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ex. let A, Be a set have 10 elements. How many diff. binary relation on A is possible?
How many of them are reflexive?
How many of them are symmetric?

ex let R be a binary relation on set of all ~~possible~~ (+ve) integers such that $R = \{(a,b) \mid a-b \text{ is odd positive integer}\}$
Find the nature of the relation.

→ No Reflexive &
No Symmetric
No transitive [(10,5) − (5,2) = (10,2) which is not odd]

Exam
Ex. Prove that for any positive integer $n > 1$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

→ Basis : $n=2$

$$L.H.S. = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + 0.707 = \boxed{1.707}$$

$$R.H.S. = \sqrt{2} = 1.41$$

Here, $1.707 > 1.41$ hence state is true for $n=2$

Hypothesis : $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{K}} > \sqrt{K}$

$$\rightarrow \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}} > \sqrt{K} + \frac{1}{\sqrt{K+1}} \quad \text{——(1)}$$

Here, $K+1 > K$

$\Rightarrow \sqrt{K+1} > \sqrt{K}$

$\Rightarrow 1 > \frac{\sqrt{K}}{\sqrt{K+1}}$

from $-(1)$

$> \frac{K}{\sqrt{K+1}} + \frac{1}{\sqrt{K+1}}$

$> \frac{K+1}{\sqrt{K+1}} > \sqrt{K+1}$

Hence, proved.

- **Equivalence Relation**

$\Rightarrow$ It is a binary relation that is Reflexive, symmetric & transitive.

Ex. $A = \{a, b, c, d, e, f\}$

	a	b	c	d	e	f
a	✓	✓				
b	✓	✓				
c			✓			
d				✓	✓	✓
e				✓	✓	✓
f				✓	✓	✓

General eq^n $R = \{(a,b) \mid a=b\}$

Set Partitions

$S = \{1,2,3,4\}$

- set can be partitioned into non-empty subsets such that every element is included in exactly one subset.

- union of all subsets will be original set and intersection will be null ($\emptyset$).

ex. $S = \{1,2,3\}$

partitions $\Rightarrow$ $\{1,2\}, \{3\}$ — $\{\overline{12}, \overline{3}\}$

$\Rightarrow$ $\{1\}, \{2,3\}$ — $\{\overline{1}, \overline{23}\}$

$\Rightarrow$ $\{2\}, \{1,3\}$ — $\{\overline{2}, \overline{13}\}$

$\Rightarrow$ $\{1\}, \{2\}, \{3\}$ — $\{\overline{1}, \overline{2}, \overline{3}\}$

(partitions.)

Notation for set partition

ex $\{1,2\}, \{3\} \rightarrow \{\overline{12}, \overline{3}\}$

→ From the equivalence relation on set A, we can define the partition of A, so that every two elements in a block are related and every two elements in different blocks are not related. This partition is called partition induced by equivalence relation.

ex. $S = \{1,2,3\}$

$\Rightarrow \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

$\{\{1,2\}, \{3\}\} \Longrightarrow \{\overline{12}, \overline{3}\}$

- 1 & 2 are related with each other and 3 is not related with either 1 or 2.

Similarly, from the partitions of the set we can define equivalence relation.

ex partition = $\{\overline{12}, \overline{3}\}$

$R = \{(1,1), (2,2), (1,2), (2,1), (3,3)\}$

ex P is a partition induced by equivalence relation $P = \{\{a,b,c\}, \{d\}, \{e\}\}$

list all pairs in equivalence relation R

$R = \{(a,a), (b,b), (c,c), (a,b), (a,c), (b,c)$
$(b,a), (c,a), (c,b), (d,d), (e,e)\}$

- Equivalence Class

An equivalence class of x, where x is an element, is defined as $[x]$ such that

$x \rightarrow [x]$

$[x] = \{y \mid y \in A, (x,y) \in R\}$

Equivalence class

ex let $A = \{1,2,3,4,5\}$

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2),$
$(2,1), (4,5), (5,4)\}$

3 classes,

$[1] = \{1,2\}$] C_1
$[2] = \{2,1\}$]

$[3] = \{3\}$] C_2

$[4] = \{4,5\}$] C_3
$[5] = \{5,4\}$]

$C_1 = \{1,2\}$

$C_2 = \{3\}$

$C_3 = \{4,5\}$

partition = $\{\{1,2\}, \{3\}, \{4,5\}\}$

Que. If set A contains n elements, How many minimum & maximum equivalence class can be possible?

Ans. Maximum $= n$

Minimum =

- Congruence Modulo Relation ($\equiv$)

Two integers ~~A~~ a & ~~B~~ b are congruent modulo if and only if (iff) they have the same ~~reminder~~ remainder when divided by some number m,

$$a \equiv b \pmod m$$

a is congruent to b modulo m.

ex $16 \equiv 13 \pmod 3$

$\frac{16}{3}$ modulo $=1$ & $\frac{13}{3}$ modulo $=1$

this means that m divides $\boxed{(a-b)}$

ex $29 \equiv 8 \pmod 7$

→ Show that the relation congruence modulo m over the set of positive numbers is an equivalence relation.

