AAM

DATE:
PAGE:

Modeling Computation

→ set - unorderd collection of objects (distinct)

→ Representation of set
ex {9, 16, 25, 49}
{x | x is a perfect square 5 < x < 50}

→ $\mathbb{N}$ - countable definite infinite
$\mathbb{R}$ - uncountable definite infinite

→ $R = \{x \mid \{a, b\} \subseteq x\}$
R is a set of set.
$R = \{\{2,3\}, \{1,2,3\}, \{1,2,3,9\}, \{8,2,3\}\}$
$= \{x \mid \{2,3\} \subseteq x\}$

→ $R = \{x \mid x \notin x\}$ $R \in R \rightarrow R \notin R$
$\{1,2\} \notin \{1,2\}$ $R \notin R \rightarrow R \in R$
$\{1,2\} \in \{1, 2, \{1,2\}\}$

→ $A = \{$highest weight, highest height$\}$
$\{a, b\} \neq \{b, a\}$ ← ordered set
$\{a, a\}$
$\{\{\{a, b\}, c\}, d\} = \{a, b, c, d\}$

→ Languages:

* $A = \{a, b, c, ---, z\}$
strings/sentences of language are defined over A

L over $A^2 = \{aa, aj, ak, \ldots\}$ - known as 2-tupl
L over $A^5 = \{aaaaa, aabaa, \ldots\}$

DATE:
PAGE:

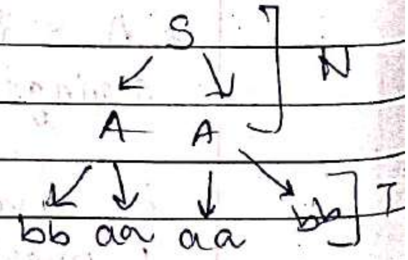
A^* = all possibe $A^1, A^2, ----$
(1 or more)
A^* = 0 or more accurence
T alphabet

ex. $A = \{a, b, c\}$, L over A^*
$L_1 = \{a, aa, aaa, baa\}$ – finite lang
$L_2 = \{cab, ccb, bcc, ---\}$ – infinite lang.
$L_3 = \{a^i c b^i c^i \mid i \geq 1\}$
$L_4 = \{ \}$ (Assume empty lang.)

ex. $B = \{a, b\}$, M over B^* (language)
M_1 (a is more than b)
M_2 (b is more than a)
M_3 (either a is higher or b is higher but they have not same)

→ To represent a language – we use grammar and that grammar is known as Phrase structure grammar.

eg 1. $L = \{aaaa, bbbb, aabb, bbaa\}$
List of alphabets $\Sigma = \{a, b\}$
P ⇒ $S = AA$ (Beggining)
Production $A \rightarrow aa \mid bb$
or



Terminals T $\{a, b\}$
Non-terminal N $\{S, A\}$
S is a starting symbol

DATE:
PAGE:

→ T — Terminals — makes up sentences
N — Non terminals — specifies the structure of a lang.
(provide templete)
P — Productions Rules $\alpha \to \beta$
S — Special symbol (starting symbol)

eg.2 Construct a grammar for the language having any no. of a's over the alphabet $\Sigma = \{a\}$

→ $L = \{\varepsilon, a, aa, aaa, \ldots\}$ $\boxed{\varepsilon = \text{empty set}}$

$T = \{a\}$ $N = \{S\}$
$P \Rightarrow S \to a \mid aS \mid \varepsilon$
$S \to aS \mid \varepsilon$
S is a starting symbol

eg.3 $L = \{a^i b^{2i} / i \geq 1\}$
$L = \{abb, aabbbb, aaabbbbbb, \ldots\}$

$T = \{a, b\}$ N
$P \Rightarrow S \to aSA \mid \varepsilon$, $A \to bb$
N $\{S, A\}$
S is a starting symbol

eg. Grammar for the language
having any no. of a's and
any no. of b's. $\Sigma = \{a, b\}^*$

→ $L = \{\varepsilon, a, b, aa, bb, ab, baa, bab, \ldots\}$

DATE:
PAGE:

$S \rightarrow aS \mid bS \mid \varepsilon$ or (i) $S \rightarrow aS$
(ii) $S \rightarrow bS$
(iii) $S \rightarrow \varepsilon$

* eg. language having at least two a's, preceded and followed by any no. of a's and b's

$\rightarrow$ $L = \{aa, aaa, aba, baaba, \dots\}$

$\cancel{(a+b)^* aa (a+b)^*}$ ← a or b
$(a+b)^* a (a+b)^* a (a+b)^*$

$S \rightarrow AaAaA$ $\quad T = \{a, b\}$
$A \rightarrow aA \mid bA \mid \varepsilon$ $\quad N = \{S, A\}$

eg $L = \{ wcw^T \mid w \in \{a+b\}^* \}$

$L = \{c, aabcbaa, baaacaaab, \dots\}$

$\cancel{S \rightarrow ScS^T \mid \varepsilon}$
$\cancel{S \rightarrow \varepsilon}$
$\cancel{S \rightarrow aScSa \mid bScSb}$

$S \rightarrow c$ $\quad T = \{a, b, c\}$
$S \rightarrow aSa$ $\quad N = \{S\}$
$S \rightarrow bSb$

eg $S \rightarrow aB$ $\quad N = \{S, A, B\}$
$S \rightarrow bA$ $\quad T = \{a, b\}$
$A \rightarrow a \mid aS \mid bAA$ $\quad$ S is a starting symbol.
$B \rightarrow b \mid bS \mid aBB$

DATE:
PAGE:

→ $L = \{ab, ba, baab, abba, bbaa, aabb, bbaabaab \ldots\}$

$L = \{a^i b^j, i = j\}$

eg. $L = \{a^n b^m c^n \mid n, m \geq 1\}$

$L = \{abc, aabcc, aabbbcc, \ldots\}$

$S \rightarrow aSC$

$S \rightarrow aAC$

$A \rightarrow bA \mid b$

$T = \{a, b, c\}$

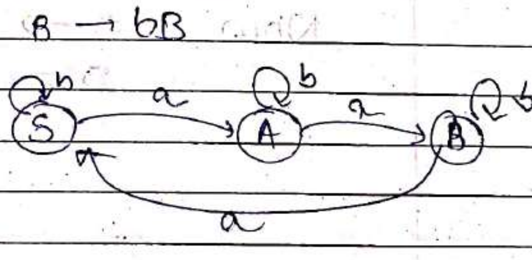
$N = \{S, A\}$

eg. $L = \{x \mid x \in \{a,b\}^*$, the member of a's in x is a multiple of 3$\}$

★

$L = \{aaab, abaa, ababa, aaa, \varepsilon, \ldots\}$

1. $S \rightarrow b \mid bS$
2. $S \rightarrow aA$
3. $A \rightarrow aB$
4. $A \rightarrow bA$
5. $B \rightarrow a \mid aS$
6. $B \rightarrow bB$



eg. $L = \{a^{m+n} b^m c^n \mid n, m \geq 1\}$

~~$L = \{aabc$~~

$L = \{a^m a^n b^m c^n \mid n, m \geq 1\}$

$= \{a^n a^m b^m c^n \mid n, m \geq 1\}$

$S \to aSc$ $\quad N = \{A, S\}$
$S \to aAc$ $\quad T = \{a, b, c\}$
$A \to ab$
$A \to aAb$

eg. $L = \{a^i b^j \mid i, j \geq 1, i \neq j\}$

$L_1 = \{a^i b^j \mid i > j, i, j \geq 1\}$

$L_2 = \{a^i b^j \mid i < j, i, j \geq 1\}$

$L = L_1 \cup L_2$

For L_1,
$A \to aA$
$A \to aB$
$B \to ab$
$B \to aBb$

For L_2,
$A \to Ab$
$A \to Bb$
$B \to ab$
$B \to aBb$

Now, $S \to A$
$S \to B$

$A \to aA$
$A \to aB$
$B \to ab$
$B \to aBb$
$C \to Cb$
$C \to Bb$

$T = \{a, b\}$
$N = \{S, A, B, C\}$

DATE:
PAGE:

→ Types of grammar & languages

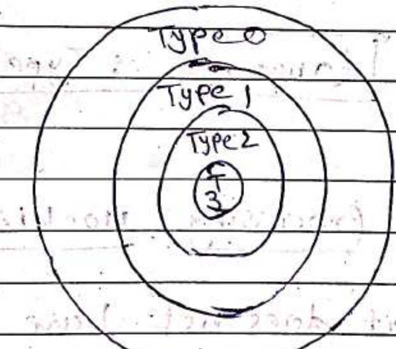
- Type 3 (regular gram.) $A \to a$, $A \to aB \mid Ba$ — a, b: Terminals; A, B: Non-Terminals; α, β: string of terminals & non-terminals

- Type 2 (context free gram) CFG — $A \to \alpha$

- Type 1 (context sensitive) CSG — $\alpha \to \beta$, length of $\beta \geq$ length of α

  ex. $AA \to a$, $AaB \to cC$ } → Not Valid

- Type 0 Universal (Recursivly enumerable gram.)



→ If a language is specified by type i grammar but not by type $(i+1)$ grammar, it is type i language.

$S \to ABC$
$A \to a$
$A \to b$
$aB \to b$
$bB \to a$
$bC \to a$, $aC \to b$

- Grammar is Type 0. | - Language is Type 3.

$aBC \rightarrow bC \rightarrow a$

$S \rightarrow ABC$

$bBC \rightarrow aC \rightarrow b$

$\boxed{\begin{array}{l} S \rightarrow a \\ S \rightarrow b \end{array}}$

Type 3

eg. $S \rightarrow AB$
$AB \rightarrow BA$
$A \rightarrow a$
$B \rightarrow b$

Grammar is of type 1.

$S \rightarrow ab \mid ba$: Type 3
$A \rightarrow b$: Type 2
$B \rightarrow a$
$S \rightarrow AB$: Type 1
$AB \rightarrow ba$

language : Type 3

Information Processing Machine

Machine that does not have memory, output depends on current input.

A finite state machine is composed of following

1. A finite set of states $S = \{s_0, s_1, s_2, \ldots\}$
2. A special state s_0, initial state
3. A set of input letters
   $I = \{i_1, i_2, i_3, \ldots\}$
4. A set of output letters
   $O = \{o_1, o_2, o_3, \ldots\}$

5. A function f from S X I to S, a transition function.

6. A function g from S to O, on output function

eg. Make Modulo 3 sum of input letters {0, 1, 2}

→ Sum of input = 3 % 3 = 0 = 3K
if sum = 3K+1 , mod 3 ⇒ 1
= 3K+2 , mod 3 ⇒ 2

Tabular way:

f State	Inputs 0	1	2	Outputs
A	A	B	C	0
B	B	C	A	1
C	C	A	B	2

(2+2) % 3 = 1 (B).

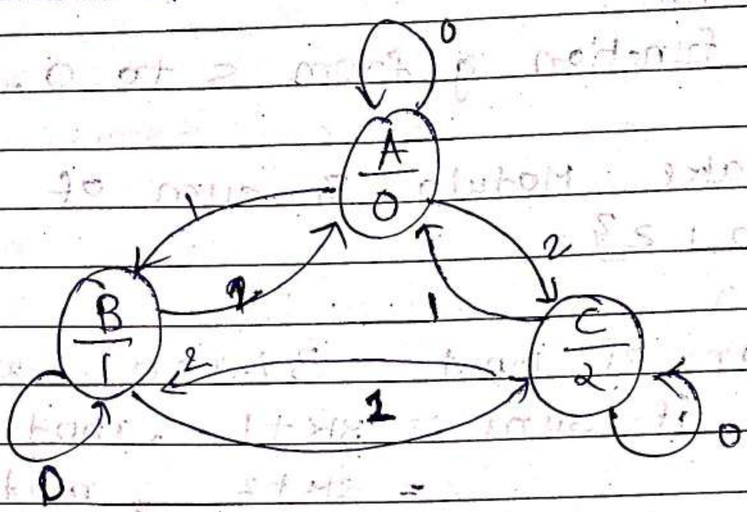
transition f^n:

f(B, 2) = A
f(C, 0) = C

Output f^n:

g(A) = 0
g(B) = 1
g(C) = 2.

Graphical way :



ATM, vending machine, calculator are example of finite state machine.

