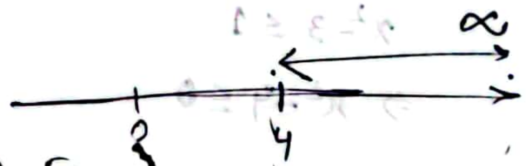
We have find their domain individually and the intersect them to find the real domain.

And where $f(x) = f/g$, in that case the domain calculation will be same but Domain of $g \neq 0$

Now, $f(x) = \sqrt{x-4}\,\sqrt{x+4}$

for, $\sqrt{x-4}$, $x - 4 \geq 0$

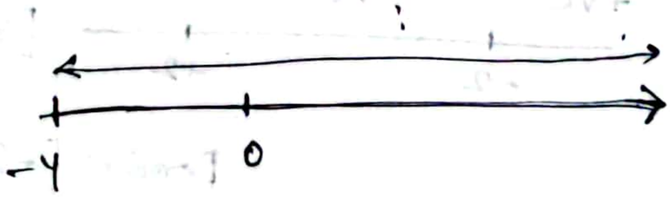
$\Rightarrow x \geq 4$



Domain $[4, \infty)$

for, $\sqrt{x+4}$, $x + 4 \geq 0$

$\Rightarrow x \geq -4$



Domain $[-4, \infty)$

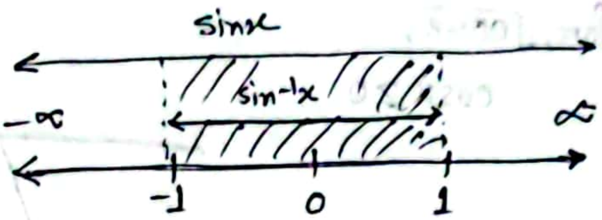
Overall Domain $[4, \infty)$ → The common between them.

Problem 07: $f(x) = \sin x + \sin^{-1} x$

for, $\sin x$, Domain : All real number → $(-\infty, \infty)$

for, $\sin^{-1} x$, Domain : $[-1, 1]$

↳ $-1 \leq x \leq 1$



Their common part → $[-1, 1]$

So, Overall Domain : $[-1, 1]$

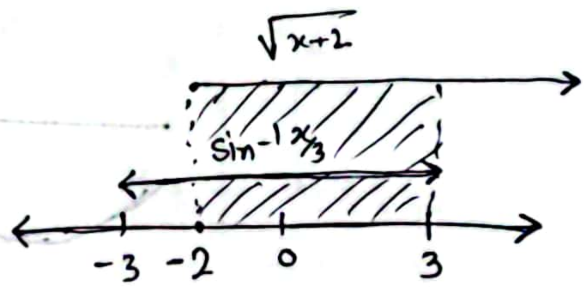
Problem 08: $f(x) = \sin^{-1}\left(\frac{x}{3}\right) + \sqrt{x+2}$

