Solving Equations using Lineare Elimination Method: (2 equations)

$5a + b = 17 \longrightarrow$ (i)

$4a - 3b = 6 \longrightarrow$ (ii)

Making (i)$\times 3 \to$

~~5a + b =~~ $15a + \cancel{3b} = 51$

$4a - \cancel{3b} = 6$

$19a = 57$

$a = 3 \longrightarrow$ (iii)

Now, $4a - 3b = 6$

$\Rightarrow 12 - 3b = 6$

$\Rightarrow 3b = 6$

$\therefore b = 2$

$\therefore (a, b) = (3, 2)$

(2)

$2a + 5b = 46 \rightarrow$ (i)

$8a + b = 32 \rightarrow$ (ii)

(ii)$\times 5 \rightarrow$ $40a + 5b = 160 \rightarrow$ (iii)

(iii) $-$ (i) $\rightarrow$ $40a + \cancel{5b} = \cancel{32}\ 160$

~~8a +~~ $2a + \cancel{5b} = 46$

$38a = 114$

$a = \frac{114}{38} = 3$

$8a + b = 32$

$24 + b = 32$

$\therefore b = 8$

$\therefore (a, b) = (3, 8)$

Solving equations using elemination method:

$a + b + 2c = 12 \rightarrow (I)$

$3a - 3b - c = 3 \rightarrow (II)$

$2a - b + 6c = 24 \rightarrow (III)$

$(III) - 2 \times (I) \rightarrow$

$$\begin{array}{r} \cancel{2a} - b + 6c = 24 \\ \cancel{2a} + 2b + 4c = 24 \\ (-) \quad (-) \quad (-) \quad (-) \\ \hline -3b + 2c = 0 \end{array} \rightarrow (IV)$$

$(II) - 3 \times (I) \rightarrow$

$$\begin{array}{r} \cancel{3a} - 3b - c = 3 \\ \cancel{3a} + 3b + 6c = 36 \\ (-) \quad (-) \quad (-) \quad (-) \\ \hline -6b - 7c = -33 \end{array} \rightarrow (V)$$

$2 \times (IV) - (V) \rightarrow$

$$\begin{array}{r} -\cancel{6b} + 4c = 0 \\ -\cancel{6b} - 7c = -33 \\ (+) \quad (+) \quad (+) \\ \hline 11c = 33 \end{array}$$

$\therefore c = 3$

Now, $-3b + 2c = 0$

$\Rightarrow -3b = -6$

$\therefore b = 2$

From, $(I) \rightarrow$

$a + 2 + 6 = 12$

$\Rightarrow a = 12 - 8$

$\Rightarrow a = 4$

$\therefore (a, b, c) = (4, 2, 3)$

An

Matrix Row Reduction → (Gaussian Elemination)

Another method to solve 3 variable eqn :

$a+b+2c=12 \rightarrow$ (I)

$3a-3b-c=3 \rightarrow$ (II)

$2a-b+6c=24 \rightarrow$ (III)

Rules:

(1) Only first eqn will contain a variable. Remove a from others.

→ Devide each row with the coefficient a

$a+b+2c=12 \longrightarrow$ (IV)

$a-b-\frac{c}{3}=1 \longrightarrow$ (V)

$a-\frac{b}{2}+3c=12 \longrightarrow$ (VI)

Now use (IV) eqn to remove a from the other eqntions

(V) − (IV) →

$$\cancel{a}-b-\frac{c}{3}=1$$

$$(-)\Rightarrow \cancel{a}\underset{(-)}{+}b\underset{(-)}{+}2c\underset{(-)}{=}12$$

$$-2b-\left(\frac{c}{3}+2c\right)=-11$$

$$\Rightarrow -2b-\frac{7c}{3}=-11$$

$$\Rightarrow 2b+\frac{7c}{3}=11 \rightarrow \text{(VII)}$$

VI − IV →

$$a - \frac{b}{2} + 3c = 12$$
$$\underset{(-)}{a} \underset{(-)}{+} \underset{(-)}{b} + 2c \underset{(-)}{=} 12$$

$$-\left(\frac{b}{2} + b\right) + c = 0$$
$$\Rightarrow -\frac{3b}{2} + c = 0 \longrightarrow \text{VIII}$$

So, the equations →

$$a + b + 2c = 12 \rightarrow \text{IV}$$
$$2b + \frac{7c}{3} = 11 \rightarrow \text{VII}$$
$$-\frac{3b}{2} + c = 0 \rightarrow \text{VIII}$$

Now again from VII and VIII remove coefficient of b.

