1 Non-Flat Mutation Rates INITIAL

$$\begin{aligned}
\bar{r}\dot{x}_i &= u_i r_{i-1} x_{i-1} + x_i r_i (1 - u_{i+1}) - \bar{r}) \\
r &= \text{const.} \\
\dot{\rho}_i &= u_i \rho_{i-1} - \rho_i u_{i+1} \\
\dot{\rho}(x,t) &= u(x)\rho(x - \Delta x, t) - \rho(x,t) u(x + \Delta x, t) \\
&= u(x)\left(\rho(x) - \Delta x \frac{\partial \rho}{\partial x} + \Delta x^2 \frac{\partial^2 \rho}{\partial x^2}\right) - \rho(x,t)\left(u(x) + \Delta x \frac{\partial u}{\partial x} + \Delta x^2 \frac{\partial^2 u}{\partial x^2}\right) \\
\Delta x &= 1 \\
\dot{\rho}(x,t) &= -u\frac{\partial \rho}{\partial x} + u\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right)
\end{aligned}$$

Collecting up some terms.

$$\dot{\rho} = -\frac{\partial}{\partial x}(\rho u) + u\frac{\partial^2 \rho}{\partial x^2} - \rho\frac{\partial^2 u}{\partial x^2}$$

Fourier transform convention

$$\tilde{\rho} = \int_{-\infty}^{\infty} \rho e^{-ikx} dx \tag{1}$$

$$\dot{\tilde{\rho}} = -\int \frac{\partial}{\partial x}(\rho u) e^{-ikx} dx + \int u \frac{\partial^2 \rho}{\partial x^2} e^{-ikx} dx - \int \rho \frac{\partial^2 u}{\partial x^2} e^{-ikx} dx$$

$$\begin{aligned}
\int \frac{\partial}{\partial x}(\rho u) e^{-ikx} dx &= \left[e^{-ikx}\rho u\right]_{-\infty}^{\infty} - \int (-ik)\rho u e^{-ikx} dx \\
&= ik \int \rho u e^{-ikx} dx
\end{aligned}$$

Applying the convolution theorem:

$$\begin{aligned}
F(f \cdot g) &= F(f) * F(g) \\
f * g &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x - y) dy
\end{aligned}$$

Need to prove these.
Applying this to the new differential equation.

$$\dot{\tilde{\rho}} = -ik(\mathcal{F}(\rho) * \mathcal{F}(u)) + (\mathcal{F}(u) * \mathcal{F}(\frac{\partial^2 \rho}{\partial x^2})) - (\mathcal{F}(\rho) * \mathcal{F}(\frac{\partial^2 u}{\partial x^2}))$$

Using what we know for the differentials and Fourier bits.

$$\dot{\tilde{\rho}} = -ik\left(\tilde{\rho} * \tilde{u}\right) + \left(\tilde{u} * (-k^2)\tilde{\rho}\right) - \left(\tilde{\rho} * (-k^2)\tilde{u}\right)$$

We can take out the factors of k^2.

$$\begin{aligned}\dot{\tilde{\rho}} &= -ik(\tilde{\rho} * \tilde{u}) - k^2\left((\tilde{u} * \tilde{\rho}) + (\tilde{\rho} * \tilde{u})\right) \\ &= (-ik - 2k^2)(\tilde{\rho} * \tilde{u})\end{aligned}$$

Convolution is commutative.
Chose a simple $u = \cos(\frac{\pi}{2M}x)$.

$$\begin{aligned}\tilde{u} &= \int \cos(\frac{\pi}{2M}x)e^{-ikx}dx \\ &= \frac{1}{2}\int (e^{\frac{i\pi}{2M}x} + e^{-\frac{i\pi}{2M}x})e^{-ikx})dx \\ &= \frac{1}{2}\left(\delta(k - \frac{\pi}{2M}) + \delta(k + \frac{\pi}{2M}\right)\end{aligned}$$

Convolution of a delta function just returns the same function with the varaible shifted.

