Ollscoil na hÉireann
The National University of Ireland

Coláiste na hOllscoile, Corcaigh
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CS4407 Algorithm Analysis

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Attempt all questions

Total marks: 100

90 minutes

Please answer all questions
Points for each question are indicated by [xx]

1. **[15]** Consider the *UniqueElements* problem, where we check whether all the elements in a given array are distinct.
   a. **[10]** Use the loop invariance approach to analyse this algorithm.
   We assume that we have an array A[0 …n-1] of n elements. Starting from the first element, we check whether this element occurs in the remainder of the array.

```
UniqueElements (A)
        n ← length(A)
        for i ← 0 to n-2 do
            for j ← i+1 to n-1 do
                If  A[i]=A[j] return false
        Return true
```

   Pre-condition: show the loop invariant holds before the first iteration, when i=0. Here, the subarray consists of just A[1], which is (trivially) unique since it has not been compared to anything else.

   Exit step: we exit when we have compared the next-to-last and last elements. Hence the entire array is now checked for uniqueness.

   Post-condition: when we exit, the entire array is now checked for uniqueness.

   Induction step: the body of the outer for loop works by comparing A[i] to A[i+1], A[i+2], A[i+3] and so on by one position to the right until the uniqueness of A[i] is established. This is true for all i from 0 to n-2.

   b. **[5]** Use this approach to specify the complexity of the algorithm.

   In the worst case, the number of array-element comparisons is

$$T_{worst}(n) = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=1}^{n-1} i = (n-1)n/2 \in O(n^2)$$

2. **[15]** Solve the following recurrence relation using repeated substitution. Do an inductive proof to show your formula is correct.

$$T(1) = 1$$
$$T(n) = T(n-1) + O(n)$$

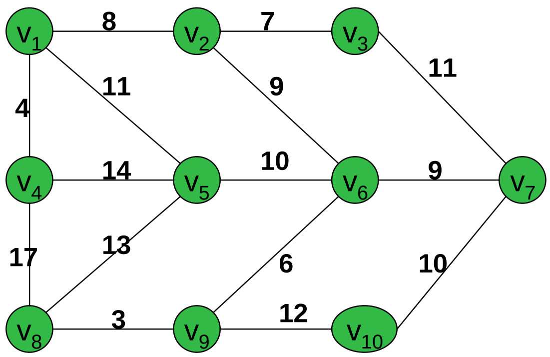
By repeated substitution, we can solve T(n) as:

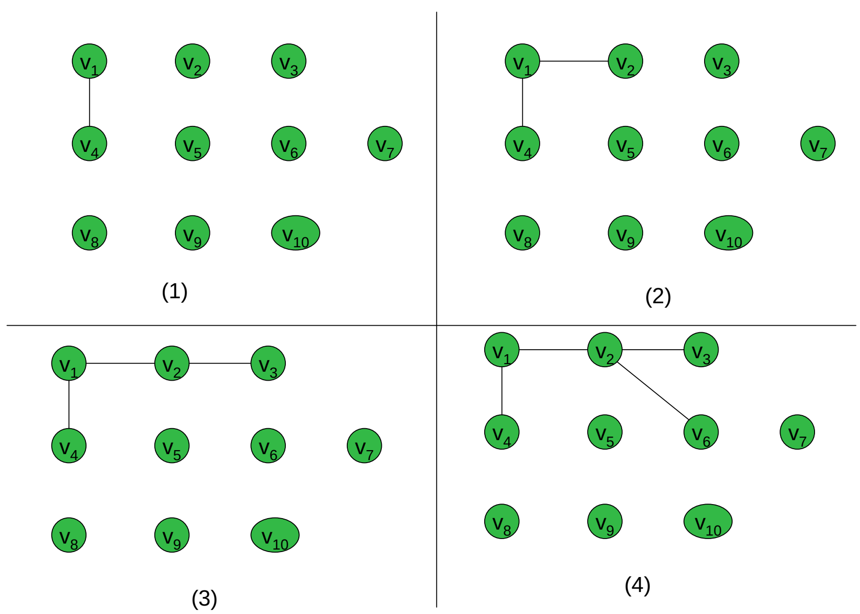
$$\begin{aligned} T(n) &= T(n-1) + O(n) \\ &= (T(n-2) + O(n-1)) + O(n) \\ &= T(n-2) + O(n-1) + O(n) \\ &= T(n-3) + O(n-2) + O(n-1) + O(n) \end{aligned}$$