for, $\sin^{-1}\frac{x}{3}$, ~~Domain : [-1~~

↳ $-1 \leq \frac{x}{3} \leq 1$

$\Rightarrow -3 \leq x \leq 3$

Domain : $[-3, 3]$



for, $\sqrt{x+2}$

↳ $x + 2 \geq 0$

$\Rightarrow x \geq -2$, Domain $[-2, \infty)$

There common part = $[-2, 3]$

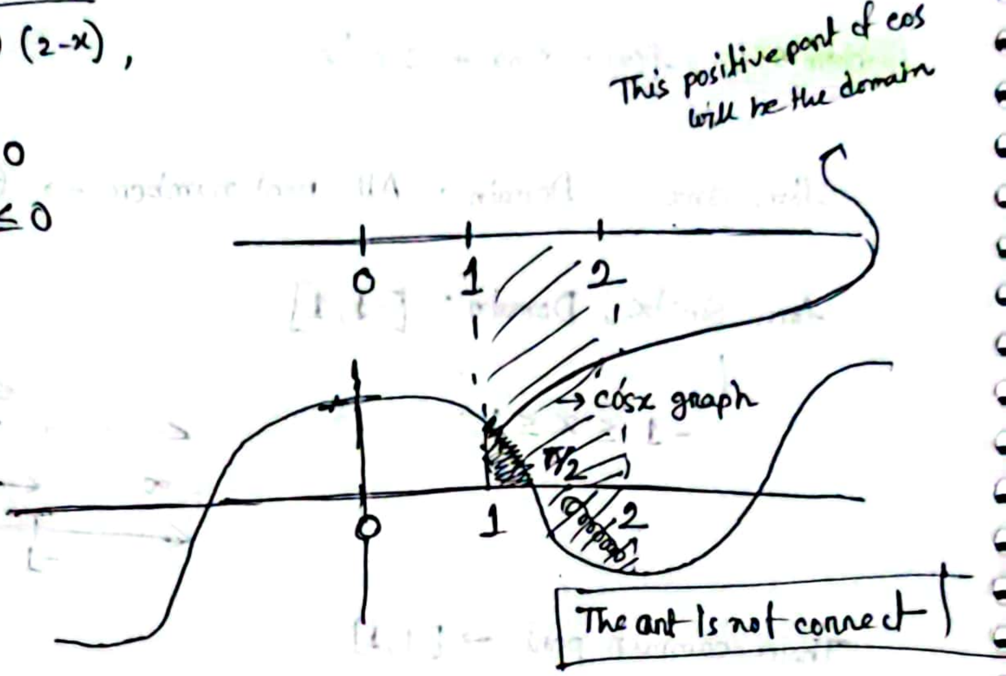
So, overall domain : $[-2, 3]$

Problem 09: $f(x) = \sqrt{\cos x} + \sqrt{(x-1)(2-x)}$

for, $\cancel{\sqrt{\cos x}}$ $\sqrt{(x-1)(2-x)}$,

