1 Generalised Transition Rate Equation

$$T^{j\to i} = \left(u_i r_{i-1}\frac{n_{i-1}}{N} + (1-u_{i+1})\, r_i \frac{n_i}{N}\right)\frac{n_j}{\bar{r}} \tag{1}$$

When setting up the system, $u_0 = 0$ and $u_{N+1} = 0$ must be defined in order to stop cells mutating past the maximum number of mutations.

$\frac{n_j}{\bar{r}}$ is selected to die. $u_i r_{i-1}\frac{n_{i-1}}{N}$ is selected to mutate and gain a mutation. I.e. a cell with one less mutation is birthed with a mutation. $(1-u_{i+1})\, r_i \frac{n_i}{N}$ A n_i cell is born and doesn't obtain a mutation.

1.1 Diffusion with Drift Derivation

Taking the large N limit.

$$\dot{x}_i = \frac{1}{N}\left[\sum_j T^{j\to i} - T^{i\to j}\right]$$

Using the generalised transition rate equation:

$$\begin{aligned}
\bar{r}\dot{x}_i &= \sum_j u_i r_{i-1} x_{i-1} x_j - u_j r_{j-1} x_{j-1} x_i + x_j x_i \left[(1-u_{i+1})r_i - (1-u_{j+1})r_j\right] \\
\alpha &= u_i r_{i-1} \\
\beta &= u_j r_{j-1} \\
\gamma &= (1-u_{i+1})r_i - (1-u_{j+1})r_j) \\
\bar{r}\dot{x}_i &= \sum_j \alpha x_j x_{i-1} - \beta x_{j-1} x_i + \gamma x_j x_i
\end{aligned}$$

In a flat fitness and mutation landscape, away from the absorbing state $u_i, r_i = 1 \forall i$. From this $\alpha = 1, \beta = 1, \gamma = 0, \bar{r} = 1$.

$$\dot{\rho}_i = \sum_j \rho_j \rho_{i-1} - \rho_{j-1}\rho_i$$

We now want to expand this sum in order to simplify;

$$\begin{aligned}
\dot{\rho}_i &= \rho_0\rho_{i-1} + (\rho_1\rho_{i-1} - \rho_0\rho_i) + (\rho_2\rho_{i-1} - \rho_1\rho_i) + ... + (\rho_{N-1}\rho_{i-1} - \rho_{N-2}\rho_i) + (\rho_N\rho_{i-1} - \rho_{N-1}\rho_i) \\
&= \rho_0(\rho_{i-1} - \rho_i) + \rho_1(\rho_{i-1} - \rho_i) + ... + \rho_N\rho_{i-1} \\
&= \sum_k^{N-1} (\rho_{i-1} - \rho_i)\rho_k + \rho_N\rho_{i-1} \\
&= (\rho_{i-1} - \rho_i)(1 - \rho_N) + \rho_N\rho_{i-1} \\
&= \rho_{i-1} + (\rho_N - 1)\rho_i \\
\dot{\rho}_i &= \rho_{i-1} + (\rho_N - 1)\rho_i
\end{aligned}$$

In the large mutation limit $\rho_N \to 0$.

$$\dot{\rho}_i = \rho_{i-1} - \rho_i \tag{2}$$

We might be breaking our previous assumptions, but for $i = 0$:

$$\begin{aligned}
\dot{\rho}_0 &= 0 - \rho_0 \\
\rho_0 &= Ae^{-t} \\
A &= 1 \\
\rho_0 &= e^{-t}
\end{aligned}$$

This gives us the first step in solving generally.

$$\begin{aligned}
\dot{\rho}_1 &= \rho_0 - \rho_1 \\
&= e^{-t} - \rho_1 \\
\rho_1 &= te^{-t} \\
\dot{\rho}_2 &= \rho_1 - \rho_2 \\
\rho_2 &= \frac{1}{2}e^{-t}t^2
\end{aligned}$$

From this we can see that the general solution for ρ_i;

$$\rho_i = \frac{t^i}{i!}e^{-t} \tag{3}$$

2 Tobias' Model

Go from state n to $n+1$ with rate $a+b$. Go from state n to $n-1$ with rate a.

$$\begin{aligned}
\dot{p_i} &= -ap_i - (a+b)p_i + (a+b)p_{i-1} + ap_{i+1} \\
&= -(2a+b)p_i + (a+b)p_{i-1} + ap_{i+1}
\end{aligned}$$

Lets start with a generating function:

$$
\begin{aligned}
\Phi &= \sum_{n=0}^{\infty} z^n p_n(t) \\
\dot{\Phi} &= \sum_{n=0}^{\infty} z^n \dot{p}_n \\
&= \sum_{n=0}^{\infty} z^n \left((a+b)p_{n-1} - (2a+b)p_n + ap_{n+1}\right) \\
&= -(2a+b)\Phi + a\sum_{n=0}^{\infty} z^n p_{n+1} + (a+b)\sum_{n=0}^{\infty} z^n p_{n-1} \\
&= -(2a+b)\Phi + \frac{a}{z}\Phi + (a+b)z\Phi
\end{aligned}
$$

Need to get a differential out of this equation though?

