1 Non-Flat Mutation Rates INITIAL

$$\begin{aligned}
\bar{r}\dot{x}_i &= u_i r_{i-1} x_{i-1} + x_i r_I (1 - u_{i+1}) - \bar{r}) \\
r &= \text{const.} \\
\dot{\rho}_i &= u_i \rho_{i-1} - \rho_i u_{i+1} \\
\dot{\rho}(x,t) &= u(x)\rho(x-\Delta x, t) - \rho(x,t)u(x+\Delta x, t) \\
&= u(x)\left(\rho(x) - \Delta x \frac{\partial \rho}{\partial x} + \Delta x^2 \frac{\partial^2 \rho}{\partial x^2}\right) - \rho(x,t)\left(u(x) + \Delta x \frac{\partial u}{\partial x} + \Delta x^2 \frac{\partial^2 u}{\partial x^2}\right) \\
\Delta x &= 1 \\
\dot{\rho}(x,t) &= -u\frac{\partial \rho}{\partial x} + u\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right)
\end{aligned}$$

Collecting up some terms.

$$\dot{\rho} = -\frac{\partial}{\partial x}(\rho u) + u\frac{\partial^2 \rho}{\partial x^2} - \rho\frac{\partial^2 u}{\partial x^2}$$

Fourier transform convention

$$\tilde{\rho} = \int_{-\infty}^{\infty} \rho e^{-ikx} dx \tag{1}$$

$$\dot{\tilde{\rho}} = -\int \frac{\partial}{\partial x}(\rho u)e^{-ikx}dx + \int u\frac{\partial^2 \rho}{\partial x^2}e^{-ikx}dx - \int \rho\frac{\partial^2 u}{\partial x^2}e^{-ikx}dx$$

$$\begin{aligned}
\int \frac{\partial}{\partial x}(\rho u)e^{-ikx}dx &= \left[e^{-ikx}\rho u\right]_{-\infty}^{\infty} - \int (-ik)\rho u e^{-ikx}dx \\
&= ik\int \rho u e^{-ikx}dx
\end{aligned}$$

Applying the convolution theorem:

$$\begin{aligned}
F(f \cdot g) &= F(f) * F(g) \\
f * g &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)f(x-y)dy
\end{aligned}$$

Need to prove these.
Applying this to the new differential equation.

$$\dot{\tilde{\rho}} = -ik(\mathcal{F}(\rho) * \mathcal{F}(u)) + (\mathcal{F}(u) * \mathcal{F}(\frac{\partial^2 \rho}{\partial x^2})) - (\mathcal{F}(\rho) * \mathcal{F}(\frac{\partial^2 u}{\partial x^2}))$$

Using what we know for the differentials and Fourier bits.

$$\dot{\tilde{\rho}} = -ik\,(\tilde{\rho} * \tilde{u}) + \left(\tilde{u} * (-k^2)\tilde{\rho}\right) - \left(\tilde{\rho} * (-k^2)\tilde{u}\right)$$

We can take out the factors of k^2.

$$\begin{aligned}\dot{\tilde{\rho}} &= -ik(\tilde{\rho} * \tilde{u}) - k^2\left((\tilde{u} * \tilde{\rho}) + (\tilde{\rho} * \tilde{u})\right) \\ &= (-ik - 2k^2)(\tilde{\rho} * \tilde{u})\end{aligned}$$

Convolution is commutative.
Chose a simple $u = \cos(\frac{\pi}{2M}x)$.

$$\begin{aligned}\tilde{u} &= \int \cos(\frac{\pi}{2M}x)e^{-ikx}dx \\ &= \frac{1}{2}\int (e^{\frac{i\pi}{2M}x} + e^{-\frac{i\pi}{2M}x})e^{-ikx})dx \\ &= \frac{1}{2}\left(\delta(k - \frac{\pi}{2M}) + \delta(k + \frac{\pi}{2M}\right)\end{aligned}$$

Convolution of a delta function just returns the same function with the varaible shifted.

$$f(x) * \delta(x \pm a) = f(x \pm a) \tag{2}$$

$$\begin{aligned}\tilde{\rho} * \tilde{u} &= \tilde{\rho} * \frac{1}{2}\left(\delta(k - \frac{\pi}{2M}) + \delta(k + \frac{\pi}{2M}\right) \\ &= \frac{1}{2}\left(\tilde{\rho}(k - \frac{\pi}{2M}) + \tilde{\rho}(k + \frac{\pi}{2M})\right)\end{aligned}$$