$(x-1)(2-x) \geq 0$

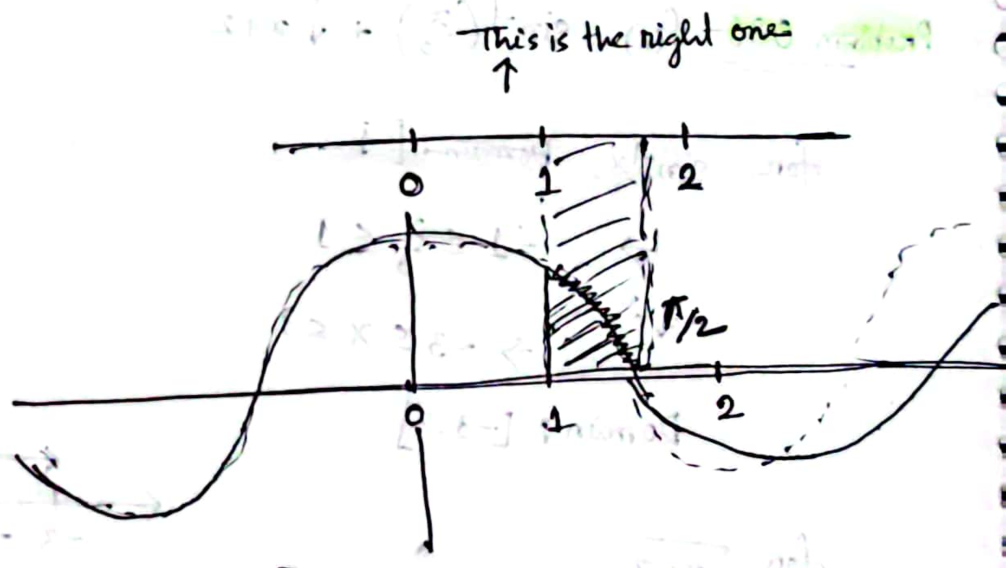
$\Rightarrow (x-1)(x-2) \leq 0$



for, $\sqrt{\cos x}$,

$\cos x \geq 0$

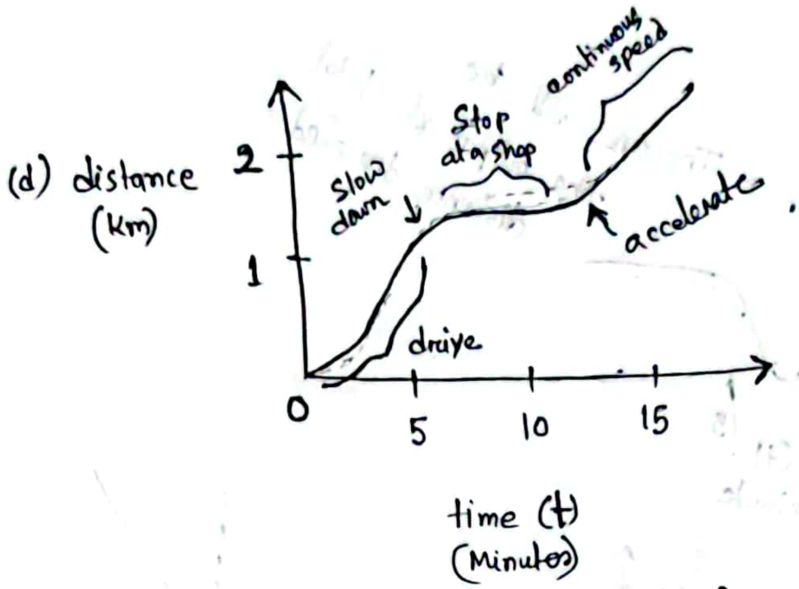
Overall Domain $[1, \pi/2]$

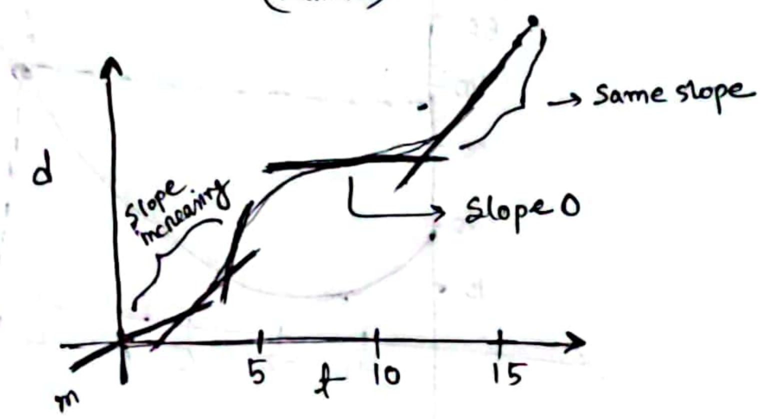


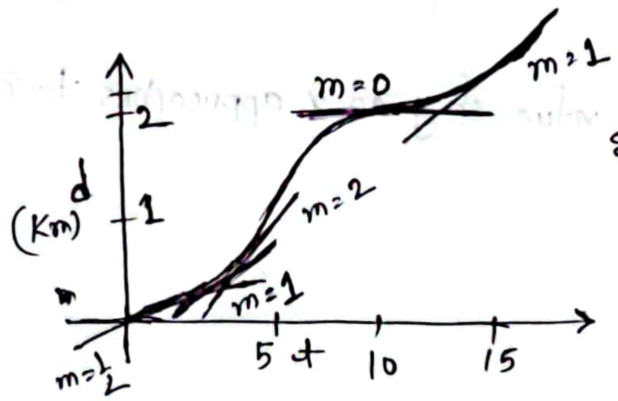
Overall Domain $[1, \pi/2]$

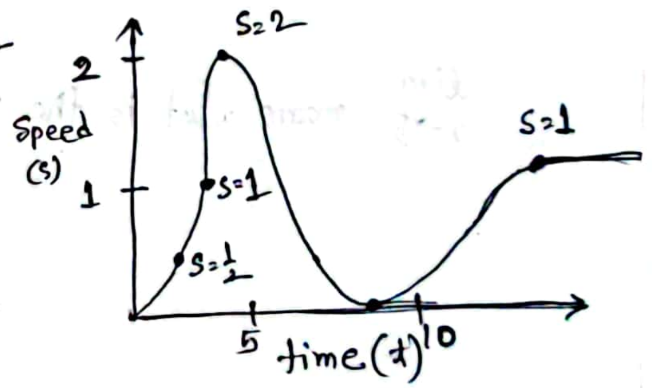
Differential Calculus:

- Calculus is the study of Rate of change



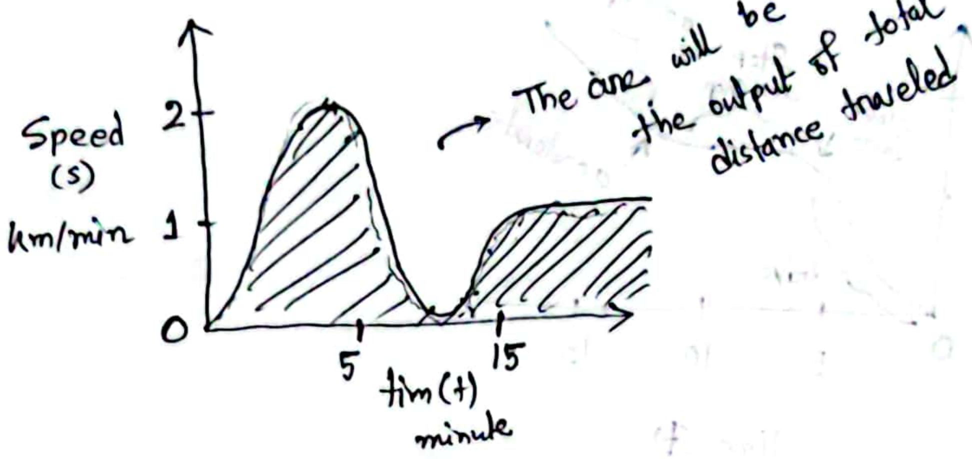




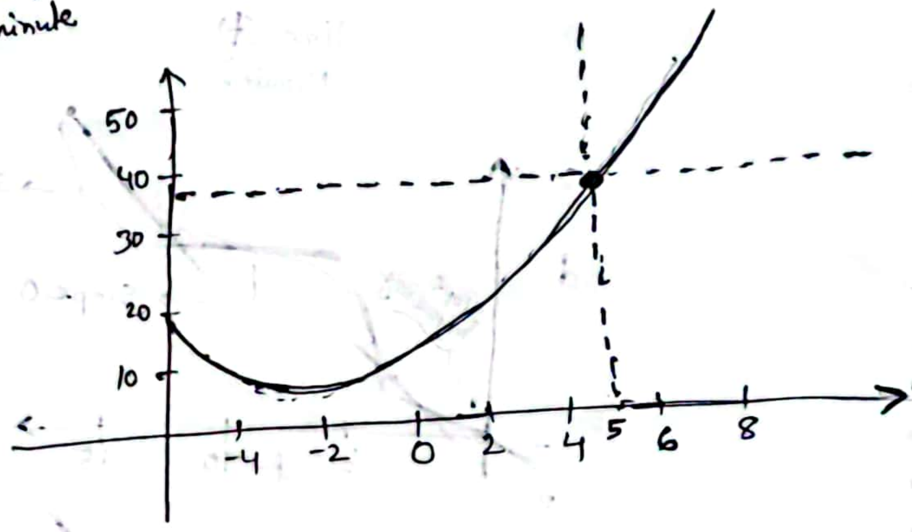


Integral Calculus:

→ The study of area under the curve.



Limits:



$$\lim_{x \to 5} x^2+2x+2$$

$\Rightarrow 5^2+2\times5+2$

$\Rightarrow 25+10+2$

$\Rightarrow 37$

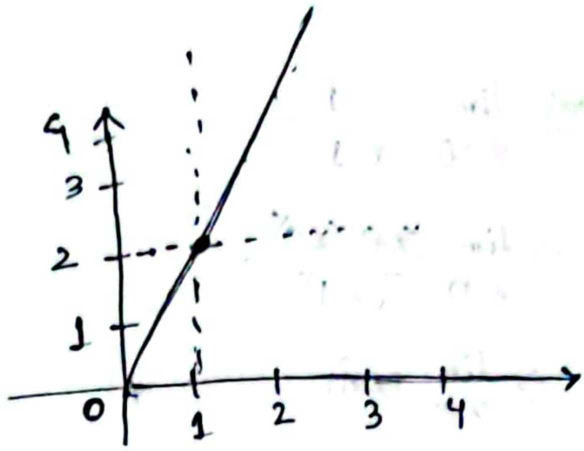
$\lim_{x \to 5}$ means what is the value of y as x approaches to 5.

