→ Discrete Maths deals with the objects which have distinct values and can be counted.

→ List of topics : set
function & relation
graph & tree
boolean algebra

→ Appⁿ of Discrete maths :
1. Mathematical modeling of the problem
2. Security and Encryption
3. Algorithm Analysis
4. Proof techniques

1. Set & Proposition

→ Proposition — Any statement that can hold true or false is called proposition.

→ set is a collection of distinct discrete objects

→ Membership of the set :
- It is denoted by $\in$

ex $R = \{1,2,3\}$
$1 \in R$, $2 \in R$, $7 \notin R$
$R = \{1, \{2,3\}\}$
$2 \notin R$, $\{2,3\} \in R$

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→ ~~Prop~~ Subset : P is a subset of Q if every element of P is also element of Q.

ex. $P \subseteq Q$

→ Proper subset : P is proper subset of Q if P is subset of Q and $P \neq Q$.

ex. $P = \{1, 2\}$, $Q = \{1, 2, 3\}$

Here, $P \subseteq Q$ and $Q \neq P$.

→ Set Properties :

- For any set P, P is subset of P. (set is subset of itself).

- The empty set $\emptyset$ is subset of any set. However, it may not be element of every set.

ex $A = \{1, 2, 3\}$

$\emptyset \subset A$ but $\emptyset \notin A$

→ No. of elements in a set is called Cardinality of the set and it is denoted by '$|A|$'.

ex $\{\{\}\}$ → cardinality is 1 because it contains 1 element.

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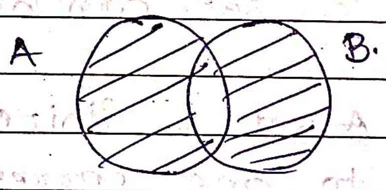
Set operations :

① Unionn → Finding all elements of two sets.

② Intersection – Finding common elements bet^n two sets.

③ Difference – (A – B) is set of all elements of set A which are not in B.

④ Symmetric difference – $(A \oplus B)$ it includes all elements of A & B but common elements should be skiped.



Properties related to set cardinality :

① $|P \cup Q| \leq |P| + |Q|$

② $|P \cap Q| \leq \min(|P|, |Q|)$

③ $|P \oplus Q| = |P| + |Q| - 2|P \cap Q|$

④ $|P - Q| \geq |P| - |Q|$

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Principle of Inclusion and Exclusion:

$|P \cup Q| = |P| + |Q| - |P \cap Q|$

$|P \cup Q \cup R| = |P| + |Q| + |R| - |P \cap Q| - |Q \cap R| - |P \cap R| + |P \cap Q \cap R|$

One to one correspondence :

One to one correspondence betn two sets exists if all elements of first set can be uniquely paired with all elements of second set.

- cardinality of a set can be finite or infinite ~~a set~~

- finite set : A set is finite if its elements have one to one correspondence with some other set whose cardinality is $K \in N$

ex. $\{1, 2, 3, 4\}$ and $\{10, 20, 30, 40\}$

$1 \rightarrow 10$, $2 \rightarrow 20$, $3 \rightarrow 30$, $4 \rightarrow 40$.

- Infinite set : It can be countable or uncountable.

- Countable infinite - If set has one to one correspondance betn its elements and elements of N. then it is called countable infinite

eg. $X = \{3, 6, 9, 12, \dots\}$

$N = \{1, 2, 3, \dots\}$

eg. Find the no. of integers which are divisible by any number 2, 3, 5 or 7 in the range 1 to 250.

→ No. of integers that are divisible by $2 = \left\lfloor \frac{250}{2} \right\rfloor = 125$

$3 = \left\lfloor \frac{250}{3} \right\rfloor = 83$

$\cancel{4 = \left\lfloor \frac{250}{4} \right\rfloor = 62}$, $5 = \left\lfloor \frac{250}{5} \right\rfloor = 50$

$7 = \left\lfloor \frac{250}{7} \right\rfloor = 35$

Intersection $2 \& 3 = \left\lfloor \frac{250}{6} \right\rfloor = 41$

$3 \& 5 = \left\lfloor \frac{250}{15} \right\rfloor = 16$, $5 \& 7 = \left\lfloor \frac{250}{35} \right\rfloor = 7$