Combining everything in one.

$$\dot{\tilde{\rho}} = -\frac{(ik + k^2)}{2}\left(\tilde{\rho}(k - \frac{\pi}{2M}) + \tilde{\rho}(k + \frac{\pi}{2M})\right)$$

Improved

$$
\begin{aligned}
\dot{\rho} &= u\rho(x-\Delta x) - \rho u(x+\Delta x) \\
&= u\left(\rho(x) - \Delta x\frac{\partial \rho}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 \rho}{\partial x^2}\right) - \rho\left(u(x) + \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}\right) \\
&= -u\frac{\partial \rho}{\partial x}\Delta x + u\frac{\Delta x^2}{2}\frac{\partial^2 \rho}{\partial x^2} - \Delta x\rho\frac{\partial u}{\partial x} - \frac{\Delta x^2}{2}\rho\frac{\partial^2 u}{\partial x^2}
\end{aligned}
$$

Introduce a new scaled variable $\hat{u} = u(x) \cdot \Delta x$.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\frac{\partial \hat{u}}{\partial x} - \frac{\Delta x}{2}\rho\frac{\partial^2 \hat{u}}{\partial x^2}$$

This collapses down into our old equation with an additional term.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)$$

In the limit $\Delta x \to 0$ we get the normal advection equation

$$\lim_{\Delta x \to 0} \dot{\rho} = -\frac{\partial(\rho\hat{u})}{\partial x} \tag{3}$$

2 Numerically Solving the PDE

Start with the PDE and initial condition.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \tag{4}$$

$$\rho(x,0) = \delta(x - x_0) \tag{5}$$

We discretise with $x_i = ih$ and $t_j = jk$.

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= \frac{\rho(x_i, t_{j+1}) - \rho(x_i, t_j)}{k} \\
\frac{\partial \rho}{\partial x} &= \frac{\rho(x_{i+1}, t) - \rho(x_i, t_j)}{h} \\
\frac{\partial^2 \rho}{\partial x^2} &= \frac{\rho(x_{i+1}, t_j) - 2\rho(x_i, t_j) + \rho(x_{i-1}, t_j)}{h^2}
\end{aligned}$$

Inserting these into the PDE

$$\begin{aligned}
\frac{\rho(x_i, t_{j+1}) - \rho(x_i, t_j)}{k} &= -\hat{u}\left(\frac{\rho(x_{i+1}, t_j) - \rho(x_i, t_j)}{h}\right) + \frac{\hat{u}\Delta x}{2}\left(\frac{\rho(x_{i+1}, t_j) - 2\rho(x_i, t_j) + \rho(x_{i-1}, t_j)}{h^2}\right) \\
&\quad - \rho(x_i, t_j) f(\hat{u}', \hat{u}'', \Delta x) \\
f(\hat{u}', \hat{u}'', \Delta x) &= \left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)
\end{aligned}$$

$$\begin{aligned}
\rho(x_i, t_{j+1}) &= \rho - \frac{\hat{u}k}{h}\left(\rho(x_{i+1}, t_j) - \rho\right) + \frac{\hat{u}\Delta x k}{2h^2}\left(\rho(x_{i+1}, t_j) - 2\rho + \rho(x_{i-1}, t_j)\right) - k\rho f \\
&= \rho\left(1 + \frac{\hat{u}k}{h} - \frac{\hat{u}\Delta x k}{h^2} - kf\right) + \frac{\hat{u}\Delta x k}{2h^2}\rho(x_{i-1}, t_j) + \rho(x_{i+1}, t_j)\left(\frac{\hat{u}\Delta x k}{2h^2} - \frac{\hat{u}k}{h}\right) \\
&= \rho\left(1 + \frac{\hat{u}k}{h}\left(1 - \frac{\Delta x}{h}\right) - k\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right)\right) + \frac{\hat{u}\Delta x k}{2h^2}\rho(x_{i-1}, t_j) + \rho(x_{i+1}, t_j)\left(\frac{\Delta x}{2h} - 1\right)\frac{\hat{u}k}{h}
\end{aligned}$$