$$f(x) * \delta(x \pm a) = f(x \pm a) \tag{2}$$

$$\begin{aligned}\tilde{\rho} * \tilde{u} &= \tilde{\rho} * \frac{1}{2}\left(\delta(k - \frac{\pi}{2M}) + \delta(k + \frac{\pi}{2M}\right) \\ &= \frac{1}{2}\left(\tilde{\rho}(k - \frac{\pi}{2M}) + \tilde{\rho}(k + \frac{\pi}{2M})\right)\end{aligned}$$

Combining everything in one.

$$\dot{\tilde{\rho}} = -\frac{(ik + k^2)}{2}\left(\tilde{\rho}(k - \frac{\pi}{2M}) + \tilde{\rho}(k + \frac{\pi}{2M})\right)$$

Improved

$$
\begin{aligned}
\dot{\rho} &= u\rho(x-\Delta x) - \rho u(x+\Delta x) \\
&= u\left(\rho(x) - \Delta x\frac{\partial \rho}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 \rho}{\partial x^2}\right) - \rho\left(u(x) + \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}\right) \\
&= -u\frac{\partial \rho}{\partial x}\Delta x + u\frac{\Delta x^2}{2}\frac{\partial^2 \rho}{\partial x^2} - \Delta x \rho\frac{\partial u}{\partial x} - \frac{\Delta x^2}{2}\rho\frac{\partial^2 u}{\partial x^2}
\end{aligned}
$$

Introduce a new scaled variable $\hat{u} = u(x) \cdot \Delta x$.

$$
\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\frac{\partial \hat{u}}{\partial x} - \frac{\Delta x}{2}\rho\frac{\partial^2 \hat{u}}{\partial x^2}
$$

This collapses down into our old equation with an additional term.

$$
\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)
$$

In the limit $\Delta x \to 0$ we get the normal advection equation

$$
\lim_{\Delta x \to 0} \dot{\rho} = -\frac{\partial(\rho\hat{u})}{\partial x} \tag{3}
$$

2 Numerically Solving the PDE

Start with the PDE and initial condition.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \tag{4}$$

$$\rho(x,0) = \delta(x - x_0) \tag{5}$$

We discretise with $x_i = ih$ and $t_j = jk$.

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{\rho(x_i, t_{j+1}) - \rho(x_i, t_j)}{k} \\
\frac{\partial \rho}{\partial x} &= \frac{\rho(x_{i+1}, t) - \rho(x_i, t_j)}{h} \\
\frac{\partial^2 \rho}{\partial x^2} &= \frac{\rho(x_{i+1}, t_j) - 2\rho(x_i, t_j) + \rho(x_{i-1}, t_j)}{h^2}
\end{aligned}$$

Inserting these into the PDE

$$\begin{aligned}
\frac{\rho(x_i, t_{j+1}) - \rho(x_i, t_j)}{k} &= -\hat{u}\left(\frac{\rho(x_{i+1}, t_j) - \rho(x_i, t_j)}{h}\right) + \frac{\hat{u}\Delta x}{2}\left(\frac{\rho(x_{i+1}, t_j) - 2\rho(x_i, t_j) + \rho(x_{i-1}, t_j)}{h^2}\right) \\
&\quad - \rho(x_i, t_j) f(\hat{u}', \hat{u}'', \Delta x) \\
f(\hat{u}', \hat{u}'', \Delta x) &= \left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)
\end{aligned}$$

$$\begin{aligned}
\rho(x_i, t_{j+1}) &= \rho - \frac{\hat{u}k}{h}\left(\rho(x_{i+1}, t_j) - \rho\right) + \frac{\hat{u}\Delta x k}{2h^2}\left(\rho(x_{i+1}, t_j) - 2\rho + \rho(x_{i-1}, t_j)\right) - k\rho f \\
&= \rho\left(1 + \frac{\hat{u}k}{h} - \frac{\hat{u}\Delta x k}{h^2} - kf\right) + \frac{\hat{u}\Delta x k}{2h^2}\rho(x_{i-1}, t_j) + \rho(x_{i+1}, t_j)\left(\frac{\hat{u}\Delta x k}{2h^2} - \frac{\hat{u}k}{h}\right) \\
&= \rho\left(1 + \frac{\hat{u}k}{h}\left(1 - \frac{\Delta x}{h}\right) - k\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)\right) + \frac{\hat{u}\Delta x k}{2h^2}\rho(x_{i-1}, t_j) + \rho(x_{i+1}, t_j)\left(\frac{\Delta x}{2h} - 1\right)\frac{\hat{u}k}{h}
\end{aligned}$$