Problem: $\lim_{x \to 1} \frac{x^2-1}{x-1}$

$\Rightarrow \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)}$

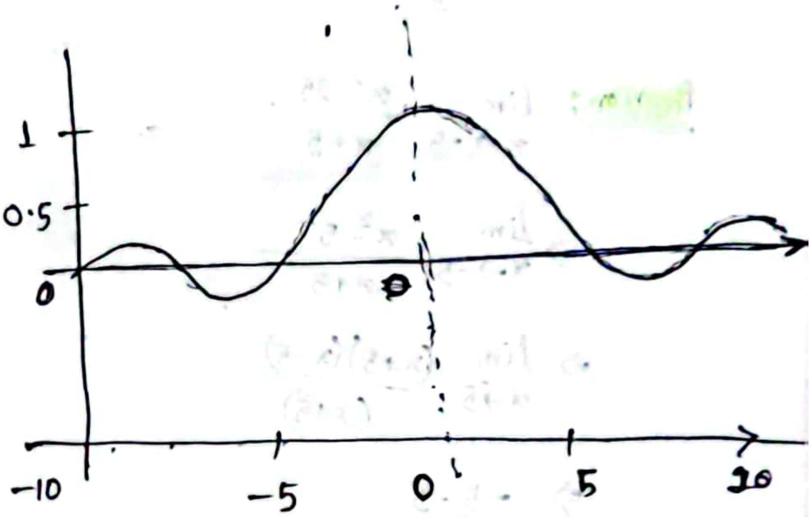
$\Rightarrow \lim_{x \to 1} (x+1)$

$\Rightarrow 2$



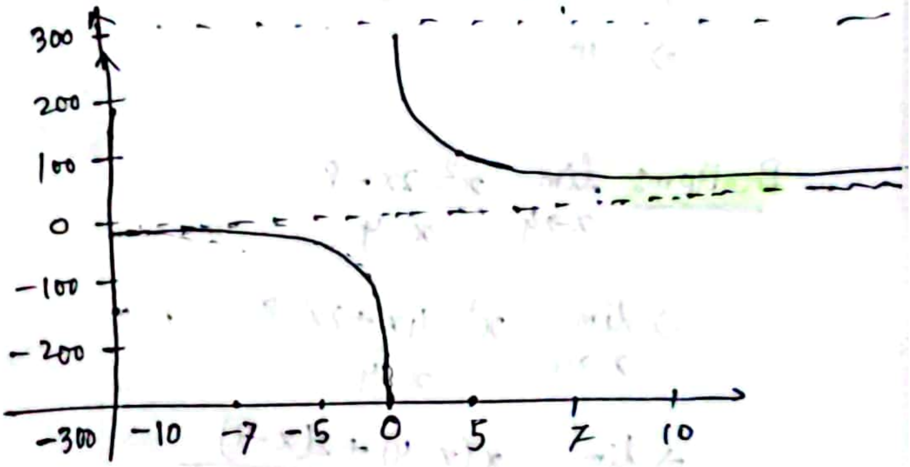
Problem: $\text{limit}_{x \to 0} \frac{\sin x}{x}$