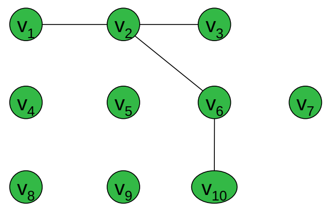
...
$= T(1) + O(2) + ... + O(n-1) + O(n)$
$= O(1 + 2 + ... + n-1 + n)$
$= O(n^2)$

3. **[20]** Given the graph G shown below,

   a. **[15]** Find a minimum spanning tree (MST) for G; show the steps of generating the MST.
   b. **[5]** What is the complexity of this algorithm?







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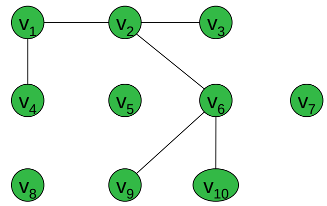

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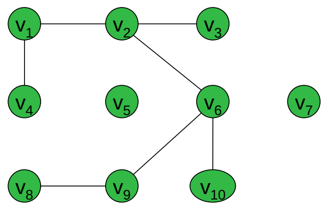

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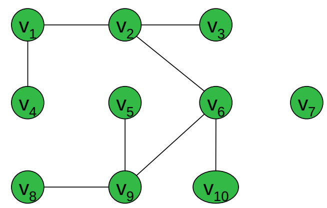

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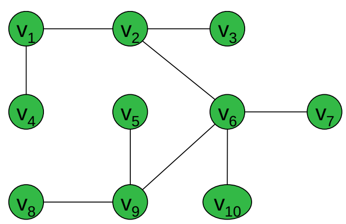

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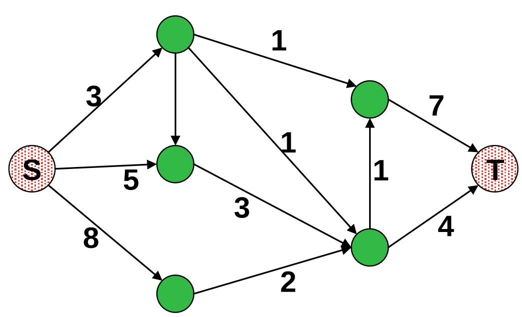
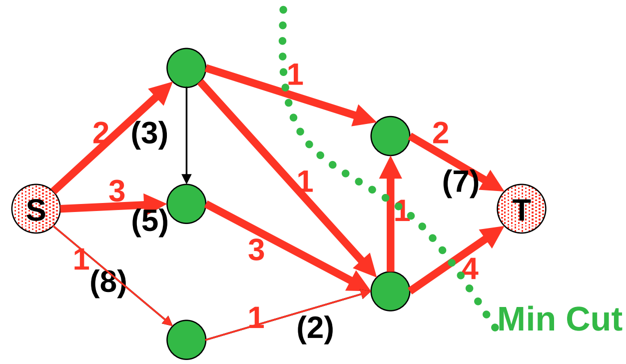
4. **[15]** Consider a graph G(V,E), with source node S and sink node T.

   a. **[8]** For the instance of a flow network shown below, compute the maximum flow. Give the actual flow as well as its value. Justify your answer.





c. **[4]**We show the solution here, with the flows on each edge defined. Our total flow is 6. We know that 6 is the Max flow, as we can identify a min-cut of 6, and by the max-flow/min-cut theorem, we know that max-flow=min-cut.

d. **[3]**Consider a decision problem defined for such a flow network: Flow:= {(G,S,T,k)|G(V,E) is a flow network, S,T∈V, and the value of a optimal flow from S to T in G is k}.

Is *Flow* in NP? Is *Flow* in P? Justify your answer.

Flow is in NP, since we can explicitly compute an optimal max-flow using a polynomial time algorithm, like Edmonds-Karp. Since P⊆NP, Flow must be in NP. Flow is also in P.

5. **[20]** Prove that SET PACKING (SP) is NP-complete.
INSTANCE: A collection *C* of finite sets over a universal set *U*, and integer *k*.
QUESTION: Does *C* contain *k* disjoint sets?

