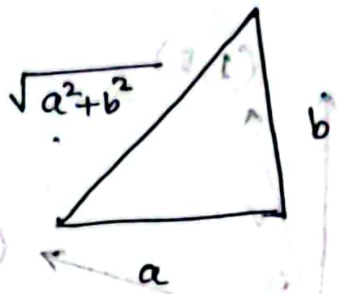
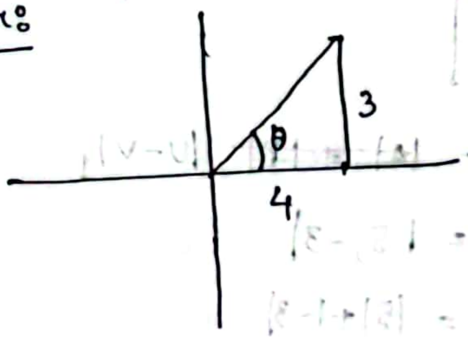
Vectors and Matrices

① Pythagorean theorem:



② Direction of a vector:

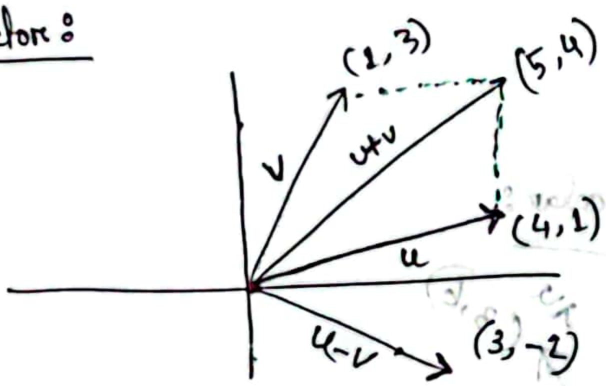


$\tan\theta = 3/4$

$\Rightarrow \theta = \tan^{-1} 3/4$

$= 36.87^{\circ}$

③ Sum of vector:



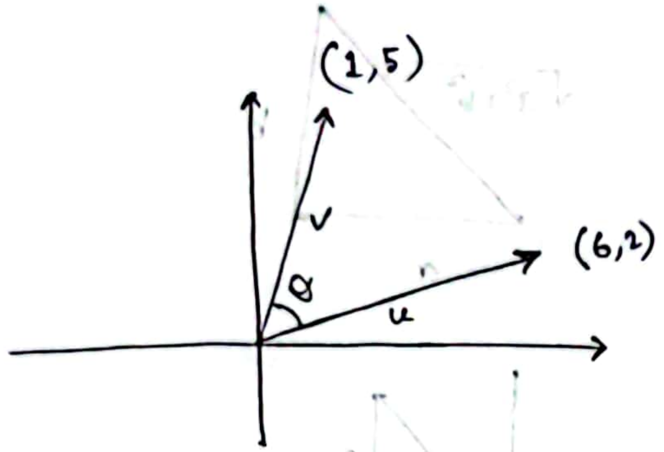
$u+v = (4+1, 1+3)$

$= (5,4)$

$u-v = (4-1, 1-3)$

$= (3,-2)$

(4) Distance between vectors:



L1 - Norm distance = ~~|A| + |B|~~ $|u-v|_1$

$= |5, -3|$

$= |5| + |-3|$

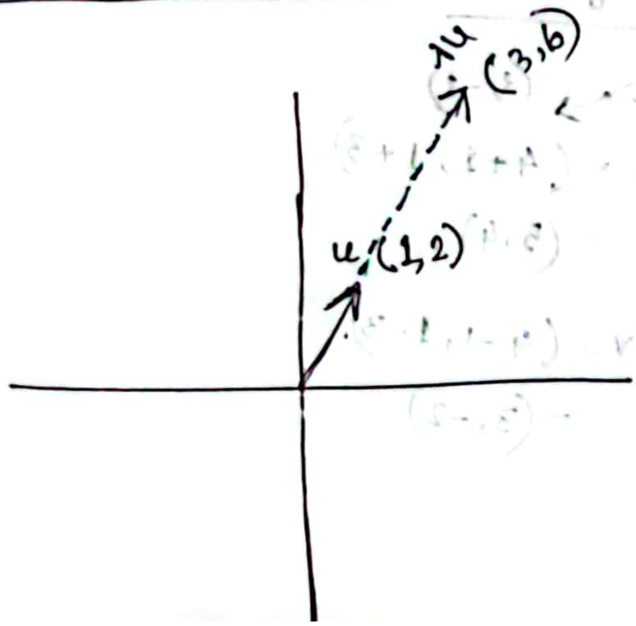
$= 8$

L2 - Norm distance $= \sqrt{5^2 + 3^2}$ ($|u-v|_2$ → representation of L2)

uvcosθ

cosine distance = cos(θ)

