Permutations & Combinations

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Genereting Function

Ex. →

set 1	set 2
2, 5, 7	10, 5, 1

Que How many combinations will generate a sum 12?
2. {(2,10), (7,5)}

Represet these sets as equation
$(x^2+x^5+x^7)$ and $(x^{10}+x^5+x^1)$

$\Rightarrow$ Multiply them,
$x^{12}+x^7+x^3+x^{15}+x^{10}+x^6+x^{17}+x^{12}+x^8$

$\Rightarrow x^3+x^6+x^7+x^8+x^{10}+(x^{12})\cdot 2+x^{15}+x^{17}$

lowest possible sum

two pairs available which make sum pair of 12.

highest possible sum

Ex. →

A	B
3	0
0	3
1	2
2	1

3 roses, wants to distribute among 2 persons

A	B	
[0,1,2,3]	[0,1,2,3]	← possible roses can have.
$(x^0+x^1+x^2+x^3)$	$(x^0+x^1+x^2+x^3)$	

$\Rightarrow x^0x^0+x^0x^1+x^0x^2+\boxed{x^0x^3}+x^1x^0+x^1x^1+\boxed{x^1x^2}+x^1x^3$
$+x^2x^0+\boxed{x^2x^1}+x^2x^2+x^2x^3+\boxed{x^3x^0}+x^3x^1+x^3x^2$
$+x^3x^3$

Valid → where sum is 3.

→ used – to solving counting problems.

- 1 to 1 mapping
2 persons A, B ⇒ 3 objects distribute
[0,1,2,3] [0,1,2,3]
$(x^0+x^1+x^2+x^3)\ (x^0+x^1+x^2+x^3)$

- Recurrence Relation – we can use generating fun^c.

- Downside – complex problem it would be lengthy. of generating fun^c.

two types
① Ordinary – combination problem (order doesn't matter)
② Exponential – to solve permutation problem

• Generating Function :

Let $(a_0, a_1, \ldots, a_n)$ be a symbolic representation of an event, or let it be simply a sequence of numbers,
The function $f(x) = a_0 u_0(x) + a_1 u_1(x) + \ldots + a_n u_n(x)$
is called ordinary generating function.
Here, $u_0(x), u_1(x), \ldots, u_n(x)$ is a sequence of function of x used as indicators.

ex. variables = (3,2,6,0,0,0) [seq. of events]
indicators = $1, 1+x, 1-x, 1+x^2, 1-x^2, \ldots$

OGF = $f(x) = a_0x^0 + a_1x^1 + \ldots + a_nx^n$
ordinary generating fun^c
$= \sum^{n} a_r x^r$

$= 3(1) + 2(1+x) + 6(1-x)$

$= \boxed{11 - 4x}$

→ whenever we apply diff. sequence of events and if our OGF produce same output then our OGF is faulty.
we need to change indicator function.

→ whenever we apply diff. sequence of events on diff. Indicators then, it should produce all diff. OGFs.

ex. events = (1, 1, 1, ...)

→ $OGF = f(x) = a_0x^0 + a_1x^1 + a_2x^2 + \dots + a_nx^n$

$$= \sum_{r=0}^{n} a_r x^r$$

$= (1)x^0 + (1)x^1 + (1)x^2 + \dots$

$= 1 + x + x^2 + \dots$

$$\boxed{f(x) = \frac{1}{1-x}}$$

($\because$ Geometric series $a + ar + \dots \infty = \frac{a}{1-r}$)

ex. events = (1, 1, 3, 1, ...)

→ $OGF = (1)x^0 + (1)x^1 + (3)x^2 + (1)x^3 + \dots$

$= 1 + x + 3x^2 + x^3 + x^4 + \dots$

~~$= 1 + x + 3x^2 + x^3(1 + x + x^2 + \dots)$~~

~~$= 1 + x + 3x^2 + x^3\left(\frac{1}{1-x}\right)$~~

$= 2x^2 + (1 + x + x^2 + x^3 + \dots)$

$= \boxed{2x^2 + (1-x)^{-1}}$

ex. events = $(1, -1, 1, -1, 1, -1, \dots)$

$\longrightarrow$ OGF $= (1+x)^{-1}$

ex. events = $(1, 2, 3, 4, \dots)$

$\longrightarrow$ OGF $= (1)x^0 + 2x^1 + 3x^2 + 4x^3 + \dots$

$= 1 + 2x + 3x^2 + 4x^3 + \dots$ — (1) (AP + GP)

