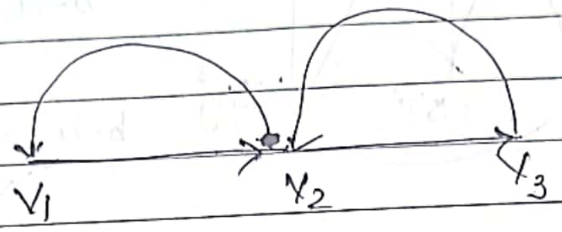
Sessional – III

~~Graph Tree.~~

Theorem 1 A directed graph contains an Eulerian circuit iff it is connected $\boxed{\text{and}}$ the incomming degree of every vertex is equal to outgoing degree.

Eulerian circuit.

Eg



Euler circuit : $V_1 - V_2 - V_3 - V_2 - V_1$

$In(V_1) = 1$ $\quad$ $In(V_2) = 2$

$Out(V_1) = 1$ $\quad$ $Out(V_2) = 2$

$In(V_3) = 1$

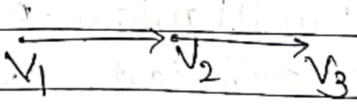
$Out(V_3) = 1$

∴ Theorem is true for above graph. Hence it contains Eulerian circuit.

Theorem 2 A directed graph contains an Eulerian path iff it is connected $\boxed{\text{and}}$ the indegree of each vertex is equal to outdegree with possible exceptions of two vertex. For these vertices, there is a difference of 1 in the indegree and out degree

eulerian path.

Eg



$In(V_1) = 0$, $Out(V_1) = 1$

$In(V_2) = 1$, $In(V_2) = 1$ ← same

$In(V_3) = 1$, $Out(V_3) = 0$.

for v_1 & v_2 Indegree & outdegree are diff but there difference is 1. so it contains Eulerian path.

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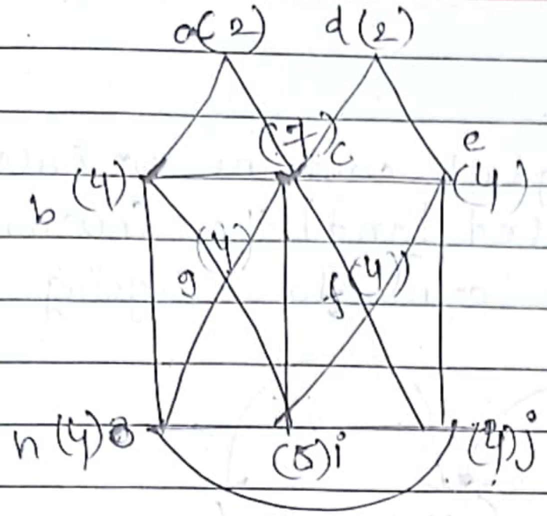
Eg



Euler path – ✓
Euler circuit – ✗.
Hamiltonian path – ✓

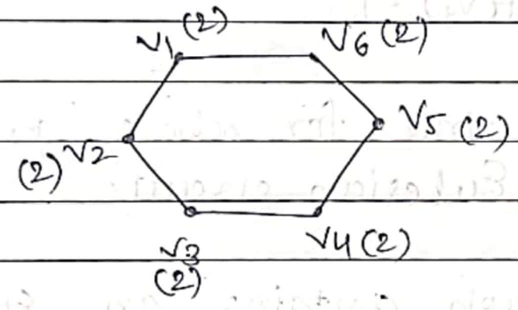
b–a–c–d–e–f–j–i–g–h
– HP

b–a–c–d–e–f–j–i–g–h–b
→ HC

Theorem3. G is graph with n vertices. if the sum of degrees of each pair of vertices is $(n-1)$ or larger, then there exist a Hamiltonian path.

this is only sufficient condn

Eg



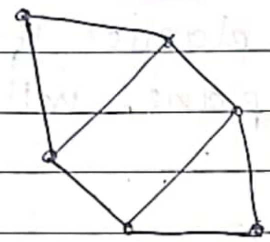
- The above theorem is sufficient condn for detection of Hamiltonian path.
  It means that even if the condn is not satisfied there may be still hamiltonian path.

→ For finding Hamiltonian path,
we only have sufficient condn but not be sufficient & necessary. this is the problem.