$$a + b + 2c = 12 \rightarrow \text{IV}$$
$$b + \frac{7c}{6} = \frac{11}{2} \rightarrow \text{IX}$$
$$b - \frac{2c}{3} = 0 \rightarrow \text{X}$$

IX + X →

$$\cancel{b} + \frac{7c}{6} = \frac{11}{2}$$
$$-\cancel{b} - \frac{2c}{3} = 0$$

$$\frac{7c}{6} - \frac{2c}{3} = \frac{11}{2}$$
$$\Rightarrow \frac{7c - 4c}{6} = \frac{11}{2}$$
$$\Rightarrow \frac{3c}{2} = \frac{11}{2} \Rightarrow c = 11$$

IX − X →

$$\cancel{b} + \frac{7c}{6} = \frac{11}{2}$$
$$\underset{(-)}{b} \underset{(+)}{-} \frac{2c}{3} \underset{(-)}{=} 0$$

$$\frac{7c}{6} + \frac{2c}{3} = 11/2$$
$$\Rightarrow \frac{7c + 4c}{6} = 11/2$$
$$\Rightarrow \frac{11c}{6} = \frac{11}{2}$$
$$\therefore c = 3$$

Now, again eqn remaining.

$$a + b + 2c = 12 \rightarrow (IV)$$

$$b + \frac{7c}{6} = \frac{11}{2} \rightarrow (IX)$$

$$c = 3$$

From, (IX) $\rightarrow$ Replace ~~value~~ c with it's value

$$b + \frac{7 \times 3}{6} = \frac{11}{2}$$

$$\Rightarrow b + \frac{21}{6} = \frac{11}{2}$$

$$\Rightarrow b = \frac{11}{2} - \frac{21}{6}$$

$$\Rightarrow b = \frac{33 - 21}{6}$$

$$\Rightarrow b = 2$$

Now, from (IV) replace b and c with values

$$a + 2 + 6 = 12$$

$$\Rightarrow a = 4$$

The soln $\rightarrow$ $(a, b, c) = (4, 2, 3)$

Original system

$5a + b = 17 \rightarrow (I)$

$4a + 3b = 6 \rightarrow (II)$

Devide each row with their coffitient of a

$a + \frac{b}{5} = \frac{17}{5} \rightarrow (III)$

$a + \frac{3b}{4} = \frac{3}{2} \rightarrow (IV)$

$(IV) - (III) \rightarrow$

$$\cancel{a} + \frac{3b}{4} = \frac{3}{2}$$

$$\cancel{a} + \frac{b}{5} = \frac{17}{5}$$

$$(-) \quad (-) \quad (-)$$

$$\left(\frac{3b}{4} - \frac{b}{5}\right) = \frac{3}{2} - \frac{17}{5}$$

$$\Rightarrow -\frac{19b}{20} = \frac{15-34}{10}$$

$$\Rightarrow -\frac{19}{20}b = \frac{-19}{10}$$

$$\Rightarrow b = 2$$

So,

Intermediate System

~~5a + b = 17~~

~~b = 2~~

$a + \frac{b}{5} = \frac{17}{5}$

$\Rightarrow a + 0.2b = 3.4 \rightarrow (III)$

$b = 2$

if we put value of b in (III)

$$a + \frac{2}{5} = \frac{17}{5}$$

$$\Rightarrow \frac{5a+2}{5} = 17$$

$$\Rightarrow a = 3$$

Solved System

$a = 3$

$b = 2$

Now, if we potray the three system into matrix

Original system

5	1
4	-3

$$\begin{pmatrix} 5a + b \\ 4a - 3b \end{pmatrix}$$

Intermediate System

1	0·2
0	1

$$\begin{pmatrix} a + \frac{1}{5} b \\ b \end{pmatrix}$$

[Row echelon Form]

Solved system

1	0
0	1

$$\begin{pmatrix} a \\ b \end{pmatrix}$$

[Reduced Row echelon form]

[This above eqn on matrix was for non singular]

Now let us check for singular system:

Original System

$a + b = 10 \rightarrow$ (i)

$2a + 2b = 20 \rightarrow$ (ii)

(i) $\times 2$ − (ii) $\rightarrow$

$$\begin{array}{r} 2a + 2b = 20 \\ \underset{(-)}{} 2a \underset{(-)}{+} 2b \underset{(-)}{=} 20 \\ \hline 0 + 0 = 0 \end{array}$$

$\Rightarrow 0a + 0b = 0$

Intermediate system

$a + b = 10 \rightarrow$ (i)

$0a + 0b = 0$

Original Matrix

1	1
2	2

Upper diagonal matrix (Intermediate)