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→ Proof for congruence modulo m is an equivalence relation.

(1) Reflexive relation
- since m divides zero we can say that
$a \equiv a \pmod{m}$.

(2) Symmetric relation
- Assume $a \equiv b \pmod{m}$.
So, $(a-b) = k \cdot m$
$\Rightarrow (b-a) = -k \cdot m$
this result means that $(b-a)$ is also divisible by m. so $b \equiv a \pmod{m}$

(3) Transitivity relation
- Assume $a \equiv b \pmod{m}$ & $b \equiv c \pmod{m}$
$\boxed{a-b = k \cdot m}$ — (1) $\boxed{b-c = l \cdot m}$ — (2)

Add (1) & (2)
$a - c = km + lm$
$a - c = (k+l)m$
this result means that $(a-c)$ is divisible by m. so, $a \equiv c \pmod{m}$.

Que $A = \{1,2,3\}$ How many diff eq. relation are possible?
1. $\{(1,1), (2,2), (3,3)\}$
2. $\{(1,1), (2,2), (3,3), (1,2), (2,1)\}$
3. $\{(1,1), (2,2), (3,3), (1,3), (3,1)\}$
4. $\{(1,1), (2,2), (3,3), (2,3), (3,2)\}$
5. $\{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$

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Partial Order

- A relation is partial order if it is ~~a re~~ (a) reflexive, (b) anti-symmetric (c) transitive

eg. $(x, y) \in R \mid x \geq y$.
$A = \{1, 2, 3\}$
Partial $= \{(1,1), (2,2), (3,3), (3,1), (2,1), (3,2)\}$

eg. $(a, b) \Rightarrow a$ divides b

let R be a relation
$R = \{(a,b) \mid a \subseteq b\}$ it defines on power set ~~(set of sets)~~

H-W.

→ $S = \{1, 2, 3\}$ R is defined on power set of S.

Partial order set (POSET)

- if R is a partial order relation defined on set S.
- set S along with relation R is called partially ordered set and denoted as (S, R) or $(S, \leq)$ ← notation

eg. consider set $S = \{1, 2, 3, 5, 6, 10, 15, 30\}$
construct a partial order set in $(S, /)$ relation.

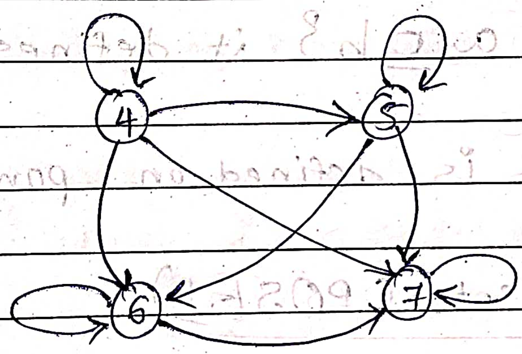
→ $R = \{(1,1), (2,2), (3,3) \ldots (30,30),$
$= (1,2)\ (1,3),\ (1,5) \ldots$
$(2,6), (2,10), (2,30), (3,6), (3,15), (3,30).$

→ elements a & b are called comparable if $(a,b) \in R$ otherwise they are called non-comparable.

• <u>Graphical representation of POSET</u>

<u>eg</u> $A = \{4,5,6,7\}$
$R = \leq$

→ $R = \{(4,4), (4,5), (4,6), (4,7), (5,5), (5,6), (5,7), (6,6), (6,7), (7,7)\}$

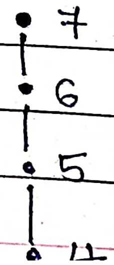


can be converted into <u>Hasse Diagram</u>

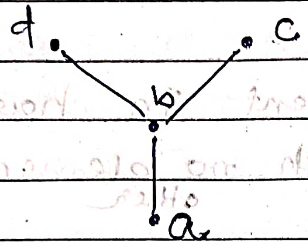
To convert ~~the~~ directed graph into hasse diagram:
1. Remove all self loops.
2. Remove transitive edges.
3. Remove directed edges & use undirected edges.
4. Instead of nodes use '•' to represent element

Hasse diagram ⟹

(bottom–top)



eg

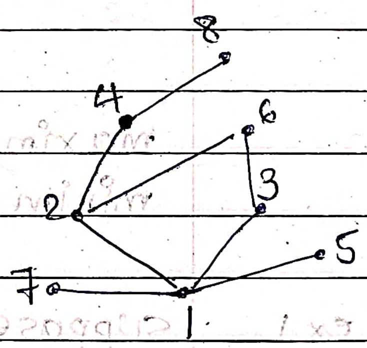


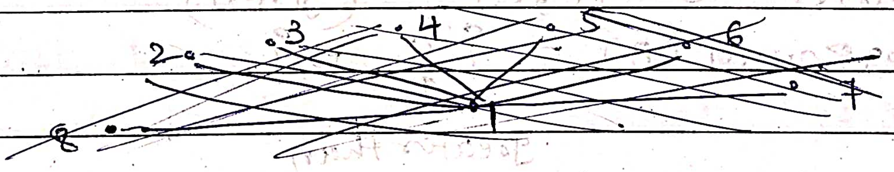
$\{(a,b), (b,d), (b,c), (a,d), (a,c)\}$

Que construct hasse diagram for given poset
$R = \{(a,b) \mid a \text{ divides } b\}$
on set
$S = \{1,2,3,4,5,6,7,8\}$