3 Our Cancer Model Version 2

$$
\begin{aligned}
\dot{x}_i &= \left(u_i r_{i-1}\frac{n_{i-1}}{N} + (1-u_{i+1})\, r_i \frac{n_i}{N}\right)\frac{n_j}{\bar{r}} - \left(u_j r_{j-1}\frac{n_{j-1}}{N} + (1-u_{j+1})\, r_j \frac{n_j}{N}\right)\frac{n_i}{\bar{r}} \\
\bar{r}\dot{x}_i &= \left(u_i r_{i-1} x_{i-1} + (1-u_{i+1})\, r_i x_i\right) x_j - \left(u_j r_{j-1} x_{j-1} + (1-u_{j+1})\, r_j x_j\right) x_i
\end{aligned}
$$

$$
\bar{r}\dot{x}_0 = \left((1-u_1)\, r_0 x_0\right) x_j - \left(u_j r_{j-1} x_{j-1} + (1-u_{j+1})\, r_j x_j\right) x_0
$$

4 Diffusion equation attempt

$$
\bar{r}\dot{x}_i = \sum_{j\neq i} \left(u_i r_{i-1} x_{i-1} + (1-u_{i+1})\, r_i x_i\right) x_j - \left(u_j r_{j-1} x_{j-1} + (1-u_{j+1})\, r_j x_j\right) x_i
$$

Following this explicitly can reproduce the equations in the Ashcroft paper. Now attempt to simplify it:

$$
\begin{aligned}
\bar{r}\dot{x}_i &= \sum_{j\neq i} \left(u_i r_{i-1} x_{i-1} + \left(1-u_{i+1}\right) r_i x_i\right) x_j - \left(u_j r_{j-1} x_{j-1} + \left(1-u_{j+1}\right) r_j x_j\right) x_i \\
&= \left(u_i r_{i-1} x_{i-1} + \left(1-u_{i+1}\right) r_i x_i\right) \sum_{j\neq i} x_j - x_i \sum_{j\neq i} \left(u_j r_{j-1} x_{j-1} + \left(1-u_{j+1}\right) r_j x_j\right) \\
&= \left(u_i r_{i-1} x_{i-1} + \left(1-u_{i+1}\right) r_i x_i\right) \left(1-x_i\right) - x_i \left(\sum_{j\neq i} u_j r_{j-1} x_{j-1} + \sum_{j\neq i} (1-u_{j+1}) r_j x_j\right) \\
&= \left(u_i r_{i-1} x_{i-1} + \left(1-u_{i+1}\right) r_i x_i\right) \left(1-x_i\right) - x_i \left(\sum_{j\neq i} u_j r_{j-1} x_{j-1} + \sum_{j\neq i} r_j x_j - \sum_{j\neq i} r_j x_j u_{j+1}\right) \\
&= \left(u_i r_{i-1} x_{i-1} + \left(1-u_{i+1}\right) r_i x_i\right) \left(1-x_i\right) - x_i \left(\sum_{j\neq i} u_j r_{j-1} x_{j-1} + \left(\bar{r} - r_i x_i\right) - \sum_{j\neq i} r_j x_j u_{j+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\bar{u} &= \sum_{i=1}^{N} u_i r_{i-1} x_{i-1} \\
\mathbf{u} &= (0, u_1, ..., u_{N-1}, 0) \\
\sum_{j\neq i} u_j r_{j-1} x_{j-1} - \sum_{j\neq i} u_{j+1} r_j x_j &= \left(\bar{u} - u_i r_{i-1} x_{i-1}\right) - \sum_{m\neq i+1} u_m r_{j-1} x_{j-1} \\
&= \left(\bar{u} - u_i r_{i-1} x_{i-1}\right) - \left(\bar{u} - u_{i+1} r_i x_i\right) \\
&= u_{i+1} r_i x_i - u_i r_{i-1} x_{i-1}
\end{aligned}
$$