1	1
0	0

Another singular system:

Original System

$5a + b = 11 \rightarrow (\text{I})$

$10a + 2b = 22 \rightarrow (\text{II})$

$2 \times (\text{I}) - (\text{II}) \rightarrow$

$$\begin{array}{r} 10a + 2b = 22 \\ \underset{(-)}{10a} \underset{(-)}{+ 2b} = \underset{(-)}{22} \\ \hline 0a + 0b = 0 \end{array}$$

Intermediate system

$5a + b = 11$

$\Rightarrow a + \frac{b}{5} = \frac{11}{5}$

$0a + 0b = 0$

Original Matrix

5	1
10	2

Upper diagonal Matrix
(Intermediate system)

1	0.2
0	0

↓

Row echelon form

General Row echelon form in Matrix:

1	*	*	*	*
0	1	*	*	*
0	0	0	0	0
0	0	0	0	0
0	0	0	0	0

$* \rightarrow$ Non zero values

↳ can be put right after 1 only

↳ Everything left of 1 is 0

↳ Everything left of 0 is 0

↳ Everything right of 0 is 0.

For 2x2 Matrices these echelon forms can happen

$$\begin{bmatrix}1 & * \\ 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & * \\ 0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 0 \\ 0 & 0\end{bmatrix}$$

For 3x3 Matrices →

$$\begin{bmatrix}1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1\end{bmatrix} \quad \begin{bmatrix}1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix} \quad \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$$

Row operations in matrix preserve singularity:

Means, if we perform different row operations like
- switching rows
- Multiplying rows with scalar

The singularity will remain unchanged

Suppose, $\begin{bmatrix}5 & 1 \\ 4 & 3\end{bmatrix}$ det = 15 − 4 = 11 (singular)

switch rows →

$\begin{bmatrix}4 & 3 \\ 5 & 1\end{bmatrix}$ det = 4 − 15 = −11 (singular)

Then,

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \xrightarrow{\text{1st row} \times 10} \begin{bmatrix} 50 & 10 \\ 4 & 3 \end{bmatrix} \rightarrow$$

det = 11

(singular)

det = 150 − 40

= 110

= 10 × 11

(singular)

So, row operations preserve singularity.

Anothe row operation example→

$$\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \begin{array}{l} \rightarrow \text{i} \\ \rightarrow \\ \rightarrow \text{ii} \end{array}$$

det = 11 (singular)

Sum row1 + row2, then calculate det with 1st row

$$\begin{bmatrix} 5 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ 9 & 4 \end{bmatrix} \text{(i+ii)}$$

det = 20 − 9

= 11

(Singular).

The rank of a matrix:

System 1

$a + b = 0 \longrightarrow$ (I)
$a + 2b = 0 \longrightarrow$ (II)

Matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

↳ Rank 2

· 2 equations
and provide 2 informations

So, Rank = 2

System 2

$a + b = 0$
$2a + 2b = 0$

2 equations
But provide only one information because we can get equation 2 from equation 1.

So, Rank = 1

Matrix

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \rightarrow \text{Rank } 1$$

System 3

$0a + 0b = 0$
$0a + 0b = 0$

2 equations
But provide no information,

So, Rank = 0

Matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \text{Rank } 0$$

Anothe formula for Rank is →

Rank = 2 − (Dimension of solution space) ↖ [For 2D Matrix]

How do we define singularity with Rank:

A matrix is non singular if it is fully ranked.

Like, For 2D Matrix Full rank value = 2

For, 3D " " " " = 3

So, if the ~~ra~~ rank of a 2D matrix is 2, then the matrix is non singular. Else it is singular.

Find the Rank of below Matrix:

$$\begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$$

As the above matrix is non singular because no row are dependent on each other. So, Rank would be 2 (as it is a 2d matrix)

Also, in anothe way we can say that, the largest number of linearly independent row/column is the Rank.

Here, largest number of independent rows = 2

So, Rank = 2

(2)

$$\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$$

Here 1st row can be converted to 2nd row by multiplying (−3) with first row. So, second row is dependent on first row.

So, Largest independent row number = 1

Rank = 1

How to find Rank of a matrix more quickly?