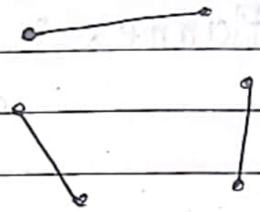
- **Factors of a graph**

K factor of a graph is defined as a spanning subgraph such that degree of each vertices is K.

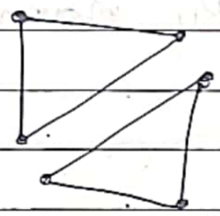
eg



graph



1 factor graph

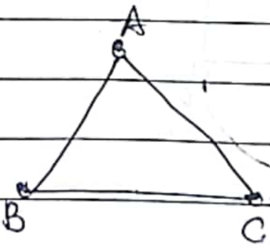


2 factor graph

- **Complete graph. (Kn)**

A graph with n vertices is called complete if there is a connectivity bet^n. every pair of vertices.

eg K_3.



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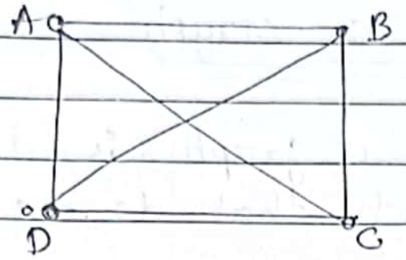
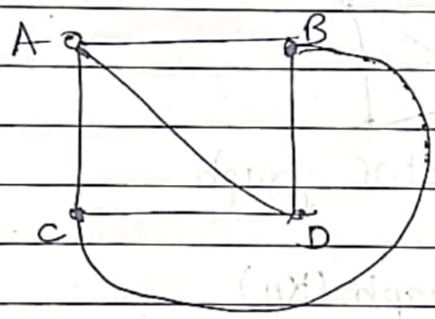
K_4

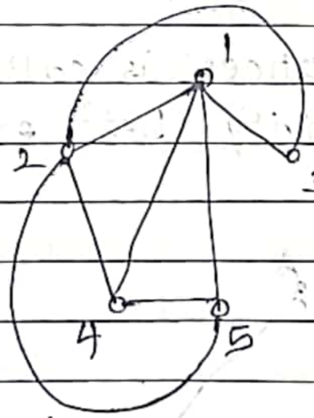


- Planner : A graph is planner if it can be drawn on a plane, without crossing of edges.

eg In previous example, K_4 is not planner graph but K_3 is planner graph.
We can make K_4 planner graph.

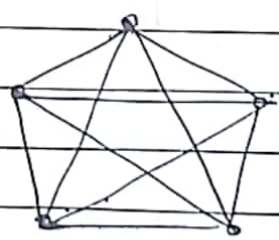


eg K_5



K_5 is non planner graph. because we can't connect 4 & 3.

K_5:



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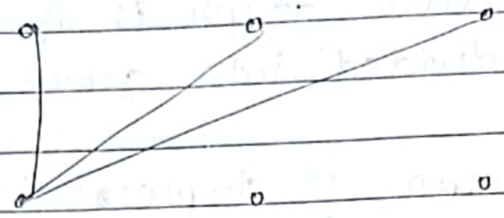
K6 :



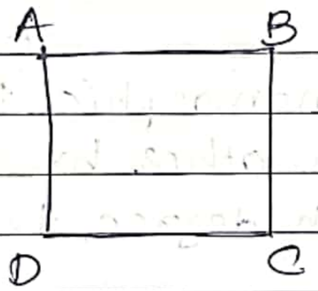
K6 is non planner

- starting from K5 all other are non-planner

- ## Bipartite Graph

A bipartite graph is a graph whose vertices can be divided into two independent sets U & V such that every edge in the graph connects the vertex from U → V or V → U.

eg

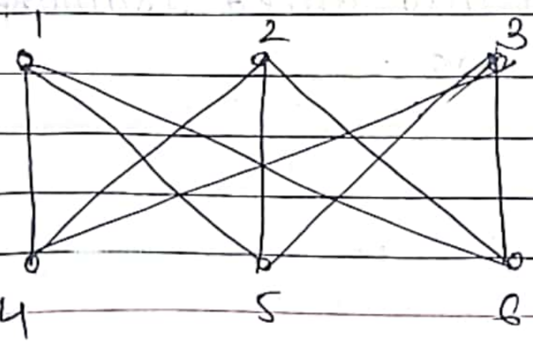


Here, U = {A, C}
V = {B, D}

- ## Complete Bipartite Graph

In Complete Bipartite Graph, every vertex of set U is connected to every vertex of set V

eg.



