1 Non-Flat Mutation Rates INITIAL

$$
\begin{aligned}
\bar{r}\dot{x}_i &= u_i r_{i-1} x_{i-1} + x_i(r_I(1-u_{i+1}) - \bar{r}) \\
r &= \text{const.} \\
\dot{\rho}_i &= u_i \rho_{i-1} - \rho_i u_{i+1} \\
\dot{\rho}(x,t) &= u(x)\rho(x-\Delta x, t) - \rho(x,t)u(x+\Delta x, t) \\
&= u(x)\left(\rho(x) - \Delta x \frac{\partial \rho}{\partial x} + \Delta x^2 \frac{\partial^2 \rho}{\partial x^2}\right) - \rho(x,t)\left(u(x) + \Delta x \frac{\partial u}{\partial x} + \Delta x^2 \frac{\partial^2 u}{\partial x^2}\right) \\
\Delta x &= 1 \\
\dot{\rho}(x,t) &= -u\frac{\partial \rho}{\partial x} + u\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right)
\end{aligned}
$$

Collecting up some terms.

$$\dot{\rho} = -\frac{\partial}{\partial x}(\rho u) + u\frac{\partial^2 \rho}{\partial x^2} - \rho\frac{\partial^2 u}{\partial x^2}$$

Fourier transform convention

$$\tilde{\rho} = \int_{-\infty}^{\infty} \rho e^{-ikx} dx \tag{1}$$

$$\dot{\tilde{\rho}} = -\int \frac{\partial}{\partial x}(\rho u) e^{-ikx} dx + \int u\frac{\partial^2 \rho}{\partial x^2} e^{-ikx} dx - \int \rho \frac{\partial^2 u}{\partial x^2} e^{-ikx} dx$$

$$
\begin{aligned}
\int \frac{\partial}{\partial x}(\rho u) e^{-ikx} dx &= \left[e^{-ikx}\rho u\right]_{-\infty}^{\infty} - \int (-ik)\rho u e^{-ikx} dx \\
&= ik \int \rho u e^{-ikx} dx
\end{aligned}
$$

Applying the convolution theorem:

$$
\begin{aligned}
F(f \cdot g) &= F(f) * F(g) \\
f * g &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x-y) dy
\end{aligned}
$$

Need to prove these.
Applying this to the new differential equation.

$$\dot{\tilde{\rho}} = -ik(\mathcal{F}(\rho) * \mathcal{F}(u)) + (\mathcal{F}(u) * \mathcal{F}(\frac{\partial^2 \rho}{\partial x^2})) - (\mathcal{F}(\rho) * \mathcal{F}(\frac{\partial^2 u}{\partial x^2}))$$

Using what we know for the differentials and Fourier bits.

$$\dot{\tilde{\rho}} = -ik\left(\tilde{\rho} * \tilde{u}\right) + \left(\tilde{u} * (-k^2)\tilde{\rho}\right) - \left(\tilde{\rho} * (-k^2)\tilde{u}\right)$$

We can take out the factors of k^2.

$$\begin{aligned}\dot{\tilde{\rho}} &= -ik(\tilde{\rho} * \tilde{u}) - k^2\left((\tilde{u} * \tilde{\rho}) + (\tilde{\rho} * \tilde{u})\right) \\ &= (-ik - 2k^2)(\tilde{\rho} * \tilde{u})\end{aligned}$$

Convolution is commutative.
Chose a simple $u = \cos(\frac{\pi}{2M}x)$.

$$\begin{aligned}\tilde{u} &= \int \cos(\frac{\pi}{2M}x)e^{-ikx}dx \\ &= \frac{1}{2}\int (e^{\frac{i\pi}{2M}x} + e^{-\frac{i\pi}{2M}x})e^{-ikx})dx \\ &= \frac{1}{2}\left(\delta(k - \frac{\pi}{2M}) + \delta(k + \frac{\pi}{2M}\right)\end{aligned}$$

Convolution of a delta function just returns the same function with the varaible shifted.

$$f(x) * \delta(x \pm a) = f(x \pm a) \tag{2}$$

$$\begin{aligned}\tilde{\rho} * \tilde{u} &= \tilde{\rho} * \frac{1}{2}\left(\delta(k - \frac{\pi}{2M}) + \delta(k + \frac{\pi}{2M}\right) \\ &= \frac{1}{2}\left(\tilde{\rho}(k - \frac{\pi}{2M}) + \tilde{\rho}(k + \frac{\pi}{2M})\right)\end{aligned}$$

Combining everything in one.