Two machines are equivalent, if we provide them same set of input signal they produce same output then Both machine are said to be equivalent.

que

	Input		
state	1	2	o/p
⇒ A	B	C	0
B	F	D	0
C	G	E	0
D	H	B	0
E	B	F	1
F	D	H	0
G	E	B	0
H	B	C	1

(A)

	Input		
state	1	2	o/p
⇒ A	B	C	0
B	C	D	0
C	D	E	0
D	E	B	0
E	B	C	1

(B)

i/p : 1122212212

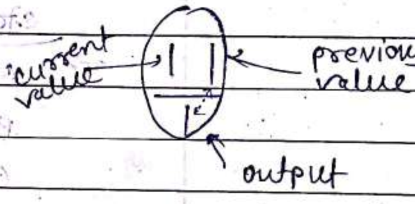
Ans ⇒

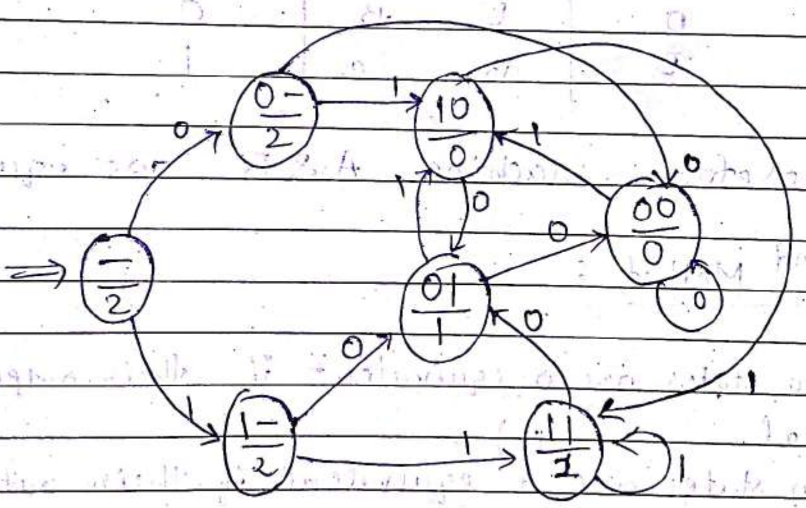
eg. Design a FSM with {0,1} as its input alphabet and {0,1,2} as its output alphabet such that for any input sequence the corresponding output sequence will consists of two 2s followed by the input sequence delayed by one time unit.

i/p 10001011001

~~o/p~~ ~~221000~~01

→ i/p 10001011001
o/p 221000101100





each state handles 2 input (0 & 1)

→ Equivalent Machines

B

i/p = 11222122 12

o/p-A = 000101000 0~~100~~

o/p-B = 0001010000

E & H are equivalent multiple of 4.
C & F are equivalent multiple of 4+2.
B 4+1
D & G are equivalent multiple of 4+3

→ Machine A :

Exhaustive Approach.

State	Input 1	Input 2	Output
⇒ A	B	C	0
B	C	D	0
C	D	E	0
D	E	B	0
E	B	C	1

Therefore, machine A & B are equivalents

⟹ 2nd Method :

- Two states are 0-equivalent if their output is equal.
- Two states are (K) 1-equivalent if their output is equal, and for their every input, its successors are 0-equivalent. (K-1)

	Input		
States	0	1	output
$\Rightarrow$ A	B	F	0
B	A	F	0
C	G	A	0
D	H	B	0
E	A	G	0
F	H	C	1
G	A	D	1
H	A	C	1

partitioning — they are equivalent

$\Pi_0 = \{\overline{ABCDE}, \overline{FGH}\}$

0-equivalent

$\Pi_1 = \{\overline{ABE}, \overline{CD}, \overline{GH}, \overline{F}\}$

1-equivalent

A's successor for 0-input is B
B's successor for 0-input is A
Is B & A are 0-equivalent → Yes.
A & B's successor for 1 input is F
which is 0-equivalent – Therefore
It is 1-equivalent.

$\Pi_2 = \{\overline{AB}, \overline{E}, \overline{CD}, \overline{F}, \overline{GH}\}$

$\Pi_3 = \{\overline{AB}, \overline{CD}, \overline{E}, \overline{F}, \overline{GH}\}$

equivalent FSM:

states	0	1	output
$\Rightarrow$ A	A	F	0
C	G	A	0
E	A	G	0
F	G	C	1
G	A	C	1

Ex.

state	Input 0	Input 1	output
⇒ A	F	B	0
B	D	C	0
C	G	B	0
D	E	A	1
E	D	A	0
F	A	G	1
G	C	H	1
H	A	H	1

$\Pi_0 = \{ \overline{ABCE}, \overline{DFGH} \}$

$\Pi_1 = \{ \overline{AB}, \overline{CE}, \overline{ABCE}, \overline{D}, \overline{FGH} \}$

$\Pi_2 = \{ \overline{AC}, \overline{BE}, \overline{D}, \overline{FGH} \}$

$\Pi_3 = \{ \overline{AC}, \overline{BE}, \overline{D}, \overline{FGH} \}$

states	Input 0	Input 1	output
A	F	B	0
B	D	A	0
D	B	A	1
F	A	F	1

DATE:
PAGE:

(a)

states	Input 0	Input 1	output
⇒ A	B	C	0
B	B	D	0
C	A	E	0
D	B	E	0
E	F	E	0
F	A	D	1
G	B	C	1

(b)

states	Input 0	Input 1	output
A	H	C	0
B	G	B	0
C	A	B	0
D	D	C	0
E	H	B	0
F	D	E	1
G	H	C	1
H	A	E	0

→ For (a), $\Pi_0 = \{\overline{ABCDE}, \overline{FG}\}$

$\Pi_1 = \{\overline{ABCD}, \overline{E}, \overline{FG}\}$

$\Pi_2 = \{\overline{AB}, \overline{CD}, \overline{E}, \overline{FG}\}$

$\Pi_3 = \{\underset{(A)}{\overline{AB}}, \underset{(B)}{\overline{CD}}, \underset{(C)}{\overline{E}}, \underset{(D)}{\overline{FG}}\}$

For (b), $\Pi_0 = \{\overline{ABCDEH}, \overline{FG}\}$

$\Pi_1 = \{\overline{ACDEH}, \overline{B}, \overline{FG}\}$

$\Pi_2 = \{\overline{ADH}, \overline{CE}, \overline{B}, \overline{FG}\}$

$\Pi_3 = \{\underset{(A)}{\overline{ADH}}, \underset{(B)}{\overline{CE}}, \underset{(C)}{\overline{B}}, \underset{(D)}{\overline{FG}}\}$

	Input 0	Input 1	output
A	A	B	0
B	A	C	0
C	D	C	0
D	A	B	1

- <u>Finite</u> <u>State</u> <u>Machine</u> <u>as</u> <u>Language</u> <u>recognizer</u>

(1) A finite set of states
$S = \{s_0, s_1, s_2, \ldots\}$

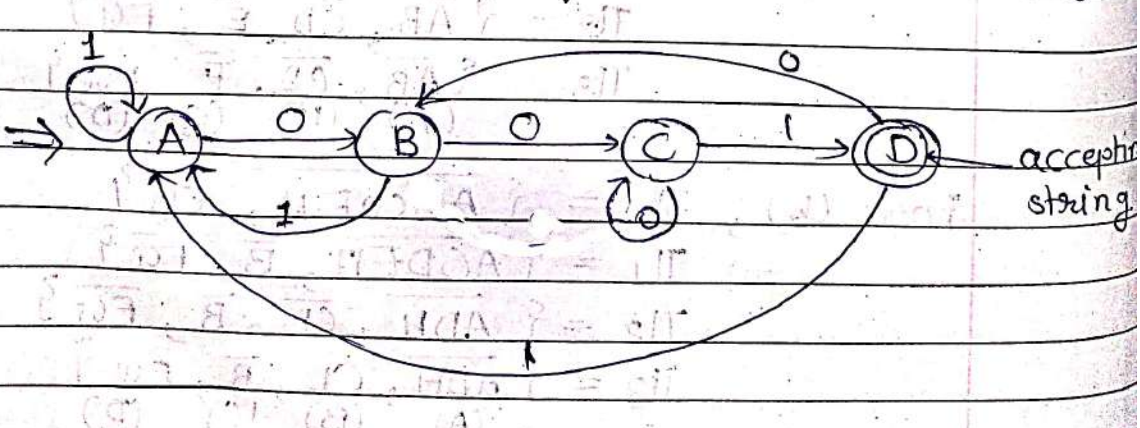
(2) A special state s_0, initial state

(3) A finite set of inputs $I = \{i_0, i_1, i_2, \ldots\}$

(4) A set F representing accepting state(s),
$F \subseteq S$

(5) A transition function f from $S \times I$ to S.

<u>Ex.</u> Design a FSM ~~with~~ which recognize the lang.
$L = \{$All binary strings that end with 001$\}$



$S = \{A, B, C, D\}$

$S_0 = A$ initial state

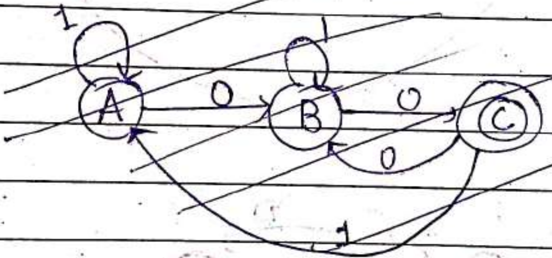
$F = \{D\}$

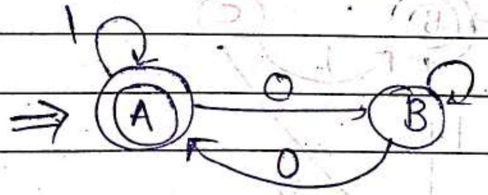
$I = \{0, 1\}$

Transition function:

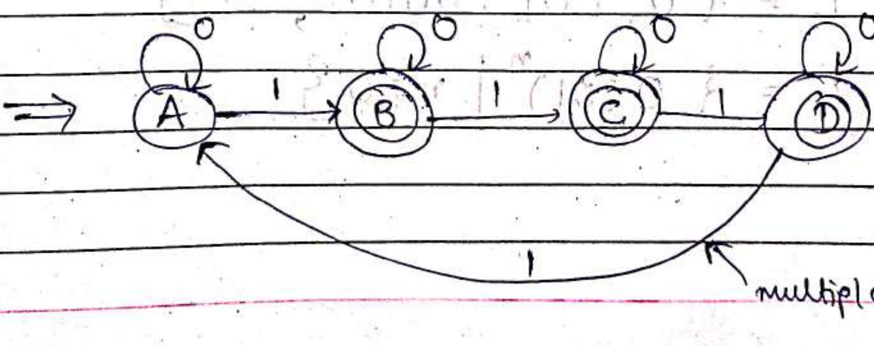
f	0	1
⇒ A	B	A
B	C	A
C	~~B~~ C	D
D	B	A

Ex. L : The set of binary strings with an even no. of 0's.





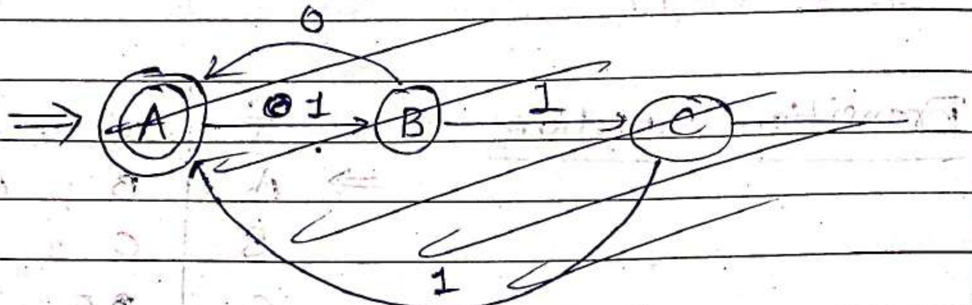
Ex. L = The set of binary strings, where no. of 1's is not multiple of 4.