$2 \& 5 = \left\lfloor \frac{250}{10} \right\rfloor = 25$, $2 \& 7 = \left\lfloor \frac{250}{14} \right\rfloor = 17$

$3 \& 7 = \left\lfloor \frac{250}{21} \right\rfloor = 11$

$2 \& 3 \& 5 = 8$, $3 \& 5 \& 7 = 2$, $2 \& 5 \& 7 = 3$

$2 \& 3 \& 7 = 5$, $2 \& 3 \& 5 \& 7 = 1$

$|2 \cup 3 \cup 5 \cup 7| = 125 + 83 + 50 + 35 - 41 - 16 - 7 - 25 - 17 - 11 + 8 + 2 + 3 + 5 - 1$

$= \cancel{159}\ \boxed{193}$

$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_3 \cap A_4| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_4| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_3 \cap A_4| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$

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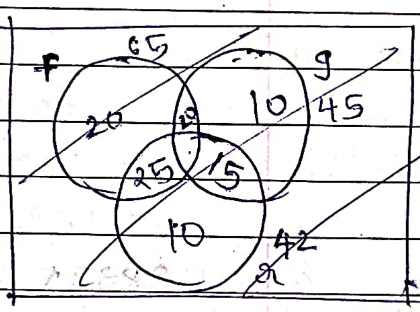
Eg. 100 out of 120 students study at least one of the languages. 65 students study french, 45 - german, 42 → Russian,
20 → french + german, 25 → french + Russian,
15 → german + Russian

(i) Find students studying only two languages but not third one
(ii) Find students who studies only one lang
(iii) Find stu. who studies all lang.

→ $|A_1| = 65$, $|A_2| = 45$, $|A_3| = 42$
$|A_1 \cap A_2| = 20$, $|A_1 \cap A_3| = 25$, $|A_2 \cap A_3| = 15$.
$|A_1 \cup A_2 \cup A_3| = 100$

(i) french + german not Russian
$|A_1 \cap A_2| - |A_1 \cap A_2 \cap A_3|$
$= 20 - 8 = 12$



(ii) Only french
$= |A_1| - |A_1 \cap A_2| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$
$= 65 - 20 - 25 + 8 = \boxed{28}$

(iii) $|A_1 \cap A_2 \cap A_3| = ?$

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$\therefore 100 = 65 + 45 + 42 - 20 - 25 - 15 + |A_1 \cap A_2 \cap A_3|$

$\therefore \boxed{|A_1 \cap A_2 \cap A_3| = 8}$

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Uncountable infinite :- there is no correspondence between its elements

⟹ Diagonal Argument (Cantor's diagonal arg)

	Red	Black	Yellow
A	yes	No	yes
B	No	No	Yes
C	Yes	Yes	Yes
New → D	No	Yes	No.

$n \times n$ grid unymmetric

The diagonal Argument states that a certain object is not one of the given objects using a fact that it is different from each of the given objects in at least one way.

⟶ Proof for all real no.s between 0 & 1 are uncountable infinite set.

— let's assume set of all no.'s bet^n 0 & 1 is countable infinite.

so there should be one to one correspondence bet^n its elements and N.

$$\begin{array}{c} \xrightarrow{\quad n \quad} \\ 0.\,a_1\ a_2\ a_3 \ldots a_n \\ 0.\,b_1\ b_2\ b_3 \ldots b_n \\ 0.\,c_1\ c_2\ c_3 \ldots c_n \\ \vdots \quad \vdots \quad \vdots \end{array}$$

$n \downarrow$

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- Suppose the list of all no.s betⁿ 0 & 1 as above.
- let's assume one no. $0.x_1 x_2 \ldots x_n$ such that x_1 is different than a_1, x_2 is different than a_2 and so on.
- So, this new no. is different than all existing no. in given list. so the list is incomplete and there is no one to one correspondence with set of Natural no. N.