~~Not~~ Notation is $K_{3,3}$

Here,
Complete Bipartite graph is of 6 vertices and vertices are divided into group of 3 vertices.

eg Which graph is simplest Non-planner graph?
($K_{3,3}$ or K_5)
$\downarrow$ $\downarrow$
9 edge 10 edges

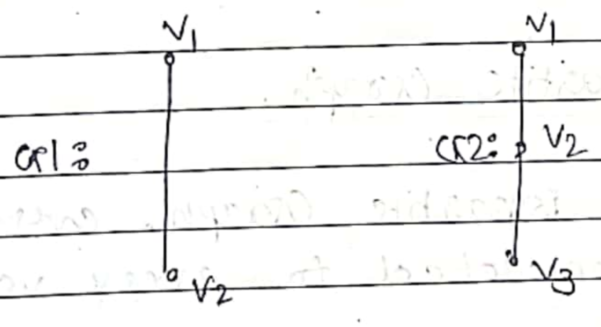
- Based upon no. of edges, we can say $K_{3,3}$ is the simplest non-planner graph.

→ $K_{3,3}$ & K_5 are called Kuratowski graphs.

- **Homomorphism**

Two graphs are called Homomorphic if they can be transformed into each other by insertion or removal of a vertex with degree two.

eg



G1 & G2 are called Homomorphic graphs.

They are also called Isomorphic ~~of~~ within degree two.

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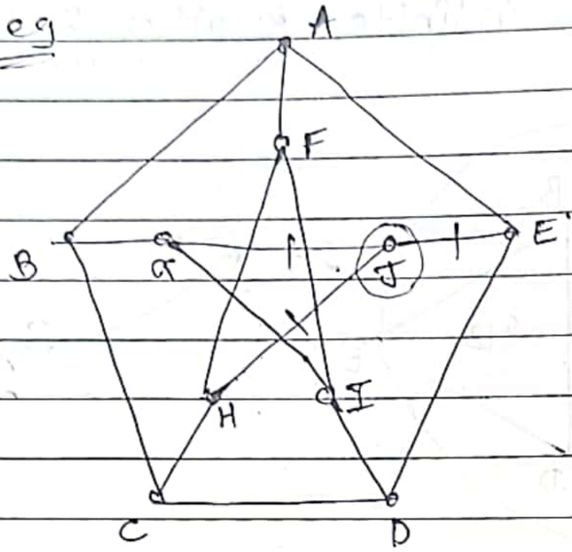
- ## Kuratowski's Theorem

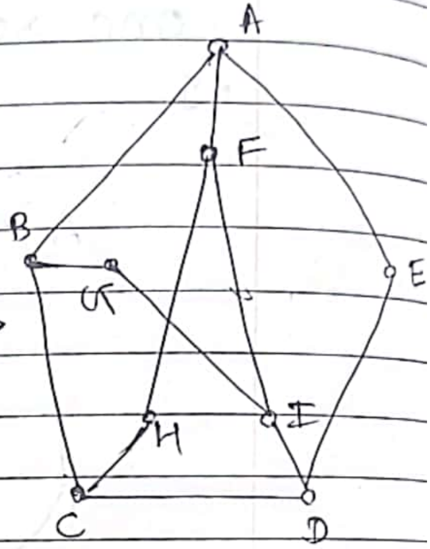
Every non-planner graph has a subgraph that is Homomorphic to K_5 or $K_{3,3}$

eg



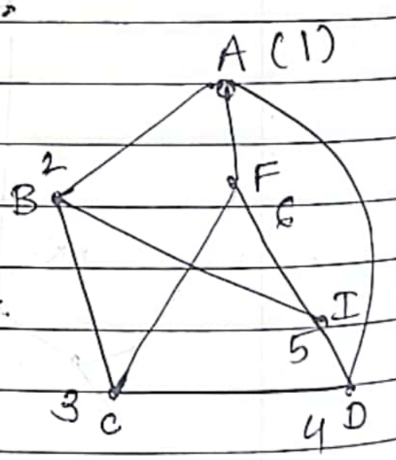
(G)

sub graph →



From subgraph, we have to delete 3 vertices of 2 degree to make isomorphic with $K_{3,3}$.