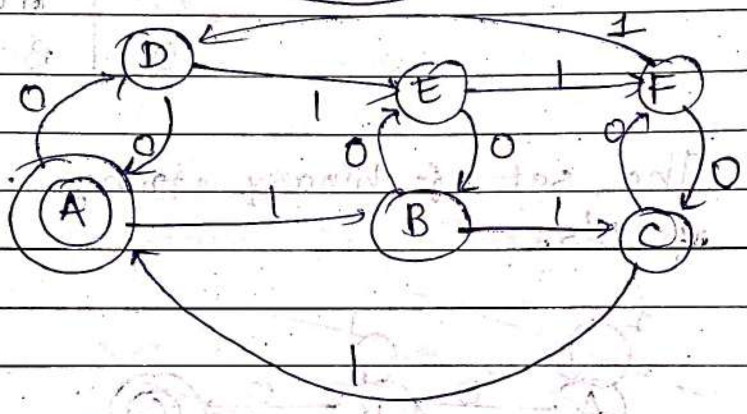


multiple of 4. (reject)

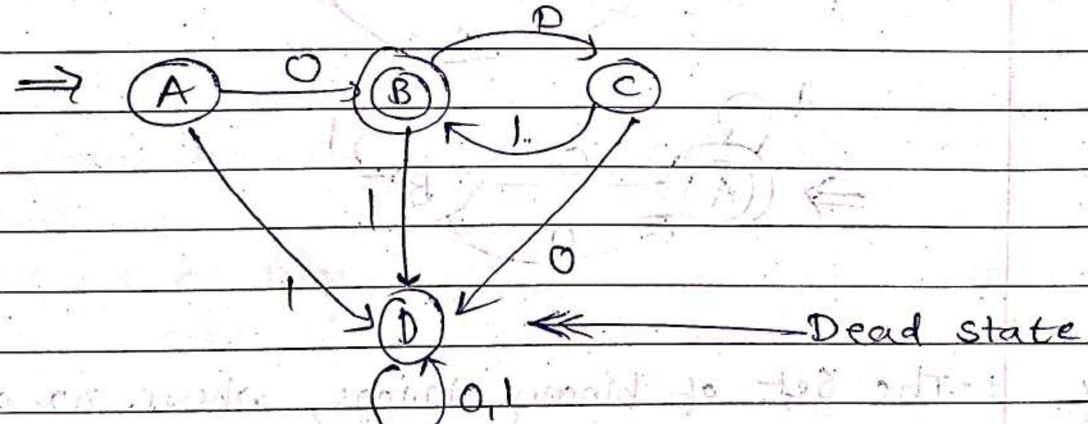
DATE:
PAGE:

Ex. L = { The set of strings of 0's and 1's in each of which the no. of 0's is even and the no. of 1's is multiple of





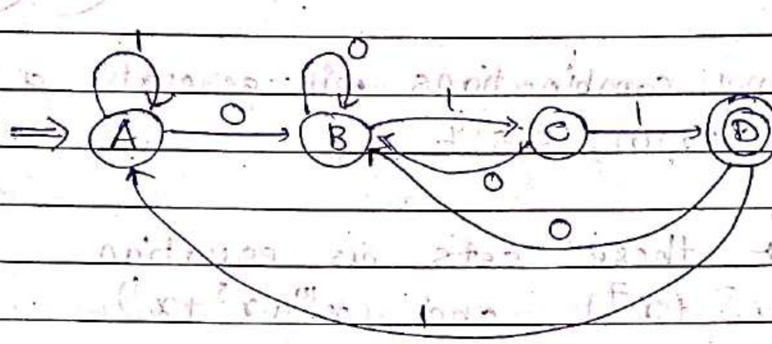
Ex.



$L = \{0, 001, 00101, \ldots\}$

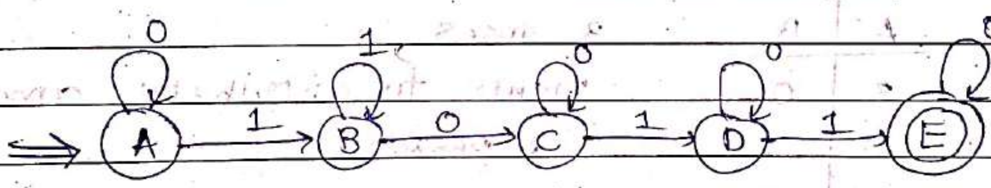
$L = \{0(01)^n \mid n \geq 0\}$

Ex. $L = \{$all binary sequences that end with the digits 011$\}$



$S = \{A, B, C, D\}$
$F = \{D\}$
$S_0 = A$
$I = \{0, 1\}$

Ex. FSM that accepts all binary sequences of the form any number of 0s, followed by one or more 1s, followed by one or more 0s, followed by 1, followed by any no. of 0s, followed by 1 and then followed by anything.



Exam sess.) $aBC \rightarrow abc$ $\Longrightarrow$ we have to check left / right of B. that's why it is context sensitive grammar.

left context right context

Que. $L = \{a^i b^i c^j \mid i \geq 1, j \geq 1\}$

$S \rightarrow AB$
$A \rightarrow ab \mid aAb$
$B \rightarrow c \mid BC$

Permutations & Combinations

DATE:
PAGE:

Generating Function

Ex. →

set 1   set 2
2, 5, 7   10, 5, 1

Que How many combinations will generate a sum 12?
2. {(2,10), (7,5)}

Represet these sets as equation
$(x^2+x^5+x^7)$ and $(x^{10}+x^5+x^1)$

$\Rightarrow$ Multiply them,
$x^{12}+x^7+x^3+x^{15}+x^{10}+x^6+x^{17}+x^{12}+x^8$
$\Rightarrow x^3+x^6+x^7+x^8+x^{10}+(x^{12})\cdot 2+x^{15}+x^{17}$

lowest possible sum

two pairs available which make sum pair of 12.

highes possible.

Ex. →

A	B
3	0
0	3
1	2
2	1

3 roses, wants to distribute among 2 persons

A   B
[0,1,2,3]   [0,1,2,3] ← possible roses can have
$(x^0+x^1+x^2+x^3)\ (x^0+x^1+x^2+x^3)$

$\Rightarrow x^0x^0+x^0x^1+x^0x^2+\boxed{x^0x^3}+x^1x^0+x^1x^1+\boxed{x^1x^2}+x^1x^3$
$+x^2x^0+\boxed{x^2x^1}+x^2x^2+x^2x^3+\boxed{x^3x^0}+x^3x^1+x^3x^2$
$+x^3x^3$

Valid → where sum is 3.

DATE:
PAGE:

→ used – to solving counting problems.

- 1 to 1 mapping
2 persons A, B ⇒ 3 objects distribute
[0, 1, 2, 3] [0, 1, 2, 3]
$(x^0+x^1+x^2+x^3)\ (x^0+x^1+x^2+x^3)$

- Recurrence Relation – we can use generating fun^c.

- Downside – complex problem it would be lengthy of generating fun^c.

Two types
① Ordinary – combination problem (order doesn't matter)
② Exponential – to solve permutation problem

• Generating Function :

def^n. Let $(a_0, a_1, \ldots, a_n)$ be a symbolic representation of an event, or let it be simply a sequence of numbers,
The function $f(x) = a_0 u_0(x) + a_1 u_1(x) + \ldots + a_n u_n(x)$
is called ordinary generating function.
Here, $u_0(x), u_1(x), \ldots, u_n(x)$ is a sequence of function of x used as indicators.

ex. variables = (3, 2, 6, 0, 0, 0) [seq. of events]
indicators = $1, 1+x, 1-x, 1+x^2, 1-x^2, \ldots$

OGF = $f(x) = a_0 x^0 + a_1 x^1 + \ldots + a_n x^n$

ordinary generating fun^c

$$= \sum_{r=0}^{n} a_r x^r$$

$= 3(1) + 2(1+x) + 6(1-x)$

$= \boxed{11-4x}$

→ whenever we apply diff. sequence of events and if our OGF produce same output then our OGF is faulty.

we need to change indicator function.

→ whenever we apply diff. sequence of events on diff. indicators then, it should produce all diff. OGFs.

ex. events = $\{1, 1, 1, \ldots\}$

→ $OGF = f(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots + a_n x^n$

$$= \sum_{r=0}^{n} a_r x^r$$

$$= (1) x^0 + (1) x^1 + (1) x^2 + \ldots$$

$$= 1 + x + x^2 + \ldots$$

$$\boxed{f(x) = \frac{1}{1-x}}$$

($\because$ Geometric seri $a + ar + \ldots \infty = \frac{a}{1-r}$)

ex. events = $\{1, 1, 3, 1, \ldots\}$

→ $OGF = (1)x^0 + (1)x^1 + (3)x^2 + (1)x^3 + \ldots$

$$= 1 + x + 3x^2 + x^3 + x^4 + \ldots$$

$$\cancel{= 1 + x + 3x^2 + x^3(1 + x + x^2 + \ldots)}$$

$$\cancel{= 1 + x + 3x^2 + x^3\left(\frac{1}{1-x}\right)}$$

DATE:
PAGE:

$= 2x^2 + (1 + x + x^2 + x^3 + --)$

$= \boxed{2x^2 + (1-x)^{-1}}$

ex. events $= (1, -1, 1, -1, 1, -1, --)$

$\longrightarrow$ OGF $= (1+x)^{-1}$

ex events $= (1, 2, 3, 4, ---)$

$\longrightarrow$ OGF $= (1)x^0 + 2x^1 + 3x^2 + 4x^3 + ---$

$= 1 + 2x + 3x^2 + 4x^3 + --- \quad -(1) \text{ (AP + GP)}$

$g(x) = x \cdot f(x) = x + 2x^2 + 3x^3 + 4x^4 + --- \quad -(2)$

take (1) $-$ (2)

$f(x) - x \cdot f(x) = 1 + x + x^2 + x^3 + ---$

$f(x)(1-x) = \frac{1}{1-x}$

$\boxed{f(x) = (1-x)^{-2}}$

$f(x) = x + x^2 + x^3 + x^4 + ---$

$f(x) = x(1 + x + x^2 + x^3 + --)$

$f(x) = x \cdot (1-x)^{-1}$

ex. events $= (0, 1, 2, 3, 4, ---)$

ex. $a_r = r \cdot b \cdot a^r \leftarrow$ numeric function

$\longrightarrow$ OGF $= f(x) = \sum_{r=0}^{\infty} a_r x^r$

$= 0 + bax + 2ba^2x^2 + 3b\cancel{x}a^3x^3 + ---$

$= bax + 2ba^2x^2 + 3ba^3x^3 + ---$

$\therefore f(x) = b(ax + 2a^2x^2 + 3a^3x^3 + --)$

$\therefore g(z) = \cancel{b}(z + 2z^2 + 3z^3 + ---)$

$\therefore z \cdot g(z) = \cancel{b}(z^2 + 2z^3 + 3z^4 + ---)$

we can avoid b

$\therefore g(z) - z \cdot g(z)$

$= z + z^2 + z^3 + \ldots$

~~$g(z) = \frac{1}{(1-z)^2}$~~

$g(z)(1-z) = z(1 + z + z^2 + \ldots)$

$\therefore g(z) = \frac{z}{(1-z)^2}$

$\therefore \boxed{f(x) = \frac{b \cdot ax}{(1-ax)^2}}$

ex. events = $\left\{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \ldots\right\}$ Ans $\frac{\log(1+x)}{x}$

$\longrightarrow$ OGF $= (1)x^0 - \frac{1}{2}x^1 + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \ldots$

$= 1 - \frac{1}{2}x + \frac{x^2}{3} - \frac{x^3}{4}$

$x \cdot f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$\therefore x \cdot f(x) = \log(1+x)$

$\therefore \boxed{f(x) = \frac{\log(1+x)}{x}}$

ex. $(1, -1, 1, -1, 1, \ldots)$

$\longrightarrow$ OGF $= (1)x^0 - (1)x^1 + (1)x^2 - (1)x^3 + (1)x^4 \ldots$

$= 1 - x + x^2 - x^3 + x^4 \ldots$

$= \frac{1}{1+x}$

$= \boxed{(1+x)^{-1}}$

→ $(a+b)^2$ $\quad \underline{a/b} \quad \underline{a/b}$

arranging	a	a	a^2
2 objects	a	b	ab
in 2 places	b	a	ab
	b	b	b^2

$= a^2+2ab+b^2$

→ $(a+b)^3$ $\quad \underline{a/b} \quad \underline{a/b} \quad a/b$

arranging	a	a	a
2 objects	a	a	b
in 3 places	a	b	a

$= a^3+3a^2b+3ab^2+b^3$

→ $(a+b)^2$ = 0 'a' 0 times a (both b)
= 1 'a' 1 times a (ab or ba)
= 2 'a' 2 times a (0 times b)