So, our assumption is incorrect.
Hence the set of real no. betⁿ 0 & 1 is uncountable infinite

⇒ <u>Principle of Mathematical Induction</u> : (PMI)

- For a given statement which involves natural no. n if we can show that,

(1) Statement is true for $n = n_0$ (Basis)

(2) Assume that the statement is true for $n = k$ (Hypothesis) and prove that a statement is true for $n = k+1$. (Induction step)

(3) Then the statement is true for all natural no. $n \geq n_0$

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eg: given the coins of value 3 & 5. Prove that any amount of 8 or more can be made using the available coins.

$8 = 3+5$ → case-I
$9 = 3+3+3$ → case-II
$10 = 5+5$ → case-III
$11 = 3+3+5$
$12 = 3+3+3+3$

(1) Basis of Induction

- let Base amount is 8 and since $8 = 5+3$, so basis is proved.

(2) Induction Hypothesis

- It is possible to make any amount k using coins of 3 & 5.

- Case-I If amount k is made using 3 & 5 then we can replace 5 with two 3's to get $k+1$ amount.

Case-II If amount k is made using only 3's then we can replace 3 3's with 2 5's to get $k+1$ amount.

Case-III If amount k is made using only 5's then we can replace 5 with two 3's to get $k+1$ amount.

So, according to above cases if hypothesis is true, then we can make amount k+1 using the coins of 3 & 5.

So, the statement is proved for all the numbers.

eg. Prove that $1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$; $(n \geq 1)$

→ Basis

L.H.S. $(n=1)$ $= 1^2 = 1$

RHS $= \frac{1(1+1)(2+1)}{6} = 1$

So, Basis is proved

Hypothesis

Assume the hypothesis $1^2+2^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$

we need to prove that

$$1^2+2^2+\dots+(k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

Now,

$$\begin{aligned} LHS &= 1^2+2^2+\dots+(k+1)^2 \\ &= 1^2+2^2+\dots+k^2+(k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6}+(k+1)^2 \quad (\because \text{From Hypothesis}) \end{aligned}$$

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$$= \cancel{k}(k+1)\left[\frac{(2k+1)(k)+6(k+1)}{6}\right]$$

$$=(k+1)\left[\frac{2k^2+7k+6}{6}\right]$$

$$=\frac{(k+1)\ (2k+3)(k+2)}{6}$$

$$=\frac{(k+1)\ (k+2)\ (2(k+1)+1)}{6}$$

= R.H.S.

So, ~~T~~ statement is true for $n = k+1$.
Hence the statement is true for all $n \geq 1$.

eg Prove that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for $n \geq 1$.

→ Basis $n=1$. LHS $= \frac{1}{1\cdot 2} = \frac{1}{2}$

RHS $= \frac{1}{1+1} = \frac{1}{2}$

So, Basis is proved.

→ Hypothesis Assume the Hypothesis

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

we need to prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

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Now, LHS

$$\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{(k+1)(k+2)}$$

$$=\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$ (∵ From Hypothesis)

$$=\frac{1}{(k+1)}\left[\frac{k(k+2)+1}{(k+2)}\right]$$

$$=\frac{(k^2+2k+1)}{(k+1)(k+2)}=\frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}$$

$$=\frac{k+1}{k+2}=\text{RHS}$$

So, Statement is true for $n=k+1$

Hence, the statement is true for all $n\geq1$.

eg. Show that for $n\geq1$,

n^3+2n is divisibe by 3.

→ Basis $n=1$

$n^3+2n=3$ is divisible by 3

Hypothesis for $n=k$ Assume that

k^3+2k is divisible by 3.

need to prove that $(k+1)^3+2(k+1)$ is divisible by 3

$(k+1)^3 + 2(k+1)$

$= k^3 + 1 + 3k^2 + 3k + 2k + 2$

$= \underline{k^3 + 2k} + 3k^2 + 3k + 3$

$= 3(k^2 + k + 1) + k^3 + 2k$ (from hypothesis $k^3 + 2k$ is divisible by

$= 3m + k^3 + 2k$

since $k^3 + 2k$ is divisible by 3 as per hypothesis and the other quantity is also divisible by 3. so the induction is proved.

eg Prove that $n^4 - 4n^2$ is divisible by 3. for $n \geq 2$.

→ Basis $n = 2$

LHS $= 16 - 16 = 0$ is divisible by 3.