$g(x) = x \cdot f(x) = x + 2x^2 + 3x^3 + 4x^4 + \dots$ — (2)

take (1) − (2)

$f(x) - x \cdot f(x) = 1 + x + x^2 + x^3 + \dots$

$f(x)(1-x) = \frac{1}{1-x}$

$\boxed{f(x) = (1-x)^{-2}}$

ex. events = $(0, 1, 2, 3, 4, \dots)$

ex. $a_r = r \cdot b \cdot a^r$ ← numeric function

$\longrightarrow$ OGF $= f(x) = \sum_{r=0}^{n} a_r x^r$

$= 0 + bax + 2ba^2x^2 + 3ba^3x^3 + \dots$

$= bax + 2ba^2x^2 + 3ba^3x^3 + \dots$

$\therefore f(x) = b(ax + 2a^2x^2 + 3a^3x^3 + \dots)$

$\therefore g(z) = \cancel{b}(z + 2z^2 + 3z^3 + \dots)$

$\therefore z \cdot g(z) = \cancel{b}(z^2 + 2z^3 + 3z^4 + \dots)$

we can avoid b

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$\therefore g(z) - z \cdot g(z)$
$= z + z^2 + z^3 + \ldots$

$\cancel{g(z) = \frac{1}{(1-z)^2}}$

$g(z)(1-z) = z(1 + z + z^2 + \ldots)$

$\therefore g(z) = \frac{z}{(1-x)^2}$

$$\therefore \boxed{f(x) = \frac{b \cdot ax}{(1-ax)^2}}$$

ex. events $= \left\{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \ldots\right\}$ Ans $\frac{\log(1+x)}{x}$

$\longrightarrow$ OGF $= (1)x^0 - \frac{1}{2}x^1 + \frac{1}{3}x^2 - \frac{1}{4}x^3 + \ldots$

$= 1 - \frac{1}{2}x + \frac{x^2}{3} - \frac{x^3}{4}$

$x \cdot f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$

$x \cdot f(x) = \log(1+x)$

$$\therefore \boxed{f(x) = \frac{\log(1+x)}{x}}$$

ex. $(1, -1, 1, -1, 1, \ldots)$

$\longrightarrow$ OGF $= (1)x^0 - (1)x^1 + (1)x^2 - (1)x^3 + (1)x^4 \ldots$

$= 1 - x + x^2 - x^3 + x^4 \ldots$

$= \frac{1}{1+x}$

$= \boxed{(1+x)^{-1}}$

$(a+b)^2$ — arranging 2 objects in 2 places

	a/b	a/b	
	a	a	a^2
	a	b	ab
	b	a	ab
	b	b	b^2

$= a^2+2ab+b^2$

$(a+b)^3$ — arranging 2 objects in 3 places

a/b	a/b	a/b
a	a	a
a	a	b
a	b	a

$= a^3+3a^2b+3ab^2+b^3$

$(a+b)^2$ = 0 'a' — 0 times a (both b)

= 1 'a' — 1 times a (ab or ba)

= 2 'a' — 2 times a (0 times b)

$= {}^2C_0\, a^0 b^2 + {}^2C_1\, a^1 b^1 + {}^2C_2\, a^2 b^0$

$= \boxed{b^2 + 2ab + a^2}$

$(a+b)^2 = \binom{2}{0} a^0 b^2 + \binom{2}{1} a^1 b^1 + \binom{2}{2} a^2 b^0$

$(a+b)^3 = \binom{3}{0} a^0 b^3 + \binom{3}{1} a^1 b^2 + \binom{3}{2} a^2 b^1 + \binom{3}{3} a^3 b^0$

Ordinary Binomial expansion

$(a+b)^n = \binom{n}{0} a^0 b^n + \binom{n}{1} a^1 b^{n-1} + \binom{n}{2} a^2 b^{n-2} + \dots + \binom{n}{n} a^n b$

Pascal's triangle

Pascal's identity

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

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→ Pascal's identity $^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$

Proof: $R.H.S. = C(n-1, r-1) + C(n-1, r)$

$$= \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{(r)!\,(n-r-1)!}$$

$$= \frac{r\cdot(n-1)!}{r!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!}$$

$$= \frac{(n-1)!}{r!}\left[\frac{r}{(n-r)!} + \frac{1}{(n-r-1)!}\right]$$

$$= \frac{(n-1)!}{r!}\left[\frac{r}{(n-r)!} + \frac{(n-r)}{(n-r)!}\right]$$

$$= \frac{n\cdot(n-1)!}{r!\,(n-r)!} = \frac{n!}{r!\,(n-r)!}$$

$$= {}^{n}C_{r}$$

$$= L.H.S.$$

Hence, it is proved.