$= 2C_0\, a^0b^2 + 2C_1\, a^1b^1 + 2C_2\, a^2b^0$

$= \boxed{b^2 + 2ab + a^2}$

→ $(a+b)^2 = \binom{2}{0} a^0b^2 + \binom{2}{1} a^1b^1 + \binom{2}{2} a^2b^0$

Ordinary Binomial expansion

$(a+b)^3 = \binom{3}{0} a^0b^3 + \binom{3}{1} a^1b^2 + \binom{3}{2} a^2b^1 + \binom{3}{3} a^3b^0$

$(a+b)^n = \binom{n}{0} a^0b^n + \binom{n}{1} a^1b^{n-1} + \binom{n}{2} a^2b^{n-2} + \dots + \binom{n}{n} a^nb^0$

→ Pascal's triangle

Pascal's identity

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

Date:
Page:

→ Pascal's identity $^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$

Proof: $R.H.S. = C(n-1, r-1) + C(n-1, r)$

$$= \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{(r)!\,(n-r-1)!}$$

$$= \frac{r\cdot(n-1)!}{r!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!}$$

$$= \frac{(n-1)!}{r!}\left[\frac{r}{(n-r)!} + \frac{1}{(n-r-1)!}\right]$$

$$= \frac{(n-1)!}{r!}\left[\frac{r}{(n-r)!} + \frac{(n-r)}{(n-r)!}\right]$$

$$= \frac{n\cdot(n-1)!}{r!\,(n-r)!} = \frac{n!}{r!\,(n-r)!}$$

$$= {}^{n}C_{r}$$

$= L.H.S.$

Hence, it is proved.

→ $\boxed{r^{th} \text{ term} = \binom{n}{r} a^{r} b^{n-r}}$

Que what is the coefficient of x^{15} and x^{12} in $(x^2 + x^{-3})^{10}$

→ r^{th} term $= \binom{10}{r}(x^2)^{r}(x^{-3})^{10-r}$

$$= \frac{10!}{r!\,(10-r)!}\, x^{2r} \cdot x^{-30+3r}$$

$$= \binom{10}{r} x^{5r-30}$$

Now, $x^{15} = x^{5r-30}$

$\Rightarrow 5r - 30 = 15$

$\Rightarrow \boxed{r = 9}$

coefficient of $x^{15} = \binom{10}{r} = \binom{10}{9}$

Similarly,

$x^{12} = x^{5r-30}$

$\Rightarrow 12 = 5r - 30$

$\Rightarrow \boxed{r = \frac{42}{5}}$ but $r \notin Z^{+}$ (r is fractional)

$\therefore$ No, term with x^{12} exist in given expansion

Que. $(x + x^{-1})^{10}$ find coefficient of x^6 and x^7.

$\rightarrow$ r^{th} term $= \binom{10}{r} (x)^{r} (x^{-1})^{10-r}$

$$= \binom{10}{r} x^{r} \cdot x^{r-10}$$

$$= \binom{10}{r} x^{2r-10}$$

$2r - 10 = 6$

$2r = 16$

$\boxed{r = 8}$

Coeff. is $\binom{10}{8}$

And

$2r - 10 = 7$

$2r = 17$

$\boxed{r = 17/2} \notin Z^{+}$

No coeff. & x^7 exist

$x^2y^5z^3$ in $(2x+y+3z)^{10}$

General term $= \frac{n!}{p!\,q!\,r!}(2x)^p(y)^q(3z)^r$ 　 $p=2,\ q=5,\ r=3$

DATE:
PAGE:

Que $(2x-3y)^{25}$ Find coeff. of $x^{12}y^{13}$

r^{th} term $= \binom{25}{r}(2x)^r(-3y)^{25-r}$

$= \binom{25}{r}2^r \cdot x^r \cdot (-3)^{25-r} \cdot y^{25-r}$

Now, $x^r = x^{12}$

$\boxed{r=12}$

coeff $= \binom{25}{12}2^{12} \cdot (-3)^{13}$

Que What is the coeff of $x^6y^2z^2$ in $(x+y+z)^{10}$

$= \frac{10!}{6!\,2!\,2!} = \frac{10 \times 9 \times 8 \times 7}{4} = \boxed{1260}$

Que Find term independent of x in $\left(\sqrt{x}-\frac{2}{x^2}\right)^{10}$

Que Find 5^{th} term in the expansion of $\left(2x^2+\frac{3}{2x}\right)^8$

- Extended binomial expansion

$(a+b)^n$

$= \binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \binom{n}{2}a^2b^{n-2} + \dots + \binom{n}{n}a^nb^{n-n}$

$$= \frac{n!}{0!(n-0)!} a^0 b^n + \frac{n!}{1!(n-1)!} a^1 b^{n-1} + \frac{n!}{2!(n-2)!} a^2 b^{n-2}$$

$$+ \dots + \frac{n!}{n!(n-n)!} a^n.$$

put $\boxed{a = x, b = 1}$

$$= \frac{n!}{0!(n-0)!} x^0 1^n + \frac{n!}{1!(n-1)!} x^1 1^{n-1} + \frac{n!}{2!(n-2)!} x^2 1^{n-2}$$

$$+ \dots + \frac{n!}{n!(n-n)!} x^n.$$

$$= 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3$$

$$+ \dots + x^n$$ (finite series)

$$= (1+x)^n.$$

→ 3 standard ways to write combinations

(1)	(2)	(3)
^nC_r	$\binom{n}{r}$	$C(n,r)$
(selecting r from n items)	valid for n is (−ve) or fractional	(n is +ve)
only valid when n is (+ve)		

$(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}x^3 + \ldots$

[$\because$ $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$

~~double prime~~ first — double prime — infinite series because Maclaurian's series is infinite.

Maclaurian series]

Assume, $0 < |x| < 1$ → so that we can terminate it.
→ so that it can become finite.

assume $(-n) = p$,

$(1+x)^p$

$= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$

$= \sum_{r=0}^{\infty} \binom{p}{r} x^r$

$\therefore (1+x)^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} x^r$

$\therefore (a+b)^{-n} = b^{-n}\left(\frac{a}{b}+1\right)^{-n}$

$= \boxed{b^{-n} \sum_{r=0}^{\infty} \binom{-n}{r} \left(\frac{a}{b}\right)^r}$

where,
$0 < |x| < 1$
$0 < \left|\frac{a}{b}\right| < 1$
$|b| > |a|$

DATE:
PAGE:

$$= b^{-n} \boxed{\sum_{r=0}^{\infty} \binom{-n}{r} a^{r} \cdot b^{-n-r}} \quad |b| > |a|$$

- Find a relation betⁿ ordinary binomial coeff. and extended binomial coeff.

→ Ordinary binomial coeff. $= C(n, r)$
extended binomial coeff. $= \binom{-n}{r}$

$$C(3,2) = \frac{3!}{2!\,(1!)} = 3$$

$$C(6,2) = \frac{6!}{2!\,4!} = \frac{6 \times 5}{2!}$$

$$C(10,3) = \frac{10 \times 9 \times 8}{3!}$$

$$C(16,4) = \frac{16 \times 15 \times 14 \times 13}{4!}$$

★ ⟹ $$\binom{n}{r} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-(r-1))}{r!}$$

$$\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2) \cdots (-n-(r-1))}{r!}$$

$$= \frac{(-1)^{r}\, n \cdot (n+1) \cdot (n+2) \cdots (n+(r-1))}{r!}$$

$$= \frac{(-1)^{r}\,(n+(r-1))\,(n+(r-2)) \cdots (n+2)(n+1)\,n}{r!}$$

$$= \frac{(-1)^{r}\,(n+(r-1))\,(n+(r-2)) \cdots (n+1)\, n\, (n-1)!}{r!\,(n-1)!}$$

$$\binom{-n}{r} = \frac{(-1)^{r}\,(n+(r-1))!}{r!\,(n-1)!}$$

$$^{n+r-1}C_r = \frac{(n+r-1)!}{r!\,(n-1)!}$$

$$\boxed{\binom{-n}{r} = (-1)^r \cdot {}^{n+r-1}C_r} \leftarrow \text{Show symbol}$$

Ex What is the coefficient of x^9 in power series $(1+x^3+x^6+x^9+\dots)^3$

$\longrightarrow$

$$S = (1+x^3+x^6+x^9+\dots)^3$$

$$= \left(\frac{1}{1-x^3}\right)^3 \qquad \left(\because \text{GP series} = \frac{a}{1-r}\right)$$

$$= (1-x^3)^{-3} \qquad (\because 0<|x|<1)$$

$$= \sum_{r=0}^{\infty} \binom{-3}{r} (-x^3)^r \qquad (\because |b|>|a|)$$

$$= \sum_{r=0}^{\infty} \binom{-3}{r} (-1)^r x^{3r}$$

put $r=3$,

coeff. of x^9 is $= \binom{-3}{3}(-1)^3$

$$= (-1)^3 \cdot {}^{3+3-1}C_3 \cdot (-1)^3$$

$$= (-1)^6 \cdot {}^5C_3$$

$$= \frac{5\times4\times3}{3!}$$

$$= \boxed{10}$$

$$\boxed{{}^{n}C_r = {}^{n}C_{n-r}}$$

DATE:
PAGE:

Ex. $S = (x^2 + x^3 + x^4 + x^5 + \dots)^8$. coeff. of x^9.

* Discrete numeric function

$f : \mathbb{Z}^+ \longrightarrow \mathbb{R}$

$\uparrow$ set of +ve integers (i/p)

$\uparrow$ set of real numbers (o/p).

$\sum_{r=0}^{n} a_r x^r \longrightarrow$ a_r is discrete numeric function

eg. $(1, 2, 9, 28, \dots)$

$a_r = (r^3 + 1) \quad r \geq 0.$

Ex. Find discrete numeric function if generating function is $f(x) = \dfrac{1}{(1-ax)^K}$

$\longrightarrow$ $f(x) = (1-ax)^{-K}$

$= \sum_{r=0}^{\infty} \binom{-K}{r} (-ax)^r$

$= \sum_{r=0}^{\infty} \binom{-K}{r} (-a)^r x^r$

$a_r = \binom{-K}{r} (-a)^r$

$= \binom{-K}{r} (-1)^r a^r = (-1)^r \, {}^{K+r-1}C_r (-1)^r a^r$

$=$

DATE:
PAGE:

$a_r = {}^{k+r-1}C_r \, a^r$

$a_r = \left({}^{k-1}C_0, {}^{k}C_1 a, {}^{k+1}C_2 a^2, \ldots \right)$

discrete numeric function.

Ex. $f(x) = (1-ax)^{-1}$

$= \sum_{r=0}^{\infty} \binom{-1}{r} (-ax)^r$

$= \sum_{r=0}^{\infty} \binom{-1}{r} (-a)^r x^r$

Therefore,

$a_r = \binom{-1}{r} (-a)^r$

$= (-1)^r \cdot {}^{1+r-1}C_r \, (-1)^r \cdot a^r$

$= {}^{r}C_r \, a^r$

$\boxed{a_r = a^r}$

$\therefore a_r = (1, a, a^2, a^3, \ldots)$

Homework Questions

(1) find term independent of x in $\left(\sqrt{x} - \frac{2}{x^2}\right)^{10}$

r^{th} term $= \binom{10}{r} (\sqrt{x})^r \left(\frac{-2}{x^2}\right)^{10-r}$

DATE:
PAGE:

$$= \binom{10}{r} x^{\frac{r}{2}} \cdot (-2)^{10-r} (x^{-2})^{10-r}$$

$$= \binom{10}{r} (-2)^{10-r} \cdot x^{\frac{r}{2}+2r-20}$$

Now, $x^0 = x^{\frac{r}{2}+2r-20}$

$$\Rightarrow \frac{r}{2}+2r-20=0$$

$$\Rightarrow r+4r-40=0$$

$$\Rightarrow 5r=40 \Rightarrow \boxed{r=8}$$

$\therefore$ $\boxed{9^{th} \text{ term}}$ is independent from x.