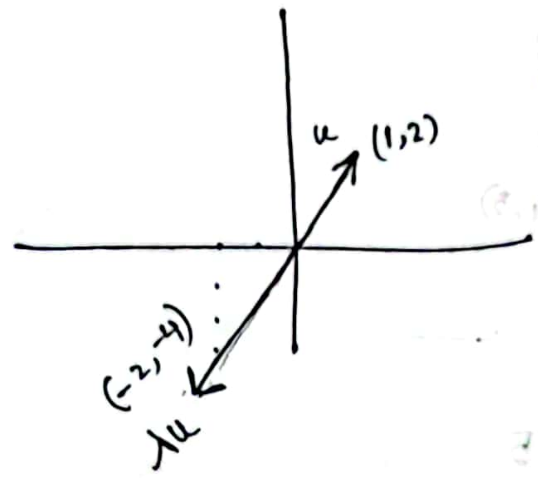
(5) Multiplying a vector by a scalar:



$u = (1,2)$

$\lambda = 3$

$\lambda u = (3,6)$



$u = (1,2)$
$\lambda = -2$
$\lambda u = (-2,-4)$

⑥ Dot product example:

Quantities	Prices
2 apples	apples: \$3
4 bananas	bananas: \$5
1 cherry	cherris : \$2

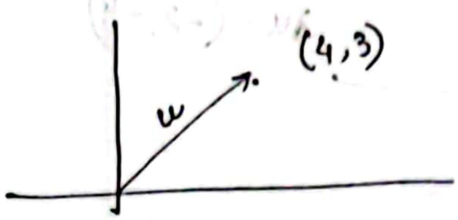
$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \rightarrow \begin{matrix} 2\times 3 = \\ 4\times 5 = \\ 1\times 2 = \end{matrix} \left.\begin{bmatrix} 6 \\ 20 \\ 2 \end{bmatrix}\right\}$$

Total price
$6 + 20 + 2 = 28$

$$= 2\times3 + 4\times5 + 1\times2 \rightarrow \begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = 28$$

$= 28$

(7) Norm of a vector using dot product:



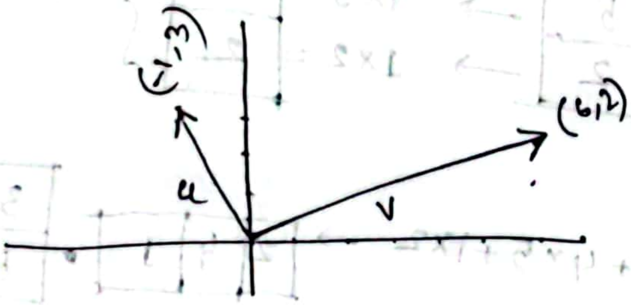
L2-Norm → $\sqrt{4^2+3^2} = \sqrt{25} = 5$

and → $\begin{bmatrix} 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 25$

$\therefore$ L2-Norm = $\sqrt{\text{dot product } (u,u)}$

$\|u\|_2 = \sqrt{\langle u,u \rangle}$

(8) Orthogonal vectors have a dot product of 0:



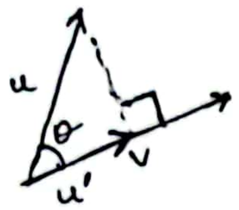
Their dot product = $\begin{bmatrix} -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \end{bmatrix} = -1 \times 6 + 3 \times 2$