Delete, vertices E, G; H
New graph,
because it has degree 2.



(G1)

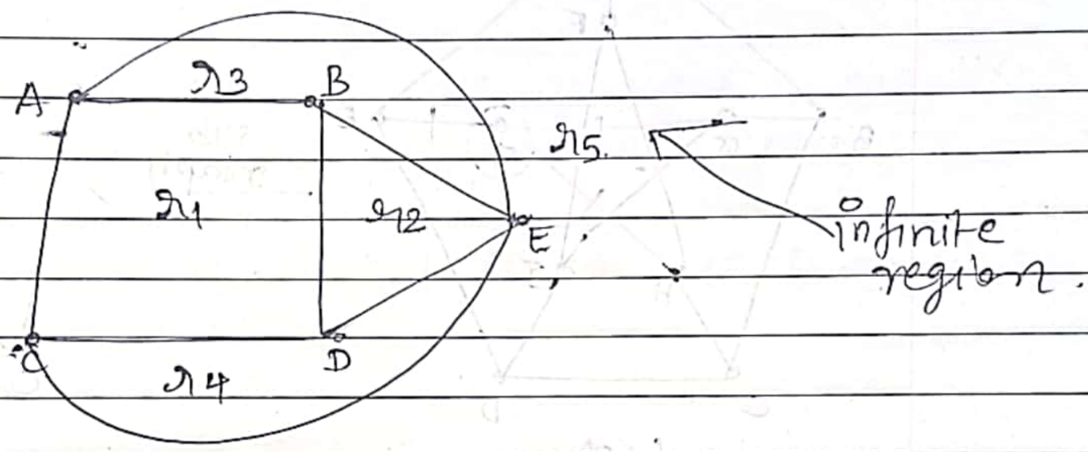
$U = \{1, 5, 3\}$
$V = \{2, 4, 6\}$

graph G is non-planner graph and it is Homomorphic to $K_{3,3}$.

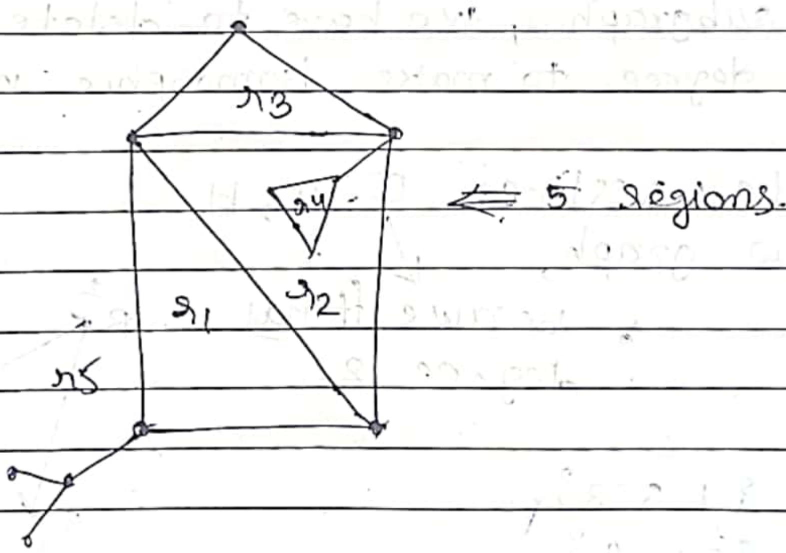
- ## Region OF a Graph

A Region of the planner graph is an area of the plane that is bounded by edges and cannot be further sub divided. Every graph has one region is infinite & other are finite.

eg



eg



→ If the graph has E edges v vertices & r regions then,

$$\boxed{v - E + r = 2}$$ ← Euler's formula for planner graph.