(2) Find 5th term in $\left(2x^2+\frac{3}{2x}\right)^8$

$$r^{th} \text{ term} = \binom{8}{r} (2x^2)^r \cdot \left(\frac{3}{2x}\right)^{8-r}$$

$$= \binom{8}{r} 2^r \cdot x^{2r} \cdot 3^{8-r} (2x)^{r-8}$$

$$= \binom{8}{r} 2^{2r-8} \cdot 3^{8-r} \cdot x^{3r-8}$$

take $r=4$,

$$= \binom{8}{4} 2^0 \cdot 3^4 \cdot x^4 = \boxed{5670x^4}$$

(3) Find coeff. of x^9 in $(x^2+x^3+x^4+\dots)^3$

$$= \left[x^2(1+x+x^2+\dots)\right]^3$$

DATE:
PAGE:

$$= x^6 (1+x+x^2+x^3+\dots)^3$$

$$= x^6 (1-x)^{-3}$$

$$= x^6 \sum_{r=0}^{\infty} \binom{-3}{r} (-x)^r$$

$$= x^6 \sum_{r=0}^{\infty} \binom{-3}{r} (-1)^r x^r$$

put $r=3$

coeff. of x^9 is $= \binom{-3}{3} (-1)^3$

$$= (-1)^3 \, {}^{3+3-1}C_3 \, (-1)^3$$

$$= {}^5C_3$$

$$= \boxed{10}$$

Ex. →

$$f(z) = (a-bz)^{-1}$$

$$= a^{-1}\left(1-\frac{bz}{a}\right)^{-1}$$

$$= a^{-1} \sum_{r=0}^{\infty} \binom{-1}{r} \left(\frac{-bz}{a}\right)^r$$

$$= a^{-1} \sum_{r=0}^{\infty} (-1)^r \, {}^rC_r \, (-1)^r \frac{b^r}{a^r} z^r$$

$$\boxed{f(z) = \sum_{r=0}^{\infty} \frac{b^r}{a^{r+1}} z^r}$$

$$\boxed{a_r = \frac{b^r}{a^{r+1}}} = \left\{\frac{1}{a}, \frac{b}{a^2}, \frac{b^2}{a^3}, \dots\right\}$$

The coeff. of z^r is discrete generating funᶜ

Ans: $2(-1)^r + 3^r$

ex $f(x) = \dfrac{3-5x}{1-2x-3x^2}$

$f(x) = \dfrac{5x-3}{3x^2+2x-1}$

$= \dfrac{5x-3}{(3x-1)(x+1)} = \dfrac{A}{x+1} + \dfrac{B}{3x-1}$

$\Rightarrow A(3x-1) + B(x+1) = 5x-3$

take $x=-1$	take $x=0$
$-4A = 5\cdot(-1)-3$	$2(-1)+B=-3$
$\boxed{A=2}$	$\boxed{B=-1}$

$$\boxed{f(x) = \frac{2}{x+1} - \frac{1}{3x-1}}$$

$= 2(1+x)^{-1} + (1-3x)^{-1}$

$= 2\sum_{r=0}^{\infty}\binom{-1}{r}x^r + \sum_{r=0}^{\infty}\binom{-1}{r}(-3x)^r$

$= \sum_{r=0}^{\infty} 2\cdot(-1)^r\cdot {}^rC_r\cdot x^r + \sum_{r=0}^{\infty}(-1)^r\cdot {}^rC_r\cdot(-1)^r\cdot 3^r\cdot x^r$

$= \sum_{r=0}^{\infty} \underline{2\cdot(-1)^r}\, x^r + \sum_{r=0}^{\infty} \underline{3^r}\cdot x^r$

therefore discrete numeric function is,

$$\boxed{a_r = 2\cdot(-1)^r + 3^r}$$

DATE:
PAGE:

→ Open form of eq^n $a_n = a_{n-1} + a_{n-2}$
closed form of eq^n $a_r = 2(-1)^r + 3^r$

- Advantage of discrete ~~nur~~ numeric generating fun^c:
  - we can get eq^n which is "closed form".

• Applications of generating functions

- combinatoring functions (to solve)
- solving counting problems.

Que. Find number of solution of
$e_1 + e_2 + e_3 = 17$
$2 \le e_1 \le 5$, $3 \le e_2 \le 6$, $4 \le e_3 \le 7$

→ for e_1, $(x^2 + x^3 + x^4 + x^5)$ —— (1)
for e_2, $(x^3 + x^4 + x^5 + x^6)$ —— (2)
for e_3, $(x^4 + x^5 + x^6 + x^7)$ —— (3)

Multiply 1 & 2 & 3

$$x^2(1+x+x^2+x^3) \cdot x^3(1+x+x^2+x^3)\, x^4(1+x+x^2+x^3)$$

$$= x^9(1+x+x^2+x^3)^3$$

$$= x^9 \left[\frac{1(x^4-1)^3}{(x-1)^3}\right] \qquad \left(\because\ a_r = \frac{a(r^{n+1}-1)}{r-1}\right)$$

$$= x^9 \frac{(1-x^4)^3}{(1-x)^3}$$

$$= x^9 (1-x^4)^3 (1-x)^{-3}$$

$$= x^9 \sum_{r=0}^{3} \binom{3}{r} (-x^4)^r \sum_{r=0}^{\infty} \binom{-3}{r} (-x)^r$$

$$= \sum_{r=0}^{3} {}^3C_r (-1)^r x^{4r+9} \cdot \sum_{r=0}^{\infty} (-1)^r \, {}^{r+3-1}C_r (-1)^r x^r$$

$$= \sum_{r=0}^{3} {}^3C_r (-1)^r x^{4r+9} \sum_{r=0}^{\infty} {}^{r+2}C_2 \, x^r$$

$$\left({}^nC_r = {}^nC_{n-r} \right)$$

For x^{17},

i) $[0, 8]$, $\Rightarrow {}^3C_0 \cdot {}^{10}C_2$

$r=0$ in 1st part x^9 , $r=8$ in 2nd part x^8

ii) $[1, 4] \Rightarrow {}^3C_1 (-1) \cdot {}^6C_2$

iii) $[2, 0] \Rightarrow {}^3C_2 \cdot (1) \cdot {}^2C_2$

$$\Rightarrow {}^3C_0 \cdot {}^{10}C_2 + {}^3C_1 \cdot (-1) \cdot {}^6C_2 + {}^3C_2 \cdot {}^2C_2$$
$$= (1) \cdot (45) - (3) \cdot (15) + (3) \cdot (1)$$
$$= 45 - 45 + 3 = \boxed{3}$$

Que. In how many diff. ways can 8 identical cookies be distributed among 3 distinct children if each child receives at least 2 cookies ~~no~~ and no more than 4 cookies.

DATE:
PAGE:

Que Find the no. of ways of selecting 'r' objects from 'n' objects with unlimited repetitions.

→ $e_1 + e_2 + e_3 + \dots + e_n = r$

$[0, 1, 2, \dots]$

$x^0 + x^1 + x^2 + \dots$

$f(x) = (x^0 + x^1 + x^2 + \dots)^n$ ← total no. of objects

$= (1 + x + x^2 + \dots)^n$

$= \dfrac{1 \cdot (1 - x^{\infty})^n}{(1-x)^n}$ $\quad (\because 0 < |x| < 1)$

$= \dfrac{1}{(1-x)^n} = \boxed{(1-x)^{-n}}$

$= \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r$

$= \sum_{r=0}^{\infty} {}^{r+n-1}C_r \, (-1)^r \cdot (-1)^r \, x^r$

$= \sum_{r=0}^{\infty} \underline{{}^{r+n-1}C_r} \cdot x^r$

with repeatations

Que How many ways we choose a commitetee of 9 members from 3 political parties so that no party has absolute majority in the commitytee?

DATE:
PAGE:

• Exponential Generating functions :

→ We want to choose 3 letters from 2 letters 'a' & 'b'.

_ _ _ 'a' & 'b'

$(x^0+x^1+x^2+x^3)(x^0+x^1+x^2+x^3)$

$= x^0x^3 + x^1x^2 + x^2x^1 + x^3x^0$ — 4 ways to choose.

bbb abb aab aaa

total possible words (order matters).

(1) $bbb = \frac{3!}{0!\,3!} = 1$

for a, for b

(2) $abb = \frac{3!}{1!\,2!} = 3$

(3) $aab = \frac{3!}{2!\,1!} = 3$

(4) $aaa = \frac{3!}{3!\,0!} = 1$

$= \frac{3!}{0!\,3!} + \frac{3!}{1!\,2!} + \frac{3!}{2!\,1!} + \frac{3!}{3!\,0!}$

$= 3!\left(\frac{1}{0!\,3!} + \frac{1}{1!\,2!} + \frac{1}{2!\,1!} + \frac{1}{3!\,0!}\right)$

$= \boxed{8}$

Que. No. of words of length 3 from unlimited supply of 'a' & 'b'.

DATE:
PAGE:

$$\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)$$

$$= \frac{x^0x^3}{0!\,3!} + \frac{x^1x^2}{1!\,2!} + \frac{x^2x^1}{2!\,1!} + \frac{x^3x^0}{3!\,0!}$$

$$= x^3\left(\frac{1}{0!\,3!} + \frac{1}{1!\,2!} + \frac{1}{2!\,1!} + \frac{1}{3!\,0!}\right)$$

$$= x^3 \underbrace{A_3}_{\text{coeff. of } x^3}$$

$$\longrightarrow \quad f(x) = x^0A_0 + x^1A_1 + x^2A_2 + \dots$$

$$= \sum_{r=0}^{\infty} A_r x^r$$

$$= \sum_{r=0}^{\infty} r!\,A_r \frac{x^r}{r!}$$

$$\boxed{f(x) = \sum_{r=0}^{\infty} a_r \frac{x^r}{r!}}$$

Def^n^: let $(a_0, a_1, a_2, \dots)$ be a symbolic representat of a sequence of an event or let it be a sequence of a numbers, the function

$$f(x) = a_0\frac{x^0}{0!} + a_1\frac{x^1}{1!} + a_2\frac{x^2}{2!} + \dots$$

called exponential generating function of the sequence $(a_0, a_1, a_2, \dots)$

DATE:
PAGE:

Ex. Find the exponential generating fun^c for a_r the no. of diff. arrangments of r objects from 4 diff. types of objects with each type of object available at least 2 times and no more than 5 times.

→ $O_1 + O_2 + O_3 + O_4 = r$

$[2, 3, 4, 5]$

$$\boxed{\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\right)^4}$$

Ex. Find the exp. generating function for the no. of ways to place r distinct people into 3 rooms with at least one person in each room.

→

$$\begin{array}{cccc} & r_1 & r_2 & r_2 \\ \text{persons} & \begin{bmatrix} a & d & c \\ d & a & c \end{bmatrix} \end{array}$$

⟹ diff. arrangement. Therefore it is permutation problem.

$r_1 \; r_2 \; r_3 = r$

$[1, 2, 3, \ldots]$

Exponential generating function is

$$\left(\frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^3$$

- What if we want an even no. of people in each room?

DATE:
PAGE:

$r_1 + r_2 + r_3 = r$

$[2, 4, 6, \ldots]$

Exp. generating fun^c is,

$$\boxed{\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\right)^4.}$$

→ Expansion of e^x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^{nx} = 1 + nx + \frac{n^2x^2}{2!} + \frac{n^3x^3}{3!} + \ldots$$

$$= \boxed{\sum_{r=0}^{\infty} \frac{n^r x^r}{r!}}$$

$$\frac{1}{2}\left[e^{-x} + e^{x}\right] = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\frac{1}{2}\left(e^{x} - e^{-x}\right) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

Ex. Find the no. of diff. r arrangements of objects chosen from an unlimited supply of n types of objects.

→ $\underset{n \; n \; n \; n \; n}{r \text{ places}} = \boxed{n^r}$

$O_1 + O_2 + O_3 + \ldots + O_n = r$

$[0, 1, \ldots, r]$

DATE:
PAGE:

$$\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots\right)^n$$

$$=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^n$$

$$=e^{xn}$$

$$=\sum_{r=0}^{\infty} n^r\cdot\frac{x^r}{r!} \qquad \boxed{Ans = n^r}$$

Ex. Find the no. of r digit quaternary (0, 1, 2, 3) sequences with an even no. of 0's and odd no. of 1's.

