$$\dot{\rho}(x,t) = a(x)\,\partial_x \rho + b(x)\,\partial_x^2 \rho + c(x)\,\rho$$

1 Non-Flat Mutation Rates

$\dot{x} = a(x) \cdot x$

$$\begin{aligned}
\bar{r}\dot{x}_i &= u_i r_{i-1} x_{i-1} + x_i(r_I(1-u_{i+1}) - \bar{r}) \\
r &= \text{const.} \\
\dot{\rho}_i &= u_i \rho_{i-1} - \rho_i u_{i+1} \\
\dot{\rho}(x,t) &= u(x)\rho(x-\Delta x, t) - \rho(x,t)u(x+\Delta x, t) \\
&= u(x)\left(\rho(x) - \Delta x\frac{\partial \rho}{\partial x} + \Delta x^2 \frac{\partial^2 \rho}{\partial x^2}\right) - \rho(x,t)\left(u(x) + \Delta x \frac{\partial u}{\partial x} + \Delta x^2 \frac{\partial^2 u}{\partial x^2}\right) \\
\Delta x &= 1 \\
\dot{\rho}(x,t) &= -u\frac{\partial \rho}{\partial x} + u\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right)
\end{aligned}$$

$$\frac{\partial^2}{\partial x^2}(u\rho) - 2\cdot\underbrace{\frac{\partial \rho}{\partial x}\frac{\partial u}{\partial x}}$$

Collecting up some terms.

$$\dot{\rho} = -\frac{\partial}{\partial x}(\rho u) + u\frac{\partial^2 \rho}{\partial x^2} - \rho\frac{\partial^2 u}{\partial x^2}$$

Fourier transform convention

$$\tilde{\rho} = \int_{-\infty}^{\infty} \rho e^{-ikx} dx \tag{1}$$

$$\dot{\tilde{\rho}} = -\int \frac{\partial}{\partial x}(\rho u) e^{-ikx} dx + \int u \frac{\partial^2 \rho}{\partial x^2} e^{-ikx} dx - \int \rho \frac{\partial^2 u}{\partial x^2} e^{-ikx} dx$$

$$\begin{aligned}
\int \frac{\partial}{\partial x}(\rho u) e^{-ikx} dx &= \left[e^{-ikx}\rho u\right]_{-\infty}^{\infty} - \int (-ik)\rho u e^{-ikx} dx \\
&= ik \int \rho u e^{-ikx} dx
\end{aligned}$$

Applying the convolution theorem:

$$\begin{aligned}
F(f \cdot g) &= F(f) * F(g) \\
f * g &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y) f(x-y) dy
\end{aligned}$$

Need to prove these.
Applying this to the new differential equation.

$$\dot{\tilde{\rho}} = -ik(\mathcal{F}(\rho) * \mathcal{F}(u)) + (\mathcal{F}(u) * \mathcal{F}(\frac{\partial^2 \rho}{\partial x^2})) - (\mathcal{F}(\rho) * \mathcal{F}(\frac{\partial^2 u}{\partial x^2}))$$

Using what we know for the differentials and Fourier bits.

$(-k^2) \cdot \mathcal{F}(u)$

$$\dot{\tilde{\rho}} = -ik\left(\tilde{\rho} * \tilde{u}\right) + \left(\tilde{u} * (-k^2)\tilde{\rho}\right) - \left(\tilde{\rho} * (-k^2)\tilde{u}\right)$$

We can take out the factors of k^2.

$\tilde{u} = \delta(k)$

$$\begin{aligned}\dot{\tilde{\rho}} &= -ik(\tilde{\rho} * \tilde{u}) - k^2\left((\tilde{u} * \tilde{\rho}) + (\tilde{\rho} * \tilde{u})\right) \\ &= (-ik - 2k^2)\underbrace{(\tilde{\rho} * \tilde{u})}_{\tilde{\xi}}\end{aligned}$$

Convolution is commutative.
Chose a simple $u = \cos(\frac{\pi}{2M}x)$.

$$\begin{aligned}\tilde{u} &= \int \cos(\frac{\pi}{2M}x)e^{-ikx}dx \\ &= \frac{1}{2}\int (e^{\frac{i\pi}{2M}x} + e^{-\frac{i\pi}{2M}x})e^{-ikx})dx \\ &= \frac{1}{2}\left(\delta(k - \frac{\pi}{2M}) + \delta(k + \frac{\pi}{2M}\right)\end{aligned}$$

Convolution of a delta function just returns the same function with the varaible shifted.

$$f(x) * \delta(x \pm a) = f(x \pm a) \tag{2}$$

$$\begin{aligned}\tilde{\rho} * \tilde{u} &= \tilde{\rho} * \frac{1}{2}\left(\delta(k - \frac{\pi}{2M}) + \delta(k + \frac{\pi}{2M}\right) \\ &= \frac{1}{2}\left(\tilde{\rho}(k - \frac{\pi}{2M}) + \tilde{\rho}(k + \frac{\pi}{2M})\right)\end{aligned}$$

Combining everything in one.

$$\dot{\tilde{\rho}}_{k} = -\frac{(ik + k^2)}{2}\left(\tilde{\rho}(\underbrace{k - \frac{\pi}{2M}}_{k-1}) + \tilde{\rho}(\underbrace{k + \boxed{\frac{\pi}{2M}}}_{k+1})\right)$$

$$\underline{\rho} = \begin{pmatrix} \vdots \\ \rho_{k-2\pi/2M} \\ \rho_{k-\pi/2M} \\ \rho_k \\ \vdots \end{pmatrix}$$

$$\dot{\rho}_k = (\ldots)\ \rho_{k-1} \quad \rho_{k+1}$$

$$\dot{\underline{\rho}} = \begin{pmatrix} \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix} \underline{\rho}$$