Can then solve this the normal computational way. $\alpha = \frac{\hat{u}k}{h}$

$$\begin{aligned}
\rho(x_i, t_{j+1}) &= \rho\left(1 + \alpha\left(1 - \frac{\Delta x}{h}\right) - k\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right)\right) \\
&\quad + \frac{\Delta x}{2h}\alpha\rho(x_{i-1}, t_j) \\
&\quad + \alpha\left(\frac{\Delta x}{2h} - 1\right)\rho(x_{i+1}, t_j)
\end{aligned}$$

3 PDE Analysis

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \tag{6}$$

Re-arrange this to the stardard form.

$$\frac{\Delta x \hat{u}}{2}\frac{\partial^2 \rho}{\partial x^2} - \hat{u}\frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial t} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) = 0$$

Classification of this PDE is based on the coeffeicents of the double derivatives.

$$\Delta(x, y) = 0^2 - 0^2 = 0$$

As there are know terms with the cross derivative and no terms with the second differential of t, the determinate is zero and the equation is parabolic.

4 Mutation Function analysis

We are free to chose a function for the mutation rate.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \tag{7}$$

The degree of freedom comes in the last term, so we define a function for ease of notation.

$$\mathcal{U}(x, \hat{u}, \hat{u}^{(n)}) = \frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2} \tag{8}$$

4.1 $\mathcal{U} = \hat{u}$

By setting the function equal to the mutation function then it can be taken as a common factor from all the terms in the PDE.

$$\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2} = \hat{u}$$

$\hat{u}$ is just a function of x, therefore this is a linear, second order differential equation with auxillary equation;

$$\lambda = \frac{-1 \pm \sqrt{1 + 2\Delta x}}{\Delta x},$$

therefore, is has solutions

$$\hat{u} = A\exp\left(\left(-\frac{1}{\Delta x} + \frac{\sqrt{1+2\Delta x}}{\Delta x}\right)x\right) + B\exp\left(-\left(\frac{1}{\Delta x} + \frac{\sqrt{1+2\Delta x}}{\Delta x}\right)x\right)$$

A and B are constants that determine the scale of mutation probabilities.

4.2 $\mathcal{U} =$ polynomial

$$\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2} = \sum_{-\infty}^{\infty} c_n x^n$$

$$v = \frac{\partial \hat{u}}{\partial x}$$
$$v + \frac{\Delta x}{2}\frac{\partial v}{\partial x} = \sum_{-\infty}^{\infty} c_n x^n$$

This has complimentary function (correct phrase?)

$$v = Be^{-\frac{2}{\Delta x}x}$$

Trial form for the particular integral;

$$\begin{aligned} v(x) &= Be^{-\frac{2}{\Delta x}x} + \sum c_\alpha x^\alpha \\ v' &= \frac{-2B}{\Delta x} e^{-\frac{2}{\Delta x}x} + \sum \alpha c_\alpha x^{\alpha-1} \end{aligned}$$

$$\begin{aligned} Be^{-\frac{2}{\Delta x}x} + \sum c_\alpha x^\alpha + \frac{\Delta x}{2}\left(\frac{-2B}{\Delta x} e^{-\frac{2}{\Delta x}x} + \sum \alpha c_\alpha x^{\alpha-1}\right) &= \sum_{-\infty}^{\infty} c_n x^n \\ \sum c_\alpha x^\alpha + \frac{\Delta x}{2} \sum \alpha c_\alpha x^{\alpha-1} &= \sum_{-\infty}^{\infty} c_n x^n \\ \sum c_\alpha x^\alpha + \frac{\Delta x}{2} \sum (\alpha+1) c_{\alpha+1} x^{\alpha} &= \sum_{-\infty}^{\infty} c_n x^n \end{aligned}$$