Can then solve this the normal computational way. $\alpha = \frac{\hat{u}k}{h}$

$$\begin{aligned}
\rho(x_i, t_{j+1}) &= \rho\left(1 + \alpha\left(1 - \frac{\Delta x}{h}\right) - k\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right)\right) \\
&\quad + \frac{\Delta x}{2h}\alpha\rho(x_{i-1}, t_j) \\
&\quad + \alpha\left(\frac{\Delta x}{2h} - 1\right)\rho(x_{i+1}, t_j)
\end{aligned}$$

3 PDE Analysis

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \tag{6}$$

Re-arrange this to the stardard form.

$$\frac{\Delta x \hat{u}}{2}\frac{\partial^2 \rho}{\partial x^2} - \hat{u}\frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial t} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) = 0$$

Classification of this PDE is based on the coeffeicents of the double derivatives.

$$\Delta(x, y) = 0^2 - 0^2 = 0$$

As there are know terms with the cross derivative and no terms with the second differential of t, the determinate is zero and the equation is parabolic.

4 Mutation Function analysis

We are free to chose a function for the mutation rate.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x}{2}\hat{u}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2}\right) \tag{7}$$

The degree of freedom comes in the last term, so we define a function for ease of notation.

$$\mathcal{U}(x, \hat{u}, \hat{u}^{(n)}) = \frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2} \tag{8}$$

4.1 $\mathcal{U} = \hat{u}$

By setting the function equal to the mutation function then it can be taken as a common factor from all the terms in the PDE.

$$\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2} = \hat{u}$$

$\hat{u}$ is just a function of x, therefore this is a linear, second order differential equation with auxillary equation;

$$\lambda = \frac{-1 \pm \sqrt{1 + 2\Delta x}}{\Delta x},$$

therefore, is has solutions

$$\hat{u} = A\exp\left(\left(-\frac{1}{\Delta x} + \frac{\sqrt{1+2\Delta x}}{\Delta x}\right)x\right) + B\exp\left(-\left(\frac{1}{\Delta x} + \frac{\sqrt{1+2\Delta x}}{\Delta x}\right)x\right)$$

A and B are constants that determine the scale of mutation probabilities.

4.2 $\mathcal{U}$ = polynomial

$$\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2} = \sum_{-\infty}^{\infty} c_n x^n$$

$$v = \frac{\partial \hat{u}}{\partial x}$$
$$v + \frac{\Delta x}{2}\frac{\partial v}{\partial x} = \sum_{-\infty}^{\infty} c_n x^n$$

This has complimentary function (correct phrase?)

$$v = Be^{-\frac{2}{\Delta x}x}$$

Trial form for the particular integral;

$$\begin{aligned} v(x) &= Be^{-\frac{2}{\Delta x}x} + \sum c_\alpha x^\alpha \\ v' &= \frac{-2B}{\Delta x} e^{-\frac{2}{\Delta x}x} + \sum \alpha c_\alpha x^{\alpha-1} \end{aligned}$$

$$\begin{aligned} Be^{-\frac{2}{\Delta x}x} + \sum c_\alpha x^\alpha + \frac{\Delta x}{2}\left(\frac{-2B}{\Delta x} e^{-\frac{2}{\Delta x}x} + \sum \alpha c_\alpha x^{\alpha-1}\right) &= \sum_{-\infty}^{\infty} c_n x^n \\ \sum c_\alpha x^\alpha + \frac{\Delta x}{2}\sum \alpha c_\alpha x^{\alpha-1} &= \sum_{-\infty}^{\infty} c_n x^n \\ \sum c_\alpha x^\alpha + \frac{\Delta x}{2}\sum (\alpha+1) c_{\alpha+1} x^{\alpha} &= \sum_{-\infty}^{\infty} c_n x^n \end{aligned}$$

From orthogonality of the powers

$$c_\alpha + \frac{\Delta x}{2}(\alpha+1)c_{\alpha+1} = c_n$$

OR

$$\begin{aligned} c_\alpha + \frac{\Delta x}{2}\frac{\alpha c_\alpha}{x} &= c_n \\ c_\alpha &= \frac{c_n}{1 + \frac{\Delta x \alpha}{2x}} \end{aligned}$$