$= 0$

So, two vectors are orthogonal if their dot product = 0

$\Rightarrow \langle u,u\rangle = |u|^2 = |u|\cdot|u|$

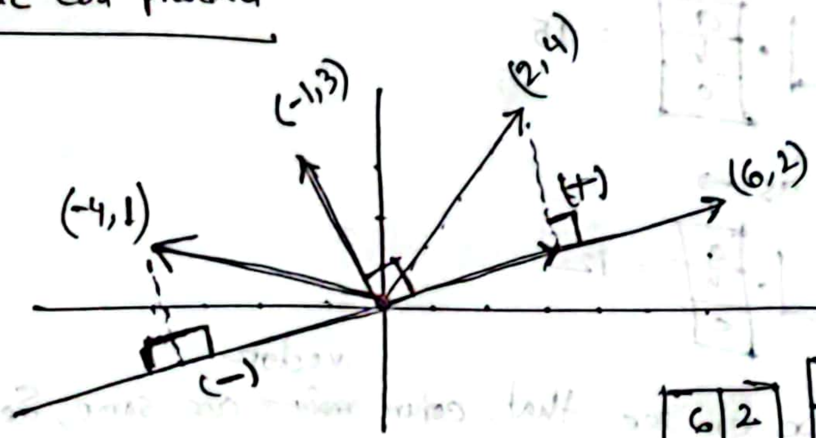
$\Rightarrow \langle u,v\rangle = |u|\cdot|v|$



$\Rightarrow \langle u,v\rangle = |u'|\cdot|v|$

$= |u||v|\cos\theta$

(9) Geometric dot product



$\begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 20$, positive

$\begin{bmatrix} 6 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 0$, peperdicular

$\begin{bmatrix} 6 & 2 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix} = -22$, Negative

(10) Multiplying a ~~matrix~~ ~~wit~~ by a vector

$a+b+c=10 \rightarrow$ (I)

$a+2b+c=15 \rightarrow$ (II)

$a+b+2c=12 \rightarrow$ (III)

(I) can be represent as →

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 10 \quad \text{(dot product)}$$

(II) can be represent as →

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 15$$

(III) Can be represent as →

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 12$$

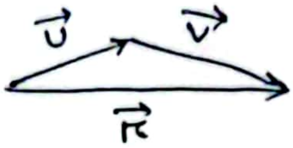
From (I), (II), (III) we can see that, colum vector ~~matrix~~ are same, so we can write like this below →

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 12 \end{bmatrix}$$

if $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$, then $\vec{a} = 0$ and $\vec{b}$ = any other vector.

(11) Vector operations:

Addition:

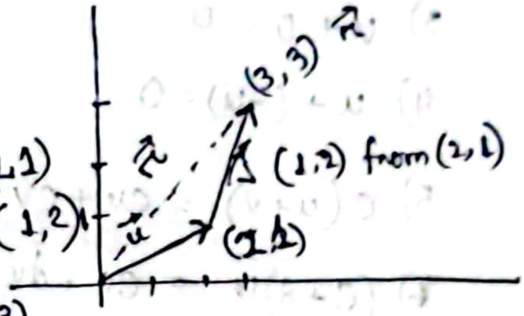


$\vec{r} = \vec{u} + \vec{v}$

$\vec{u} = (2,1)$

$\vec{v} = (1,2)$

$\vec{r} = (3,3)$

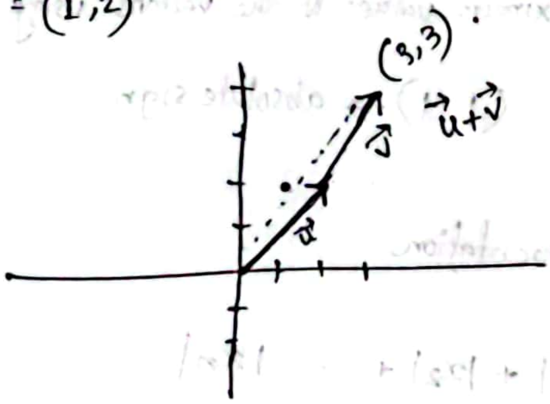


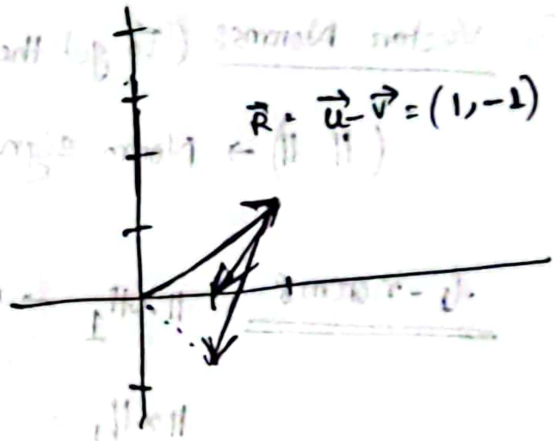
Subtraction:

$\vec{u} = (2,1)$

$\vec{v} = (1,2)$

$\vec{u} - \vec{v} =$



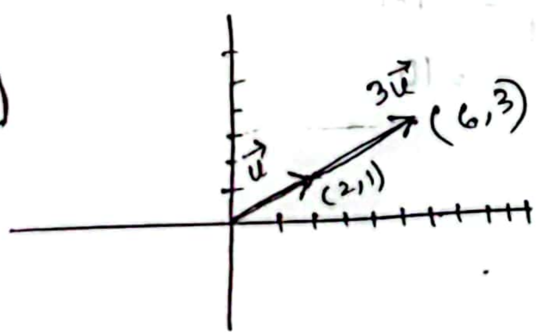


Scalar Multiplication:

$u = [2,1]$

$3\vec{u} = 3 \times [2,1]$

$= [6,3]$



(12) Vector operations rules:

1) $U+V = V+U$
2) $(U+V)+W = U+(V+W)$
3) $u+0 = u$
4) $u+(-u) = 0$
5) $c(u+v) = cu+cv$
6) $(c+d)u = cu+du$
7) $c(du) = (cd)u$
8) $1(u) = u$

(13) Vector Norms: (To get the maximum values of the vectors using ~~no~~ normalization)

($\| \ \|$) → Norm sign ($| \ |$) → absolute sign

l_1-norm: $\|x\|_1$ → representation

$$\|x\|_1 = |x_1| + |x_2| + \dots |x_n|$$

$$x = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \rightarrow \|x\|_1 = |2| + |5| + |-3|$$

$$= 10$$

(14) **l_2-norm :** This is also called Euclidean Norm.

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}$$

$$x = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \rightarrow \|x\|_2 = \sqrt{2^2 + 5^2 + (-3)^2}$$

$$= \sqrt{4 + 25 + 9}$$

$$= \sqrt{38}$$

Another way to represent $\|x\|_2$

$$\|x\|_2 = \sqrt{x^T x}$$

$$\Rightarrow x^T = \begin{bmatrix} 2 & 5 & -3 \end{bmatrix}, \quad x = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \sqrt{\begin{bmatrix} 2 & 5 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}}$$

$$\Rightarrow \sqrt{2 \times 2 + 5 \times 5 + ((-3) \times (-3))}$$

$$\Rightarrow \|x\|_2 = \sqrt{38}$$

(15) **l_∞-form :**

$$\|x\|_\infty = \max_{1 \le x_i \le n} |x_i|$$

$$x = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \Rightarrow |2|, |5|, |-3|$$

max . 5

$$\therefore \|x\|_\infty = 5 \text{ Ans}$$

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16. **Unit vector:** (used to represent the direction of a vector)

It is a vector with magnitude 1 in a given direction

$\vec{a} = [3, 4]$

$\|a\|_2 = \sqrt{3^2 + 4^2}$

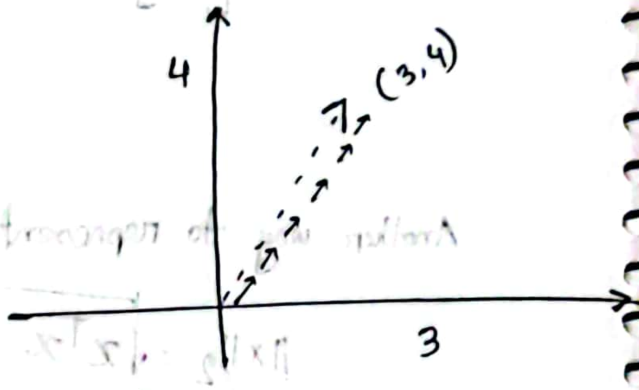
$= 5$

Unit vector, $\hat{a} = \dfrac{a_x}{\|a\|_2}, \dfrac{a_y}{\|a\|_2}$

$= \dfrac{3}{5}, \dfrac{4}{5}$

$\|\hat{a}\|_2 = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$

$= 1$



(These small arrows represent unit vector)