$$
\begin{aligned}
\bar{r}\dot{x}_i &= \left(u_i r_{i-1} x_{i-1} + \left(1-u_{i+1}\right) r_i x_i\right) \left(1-x_i\right) - x_i \left(u_{i+1} r_i x_i - u_i r_{i-1} x_{i-1} + \left(\bar{r} - r_i x_i\right)\right) \\
&= \left(u_i r_{i-1} x_{i-1}(1-x_i) + \left(1-u_{i+1}\right)\left(1-x_i\right) r_i x_i\right) - \big(u_{i+1} r_i x_i^2 - u_i r_{i-1} x_{i-1} x_i + x_i(\bar{r} - r_i x_i)\big) \\
&= \left(u_i r_{i-1} x_{i-1}(1-x_i) + \left(1-u_{i+1}\right)\left(1-x_i\right) r_i x_i\right) - u_{i+1} r_i x_i^2 + u_i r_{i-1} x_{i-1} x_i - x_i(\bar{r} - r_i x_i) \\
&= \left(u_i r_{i-1} x_{i-1}(1-x_i) + \left(1-u_{i+1}\right)\left(1-x_i\right) r_i x_i\right) - u_{i+1} r_i x_i^2 + u_i r_{i-1} x_{i-1} x_i - x_i \bar{r} + r_i x_i^2 \\
&= \left(u_i r_{i-1} x_{i-1} - u_i r_{i-1} x_{i-1} x_i + \left(1-u_{i+1}\right)\left(1-x_i\right) r_i x_i\right) - u_{i+1} r_i x_i^2 + u_i r_{i-1} x_{i-1} x_i - x_i \bar{r} + r_i x_i^2 \\
&= \left(u_i r_{i-1} x_{i-1} - u_i r_{i-1} x_{i-1} x_i + \left(1-u_{i+1} - x_i + u_{i+1} x_i\right) r_i x_i\right) - u_{i+1} r_i x_i^2 + u_i r_{i-1} x_{i-1} x_i - x_i \bar{r} + r_i x_i^2 \\
&= u_i r_{i-1} x_{i-1} - u_i r_{i-1} x_{i-1} x_i + r_i x_i - r_i x_i u_{i+1} - r_i x_i^2 + r_i x_i^2 u_{i+1} - u_{i+1} r_i x_i^2 + u_i r_{i-1} x_{i-1} x_i - x_i \bar{r} + r_i x_i^2 \\
&= u_i r_{i-1} x_{i-1} - u_i r_{i-1} x_{i-1} x_i + r_i x_i - r_i x_i u_{i+1} - r_i x_i^2 + u_i r_{i-1} x_{i-1} x_i - x_i \bar{r} + r_i x_i^2 \\
&= u_i r_{i-1} x_{i-1} - u_i r_{i-1} x_{i-1} x_i + r_i x_i - r_i x_i u_{i+1} + u_i r_{i-1} x_{i-1} x_i - x_i \bar{r} \\
&= u_i r_{i-1} x_{i-1} + r_i x_i - r_i x_i u_{i+1} - x_i \bar{r} \\
&= u_i r_{i-1} x_{i-1} + x_i \left(r_i \left(1-u_{i+1}\right) - \bar{r}\right)
\end{aligned}
$$

$$
\bar{r}\dot{x}_i = u_i r_{i-1} x_{i-1} + x_i \left(r_i \left(1-u_{i+1}\right) - \bar{r}\right) \tag{4}
$$

This reproduces the Ashcroft results as expected. So we can proceed.

$$\bar{r}\dot{x}_0 = x_0\left(r_0\left(1-u_1\right)-\bar{r}\right) \tag{5}$$
$$\bar{r}\dot{x}_i = u_i r_{i-1}x_{i-1} + x_i\left(r_i\left(1-u_{i+1}\right)-\bar{r}\right) \tag{6}$$
$$\bar{r}\dot{x}_N = u_N r_{N-1}x_{N-1} + x_N\left(r_N-\bar{r}\right) \tag{7}$$

Assume a constant r and u landscape and just sticking with the middle equation:

$$\begin{aligned}
r\dot{\rho}_i &= ur\rho_{i-1} + \rho_i(r(1-u)-r)\\
\dot{\rho}_i &= u\rho_{i-1} + \rho_i(1-u) - \rho_i\\
&= u(\rho_{i-1}-\rho_i)
\end{aligned}$$

We then make the change of variables $\rho_i \to \rho(t,x)$.

$$\begin{aligned}
\dot{\rho}(t,x) &= u\left(\rho(t,x-\Delta x)-\rho(t,x)\right)\\
&= u\left(\rho(t,x) - \Delta x\frac{\partial\rho}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2\rho}{\partial x^2} - \rho(t,x)\right)\\
&= \frac{u\Delta x^2}{2}\frac{\partial^2\rho}{\partial x^2} - u\Delta x\frac{\partial\rho}{\partial x}
\end{aligned}$$

This resembles the Fokker Plank Equation.