Hypothesis for $n = k$ Assume that $k^4 - 4k^2$ is divisible by 3.

need to prove that $(k+1)^4 - 4(k+1)^2$ is divisible by 3

$k^2(k^2 - 4)$

$= (k+1)^4 - 4(k+1)^2$

$= (k+1)^2 [(k+1)^2 - 4]$

$= (k+1)^2 [k^2 + 2k - 3]$

$= (k+1)^2 (k+3)(k-1)$

$= (k^2 + 2k + 1)(k^2 + 2k - 3)$

$= k^4 + 2k^3 - 3k^2 + 2k^3 + 4k^2 - 6k + k^2 + 2k - 3$

$= \underline{k^4} + 4k^3 - \underline{4k^2} + 4k^2 + 2k^2 - 3 - 4k$

$= 4k^3 + 6k^2 - 3 + (k^4 - 4k^2) - 4k$

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$$= \cancel{3\left(\frac{4}{3}k^3+2k^2\right)-3+(k^4-4k^2)}$$

$$= (k^4-4k^2)+(4k^3-4k)+(6k^2-3)$$

$$= 4(k+1)k(k-1)+(k^4-4k^2)+(6k^2-3)$$

Second quantity is divisible by 3 as per hypothesis. first quantity is divisible by 3 because it is product of three consecutive number and third quantity is of form 3(m) that is also divisible by 3.

Hence the induction is proved.

eg. Show that, any integer composed of 3^n identical digits is divisibe by 3^n.

→ Bases : $n=1$

$3^1 = 3$ is divisible by 3.

ex 111 is divisible by 3.

for $n=1$ all no.'s with identical 3 digits are divisible by 3. such 9 no.s are possible

Hypothesis :

Any no. with 3^k identical digits is divisible by 3^k. prove that, if a no. contains 3^{k+1} identical digits then it is divisible by 3^{k+1}.

let's assume a number with 3^{k+1} identical digits.
Now, assume that number is product of x and y.
where x is a number containing 3^k identical numbers

$$3^{k+1} = x \cdot y$$

(ex. 222 222 222 $\Rightarrow$ $x = 222$
$y = 1001001$)

x is divisible by 3^k by hypothesis.
Because sum of digits in y is equal to 3 so
y is divisible by 3. So the assumed no. is ~~also~~
divisible by 3^{k+1}

Eg Show that any integer n, $(11)^{n+2} + (12)^{2n+1}$ is
divisible by 133.

→ Basis $n = 1$

$$(11)^3 + 12^3 = 1331 + 1728$$
$$= 3059 \text{ is divisible by } 133.$$

Hypothesis for any integer $n = k$
$(11)^{k+2} + (12)^{2k+1}$ is divisible by 133.

we need to prove that $(11)^{k+3} + (12)^{2(k+1)+1}$
is divisible by 133.

$$= (11)^{k+3} + (12)^{2(k+1)+1}$$
$$= (11)^{k+2} \cdot (11) + (12)^{2k+3}$$
$$= (11)^{k+2} \cdot (11) + (12)^{2k+1} (12)^2$$
$$= (11)^{k+2} \cdot (11) + (12)^{2k+1} (133 + 11)$$

$= (11)(11)^{K+2} + (11)(12)^{2K+1} + (133)(12)^{2K+1}$

$= (11)\left[11^{K+2} + 12^{2K+1}\right] + 133\,(12)^{2K+1}$

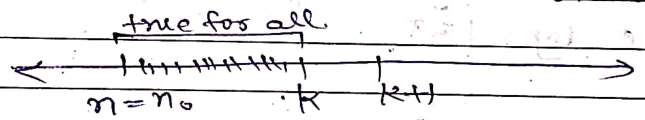
$= 11\,(133y) + 133\,(x)$ ($\because$ From Hypothesis)

$= 133\,(11y + x)$

$= 133 \cdot m$ is divisible by 133.

Eg $\Rightarrow$ Strong Mathematical Induction.

- In strong Mathematical Ind^n the hypothesis is assumed to be true for $n_0 \le n \le K$. Other details are same as normal Ind^n.