→ r^{th} term $= \binom{n}{r} a^{r} b^{n-r}$

Que what is the coefficient of x^{15} and x^{12} in $(x^2 + x^{-3})^{10}$

→ r^{th} term $= \binom{10}{r} (x^2)^{r} (x^{-3})^{10-r}$

$$= \frac{10!}{r!\,(10-r)!} x^{2r} \cdot x^{-30+3r}$$

$$= \binom{10}{r} x^{5r-30}$$

Now, $x^{15} = x^{5r-30}$

$\Rightarrow 5r - 30 = 15$

$\Rightarrow \boxed{r = 9}$

coefficient of $x^{15} = \binom{10}{r} = \binom{10}{9}$

Similarly,

$x^{12} = x^{5r-30}$

$\Rightarrow 12 = 5r - 30$

$\Rightarrow \boxed{r = \frac{42}{5}}$ but $r \notin Z^{+}$ (r is fractional)

$\therefore$ No, term with x^{12} exist in given expansion

Que: $(x + x^{-1})^{10}$ find coefficient of x^6 and x^7.

$\rightarrow$ r^{th} term $= \binom{10}{r} (x)^{r} (x^{-1})^{10-r}$

$$= \binom{10}{r} x^{r} \cdot x^{r-10}$$

$$= \binom{10}{r} x^{2r-10}$$

$2r - 10 = 6$

$2r = 16$

$\boxed{r = 8}$

Coeff. is $\binom{10}{8}$

And

$2r - 10 = 7$

$2r = 17$

$\boxed{r = 17/2} \notin Z^{+}$

No coeff. of x^7 exist

Que $(2x-3y)^{25}$ Find coeff. of $x^{12}y^{13}$

r^{th} term $= \binom{25}{r}(2x)^r(-3y)^{25-r}$

$= \binom{25}{r} 2^r \cdot x^r \cdot (-3)^{25-r} \cdot y^{25-r}$

Now, $x^r = x^{12}$

$\boxed{r=12}$

coeff $= \binom{25}{12} 2^{12} \cdot (-3)^{13}$

Que What is the coeff of $x^6y^2z^2$ in $(x+y+z)^{10}$

$= \frac{10!}{6!\,2!\,2!} = \frac{10\times9\times\overset{2}{\cancel{8}}\times7}{\cancel{4}} = \boxed{1260}$

Que Find term independent of x in $\left(\sqrt{x}-\frac{2}{x^2}\right)^{10}$

Que Find 5^{th} term in the expansion of $\left(2x^2+\frac{3}{2x}\right)$

- Extended binomial expansion

$(a+b)^n$

$= \binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \binom{n}{2}a^2b^{n-2} + \dots + \binom{n}{n}a^nb$

$$= \frac{n!}{0!(n-0)!} a^0 b^n + \frac{n!}{1!(n-1)!} a^1 b^{n-1} + \frac{n!}{2!(n-2)!} a^2 b^{n-2}$$

$$+ \dots + \frac{n!}{n!(n-n)!} a^n$$

put $a = x, b = 1$

$$= \frac{n!}{0!(n-0)!} x^0 1^n + \frac{n!}{1!(n-1)!} x^1 1^{n-1} + \frac{n!}{2!(n-2)!} x^2 1^{n-2}$$

$$+ \dots + \frac{n!}{n!(n-n)!} x^n$$

$$= 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3$$

$$+ \dots + x^n$$ (finite series)

$$= (1+x)^n$$

→ 3 standard ways to write combinations

①	②	③
^nC_r (selecting r from n items) only valid when n is (+ve)	$\binom{n}{r}$ valid for n is (-ve) or fractional	$C(n, r)$ (n is +ve)

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$$(1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}x^3 + \ldots$$

[$\because$ $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$

~~double prime~~ first   double prime

Maclaurian series]

infinite series because Maclaurian series is infinite.

Assume, $0 < |x| < 1$ → so that we can terminate it.
→ so that it can become finite.

assume $(-n) = p$,

$(1+x)^p$

$$= 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots$$

$$= \sum_{r=0}^{\infty} \binom{p}{r} x^r$$

$$\therefore (1+x)^{-n} = \sum_{r=0}^{\infty} \binom{-n}{r} x^r$$

$$\therefore (a+b)^{-n} = b^{-n}\left(\frac{a}{b}+1\right)^{-n}$$

$$= \boxed{b^{-n}\sum_{r=0}^{\infty}\binom{-n}{r}\left(\frac{a}{b}\right)^r}$$

where,
$0 < |x| < 1$
$0 < \left|\frac{a}{b}\right| < 1$
$|b| > |a|$

$$= b^{-n} \boxed{\sum_{r=0}^{\infty} \binom{-n}{r} a^{r} b^{-n-r}}$$ $|b|>|a|$

- Find a relation betn ordinary binomial coeff. and extended binomial coeff.