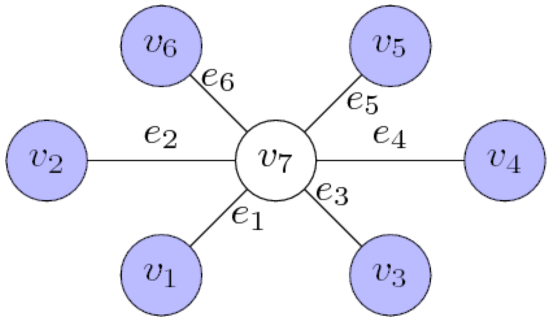
(Assume that you need to define a reduction from one of the following NP-complete problems: HAMILTON CIRCUIT, CLIQUE, INDEPENDENT SET, 3-SAT)

(a) Prove that SET PACKING (SP) is in NP. A certificate consists of a subset $S=\{S_1,\ldots,S_k\}$ of C. We must check 2 things. We must check if $S_i \cap S_j = \varnothing$, $i \neq j$, and $S_i, S_j \in S$.We can clearly do this check in $O(n^2)$ time.

(b) Define a reduction from the INDEPENDENT SET (IS) problem, which is NP-complete.

INSTANCE: Undirected finite graph *G(V,E)* and integer *k*.
QUESTION: Does G contain a set of *k* independent vertices?

Given an instance *G(V,E)* of IS, we generate an instance of SP as follows. First, we generate an element U_i of U corresponding to every edge E_i; and second, we create the set S_i consisting of the edges incident to vertex V_i in IS. Finally, we set the size of Set Packing to be k as well. Clearly, this reduction can be performed in time polynomial in the size of V and E.



Example: Graph of Independent Set

Example: From the construction, a Set Packing of the example above would be $S_1 = \{e_1\}, S_2 = \{e_2\}, S_3 = \{e_3\}, S_4 = \{e_4\}, S_5 = \{e_5\}, S_6 = \{e_6\}, S_7 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. So, the $S = \{1,2,3,4,5,6\}$.

We need to show that Independent Set $\leq_p$ Set Packing (This construction of Independent Set leads to is a special case of Set Packing).
IS is a Independent Set of size k in G if and only if S is a Set Packing of at most k S_i's, such that $S_i \cap S_j = \emptyset$, $i \neq j$, and $i, j \in S$.

$\Rightarrow$IS has an Independent Set of size k in G if S is a Set Packing of at most k S_i's, such that $S_i \cap S_j = \emptyset$.

Suppose G has an independent set of size at least k, call it IS. Construct S by including exactly the S_i with $v_i \in$IS. Obviously, the size of S is equal to the size of IS, k. Furthermore, since no two vertices in S can share an edge, the sets S_i we picked must be pairwise disjoint, so we have found a valid set packing of at least k sets.

$\Leftarrow$ S is a Set Packing of at most k S_i's, such that $S_i \cap S_j = \emptyset$ if IS has an Independent Set of size k in G

Assume that we are given the new instance of set packing S of size at least k. Then, we define a vertex set IS as consisting exactly of those v_i for which $i \in S$. The size of IS is the same as that of S, k. For any edge e, at most one set S_i with $i \in S$ may contains e, so at most one node v_i can be incident on e. Thus, no two selected nodes are connected by an edge, and IS is in fact an independent set of size at least k.

6. **[20]** Consider a class of graphs G(V,E) which contain an independent set of size ¾|V|. An independent set is a subset V' of vertices such that no two vertices in V' are connected by an edge of G.

   i. **[10]** Provide an approximation algorithm for G that can provably compute an independent set of size at least ½|V|.
   ii. **[10]** Prove that your algorithm can meet such bounds.

(Hint: you may make use of the 2-approximation algorithm for vertex cover that was described in class, i.e., you may assume that this algorithm exists and can be called as a subroutine. A vertex cover of a graph G is a subset of vertices V' such that all edges in G are adjacent to at least one node of V'.)

Algorithm:
//Input: G(V,E)//
Compute complement G'of G
VC←*2-Approx*(G') //return approximate vertex cover of G'//
IndepSet←VC' //compute complement of approx. vertex cover VC//
Return IndepSet

Proof: G(V,E) contains an independent set of size ¾|V|. The complement G'of G thus has a vertex cover of size ¼|V|, by definition of independent set and vertex cover.
We can use the 2-approximation algorithm for vertex cover to find a vertex cover of size at most ½|V| in G'. The complement of this vertex cover is an independent set of size at least ½|V|.