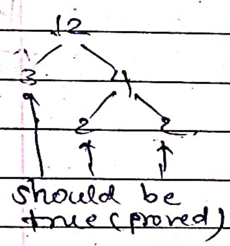
(assumes more)

Ex. Show that any positive integer $n \geq 2$ is either prime no. or product of two prime numbers.

→ Basis of Induction $n = 2$
since 2 is prime number, basis is true.

Hypothesis (Strong induction)

The statement is true for any n such that ~~two~~ $2 \le n \le K$.



Induction step:

Assume a no. $K+1$

case-I $(K+1)$ is prime

then the statement is true

case-II If $(K+1)$ is not prime, then it is can be always expressed as product of two no.s p and q, such that $[p, q \leq K]$

$= p \cdot q \quad [p, q \leq K]$

So As per hypothesis p & q are either prime no.s or product of prime no.s. So the no: $K+1$ is also either prime or product of prime no.s.

Ex. Show that $2^n > n^3$ for $n \geq 10$.

→ Basis $n = 10$

LHS $= 2^{10} = 1024 > 10^3 = $ RHS.

Hypothesis

statement is true for any k such that

$n = k$, $2^k > k^3$

need to prove for $2^{k+1} > (k+1)^3$

$$2^k \cdot 2 > k^3 + 1 + 3k^2 + 3k$$

Now, $2^k > k^3$

$2^{k+1} > 2k^3$

if we prove $2k^3 > (k+1)^3$

$\Rightarrow \quad 2K^3 > K^3 + 3K^2 + 3K + 1$

$\Rightarrow \quad K^3 > 3K^2 + 3K + 1$

$\Rightarrow \quad 1 > \frac{3}{K} + \frac{3}{K^2} + \frac{1}{K^3}$ ($\because$ Divide by K^3)

put $K = 10$

$1 > \frac{3}{10} + \frac{3}{10^2} + \frac{1}{10^3}$

$1 > 0.333$

for $K=10$ statement is true. $1 > 0.333$

So, for all values $K \geq 10$ the statement is true.

$2^{K+1} > 2K^3$ —— (1)

$2K^3 > (K+1)^3$ —— (2)

By transitivity, $2^{K+1} > (K+1)^3$ is also proved.

Proposition

- The use of proposition logic is to convert natural language into mathematical statement.
- Proposition is a declarative sentence - either true or false.
- Some propositions are always true. A proposition which is always true - Tautology.
- A proposition which is always false - Contradiction.
- Evaluation of proposition means assigning true or false value to the proposition.

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- Two propositions can be combined.

- Two or more propositions can be combined using connectives.

- Connectives:

(1) Disjunction ($p \vee q$)
(2) Conjunction ($p \wedge q$)

p	q	$p \vee q$	$p \wedge q$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

(3) Negation ($\sim p$)

p	$\tilde{p}$
T	F
F	T

(4) Exclusive-OR ($p \oplus q$)

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

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→ A proposition which is obtained from the combination of the other propositions is called "Compound proposition."

→ A proposition which is not a combination is called "Atomic proposition".

p	q	$(p \wedge q) \wedge \tilde{p}$
T	T	F
T	F	F
F	T	F
F	F	F

→ Compound proposition are of two types:
(i) Conditional $(p \rightarrow q)$
(ii) Biconditional $(p \leftrightarrow q)$

(i) Conditional statement is defined as $p \rightarrow q$ (If p then q)

(ii) Biconditional statement is defined as $p \leftrightarrow q$ (p if and only if q) $[(p \rightarrow q) \wedge (q \rightarrow p)]$

→ <u>Necessary</u> & <u>Sufficient Cond^n</u>:

p is necessary for q means if q is true p must be true but p is true that doesn't mean q is true.

<u>ex</u>. p = a person is 18 years old.
q = a person is president of india.

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Sufficient Cond^n:

p is sufficient for q means that if p is true we can always conclude that q is also true

ex. p = person has voter card.
q = person is 18 years old.