→ Ordinary binomial coeff. $= C(n,r)$
extended binomial coeff. $= \binom{-n}{r}$

$$C(3,2) = \frac{3!}{2!\,(1!)} = 3$$

$$C(6,2) = \frac{6!}{2!\,4!} = \frac{6\times5}{2!}$$

$$C(10,3) = \frac{10\times9\times8}{3!}$$

$$C(16,4) = \frac{16\times15\times14\times13}{4!}$$

$$\binom{n}{r} = \frac{n\cdot(n-1)\cdot(n-2)\ldots(n-(r-1))}{r!}$$

$$\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\ldots(-n-(r-1))}{r!}$$

$$= \frac{(-1)^{r}\, n\cdot(n+1)\cdot(n+2)\ldots(n+(r-1))}{r!}$$

$$= \frac{(-1)^{r}(n+(r-1))\,(n+(r-2))\ldots(n+2)(n+1)}{r!}$$

$$= \frac{(-1)^{r}(n+(r-1))\,(n+(r-2))\ldots(n+1)\,n\,(n-1)!}{r!\,(n-1)!}$$

$$\binom{-n}{r} = \frac{(-1)^{r}(n+(r-1))!}{r!\,(n-1)!}$$

$$^{n+r-1}C_r = \frac{(n+r-1)!}{r!\,(n-1)!}$$

$$\boxed{\binom{-n}{r} = (-1)^r \cdot {}^{n+r-1}C_r} \leftarrow \text{show symbol}$$

Ex What is the coefficient of x^9 in power series $(1+x^3+x^6+x^9+\ldots)^3$

$\longrightarrow$

$$S = (1+x^3+x^6+x^9+\ldots)^3$$

$$= \left(\frac{1}{1-x^3}\right)^3 \qquad \left(\because \text{GP series} = \frac{a}{1-r}\right)$$

$$= (1-x^3)^{-3} \qquad (\because 0<|x|<1)$$

$$= \sum_{r=0}^{\infty} \binom{-3}{r} (-x^3)^r \qquad (\because |b|>|a|)$$

$$= \sum_{r=0}^{\infty} \binom{-3}{r} (-1)^r x^{3r}$$

put $r=3$,

coeff. of x^9 is $= \binom{-3}{3}(-1)^3$

$$= (-1)^3 \cdot {}^{3+3-1}C_3 \cdot (-1)^3$$

$$= (-1)^6 \cdot {}^5C_3$$

$$= \frac{5\times4\times3}{3!}$$

$$= \boxed{10}$$

$\boxed{{}^nC_r = {}^nC_{n-r}}$

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Ex. $S = (x^2 + x^3 + x^4 + x^5 + \dots)^3$. coeff. of x^9.

* **Discrete numeric function**

$f : \mathbb{Z}^+ \longrightarrow \mathbb{R}$

set of +ve integers (i/p) — set of real numbers (o/p).

$$\sum_{r=0}^{n} a_r x^r \longrightarrow a_r \text{ is discrete numeric function}$$

eg. $(1, 2, 9, 28, \dots)$

$a_r = (r^3 + 1) \qquad r \geq 0.$

Ex. Find discrete numeric function if generating function is $f(x) = \dfrac{1}{(1-ax)^K}$

$\longrightarrow$ $f(x) = (1-ax)^{-K}$

$$= \sum_{r=0}^{\infty} \binom{-K}{r} (-ax)^r$$

$$= \sum_{r=0}^{\infty} \binom{-K}{r} (-a)^r x^r.$$

$$a_r = \binom{-K}{r} (-a)^r.$$

$$= \binom{-K}{r} (-1)^r a^r = (-1)^r \, {}^{K+r-1}C_r \, (-1)^r a^r$$

$=$

$$a_r = {}^{k+r-1}C_r\, a^r.$$

$$a_r = \left({}^{k-1}C_0,\ {}^{k}C_1 a,\ {}^{k+1}C_2 a^2, \ldots \right)$$

discrete numeric function

<u>Ex</u> $f(x) = (1-ax)^{-1}$