→

#0	#1	#2	#3
c_1	c_2	c_3	c_4
[0, 2, 4, ...]	[1, 3, 5, ...]	[0, 1, 2, ...]	

$$\left(\frac{x^0}{0!}+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(\frac{x^1}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots\right)^2$$

$$=\frac{1}{2}(e^x+e^{-x})\cdot\frac{1}{2}(e^x-e^{-x})\cdot(e^x)^2$$

$$=\frac{1}{4}(e^{2x}-e^{-2x})\cdot e^{2x}=\frac{1}{4}\left[e^{4x}-1\right]$$

$$=\frac{1}{4}\left(\sum_{r=0}^{\infty}4^r\frac{x^r}{r!}-1\right)=\sum_{r=0}^{\infty}4^{r-1}\frac{x^r}{r!}-\frac{1}{4}.$$

$$\boxed{Ans = 4^{r-1}}$$

closed
(compact form of eq^n)

Que Solve the recurrence relation using generating function

$$a_{n+2} - 3a_{n+1} + 2a_n = 0 \; ; \; a_0 = 2, \; a_1 = 3$$

→ Multiply with x^n and $\sum_{n=0}^{\infty}$ → therefore we can write a_0 putting $n=0$

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 3\sum_{n=0}^{\infty} a_{n+1}x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

Expanding ... we ~~can~~ get

$$= (a_2 + a_3x + a_4x^2 + \dots) - 3(a_1 + a_2x + a_3x^2 + \dots)$$
$$+ 2(a_0 + a_1x + a_2x^2 + \dots) = 0.$$

$$\Big[a_2 + a_3x + a_4x^2 + \dots$$
$$= \frac{a_2x^2 + a_3x^3 + a_4x^4 + \dots}{x^2}$$
$$= \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + \dots - a_0 - a_1x}{x^2}$$
$$= \frac{G(x) - a_0 - a_1x}{x^2} \Big]$$

= Rewrite eq^n in form of $G(x)$

$$= \frac{G(x) - a_0 - a_1x}{x^2} - 3\left[\frac{G(x) - a_0}{x}\right] + 2G(x) = 0$$

using initial values,

$$= \frac{G(x) - 2 - 3x}{x^2} - 3\,\frac{G(x) - 2}{x} + 2G(x) = 0$$

$$= G(x) - 2 - 3x - 3xG(x) + 6x + 2x^2G(x) = 0$$

DATE:
PAGE:

$\Rightarrow G(x)\left[2x^2-3x+1\right]=2-3x$

$\Rightarrow G(x)=\dfrac{2-3x}{2x^2-3x+1}=\dfrac{2-3x}{2x(x-1)-1(x-1)}$

$\Rightarrow \dfrac{2-3x}{(2x-1)(x-1)}=\dfrac{A}{x-1}+\dfrac{B}{2x-1}$

$\Rightarrow A(2x-1)+B(x-1)=2-3x$

take $x=1/2$

$B(-1/2)=2-3/2 \Rightarrow \boxed{B=-1}$

$A(2x-1)+(1-x)=2-3x$

$A(2x-1)=1-2x \Rightarrow \boxed{A=-1}$

$G(x)=\dfrac{1}{1-x}+\dfrac{1}{1-2x}$

$=(1-x)^{-1}+(1-2x)^{-1}$

$=\sum_{r=0}^{\infty}\binom{-1}{r}(-x)^r+\sum_{r=0}^{\infty}\binom{-1}{r}(-2x)^r$

$=\sum_{r=0}^{\infty}(-1)^r\cdot(-1)^r\cdot x^r+\sum_{r=0}^{\infty}(-1)^r\cdot(-1)^r\cdot 2^r\cdot x^n$

$\boxed{G(x)=1+2^r}$

Que Solve the recurrence relation using Generating function $u_n=6u_{n-1}+2^{n-1}$; $u_0=1$

$\rightarrow$ Multiply with x^n and $\sum_{n=1}^{\infty}$

$\Rightarrow \sum_{n=1}^{\infty}u_nx^n=6\sum_{n=1}^{\infty}u_{n-1}x^n+\sum_{n=1}^{\infty}2^{n-1}x^n$

Keep exponent as same as subscript of coefficient inside Σ.

$G(x)=a_0+a_1x+a_2x^2+\dots$

DATE:
PAGE:

$$\Rightarrow \sum_{n=1}^{\infty} u_n x^n = 6x \sum_{n=1}^{\infty} u_{n-1} x^{n-1} + x \cdot \sum_{n=1}^{\infty} 2^{n-1} \cdot x^{n-1}$$

Expanding we get

$$\Rightarrow (u_1 x + u_2 x^2 + \ldots) = 6x(u_0 + u_1 x + u_2 x^2 + \ldots) + x(1 + 2x + 2^2 x^2 + \ldots)$$

Rewrite in terms of $G(x)$ → using G.P. series.

$$G(x) - u_0 = 6x(G(x)) + x \cdot \frac{1}{1-2x}$$

$$\Rightarrow G(x) - 1 = 6x \cdot G(x) + \frac{x}{1-2x}$$

$$\Rightarrow G(x)(1-6x) = 1 + \frac{x}{1-2x}$$

$$\Rightarrow G(x) = \frac{1}{1-6x} + \frac{x}{(1-2x)(1-6x)}$$

$$\therefore \boxed{G(x) = 6^r + 2^r}$$

Que $u_n = u_{n-1} + 2u_{n-2}$; $u_0 = 3$, $u_1 = 7$

Que $a_n - 7a_{n-1} + 12a_{n-2} = 0$

→ Multiply with x^n and $\sum_{n=2}^{\infty}$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7 \sum_{n=2}^{\infty} a_{n-1} x^n + 12 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Expanding we get,

$\cancel{\Rightarrow (a_0 + a_1 x + a_2 x^2 + \ldots) - 7(a_1 + a_2 x^2 + \ldots}$

$$\Rightarrow (a_2x^2 + a_3x^3 + \ldots) - 7(a_1x^2 + a_2x^3 + \ldots) + 12(a_0x^2 + a_1x^3 + \ldots) = 0$$

$\Rightarrow$ Rewrite eq^n in form of $G(x)$

$$\Rightarrow (G(x) - a_0 - a_1x) - 7x(G(x) - a_0) + 12x^2(G(x)) = 0$$

$$\Rightarrow G(x) - a_0 - a_1x - 7xG(x) + 7xa_0 + 12x^2G(x) = 0$$

$$\Rightarrow G(x)[12x^2 - 7x + 1] + (7x-1)a_0 - a_1x = 0$$

$$\Rightarrow G(x)(3x-1)(4x-1) + (7x-1)a_0 - a_1x = 0$$

$$\Rightarrow \boxed{G(x) = \frac{a_1x - (7x-1)a_0}{(3x-1)(4x-1)}}$$

DATE:
PAGE:

Sessional - III.

- **Experiment**

\# outcome of exp1 m \# outcome of exp2 n

- (both happen) total # of outcomes $= m * n$ (rule of product)
- any one of them happen then total # of outcomes $= m + n$ (rule of sum)

Ex → **Permutation**

r distinctly coloured balls,
n distinctly numbered boxes.

Q. How many ways are possible?

$n \cdot (n-1) \cdot \ldots \cdot (n-(r-1))$

$= n \cdot n-1 \cdot n-2 \cdot \ldots \quad n-r+1$

$$P(n, r) = \frac{n!}{(n-r)!}$$

Ex →

rooms $= 7$
officies of programmers $= 4$
officies of terminals $= 3$

Q. How many ways we can distribute these 7 rooms

$\rightarrow 7 \cdot 6 \cdot 5 \cdot 4 \cdot 1$

3rd

4th.

2nd.

for terminals

1st programmer can chose out of 7 rooms.

DATE:
PAGE:

Ex → no. of four digit numbers without repeatitions

$\underline{9}\ \underline{9}\ \underline{8}\ \underline{7} = 9 \times 9 \times 8 \times 7$

~~exp~~ except 0

one is used out of 10.

Ex → # of exams = 3
# of days = 5
no constraints on # of exams on a day.

$\underline{5}\ \underline{5}\ \underline{5} = 5 * 5 * 5 = \boxed{125}$

1st exam we can schedule on any of the day.

Ex →
(0,1,2,3,4)
How many r-digit quintary sequences have an even no. of 1's ?

- 0, 1, 2, 3, 4

$2, 3, 4 = 3^r$

$\underline{3}\ \underline{3}\ \underline{3}\ \_$ --- r digit

(2,3,4) (2,3,4)

total no. of sequences = 5^r

Now, $\underbrace{5^r - 3^r}$

contain all digits

Therefore,

$$\boxed{\frac{5^r - 3^r}{2} + 3^r}$$

even no. of 1's

only 2, 3, 4.

eg 234XX334X23
(odd) 2840033412 3
(even) 2840133412 3

half of them would be even and half is odd. (x can be 0 or 1)

DATE:
PAGE:

Ex
$\Longrightarrow$ The total no. of distinct slips needed to print all five-digit numbers on slip of paper with one no. of on each slip of paper

- # numbers = 10^5

① 1 0 6 9 8 — 1 0 9 6 8

② 1 6 0 8 1 — 1 8 0 9 1 (we can use same slip)

③ 1 6 0 9 1 — 1 6 0 9 1

in 3^{rd} case

1 6 0 9 1 — 1 6 0 9 1

$3 * 5 * 1 * 5 * 1$ → same for 2^{nd} & 4^{th} place

Middle number can 1, 0, 8 — first place we can have 6 — last number become fix

1 — 1, 8 — 8, 6 — 9,
0 — 0, 9 — 6 (this is fix)

$$= \boxed{10^5 - \frac{5^5 - 3\cdot 5^2}{2}}$$

$5^5 - 3\cdot 5^2$ we need 2 slips to represent numbers II^{nd} case.

Ex
$\Longrightarrow$ r colored balls
n numbered boxes
r_1 of balls are of same color
r_2 of balls are of same color

How many ways you can distribut r balls in n boxes

$$\boxed{\text{Ans.} = \frac{P(n, r)}{r_1!\, r_2!}}$$

DATE:
PAGE:

EX →

3 dash 2 dots

we want to identify message which contains 3 dashes & 2 dots

$\boxed{\text{Ans} \quad \frac{5!}{3!\,2!}}$

EX →

r balls of same color — combination problem

n numbered boxed

$$= \frac{n\cdot(n-1)\cdot(n-2)\ \ldots\ (n-(r-1))}{r!}$$

$$= \frac{n\cdot(n-1)\cdot(n-2)\ \ldots\ (n-r+1)}{r!}$$

$$\boxed{C(n,r) = \frac{n!}{(n-r)!\,r!}}$$

EX →

11 MLAs

A committee of 5.

Ans $C(11,5)$

- MLA – A is always there
  $C(10,4)$
- MLA – A is not there always
  $C(10,5)$
- MLA – A, B (MLA A or MLA B can be there)
  $C(9,4) + C(9,4) + C(9,3)$

either A or B can be there

both are not there

$C(9,4)$ = A is fixed, B is not there

∴ remaining is 9.

we have to select 4 out of 9.