– p can be both sufficient and necessary cond^n for q.

ex. p = person is born in some country.
q = person is citizen of that country
– p is sufficient cond^n for q.

ex p = no. is multiple of 9.
q = no. is multiple of 3.
– p is sufficient cond^n for q.

ex. p = two lines are parallel
q = two lines are not intersacting.
– p is sufficient & necessary for q. $(p \leftrightarrow q)$

p	q	$p \to q$	$p \leftrightarrow q$ (X-NOR)
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	T

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Ex. An island has two tribes of natives. Any native from the first tribe always tells the truth, while any native from the other tribe always lies. You arrive at the island and ask a native if there is gold on the island. He answers, "There is gold on the island if and only if I always tell the truth". Which tribe is he from? Is there gold on the island?

→ p = he always tells the truth
q = there is gold on the island.

Thus, his answer is $p \leftrightarrow q$

Suppose that ~~his answer to a~~ he always tells the truth. that is, p is true.
Therefore, q must be true. that is his answer to our question must be true.

$\therefore$ $p \leftrightarrow q$ is true

Suppose he always lies, that is p is false.
Also, his answer to our question is a lie, which means that $p \leftrightarrow q$ is false.
Consequently, q must be true.
$(F, T = F)$

Thus, in both cases we can conclude that there is gold on the island, although the native could have been from either tribe.

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Ex. p = the food is good
q = the service is good
r = the rating is 3 star

(1) Either the food is good or service is good or both. $\longrightarrow p \vee q$

(2) Either the food is good or service is good but not both $\longrightarrow p \oplus q$

(3) Food is good but the service is poor. $\longrightarrow p \wedge \tilde{q}$

(4) It is not the case that food is good and rating is 3 star $\longrightarrow (\overline{p \wedge r})$

(5) If both food and service is good then rating is 3 star. $\longrightarrow (p \wedge q) \longrightarrow r$

(6) It is not true that ~~st~~ 3 stare always means good food and good service

$[r \longrightarrow (p \wedge q)]'$

→ For proposition $p \rightarrow q$, the proposition $q \rightarrow p$ is called converse.

→ The proposition $\bar{q} \rightarrow \bar{p}$ is called contra positive.

→ The proposition $\bar{p} \rightarrow \bar{q}$ is called inverse.

Que. Find truth values of all above.

p	q	$p \to q$	$q \to p$	$\bar{p} \to \bar{q}$	$\bar{q} \to \bar{p}$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

$p \to q$ & $\bar{q} \to \bar{p}$ is same.

$q \to p$ & $\bar{p} \to \bar{q}$ is same.

Ex. Give the converse, contrapositive and inverse of the foll^{n} implication : "If it rains today, I will go to college tomorrow."

→ Converse : If I will go to college tomorrow, then it would have rained today.

Contrapositive : If I do not go to college tomorrow, then it will not have rained today.

Inverse : If it does not rain today, then I will not go to college tomorrow

3. Relation & Function

• Relation:

→ In a set, the elements are sometimes related with each other, such elements have some comman factor betⁿ them.

$S = \{1, 2, 3, 4, 5, 6\}$

$R = \{(2, 4, 6), (1, 3, 5)\}$ ← called relation.

→ Define cartesian Product:

$A = \{a, b\}$, $B = \{c, d\}$

$A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$

$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$

→ Binary relation from set A to B is subset of cartesian product $A \times B$.

$A = \{1, 2\}$ $B = \{1, 3\}$

$A \times B = \{(1,1), (1,3), (2,1), (2,3)\}$

If Relation is defined as "less than" then,

$R = \{(1,3), (2,3)\}$

→ aRb — a is related with b

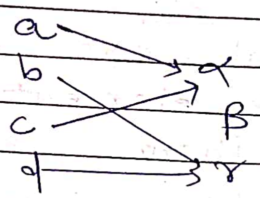
→ Relation is always ordered pair.

→ Tabular Form of Relation:

$R = \{(a, \alpha), (c, \alpha), (b, \gamma), (c, \gamma)\}$

	α	β	γ
a	✓		-
b			✓
c	✓		✓
d			

⟹ <u>Graphical form of Relation</u>



⟶ Just like a set relation has foll^{n} operations
$R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_1 \oplus R_2$

⟶ Ternary relation can be defined as subset of Cartesian product of $(A \times B)$ and C

eg. $R = \{((a,\alpha),1), ((a,\alpha),2) \ldots\}$

• <u>Properties of Binary Relation</u>

(1) ~~Reflexibility~~ Reflexivity
– Relation R is defined on set A, for $a \in A$, if $(a,a) \in R$ then R is called Reflexive Relation.