From orthogonality of the powers

$$c_\alpha + \frac{\Delta x}{2}(\alpha+1)c_{\alpha+1} = c_n$$

OR

$$\begin{aligned} c_\alpha + \frac{\Delta x}{2}\frac{\alpha c_\alpha}{x} &= c_n \\ c_\alpha &= \frac{c_n}{1 + \frac{\Delta x \alpha}{2x}} \end{aligned}$$

So the full solution is

$$v(x) = Be^{\frac{-2x}{\Delta x}} + \sum \frac{c_n x^n}{1 + \frac{\Delta x n}{2x}}$$

Unwinding the substitution

$$\begin{aligned} \frac{\partial \hat{u}}{\partial x} &= Be^{\frac{-2x}{\Delta x}} + \sum \frac{c_n x^n}{1 + \frac{\Delta x n}{2x}} \\ \hat{u} &= -\frac{B\Delta x}{2} e^{\frac{-2x}{\Delta x}} + \int \sum \frac{2c_n x^{n+1}}{2x + \Delta x n} \mathrm{d}x \end{aligned}$$

Testing by differentiation and subbing back in.

$$
\begin{aligned}
\hat{u} &= -\frac{B\Delta x}{2}e^{\frac{-2x}{\Delta x}} + \int \sum \frac{2c_n x^{n+1}}{2x+\Delta x n}\mathrm{d}x \\
\hat{u}' &= Be^{-\frac{2x}{\Delta x}} + \sum \frac{2c_n x^{n+1}}{2x+\Delta x n} \\
\hat{u}'' &= \frac{-2B}{\Delta x}e^{-\frac{2x}{\Delta x}} + \sum \left(\frac{2c_n(n+1)x^n}{(2x+\Delta x n)} - \frac{4c_n x^{n+1}}{(2x+\Delta x n)^2} \right) \\
\hat{u}' + \frac{\Delta x}{2}\hat{u}'' &= \sum \frac{2c_n x^{n+1}}{2x+\Delta x n} + \frac{\Delta x}{2}\left(\sum \left(\frac{2c_n(n+1)x^n}{(2x+\Delta x n)} - \frac{4c_n x^{n+1}}{(2x+\Delta x n)^2} \right) \right)
\end{aligned}
$$

In the limit of $\Delta x \to 0$ this is correct.

Example

$$
\begin{aligned}
\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2} &= \sum_{-\infty}^{\infty} c_n x^n \\
\sum_{-\infty}^{\infty} c_n x^n &= x \\
c_n &= \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases} \\
\hat{u} &= -\frac{B\Delta x}{2}e^{\frac{-2x}{\Delta x}} + \int \sum \frac{2c_n x^{n+1}}{2x+\Delta x n}\mathrm{d}x \\
&= -\frac{B\Delta x}{2}e^{\frac{-2x}{\Delta x}} + \int \left(\frac{2x^2}{2x+\Delta x} \right)
\end{aligned}
$$

4.3 FKKP Methods

"Wave of Advance of Advantageous Genes" Ronald Fisher.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x \hat{u}}{2}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial u}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}\right)$$

We want a wave solution, travelling at speed v, therefore $\frac{\partial \rho}{\partial t} = -v\frac{\partial \rho}{\partial x}$

$$\frac{\Delta x \hat{u}}{2}\rho^{''} + (v-\hat{u})\rho^{'} - \rho\left(u^{'} + \frac{\Delta x}{2}u^{''}\right) = 0 \tag{9}$$

There is no explicit x dependence in the equation, therefore we can write the concentration as the derivative of another function.

$$g = -\frac{d\rho}{dx}$$
$$\frac{\partial g}{\partial x} = -\frac{\partial^2 \rho}{\partial x^2}$$
$$\frac{\partial^2 \rho}{\partial x^2} = g\frac{\partial g}{\partial \rho}$$

$$\frac{\Delta x \hat{u}}{2}g\frac{\partial g}{\partial \rho} + (v-\hat{u})(-g) - \rho\left(\frac{\partial u}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{10}$$

$$\lim_{\rho \to 0}\frac{g}{\rho} = \omega$$

$$\frac{\partial g}{\partial \rho} = \omega$$

$$\frac{\hat{u}\Delta x}{2}\omega^2 + (v-\hat{u})(-\omega) - \left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) = 0 \tag{11}$$