5 Solution of Master Equation with Poissonian

$$\begin{aligned}
\dot{\rho}_0 &= -u\rho_0\\
\rho_0 &= Ae^{-ut}\\
A &= 1
\end{aligned}$$

Then solve for ρ_1

$$\begin{aligned}
\dot{\rho}_1 &= u(\rho_0-\rho_1)\\
&= u\left(e^{-ut}-\rho_1\right)\\
\dot{\rho}_1 + u\rho_1 &= ue^{-ut}\\
\frac{d}{dt}\left(e^{ut}\rho\right) &= u\\
\rho_1 &= (ut)e^{-ut}
\end{aligned}$$

Doing a proof by induction and we arrive at the standard Poissonian;

$$\rho_n = \frac{(ut)^n}{n!}e^{-ut}$$

Solution with Fourier

$$\dot{\rho} = \frac{\tilde{u}\Delta x}{2}\frac{\partial^2}{\partial x^2}\rho - \tilde{u}\frac{\partial}{\partial x}\rho$$
$$\tilde{u} = u\Delta x$$

$$\tilde{\rho} = \int_{-\infty}^{\infty} \rho e^{-ikx} dx$$
$$\tilde{\rho}_x = \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x}\rho\right) e^{-ikx} dx$$
$$\tilde{\rho}_{xx} = \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial x^2}\rho\right) e^{-ikx} dx$$

Proof of the differentials Fourier transformed;

$$\begin{aligned}
\tilde{\rho}_x &= \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x}\rho\right) e^{-ikx} dx \\
&= \left[\rho e^{-ikx}\right]_{-\infty}^{\infty} + ik\tilde{\rho}
\end{aligned}$$

$$\begin{aligned}
\tilde{\rho}_{xx} &= \int_{-\infty}^{\infty} \left(\frac{\partial^2}{\partial x^2}\rho\right) e^{-ikx} dx \\
&= \left[\rho_x e^{-ikx}\right]_{-\infty}^{\infty} + ik\int e^{-ikx}\rho_x dx \\
&= \left[\rho_x e^{-ikx}\right]_{-\infty}^{\infty} + ik\left(\left[\rho e^{-ikx}\right]_{-\infty}^{\infty} + ik\tilde{\rho}\right) \\
&= \left[\rho_x e^{-ikx}\right]_{-\infty}^{\infty} + ik\left[\rho e^{-ikx}\right]_{-\infty}^{\infty} - k^2\tilde{\rho}
\end{aligned}$$

$$\begin{aligned}
\dot{\tilde{\rho}} &= -\frac{\tilde{u}\Delta x k^2}{2}\tilde{\rho} - ik\tilde{u}\tilde{\rho} \\
\tilde{\rho} &= A\exp\left(-\tilde{u}k\left(\frac{k\Delta x}{2} + i\right)t\right) \\
\tilde{\rho}(k,0) &= A \\
\tilde{\rho}(k,0) &= \int_{-\infty}^{\infty} \rho(x,0)e^{-ikx}dx \\
&= \int_{-\infty}^{\infty} \delta(x-x_0)e^{-ikx}dx \\
&= e^{-ikx_0}
\end{aligned}$$

$$\tilde{\rho} = e^{-ikx_0}\exp\left(-\tilde{u}k\left(\frac{k\Delta x}{2} + i\right)t\right) \tag{8}$$

Need to do the inverse Fourier transform to get back to concentration space.

$$\begin{aligned}
\rho &= \int_{-\infty}^{\infty} \exp\left(-\tilde{u}k\left(\frac{k\Delta x}{2} + i\right)t - ikx_0\right) e^{ikx} dk \\
&= \sqrt{\frac{2\pi}{\Delta x \tilde{u} t}} \exp\left(-\frac{(\tilde{u}t - x + x_0)^2}{2\Delta x \tilde{u} t}\right)
\end{aligned}$$

This is the solution, which gives a Gaussian shape propagating.

