1 Generalised Transition Rate Equation

$$T^{j\to i} = \left(u_i r_{i-1}\frac{n_{i-1}}{N} + (1-u_{i+1})\, r_i \frac{n_i}{N}\right)\frac{n_j}{\bar{r}} \tag{1}$$

When setting up the system, $u_0 = 0$ and $u_{N+1} = 0$ must be defined in order to stop cells mutating past the maximum number of mutations.

$\frac{n_j}{\bar{r}}$ is selected to die. $u_i r_{i-1}\frac{n_{i-1}}{N}$ is selected to mutate and gain a mutation. I.e. a cell with one less mutation is birthed with a mutation. $(1-u_{i+1})\, r_i \frac{n_i}{N}$ A n_i cell is born and doesn't obtain a mutation.

1.1 Diffusion with Drift Derivation

Taking the large N limit.

$$\dot{x}_i = \frac{1}{N}\left[\sum_j T^{j\to i} - T^{i\to j}\right]$$

Using the generalised transition rate equation:

$$\begin{aligned}
\bar{r}\dot{x}_i &= \sum_j u_i r_{i-1} x_{i-1} x_j - u_j r_{j-1} x_{j-1} x_i + x_j x_i \left[(1-u_{i+1}) r_i - (1-u_{j+1}) r_j\right] \\
\alpha &= u_i r_{i-1} \\
\beta &= u_j r_{j-1} \\
\gamma &= (1-u_{i+1}) r_i - (1-u_{j+1}) r_j) \\
\bar{r}\dot{x}_i &= \sum_j \alpha x_j x_{i-1} - \beta x_{j-1} x_i + \gamma x_j x_i
\end{aligned}$$

In a flat fitness and mutation landscape, away from the absorbing state $u_i, r_i = 1 \forall i$. From this $\alpha = 1, \beta = 1, \gamma = 0, \bar{r} = 1$.

$$\dot{\rho}_i = \sum_j \rho_j \rho_{i-1} - \rho_{j-1}\rho_i$$

We now want to expand this sum in order to simplify;

$$\begin{aligned}
\dot{\rho}_i &= \rho_0\rho_{i-1} + (\rho_1\rho_{i-1} - \rho_0\rho_i) + (\rho_2\rho_{i-1} - \rho_1\rho_i) + ... + (\rho_{N-1}\rho_{i-1} - \rho_{N-2}\rho_i) + (\rho_N\rho_{i-1} - \rho_{N-1}\rho_i) \\
&= \rho_0(\rho_{i-1} - \rho_i) + \rho_1(\rho_{i-1} - \rho_i) + ... + \rho_N\rho_{i-1} \\
&= \sum_k^{N-1} (\rho_{i-1} - \rho_i)\rho_k + \rho_N\rho_{i-1} \\
&= (\rho_{i-1} - \rho_i)(1 - \rho_N) + \rho_N\rho_{i-1} \\
&= \rho_{i-1} + (\rho_N - 1)\rho_i \\
\dot{\rho}_i &= \rho_{i-1} + (\rho_N - 1)\rho_i
\end{aligned}$$

In the large mutation limit $\rho_N \to 0$.

$$\dot{\rho}_i = \rho_{i-1} - \rho_i \tag{2}$$

We might be breaking our previous assumptions, but for $i = 0$:

$$\begin{aligned}
\dot{\rho}_0 &= 0 - \rho_0 \\
\rho_0 &= Ae^{-t} \\
A &= 1 \\
\rho_0 &= e^{-t}
\end{aligned}$$

This gives us the first step in solving generally.

$$\begin{aligned}
\dot{\rho}_1 &= \rho_0 - \rho_1 \\
&= e^{-t} - \rho_1 \\
\rho_1 &= te^{-t} \\
\dot{\rho}_2 &= \rho_1 - \rho_2 \\
\rho_2 &= \frac{1}{2}e^{-t}t^2
\end{aligned}$$

From this we can see that the general solution for ρ_i;

$$\rho_i = \frac{t^i}{i!}e^{-t} \tag{3}$$

2 Tobias' Model

Go from state n to $n+1$ with rate $a+b$. Go from state n to $n-1$ with rate a.

$$\begin{aligned}
\dot{p_i} &= -ap_i - (a+b)p_i + (a+b)p_{i-1} + ap_{i+1} \\
&= -(2a+b)p_i + (a+b)p_{i-1} + ap_{i+1}
\end{aligned}$$

Lets start with a generating function:

$$
\begin{aligned}
\Phi &= \sum_{n=0}^{\infty} z^n p_n(t) \\
\dot{\Phi} &= \sum_{n=0}^{\infty} z^n \dot{p}_n \\
&= \sum_{n=0}^{\infty} z^n \left((a+b)p_{n-1} - (2a+b)p_n + ap_{n+1}\right) \\
&= -(2a+b)\Phi + a\sum_{n=0}^{\infty} z^n p_{n+1} + (a+b)\sum_{n=0}^{\infty} z^n p_{n-1} \\
&= -(2a+b)\Phi + \frac{a}{z}\Phi + (a+b)z\Phi
\end{aligned}
$$

Need to get a differential out of this equation though?

3 Our Cancer Model Version 2

$$
\begin{aligned}
\dot{x}_i &= \left(u_i r_{i-1}\frac{n_{i-1}}{N} + (1-u_{i+1})\, r_i \frac{n_i}{N}\right)\frac{n_j}{\bar{r}} - \left(u_j r_{j-1}\frac{n_{j-1}}{N} + (1-u_{j+1})\, r_j \frac{n_j}{N}\right)\frac{n_i}{\bar{r}} \\
\bar{r}\dot{x}_i &= \left(u_i r_{i-1} x_{i-1} + (1-u_{i+1})\, r_i x_i\right) x_j - \left(u_j r_{j-1} x_{j-1} + (1-u_{j+1})\, r_j x_j\right) x_i
\end{aligned}
$$

$$
\bar{r}\dot{x}_0 = \left((1-u_1)\, r_0 x_0\right) x_j - \left(u_j r_{j-1} x_{j-1} + (1-u_{j+1})\, r_j x_j\right) x_0
$$