By examining the determinant of this quadratic equation we can arrive at an equation for v.

$$(v-\hat{u})^2 - 4\frac{\hat{u}\Delta x}{2}\left(-\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right)\right) \geq 0$$
$$v^2 - 2v\hat{u} + \hat{u}^2 + 2\hat{u}\Delta x\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) \geq 0$$
$$v^2 - 2v\hat{u} + \hat{u}^2 + 2\hat{u}\Delta x\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) = C$$
$$C \geq 0$$

$$v^2 - 2v\hat{u} + \hat{u}^2 + 2\hat{u}\Delta x\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) - C = 0$$

$$v = 2\hat{u} \pm \sqrt{4\hat{u}^2 - 4\left(\hat{u}^2 + 2\hat{u}\Delta x\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) - C\right)}$$
$$= 2\hat{u} \pm \sqrt{-8\hat{u}\Delta x\left(\hat{u}' + \frac{\Delta x\hat{u}''}{2}\right) + 4C}$$

As the velocity is a physical quantity, this gives us some constraints on the function inside the square brackets.

$$C - 2\hat{u}\Delta x\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) \geq 0 \tag{12}$$

This gives us an ODE for $\hat{u}$:

$$\hat{u}\Delta x\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) + D = 0$$

Where D is just the redefining of the constant.
This is very non linear and doesn't have an analytic solution.
NEED TO SOLVE IT NUMERICALLY.

5 Varying u(x) and r(x)

Starting from

$$\bar{r}\dot{x}_i = u_i r_{i-1}x_{i-1} + x_i(r_i(1 - u_{i+1}) - \bar{r})) \tag{13}$$
$$= u_i r_{i-1}x_{i-1} + x_i r_i - x_i r_i u_{i+1} - x_i\bar{r} \tag{14}$$
$$\tag{15}$$

taking this in the continuous limit we get

$$\bar{r}\rho(\dot{x}, t) = u(x)r(x - \Delta x)\rho(x - \Delta x, t) + \rho(x, t)r(x) - \rho(x, t)r(x)u(x + \Delta x) \tag{16}$$

which leads to (after taylor expansion)

$$\begin{aligned}
\bar{r}\dot{\rho}(x, t) &= u(x)[r(x) - \Delta x\frac{\partial r}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 r}{\partial^2 x}][\rho(x, t) - \Delta x\frac{\partial \rho}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 \rho}{\partial^2 x}] + \rho(x, t)r(x) \\
&\quad - \rho(x, t)r(x)[u(x) + \partial_x u(x) + \frac{(\Delta x)^2}{2}\partial_x^2 u(x)] - \rho(x, t)\bar{r} \\
&= u(x)[r(x)\rho(x, t) - \rho(x, t)\Delta x\partial_x r + \frac{(\Delta x)^2}{2}\rho\partial_x^2 r - r(x)\Delta\partial_x\rho + (\Delta x)^2\partial_x r\partial_x\rho + r(x)\frac{(\Delta x)^2}{2}\partial_x^2\rho] \\
&\quad + \rho(x, t)r(x) - \rho(x, t)u(x)r(x) - \Delta x\rho(x, t)r(x)\partial_x u(x) - \frac{(\Delta x)^2}{2}\rho(x, t)r(x)\partial_x^2 u(x) - \rho(x, t)\bar{r} \\
\bar{r}\dot{\rho} &= -u\rho\Delta x\partial_x r - ur\Delta x\partial)x\rho - \Delta x\rho r\partial_x u + \frac{(\Delta x)^2}{2}(u\rho\partial_x^2 r + ur\partial_x^2\rho - \rho r\partial_x^2 u + 2u\partial_x r\partial_x\rho) + \rho(r - \bar{r}) \\
&= -\Delta x\partial_x(ur\rho) + \rho(r - \bar{r}) + \frac{(\Delta x)^2}{2}\partial_x^2(ur\rho) - 2r\rho\partial_x^2 u - 2\rho\partial_x u\partial_x r - 2r\partial_x u\partial_x\rho
\end{aligned}$$