→ $R = \{(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8),$
poset→ $(2,4), (2,6), (2,8), (3,6), (4,8),$
self loops$\}$

hasse diagram :





→ Maximal Element
It is an element of poset which is not related to any other element.
eg In above example, 8, 6, ~~4~~, 5, 7 ~~is~~ are maximal elements

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Minimal Element:

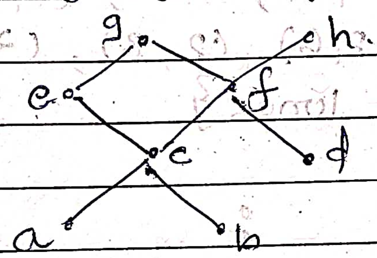
The lowest element in hasse diagram or the element with which no other elements are related.

eg No other ~~w~~ element is there. (x, 1).
1 is minimal element.

Upper bound:

Consider B as a subset of partial order set A. An element $x \in A$ is called upper bound of B if $y \leq x$, for y in B.

eg



maximal elements = g, h
minimal elements = a, b, d

ex 1 suppose, B = {e, c} (given)
(i) UpperBound (B) = {g, e}
greater than e&c

h is not related with e so it shouldn't add

(ii) LowerBound (B) = {a, b, c}

ex.2 B = {c, f, d}
(i) UB (B) = {g, h, f}
LB (B) = ~~{a, b, c, d}~~ ϕ

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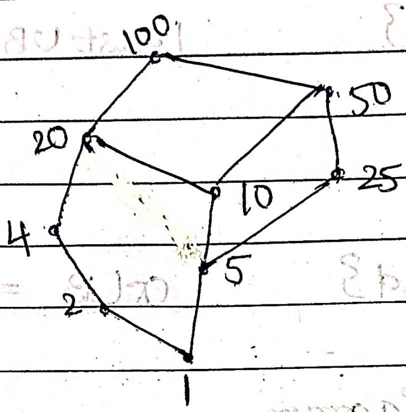
Ex. Draw hasse diagram for given ~~lattice~~ relation R is devise relation defined on set S
$S = \{1, 2, 4, 5, 10, 20, 25, 50, 100\}$

→



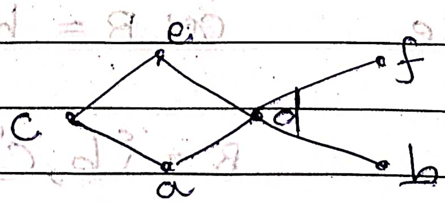
(i) $B = \{5, 10\}$
$UB(B) = \{20, 50, 100, 10\}$
$LB(B) = \{1, 5\}$

(ii) $B = \{5, 10, 2, 4\}$
$UB(B) = \{20, 100\}$
$LB(B) = \{1\}$

→ Least upper bound / (Supreme, join point)
It is the min. element upper bound

→ Greatest lower bound / (~~Grea~~ infimum, meet)
It is the greatest ele. in the lower bound.

eg.



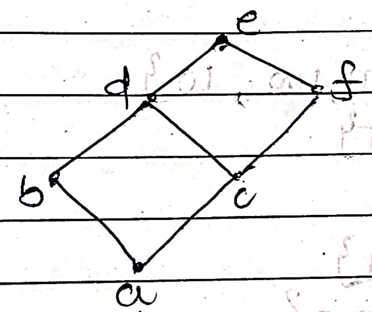
(i) $B = \{c, d\}$

UB(B) = {e}
LB(B) = {a}

(ii) B = {a, b}
UB(B) = {d, e, f}   Least UB = {d}
LB(B) = ϕ

(iii) B = {e, f}
LB(B) = {a, b, d}   GLB = d

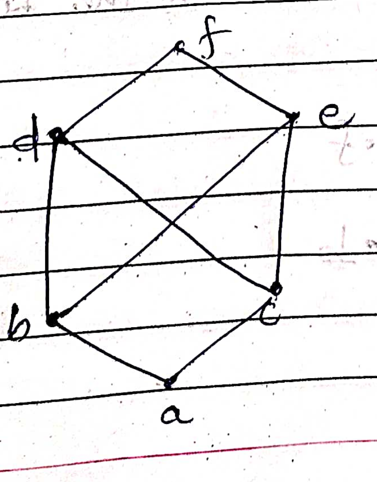
Que Draw hasse diagram.



(i) B = {a, e, f}
LB(B) = {a}   LUB = a
UB(B) = {e}   GLB = e.

(ii) B = {d, c}
LB(B) = {a, c}   GLB = c , LUB = d.

Que



B = {d, e}
GLB = b, c , LUB = d, e

B = {b, c}
GLB = a , LUB = b, c