5.1 Sanity check with differentiation

$$\begin{aligned}
\rho &= \sqrt{\frac{2\pi}{\Delta x \tilde{u} t}} \exp\left(-\frac{(\tilde{u}t - x + x_0)^2}{2\Delta x \tilde{u} t}\right) \\
&= \frac{C}{\sqrt{t}} \exp\left(-\frac{(\tilde{u}t - x + x_0)^2}{Bt}\right) \\
&= \frac{C}{\sqrt{t}} \exp\left(-\frac{(\tilde{u}t)^2 + (x_0 - x)^2 + 2\tilde{u}t(x_0 - x)}{Bt}\right) \\
&= \frac{C}{\sqrt{t}} \exp\left(-\left[\frac{\tilde{u}^2 t}{B} + \frac{(x_0 - x)^2}{Bt} + \frac{2\tilde{u}(x_0 - x)}{B}\right]\right) \\
&= \frac{C}{\sqrt{t}} e^{-\alpha} e^{-\beta} e^{-\gamma}
\end{aligned}$$

$$\begin{aligned}
\partial_t \rho &= \rho\left(-\frac{1}{2t} - \frac{\tilde{u}^2}{B} + \frac{(x_0 - x)^2}{Bt^2}\right) \\
\partial_x \rho &= \rho\left(\frac{2(x_0 - x)}{Bt} + \frac{2\tilde{u}}{Bt}\right) \\
\partial_{xx} \rho &= \rho\left(\left[\frac{2(x_0 - x)}{Bt} + \frac{2\tilde{u}}{Bt}\right]^2 - \frac{2}{Bt}\right)
\end{aligned}$$

$$\dot{\rho} = \frac{\tilde{u}\Delta x}{2}\partial_{xx}\rho - \tilde{u}\partial_x \rho$$

$$\begin{aligned}
\left(-\frac{1}{2t} - \frac{\tilde{u}^2}{B} + \frac{(x_0 - x)^2}{Bt^2}\right) &= \frac{\tilde{u}\Delta x}{2}\left(\frac{2(x_0 - x)}{Bt} + \frac{2\tilde{u}}{Bt}\right) - \tilde{u}\left(\left[\frac{2(x_0 - x)}{Bt} + \frac{2\tilde{u}}{Bt}\right]^2 - \frac{2}{Bt}\right) \\
\left(-\frac{1}{2t} - \frac{\tilde{u}^2}{B} + \frac{(\tilde{x})^2}{Bt^2}\right) &= \frac{B}{4}\left(\frac{2(\tilde{x})}{Bt} + \frac{2\tilde{u}}{Bt}\right) - \tilde{u}\left(\left[\frac{2(\tilde{x})}{Bt} + \frac{2\tilde{u}}{Bt}\right]^2 - \frac{2}{Bt}\right) \\
\left(-\frac{1}{2t} - \frac{\tilde{u}^2}{B} + \frac{(\tilde{x})^2}{Bt^2}\right) &= \frac{1}{2t}(\tilde{x} + \tilde{u}) - \tilde{u}\left(\frac{4}{B^2t^2}(\tilde{x}^2 + \tilde{u}^2 + 2\tilde{u}\tilde{x}) - \frac{2}{Bt}\right)
\end{aligned}$$

It doesn't work! something has gone wrong somewhere.

Constraints on the Solution

From the differentiation not working, can imply that this imposes some constraints on the constants.

The concentration must also also be conserved;

$$\int_0^\infty \rho(x,t)dx = 1 \quad \forall t \tag{9}$$

$$\dot{\rho} = \frac{\tilde{u}\Delta x}{2}\frac{\partial^2}{\partial x^2}\rho - \tilde{u}\frac{\partial}{\partial x}\rho$$

From the Fourier analysis we assume a solution;

$$\rho = \frac{C}{\sqrt{t}}\exp\left(-\frac{(At-(x-x_0)^2}{Bt}\right) \tag{10}$$

Doing the integral thing

$$\begin{aligned}\frac{1}{2}C\sqrt{\pi B}\left(\text{Erf}\left(\frac{x_0+At}{\sqrt{B}\sqrt{t}}\right)+1\right) &= 1\\ x_0 &= 0\\ C\sqrt{\pi B}\left(\text{Erf}\left(\frac{A\sqrt{t}}{\sqrt{B}}\right)+1\right) &= 2\end{aligned}$$

The error function tends to 1 quickly, so even after a small amount of time, its reasonable to replace it with 1.

$$C\sqrt{B} = \frac{1}{\sqrt{\pi}} \tag{11}$$