$= 1$



Problem: $\lim_{x \to \infty} \frac{25}{x}$

$= 0$



Problem: $\lim_{x\to 0} \frac{x^2-1}{x-1}$

$\Rightarrow \lim_{x\to 0} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$

$\Rightarrow \lim_{x\to 0} (x+1)$

$\Rightarrow 1$

An

Problem: $\lim_{x\to -5} \frac{x^2-25}{x+5}$

$\Rightarrow \lim_{x\to -5} \frac{x^2-5^2}{x+5}$

$\Rightarrow \lim_{x\to -5} \frac{\cancel{(x+5)}(x-5)}{\cancel{(x+5)}}$

$\Rightarrow -5-5$

$\Rightarrow -10$

Problem: $\lim_{x\to 4} \frac{x^2-2x-8}{x-4}$

$\Rightarrow \lim_{x\to 4} \frac{x^2-4x+2x-8}{x-4}$

$\Rightarrow \lim_{x\to 4} \frac{x(x-4)+2(x-4)}{(x-4)}$

$\Rightarrow \lim_{x\to 4} \frac{\cancel{(x-4)}(x+2)}{\cancel{(x-4)}}$

$\Rightarrow \lim_{x\to 4} (x+2)$

$\Rightarrow 4+2$

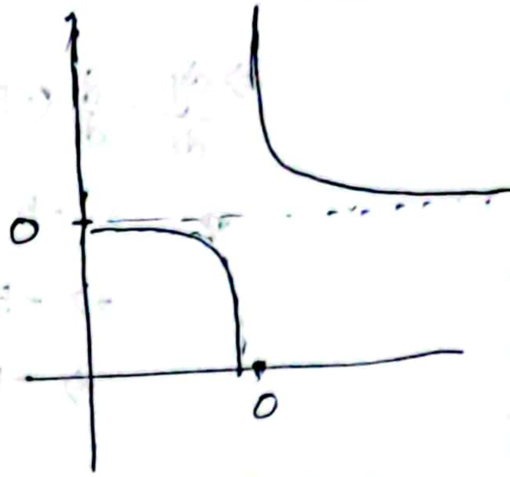
$\Rightarrow 6$

Problem: $\lim_{x\to\infty} \frac{25}{x}$

$= 0$

Problem: $\lim_{x\to 0} \frac{25}{x}$

$f(x) = \frac{25}{x}$



$\lim_{x\to 0^+} f(x) = \infty$ (Right-hand limit)

$\text{limit}_{x\to 0^-} f(x) = -\infty$ (left-handed limit)

Most common differentiation equation: $\frac{dy}{dx} = \lim_{\Delta x\to 0} \cancel{f(x)+\delta} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Derivative Rules (Formulas):

① $\frac{d}{dx} c = 0$ [c = constant]

The sum Rule: $\frac{d}{dx}(u+v) = \frac{d}{dx}u + \frac{d}{dx}v$

Problem: $\frac{d}{dx}(x^4+x^9)$

$\Rightarrow \frac{d}{dx}(x^4) + \frac{d}{dx}(x^9)$

$\Rightarrow 4x^3 + 9x^8$

Problem: $y = -5x^3$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(-5x^3)$

$= -5 \times \frac{d}{dx}(x^3)$

$\Rightarrow -5 \times 3x^2$

$\Rightarrow -15x^2$

Ans

Problem: $y = 2x^2 + 2x + 2$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(2x^2 + 2x + 2)$

$= \frac{d}{dx}(2x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}(2)$

$= 2 \cdot 2x + 2 + 0$

$= 4x + 2$

$\cancel{= 4(x+1)}$

Ans

Problem: $y = 10x^5 - 6x^3 - x - 1$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(10x^5 - 6x^3 - x - 1)$

$= \frac{d}{dx}(10x^5) - \frac{d}{dx}(6x^3) - \frac{d}{dx}(x) - \frac{d}{dx}(1)$

$= 10 \cdot 5x^4 - 6 \cdot 3x^2 - 1 - 0$

$= 50x^4 - 18x^2 - 1$

Ans

Problem: Find the slope of the curve $y = x^2 + 2x + 2$ where $x = 2$, and $x = -1$

$y = x^2 + 2x + 2$

$\Rightarrow \frac{d}{dx} y = 2x + 2$

for, $x = 2$, slope $(m_1) = 2 \times 2 + 2 = 6$

for, $x = -1$, slope $(m_2) = 2(-1) + 2 = 0$

Product Rule: $y = (6x^3)(7x^4)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(6x^3)(7x^4)$$

$$= 6x^3 \frac{d}{dx}(7x^4) + 7x^4 \frac{d}{dx}(6x^3)$$

$$= 6x^3 \cdot 28x^3 + 7x^4 \cdot 18x^2$$

$$= 168x^6 + 126x^6$$

$$= 294x^6$$

Ans

The Quotient Rule: $\frac{d}{dx}(u/v) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$