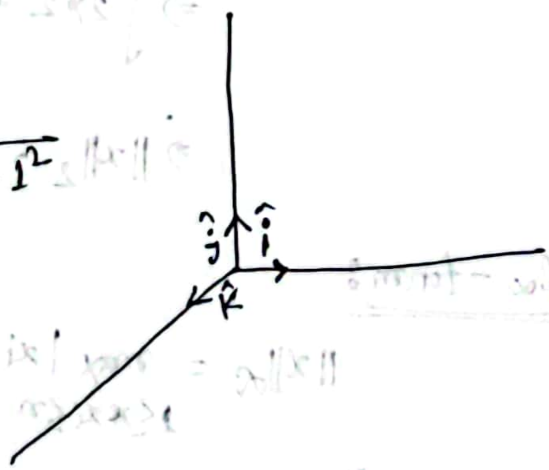
17. **For 3 dimension:**

(A vector that starts from the origin, is a position vector)

$\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$

Magnitude of $\vec{a}$ = $\|a\|_2 = \sqrt{2^2 + 3^2 + 1^2}$

$= \sqrt{14}$



Unit vector $(\hat{a})$ in the direction of

$\vec{a} = \dfrac{1}{\text{~~my~~ magnitude of } \vec{a}} \times \cancel{d}\ \vec{a}$

$$\Rightarrow \hat{a} = \frac{1}{\sqrt{14}} [2\hat{i} + 3\hat{j} + \hat{k}]$$

We know, $\hat{a} = 1$, So, $\hat{a} = \frac{1}{\sqrt{14}} [2\hat{i} + 3\hat{j} + \hat{k}] = 1$

$$\Rightarrow \hat{a} = \frac{2\hat{i}}{\sqrt{14}} + \frac{3\hat{j}}{\sqrt{14}} + \frac{\hat{k}}{\sqrt{14}} = 1$$

(18) Dot product of 2 vectors :

$$\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{a \cdot b}{\|a\| \|b\|}$$

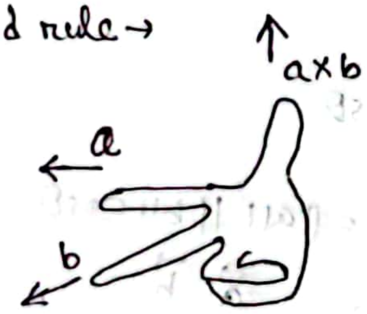
$$\therefore \theta = \cos^{-1} \frac{a \cdot b}{\|a\| \|b\|}$$

(19) Cross Product

$$a \times b = \|a\| \|b\| \sin(\theta) \hat{n}$$

Cross product resultant vector direction →

Right hand rule →



$\vec{a} = (3, -3, 1), \ \vec{b} = (4, 9, 2)$

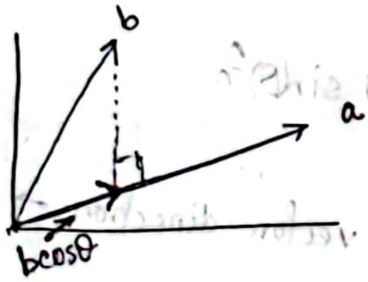
Cross product $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & -3 & 1 \\ 4 & 9 & 2 \end{vmatrix}$

$= i((-3\times 2) - (1\times 9)) - j((3\times 2) - (1\times 4))$
$\quad + k((3\times 9) - (-3\times -4))$

$= -15\hat{i} - 2\hat{j} + 39\hat{k}$

20 Vector Projections:

Projection of vector b on a:



Projection $\underset{\vec{a}}{\vec{b}}$: $\|b\|\cos\theta$

We know that $\Rightarrow \vec{a}\cdot\vec{b} = \|a\|\,\|b\|\cos\theta$

$$\|b\|\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|a\|}$$

$$\therefore \underset{\vec{a}}{\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{\|a\|}$$

Projection of vector a on b:

Similarly $\underset{\vec{b}}{\vec{a}} = \frac{\vec{a}\cdot\vec{b}}{\|b\|}$

Example: Find the projection of vector $\vec{a} = [1,2]$, on vector $\vec{b} = [3,4]$

$$\vec{a} \cdot \vec{b} = (1 \times 3 + 2 \times 4)$$
$$= 11$$

magnitude of vector $||b|| = \sqrt{3^2 + 4^2}$
$$= 5$$

vector projection $proj_{\vec{b}} \vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{||b||}$

$$= \frac{11}{25}$$

(21) Linear Combination of vectors:

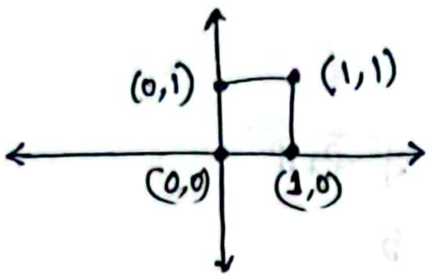
If one vector is equal to the sum of scalar multiples of other vectors, it is said to be a linear combination of the other vectors

Example:

$$\underset{A}{\begin{bmatrix} 11 \\ 6 \end{bmatrix}} = 2 \underset{B}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + 3 \underset{C}{\begin{bmatrix} 3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 2 \times 1 + 3 \times 3 \\ 2 \times 2 + 3 \times 4 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

Here, 2B is a scalar multiple, so as 3C. A is a linear combination of B and C.

22 Matrices Linear Transformation : (Simply defined as a change of coordinates)



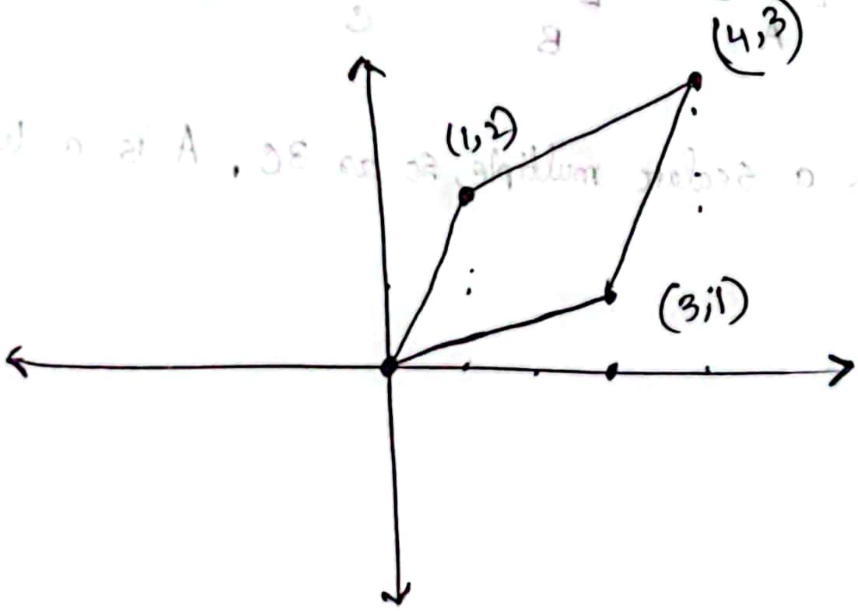
Matrix → $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ $(0,0) \to (0,0)$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \cancel{\begin{bmatrix} 0 \\ 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$ $(1,0) \to (3,1)$

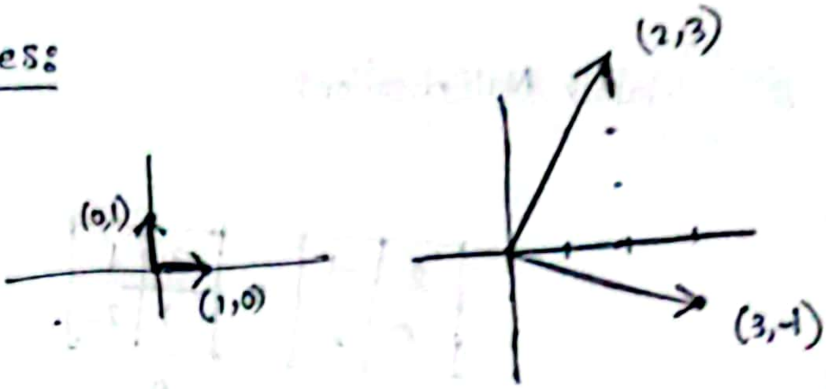
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ $(0,1) \to (1,2)$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$ $(1,1) \to (4,3)$