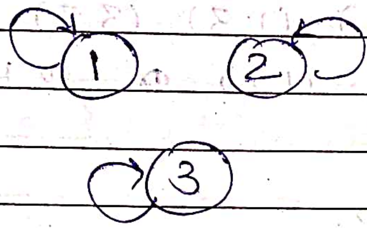
ex $A = \{1,2,3\}$
$R = \{(1,1),(2,2),(2,3),(3,3)\}$ is the example of reflexive relation.

Because, $(1,1),(2,2),(3,3) \in R$

– In the relation matrix, Reflexivity is indicated by diagonal cells.

R	1	2	3
1	—		
2		—	
3			—

- In the directed graph, self loop represent the reflexivity.

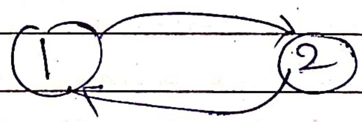


(2) Symmetric Relation

- Relation R is defined on set A, for each pair $(a,b) \in R$ if $(b,a) \in R$ then relation R is symmetric.

ex: $R = \{(1,2),(2,1)\}$ $R = \{(a,b) \mid a=b\}$

- In the relation graph, the cycle indicates the symmetric relation.



(3) Anti-symmetric Relation

- R is relation on A, if $(a,b) \in R$ and $(b,a) \in R$ implies ($\Rightarrow$) $a=b$ then, R is Anti-symmetric Relation.

- Anti-symmetric $\neq$ Not symmetric

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ex. $R = \{(1,1), (1,2), (1,3), (3,3), (2,1) (2,2)\}$
$A = \{1,2,3\}$

R is Reflexive
R is not symmetric [(1,3)]
R is not anti-symmetric

For make it Anti-symmetric:
$R = \{(1,1), (1,2), (1,3), (3,3), (2,2)\}$
(Remove either (1,2) or (2,1))

Ex. $A = \{a,b\}$
$R = \{(a,a), (b,b)\}$

R is Reflexive, Symmetric, Anti-symmetric

Ex. $A = \{a,b\}$
$R = \{(a,b), (a,c), (c,a)\}$
[(a,c) & (c,a)]
R is not Reflexive, symmetric, Anti-symmetric

Q.1 Is every Reflexive Relation is Anti-symmetric.

$A = \{1,2,3\}$
$R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

R is Reflexive but not Anti-symmetric
because (1,2) and also (2,1) is present.

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(4) Transitivity

- R is relation defined on set A, if $(a,b) \in R$ and $(b,c) \in R$ ~~then~~ implies $(a,c) \in R$ then the relation is transitivity for every pair.

ex $R = \{(a,a), (a,b), (a,c), (b,c)\}$.
R is transitivity.

$R = \{(a,b) \mid a > b\}$ is transitivity.

→ Closure of Relation

(1) Reflexive closure

The Reflexive closure of Relation R is smallest Reflexive Relation which contains R.

ex $R = \{(1,1), (2,2), (2,3)\}$ defined on set $A = \{1,2,3\}$

$R_R^* = \{(1,1), (2,2), (2,3), (3,3)\}$

Reflexive closure

cannot remove any pair.

$R_R^* = R \cup \{(a,a) \mid a \in A\}$

(2) Symmetric Closure

It is smallest Symmetric Relation which contains R.

ex. $R = \{(0,1), (1,2), (2,1)\}$
$A = \{0,1,2\}$

$$R_s^* = \{(0,1), (1,2), (2,1), (1,0)\}$$

$$R_s^* = R \cup \{(b,a) \mid (a,b) \in R\}$$

(3) Transitive Closure

Transitive closure is the smallest transitive Relation that contains R.

ex. $A = \{1,2,3\}$
$R = \{(1,1), (2,3), (3,1)\}$

$$R_t^* = \{(1,1), (2,3), (3,1), (2,1)\}$$

In general, $R_t^* = R \cup \{(a,c) \mid (a,b) \in R \ \& \ (b,c) \in R\}$

Ex. $R = \{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3)\}$
$A = \{1,2,3,4\}$

$$R_t^* = \{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3), (2,4), (3,3), (4,4)\}$$

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Warshall's Algorithm to find the transitive closure.