→ Using row echelon matrix

Original Matrix

5	1
4	−3

→ Devide each row by the left-most co efficient (non zero)

1	0.2	→ (I)
1	−0.75	→ (II)

Now subtract row1 from row2

$$\begin{array}{rr} 1 & -0.75 \\ (-)\,1 & (\pm)\,0.2 \\ \hline 0 & -0.95 \end{array} \rightarrow \text{(III)}$$

~~Add it~~ 1

$$\begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix}$$

→ Devide the second row by the left most non zero co efficient.

$$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$ → (Row echelon form)

Another matrix

Original matrix

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$$ → Devide each row by the leftmost co efficient

$$\begin{bmatrix} 1 & 0.2 \\ 1 & 0.2 \end{bmatrix}$$ → Do Row2 - Row1

$$\begin{bmatrix} 0 & 0 \end{bmatrix}$$ → ~~Add it row~~ Add it as a matrix with row1 as row 2 instead of previous row 2

$$\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$ → Devide the second row by the left most non zero co efficient

→ No, non zero cofficient in row 2

So $$\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$$ → ~~Ech~~ Row echelon matrix

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Another example

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow$ Can't be devide with co-efficient

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow$ Row echelon form

Connection of Row echelon form, Rank and Singularity:

Original Matrix	Row echelon form	Rank	Singularity
$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$	2	non singular Rank 2 (2D Matrix)
$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$	1	Singular Rank 1 (2D Matrix)
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	0	Singular Rank 0 (2D Matrix)

For 3D Matrix

1	1	1
3	-3	-1
2	-1	6

→ Devide each row with leftmost coefficient

1	1	1
1	-1	-0.3
1	-0.5	3

→ Subtrac row2 - row1

→ " row 3 - row 1

r2 - r1 → | 0 | -2 | -1.3 |

r3 - r1 → | 0 | -1.5 | 2 |

} Add this rows with existing row 1

1	1	1
0	-2	-1.3
0	-1.5	2

Now, r2 × 1.5 and r3 × 2 →

0	-3	-1.95
0	-3	4

r3 - r_2 = | 0 | 0 | 5.95 |

add this with previous matrix

1	1	1
0	-2	-1.3
0	0	5.95

→ Devide leftmost co-efficient

Row echelon form →

1	1	1
0	1	0.65
0	0	1

Number of pivots = 3

Rank = 3

$\boxed{\text{Example 2}}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \quad \begin{array}{l} \rightarrow \text{(ii)} - \text{(i)} \\ \rightarrow \text{(iii)} - \text{(i)} \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \rightarrow \text{(ii)} \times 2 - \text{(iii)}$$

Row echelon form $\rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \begin{array}{l} \text{Number of pivots} = 2 \\ \text{Rank} = 2 \end{array}$$

$\boxed{\text{Example - 3}}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \quad \begin{array}{l} \rightarrow \text{(i)} \times 2 - \text{(ii)} \\ \rightarrow \text{(i)} \times 3 - \text{(iii)} \end{array}$$

Row echelon form $\rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \rightarrow \begin{array}{l} \text{Number of pivots} = 1 \\ \text{Rank} = 1 \end{array}$$

Example 4

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

→ Can't do any additional thing

number of pivot = 0

Rank = 0

~~Reduced~~ Row Echelon form : (General)

$$\begin{bmatrix} 2 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & -5 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Rank = 5

$$\begin{bmatrix} 3 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & -4 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

→ Zero rows at the bottom.

→ Each rows has a pivot (Leftmost non-zero entry)

→ Every pivot is right of the pivots on the rows above

→ Rank of the matrix is the number of pivots

Reduced Row Echelon form:

Original system	Intermediate system	Solved System
$5a + b = 17$ $4a - 3b = 6$	$a + 0.2b = 3.4$ $b = 2$	$1a + 0b = 3$ $0a + 1b = 2$

Original Matrix

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$$

Upper diagonal Matrix

$$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

Row echelon form

Diagonal Matrix →

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reduced row echelon form

Row echelon form → $\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$ → (ii) × 0.2 – (i)

→ $\begin{bmatrix} 0 & 0.2 \end{bmatrix}$ → (iii)

(i) and (iii) →

$$\begin{bmatrix} 1 & 0.2 \\ 1 & 0 \end{bmatrix} \rightarrow \text{(i) – (iii)}$$

→ $\begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix}$

Now, ① - ⅲ → $\begin{bmatrix} 1 & 0 \end{bmatrix}$ → exchang with ①

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ → Reduced row echelon form

Reduced row echelon form in general :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

→ is in row echelon form

→ Each pivot is 1

→ Any number above the pivot is 0

$$\begin{bmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Make reduced row echelon form from a row echelon form

row echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

as above ~~on~~ 1, only 0 can stay in reduced row echelon form

subtract ~~xxx~~ $r_4 = r_4 - 2 \times r_2$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$r_4 = 5r_3 + r_4$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$r_2 =$ ~~xx~~ $4r_3 - r_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

reduced row echelon form