So the full solution is

$$v(x) = Be^{\frac{-2x}{\Delta x}} + \sum \frac{c_n x^n}{1 + \frac{\Delta x n}{2x}}$$

Unwinding the substitution

$$\begin{aligned} \frac{\partial \hat{u}}{\partial x} &= Be^{\frac{-2x}{\Delta x}} + \sum \frac{c_n x^n}{1 + \frac{\Delta x n}{2x}} \\ \hat{u} &= -\frac{B\Delta x}{2} e^{\frac{-2x}{\Delta x}} + \int \sum \frac{2c_n x^{n+1}}{2x + \Delta x n} \mathrm{d}x \end{aligned}$$

Testing by differentiation and subbing back in.

$$\begin{aligned}
\hat{u} &= -\frac{B\Delta x}{2}e^{\frac{-2x}{\Delta x}} + \int \sum \frac{2c_n x^{n+1}}{2x+\Delta xn}\mathrm{d}x \\
\hat{u}' &= Be^{-\frac{2x}{\Delta x}} + \sum \frac{2c_n x^{n+1}}{2x+\Delta xn} \\
\hat{u}'' &= \frac{-2B}{\Delta x}e^{-\frac{2x}{\Delta x}} + \sum \left(\frac{2c_n(n+1)x^n}{(2x+\Delta xn)} - \frac{4c_n x^{n+1}}{(2x+\Delta xn)^2}\right) \\
\hat{u}' + \frac{\Delta x}{2}\hat{u}'' &= \sum \frac{2c_n x^{n+1}}{2x+\Delta xn} + \frac{\Delta x}{2}\left(\sum \left(\frac{2c_n(n+1)x^n}{(2x+\Delta xn)} - \frac{4c_n x^{n+1}}{(2x+\Delta xn)^2}\right)\right)
\end{aligned}$$

In the limit of $\Delta x \to 0$ this is correct.

Example

$$\begin{aligned}
\frac{\partial \hat{u}}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 \hat{u}}{\partial x^2} &= \sum_{-\infty}^{\infty} c_n x^n \\
\sum_{-\infty}^{\infty} c_n x^n &= x \\
c_n &= \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases} \\
\hat{u} &= -\frac{B\Delta x}{2}e^{\frac{-2x}{\Delta x}} + \int \sum \frac{2c_n x^{n+1}}{2x+\Delta xn}\mathrm{d}x \\
&= -\frac{B\Delta x}{2}e^{\frac{-2x}{\Delta x}} + \int \left(\frac{2x^2}{2x+\Delta x}\right)
\end{aligned}$$

4.3 FKKP Methods

"Wave of Advance of Advantageous Genes" Ronald Fisher.

$$\dot{\rho} = -\hat{u}\frac{\partial \rho}{\partial x} + \frac{\Delta x \hat{u}}{2}\frac{\partial^2 \rho}{\partial x^2} - \rho\left(\frac{\partial u}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}\right)$$

We want a wave solution, travelling at speed v, therefore $\frac{\partial \rho}{\partial t} = -v\frac{\partial \rho}{\partial x}$

$$\frac{\Delta x \hat{u}}{2}\rho^{''} + (v - \hat{u})\rho^{'} - \rho\left(u^{'} + \frac{\Delta x}{2}u^{''}\right) = 0 \tag{9}$$

There is no explicit x dependence in the equation, therefore we can write the concentration as the derivative of another function.

$$\begin{aligned} g &= -\frac{d\rho}{dx} \\ \frac{\partial g}{\partial x} &= -\frac{\partial^2 \rho}{\partial x^2} \\ \frac{\partial^2 \rho}{\partial x^2} &= g\frac{\partial g}{\partial \rho} \end{aligned}$$

$$\frac{\Delta x \hat{u}}{2}g\frac{\partial g}{\partial \rho} + (v - \hat{u})(-g) - \rho\left(\frac{\partial u}{\partial x} + \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{10}$$

$$\lim_{\rho \to 0}\frac{g}{\rho} = \omega$$

$$\frac{\partial g}{\partial \rho} = \omega$$

$$\frac{\hat{u}\Delta x}{2}\omega^2 + (v - \hat{u})(-\omega) - \left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) = 0 \tag{11}$$

By examining the determinant of this quadratic equation we can arrive at an equation for v.