(23) Linear Transformation into Matrices:

$(0,0) \rightarrow (0,0)$
$(1,0) \rightarrow (3,-1)$
$(0,1) \rightarrow (2,3)$
$(1,1) \rightarrow (5,2)$



We would only need $(1,0) \rightarrow (3,-1)$

$(0,1) \rightarrow (2,3)$

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\rightarrow \quad \begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(24) Matrix Multiplication:

$$\underset{A}{\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}} \times \underset{B}{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} 2\times3+(-1)\times1 & 2\times1+(-1)\times2 \\ 0\times3+2\times1 & 0\times1+2\times2 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}$$

(25) Identity Matrix:

If this matrix is multiplied by some other matrix (x)
→ It will give result x

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{Identity matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

(26) Inverse of a Matrix:

If we dot product matrix $A \cdot A^{-1}$ = identity matrix

$$\underset{A}{\begin{bmatrix}3&1\\1&2\end{bmatrix}}\underset{A^{-1}}{\begin{bmatrix}a&b\\c&d\end{bmatrix}}=\underset{\text{Identity matrix}}{\begin{bmatrix}1&0\\0&1\end{bmatrix}}$$

$$\begin{bmatrix}3&1\end{bmatrix}\cdot\begin{bmatrix}a\\c\end{bmatrix} \Rightarrow 3a+c=1 \rightarrow (i)$$

$$\begin{bmatrix}3&1\end{bmatrix}\begin{bmatrix}b\\d\end{bmatrix} \Rightarrow 3b+d=0 \rightarrow (ii)$$

$$\begin{bmatrix}1&2\end{bmatrix}\begin{bmatrix}a\\c\end{bmatrix} \Rightarrow a+2c=0 \rightarrow (iii)$$

$$\begin{bmatrix}1&2\end{bmatrix}\begin{bmatrix}b\\d\end{bmatrix} \Rightarrow b+2d=1 \rightarrow (iv)$$

$(i) - 3\times(iii) \rightarrow$

$$\begin{aligned}3a+c&=1\\ \underset{(-)}{3a}\ \underset{(-)}{+}\ \underset{(-)}{6c}&=0\\ \hline -5c&=1\\ \therefore c&=-\tfrac{1}{5}\end{aligned}$$

$(iii) \rightarrow$

$$a+2\times\left(-\tfrac{1}{5}\right)=0$$

$$\Rightarrow a=\frac{2}{5}$$

$(ii) - 3\times(iv) \rightarrow$

$$\begin{aligned}3b+d&=0\\ \underset{(-)}{3b}\ \underset{(-)}{+6d}&=\underset{(-)}{3}\\ \hline -5d&=-3\\ \therefore d&=\tfrac{3}{5}\end{aligned}$$

$(ii) \rightarrow$

$$3b+\frac{3}{5}=0$$

$$\Rightarrow b=-\frac{1}{5}$$

$$\therefore A^{-1}=\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}\frac{2}{5}&-\frac{1}{5}\\-\frac{1}{5}&\frac{3}{5}\end{bmatrix}$$

$$\begin{bmatrix}2/5&-1/5\\-1/5&3/5\end{bmatrix}$$

Is inverse possible of this Matrix → $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$a+b=1$

$b+d=0$

$2a+2b=0$

2

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ c \end{bmatrix} \rightarrow a+c=1$$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = b+d=0$$

$$\begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = 2a+2c=0$$

$$\begin{bmatrix} 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} b \\ d \end{bmatrix} = 2b+2d=1$$

Here equations are contradictory

if $a+c=1$

$\Rightarrow 2a+2c =$ can't be 0

Also goes for $2b+2b=1$, and $b+d=0$, they contradict

So, inverse can't happen in this equation

(27) So, which matrix can have an inverse?

In one word, the answer is it depends on singular and non singular matrices.

If matrix is non singular → inverse possible

If matrix is singular → inverse not possible

Also, it can be said that →

if determinant is not $= 0$, → inverse possible

if determinant $== 0$, → inverse not possible