Warshall's Algorithm is used to find transitive closure of the relation R.

Step:

(1) If the set on which the relation is defined contains 'n' elements then create a matrix of n columns 2 rows

first row, is for columns (C)
second row, is for row (R)

	I	II	III
C			
R			

(2) In the ith cell of the column section write the elements which are present in the particular column of Relation matrix.

In the ith cell of the row section, write the elements which are present in the particular row of Relation matrix.

(3) For each column, perform the cartesian product of the elements in the first & second row
repeat this for all columns.
Whichever pairs are result of cartesian product they will be included in the transitive closure.

Ex. $A = \{1, 2, 3, 4\}$

$R = \{(2,1), (2,3), (3,1), (3,4), (4,1) (4,3)\}$

→ Relation Matrix :

	1	2	3	4
1	0	0	0	0
2	1	0	1	0
3	1	0	0	1
4	1	0	1	0

	I	II	III	IV
C	$\{2,3,4\}$	ϕ	$\{2,4\}$	$\{3,$ [illegible]$\}$
R	ϕ	$\{1,3\}$	$\{1,4\}$	$\{1,3\}$
cross product	ϕ	ϕ	$\{(2,1), (2,4), (4,1), (4,4)\}$	$\{(3,1), (3,3)\}$

Add into the original Matrix

	1	2	3	4
1	0	0	0	0
2	1	0	1	1
3	1	0	1	1
4	1	0	1	1

$R_t^{*} = \{(2,1), (2,3), (2,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4)\}$

Ex: Find a transitive closure using Warshall's Algo.

$A = \{a, b, c, d, e\}$

$R = \{(a,c), (b,d), (c,a), (d,b), (e,d)\}$

→ Relation Matrix :

	a	b	c	d	e
a	0	0	1	0	0
b	0	0	0	1	0
c	1	0	0	0	0
d	0	1	0	0	0
e	0	0	0	1	0

	I	II	III	IV	V
C	{c}	{d}	{a}	{b, e}	ϕ
R	{c}	{d}	{a}	{b}	{d}
	{(c,c)}	{(d,d)}	{(a,a)}	{(b,b), (e,b)}	ϕ

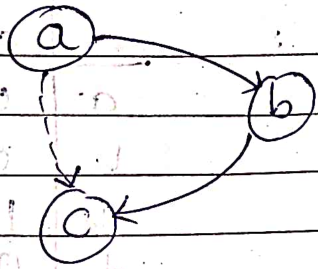
R =

	a	b	c	d	e
a	1	0	1	0	0
b	0	1	0	1	0
c	1	0	1	0	0
d	0	1	0	1	0
e	0	1	0	1	0

$R_t^* = \{(a,a), (a,c), (b,b), (b,d), (c,a), (c,c), (d,b), (d,d), (e,b), (e,d)\}$

→ transitive closure indicates the reachability in the graph.

$R = \{(a,b), (b,c)\}$



Graph says, that we dont have path from a to c directly.

If we apply transitive closure then we can get path from a to c via b.

ex. Find the smallest relation containing the relation $R = \{(1,2), (1,4), (3,3), (4,1)\}$ that is Reflexive and transitive.

→ $R_R^* = \{(1,2), (1,4), (3,3), (4,1), (1,1), (2,2), (4,4)\}$

$R_t^* = \{(1,2), (1,4), (3,3), (4,1), (1,1), (4,4), (4,2)\}$

$R = R_R^* \cup R_t^*$

$= \{(1,2), (1,4), (3,3), (4,1), (1,1), (2,2), (4,4), (4,2)\}$

ex. let A, Be a set have 10 elements. How many diff binary relation on A is possible?
How many of them are reflexive?
How many of them are symmetric?

ex. let R be a binary relation on set of all ~~possible~~ (+ve) integers such that $R = \{(a,b) \mid a-b \text{ is odd positive integer}\}$
Find the nature of the relation.

→ No Reflexive &
No Symmetric
No transitive [(10,5) − (5,2) = (10,2) which is not odd]