$$\dot{\tilde{\rho}} = -\frac{(ik + k^2)}{2}\left(\tilde{\rho}(k - \frac{\pi}{2M}) + \tilde{\rho}(k + \frac{\pi}{2M})\right)$$

Improved

$$\begin{aligned}\dot{\rho} &= u\rho(x-\Delta x) - \rho u(x+\Delta x) \\ &= u\left(\rho(x) - \Delta x\frac{\partial \rho}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 \rho}{\partial x^2}\right) - \rho\left(u(x) + \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}\right) \\ &= -u\frac{\partial \rho}{\partial x}\Delta x + u\frac{\Delta x^2}{2}\frac{\partial^2 \rho}{\partial x^2} - \Delta x\rho\frac{\partial u}{\partial x} - \frac{\Delta x^2}{2}\rho\frac{\partial^2 u}{\partial x^2}\end{aligned}$$

Introduce a new scaled variable $\hat{u} = u(x) \cdot \Delta x$.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\frac{\partial \hat{u}}{\partial x} - \frac{\Delta x}{2}\rho\frac{\partial^2 \hat{u}}{\partial x^2}$$

This collapses down into our old equation with an additional term.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)$$

In the limit $\Delta x \to 0$ we get the normal advection equation

$$\lim_{\Delta x \to 0} \dot{\rho} = -\frac{\partial(\rho\hat{u})}{\partial x} \tag{3}$$

2 Numerically Solving the PDE

Start with the PDE and initial condition.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \tag{4}$$

$$\rho(x, 0) = \delta(x - x_0) \tag{5}$$

We discretise with $x_i = ih$ and $t_j = jk$.

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{\rho(x_i, t_{j+1}) - \rho(x_i, t_j)}{k} \\
\frac{\partial \rho}{\partial x} &= \frac{\rho(x_{i+1}, t) - \rho(x_i, t_j)}{h} \\
\frac{\partial^2 \rho}{\partial x^2} &= \frac{\rho(x_{i+1}, t_j) - 2\rho(x_i, t_j) + \rho(x_{i-1}, t_j)}{h^2}
\end{aligned}$$

Inserting these into the PDE

$$\begin{aligned}
\frac{\rho(x_i, t_{j+1}) - \rho(x_i, t_j)}{k} &= -\hat{u}\left(\frac{\rho(x_{i+1}, t_j) - \rho(x_i, t_j)}{h}\right) + \frac{\hat{u}\Delta x}{2}\left(\frac{\rho(x_{i+1}, t_j) - 2\rho(x_i, t_j) + \rho(x_{i-1}, t_j)}{h^2}\right) \\
&\quad - \rho(x_i, t_j) f(\hat{u}', \hat{u}'', \Delta x) \\
f(\hat{u}', \hat{u}'', \Delta x) &= \left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)
\end{aligned}$$

$$\begin{aligned}
\rho(x_i, t_{j+1}) &= \rho - \frac{\hat{u}k}{h}\left(\rho(x_{i+1}, t_j) - \rho\right) + \frac{\hat{u}\Delta x k}{2h^2}\left(\rho(x_{i+1}, t_j) - 2\rho + \rho(x_{i-1}, t_j)\right) - k\rho f \\
&= \rho\left(1 + \frac{\hat{u}k}{h} - \frac{\hat{u}\Delta x k}{h^2} - kf\right) + \frac{\hat{u}\Delta x k}{2h^2}\rho(x_{i-1}, t_j) + \rho(x_{i+1}, t_j)\left(\frac{\hat{u}\Delta x k}{2h^2} - \frac{\hat{u}k}{h}\right) \\
&= \rho\left(1 + \frac{\hat{u}k}{h}\left(1 - \frac{\Delta x}{h}\right) - k\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)\right) + \frac{\hat{u}\Delta x k}{2h^2}\rho(x_{i-1}, t_j) + \rho(x_{i+1}, t_j)\left(\frac{\Delta x}{2h} - 1\right)\frac{\hat{u}k}{h}
\end{aligned}$$

Can then solve this the normal computational way.

3 PDE Analysis

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \tag{6}$$

Re-arrange this to the stardard form.

$$\frac{\Delta x \hat{u}}{2}\frac{\partial^2 \rho}{\partial x^2} - \hat{u}\frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial t} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) = 0$$

Classification of this PDE is based on the coeffeicents of the double derivatives.

$$\Delta(x, y) = 0^2 - 0^2 = 0$$

As there are know terms with the cross derivative and no terms with the second differential of t, the determinate is zero and the equation is parabolic.