Again, using the previous notation of $\hat{u} = u(x)\Delta x$.

$$\bar{r}\dot{\rho} = \rho\left(r - \bar{r} - \frac{\partial}{\partial x}(\hat{u}r) + \frac{\Delta x}{2}(\hat{u}\frac{\partial^2 r}{\partial x^2} - r\frac{\partial^2 \hat{u}}{\partial x^2})\right) + \frac{\partial \rho}{\partial x}\left(-\hat{u}r + \frac{\Delta x}{2}\hat{u}\frac{\partial r}{\partial x}\right) + \frac{\Delta x}{2}\hat{u}r\frac{\partial^2 \rho}{\partial^2 x}$$

This collapse into the flat case for constant u and r.
Taking the small Δx limit without forming the $\hat{u}$.

$$\begin{aligned}
\bar{r}\dot{\rho} &= \rho\left(r - \bar{r} - \Delta x\frac{\partial}{\partial x}(ur) + \frac{\Delta x^2}{2}(u\frac{\partial^2 r}{\partial x^2} - r\frac{\partial^2 u}{\partial x^2})\right) + \frac{\partial \rho}{\partial x}\left(-\Delta x ur + \frac{\Delta x^2}{2}u\frac{\partial r}{\partial x}\right) + \frac{\Delta x^2}{2}ur\frac{\partial^2 \rho}{\partial^2 x} \\
\bar{r}\dot{\rho} &= \rho\left(r - \bar{r}\right) \\
\bar{r} &= \int \rho(x, t = t_i) r(x) dx
\end{aligned}$$

Want to remove this integral, is this achieved by differentiating?

$$\begin{aligned}
\bar{r}\left(\dot{\rho} + \rho\right) &= \rho r \\
\bar{r} &= \frac{\rho r}{(\dot{\rho} + \rho)} \\
\int r(x)\rho(x, t)\mathrm{d}x &= \frac{\rho r}{(\dot{\rho} + \rho)} \\
r\rho &= \frac{\partial}{\partial x}\left(\frac{\rho r}{\dot{\rho} + \rho}\right) \\
&= \frac{1}{\dot{\rho} + \rho}\left(\frac{\partial \rho}{\partial x}r + \frac{\partial r}{\partial x}\rho\right) - \rho r\left(\frac{1}{(\dot{\rho} + \rho)^2}\right)\left(\frac{\partial^2 \rho}{\partial x \partial t} + \frac{\partial \rho}{\partial x}\right)
\end{aligned}$$

Our equation stability analysis

From our equation with varying u and r we will do some stability analysis of the fixed points. The coeffecients are just place holders at the minute.

$$\begin{aligned}
\dot{\rho} &= a\rho + b\frac{\partial \rho}{\partial x} + c\frac{\partial^2 \rho}{\partial x^2} \\
\rho(z) &\equiv U(x - vt) \\
\dot{\rho} &= -v\frac{dU}{dz} \\
\frac{\partial \rho}{\partial x} &= \frac{dU}{dz} \\
\frac{\partial^2 \rho}{\partial x^2} &= \frac{d^2U}{dz^2}
\end{aligned}$$

$$\begin{aligned}
-vU' &= aU + bU' + cU'' \\
cU'' &= -(b+v)U' - aU
\end{aligned}$$

Now introduce a new variable to change this from a second order to two coupled first order equations.

$$\begin{aligned}
V &\equiv U' \\
V' &= -\frac{(b+v)}{c}V - \frac{a}{c}U
\end{aligned}$$

This is a matrix equation

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{a}{c} & -\frac{(b+v)}{c} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} \tag{17}$$

This shows that there is only one fixed point. $(0, 0)$. To calculate it's stability we need to evaluate the eigenvalues of this matrix.

$$\begin{aligned}
-\lambda\left(-\frac{(b+v)}{c} - \lambda\right) + \frac{a}{c} &= 0 \\
&= \lambda^2 + \frac{(b+v)}{c} + \frac{a}{c}
\end{aligned}$$

$$\begin{aligned}
\lambda &= -\frac{(b+v)}{2c} \pm \frac{1}{2}\sqrt{\frac{(b+v)^2}{c^2} - \frac{4a}{c}} \\
&= -\frac{(b+v)}{2c}\left(1 \mp \sqrt{1 - \frac{4ac}{(b+v)^2}}\right)
\end{aligned}$$