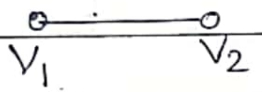
Result: In a connected planner graph, with e edges & v vertices

$$\boxed{3v - e \geq 6}$$

- **Proof of Euler's formula for planner graph:**

step: 1 **Basis of Induction:**

for given graph,



$e = 1$ (~~Assume~~ Let)
$v = 2$
$r = 1$

step: 2 **Induction Hypothesis:**

Assume that, formula is correct for $e = k$

$$v - e_k + r = 2$$

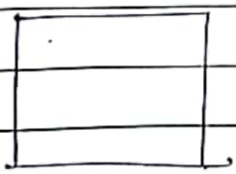
Induction step:

Prove the statement for $e = k+1$

case-I. In this case, Adding a new edge increases the no. of regions by 1. so,

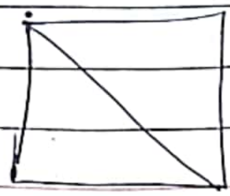
$$v_{k+1} - e_{k+1} + r_{k+1}$$
$$= v_k - (e_k + 1) + r_k + 1$$
$$= v_k - e_k + r_k = 2$$

Eg



$e = 4$
$v = 4$
$r = 2$

And now,



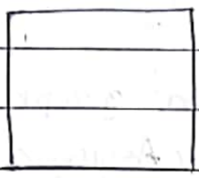
$e = 5$
$v = 4$
$r = 3$

Case-II In this case, adding a new edge increases the no. of vertices by 1, ~~and~~ so,

$$v_{k+1} - e_{k+1} + r_{k+1}$$
$$= v_{k+1} - (e_k + 1) + r_k$$
$$= v_k - e_k + r_k = 2.$$

Hence, Hypothesis is proved.

∴ Eg.



$v=4$
$e=4$
$r=2$

And now

$v=5$
$e=5$
$r=2$

Hence, $v_k - e_k + r_k = 2$ is proved.

Tree And Cutset

- Tree is a connected acyclic undirected graph.
- Tree is always undirected.
- Tree represents hierarchical relationship bet^n individual elements.

• Properties of the Tree:

(1) There is an unique path bet^n every two vertices in the tree.

(2) The no. of vertices in a tree is one more than the no. of edges. ($\because e = v-1$)

(3) A tree with two or more vertices has at least two leaves.

~~proof~~ : let the no. of total vertices $= n$
and out of that leaves $= K$.
$\therefore$ Non-leaves $= n-K$.

sum of degrees of all vertices $\sum_{i=1}^{n} d(v_i) = K(1) + (n-K) \cdot y$

y is degree of non-leaves.

And $y \geq 2$, So,

$$\sum d(v_i) \geq K + (n-K) \cdot 2$$ ($\because$ Principle of inequality)

$$\boxed{\sum_{i=1}^{n} (d(v_i)) \geq 2n - K}$$

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Now, we know;

$$\sum_{i=1}^{n} d(v_i) = 2e \quad (\because \text{Hand Shaking theorem})$$

$$= 2(n-1)$$ according to previous property

So,

$$2(n-1) \geq 2n-k$$
$$2n-2 \geq 2n-k$$
$$\therefore \boxed{k \geq 2}$$

Hence, it is proved.

- Characteristics of the Tree :

1. A graph in which there is an unique (acyclic) path bet^n every pair of vertices is a tree.

2. A connected graph with $e = v-1$ edges is a tree

proof : Suppose, a connected graph of v vertices and $v-1$ edges has a cycle

Now,
to ~~make the~~ remove the cycle we remove one edge from the graph so the no. of edges will be $v-2$. but this will make the graph disconnected. which is contradiction to the original statement. So, the graph does not have any cycle & it is a tree.

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(3) A graph with "$e = v-1$" edges that has no circuit is a tree.

- ## Rooted Trees

— A directed graph is said to be directed tree if it becomes a tree when directions are ignored.

— A directed tree is called rooted tree if there is exactly one vertex whose indegree is 0 & all other nodes directly or indirectly originate from that node whose indegree is 0.

— In a directed tree all vertices whose outdegree is non-zero are called branch or internal nodes.

- ## Ordered Tree

An ordered tree is a rooted tree in which the edges originating from each branch is labeled as 1, 2, 3, ... i

→ An ordered tree in which every branch has at most m children is called m-ary tree.

→ An m-ary tree is regular if every brach node has exactly m children.

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Eg

regular 3-ary