Problem: $y = \dfrac{4x^2}{x^3+1}$

$$\Rightarrow \frac{dy}{dx} = \left\{(x^3+1)\frac{d}{dx}(4x^2) - 4x^2\frac{d}{dx}(x^3+1)\right\} / (x^3+1)^2$$

$$= \left\{(x^3+1)\cdot 8x - 4x^2 \cdot 3x^2\right\} / (x^3+1)^2$$

$$= \left\{8x^4 + 8x - 12x^4\right\} / (x^3+1)^2$$

$$= (-4x^4 + 8x) / (x^3+1)^2$$

The chain Rule: When y is a function of u and u is a function of x.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Problem: $y = (2x^2+8)^2$

$$\frac{dy}{du} = u^2 \qquad\qquad u = 2x^2+8$$

$$\Rightarrow \quad = 2u \qquad\qquad \Rightarrow \frac{du}{dx} = 4x$$

$$= 2(2x^2+8)$$

$$= 4x^2+16$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= (4x^2+16)\cdot(4x)$$

$$= 16x^3 + 64x$$

Problem: $y = (2x^2+6x)(2x^3+5x^2)$

$$\Rightarrow \frac{dy}{dx} = (2x^2+6x)\frac{d}{dx}(2x^3+5x^2) + (2x^3+5x^2)\frac{d}{dx}(2x^2+6x)$$

$$= (2x^2+6x)\cdot(6x^2+10x) + (2x^3+5x^2)(4x+6)$$

$$= (12x^4+20x^3+36x^3+60x^2) + (4x^4+12x^3+20x^3+30x^2)$$

$$= 12x^4+56x^3+60x^2+4x^4+32x^3+30x^2$$

$$= 16x^4+88x^3+90x^2$$

Problem: $y = \dfrac{6x^2}{2-x}$

$$\Rightarrow \frac{dy}{dx} = \frac{(2-x)\frac{d}{dx}(6x^2) - \cancel{(2-x)}\frac{d}{dx}\overset{6x^2}{\cancel{(6x^2)}}(2-x)}{(2-x)^2}$$

$$= \frac{(2-x)\cdot 12x - \cancel{(2-x)}\,6x^2(-1)}{(2-x)^2}$$

$$= \frac{24x-12x^2+6x^2}{(2-x)^2}$$

$$= \frac{-6x^2+24x}{(2-x)^2}$$

Problem: $y = (3x+1)^2$ | let say, $u = 3x+1$

$\Rightarrow \frac{dy}{du} = u^2$

$= 2u$

$= 2(3x+1)$

$= 6x+2$

$u = 3x+1$

$\Rightarrow \frac{du}{dx} = \frac{d}{dx}(3x+1)$

$= 3$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= (6x+2) \cdot 3$

$= 18x+6$ Ans

Problem: $y = (x^2+5x)^6$ | let say, $u = x^2+5x$

$\Rightarrow \frac{dy}{du} = u^6$

$= 6u^5$

$= 6(x^2+5x)^5$

$= 6x^7 + 30x^5$

$u = x^2+5x$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x^2+5x)$

$= 2x+5$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$= (6x^7 + 30x^5) \cdot (2x+5)$

$= 12x^8 + 30x^7 + 60x^6 + 150x^5$

Problem: $y = \cancel{(x^2+5x)} \dfrac{1}{(x^4+1)^5+7}$

let, say, $u = (x^4+1)^5+7$ and $t = x^4+1$

$= t^5+7$

$$\frac{dy}{du} = \frac{1}{u}$$
$$= u^{-1}$$
$$= -u^{-2}$$
$$= -(t^5+7)^{-2}$$
$$= -\left((x^4+1)^5+7\right)^{-2}$$

$u = t^5+7$

$\Rightarrow \dfrac{du}{dt} = 5t^4 \cancel{+7}$

$\cancel{= 5(x^4+1)}$

$= 5t^4 \cancel{+0}$

$= 5(x^4+1)^4$

$t = x^4+1$

$\Rightarrow \dfrac{dt}{dx} = 4x^3$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dt} \cdot \frac{dt}{dx}$$
$$= -\left((x^4+1)^5+7\right)^{-2} \cdot 5(x^4+1)^4 \cdot 4x^3$$
$$= \frac{20(x^4+1)^4 x^3}{-\left((x^4+1)^5+7\right)}$$

Ans

Power rule on a function chain: $\dfrac{d}{dx} u^n = n u^{n-1} \cdot \dfrac{du}{dx}$

Problem: $y = (3x+1)^2$ | Here, $n = 2$, $u = 3x+1$

$$\frac{dy}{dx} = 2(3x+1) \cdot \frac{d}{dx}(3x+1)$$
$$= (6x+2) \cdot 3$$
$$= 18x+6$$

Ans