$$\begin{aligned} (v - \hat{u})^2 - 4\frac{\hat{u}\Delta x}{2}\left(-\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right)\right) &\geq 0 \\ v^2 - 2v\hat{u} + \hat{u}^2 + 2\hat{u}\Delta x\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) &\geq 0 \\ v^2 - 2v\hat{u} + \hat{u}^2 + 2\hat{u}\Delta x\left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) &= C \\ C &\geq 0 \end{aligned}$$

$$v^2 - 2v\hat{u} + \hat{u}^2 + 2\hat{u}\Delta x \left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) - C = 0$$

$$\begin{aligned} v &= 2\hat{u} \pm \sqrt{4\hat{u}^2 - 4\left(\hat{u}^2 + 2\hat{u}\Delta x \left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) - C\right)} \\ &= 2\hat{u} \pm \sqrt{-8\hat{u}\Delta x \left(\hat{u}' + \frac{\Delta x \hat{u}''}{2}\right) + 4C} \end{aligned}$$

As the velocity is a physical quantity, this gives us some constraints on the function inside the square brackets.

$$C - 2\hat{u}\Delta x \left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) \geq 0 \tag{12}$$

This gives us an ODE for $\hat{u}$:

$$\hat{u}\Delta x \left(\hat{u}' + \frac{\Delta x}{2}\hat{u}''\right) + D = 0$$

Where D is just the redefining of the constant.
This is very non linear and doesn't have an analytic solution.
NEED TO SOLVE IT NUMERICALLY.

5 Varying u(x) and r(x)

Starting from

$$\begin{aligned} \bar{r}\dot{x}_i &= u_i r_{i-1} x_{i-1} + x_i(r_i(1 - u_{i+1}) - \bar{r})) \qquad (13) \\ &= u_i r_{i-1} x_{i-1} + x_i r_i - x_i r_i u_{i+1} - x_i \bar{r} \qquad (14) \\ & \qquad (15) \end{aligned}$$

taking this in the continuous limit we get

$$\bar{r}\rho(\dot{x}, t) = u(x)r(x - \Delta x)\rho(x - \Delta x, t) + \rho(x,t)r(x) - \rho(x,t)r(x)u(x + \Delta x) \tag{16}$$

which leads to (after taylor expansion)

$$\begin{aligned} \bar{r}\dot{\rho}(x,t) &= u(x)[r(x) - \Delta x\frac{\partial r}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 r}{\partial^2 x}][\rho(x,t) - \Delta x\frac{\partial \rho}{\partial x} + \frac{(\Delta x)^2}{2}\frac{\partial^2 \rho}{\partial^2 x}] + \rho(x,t)r(x) \\ &\quad - \rho(x,t)r(x)[u(x) + \partial_x u(x) + \frac{(\Delta x)^2}{2}\partial_x^2 u(x)] - \rho(x,t)\bar{r} \\ &= u(x)[r(x)\rho(x,t) - \rho(x,t)\Delta x\partial_x r + \frac{(\Delta x)^2}{2}\rho\partial_x^2 r - r(x)\Delta\partial_x\rho + (\Delta x)^2\partial_x r\partial_x\rho + r(x)\frac{(\Delta x)^2}{2}\partial_x^2\rho] \\ &\quad + \rho(x,t)r(x) - \rho(x,t)u(x)r(x) - \Delta x\rho(x,t)r(x)\partial_x u(x) - \frac{(\Delta x)^2}{2}\rho(x,t)r(x)\partial_x^2 u(x) - \rho(x,t)\bar{r} \\ \bar{r}\dot{\rho} &= -u\rho\Delta x\partial_x r - ur\Delta x\partial)x\rho - \Delta x\rho r\partial_x u + \frac{(\Delta x)^2}{2}(u\rho\partial_x^2 r + ur\partial_x^2\rho - \rho r\partial_x^2 u + 2u\partial_x r\partial_x\rho) + \rho(r - \bar{r}) \\ &= -\Delta x\partial_x(ur\rho) + \rho(r - \bar{r}) + \frac{(\Delta x)^2}{2}\partial_x^2(ur\rho) - 2r\rho\partial_x^2 u - 2\rho\partial_x u\partial_x r - 2r\partial_x u\partial_x\rho \end{aligned}$$