A is fixed

DATE:
PAGE:

sess-II

(f) $\binom{-1/2}{n} = \dfrac{\binom{2n}{n}}{(-4)^n}$

$$\text{L.H.S.} = \binom{-1/2}{n} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(\frac{1-2n}{2}\right)}{n!}$$

$$= \frac{(-1)^n\,(1\cdot 3\cdot 5\cdots(2n-1))}{2^n\, n!}$$

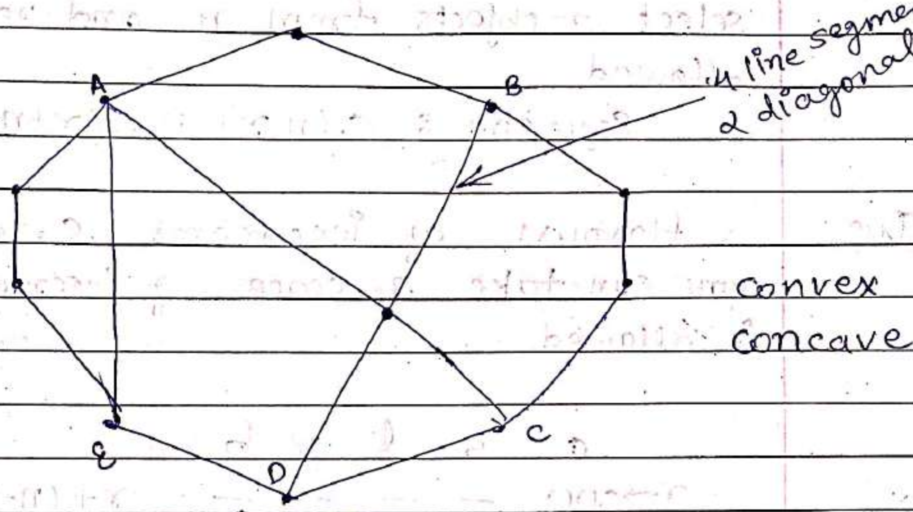
Now, multiply - divide $(2\cdot 4\cdot 6 \cdots 2n)$

$$= \frac{(-1)^n (2n)!}{2^n \cdot n!\; 2^n \cdot n!}$$

take 2 common

$$= \frac{\binom{2n}{n}}{(-4)^n} = \text{R.H.S.}$$

Continue



Decagon

→ 3 diagonals never pass through single intersection point

Que → How many line segments we have when diagonals intersect with eachother ?

- total diagonals = C(10,2) − 10 → sides.

$= \boxed{35}$

$\boxed{\text{Total diagonals} = {}^nC_2 - n}$

k intersection can create k+1 segments.

- # intersection points = C(10,4) = 210
↳ 1 intersection needs 4 points (ABCD)

# of line segments = 210 * 2 + 35 → one diagonal is not intersecting with any other (AF)

k+1 on one diagonal; k+1 on second diagonal select r objects from n and repetation is allowed.

Equation : $C(n,r) = C(r+n-1, r) = \frac{(r+n-1)!}{r!\,(n-1)!}$

Que 5 flavours of icecreams c, s, l, v, b
you can take 3 scoops of icecream and repeatation is allowed.

c s l v b

$r \rightarrow \underline{000}$ — — — — $r+(n-1)$.

— 0 — — 0 — 0

— — — 00 — 0 —

When we have repeatation ~~is allowed~~
choices increase from n to $r+(n-1)$,

Que types of cookies = 4
6 cookies to select

→ $C(r+n-1, r)$ $n=4$, $r=6$ Here $n < r$.
therefore repetition we need to consider

$= C(6+4-1, 6)$
$= C(4,6)$

Que 5 girls, 12 chairs.
5 girls to sit in 12 chairs. How many ways are possible?

→ ? # ? # ? # ? # ? # ? ?= places

5 chairs and 7 are left.

Here $r=7$, $n=6$

No. of ways $= 5! * C(6+7-1, 7)$

5 girls can exchange places

Que In same que above, ~~how~~ no girls can ~~seat~~ sit next to each other

- # - # - # -

→ girls (5)

– → chairs (4)

∴ remaining places $r=3$

$n=6$

No. of ways $= 5! * C(3+6-1, 3)$

Que $2t+1$ → 3 distinguisable balls put them 3 boxes such that 2 boxes will contain ~~no~~ more balls then 3rd one.

→ distribution of $2t+1$ into 3 boxes

$C(r+n-1, r) = C(2t+1+3-1, 2t+1)$

$= C(2t+3, 2t+1)$

Sub Que In same, 1 box will contain more balls then remaining 2 boxes.

$2t+1 = t + (t+1)$ distribute t in all boxes

$3 \cdot C(t+3-1, t) \Rightarrow 3C(t+2, t)$

$\therefore$ 2 boxes > 1 box
$= \boxed{C(2t+3, 2t+1) - 3 \cdot C(t+2, t)}$

- Generation of Permutation and combination
(order matters)

- combination and subset are same thing

- TSP (Travelling Sales Person Problem).
we generate all possible combinations and also follow "lexicographical order".
eg 1, 10, 11, 2 ← lexicographical order.

→ Algorithm to generate all possible lexicograph permutation:

given : $a_1, a_2, \ldots, a_n$
next : $b_1, b_2, \ldots, b_n$

1. $a_i = b_i$, $1 \leq i \leq m-1$, and $a_m < b_m$ for the largest possible m

2. b_m is the smallest element among a_{m+1}, a_{m+2} that is larger than a_m.

3. $b_{m+1} < b_{m+2} < \ldots < b_n$.

eg We want to find the next sequence of 124653

a_m = from this element start we change the sequence

DATE:
PAGE:

Need to find a_m from right side (LSB)

1 2 <u>4</u> 6 5 3 (a_m = 4)	← Here 4 < 6 Hence it is a_m.
1 2 5 3 4 6	put all a_{m-1} elements as it is.
1 2 5 3 6 4	Now, maximum than a_m, write it
1 2 5 4 3 6	
1 2 5 4 6 3	

eg. $\{1, 2, 3, 4\} \rightarrow$ 1234, 1243, 1324, ⋮ 1342, 1423, 1432, 2134, 2143

- Generating r combinations / K-subsets

$S = \{1, 2, \ldots, n-1, n\}$

Begin with $12\ldots r$

while $a_1 a_2 \ldots a_r \, != (n-r+1) \ldots (n-1)n$

do

1. Find the largest integer k such that $a_k < n$ and a_{k+1} is not in the $a_1 a_2 \ldots a_r$

2. Replace $a_1 a_2 \ldots a_r$ with $a_1 a_2 \ldots a_{k-1} (a_k+1)(a_k+2) \ldots (a_k + r - k + 1)$

eg $\{1,2,3,4,5,6\}$ How many ways 4 elements can be chosen from $\{1,2,3,4,5,6\}$

$n=6$, $r=4$

$- {}^6C_4 = 15$ subsets

we find.

a_k

1234

1235

1236 Here $a_k = 3 < n=6$

1245 a_k a_{k+1}

1246 Here $a_k = 4$ and a_{k+1} is not in sequence.

1256

1345 $a_k=2$, $a_{k+1}=3$ and remaining a_{k+1}, a_{k+2}

1346

1356

1456 $a_k=1$

2345

2346

2356

2456

3456.

- Discrete Probability

→ Outcomes can be mutually exclusive (all outcome are not there)
mutually exhaustive (either of o/com are there)

→ Experiment : Something which has outcome. eg. toss a coin

→ sample space : It is a space where all possible outcomes are listed.

eg $S = \{HH, HT, TH, TT\}$

DATE:
PAGE:

→ Discrete Sample space :

Sample space where we have finite no. of outcomes or ~~com~~ ~~countabilly~~ countably infinite no. of outcomes.

eg $S = \{HH, HT, TH, TT\}$ → finite

$S = \{h, mh, mmh, mmmh, \ldots\}$ → countably infinite.

↳ hit & miss

→ Probability is a real number which is allocated to every outcome.

→ probability signifies frequency.

→ (i) $\sum_{x_i \in S} P(x_i) = 1$

(ii) $0 \leq P(x_i) \leq 1$

→ Pillars

ex $S = \{hh, ht, th, tt\}$

$P(hh) = \frac{4}{9}$, $P(tt) = \frac{2}{9}$ Unfair coin

$P(ht) = \frac{1}{9}$, $P(th) = \frac{2}{9}$

ex $S = \{h, mh, mmh, mmmh, \ldots\}$

$P(h) = \frac{1}{2}$, $P(mh) = \frac{1}{4}$

either miss or hit

$P(\underbrace{mmm \ldots h}_{K}) = 2^{-K}$.

→ Event - subset of a sample space.

Event $\subseteq S$

simple — only one outcome is guranting.

Compound — more then one sample.

Eg. 2 dice.

$S = \{(1,1), \dots, (1,6)$
$(2,1), \dots, (6,6)\}$

$|S| = 36$

Event : The sum of the rolling dice is 9.

$P(E) = \frac{4}{36}$.

Eg 8 students

I II III IV

two of them are belonging in I^{st} year Similar
two stu → II^{nd} year
two stu → III^{rd} year
two stu → IV^{th} year

$P(E) = \dfrac{8!}{2! \times 2! \times 2! \times 2! \times 4^8}$

two stu. I^{st} year

sample space

1^{st} stu can belong to I, II, III, IV year eg

People $P = 100,000$ find, $P(fb)$

Female $F = 51,500$ $P(mb)$

Male $M = 48,500$ $P(mh)$

$F\cancel{M}B = 9000$ $P(\cancel{m}fh)$

$MB = 30200$

$F\cancel{W}^{H} = 42500$

$M\cancel{W}_{H} = 18300$

$\longrightarrow$ $P(fb) = \dfrac{9000}{100,000}$

$P(mb) = \dfrac{30200}{100,000}$

$P(mh) = \dfrac{18300}{100,000}$

$P(fh) = \dfrac{42,500}{100,000}$

eg

A : A bald person was chosen.

B : A female was chosen

$P(A) = P(mb) + P(fb)$

$=$

$P(B) = P(fb) + P(fh)$

$=$

$P(A \cup B) = P(mb) + P(fb) + P(fh)$

$P(A \oplus B) = A\bar{B} + \bar{A}B = P(mb) + P(fh)$

$P(A \cap B) = P(fb)$

$P(A - B) = P(mb)$

$P(B - A) = P(fh)$

Cumilitive dist^n fun^n → also known as = dist^n fun^c
Probability dist^n fun^n → also known as → Prob. fun^c

DATE:
PAGE:

- **Discrete Random Variable**

Discrete random number is defined over discrete ~~random~~ sample space.

Eg. $rv = X$ ← no. of heads on 4 tosses of coin
$S = 2^4$

$\therefore X = \{0, 1, 2, 3, 4\}$
1 head, 2 head

- random variable represents sum of numbers when 4 ~~coins~~ dice are rolled together

$|S| = 6^4$

- $y = \{4, \ldots, 24\}$

Eg. 3 dice are rolled together. sum should be even.

random variable $X = \{4, \ldots, 18\}$ → (6+6+6)
$|S| = 6^3$

minimum (1+1+2)
list all in exam

- **Probability Distribution Function**

If x is a drv with sample space S then a function denoted by $f(x)$ or $P(X=x)$ and defined as $f(x) = P(X=x)$ = Probability for the random variable $X = x$ is called the probability distribution function or pdf or probability function density → distribution is continuous

eg 4 coins are toss

X = {no. of heads observer over 4 tosses of coins}

$X = \{0, 1, 2, 3, 4\}$, $|S| = 2^4$

Now, Prob. dist^n func,

$f(0) = P(X=0) = \frac{4C_0}{16}$

$f(1) = P(X=1) = \frac{4C_1}{16}$

$f(2) = P(X=2) = \frac{4C_2}{16}$

$f(3) = P(X=3) = \frac{4C_3}{16}$

Now, $f(X) \geq 0$

$$\sum_{x_i \in X} f(x) = 1$$

eg 2 dice once, sum = 2, 3, 4, 5 & 6

So,

X = sum resulting from rolling of 2 dice once

$= \{2, \ldots, 12\}$

$|S| = 6^2$

DATE:
PAGE:

$P(X=2) = \frac{1}{36}$ // (1,1)

$f(X=3) = \frac{2}{36}$ // (1,2) (2,1)

$f(X=4) = \frac{3}{36}$ // (1,3), (2,2), (3,1)

$f(5) = \frac{4}{36}$ // (1,4), (2,3), (3,2), (4,1)

$f(6) = \frac{5}{36}$ // (1,5), (2,4), (3,3), (4,2), (5,1)

Eg $2P(X=1) = 3P(X=2) = P(X=3) = 5(P(X=4))$

$X = \{1, 2, 3, 4\}$

LCM $(2, 3, 1, 5) = 30$

$2P(X=1) = 30K$ $\quad \therefore P(X=2) = 10K$

$\Rightarrow \boxed{P(X=1) = 15K}$ $\quad \therefore P(X=3) = 30K$

$\therefore P(X=4) = 6K$

Wkt, $\sum_{x_i \in X} f(x_i) = 1 \Rightarrow 61K = 1 \Rightarrow \boxed{K = \frac{1}{61}}$

So, $P(1) = \frac{15}{61}$, $P(2) = \frac{10}{61}$, $P(3) = \frac{30}{61}$, $P(4) = \frac{6}{61}$

DATE:
PAGE:

- ~~Discrete~~ Distribution function (CDF)

If X is drv defined over the sample space S with pdf(x) then a function given by,

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

is called the distribution function.

eg Prev. Example

$$F(1) = \sum_{x \leq 1} f(x) = f(1) = \frac{15}{61}$$

$$F(2) = \sum_{x \leq 2} f(x) = f(1) + f(2) = \frac{25}{61}$$

$$F(3) = \sum_{x \leq 3} f(x) = f(1) + f(2) + f(3) = \frac{55}{61}$$

$$F(4) = \sum_{x \leq 4} f(x) = f(1) + f(2) + f(3) + f(4) = \frac{61}{61} = 1$$

~~Eg.~~

$$F(X) = \begin{cases} 0 & ; x < 1 \\ 15/61 & ; 1 \leq x \leq 2 \\ 25/61 & ; 2 \leq x < 3 \\ 55/61 & ; 3 \leq x < 4 \\ 1 & ; 4 \leq x < 5 \\ 1 & ; x > 4 \end{cases}$$

Above is cumilitive dist^n fun^c

Eg $f(0) = f(1) = \frac{1}{3}$ & $f(2) = f(3) = \frac{1}{6}$

→ $\overset{\text{rv.}}{X} = \{0, 1, 2, 3\}$

$F(0) = \sum_{x \leq 0} f(x) = f(0) = \frac{1}{3}$

$F(1) = \sum_{x \leq 1} f(x) = f(0) + f(1) = \frac{2}{3}$

$F(2) = \sum_{x \leq 2} f(x) = f(0) + f(1) + f(2) = \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = \frac{5}{6}$

$F(3) = \sum_{x \leq 3} f(x) = f(0) + f(1) + f(2) + f(3) = 1$

Cdf : $F(X) = \begin{cases} 0 & ; \ x < 0 \\ 1/3 & ; \ 0 \leq x < 1 \\ 2/3 & ; \ 1 \leq x < 2 \\ 5/6 & ; \ 2 \leq x < 3 \\ 1 & ; \ 3 \leq x < 4 \\ 1 & ; \ x \geq 4 \end{cases}$

Eg Suppose when a rv can take only two values each with prob. 1/2. Find its distⁿ funᶜ

- Conditional Probability

$P(4) = \frac{1}{6}$

$P(4 \mid \text{even}) = \frac{1}{3}$

$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

DATE:
PAGE:

eg. coin was chosen at random. prob. that chosen is a fair one and it shows head is 1/5.
~~given that fair coin was chosen.~~
A fair coin was chosen and tail shows 1/3. Prob. that unfair was chosen & head shows 1/12 & unfair was chosen & tail shows 1/4. find P(unfair coin | head) & P(head | unfair)

→ $P(\text{fair} \cap \text{head}) = 1/3$
$P(\text{fair} \cap \text{tail}) = 1/3$
$P(\text{unfair} \cap \text{head}) = 1/12$
$P(\text{unfair} \cap \text{tail}) = 1/4$

$P(\text{head}) = \frac{1}{3} + \frac{1}{12}$

$P(\text{unfair}) = \frac{1}{12} + \frac{1}{4}$

i) $P(\text{unfair} \mid \text{head}) = \frac{P(\text{unfair} \cap \text{head})}{P(\text{head})}$

$= \frac{1/12}{1/3 + 1/12}$

ii) $P(\text{head} \mid \text{unfair}) = \frac{P(\text{unfair} \cap \text{head})}{\frac{1}{12} + \frac{1}{4} \quad P(\text{unfair})}$

$= \frac{1/12}{\frac{1}{4} + \frac{1}{12}}$.

Eg. 3 dice given that no two ~~phase~~ faces were same. What is the prob. that there was an ace?

$\longrightarrow$ $P\left(\begin{array}{c}\text{there is an} \\ \text{ace}\end{array} \middle| \begin{array}{c}\text{no two} \\ \text{faces are same}\end{array}\right) = P(A|B)$

$P(\text{No two faces are same}) = \dfrac{P(6,3)}{6^3}$

$\therefore P\left(\begin{array}{c}\text{there is an ace)} \\ \text{and no two are} \\ \text{same}\end{array}\right.$ $(A \cap B)$ $= \dfrac{\overset{3 \text{ ways}}{3} \cdot \overset{\text{ace}}{1} \cdot P(5,2)}{6^3}$ remaing 2 select from 5

Topic: Information

$\therefore P\left(\begin{array}{c}\text{there is an} \\ \text{ace}\end{array} \middle| \begin{array}{c}\text{no two} \\ \text{faces are same}\end{array}\right) = \boxed{\dfrac{3 \cdot 1 \cdot P(5,2)}{P(6,3)}}$

Here, A: There is an ace.

B: No two faces of the rolled dice are same

AAM

Date: / /
Page No.

Baye's Theorem

→ Prior Information

↓

Further Information ⇒ (In Machine learning it is similar to 'Reinforcement')

↓

Revising Probabilities

defⁿ

→ If $A_1, A_2, \ldots, A_n$ are mutually disjoint events with $P(A_i) \neq 0$, then for any arbitrary event E which is subset of $\bigcup_{i=1}^{n} A_i$, such that $P(E) > 0$

we have,

$$P(A_i \mid E) = \frac{P(A_i) \cdot P(E \mid A_i)}{\sum_{i=1}^{n} P(A_i) P(E/A_i)}$$

A_i: cause, E: effect

(Effect is already occured & we want to know about its cause)

→ From defⁿ,

$A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n = \emptyset$

$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$

Now, $E \subseteq \bigcup_i A_i$

$\Rightarrow E = E \cap \bigcup_i A_i$ $\quad$ ($\because A \subset B \Rightarrow A = A \cap B$)

$= \bigcup_{i=1}^{n} (E \cap A_i)$ $\quad$ ($\because$ because it is disjoint)

Date: / /
Page No.

$$P(E) = P\left(\bigcup_{i=1}^{n} (E \cap A_i)\right)$$

$$\Rightarrow P(E) = \sum_{i=1}^{n} \left(P(E \cap A_i)\right)$$

$$\Rightarrow P(E) = \sum_{i} P(A_i) \cdot P(E/A_i)$$ ($\because$ from cond^n prob. eq^n).

Now, $P(A_i|E) = \frac{P(A_i \cap E)}{P(E)}$

$$\boxed{P(A_i|E) = \frac{P(A_i) \cdot P(E|A_i)}{\sum_{i=1}^{n} P(A_i) \cdot P(E|A_i)}}$$

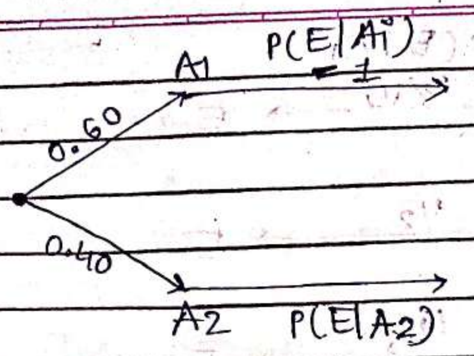
posterior prob.

Examples:

1) A student knew only 60% of the questions in a test. Each with 5 answers. He simply guessed while answering the test. What is the probability that he knew the answer to a question given that he answered it correctly?

$\longrightarrow$ A_1 : student knew the answer] causes
A_2 : student guess the answer
E : He answered it correctly

Find $P(A_1|E) = ?$.



$= 1/5$ (we have 5 options)

$P(E) = 0.60 \times 1 + 0.40 \times (1/5)$

$\boxed{P(E) = 0.68}$

Now, $P(A1|E) = \dfrac{P(A1) \cdot P(E|A1)}{\sum_i P(Ai) P(E|Ai)}$

$= \dfrac{(0.60)(1)}{0.68} = \boxed{}$

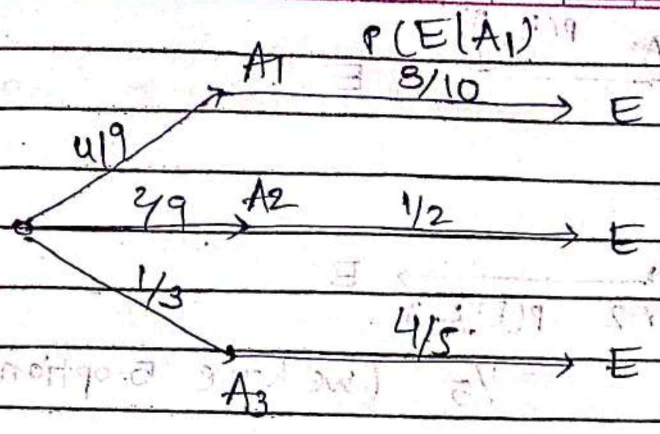
2) The probabilities of X, Y, Z becoming managers is 4/9, 2/9, 1/3 respectively. The probabilitie that the bonus scheme will be introduced if X, Y, Z becomes manager are 3/10, 1/2, 4/5 respectively. If a the bonus is introduced what is the prob. the manager appointed was Y.

→ Here, A_1 : The manager is X
A_2 : The manager is Y
A_3 : The manager is Z

$P(A_1) = \dfrac{4}{9}$, $P(A_2) = \dfrac{2}{9}$, $P(A_3) = \dfrac{1}{3}$

Date: / /

Page No.



E : The bonus is introduced.

$P(A_2 \mid E) = ?$

$$\therefore P(A_2 \mid E) = \frac{P(A_2) \cdot P(E \mid A_2)}{\sum_i P(A_i)\, P(E \mid A_i)}$$

$$= \frac{\left(2/9\right)\left(1/2\right)}{\left(\frac{4}{9}\right)\left(\frac{3}{10}\right) + \left(\frac{2}{9}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{3}\right)\left(\frac{4}{5}\right)}$$

$$= \boxed{}$$

3) From a bag containing 3 white & 5 black balls. 4 balls are transfered into an empty bag. From this bag a ball is drawn (chosen) and it is found to be white. What is the prob. that out of 4 balls transfered 3 are white & 1 is black?

→ B1 : 3 white & 5 black — transfered 4 balls

B2 : Initially empty

causes A_1 : 0W 4B

A_2 : 1W 3B

A_3 : 2W 2B

A_4 : 3W 1B

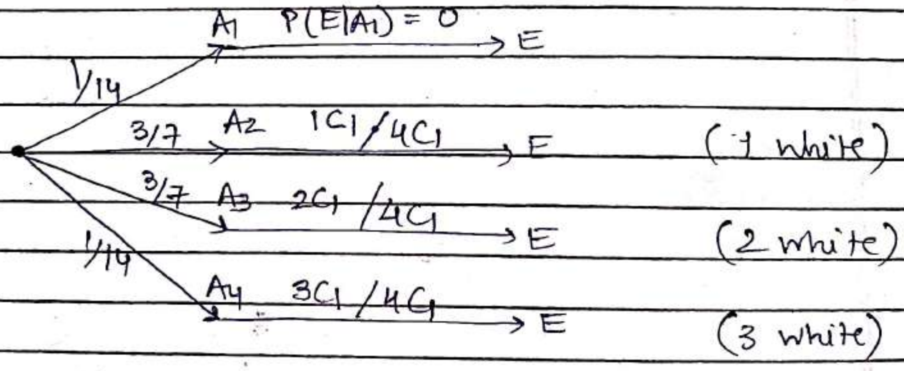
E : The chosen ball is white

$P(A_4|E) = ?$

Now, $P(A_1) = \dfrac{{}^3C_0 \cdot {}^5C_4}{{}^8C_4}$

$P(A_2) = \dfrac{{}^3C_1 \cdot {}^5C_3}{{}^8C_4}$

$P(A_3) = \dfrac{{}^3C_2 \cdot {}^5C_2}{{}^8C_4}$, $P(A_4) = \dfrac{{}^3C_3 \cdot {}^5C_1}{{}^8C_4}$



$$P(A_4|E) = \frac{P(A_4)\cdot P(E|A_4)}{\sum_i P(A_i)\, P(E|A_i)}$$

$$= \frac{(1/14)\,({}^3C_1/{}^4C_1)}{(\frac{3}{7})({}^1C_1/{}^4C_1) + (\frac{3}{7})({}^2C_1/{}^4C_1) + (1/14)({}^3C_1/{}^4C_1)} = \boxed{\frac{1}{7}}$$