From the Fisher wave analysis we know that the $(0,0)$ fixed point needs to be a stable node.

$$\begin{aligned}1 - \frac{4ac}{(b+v)^2} &\geq 0 \\ (b+v)^2 &\geq 4ac \\ v &\geq \pm 2\sqrt{ac} - b\end{aligned}$$

Can ignore the minus solution if the positive is satisfied so is the minus.

$$v \geq 2\sqrt{ac} - b \tag{18}$$

Also, for the node to be stable:

$$1 - \sqrt{1 - \frac{4ac}{(b+v)^2}} > 0 \tag{19}$$

$$\begin{aligned}1 - \sqrt{1 - \frac{4ac}{(b+v)^2}} &> 0 \\ \sqrt{1 - \frac{4ac}{(b+v)^2}} &< 1 \\ 1 - \frac{4ac}{(b+v)^2} &< 1\end{aligned}$$

This is satisfied by the wave speed constraints, therefore so long as the wave speed constraint above is satisfied then the node will be stable.

Fisher waves are the connection between two points though, without the two fixed points implicit in the equation its not brilliant.

Wave Solution following Fisher

$$\begin{aligned}\dot{\rho} &= a\rho + b\frac{\partial \rho}{\partial x} + c\frac{\partial^2 \rho}{\partial x^2} \\ \dot{\rho} &= -v\frac{\partial \rho}{\partial x} \\ a\rho + (b+v)\frac{\partial \rho}{\partial x} + c\frac{\partial^2 \rho}{\partial x^2} &= 0 \\ g &= -\frac{\partial \rho}{\partial x} \\ a\rho - (b+v)g + cg\frac{\partial g}{\partial \rho} &= 0 \\ \lim_{\rho \to 0} \frac{g}{\rho} &= u \\ a - (b+v)u + cu^2 &= 0\end{aligned}$$

Quadratic equation with solutions:

$$
\begin{aligned}
u &= \frac{(b+v) \pm \sqrt{(b+v)^2 - 4ac}}{2a} \\
(b+v)^2 &\geq 4ac \\
v \geq& 2\sqrt{ac} - b
\end{aligned}
$$

Which is the same result as the stability analysis.

Flat r and Flat u

$$
\begin{aligned}
a &= 0 \\
b &= -\tilde{u}
\end{aligned}
$$

This shows that there is a wave solution so long as the velocity is greater than the mutation probability.

Flat r, variable u

$$
\begin{aligned}
a &= -\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \\
b &= -\hat{u} \\
c &= \frac{\Delta x}{2}\hat{u}
\end{aligned}
$$

$$
v \geq 2\sqrt{-\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)\frac{\Delta x}{2}\hat{u}} + \hat{u}
$$

$$
\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \leq 0
$$

Find a family of solutions for this spectrum of mutation rates and see if there is a change in the way it behaves.

Variable r, variable u

$$
\begin{aligned}
a &= \left(r - \bar{r} - \Delta x\frac{\partial (ur)}{\partial x} + \frac{\Delta x^2}{2}\left(u\frac{\partial^2 r}{\partial x^2} - r\frac{\partial^2 u}{\partial x^2}\right)\right)\frac{1}{\bar{r}} \\
b &= \left(\frac{\Delta x^2}{2}u\frac{\partial r}{\partial x} - \Delta x ur\right)\frac{1}{\bar{r}} \\
c &= \frac{\Delta x^2}{2\bar{r}}ur
\end{aligned}
$$

Phase Portrait of Our System

$$\dot{\rho} = a\rho + b\frac{\partial \rho}{\partial x} + c\frac{\partial^2 \rho}{\partial x^2} \tag{20}$$

$$\begin{aligned} -vU' &= aU + bU' + cU'' \\ 0 &= aU + (b+v)U' + cU'' \end{aligned}$$

$$\begin{aligned} V &= U' \\ V' &= -\frac{(b+v)}{c}V - \frac{a}{c}U \end{aligned}$$